# Analysis of Electrical Circuits and Mechanical Systems with Fractional-order Elements: A Systems Theory Approach

Dissertation-I

By

**Yogesh M. Gulekar**

Under the Guidance of

**Prof. M. D. Patil**

# Outline

1. Introduction.
2. What is Fractional Calculus.
3. Fractional Order (FO) Elements.
4. Circuit With FO Elements.
5. Fractional Order Transfer Function.
6. Stability Of FO System.
7. Problem Definition.

# Introduction

- Integer Order System and its limitations.
- Fractional Order System and its approach.
- Basics of Fractional Order System.
- FO elements and its applications.
- Problem Definition.
- Conclusion.

## Fractional Calculus

Real order generalization of fractional calculus

$$D^{\alpha} = \begin{cases} \frac{d^{\alpha}}{dt^{\alpha}} & \alpha > 0 \\ 1 & \alpha = 0 \\ \int_a^t (d\tau)^{\alpha} & \alpha < 0 \end{cases} \tag{1}$$

with $\alpha \in \mathcal{R}$.

# Riemann-Liouville:

**Integral:**

$$J_c{}^{\alpha} f(t) = \frac{1}{\Gamma\alpha} \int \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{2}$$

**Derivative:**

$$D^{\alpha} f(t) = \frac{d^m}{dt^m}[\frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau], m \in Z^+, m-1 < \alpha \leq m. \tag{3}$$

## Grunwald-Letnikov:

**Integral:**

$$D^{-\alpha} = \lim_{h \longrightarrow 0} h^{\alpha} \sum_{m=0}^{(t-a)/h} \frac{\Gamma(\alpha + m)}{m!\Gamma(\alpha)} f(t - mh), \tag{4}$$

**Derivative:**

$$D^{\alpha} = \lim_{h \longrightarrow 0} \frac{1}{h^{\alpha}} \sum_{m=0}^{(t-a)/h} (-1)^m \frac{\Gamma(\alpha + 1)}{m!\Gamma(\alpha - m + 1)} f(t - mh), \tag{5}$$

# Caputo:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \tag{6}$$

- Inductor or lossy coil have hysteresis and eddy current losses.
- Capacitors produces dielectric losses.
- FO inductors or FO capacitors yield a more accurate description.

# FO Elements

Schematic of FO Capacitor

FOmodelRpztnOfossyC.pdf

The voltage-current relationships for an FO capacitor are

$$i_c(t) = C_\alpha \frac{d^\alpha}{dt^\alpha} v_c(t), \tag{7}$$

or

$$v_c(t) = \frac{1}{C_\alpha} \; {}_0{J_t}^\alpha i_c(t), \tag{8}$$

where the constant $0 < \alpha < 1, \alpha \in \mathbf{R}$ is a measure of the losses in the capacitor.

$$j^{-\alpha} = \cos(\alpha\pi/2) - j\sin(\alpha\pi/2) = e^{-j\alpha\pi/2} \tag{9}$$

## FO Elements

Phase Relationship Between V and I for FO capacitor

PhaseRelationBtwnVandIforFOC.pdf

# FO Elements

Schematic of FO Inductor

FOmodelofL.pdf

For an FO inductor we have,

$$V_L(t) = L_\beta \frac{d^\beta}{dt^\beta} i_L(t) \tag{10}$$

where the fractional power $\beta$ is related to the phenomenon of proximity effect. In frequency domain,

$$V_L(s) = s^\beta L_\beta I_L(s) \tag{11}$$

$$j = \cos(\pi/2) + j\sin(\pi/2) = e^{j\pi/2} \tag{12}$$

$$j^\beta = \cos(\beta\pi/2) + j\sin(\beta\pi/2) = e^{j\beta\pi/2} \tag{13}$$

# FO Elements

Phase Relationship Between V and I for FO inductor

PhasorDiagForFOL.pdf

# Stability of FO System

- Fractional Domain
- Mapping of S-plane to F-plane.
- $\alpha$ increases the stable F-domain regions decreases.
- Stability decreases for $\alpha$ >2

## Stability of FO

stability2.pdf

## Problem Definition

R-Lckt.pdf

# Transfer Function of FO

$$\frac{I(s)}{V(s)} = \frac{\frac{1}{L_\beta}}{S^\beta + \frac{R}{L_\beta}}. \tag{14}$$

# Future Work

- Analysis of electrical circuits and mechanical systems with fractional-order elements.
- Development of various linear models.
- Stability analysis using Riemann sheet concept.
- Analysis of controllability, observability, reacheability, and solution of the state-equation.
- Design of linear compensators and controllers.
- Development of equivalent relationship between FO electrical and mechanical systems.

# Thank You...