Appendices

1 Proofs

1.1 Proposition 1

$\rightarrow$

Proof. First, consider the relevant objects from the Anscombe-Aumann expected utility representation theorem, for which the **?** theorem is an extension of (referred to as GS from here on out):

$$L = \left\{ p : Y \mapsto [0,1] \;\middle|\; \# \; \{y|p(y) > 0\} < \infty, \sum_{y \in Y} p(y) = 1 \right\}.$$

Where L is the choice set from the vNM-EU model, and F is from the GS-EU set-up. First, recall the GS-EU theorem:

Theorem 1. *$\succsim$ satisfies AA1, AA2, C-Independence, AA4, AA5, and Uncertainty aversion if and only if there exists a closed and convex set of probabilities on S, $C \subset \Delta(S)$, and a non-constant function $U : Y \to \mathbb{R}$ such that, for every $f, f^* \in F$,*

$$f \succsim f^* \Longleftrightarrow \min_{\lambda \in \Delta(S)} \int_S (\mathbb{E}_{p(s)} u) d\lambda \geq \min_{\lambda \in \Delta(S)} \int_S (\mathbb{E}_{p^*(s)} u) d\lambda.$$

$$\Longleftrightarrow \min_{\lambda \in \Delta(S)} \int_S \int_0^{\bar{y}} U(y)p(s)(y) dy d\lambda \geq \min_{\lambda \in \Delta(S)} \int_S \int_0^{\bar{y}} U(y)p^*(s)(y) dy d\lambda$$

For simplicity, consider the discrete version of the implication of this expected utility representation result. Namely,

$$\Longleftrightarrow \min_{\lambda \in \Delta(S)} \sum_S \sum_{y \in Y} U(y)p(s)(y)\lambda(s) \geq \min_{\lambda \in \Delta(S)} \sum_S \sum_{y \in Y} U(y)p^*(s)(y)\lambda(s).$$

The key observation in this proof can be observed upon fixing some state $s' \in S$. By the definition of an act $f \in F$, for each $f(w) = p(s')(y)$ can be written as $P(y) \in L$. Thus, denote $\underline{\lambda}$ as the value of $\lambda \in \Delta(S)$ that minimizes $\sum_S \sum_{y \in Y} U(y)p(s)(y)$. Then,

$$f \succsim f^* \Longleftrightarrow \sum_S \sum_{y \in Y} U(y)p(s)(y)\underline{\lambda}(s) \geq \sum_S \sum_{y \in Y} U(y)p^*(s)(y)\underline{\lambda}(s).$$

But we know that the vNM-EU representation is unique up to positive, affine (linear) transformations! That is,

$$\sum_S \sum_{y \in Y} U(y)p(s)(y)\underline{\lambda}(s) \geq \sum_S \sum_{y \in Y} U(y)p^*(s)(y)\underline{\lambda}(s) \Longleftrightarrow \sum_S \sum_{y \in Y} U(y)p(s)(y) \geq \sum_S \sum_{y \in Y} U(y)p^*(s)(y).$$

Finally, use the previous observation and rewrite the above expression as

$$\sum_{y\in Y} U(y)P(y) \geq \sum_{y\in Y} U(y)P^*(y).$$

Thus, we see that the problem has been reduced to the case of **?**[1], where thee condition permitting a partial order on income frequency distributions is given by

Proposition 1. *A distribution $f(y)$ will be preferred to another distribution $f^*(y)$ according to W for all $U(y)(U' > 0, U'' \leq 0)$ if and only if*

$$\int_0^x [F(y) - F^*(y)]dy \leq 0 \quad \textit{for all } z, \quad 0 \leq z \leq \bar{y}$$

and

$$F(y) \neq F^*(y) \quad \textit{for some } y,$$

where $F(y) = \int_0^y f(y)dy$.

Thus, the second order dominance result must hold in each state $s' \in S$ for the partial ranking over wealth distributions to be achieved.

$\square$

$\leftarrow$

Proof. Suppose that $\{p_s(y)\}_{s\in S}$ is ordered by S.O.S.D, for all $s \in S$. Fix a state $s' \in S$. Then, for all $y \in (0, \bar{y})$,

$$\Longleftrightarrow \min_{\lambda\in\Delta(S)} \int_S \int_0^{\bar{y}} U(y)p(s)(y)dyd\lambda \geq \min_{\lambda\in\Delta(S)} \int_S \int_0^{\bar{y}} U(y)p^*(s)(y)dyd\lambda.$$

A key observation is that $\lambda(s)$ is a probability measure over the state space S. Consequently, the double-expectation

$$\mathbb{E}_\lambda\left(\mathbb{E}_{p(s)}u\right)$$

is linear in the probabilities $(\lambda(s_1), \lambda(s_2), \ldots, \lambda(s_n)) = \lambda \in \Delta(S)$. In other words, the preferences represented by the "inner expectation" will be *invariant to linear (monotone) transformations.* Thus, $\forall\lambda \in \Delta(S)$,

$$\int_S \int_0^{\bar{y}} U(y)p(s)(y)dyd\lambda \geq \int_S \int_0^{\bar{y}} U(y)p^*(s)(y)dyd\lambda.$$

[1] I've switched the notation from F to P, since the objective-subjective uncertainty literature using the former to define the set of acts.

$$\int_0^{\bar{y}} U(y)p(s)(y)dy \geq \int_0^{\bar{y}} U(y)p^*(s)(y)dy.$$

Next, we can exploit the "change of variable" seen in the proof of the "$\rightarrow$" direction, which was permitted upon fixing a particular state $s' \in S$:

$$\Longleftrightarrow \int_0^{\bar{y}} U(y)p(y)dy \geq \int_0^{\bar{y}} U(y)p^*(y)dy$$

for all $y \in [0, \bar{y}]$, by the S.O.S.D. result, where $U(y)$ such that $U(y)(U' > 0, U'' \leq 0)$. To see the argument, first consider the "double" integration by parts procedure:

$$\int_0^{\bar{y}} U(y)p(y) = U(y)p(y)\Big|_0^{\bar{y}} - \int_0^{\bar{y}} U'(y)P(y)dy = U(\bar{y}) - \int_0^{\bar{y}} U'(y)P(y)dy.$$

And the second round of IBP, define $\hat{P}(y) = \int_0^{\bar{y}} P(y)dy$:

$$= U(\bar{y}) - \Big|_0^{\bar{y}} U'(y)P(y)dy + \int_0^{\bar{y}} U''(y)\hat{P}(y)dy = U(\bar{y}) - U'(\bar{y})\hat{P}(\bar{y}) + \int_0^{\bar{y}} U''(y)\hat{P}(y)dy.$$

With this expression at our disposal, return to the S.O.S.D assumption on the family of wealth distributions, given we fix some state $s' \in S$:

$$\Longleftrightarrow \int_0^{\bar{y}} U(y)p(y)dy - \int_0^{\bar{y}} U(y)p^*(y)dy \geq 0$$

$$\Longleftrightarrow [U(\bar{y}) - U'(\bar{y})\hat{P}(\bar{y}) + \int_0^{\bar{y}} U''(y)\hat{P}(y)dy] - [U(\bar{y}) - U'(\bar{y})\hat{P^*}(\bar{y}) + \int_0^{\bar{y}} U''(y)\hat{P^*}(\bar{y})dy] \geq 0$$

$$\Longleftrightarrow U'(\bar{y})[\hat{P^*}(y) - \hat{P}(\bar{y})] + \int_0^{\bar{y}} U''(y)[\hat{P}(\bar{y}) - \hat{P^*}(\bar{y})]dy \geq 0$$

Notice that $\hat{P}(\bar{y}) \succsim_{S.O.S.D} \hat{P^*}(\bar{y})$ if and only if $\hat{P}(\bar{y}) < \hat{P^*}(\bar{y})$ for all $y \in (0, \bar{y})$ and $\hat{P}(\bar{y})$ $= \hat{P^*}(\bar{y})$ when $y = 0$ and $y = 1$.

We now ask: **When is the previous expression greater than or equal to 0?**

Clearly, when $U''(y) = 0$, by S.O.S.D. Next, suppose that $U'' < 0$. Then, the term

$$\int_0^{\bar{y}} U''(y)\Big[\hat{P}(\bar{y}) - \hat{P^*}(\bar{y})\Big]dy > 0,$$

by S.O.S.D., and the term

$$U'(\bar{y})[\hat{P^*}(y) - \hat{P}(\bar{y})] \geq 0,$$

also by S.O.S.D. But this must hold for any state $s' \in S$. Thus, we have established that

$$p(s)(y) \succsim p^*(s)(y) \Longleftrightarrow f(w) \succsim f^*(w),$$

$f, f^* \in F$.

$\square$