A Decision-theoretic Approach for Alternative Measures of Wealth Inequality

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Decory Edwards[1]

The inequality literature is notable in its ability to bring together four distinct concepts: (i) measures of inequality, (ii) social welfare functions, (iii) mean values, and (iv) models of choice under uncertainty, as noted by Dalton (1920). This interplay between statistical and theoretical analysis ahs been commonly applied in the context of inequality in the distribution of *income*. It is well-known that income and wealth inequality are markedly distinct phenomena. This distinction becomes more pronounced when one makes racial considerations regarding inequality. Notably, as recent as 2019, it has been reported that "the median white household held \$188,200 in wealth – 7.8 times that of the typical Black household". This paper provides a theoretical foundation for wealth inequality that incorporates the first three of the four distinct concepts. The ranking over wealth distributions will be established through the use of a *choice under objective-subjective uncertainty* model to acheive a ranking over the defined wealth distributions. From there, the "social welfare approach" to proposing inequality measures as seen in Atkinson (1971) is applied in this context, with the future goal of comparing an alternative measure to the *black-white median wealth gap* in the U.S.

[1]Edwards: Department of Economics, Johns Hopkins University, dedwar65@jhu.edu,

1 Introduction

The distribution of outcomes such as income and wealth among individuals in a society are of interest not only to economists, but to those interested in social welfare and inequality. As Hong (1983) notes, the literature on inequality notably connects four distinct concepts; (i) measures of inequality, (ii) social welfare functions, (iii) mean values, and (iv) models of choice under uncertainty.

Measures of inequality, such as the variance and the coefficient of variation, represent the traditionally statistical approach to characterizing inequality in a society. Almost a century ago, Dalton (1920) suggested that a given measure of inequality would correspond to a social welfare function. This can be gleaned from the fact that different measures of inequality produce different rankings over distributions.

The contribution of Atkinson (1970) is two-fold. First, he provides a theoretical foundation for ranking distributions and a notion of income inequality aversion by connecting results from the choice under risk literature with Dalton's observation of underlying social welfare functions. Although not explicitly stated, he later proposes an inequality measure which highlights the connection between measures and quasilinear mean values. This is done through the use of an analogy to certainty equivalence from the choice under risk literature.

This literature is almost exclusively interested in characterizing inequality in the distribution of income. In the papers that claim to refer to wealth inequality, there is no "true" distinction, since the underlying choice under uncertainty model used only incorporates objective uncertainty. With potential interest in describing economic inequality with socially relevant components, such as racial inequality in the U.S., one may wish to make a non-trivial distinction between income and wealth inequality. The motivation for this distinction may be supported by abstraction and reasoning or by analyzing empirical observations.

This paper will assume that this distinction is reasonably motivated and defended a-priori. The goal of this paper is to produce an alternative measure inequality in the distribution of wealth using this "social welfare approach" to be compared to conventional summary statistics of racial wealth inequality, such as the black-white median wealth gap. This is achieved by exploiting connections between the aforementioned distinct concepts. First, I make use of an analogy between uncertainty aversion and wealth inequality aversion through specifications on the underlying social welfare function in a choice under objective-subjective uncertainty model. From there, I propose a familiar equally distributed measure which is a quasilinear-mean model of representative wealth.

The paper will proceed as follows. In section 2, I give the explicit formulation of the connection between the four concepts in the context of income inequality. Section 3 provides a number of arguments for the distinction between characterizing income and wealth inequality using these distinct concepts. Sections 4-7 will cover each of these concepts for the distribution of wealth in the following order: choice under objective-subjective uncertainty, social welfare functions, measures of (wealth) inequality, and mean values. Section 8 concludes the analysis with a brief discussion on an empirical application of these efforts. A short proof of the singular proposition of the paper is given in the appendix.

2 Baseline model: choice under risk

Here, I provide an explicit reformulation of the previous work done on proposing measures of income inequality using the choice under objective uncertainty literature.

2.1 Analytical framework

Denote the set of outcomes $Y = [0, \bar{y}]$. Then, the choice set is given by:

$$L = \Big\{ p : 2^{[0,\bar{y}]} \to \mathbb{R} \mid p(\cdot) \text{ is an income frequency distribution} \Big\}.$$

We may define a binary relation over both sets Y, L :

$$\succsim_y \subseteq [0, \bar{y}] \times [0, \bar{y}] \subseteq \mathbb{R} \times \mathbb{R}$$

$$\succsim L \times L.$$

Notice that $\succsim_y$ may be represented by a real-valued utility function. This is the social welfare $U(y)$ which is a key object of analysis in this paper.

2.2 Axiomatization

V 1 (Weak Order)**.** $\succsim$ *is complete and transitive.*

V 2 (Continuity)**.** *For every* $p(\cdot)$, $p*(\cdot)$, $p'(\cdot) \in L$, *if* $p(\cdot) \succ p*(\cdot) \succ p'(\cdot)$, *there exists* $\alpha, \beta \in (0, 1)$ *such that*

$$\alpha p + (1-\alpha)p' \succ p* \succ \beta p + (1-\beta)p'.$$

V 3 (Independence)**.** *For every* $p, p*, p' \in L$ *and every* $\alpha \in (0, 1)$ *such that*

$$p \succsim p* \Longrightarrow \alpha p + (1-\alpha)p' \succsim \alpha p^* + (1-\alpha)p'.$$

2.3 Expected utility representation of the ranking over income distributions

The ranking over income distributions will be represented using the vNM-expected utility representation. That is,

$$p(y) \sim \int_0^{\bar{y}} U(y)p(y)dy \equiv W.$$

Formally, note the slight modification of the vNM-EU theorem in this setting of ranking income distributions:

Theorem 1. *$\succsim \subseteq L \times L$ satisfies $(V1)$* ***weak order****, $(V2)$* ***continuity****, $(V3)$* ***independence*** *if and only if there exists $U : Y \to \mathbb{R}$ such that, for every $p(y), p*(y) \in L$,*

$$p(y) \succsim p*(y) \Longleftrightarrow \int_0^{\bar{y}} U(y)p(y)dy \geq \int_0^{\bar{y}} U(y)p*(y)dy.$$

3 Motivation

For the purposes of this paper, we will assume that ranking wealth distributions is a non-trivially distinct exercise from the previous case of ranking income distributions. The motivation for this can be approached from many directions. I will make note of them here, but a more rigorous defense of these points is beyond the scope of this paper.

3.1 A statistical argument

It is well-known that income data is more reliably collected than wealth data. Consider a quote from Saez and Zucman (2014) which provides an estimate for wealth inequality using available administrative data:

> Because of the lack of administrative data on wealth, none of the existing sources offer a definitive estimate. We see our paper as an attempt at using the most comprehensive administrative data currently available, but one that ought to be improved in at least two ways: (i) by using additional information already available at the Statistics of Income division of the IRS, and (ii) new data that the US Treasury could collect at low cost. A modest data collection effort would make it possible to obtain a better picture of the joint distributions of wealth, income, and saving, a necessary piece of information to evaluate proposals for consumption or wealth taxation.

So it is clear: there are differences in the collection of income data and wealth data. But is it enough to justify introducing subjective uncertainty in the exercise of comparing wealth distributions?

Consider next a quote from Bricker, Henriques, Krimmel, and Sabelhaus (2016) which provides a justification for their use of administrative data and their measurement of income and wealth inequality:

> In general, administrative data should provide better estimates of top income and wealth shares, because traditional random household surveys suffer from under- representation of wealthy families. Unlike most other household surveys, the SCF is designed to overcome the underrepresentation problem, because administrative data are used to select the sample, and rigorous targeting and accounting for wealthy family participation assures those families are properly represented in the survey data.

Not only is "good" data on income more accessible, but the estimation of wealth shares typically employs some combination of administrative and survey data. But survey data is by, quite literally, a subjective measure: it implicitly relies on individual's truthful and accurate reporting of their own levels of wealth.

3.2 A thought experiment

Country A has 60 individuals, each of either one of two races, black or white. Suppose that there are 10 black and 50 white individuals. There is an initial wealth distribution $f(w)$. One can consider the total level of wealth, $\bar{w} = \sum_{i=1}^{60} w_i f(w_i)$, as well as the total level of wealth within each racial group, $\bar{w}_b$ and $\bar{w}_w$.

A policymaker if tasked with comparing this initial distribution of wealth $f(w)$ with the distribution $f'(w)$ reached by the following redistribution of wealth:

> "Increase wealth of white individuals by 2 units, and decrease wealth of black individuals by 9 units"

The redistributive policy outlined above suggests redistributing wealth from black individuals who are wealthier within group to poorer white individuals within group. If the policymaker takes a *Utilitarian* view of social welfare, then they should favor this policy since:

$$9*10 < 2*50 \Longrightarrow f(\bar{w}_b - 10)U(\bar{w}_b - 10) + f(\bar{w}_w + 2)U(\bar{w}_w + 2) > f(\bar{w}_b)U(\bar{w}_b) + f(\bar{w}_w)U(\bar{w}_w)$$

However, it is not unreasonable to suspect that a policy such as this would not garner much support from individuals in practice, considering racial differences in wealth holdings in the U.S. This suggests that the choice under objective uncertainty model may be lacking in its ability to implement favorable wealth distributive policies.

A notion of subjective uncertainty can be incorporated into this thought experiment in a straightforward way: denote the finite state space S $=\{$black, white$\}$. From here, using a model which allows for one to define a notion of uncertainty aversion is a reasonable deviation from the aforementioned case.

3.3 A non-economic argument

Though "non-economic" arguments seem to be the least compelling up front, the following statements of the late philosopher John Rawls may convince readers that this is the most compelling argument for the modeling choices to come.

Rawls (1971) *theory of distributive justice* is interested in the problem of getting a group of people with different circumstances and motives to agree on a "social contract"; that is, an agreement on a system of governing for which all members of the group must abide by.

He proposes that this problem should be approached as if those deciding on the governance of society are behind a *veil of ignorance*: decision-makers are ignorant of their own circumstances. He then proposes two principles that may accompany this veil of ignorance in the delivery of justice for institutions which govern our society. Consider Rawls' final statement of the two principles of justice for institutions:

1. Each person is to have an equal right to the most extensive total system of equal basic liberties compatible with a similar system of liberty for all.
2. Social and economic inequalities are to be arranged so that they are both:
   a) to the greatest benefit of the least advantaged, consistent with the just savings principle, and
   b) attached to offices and positions open to all under conditions of fair equality of opportunity.

But how is this related to the social planner's decision problem of ranking wealth distributions? The link is that there is some underlying notion of a social contract implicit in any concept of wealth. That is, the outcome wealth is incoherent without some underlying notion of ownership! Individuals must agree on what it means to own something, and must respect that some individuals own more and/or less than they do.

If the social planner takes this fact about wealth seriously, then this may suggest alternative modeling choices than those made in the income context. Consider another quote from Rawls, which is, quite literally, the intuitive counterpart to the modeling choices I make later in the characterization of a comparison over wealth distributions:

> The maximin rule tells us to rank alternatives by their worst possible outcomes: we are to adopt the alternative the worst outcome of which is superior to the worst outcomes of the others[1].

4 Extending the model: choice under ambiguity

Consider the excerpt from Gilboa (2009), which may serve as a precursor to the modeling choices that will be made to characterize a ranking over wealth distributions:

> To make sure that we understand the structure, observe that there are two sources of uncertainty: the choice of the state s, which is sometimes referred to as "subjective uncertainty", because no objective probabilities are given on it, and the choice of x, which is done with objective probabilities once you chose your act and Nature chose a state. Specifically, if you choose $f \in F$ and Nature chooses $s \in S$, a roulette wheel is spun, with distribution $f(s)$ over the outcomes X, so that your probability to get outcome x is $f(s)(x)$.

[1] Coincidentally, Rawls' second principle of justice for institutions is often considered by economists to be interchangeable with the maximin principle, despite his belief that it is "undesirable to use the same name for two things that are so distinct.".

4.1 Analytical framework

Denote the set of outcomes $Y = [0, \bar{y}]$. Then, the choice set is given by:

$$L = \left\{ p : 2^{[0,\bar{y}]} \to \mathbb{R} \mid p(\cdot) \text{ is an income frequency distribution} \right\}.$$

We may define a binary relation over both sets Y, L :

$$\succsim_y \subseteq [0, \bar{y}] \times [0, \bar{y}] \subseteq \mathbb{R} \times \mathbb{R}$$

$$\succsim L \times L.$$

Notice that $\succsim_y$ may be represented by a real-valued utility function. This is the social welfare $U(y)$ which is a key object of analysis in this paper.

Preliminary remarks

We are concerned with the comparison of two frequency distributions $f(w)$ of an outcome w which we refer to as wealth. We seek to use the notion of *uncertainty aversion* as an analogy to the use of the notion of risk aversion in the characterization of a ranking over income distributions.

The presence of both objective and subjective uncertainty is at the heart of this analysis. Section 2 covered the analysis for objective uncertainty. Thus, the wealth frequency distribution $f(w)$ must be formalized in this abstract setting so that it explicitly captures both forms of uncertainty. Namely, each wealth distribution is associated with some relevant, underlying state space S (the source of subjective uncertainty), and its objective component $p(y) \in L$. WIth this in mind, we work under the following assumption on the functional form of wealth distributions for the remainder of this paper:

Assumption 1. *Each wealth distribution $f(w)$ can be written as $p(s)y$.*

4.2 Axiomatization

AA 1 (Weak Order)**.** $\succsim$ *is complete and transitive.*

AA 2 (Continuity)**.** *For every $f, g, h \in F$, if $f \succ g \succ h$, there exists $\alpha, \beta \in (0, 1)$ such that*

$$\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \beta)h.$$

C-Independence (C-Independence)**.** *For every $f, g \in F$, every constant $h \in F$ and every $\alpha \in (0, 1)$,*

$$f \succsim g \Longleftrightarrow \alpha f + (1 - \alpha)h \succsim \alpha g + (1 - \alpha)h.$$

AA 3 (Monotonicity)**.** *For every $f, g \in F$, $f(s) \geq g(s)$ for all $s \in S$ implies $f \geq g$.*

AA 4 (Non-trivality)**.** *There exists $f, g \in X$ such that $f \succ g$.*

Uncertainty Aversion (Uncertainty Aversion)**.** *For every $f, g \in F$, if $f \sim g$, then, for every $\alpha \in (0, 1)$,*

$$\alpha f + (1-\alpha)g \succsim f.$$

4.3 Expected utility representation of the ranking over wealth distributions

Finally, we have the representation theorem by Gilboa and Schmeidler (1989).

Theorem 2. *$\succsim$ satisfies AA1, AA2, C-Independence, AA4, AA5, and Uncertainty aversion if and only if there exists a closed and convex set of probabilities on S, $C \subset \Delta(S)$, and a non-constant function $U : Y \to \mathbb{R}$ such that, for every $f, f^* \in F$,*

$$f \succsim f^* \Longleftrightarrow \min_{\lambda\in\Delta(S)} \int_S (\mathbb{E}_{p(s)}u) d\lambda \geq \min_{\lambda\in\Delta(S)} \int_S (\mathbb{E}_{p^*(s)}u) d\lambda.$$

From here on out, we assume that wealth distributions will be ranked according to:

$$W' \equiv \min_{\lambda\in\Delta(S)} \int_S (\mathbb{E}_{p(s)}u) d\lambda$$

$$= \min_{\lambda\in C} \int_S \int_0^{\bar{y}} p(s)(y)U(y) dy d\lambda.$$

5 The social welfare approach to ranking wealth distributions

5.1 Partial ranking over wealth distributions

From here, I assume that the function $U : Y \mapsto \mathbb{R}$ is an increasing and concave function. Again, we seek the conditions for which a ranking over wealth distributions can be achieved without any further specifications on $U(y)$.

A key observation is that the assumption on the functional form of the wealth distribution $f(w) = p(s)(y)$ implicitly suggests that the objects of interest are a family of distributions $p(y) \in L$, indexed by elements of the state space $s \in S$, or:

$$\{p_s(y)\}_{s\in S}.$$

Thus, we may view each act $f \in F$ as a "state-contingent frequency distribution". Moreover, upon fixing a state $s' \in S$, the analysis will become identical to the use of the choice under objective uncertainty model that was used to reach a ranking over income frequency distributions.

Recall that the second-order stochastic dominance result was necessary and sufficient to reach a partial ordering over income distributions. Consider the following proposition below which extends this result to a partial ordering over wealth distributions for the most general class of social welfare functions $U(y)$.

Proposition 1. *A distribution $p(s)(y)$ will be preferred to another distribution $p^*(s)(y)$ according to W' for all $U(y)(U' > 0, U'' \leq 0)$ if and only if, $\forall s' \in S$,*

$$\int_0^x [P(s')(y) - P^*(s')(y)]dy \leq 0 \quad \textit{for all } z, \quad 0 \leq z \leq \bar{y}$$

and

$$P(s')(y) \neq P^*(y) \quad \textit{for some } y,$$

where $P(s')(y) = \int_0^y p(s')(y)dy$.

The proof can be found in the Appendix.

5.2 Complete ranking over wealth distributions

To reach our complete ordering over wealth distribution, we must guarantee that $U(y)$ is specified up to a linear (monotonic) transformation. Work done by Kihlstrom and Mirman (1981) extending the notion of relative risk aversion results to the case of "multidimensional commodities" implies that the restriction we are after is the class of social welfare functions representing homothetic preferences. That is,

$$U(y) = \begin{cases} A + B\frac{y^{1-\epsilon}}{1-\epsilon}, & \epsilon \neq 1 \\ \ln(y), & \epsilon = 1 \end{cases} \tag{1}$$

where $\epsilon \geq 0$ is required to preserve concavity of $U(y)$.

It is well-know that, in the risk and risk aversion literature, this functional form is associated with the class of utility functions representing constant relative risk averse (CRRA) and decreasing absolute risk averse (DARA) preferences.

6 Proposing an alternative measure of inequality

The ranking over wealth distributions via specifications on $U(y)$ will allow us to propose measures of inequality. To see the novelty in this "social welfare approach", notice that measures of inequality are generally statistical objects. Namely, the variance, coefficient of variation, and mean deviation are each calculated using collected data on the distribution of an outcome such as income or wealth.

Consider one plausible inequality measure: given some present distribution $f(w) = p(s)(y)$, the *the ratio between the level of social welfare evaluated at* $p(s)(y)$ *and the level of social welfare if everyone had the same level of wealth.* Formally,

$$D' = \frac{\min_{\lambda \in \Delta(s)} \int_S \int_0^{\bar{y}} U(y)p(s)(y)dyd\lambda}{U(\mu)}.$$

However, since this measure is not invariant with respect to linear (monotone) transformations on $U(y)$, it is possible for two decision-makers to agree on the ranking between two distributions but characterize the level of inequality between the two differently.[2]

6.1 The equally distributed equivalent measure of inequality

To propose an inequality measure without this undesirable property, we consider the following definition.

Definition 1. *The equally distributed equivalent level of wealth* w_{EDE} *is the level of wealth each person should receive such that, and equal distribution at this level of wealth per head gives the same level of social welfare as the present distribution* $f(w) = p(s)(y)$*, that is,*

$$U(w_{EDE}) \int_S \int_0^{\bar{y}} p(s)(y)dyds = \min_{\lambda \in \Delta(S)} \int_S \int_0^{\bar{y}} U(y)p(s)(y)dyd\lambda.$$

Indeed, I present the *equally distributed equivalent inequality measure*:

$$I' = 1 - \frac{w_{EDE}}{\mu}.$$

Note that this measure no longer depends on the units of measurement regarding the distribution of wealth. The measure will return a number in the interval $[0, 1]$ which will be associated with how far the current distribution is from the distribution that gives everyone the same level of wealth.

6.2 Reaching the mean-independent, equally distributed inequality measure

Using the specification on the social welfare function so that it represents preferences satisfying homotheticity found in 2, one may derive the following expression for the equally distributed equivalent level of wealth:

$$w_{EDE} = \left[\left(\frac{1-\epsilon}{B} \right) \frac{\min_{\lambda \in \Delta(S)} \int_S \int_0^{\bar{y}} A + B\frac{y^{1-\epsilon}}{1-\epsilon} p(s)(y)dyd\lambda - A}{\int_S \int_0^{\bar{y}} p(s)(y)dyds} \right].$$

[2]As show by Dalton (1920) and others.

Finally, we may present the equally distributed measure of wealth inequality, which retains the mean-independence property that many of the conventional summary statistics of inequality possess:

$$I' = 1 - \left(\frac{1-\epsilon}{B\mu^{\frac{1}{1-\epsilon}}}\right)^{1-\epsilon} \left[\frac{\min_{\lambda\in\Delta(S)} \int_S \int_0^{\bar{y}} A + B\frac{y^{1-\epsilon}}{1-\epsilon} p(s)(y)dyd\lambda - A}{\int_S \int_0^{\bar{y}} p(s)(y)dyds}\right].$$

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Consider one plausible inequality measure: given some present distribution $f(w) = p(s)(y)$, the *the ratio between the level of social welfare evaluated at* $p(s)(y)$ *and the level of social welfare if everyone had the same level of wealth.* Formally,

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Indeed, I present the *equally distributed equivalent inequality measure*:

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Finally, we may present the equally distributed measure of wealth inequality, which retains the mean-independence property that many of the conventional summary statistics of inequality possess:

$$I' = 1 - \left(\frac{1-\epsilon}{B\mu^{\frac{1}{1-\epsilon}}} \right)^{1-\epsilon} \left[\frac{\min_{\lambda \in \Delta(S)} \int_S \int_0^{\bar{y}} A + B \frac{y^{1-\epsilon}}{1-\epsilon} p(s)(y) dy d\lambda - A}{\int_S \int_0^{\bar{y}} p(s)(y) dy ds} \right].$$

7 Conclusion

We have reached an alternative measure of wealth inequality, I'. The measure both utilizes the notion of representative wealth levels and preserves the mean-independence property that many statistical measures of inequality have. The goal of this exercise is to apply it in the context of racial wealth inequality. There, the story is generally told using the aforementioned statistical measures of inequality (nost notable, the *black-white median wealth gap*).

With the arguments made in section 3, it seems reasonable to suspect that a state space such as $S = \{\text{black, white}\}$ should largely inform our measure of wealth inequality via some ranking over wealth distributions. Historical episodes such as enslavement and the era of Jim Crows laws suggest that race may be a socially-relevant feature in characterizing the level of inequality in a given wealh distribution. The main issue is whether or not the effects of these historical episodes persist over time, as well as how explanatory they are for racial differences in wealth.

Appendices

A Proofs

A.1 Proposition 1

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Proof. First, consider the relevant objects from the Anscombe-Aumann expected utility representation theorem, for which the Gilboa and Schmeidler (1989) theorem is an extension of (referred to as GS from here on out):

$$L = \left\{ p : Y \mapsto [0,1] \;\middle|\; \# \; \{y|p(y) > 0\} < \infty, \sum_{y \in Y} p(y) = 1 \right\}.$$

Where L is the choice set from the vNM-EU model, and F is from the GS-EU set-up. First, recall the GS-EU theorem:

Theorem 3. *$\succsim$ satisfies AA1, AA2, C-Independence, AA4, AA5, and Uncertainty aversion if and only if there exists a closed and convex set of probabilities on S, $C \subset \Delta(S)$, and a non-constant function $U : Y \to \mathbb{R}$ such that, for every $f, f^* \in F$,*

$$f \succsim f^* \Longleftrightarrow \min_{\lambda \in \Delta(S)} \int_S (\mathbb{E}_{p(s)} u) d\lambda \geq \min_{\lambda \in \Delta(S)} \int_S (\mathbb{E}_{p^*(s)} u) d\lambda.$$

$$\Longleftrightarrow \min_{\lambda \in \Delta(S)} \int_S \int_0^{\bar{y}} U(y) p(s)(y) dy d\lambda \geq \min_{\lambda \in \Delta(S)} \int_S \int_0^{\bar{y}} U(y) p^*(s)(y) dy d\lambda$$

For simplicity, consider the discrete version of the implication of this expected utility representation result. Namely,

$$\Longleftrightarrow \min_{\lambda \in \Delta(S)} \sum_S \sum_{y \in Y} U(y) p(s)(y) \lambda(s) \geq \min_{\lambda \in \Delta(S)} \sum_S \sum_{y \in Y} U(y) p^*(s)(y) \lambda(s).$$

The key observation in this proof can be observed upon fixing some state $s' \in S$. By the definition of an act $f \in F$, for each $f(w) = p(s')(y)$ can be written as $P(y) \in L$. Thus, denote $\underline{\lambda}$ as the value of $\lambda \in \Delta(S)$ that minimizes $\sum_S \sum_{y \in Y} U(y) p(s)(y)$. Then,

$$f \succsim f^* \Longleftrightarrow \sum_S \sum_{y \in Y} U(y) p(s)(y) \underline{\lambda}(s) \geq \sum_S \sum_{y \in Y} U(y) p^*(s)(y) \underline{\lambda}(s).$$

But we know that the vNM-EU representation is unique up to positive, affine (linear) transformations! That is,

$$\sum_S \sum_{y \in Y} U(y) p(s)(y) \underline{\lambda}(s) \geq \sum_S \sum_{y \in Y} U(y) p^*(s)(y) \underline{\lambda}(s) \Longleftrightarrow \sum_S \sum_{y \in Y} U(y) p(s)(y) \geq \sum_S \sum_{y \in Y} U(y) p^*(s)(y).$$

Finally, use the previous observation and rewrite the above expression as

$$\sum_{y\in Y} U(y)P(y) \geq \sum_{y\in Y} U(y)P^*(y).$$

Thus, we see that the problem has been reduced to the case of Atkinson (1970)[3], where thee condition permitting a partial order on income frequency distributions is given by

Proposition 2. *A distribution $f(y)$ will be preferred to another distribution $f^*(y)$ according to W for all $U(y)(U' > 0, U'' \leq 0)$ if and only if*

$$\int_0^x [F(y) - F^*(y)]dy \leq 0 \quad \text{for all } z, \quad 0 \leq z \leq \bar{y}$$

and

$$F(y) \neq F^*(y) \quad \text{for some } y,$$

where $F(y) = \int_0^y f(y)dy$.

Thus, the second order dominance result must hold in each state $s' \in S$ for the partial ranking over wealth distributions to be achieved.

□

$\leftarrow$

Proof. Suppose that $\{p_s(y)\}_{s\in S}$ is ordered by S.O.S.D, for all $s \in S$. Fix a state $s' \in S$. Then, for all $y \in (0, \bar{y})$,

$$\Longleftrightarrow \min_{\lambda\in\Delta(S)} \int_S \int_0^{\bar{y}} U(y)p(s)(y)dyd\lambda \geq \min_{\lambda\in\Delta(S)} \int_S \int_0^{\bar{y}} U(y)p^*(s)(y)dyd\lambda.$$

A key observation is that $\lambda(s)$ is a probability measure over the state space S. Consequently, the double-expectation

$$\mathbb{E}_\lambda\left(\mathbb{E}_{p(s)}u\right)$$

is linear in the probabilities $(\lambda(s_1), \lambda(s_2), \ldots, \lambda(s_n)) = \lambda \in \Delta(S)$. In other words, the preferences represented by the "inner expectation" will be *invariant to linear (monotone) transformations*. Thus, $\forall \lambda \in \Delta(S)$,

$$\int_S \int_0^{\bar{y}} U(y)p(s)(y)dyd\lambda \geq \int_S \int_0^{\bar{y}} U(y)p^*(s)(y)dyd\lambda.$$

[3] I've switched the notation from F to P, since the objective-subjective uncertainty literature using the former to define the set of acts.

$$\int_0^{\bar{y}} U(y)p(s)(y)dy \geq \int_0^{\bar{y}} U(y)p^*(s)(y)dy.$$

Next, we can exploit the "change of variable" seen in the proof of the "$\rightarrow$" direction, which was permitted upon fixing a particular state $s' \in S$:

$$\iff \int_0^{\bar{y}} U(y)p(y)dy \geq \int_0^{\bar{y}} U(y)p^*(y)dy$$

for all $y \in [0, \bar{y}]$, by the S.O.S.D. result, where $U(y)$ such that $U(y)(U' > 0, U'' \leq 0)$. To see the argument, first consider the "double" integration by parts procedure:

$$\int_0^{\bar{y}} U(y)p(y) = U(y)p(y)\Big|_0^{\bar{y}} - \int_0^{\bar{y}} U'(y)P(y)dy = U(\bar{y}) - \int_0^{\bar{y}} U'(y)P(y)dy.$$

And the second round of IBP, define $\hat{P}(y) = \int_0^{\bar{y}} P(y)dy$:

$$= U(\bar{y}) - \Big|_0^{\bar{y}} U'(y)P(y)dy + \int_0^{\bar{y}} U''(y)\hat{P}(y)dy = U(\bar{y}) - U'(\bar{y})\hat{P}(\bar{y}) + \int_0^{\bar{y}} U''(y)\hat{P}(y)dy.$$

With this expression at our disposal, return to the S.O.S.D assumption on the family of wealth distributions, given we fix some state $s' \in S$:

$$\iff \int_0^{\bar{y}} U(y)p(y)dy - \int_0^{\bar{y}} U(y)p^*(y)dy \geq 0$$

$$\iff [U(\bar{y}) - U'(\bar{y})\hat{P}(\bar{y}) + \int_0^{\bar{y}} U''(y)\hat{P}(y)dy] - [U(\bar{y}) - U'(\bar{y})\hat{P^*}(\bar{y}) + \int_0^{\bar{y}} U''(y)\hat{P^*}(\bar{y})dy] \geq 0$$

$$\iff U'(\bar{y})[\hat{P^*}(y) - \hat{P}(\bar{y})] + \int_0^{\bar{y}} U''(y)[\hat{P}(\bar{y}) - \hat{P^*}(\bar{y})]dy \geq 0$$

Notice that $\hat{P}(\bar{y}) \succsim_{S.O.S.D} \hat{P^*}(\bar{y})$ if and only if $\hat{P}(\bar{y}) < \hat{P^*}(\bar{y})$ for all $y \in (0, \bar{y})$ and $\hat{P}(\bar{y}) = \hat{P^*}(\bar{y})$ when $y = 0$ and $y = 1$.

We now ask: **When is the previous expression greater than or equal to 0?**

Clearly, when $U''(y) = 0$, by S.O.S.D. Next, suppose that $U'' < 0$. Then, the term

$$\int_0^{\bar{y}} U''(y)\Big[\hat{P}(\bar{y}) - \hat{P^*}(\bar{y})\Big]dy > 0,$$

by S.O.S.D., and the term

$$U'(\bar{y})[\hat{P^*}(y) - \hat{P}(\bar{y})] \geq 0,$$

also by S.O.S.D. But this must hold for any state $s' \in S$. Thus, we have established that

$$p(s)(y) \succsim p^*(s)(y) \Longleftrightarrow f(w) \succsim f^*(w),$$

$f, f^* \in F$.

$\square$

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