1 The Usual Theory, and a Bit More Notation

{sec:the-usual-theor

1.1 Steps

Generically, we want to think of the Bellman solution as having three steps:

1. **Arrival**: Incoming state variables (e.g., k) are known, but any shocks associated with the period have not been realized and decision(s) have not yet been made

2. **Decision**: All exogenous variables (like income shocks, rate of return shocks, and predictable income growth $\mathcal{G}$) have been realized (so that, e.g., m's value is known) and the agent solves the optimization problem

3. **Continuation**: After all decisions have been made, their consequences are revealed by evaluation of the continuing-value function at the values of the 'outgoing' state variables.

In the standard treatment in the literature, the (implicit) default assumption is that the step where the agent is solving a decision problem is the unique moment at which the problem is defined. This is what implicitly was done above, when (for example) in (**??**) we related the value v of the current decision to the explicit expectation of the future value v_{+1}. The idea, here, though, is to encapsulate the current period's problem as a standalone object, which is solvable contingent on some exogenously-provided continuation-value function $\mathrm{v}_{\rightarrow}(a)$.

When we want to refer to a specific step within period t we will do so by modifying its period subscript with an indicator character:

Stp	Indicator	State	Usage	Explanation
Arrival	$\leftarrow$ prefix	k	$\mathrm{v}_{\leftarrow}(k)$	value at entry to (before shocks)
Decision	(blank/none)	m	$\mathrm{v}(m)$	value of t-decision (after shocks)
Continuation	$\rightarrow$ suffix	a	$\mathrm{v}_{\rightarrow}(a)$	value at exit (after decision)

Notice that different steps of the problem have distinct state variables. k is the state at the beginning of the period because the shocks that yield m from k have not yet been realized. The state variable for the continuation step is a because after the consumption decision has been made the model assumes that all that matters is where you have ended up, not how you got there.

1.2 The Usual Theory, Notated

Using this new notation, the first order condition for (**??**) with respect to c is

$$\begin{aligned} \mathrm{u}^{c}(c) &= \mathbb{E}_{\rightarrow}[\beta\mathcal{R}_{+1}\mathcal{G}_{+1}^{1-\rho}\mathrm{v}_{+1}^{m}(m_{+1})] \\ &= \mathbb{E}_{\rightarrow}[\beta\mathsf{R}\quad\mathcal{G}_{+1}^{-\rho}\mathrm{v}_{+1}^{m}(m_{+1})] \end{aligned} \tag{1}$$

{eq:upceqEvtp1}

and because the `Envelope` theorem tells us that

$$\mathrm{v}^{m}(m) = \mathbb{E}_{\leftarrow}[\beta\mathsf{R}\mathcal{G}_{+1}^{-\rho}\mathrm{v}_{+1}^{m}(m_{+1})] \tag{2}$$

{eq:envelope}

we can substitute the LHS of (**??**) for the RHS of (**??**) to get

$$\mathrm{u}^{c}(c) = \mathrm{v}^{m}(m) \tag{3}$$ {eq:upcteqvtp}

and rolling forward one period,

$$\mathrm{u}^{c}(c_{+1}) = \mathrm{v}^{m}_{+1}(a\mathcal{R}_{+1} + \boldsymbol{\theta}_{+1}) \tag{4}$$ {eq:upctp1EqVpxtp

so that substituting the LHS in equation (**??**) finally gives us the Euler equation for consumption:

$$\mathrm{u}^{c}(c) = \mathbb{E}_{\rightarrow}[\beta \mathsf{R} \mathcal{G}_{+1}^{-\rho} \mathrm{u}^{c}(c_{+1})]. \tag{5}$$ {eq:cEuler}

From the perspective of the beginning of period +1 we can write the 'arrival value' function and its first derivative as

$$\begin{aligned} \mathrm{v}_{\leftarrow(+1)}(k_{+1}) &= \mathbb{E}_{\leftarrow(+1)}[\ \mathcal{G}_{t+1}^{1-\rho}\mathrm{v}_{+1}(\overbrace{\mathcal{R}_{+1}k_{+1} + \boldsymbol{\theta}_{+1}}^{m_{+1}})] \\ \mathrm{v}^{k}_{\leftarrow(+1)}(k_{+1}) &= \mathbb{E}_{\leftarrow(+1)}[\mathsf{R}\mathcal{G}_{+1}^{-\rho}\mathrm{v}^{m}_{+1}(m_{+1})] \end{aligned} \tag{6}$$ {eq:vFuncBegtpdefr

because they return the expected $t+1$ value and marginal value associated with arriving in period +1 with any given amount of $k$apital.

Finally, recalling that we obtain $\mathrm{v}_{\rightarrow}(a) = \beta \mathrm{v}_{\leftarrow(+1)}(k_{+1})$ using $a = k_{+1}$, note for future use that we can write the Euler equation (**??**) more compactly as

$$\mathrm{u}^{c}(c) = \mathrm{v}^{a}_{\rightarrow}(m - c). \tag{7}$$ {eq:upEqbetaOp}

1.3 Summing Up

{subsec:summing-up

For future reference, it will be useful here to write the full expressions for the distinct value functions at the Arrival ($\leftarrow$) and Decision steps. (Recall that $\mathrm{v}_{\rightarrow}(a)$ is provided to the solution algorithm as an input).

There is no need to use our step-identifying notation for the model's variables; k, for example, will have only one unique value over the course of the solution and therefore a notation like $k_{\rightarrow}$ would be useless; the same is true of all other variables. Given the continuation value function $\mathrm{v}_{\rightarrow}$, the problem can be written entirely without period subscripts:

$$\mathrm{v}_{\leftarrow}(k) = \mathbb{E}_{\leftarrow}[\mathrm{v}(\overbrace{k\mathcal{R} + \boldsymbol{\theta}}^{m})] \tag{8}$$ {eq:vBeg}

$$\mathrm{v}(m) = \max_{\{c\}} \ \ \mathrm{u}(c) + \mathbb{E}[\mathrm{v}_{\rightarrow}(\overbrace{m - c}^{a})] \tag{9}$$ {eq:vMid}