1 Multiple Control Variables

{sec:multiple-contro

We now consider how to solve problems with multiple control variables. Specifically, we will examine a consumer who has both a choice of how much to consume and a choice of how much of their unconsumed resources to invest in risky versus safe assets.

1.1 Theory

{subsec:MCTheory}

The portfolio-share control-variable is captured by the archaic Greek character 'stigma': ς represents the share of their available assets the agent invests in the risky asset (conventionally, the stock market). Designating the return factor for the risky asset as $\mathbf{R}$ and the share of the portfolio invested in $\mathbf{R}$ as ς, the realized portfolio rate of return $\mathfrak{R}$ as a function of the share ς is:

$$\mathfrak{R}(\varsigma) = \mathsf{R} + (\mathbf{R} - \mathsf{R})\varsigma. \tag{1}$$ {eq:Shr}

If we imagine the portfolio share decision as being made simultaneously with the c decision, the traditional way of writing the problem is (substituting the budget constraint):

$$\mathrm{v}_t(m) = \max_{\{\mathrm{c},\varsigma\}} \quad \mathrm{u}(c) + \mathbb{E}[\beta \mathrm{v}_{t+1}((m-c)\mathfrak{R}(\varsigma) + \theta_{t+1})] \tag{2}$$ {eq:Bellmanundatec

where we have deliberately omitted the period-designating subscripts for ς and the return factors to highlight the point that, once the consumption and ς decisions have been made, it makes no difference to this equation whether the risky return factor $\mathbf{R}$ is revealed a nanosecond before the end of the current period or a nanosecond after the beginning of the successor period.

1.2 Stages Within a Period

{subsec:stageswithi

In most cases it is possible to take multiple-control problems and turn them into a sequence of single-control 'stages' which can be solved sequentially. For this problem we will call the 'consumption stage' c and the 'portfolio stage' ς. Our earlier point that, substantively, the timing of the realization of the return shocks does not matter means that these could come in either order in the period: We designate the 'portfolio choice first, then consumption' version by $[\varsigma, \mathrm{c}]$ and the 'consumption choice first, then portfolio' scheme as $[\mathrm{c}, \varsigma]$.

In a problem with multiple stages, if we want to refer to a sub-step of a particular stage – say, the Arrival step of the portfolio stage – we simply add a stage-indicator subscript (in square brackets) to the notation we have been using until now. That is, the Arrival stage of the portfolio problem would be $\mathrm{v}_{\leftharpoondown[\varsigma]}$.{SB, AL, MNW: An alternative notational choice would be $\mathrm{v}_{[\leftharpoondown\varsigma]}$.} (The version where both choices are made simultaneously could be designated as a single stage named $[\mathrm{c}\varsigma]$) with arrival value function $\mathrm{v}_{\leftharpoondown[\mathrm{c}\varsigma]}$.{SB, AL, MNW: with arrival value function $\mathrm{v}_{[\leftharpoondown\mathrm{c}\varsigma]}$.}

1.2.1 The (Revised) Consumer's Problem

{subsubsec:revised-

A slight modification to the consumer's problem specified earlier is necessary to make the stages of the problem completely modular. The difficulty with the earlier formulation is that it assumed that asset returns occurred in the middle step of the consumption problem. Our revised version of the consumption problem takes as its input state the amount of bank balances that have resulted from any prior portfolio decision. The problem is therefore:

$$\begin{aligned} \mathrm{v}_{[\mathrm{c}]}(m) &= \max_{c} \quad \mathrm{u}(c) + \mathrm{v}_{[\mathrm{c}]\rightharpoondown}(\underbrace{m-c}_{a}) \\ \mathrm{v}_{\leftharpoondown[\mathrm{c}]}(b) &= \mathbb{E}_{\leftharpoondown[\mathrm{c}]}\left[\mathrm{v}_{[c]}(\overbrace{b+\theta}^{m})\right] \end{aligned} \tag{3}$$

{eq:vBalances}

1.2.2 The Investor's Problem

{subsubsec:investor

Consider the standalone problem of an 'investor' whose continuation-value function $\mathrm{v}_{[\varsigma]\rightharpoondown}$ depends on how much wealth $\acute{w}$ they end up after the realization of the stochastic $\mathbf{R}$ return.

Using the $\breve{}$ accent to designate the optimized value of the accented control, the Decision stage of this problem yields the portfolio share function:

$$\breve{\varsigma}(w) = \arg\max_{\varsigma} \mathbb{E}_{[\varsigma]}\left[\mathrm{v}_{[\varsigma]\rightharpoondown}\overbrace{(w\mathfrak{R}(\varsigma))}^{\acute{w}}\right], \tag{4}$$

{eq:shrDecision}

and the Arrival value function is the expectation of the Continuation-value function over the wealth that results from the portfolio returns obtained under the choice of portfolio share made in the Decision step of the problem, $\breve{\mathfrak{R}} = \mathsf{R} + \breve{\varsigma}(w)(\mathbf{R} - \mathsf{R})$:

$$\mathrm{v}_{\leftharpoondown[\varsigma]}(w) = \mathbb{E}_{[\varsigma]}\left[\mathrm{v}_{[\varsigma]\rightharpoondown}(w\breve{\mathfrak{R}})\right]. \tag{5}$$

{eq:vMidStgShr}

The reward for all this notational investment is that it is now clear that *exactly the same code* for solving the portfolio share problem can be used in two distinct problems: a 'beginning-of-period-returns' model and an 'end-of-period-returns' model.

1.2.3 The 'beginning-of-period returns' Problem

{subsubsec:beginnin

The beginning-returns problem effectively just inserts a portfolio choice that happens at a stage immediately before the consumption stage in the optimal consumption problem described in (3), for which we had a beginning-of-stage value function $\mathrm{v}_{\leftharpoondown[\mathrm{c}]}(b)$. The agent makes their portfolio share decision within the stage but (obviously) before the risky returns $\mathbf{R}$ for the period have been realized. So the problem's portfolio-choice stage also takes k as its initial state and solves the investor's problem outlined in section 1.2.2:

$$\begin{aligned} \mathrm{v}_{\leftharpoondown[\varsigma]}(k) &= \mathbb{E}_{\leftharpoondown[\varsigma]}[\mathrm{v}_{[\varsigma]\rightharpoondown}(\underbrace{k\breve{\mathfrak{R}}}_{b})] \\ \mathrm{v}_{[\varsigma]\rightharpoondown}(b) &= \mathrm{v}_{\leftharpoondown[\mathrm{c}]}(b) \end{aligned} \tag{6}$$

Since in this setup bank balances have been determined before the consumption problems starts, we need to rewrite the consumption stage as a function of bank balances that will have resulted from the portfolio investment b, combined with the income shocks θ:

$$\mathrm{v}_{\leftharpoondown[\mathrm{c}]}(b) = \max_{\mathrm{c}}\ \mathrm{u}(c) + \mathbb{E}_{\leftharpoondown[\mathrm{c}]}[\mathrm{v}_{[\mathrm{c}]\rightharpoondown}(\underbrace{\overbrace{b+\theta}^{m} - c}_{a})] \tag{7}$$

and since the consumption stage is the last stage in the period, the (undated) a that emerges from this equation is equivalent to the a_t characterizing the end of the period. The 'state transition' equation between t and $t+1$ is simply $k_{t+1} = a_t$ and the continuation-value function transition is $\mathrm{v}_{t\rightharpoondown}(k) \leftrightarrow \beta\mathrm{v}_{\leftharpoondown(t+1)}(k)$ which reflects the above-mentioned point that there is no substantive difference between the two problems (their $\mathrm{v}_{[\mathrm{c}]}(m)$ value functions and $\mathrm{c}(m)$ functions will be identical).

$$\mathrm{v}_{[\mathrm{c}]\rightharpoondown}(a) = \mathrm{v}_{t\rightharpoondown}(a) \tag{8}$$

(and recall that $\mathrm{v}_{t\rightharpoondown}(a)$ is exogenously provided as an input to the period's problem via the transition equation assumed earlier: $\mathrm{v}_{t\rightharpoondown}(a) = \beta\mathrm{v}_{\leftharpoondown(t+1)}(a)$).

1.2.4 The 'end-of-period-returns' Problem

If the portfolio share and risky returns are realized at the end of the period, we need to move the portfolio choice stage to immediately before the point at which returns are realized (and after the c choice has been made). This creates a slight awkwardness because the variable we have heretofore dubbed a is no longer the end-of-period state, since this money must be invested and the returns realized before the end of the period. We want to continue using a for 'assets-after-all-actions-are-accomplished' but now to include the 'actions' of the market, so we will temporarily designate the consumer's unspent market resources by $w = mNrm - c$ because defined w earlier as the input to the investor's problem. So, the portfolio stage of the problem is

$$\mathrm{v}_{\leftharpoondown[\varsigma]}(w) = \mathbb{E}_{\leftharpoondown[\varsigma]}[\mathrm{v}_{[\varsigma]\rightharpoondown}(\underbrace{\breve{\mathfrak{R}}w}_{\equiv a})] \tag{9}$$

so the continuation-value function is $\mathrm{v}_{[\varsigma]\rightharpoondown}(\underbrace{a}_{\equiv\mathfrak{R}w})$ is still a function of a (and the 'state transition' equation between t and $t+1$ remains $k_{t+1} = a_t$ and the continuation-value function transition is $\mathrm{v}_{t\rightharpoondown}(a) \leftrightarrow \beta\mathrm{v}_{\leftharpoondown(t+1)}(k)$.

(Note that we are assuming that there will be only one consumption function in the period, so no stage subscript is necessary to pick out 'the consumption function').

1.2.5 Numerical Solution

we can solve it numerically for the optimal ς at a vector of $\boldsymbol{a}$ (`aVec` in the code) and then construct an approximated optimal portfolio share function $\grave{\varsigma}(a)$ as the interpolating function among the members of the $\{\boldsymbol{a}, \boldsymbol{\varsigma}\}$ mapping. Having done this, we can now calculate a vector of values and marginal values that correspond to `aVec`:

$$\begin{aligned} \boldsymbol{v} &= \mathrm{v}_{\leftharpoondown[\varsigma]}(\boldsymbol{a}) \\ \boldsymbol{v}^{a} &= \mathrm{v}^{a}_{\leftharpoondown[\varsigma]}(\boldsymbol{a}). \end{aligned} \tag{10}$$ {eq:vShrEnd}

With the $\boldsymbol{v}^a$ approximation described in hand, we can construct our approximation to the consumption function using *exactly the same EGM procedure* that we used in solving the problem *without* a portfolio choice (see (34)):

$$\boldsymbol{c} \equiv (\boldsymbol{v}^{a})^{-1/\rho}, \tag{11}$$ {eq:cVecPort}

which, following a procedure identical to that in the EGM subsection 6.8, yields an approximated consumption function $\grave{c}_t(m)$. Thus, again, we can construct the consumption function at nearly zero cost (once we have calculated $\boldsymbol{v}^a$).

1.2.6 The Point

{subsubsec:the-poin

The upshot is that all we need to do is change some of the transition equations and we can use the same solution code (both for the ς-stage and the c-stage) to solve the problem with either assumption (beginning-of-period or end-of-period) about the timing of portfolio choice. There is even an obvious notation for the two problems: $\mathrm{v}_{\leftharpoondown t[\varsigma c]}$ can be the period-arrival value function for the version where the portfolio share is chosen at the beginning of the period, and $\mathrm{v}_{\leftharpoondown t[c\varsigma]}$ is period-arrival value for the the problem where the share choice is at the end.

What is the benefit of writing effectively the identical problem in two different ways? There are several:

- It demonstrates that, if they are carefully constructed, Bellman problems can be "modular"
  - In a life cycle model one might want to assume that at at some ages agents have a portfolio choice and at other ages they do not. The consumption problem makes no assumption about whether there is a portfolio choice decision (before or after the consumption choice), so there would be zero cost of having an age-varying problem in which you drop in whatever choices are appropriate to the life cycle stage.
- It emphasizes the flexibilty of choice a modeler has to date variables arbitrarily. In the specific example examined here, there is a strong case for preferring the beginning-returns specification because we typically think of productivity or other shocks at date t affecting the agent's state variables before the agent makes that period's choices. It would be awkward and confusing to have a productivity shock

dated $t-1$ effectively applying for the problem being solved at t (as in the end-returns specification)

- It may help to identify more efficient solution methods
  - For example, under the traditional formulation in equation (2) it might not occur to a modeler that the endogenous gridpoints solution method can be used, because when portfolio choice and consumption choice are considered simultaneously the EGM method breaks down because the portfolio choice part of the problem is not susceptible to EGM solution. But when the problem is broken into two simpler problems, it becomes clear that EGM can still be applied to the consumption problem even though it cannot be applied to the portfolio choice problem

1.3 Application

{subsec:MCApplica

In specifying the stochastic process for $\mathbf{R}_{t+1}$, we follow the common practice of assuming that returns are lognormally distributed, $\log \mathbf{R} \sim \mathcal{N}(\phi + \mathsf{r} - \sigma^2_{\mathbf{r}}/2, \sigma^2_{\mathbf{r}})$ where ϕ is the equity premium over the thin returns r available on the riskless asset.[1]

As with labor income uncertainty, it is necessary to discretize the rate-of-return risk in order to have a problem that is soluble in a reasonable amount of time. We follow the same procedure as for labor income uncertainty, generating a set of $n_{\mathbf{r}}$ equiprobable shocks to the rate of return; in a slight abuse of notation, we will designate the portfolio-weighted return (contingent on the chosen portfolio share in equity, and potentially contingent on any other aspect of the consumer's problem) simply as $\mathfrak{R}_{i,j}$ (where dependence on i is allowed to permit the possibility of nonzero correlation between the return on the risky asset and the θ shock to labor income (for example, in recessions the stock market falls and labor income also declines).

The direct expressions for the derivatives of $\underrightarrow{\mathrm{v}}$ are

$$
\begin{aligned}
\underrightarrow{\mathrm{v}}^{a}(a_t, \varsigma_t) &= \beta \left(\frac{1}{n_{\mathbf{r}} n_{\theta}}\right) \sum_{i=1}^{n_\theta} \sum_{j=1}^{n_{\mathbf{r}}} \mathfrak{R}_{i,j} \left(\mathrm{c}_{t+1}(\mathfrak{R}_{i,j} a_t + \theta_i)\right)^{-\rho} \\
\underrightarrow{\mathrm{v}}^{\varsigma}(a_t, \varsigma_t) &= \beta \left(\frac{1}{n_{\mathbf{r}} n_{\theta}}\right) \sum_{i=1}^{n_\theta} \sum_{j=1}^{n_{\mathbf{r}}} (\mathbf{R}_{i,j} - \mathsf{R}) \left(\mathrm{c}_{t+1}(\mathfrak{R}_{i,j} a_t + \theta_i)\right)^{-\rho}.
\end{aligned} \tag{12}
$$

Writing these equations out explicitly makes a problem very apparent: For every different combination of $\{a_t, \varsigma_t\}$ that the routine wishes to consider, it must perform two double-summations of $n_{\mathbf{r}} \times n_\theta$ terms. Once again, there is an inefficiency if it must perform these same calculations many times for the same or nearby values of $\{a_t, \varsigma_t\}$, and again the solution is to construct an approximation to the (inverses of the) derivatives of the $\underrightarrow{\mathrm{v}}$ function.

Details of the construction of the interpolating approximations are given below; assume for the moment that we have the approximations $\grave{\underrightarrow{\mathrm{v}}}^{a}$ and $\grave{\underrightarrow{\mathrm{v}}}^{\varsigma}$ in hand and we want

[1] This guarantees that $\mathbb{E}[\mathbf{R}] = \boldsymbol{\varphi}/\mathsf{R}$ is invariant to the choice of σ^2_ϕ; see LogELogNorm.

to proceed. As noted above in the discussion of (2), nonlinear equation solvers can find the solution to a set of simultaneous equations. Thus we could ask one to solve

$$\begin{aligned} c_t^{-\rho} &= \grave{\mathrm{v}}^{a}_{t_\rightharpoondown}(m_t - c_t, \varsigma_t) \\ 0 &= \grave{\mathrm{v}}^{\varsigma}_{t_\rightharpoondown}(m_t - c_t, \varsigma_t) \end{aligned} \tag{13}$$

{eq:FOCwrtw}

simultaneously for c and ς at the set of potential m_t values defined in `mVec`. However, as noted above, multidimensional constrained maximization problems are difficult and sometimes quite slow to solve.

There is a better way. Define the problem

$$\begin{aligned} \check{\mathrm{v}}_{t_\rightharpoondown}(a_t) &= \max_{\varsigma_t} \quad \mathrm{v}_\rightharpoondown(a_t, \varsigma_t) \\ &\text{s.t.} \\ 0 \leq &\varsigma_t \leq 1 \end{aligned}$$

where the tilde over $\check{\mathrm{v}}(a)$ indicates that this is the v that has been optimized with respect to all of the arguments other than the one still present (a_t). We solve this problem for the set of gridpoints in `aVec` and use the results to construct the interpolating function $\grave{\check{\mathrm{v}}}^a_t(a_t)$.[2] With this function in hand, we can use the first order condition from the single-control problem

$$c_t^{-\rho} = \grave{\check{\mathrm{v}}}^a_t(m_t - c_t)$$

to solve for the optimal level of consumption as a function of m_t using the endogenous gridpoints method described above. Thus we have transformed the multidimensional optimization problem into a sequence of two simple optimization problems.

Note the parallel between this trick and the fundamental insight of dynamic programming: Dynamic programming techniques transform a multi-period (or infinite-period) optimization problem into a sequence of two-period optimization problems which are individually much easier to solve; we have done the same thing here, but with multiple dimensions of controls rather than multiple periods.

1.4 Implementation

Following the discussion from section 1.1, to provide a numerical solution to the problem with multiple control variables, we must define expressions that capture the expected marginal value of end-of-period assets with respect to the level of assets and the share invested in risky assets. This is addressed in “Multiple Control Variables.”

[2] A faster solution could be obtained by, for each element in `aVec`, computing $\mathrm{v}^{\varsigma}_{\rightharpoondown}(m_t - c_t, \varsigma)$ of a grid of values of ς, and then using an approximating interpolating function (rather than the full expectation) in the `FindRoot` command. The associated speed improvement is fairly modest, however, so this route was not pursued.

1.5 Results With Multiple Controls

{subsec:results-with

Figure 1 plots the $t-1$ consumption function generated by the program; qualitatively it does not look much different from the consumption functions generated by the program without portfolio choice.

But Figure 2 which plots the optimal portfolio share as a function of the level of assets, exhibits several interesting features. First, even with a coefficient of relative risk aversion of 6, an equity premium of only 4 percent, and an annual standard deviation in equity returns of 15 percent, the optimal choice is for the agent to invest a proportion 1 (100 percent) of the portfolio in stocks (instead of the safe bank account with riskless return R) is at values of a_t less than about 2. Second, the proportion of the portfolio kept in stocks is *declining* in the level of wealth - i.e., the poor should hold all of their meager assets in stocks, while the rich should be cautious, holding more of their wealth in safe bank deposits and less in stocks. This seemingly bizarre (and highly counterfactual – see Carroll (2002)) prediction reflects the nature of the risks the consumer faces. Those consumers who are poor in measured financial wealth will likely derive a high proportion of future consumption from their labor income. Since by assumption labor income risk is uncorrelated with rate-of-return risk, the covariance between their future consumption and future stock returns is relatively low. By contrast, persons with relatively large wealth will be paying for a large proportion of future consumption out of that wealth, and hence if they invest too much of it in stocks their consumption will have a high covariance with stock returns. Consequently, they reduce that correlation by holding some of their wealth in the riskless form.

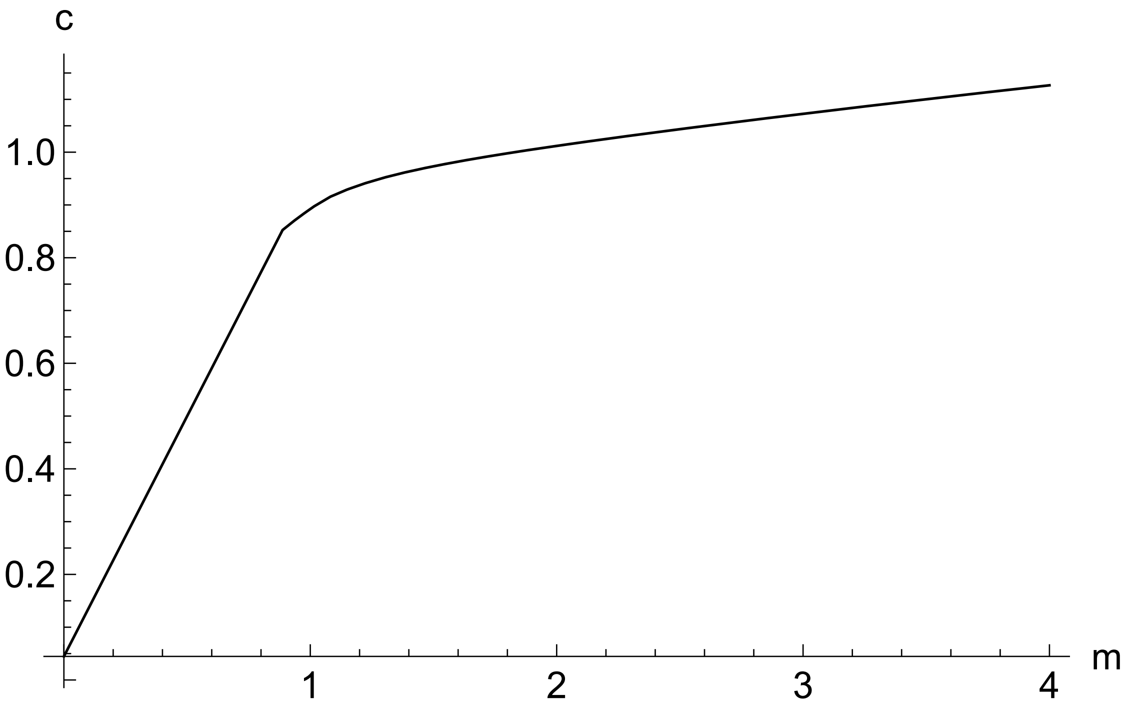

Figure 1 c(m_1) With Portfolio Choice

{fig:PlotctMultCont

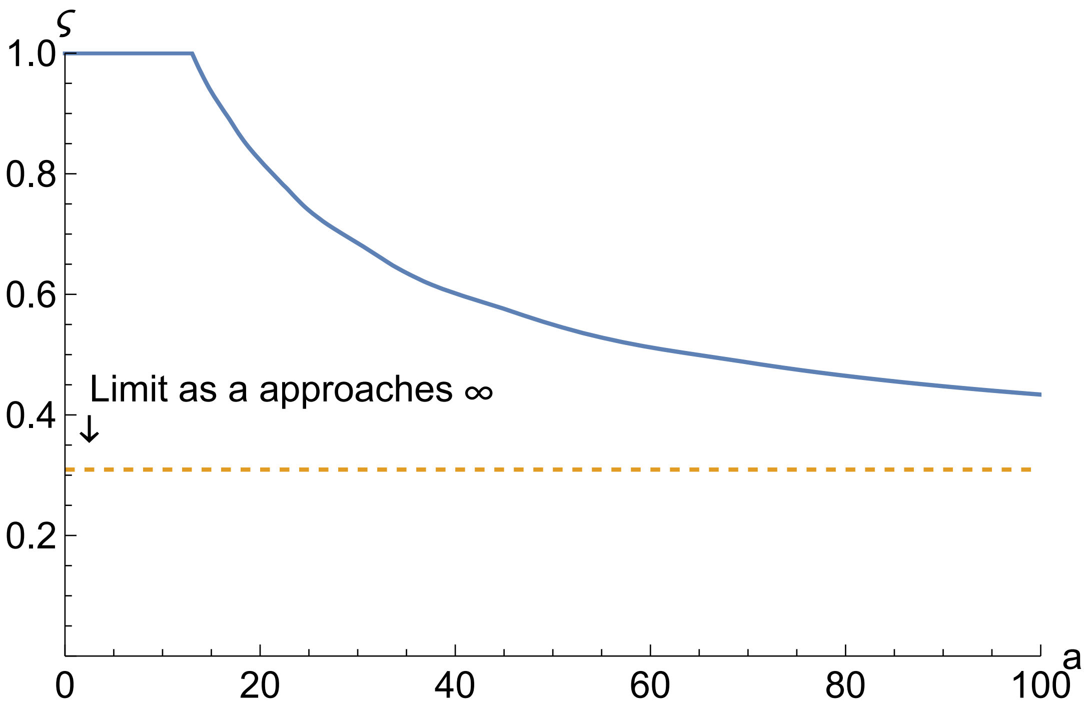

Figure 2 Portfolio Share in Risky Assets in First Period $\varsigma(a)$

{fig:PlotRiskyShare