The Method of Moderation

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Christopher D. Carroll[1]
JHU

Karsten Chipeniuk
RBNZ

Kiichi Tokuoka[2]
ECB

Weifeng Wu[3]
Fannie Mae

Abstract

In a risky world, a pessimist assumes the worst will happen. Someone who ignores risk altogether is an optimist. Consumption decisions are mathematically simple for both the pessimist and the optimist because both behave as if they live in a riskless world. A realist (that is, someone who wants to respond optimally to risk) faces a much more difficult problem, but (under standard conditions) will choose a level of spending somewhere between pessimist's and the optimist's. We use this fact to redefine the space in which the realist searches for optimal consumption rules. The resulting solution accurately represents the numerical consumption rule over the entire interval of feasible wealth values with remarkably few computations.

Keywords Dynamic Stochastic Optimization

JEL codes FillInLater

[1]Carroll: Department of Economics, Johns Hopkins University, Baltimore, MD, http://econ.jhu.edu/people/ccarroll/, ccarroll@jhu.edu [2]Tokuoka: International Monetary Fund, Washington, DC, ktokuoka@imf.org [3]Wu: Weifeng Wu, Fanie Mae, Washington DC, .

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1 Introduction

{sec:Intro}

Solving a consumption, investment, portfolio choice, or similar intertemporal optimization problem using numerical methods generally requires the modeler to choose how to represent a policy or value function. In the stochastic case, where analytical solutions are generally not available, a common approach is to use low-order polynominal splines that exactly match the function (and maybe some derivatives) at a finite set of gridpoints, and then to assume that interpolated or extrapolated versions of that spline represent the function well at the continuous infinity of unmatched points.

This paper argues that a better approach in the standard consumption problem is to rely upon the fact that without uncertainty, the optimal consumption function has a simple analytical solution. The key insight is that, under standard assumptions, the consumer who faces an uninsurable labor income risk will consume less than a consumer with the same path for expected income but who does not perceive any uncertainty as being attached to that future income. The 'realistic' consumer, who *does* perceive the risks, will engage in 'precautionary saving,' so the perfect foresight riskless solution provides an upper bound to the solution that will actually be optimal. A lower bound is provided by the behavior of a consumer who has the subjective belief that the future level of income will be the worst that it can possibly be. This consumer, too, behaves according to the convenient analytical perfect foresight solution, but his certainty is that of a pessimist perfectly confident in his pessimism.

Using results from **?**, we show how to use these upper and lower bounds to tightly constrain the shape and characteristics of the solution to problem of the 'realist.' Imposition of these constraints can clarify and speed the solution of the realist's problem.

After showing how to use the method in the baseline case, we show how refine it to encompass an even tighter theoretical bound.

2 The Realist's Problem

We assume that truly optimal behavior in the problem facing the consumer who understands all his risks is captured by

$$\max\ \mathbb{E}_t\left[\sum_{n=0}^{T-t}\beta^n \mathrm{u}(\mathbf{c}_{t+n})\right]. \tag{1}$$ {eq:MaxProb}

subject to

$$\begin{aligned}\mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t\\ \mathbf{b}_{t+1} &= \mathbf{a}_t\mathsf{R}_{t+1}\\ \mathbf{y}_{t+1} &= \mathbf{p}_{t+1}\boldsymbol{\theta}_{t+1}\\ \mathbf{m}_{t+1} &= \mathbf{b}_{t+1}+\mathbf{y}_{t+1}\end{aligned} \tag{2}$$ {DBCLevel}

where

$$
\begin{aligned}
\beth &- \text{pure time discount factor} \\
\mathbf{a}_t &- \text{assets after all actions have been accomplished in period } t \\
\mathbf{b}_{t+1} &- \text{‘bank balances’ (nonhuman wealth) at the beginning of } t+1 \\
\mathbf{c}_t &- \text{consumption in period } t \\
\mathbf{m}_t &- \text{‘market resources’ available for consumption (‘cash-on-hand’)} \\
\mathbf{p}_{t+1} &- \text{‘permanent labor income’ in period } t+1 \\
\mathsf{R}_{t+1} &- \text{interest factor } (1+\mathsf{r}_{t+1}) \text{ from period } t \text{ to } t+1 \\
\mathbf{y}_{t+1} &- \text{noncapital income in period } t+1.
\end{aligned}
$$

and the exogenous variables evolve according to

$$
\begin{aligned}
\mathbf{p}_{t+1} &= \mathcal{G}_{t+1}\mathbf{p}_t && - \text{permanent labor income dynamics} \\
\log\ \boldsymbol{\theta}_{t+n} &\sim\ \mathcal{N}(-\sigma_{\boldsymbol{\theta}}^2/2, \sigma_{\boldsymbol{\theta}}^2) && - \text{lognormal transitory shocks } \forall\ n > 0.
\end{aligned} \tag{3}
$$

{eq:permincgrow}

It turns out (see **?** for a proof) that this problem can be rewritten in a more convenient form in which choice and state variables are normalized by the level of permanent income, e.g., using nonbold font for normalized variables, $m_t = \mathbf{c}_t/\mathbf{p}_t$. When that is done, the Bellman equation for the transformed version of the consumer's problem is

$$
\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{c_t} \quad \mathrm{u}(c_t) + \beta\mathbb{E}_t[\mathcal{G}_{t+1}^{1-\rho}\mathrm{v}_{t+1}(m_{t+1})] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
k_{t+1} &= a_t \\
b_{t+1} &= \underbrace{(\mathsf{R}/\mathcal{G}_{t+1})}_{\equiv\mathcal{R}_{t+1}} k_{t+1} \\
m_{t+1} &= b_{t+1} + \boldsymbol{\theta}_{t+1},
\end{aligned} \tag{4}
$$

{eq:vNormed}

and because we have not imposed a liquidity constraint, the solution satisfies the Euler equation

$$
\mathrm{u}^c(c_t) = \mathbb{E}_{\rightarrow}[\beta\mathsf{R}\mathcal{G}_{t+1}^{-\rho}\mathrm{u}^c(c_{t+1})]. \tag{5}
$$

{eq:cEuler}

For the remainder of the paper we will assume that permanent income $\mathbf{p}_t$ grows by a constant factor $\mathcal{G}$ and is not subject to stochastic shocks. (The generalization to the case with permanent shocks is straightforward.)

3 Benchmark: The Method of Endogenous Gridpoints

For comparison to our new solution method, we use the endogenous gridpoints solution to the microeconomic problem presented in **?**. That method computes the level of consumption at a set of gridpoints for market resources m that are determined endogenously using the Euler equation. The consumption function is then constructed by linear interpolation among the gridpoints thus found.

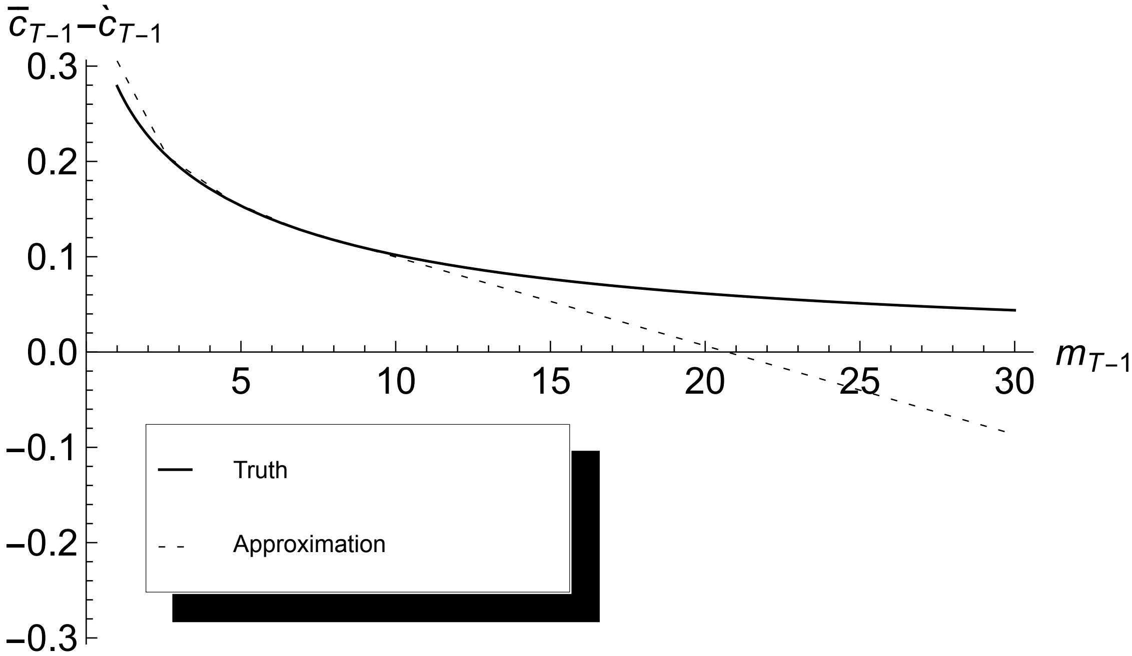


Figure 1 For Large Enough m_{t-1}, Predicted Precautionary Saving is Negative (Oops!)

{fig:ExtrapProblem

? describes a specific calibration of the model and constructs a solution using five gridpoints chosen to capture the structure of the consumption function reasonably well at values of m near the infinite-horizon target value. (See those notes for details).

Unfortunately, this endogenous gridpoints solution is not very well-behaved outside the original range of gridpoints targeted by the solution method. (Though other common solution methods are no better outside their own predefined ranges). Figure 1 demonstrates the point by plotting the amount of precautionary saving implied by a linear extrapolation of our approximated consumption rule (the consumption of the perfect foresight consumer $\bar{\mathrm{c}}_{t-1}$ minus our approximation to optimal consumption under uncertainty, $\grave{\mathrm{c}}_{t-1}$). Although theory proves that precautionary saving is always positive, the linearly extrapolated numerical approximation eventually predicts negative precautionary saving (at the point in the figure where the extrapolated locus crosses the horizontal axis).

This error cannot be fixed by extending the upper gridpoint; in the presence of serious uncertainty, the consumption rule will need to be evaluated outside of *any* prespecified grid (because starting from the top gridpoint, a large enough realization of the uncertain variable will push next period's realization of assets above that top; a similar argument applies below the bottom gridpoint). While a judicious extrapolation technique can prevent this problem from being fatal (for example by carefully excluding negative precautionary saving), the problem is often dealt with using inelegant methods whose implications for the accuracy of the solution are difficult to gauge.

4 The Method of Moderation

4.1 The Optimist, the Pessimist, and the Realist

4.1.1 The Consumption Function

As a preliminary to our solution, define $h_{\rightarrow}$ as end-of-period human wealth (the present discounted value of future labor income) for a perfect foresight version of the problem of a 'risk optimist:' a period-t consumer who believes with perfect confidence that the shocks will always take their expected value of 1, $\boldsymbol{\theta}_{t+n} = \mathbb{E}[\boldsymbol{\theta}] = 1 \ \forall \ n > 0$. The solution to a perfect foresight problem of this kind takes the form[1]

$$\bar{\mathrm{c}}_t(m_t) = (m_t + h_{\rightarrow})\underline{\kappa}_t \tag{6}$$ {eq:cFuncAbove}

for a constant minimal marginal propensity to consume $\underline{\kappa}_t$ given below.

$$\bar{\mathrm{c}}_t(m_t) = (m_t + h_{\rightarrow})\underline{\kappa}_t \tag{7}$$ {eq:cFuncAbove}

for a constant minimal marginal propensity to consume $\underline{\kappa}_t$ given below. We similarly define $\underline{h}_{\rightarrow}$ as 'minimal human wealth,' the present discounted value of labor income if the shocks were to take on their worst possible value in every future period $\boldsymbol{\theta}_{t+n} = \underline{\boldsymbol{\theta}} \ \forall \ n > 0$ (which we define as corresponding to the beliefs of a 'pessimist').

We will call a 'realist' the consumer who correctly perceives the true probabilities of the future risks and optimizes accordingly.

A first useful point is that, for the realist, a lower bound for the level of market resources is $\underline{m}_t = -\underline{h}_{\rightarrow}$, because if m_t equalled this value then there would be a positive finite chance (however small) of receiving $\boldsymbol{\theta}_{t+n} = \underline{\boldsymbol{\theta}}$ in every future period, which would require the consumer to set c_t to zero in order to guarantee that the intertemporal budget constraint holds (this is the multiperiod generalization of the discussion in section **??** explaining the derivation of the 'natural borrowing constraint' for period $T-1$, $\underline{a}_{t-1}$). Since consumption of zero yields negative infinite utility, the solution to realist consumer's problem is not well defined for values of $m_t < \underline{m}_t$, and the limiting value of the realist's c_t is zero as $m_t \downarrow \underline{m}_t$.

Given this result, it will be convenient to define 'excess' market resources as the amount by which actual resources exceed the lower bound, and 'excess' human wealth as the amount by which mean expected human wealth exceeds guaranteed minimum human wealth:

$$\begin{aligned}
\blacktriangle m_t &= m_t + \overbrace{\underline{h}_{\rightarrow}}^{=-\underline{m}_t} \\
\blacktriangle h_{\rightarrow} &= h_{\rightarrow} - \underline{h}_{\rightarrow}.
\end{aligned}$$

We can now transparently define the optimal consumption rules for the two perfect foresight problems, those of the 'optimist' and the 'pessimist.' The 'pessimist' perceives human wealth to be equal to its minimum feasible value $\underline{h}_{\rightarrow}$ with certainty, so consump-

[1]For a derivation, see **?**; $\underline{\kappa}_t$ is defined therein as the MPC of the perfect foresight consumer with horizon $T-t$.

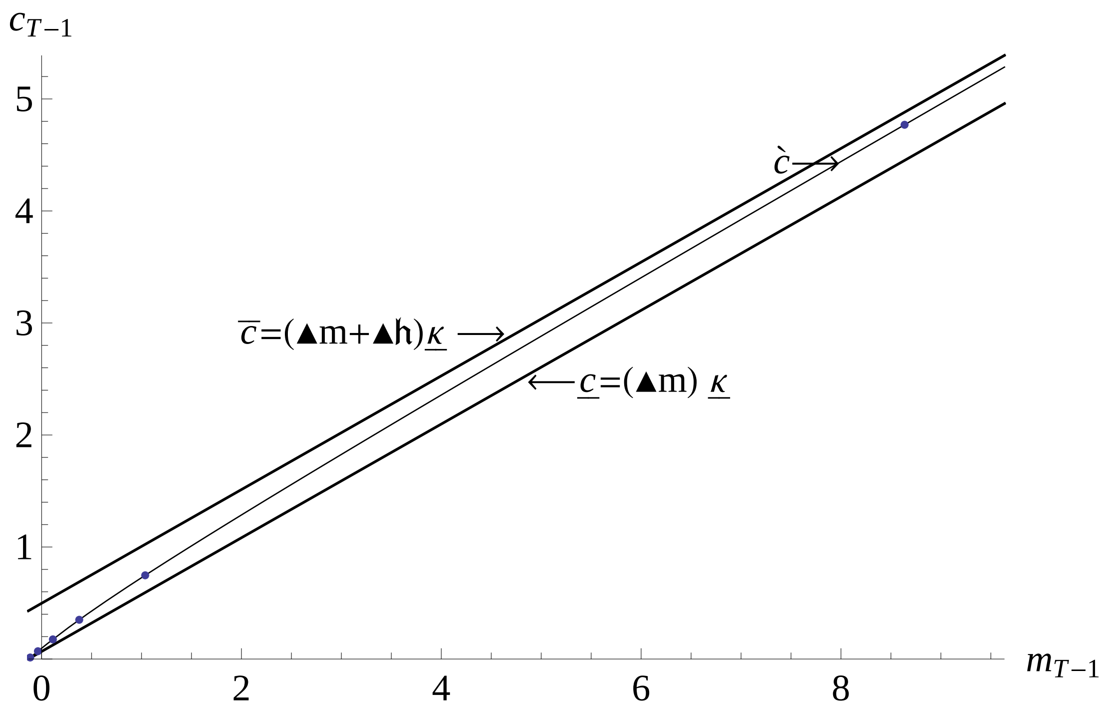


Figure 2 Moderation Illustrated: $\underline{\mathrm{c}}_{t-1} < \grave{\mathrm{c}}_{t-1} < \bar{\mathrm{c}}_{t-1}$

{fig:IntExpFOCInv

tion is given by the perfect foresight solution

$$\begin{aligned}\underline{\mathrm{c}}_t(m_t) &= (m_t + \underline{h}_{\rightarrow})\underline{\kappa}_t \\ &= \blacktriangle m_t \underline{\kappa}_t.\end{aligned}$$

The 'optimist,' on the other hand, pretends that there is no uncertainty about future income, and therefore consumes

$$\begin{aligned}\bar{\mathrm{c}}_t(m_t) &= (m_t + \underline{h}_{\rightarrow} - \underline{h}_{\rightarrow} + h_{\rightarrow})\underline{\kappa}_t \\ &= (\blacktriangle m_t + \blacktriangle h_{\rightarrow})\underline{\kappa}_t \\ &= \underline{\mathrm{c}}_t(m_t) + \blacktriangle h_{\rightarrow}\underline{\kappa}_t.\end{aligned}$$

It seems obvious that the spending of the realist will be strictly greater than that of the pessimist and strictly less than that of the optimist. Figure 2 illustrates the proposition for the consumption rule in period $T-1$.

The proof is more difficult than might be imagined, but the necessary work is done in **?** so we will take the proposition as a fact and proceed by manipulating the inequality:

$$\begin{array}{rcl}\blacktriangle m_t \underline{\kappa}_t < & \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) < (\blacktriangle m_t + \blacktriangle h_{\rightarrow})\underline{\kappa}_t & \\ -\blacktriangle m_t \underline{\kappa}_t > & -\mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & > -(\blacktriangle m_t + \blacktriangle h_{\rightarrow})\underline{\kappa}_t \\ \blacktriangle h_{\rightarrow}\underline{\kappa}_t > & \bar{\mathrm{c}}_t(\underline{m}_t + \blacktriangle m_t) - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & > 0 \\ 1 > & \underbrace{\left(\frac{\bar{\mathrm{c}}_t(\underline{m}_t + \blacktriangle m_t) - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t)}{\blacktriangle h_{\rightarrow}\underline{\kappa}_t}\right)}_{\equiv \bar{\varphi}_t} & > 0\end{array}$$

where the fraction in the middle of the last inequality is the ratio of actual precautionary saving (the numerator is the difference between perfect-foresight consumption and optimal consumption in the presence of uncertainty) to the maximum conceivable amount of precautionary saving (the amount that would be undertaken by the pessimist who consumes nothing out of any future income beyond the perfectly certain component). Defining $\mu_t = \log \blacktriangle m_t$ (which can range from $-\infty$ to ∞), the object in the middle of the last inequality is

$$\bar{\koppa}_t(\mu_t) \equiv \left(\frac{\bar{\mathrm{c}}_t(\underline{m}_t + e^{\mu_t}) - \mathrm{c}_t(\underline{m}_t + e^{\mu_t})}{\blacktriangle h_{\rightarrow}\underline{\kappa}_t}\right), \tag{8}$$ {eq:koppa}

and we now define

$$\begin{aligned} {}_t(\mu_t) &= \log\left(\frac{1-\bar{\koppa}_t(\mu_t)}{\bar{\koppa}_t(\mu_t)}\right) \\ &= \log\left(1/\bar{\koppa}_t(\mu_t) - 1\right) \end{aligned} \tag{9}$$ {eq:chi}

which has the virtue that it is linear in the limit as μ_t approaches $+\infty$.

Given , the consumption function can be recovered from

$$\bar{\mathrm{c}}_t = \bar{\mathrm{c}}_t - \overbrace{\left(\frac{1}{1+\exp({}_t)}\right)}^{=\bar{\koppa}_t} \blacktriangle h_{\rightarrow}\underline{\kappa}_t. \tag{10}$$ {eq:cFuncHi}

Thus, the procedure is to calculate ${}_t$ at the points $\boldsymbol{\mu}_t$ corresponding to the log of the $\blacktriangle\boldsymbol{m}_t$ points defined above, and then using these to construct an interpolating approximation $\grave{}_t$ from which we indirectly obtain our approximated consumption rule $\grave{\bar{\mathrm{c}}}_t$ by substituting $\grave{}_t$ for in equation (10).

Because this method relies upon the fact that the problem is easy to solve if the decision maker has unreasonable views (either in the optimistic or the pessimistic direction), and because the correct solution is always between these immoderate extremes, we call our solution procedure the 'method of moderation.'

Results are shown in Figure 3; a reader with very good eyesight might be able to detect the barest hint of a discrepancy between the Truth and the Approximation at the far righthand edge of the figure – a stark contrast with the calamitous divergence evident in Figure 1.

4.1.2 The Value Function

Often it is useful to know the value function as well as the consumption rule. Fortunately, many of the tricks used when solving for the consumption rule have a direct analogue in approximation of the value function.

Consider the perfect foresight (or "optimist's") problem in period $T-1$. Using the fact that in a perfect foresight model the growth factor for consumption is $(\mathsf{R}\beta)^{1/\rho}$, we

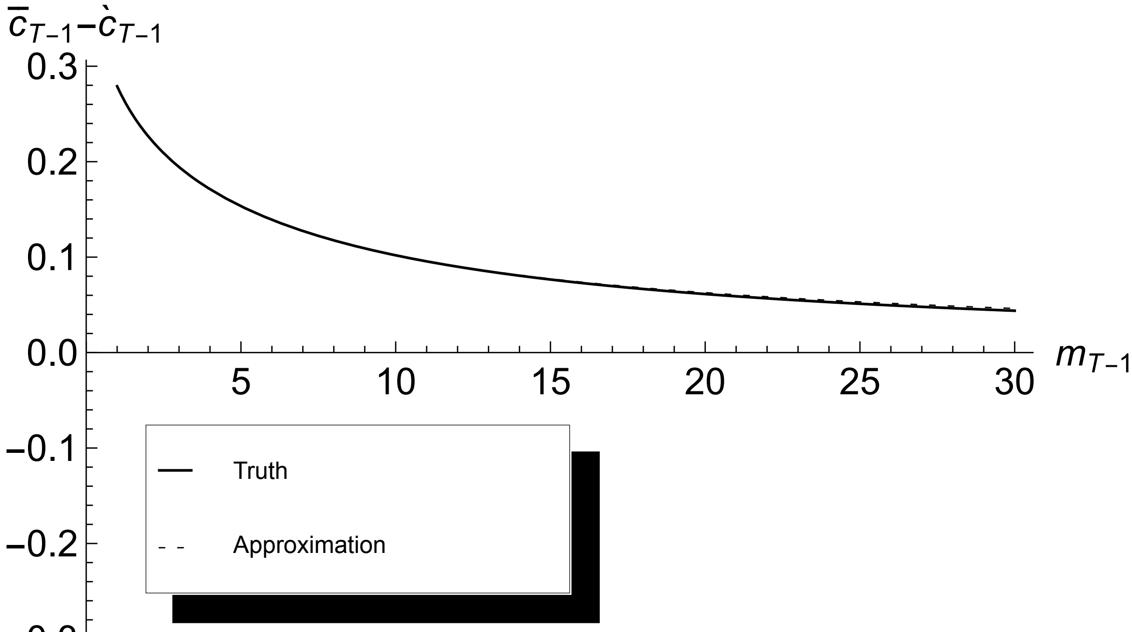


Figure 3 Extrapolated $\grave{\bar{\mathrm{c}}}_{t-1}$ Constructed Using the Method of Moderation

{fig:ExtrapProblem

can use the fact that $c_t = (\mathsf{R}\beta)^{1/\rho} c_{t-1}$ to calculate the value function in period $T-1$:

$$
\begin{aligned}
\bar{\mathrm{v}}_{t-1}(m_{t-1}) &\equiv \mathrm{u}(c_{t-1}) + \beta \mathrm{u}(c_t) \\
&= \mathrm{u}(c_{t-1}) \left(1 + \beta((\beta\mathsf{R})^{1/\rho})^{1-\rho}\right) \\
&= \mathrm{u}(c_{t-1}) \left(1 + (\beta\mathsf{R})^{1/\rho}/\mathsf{R}\right) \\
&= \mathrm{u}(c_{t-1}) \underbrace{\mathrm{PDV}_t^T(c)/c_{t-1}}_{\equiv \mathbb{C}_{t-1}^T}
\end{aligned}
$$

where $\mathbb{C}_t^T = \mathrm{PDV}_t^T(c)$ is the present discounted value of consumption, normalized by current consumption. Using the fact demonstrated in **?** that $\mathbb{C}_t = \kappa_t^{-1}$, a similar function can be constructed recursively for earlier periods, yielding the general expression

$$
\begin{aligned}
\bar{\mathrm{v}}_t(m_t) &= \mathrm{u}(\bar{c}_t)\mathbb{C}_t^T \\
&= \mathrm{u}(\bar{c}_t)\underline{\kappa}_t^{-1} \\
&= \mathrm{u}((\blacktriangle m_t + \blacktriangle h_{\rightarrow})\underline{\kappa}_t)\underline{\kappa}_t^{-1} \\
&= \mathrm{u}(\blacktriangle m_t + \blacktriangle h_{\rightarrow})\underline{\kappa}_t^{1-\rho}\underline{\kappa}_t^{-1} \\
&= \mathrm{u}(\blacktriangle m_t + \blacktriangle h_{\rightarrow})\underline{\kappa}_t^{-\rho}
\end{aligned}
\tag{11}
$$

{eq:vFuncPF}

This can be transformed as

$$\begin{aligned}\bar{\Lambda}_t &\equiv ((1-\rho)\bar{\mathrm{v}}_t)^{1/(1-\rho)} \\ &= c_t(\mathbb{C}_t^T)^{1/(1-\rho)} \\ &= (\blacktriangle m_t + \blacktriangle h_{\rightarrow})\underline{\kappa}_t^{-\rho/(1-\rho)}\end{aligned}$$

with derivative

$$\begin{aligned}\bar{\Lambda}_t^m &= (\mathbb{C}_t^T)^{1/(1-\rho)}\underline{\kappa}_t, \\ &= \underline{\kappa}_t^{-\rho/(1-\rho)}\end{aligned}$$

and since $\mathbb{C}_t^T$ is a constant while the consumption function is linear, $\bar{\Lambda}_t$ will also be linear.

We apply the same transformation to the value function for the problem with uncertainty (the "realist's" problem) and differentiate

$$\begin{aligned}\bar{\Lambda}_t &= ((1-\rho)\bar{\mathrm{v}}_t(m_t))^{1/(1-\rho)} \\ \bar{\Lambda}_t^m &= ((1-\rho)\bar{\mathrm{v}}_t(m_t))^{-1+1/(1-\rho)}\,\bar{\mathrm{v}}_t^m(m_t)\end{aligned}$$

and an excellent approximation to the value function can be obtained by calculating the values of $\bar{\Lambda}$ at the same gridpoints used by the consumption function approximation, and interpolating among those points.

However, as with the consumption approximation, we can do even better if we realize that the $\bar{\Lambda}$ function for the optimist's problem is an upper bound for the Λ function in the presence of uncertainty, and the value function for the pessimist is a lower bound. Analogously to (8), define an upper-case

$$\hat{\mathrm{Q}}_t(\mu_t) = \left(\frac{\bar{\Lambda}_t(\underline{m}_t + e^{\mu_t}) - \Lambda_t(\underline{m}_t + e^{\mu_t})}{\blacktriangle h_{\rightarrow}\underline{\kappa}_t(\mathbb{C}_t^T)^{1/(1-\rho)}}\right) \tag{12}$$ {eq:Koppa}

with derivative (dropping arguments)

$$\hat{\mathrm{Q}}_t^\mu = (\blacktriangle h_{\rightarrow}\underline{\kappa}_t(\mathbb{C}_t^T)^{1/(1-\rho)})^{-1}e^{\mu_t}\,(\bar{\Lambda}_t^m - \Lambda_t^m) \tag{13}$$ {eq:KoppaPrime}

and an upper-case version of the equation in (9):

$$\begin{aligned}\hat{\mathrm{X}}_t(\mu_t) &= \log\left(\frac{1-\hat{\mathrm{Q}}_t(\mu_t)}{\hat{\mathrm{Q}}_t(\mu_t)}\right) \\ &= \log\left(1/\hat{\mathrm{Q}}_t(\mu_t) - 1\right)\end{aligned} \tag{14}$$ {eq:Chi}

with corresponding derivative

$$\hat{\mathrm{X}}_t^\mu = \left(\frac{-\hat{\mathrm{Q}}_t^\mu/\hat{\mathrm{Q}}_t^2}{1/\hat{\mathrm{Q}}_t - 1}\right) \tag{15}$$

and if we approximate these objects then invert them (as above with the $\bar{\varrho}$ and functions) we obtain a very high-quality approximation to our inverted value function at the same points for which we have our approximated value function:

$$\hat{\Lambda}_t = \bar{\Lambda}_t - \overbrace{\left(\frac{1}{1+\exp(\hat{\mathrm{X}}_t)}\right)}^{=\hat{\Omega}_t} \blacktriangle h_{\rightarrow} \underline{\kappa}_t (\mathbb{C}_t^T)^{1/(1-\rho)} \tag{16}$$

from which we obtain our approximation to the value function and its derivatives as

$$\begin{aligned} \hat{\mathrm{v}}_t &= \mathrm{u}(\hat{\Lambda}_t) \\ \hat{\mathrm{v}}_t^m &= \mathrm{u}^c(\hat{\Lambda}_t)\hat{\Lambda}^m \\ \hat{\mathrm{v}}_t^{mm} &= \mathrm{u}^{cc}(\hat{\Lambda}_t)(\hat{\Lambda}^m)^2 + \mathrm{u}^c(\hat{\Lambda}_t)\hat{\Lambda}^{mm}. \end{aligned} \tag{17}$$

Although a linear interpolation that matches the level of Λ at the gridpoints is simple, a Hermite interpolation that matches both the level and the derivative of the $\bar{\Lambda}_t$ function at the gridpoints has the considerable virtue that the $\bar{\mathrm{v}}_t$ derived from it numerically satisfies the envelope theorem at each of the gridpoints for which the problem has been solved.

If we use the double-derivative calculated above to produce a higher-order Hermite polynomial, our approximation will also match marginal propensity to consume at the gridpoints; this would guarantee that the consumption function generated from the value function would match both the level of consumption and the marginal propensity to consume at the gridpoints; the numerical differences between the newly constructed consumption function and the highly accurate one constructed earlier would be negligible within the grid.

5 Extensions

5.1 A Tighter Upper Bound

? derives an upper limit $\bar{\kappa}_t$ for the MPC as m_t approaches its lower bound. Using this fact plus the strict concavity of the consumption function yields the proposition that

$$\mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) < \bar{\kappa}_t \blacktriangle m_t. \tag{18}$$

The solution method described above does not guarantee that approximated consumption will respect this constraint between gridpoints, and a failure to respect the constraint can occasionally cause computational problems in solving or simulating the model. Here, we describe a method for constructing an approximation that always satisfies the constraint.

Defining $m_t^{\#}$ as the 'cusp' point where the two upper bounds intersect:

$$\begin{aligned} \left(\blacktriangle m_t^{\#} + \blacktriangle h_{\rightarrow}\right)\underline{\kappa}_t &= \bar{\kappa}_t \blacktriangle m_t^{\#} \\ \blacktriangle m_t^{\#} &= \frac{\underline{\kappa}_t \blacktriangle h_{\rightarrow}}{(1-\underline{\kappa}_t)\bar{\kappa}_t} \\ m_t^{\#} &= \frac{\underline{\kappa}_t h_{\rightarrow} - \underline{h}_{\rightarrow}}{(1-\underline{\kappa}_t)\bar{\kappa}_t}, \end{aligned}$$

we want to construct a consumption function for $m_t \in (\underline{m}_t, m_t^{\#}]$ that respects the tighter upper bound:

$$\begin{aligned} \blacktriangle m_t \underline{\kappa}_t < \quad & \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) < \bar{\kappa}_t \blacktriangle m_t \\ \blacktriangle m_t(\bar{\kappa}_t - \underline{\kappa}_t) > \quad & \bar{\kappa}_t \blacktriangle m_t - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) \quad > 0 \\ 1 > \quad & \left(\tfrac{\bar{\kappa}_t \blacktriangle m_t - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t)}{\blacktriangle m_t(\bar{\kappa}_t - \underline{\kappa}_t)}\right) \quad > 0. \end{aligned}$$

Again defining $\mu_t = \log \blacktriangle m_t$, the object in the middle of the inequality is

$$\begin{aligned} \underline{\varrho}_t(\mu_t) &\equiv \frac{\bar{\kappa}_t - \mathrm{c}_t(\underline{m}_t + e^{\mu_t})e^{-\mu_t}}{\bar{\kappa}_t - \underline{\kappa}_t} \\ \underline{\varrho}_t^{\mu}(\mu_t) &= \frac{\mathrm{c}_t(\underline{m}_t + e^{\mu_t})e^{-\mu_t} - \boldsymbol{\kappa}_t^m(\underline{m}_t + e^{\mu_t})}{\bar{\kappa}_t - \underline{\kappa}_t}. \end{aligned}$$

As m_t approaches $-\underline{m}_t$, $\underline{\varrho}_t(\mu_t)$ converges to zero, while as m_t approaches $+\infty$, $\underline{\varrho}_t(\mu_t)$ approaches 1.

As before, we can derive an approximated consumption function; call it $\grave{\underline{\mathrm{c}}}_t$. This function will clearly do a better job approximating the consumption function for low values of m_t while the previous approximation will perform better for high values of m_t.

For middling values of m it is not clear which of these functions will perform better. However, an alternative is available which performs well. Define the highest gridpoint below $m_t^{\#}$ as $\bar{\check{m}}_t^{\#}$ and the lowest gridpoint above $m_t^{\#}$ as $\underline{\hat{m}}_t^{\#}$. Then there will be a unique interpolating polynomial that matches the level and slope of the consumption function at these two points. Call this function $\tilde{\mathrm{c}}_t(m)$.

Using indicator functions that are zero everywhere except for specified intervals,

$$\begin{aligned} \mathbf{1}_{\mathrm{Lo}}(m) &= 1 \text{ if } m \leq \bar{\check{m}}_t^{\#} \\ \mathbf{1}_{\mathrm{Mid}}(m) &= 1 \text{ if } \qquad \bar{\check{m}}_t^{\#} < m < \underline{\hat{m}}_t^{\#} \\ \mathbf{1}_{\mathrm{Hi}}(m) &= 1 \text{ if } \qquad\qquad \underline{\hat{m}}_t^{\#} \leq m \end{aligned}$$

we can define a well-behaved approximating consumption function

$$\grave{\mathrm{c}}_t = \mathbf{1}_{\mathrm{Lo}}\grave{\underline{\mathrm{c}}}_t + \mathbf{1}_{\mathrm{Mid}}\grave{\tilde{\mathrm{c}}}_t + \mathbf{1}_{\mathrm{Hi}}\grave{\bar{\mathrm{c}}}_t. \tag{19}$$

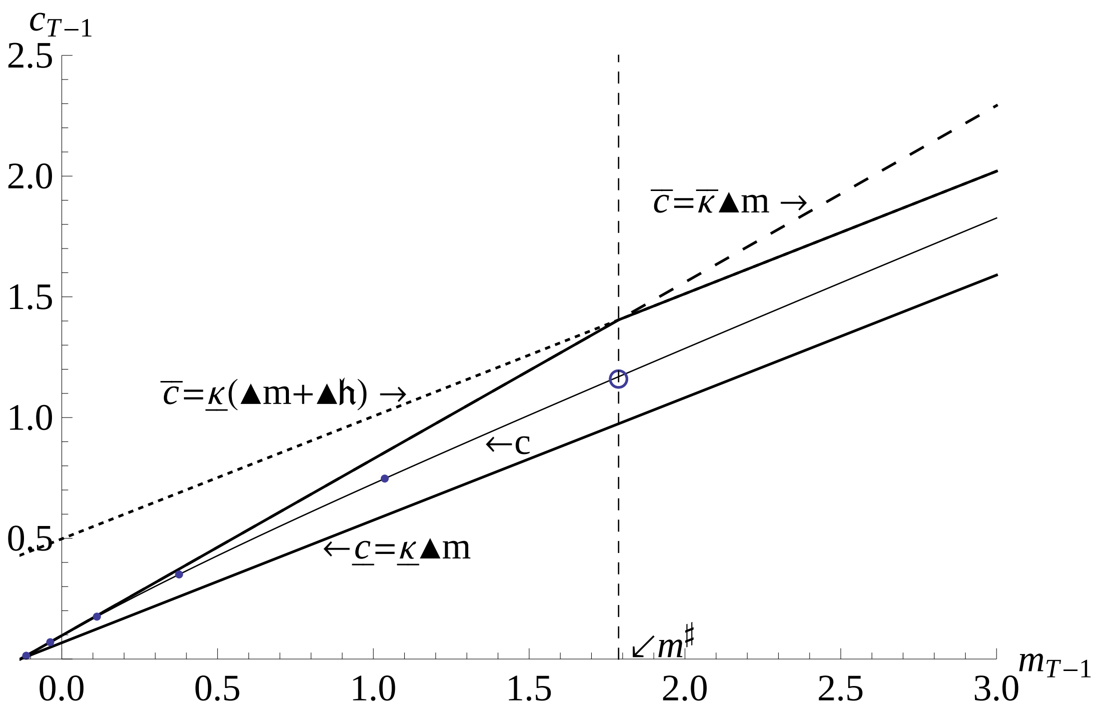
This just says that, for each interval, we use the approximation that is most appropriate. The function is continuous and once-differentiable everywhere, and is therefore well behaved for computational purposes.

We now construct an upper-bound value function implied for a consumer whose spending behavior is consistent with the refined upper-bound consumption rule.

For $m_t \geq m_t^{\#}$, this consumption rule is the same as before, so the constructed upper-bound value function is also the same. However, for values $m_t < m_t^{\#}$ matters are slightly more complicated.

Start with the fact that at the cusp point,

$$\begin{aligned} \bar{\mathrm{v}}_t(m_t^{\#}) &= \mathrm{u}(\bar{c}_t(m_t^{\#}))\mathbb{C}_t^T \\ &= \mathrm{u}(\blacktriangle m_t^{\#}\bar{\kappa}_t)\mathbb{C}_t^T. \end{aligned}$$



But for *all* m_t,

$$\bar{\mathrm{v}}_t(m) = \mathrm{u}(\bar{c}_t(m)) + \bar{\mathfrak{v}}(m - \bar{c}_t(m)),$$

and we assume that for the consumer below the cusp point consumption is given by $\bar{\kappa}\blacktriangle m_t$ so for $m_t < m_t^{\#}$

$$\bar{\mathrm{v}}_t(m) = \mathrm{u}(\bar{\kappa}_t \blacktriangle m) + \bar{\mathfrak{v}}((1 - \bar{\kappa}_t)\blacktriangle m),$$

which is easy to compute because $\bar{\mathfrak{v}}(a_t) = \beta \bar{\mathrm{v}}_{t+1}(a_t \mathcal{R} + 1)$ where $\bar{\mathrm{v}}_t$ is as defined above because a consumer who ends the current period with assets exceeding the lower bound will not expect to be constrained next period. (Recall again that we are merely constructing an object that is guaranteed to be an *upper bound* for the value that the 'realist' consumer will experience.) At the gridpoints defined by the solution of the consumption problem can then construct

$$\bar{\Lambda}_t(m) = ((1 - \rho)\bar{\mathrm{v}}_t(m))^{1/(1-\rho)}$$

and its derivatives which yields the appropriate vector for constructing $\check{\mathrm{X}}$ and $\check{\text{Ϙ}}$. The rest of the procedure is analogous to that performed for the consumption rule and is thus omitted for brevity.

6 Conclusion

The method proposed here is not universally applicable. For example, the method cannot be used for problems for which upper and lower bounds to the 'true' solution are not known. But many problems do have obvious upper and lower bounds, and in those cases (as in the consumption example used in the paper), the method may result in substantial improvements in accuracy and stability of solutions.