1 The Usual Theory, and a Bit More Notation

{sec:the-usual-theor

1.1 Tics, Tacs, Toes

For the problem specified in (**??**), the agent has only one decision to make in each tic (how much to consume). This simplifies matters because there is no need to distinguish betwen the next tac and the problem of the following tic, so we can conflate the two. (See the portfolio choice example below for the notation and analysis of multi-tac problems.)

1.2 Toes

Generically, we want to think of the Bellman solution as having three toes:

1. **Arrival**: Incoming state variables (e.g., k) are known, but any shocks associated with the period have not been realized and decision(s) have not yet been made
2. **Decision**: All exogenous variables (like income shocks, rate of return shocks, and predictable income growth $\mathcal{G}$) have been realized (so that, e.g., m's value is known) and the agent solves the optimization problem
3. **Continuation**: After all decisions have been made, their consequences are measured by evaluation of the continuing-value function at the values of the 'outgoing' state variables (sometimes called 'post-state' variables).

In the standard treatment in the literature, the (implicit) default assumption is that the toe where the agent is solving a decision problem is the unique toe at which the problem is defined. This is what was done above, when (for example) in (**??**) we related the value v of the current decision to the expectation of the future value v_{t+1}. Here, instead, we want to encapsulate the current tac's problem as a standalone object, which is solved by taking as given an exogenously-provided continuation-value function (in our case, $\mathrm{v}_{\rightarrow}(a)$).

When we want to refer to a specific toe in the one tac of t we will do so by supplementing the toe with an indicator which tracks the toe (and we need not denote the tac within the tic because we have assumed there is only one tac in the tic):

Stp	Indicator	State	Usage	Explanation
Arrival	$\leftarrow$ prefix	k	$\mathrm{v}_{\leftarrow t}(k)$	value at entry to t (before shocks)
Decision	(blank/none)	m	$\mathrm{v}(m)$	value of t-decision (after shocks)
Continuation	$\rightarrow$ suffix	a	$\mathrm{v}_{\rightarrow}(a)$	value at exit (after decision)

Notice that different toes of the tac have distinct state variables. k is the state at the beginning of the tac/tic because the shocks that yield m from k have not yet been realized. The state variable for the continuation toe is a because after the consumption decision has been made the model assumes that all that matters is where you have ended up, not how you got there.

We can now restate the problem (**??**) with our new notation:

$$\mathrm{v}(m) = \max_{c} \quad \mathrm{u}(c) + \mathrm{v}_{\rightarrow}(m - c) \tag{1}$$

whose first order condition with respect to c is

$$\mathrm{u}^{c}(c) = \mathrm{v}_{\rightarrow}^{k}(m - c) \tag{2}$$

{eq:FOCnew}

which is mathematically equivalent to the usual Euler equation for consumption.

We will revert to this formulation when we turn to section **??**.

1.3 Summing Up

{subsec:summing-u

For future reference, it will be useful here to write the full expressions for the distinct value functions at the Arrival ($\leftarrow$) and Decision toes.

There is no need to use our tic-identifying notation for the model's variables; k, for example, will have only one unique value over the course of the tic and therefore a notation like $k_{\rightarrow}$ would be pointless; the same is true of all other variables.

Recall that the continuation value function $\mathrm{v}_{\rightarrow}(a) = \beta \mathrm{v}_{\leftarrow(t+1)}(a)$ is provided as an input to the current tac Bellman problem. Since within the scope of the solution of the current tac there is only one such continuation value function, in the solution context there is no point in keeping the tic subscript when we write this function. The same point applies to all variables and functions in the tac. Given the continuation value function $\mathrm{v}_{\rightarrow}$, the problem within the tac can be written with only the toe indicators:

$$\mathrm{v}_{\leftarrow}(k) = \mathbb{E}_{\leftarrow}[\mathrm{v}(\overbrace{k\mathcal{R} + \boldsymbol{\theta}}^{m})] \tag{3}$$

{eq:vBegStp}

$$\mathrm{v}(m) = \max_{\{c\}} \quad \mathrm{u}(c) + \mathbb{E}[\mathrm{v}_{\rightarrow}(\overbrace{m - \mathrm{c}}^{a})] \tag{4}$$

{eq:vMid}