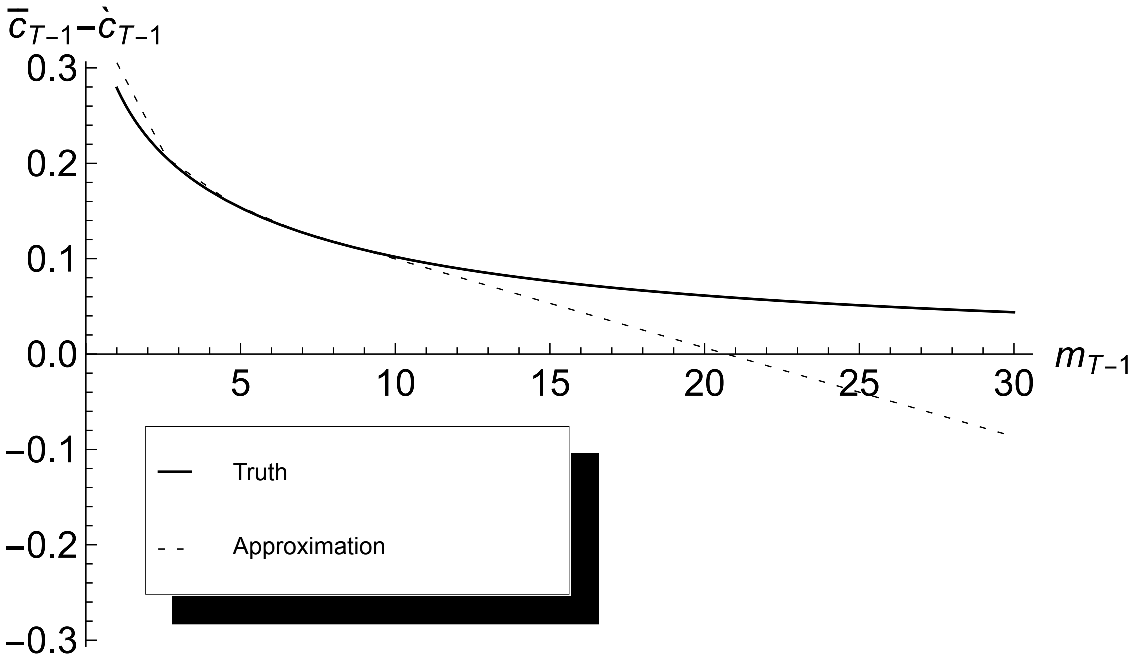

Figure 1 For Large Enough m_{t-1}, Predicted Precautionary Saving is Negative (Oops!)

{fig:ExtrapProblem

0.1 The Method of Moderation

{sec:method-of-mod

Unfortunately, this endogenous gridpoints solution is not very well-behaved outside the original range of gridpoints targeted by the solution method. (Though other common solution methods are no better outside their own predefined ranges). Figure 1 demonstrates the point by plotting the amount of precautionary saving implied by a linear extrapolation of our approximated consumption rule (the consumption of the perfect foresight consumer $\bar{\mathrm{c}}_{t-1}$ minus our approximation to optimal consumption under uncertainty, $\grave{\mathrm{c}}_{t-1}$). Although theory proves that precautionary saving is always positive, the linearly extrapolated numerical approximation eventually predicts negative precautionary saving (at the point in the figure where the extrapolated locus crosses the horizontal axis).

This error cannot be fixed by extending the upper gridpoint; in the presence of serious uncertainty, the consumption rule will need to be evaluated outside of *any* prespecified grid (because starting from the top gridpoint, a large enough realization of the uncertain variable will push next period's realization of assets above that top; a similar argument applies below the bottom gridpoint). While a judicious extrapolation technique can prevent this problem from being fatal (for example by carefully excluding negative precautionary saving), the problem is often dealt with using inelegant methods whose implications for the accuracy of the solution are difficult to gauge.

As a preliminary to our solution, define $h_{t_{\rightarrow}}$ as end-of-period human wealth (the present discounted value of future labor income) for a perfect foresight version of the problem of a 'risk optimist:' a period-t consumer who believes with perfect confidence that the shocks will always take their expected value of 1, $\theta_{t+n} = \mathbb{E}[\theta] = 1 \; \forall \; n > 0$. The solution to a perfect foresight problem of this kind takes the form[1]

$$\bar{\mathrm{c}}_t(m_t) = (m_t + h_{t\rightarrow})\underline{\kappa}_t \tag{1}$$

{eq:cFuncAbove}

for a constant minimal marginal propensity to consume $\underline{\kappa}_t$ given below.

We similarly define $\underline{h}_{t\rightarrow}$ as 'minimal human wealth,' the present discounted value of labor income if the shocks were to take on their worst possible value in every future period $\theta_{t+n} = \underline{\theta} \; \forall \; n > 0$ (which we define as corresponding to the beliefs of a 'pessimist').

We will call a 'realist' the consumer who correctly perceives the true probabilities of the future risks and optimizes accordingly.

A first useful point is that, for the realist, a lower bound for the level of market resources is $\underline{m}_t = -\underline{h}_{t\rightarrow}$, because if m_t equalled this value then there would be a positive finite chance (however small) of receiving $\theta_{t+n} = \underline{\theta}$ in every future period, which would require the consumer to set c_t to zero in order to guarantee that the intertemporal budget constraint holds (this is the multiperiod generalization of the discussion in section 5.7 explaining the derivation of the 'natural borrowing constraint' for period $T-1$, $\underline{a}_{t-1}$). Since consumption of zero yields negative infinite utility, the solution to realist consumer's problem is not well defined for values of $m_t < \underline{m}_t$, and the limiting value of the realist's c_t is zero as $m_t \downarrow \underline{m}_t$.

Given this result, it will be convenient to define 'excess' market resources as the amount by which actual resources exceed the lower bound, and 'excess' human wealth as the amount by which mean expected human wealth exceeds guaranteed minimum human wealth:

$$\begin{aligned}
\blacktriangle m_t &= m_t + \overbrace{\underline{h}_{t\rightarrow}}^{=-\underline{m}_t} \\
\blacktriangle h_{t\rightarrow} &= h_{t\rightarrow} - \underline{h}_{t\rightarrow}.
\end{aligned}$$

We can now transparently define the optimal consumption rules for the two perfect foresight problems, those of the 'optimist' and the 'pessimist.' The 'pessimist' perceives human wealth to be equal to its minimum feasible value $\underline{h}_{t\rightarrow}$ with certainty, so consumption is given by the perfect foresight solution

$$\begin{aligned}
\underline{\mathrm{c}}_t(m_t) &= (m_t + \underline{h}_{t\rightarrow})\underline{\kappa}_t \\
&= \blacktriangle m_t \underline{\kappa}_t.
\end{aligned}$$

The 'optimist,' on the other hand, pretends that there is no uncertainty about future income, and therefore consumes

$$\begin{aligned}
\bar{\mathrm{c}}_t(m_t) &= (m_t + \underline{h}_{t\rightarrow} - \underline{h}_{t\rightarrow} + h_{t\rightarrow})\underline{\kappa}_t \\
&= (\blacktriangle m_t + \blacktriangle h_{t\rightarrow})\underline{\kappa}_t \\
&= \underline{\mathrm{c}}_t(m_t) + \blacktriangle h_{t\rightarrow}\underline{\kappa}_t.
\end{aligned}$$

[1] For a derivation, see Carroll (2023b); $\underline{\kappa}_t$ is defined therein as the MPC of the perfect foresight consumer with horizon $T-t$.

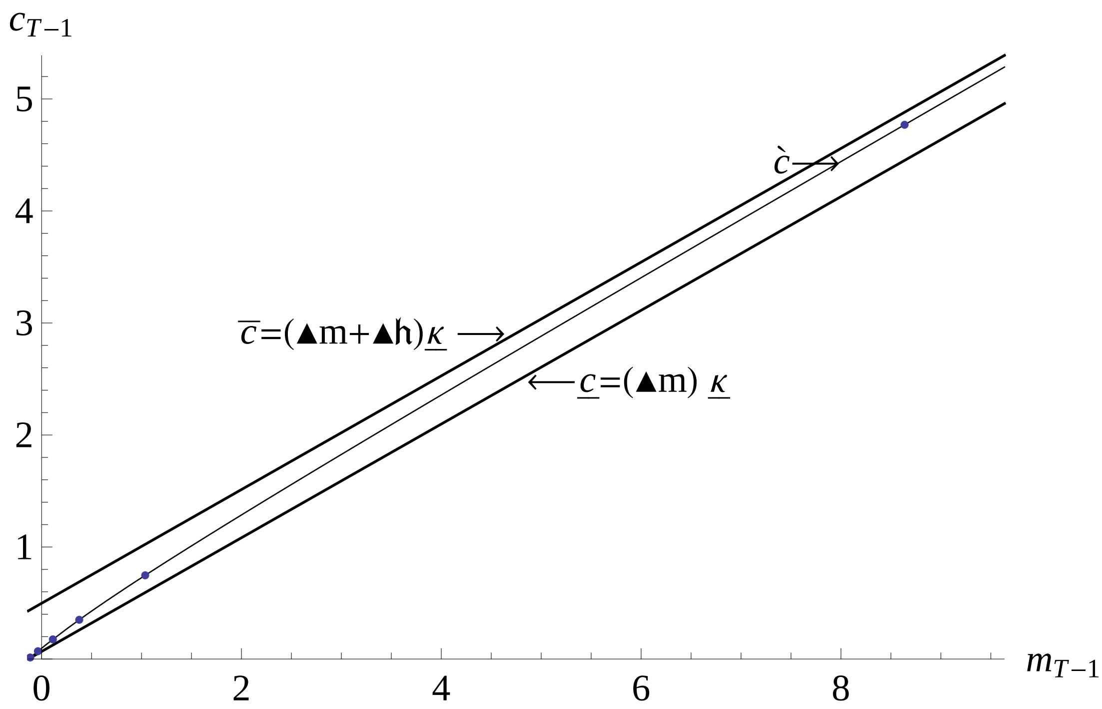

Figure 2 Moderation Illustrated: $\underline{\mathrm{c}}_{t-1} < \grave{\mathrm{c}}_{t-1} < \bar{\mathrm{c}}_{t-1}$

{fig:IntExpFOCInv

It seems obvious that the spending of the realist will be strictly greater than that of the pessimist and strictly less than that of the optimist. Figure 2 illustrates the proposition for the consumption rule in period $T-1$.

The proof is more difficult than might be imagined, but the necessary work is done in Carroll (2023b) so we will take the proposition as a fact and proceed by manipulating the inequality:

$$\begin{array}{rcl}
\blacktriangle m_t \underline{\kappa}_t < & \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) < (\blacktriangle m_t + \blacktriangle h_{t\rightarrow})\underline{\kappa}_t & \\
-\blacktriangle m_t \underline{\kappa}_t > & -\mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & > -(\blacktriangle m_t + \blacktriangle h_{t\rightarrow})\underline{\kappa}_t \\
\blacktriangle h_{t\rightarrow} \underline{\kappa}_t > & \bar{\mathrm{c}}_t(\underline{m}_t + \blacktriangle m_t) - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & > 0 \\
1 > & \underbrace{\left(\frac{\bar{\mathrm{c}}_t(\underline{m}_t + \blacktriangle m_t) - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t)}{\blacktriangle h_{t\rightarrow} \underline{\kappa}_t}\right)}_{\equiv \hat{\koppa}_t} & > 0
\end{array}$$

where the fraction in the middle of the last inequality is the ratio of actual precautionary saving (the numerator is the difference between perfect-foresight consumption and optimal consumption in the presence of uncertainty) to the maximum conceivable amount of precautionary saving (the amount that would be undertaken by the pessimist who consumes nothing out of any future income beyond the perfectly certain component).

Defining $\mu_t = \log \blacktriangle m_t$ (which can range from $-\infty$ to ∞), the object in the middle of the last inequality is

$$\hat{\koppa}_t(\mu_t) \equiv \left(\frac{\bar{\mathrm{c}}_t(\underline{m}_t + e^{\mu_t}) - \mathrm{c}_t(\underline{m}_t + e^{\mu_t})}{\blacktriangle h_{t\rightarrow} \underline{\kappa}_t}\right), \tag{2}$$

{eq:koppa}

and we now define

$$\begin{aligned} \hat{\boldsymbol{\chi}}_t(\mu_t) &= \log\left(\frac{1-\hat{\koppa}_t(\mu_t)}{\hat{\koppa}_t(\mu_t)}\right) \\ &= \log\left(1/\hat{\koppa}_t(\mu_t) - 1\right) \end{aligned} \tag{3}$$ {eq:chi}

which has the virtue that it is linear in the limit as μ_t approaches $+\infty$.

Given $\hat{\boldsymbol{\chi}}$, the consumption function can be recovered from

$$\hat{\mathrm{c}}_t = \bar{\mathrm{c}}_t - \overbrace{\left(\frac{1}{1+\exp(\hat{\boldsymbol{\chi}}_t)}\right)}^{=\hat{\koppa}_t} \blacktriangle h_{t\rightarrow}\underline{\kappa}_t. \tag{4}$$ {eq:cFuncHi}

Thus, the procedure is to calculate $\hat{\boldsymbol{\chi}}_t$ at the points $\boldsymbol{\mu}_t$ corresponding to the log of the $\blacktriangle \boldsymbol{m}_t$ points defined above, and then using these to construct an interpolating approximation $\grave{\hat{\boldsymbol{\chi}}}_t$ from which we indirectly obtain our approximated consumption rule $\grave{\hat{\mathrm{c}}}_t$ by substituting $\grave{\hat{\boldsymbol{\chi}}}_t$ for $\hat{\boldsymbol{\chi}}$ in equation (4).

Because this method relies upon the fact that the problem is easy to solve if the decision maker has unreasonable views (either in the optimistic or the pessimistic direction), and because the correct solution is always between these immoderate extremes, we call our solution procedure the 'method of moderation.'

Results are shown in Figure 3; a reader with very good eyesight might be able to detect the barest hint of a discrepancy between the Truth and the Approximation at the far righthand edge of the figure – a stark contrast with the calamitous divergence evident in Figure 1.

0.2 Approximating the Slope Too

Until now, we have calculated the level of consumption at various different gridpoints and used linear interpolation (either directly for c_{t-1} or indirectly for, say, $\hat{\boldsymbol{\chi}}_{t-1}$). But the resulting piecewise linear approximations have the unattractive feature that they are not differentiable at the 'kink points' that correspond to the gridpoints where the slope of the function changes discretely.

Carroll (2023b) proves that the true consumption function for this problem is 'smooth:' It exhibits a well-defined unique marginal propensity to consume at every positive value of m. This suggests that we should calculate, not just the level of consumption, but also the marginal propensity to consume (henceforth κ) at each gridpoint, and then find an interpolating approximation that smoothly matches both the level and the slope at those points.

This requires us to differentiate (2) and (3), yielding

$$\begin{aligned} \hat{\koppa}^{\mu}_t(\mu_t) &= (\blacktriangle h_{t\rightarrow}\underline{\kappa}_t)^{-1} e^{\mu_t}\left(\underline{\kappa}_t - \overbrace{\mathrm{c}^m_t(\underline{m}_t + e^{\mu_t})}^{\equiv \boldsymbol{\kappa}_t(m_t)}\right) \\ \hat{\boldsymbol{\chi}}^{\mu}_t(\mu_t) &= \left(\frac{-\hat{\koppa}^{\mu}_t(\mu_t)/\hat{\koppa}^2_t}{1/\hat{\koppa}_t(\mu_t) - 1}\right) \end{aligned} \tag{5}$$ {eq:koppaPrime}

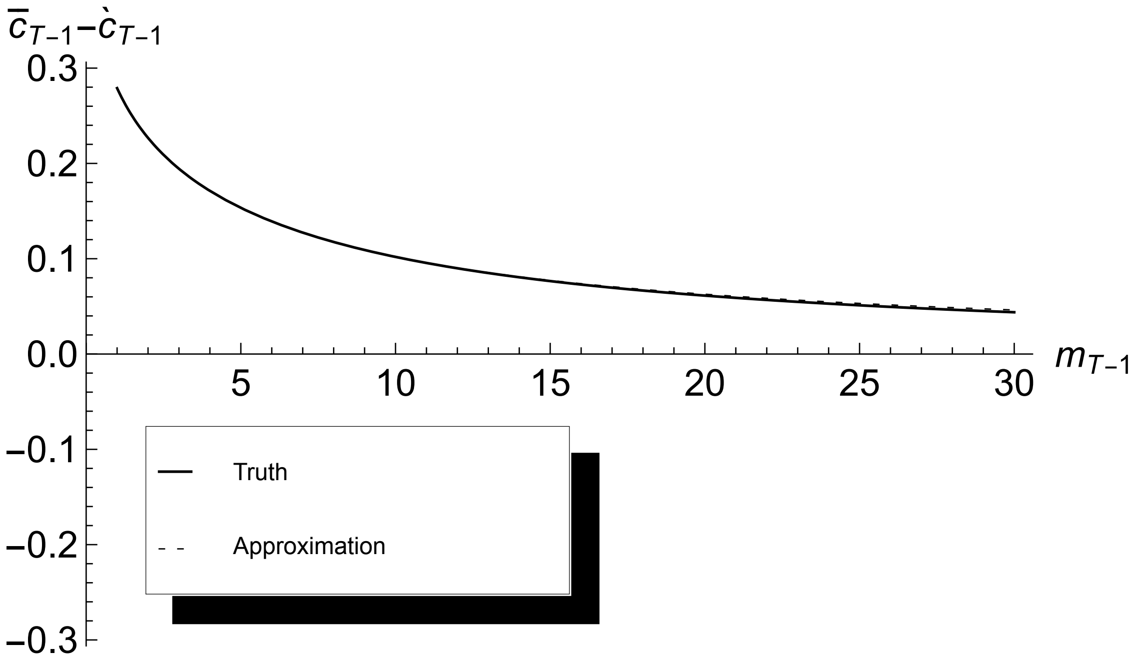

Figure 3 Extrapolated $\grave{\hat{c}}_{t-1}$ Constructed Using the Method of Moderation

{fig:ExtrapProblem

and (dropping arguments) with some algebra these can be combined to yield

$$\hat{\boldsymbol{\chi}}_t^{\mu} = \left(\frac{\underline{\kappa}_t \blacktriangle m_t \blacktriangle h_{t\rightarrow}(\underline{\kappa}_t - \kappa_t)}{(\bar{\mathrm{c}}_t - \mathrm{c}_t)(\bar{\mathrm{c}}_t - \mathrm{c}_t - \underline{\kappa}_t \blacktriangle h_{t\rightarrow})} \right). \tag{6}$$

To compute the vector of values of (5) corresponding to the points in $\boldsymbol{\mu}_t$, we need the marginal propensities to consume (designated κ) at each of the gridpoints, c_t^m (the vector of such values is $\boldsymbol{\kappa}_t$). These can be obtained by differentiating the Euler equation (12) (where we define $\mathrm{m}_{t\rightarrow}(a) \equiv \mathrm{c}_{t\rightarrow}(a) + a$, and drop the (a) arguments to reduce clutter):

$$\mathrm{u}^c(\mathrm{c}_{t\rightarrow}) = \hat{\mathrm{v}}_{t\rightarrow}^a(\mathrm{m}_{t\rightarrow} - \mathrm{c}_{t\rightarrow}), \tag{7}$$

yielding a marginal propensity to *have consumed* $\mathrm{c}_{t\rightarrow}^a$ at each gridpoint:

$$\begin{aligned} \mathrm{u}''(\mathrm{c}_{t\rightarrow})\mathrm{c}_{t\rightarrow}^a &= \hat{\mathrm{v}}_{t\rightarrow}^a(\mathrm{m}_{t\rightarrow} - \mathrm{c}_{t\rightarrow}) \\ \mathrm{c}_{t\rightarrow}^a &= \hat{\mathrm{v}}^a(\mathrm{m}_{t\rightarrow} - \mathrm{c}_{t\rightarrow})/\mathrm{u}''(\mathrm{c}_{t\rightarrow}) \end{aligned} \tag{8}$$

and the marginal propensity to consume at the beginning of the period is obtained from the marginal propensity to have consumed by differentiating the identity with respect to a:

$$\begin{aligned} \mathrm{c}_{t\rightarrow} &= \mathrm{m}_{t\rightarrow} - a \\ \mathrm{c}_{t\rightarrow}^a + 1 &= \mathrm{m}_{t\rightarrow}^a \end{aligned}$$

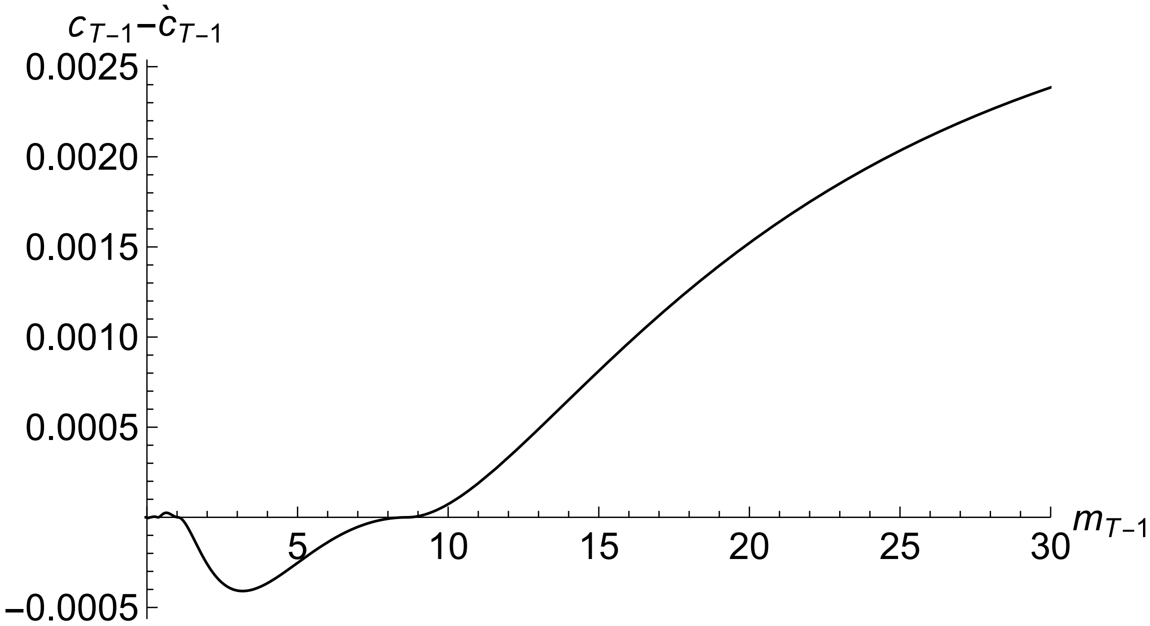

Figure 4 Difference Between True c_{t-1} and $\grave{\hat{\mathrm{c}}}_{t-1}$ Is Minuscule

{fig:IntExpFOCInv

which, together with the chain rule $\mathrm{c}^a_{t\rightarrow} = \mathrm{c}^m_t \mathrm{m}^a_{t\rightarrow}$, yields the MPC from

$$\begin{aligned} \mathrm{c}^m(\overbrace{\mathrm{c}^a_{t\rightarrow}+1}^{=\mathrm{m}^a_{t\rightarrow}}) &= \mathrm{c}^a_{t\rightarrow} \\ \mathrm{c}^m &= \mathrm{c}^a_{t\rightarrow}/(1+\mathrm{c}^a_{t\rightarrow}). \end{aligned} \tag{9}$$

{eq:MPCfromMPT

Designating $\grave{\hat{\mathrm{c}}}_{t-1}$ as the approximated consumption rule obtained using an interpolating polynomial approximation to $\hat{\chi}$ that matches both the level and the first derivative at the gridpoints, Figure 4 plots the difference between this latest approximation and the true consumption rule for period $T-1$ up to the same large value (far beyond the largest gridpoint) used in prior figures. Of course, at the gridpoints the approximation will exactly match the true function; but this figure illustrates that the approximation is quite accurate far beyond the last gridpoint (which is the last point at which the difference touches the horizontal axis). (We plot here the difference between the two functions rather than the level plotted in previous figures, because in levels the difference between the approximate and the exact function would not be detectable even to the most eagle-eyed reader.)

0.3 Value

Often it is useful to know the value function as well as the consumption rule. Fortunately, many of the tricks used when solving for the consumption rule have a direct analogue in approximation of the value function.

Consider the perfect foresight (or "optimist's") problem in period $T-1$. Using the fact that in a perfect foresight model the growth factor for consumption is $(\mathsf{R}\beta)^{1/\rho}$, we can use the fact that $c_t = (\mathsf{R}\beta)^{1/\rho}c_{t-1}$ to calculate the value function in period $T-1$:

$$
\begin{aligned}
\bar{\mathrm{v}}_{t-1}(m_{t-1}) &\equiv \mathrm{u}(c_{t-1}) + \beta \mathrm{u}(c_t) \\
&= \mathrm{u}(c_{t-1})\left(1 + \beta((\beta\mathsf{R})^{1/\rho})^{1-\rho}\right) \\
&= \mathrm{u}(c_{t-1})\left(1 + (\beta\mathsf{R})^{1/\rho}/\mathsf{R}\right) \\
&= \mathrm{u}(c_{t-1})\underbrace{\mathrm{PDV}_t^T(c)/c_{t-1}}_{\equiv \mathbb{C}_{t-1}^T}
\end{aligned}
$$

where $\mathbb{C}_t^T = \mathrm{PDV}_t^T(c)$ is the present discounted value of consumption, normalized by current consumption. Using the fact demonstrated in Carroll (2023b) that $\mathbb{C}_t = \kappa_t^{-1}$, a similar function can be constructed recursively for earlier periods, yielding the general expression

$$
\begin{aligned}
\bar{\mathrm{v}}_t(m_t) &= \mathrm{u}(\bar{c}_t)\mathbb{C}_t^T \\
&= \mathrm{u}(\bar{c}_t)\underline{\kappa}_t^{-1} \\
&= \mathrm{u}((\blacktriangle m_t + \blacktriangle h_{t\rightarrow})\underline{\kappa}_t)\underline{\kappa}_t^{-1} \\
&= \mathrm{u}(\blacktriangle m_t + \blacktriangle h_{t\rightarrow})\underline{\kappa}_t^{1-\rho}\underline{\kappa}_t^{-1} \\
&= \mathrm{u}(\blacktriangle m_t + \blacktriangle h_{t\rightarrow})\underline{\kappa}_t^{-\rho}
\end{aligned}
\tag{10}
$$

{eq:vFuncPF}

This can be transformed as

$$
\begin{aligned}
\bar{\Lambda}_t &\equiv ((1-\rho)\bar{\mathrm{v}}_t)^{1/(1-\rho)} \\
&= c_t(\mathbb{C}_t^T)^{1/(1-\rho)} \\
&= (\blacktriangle m_t + \blacktriangle h_{t\rightarrow})\underline{\kappa}_t^{-\rho/(1-\rho)}
\end{aligned}
$$

with derivative

$$
\begin{aligned}
\bar{\Lambda}_t^m &= (\mathbb{C}_t^T)^{1/(1-\rho)}\underline{\kappa}_t, \\
&= \underline{\kappa}_t^{-\rho/(1-\rho)}
\end{aligned}
$$

and since $\mathbb{C}_t^T$ is a constant while the consumption function is linear, $\bar{\Lambda}_t$ will also be linear.

We apply the same transformation to the value function for the problem with uncertainty (the "realist's" problem) and differentiate

$$
\begin{aligned}
\bar{\Lambda}_t &= ((1-\rho)\bar{\mathrm{v}}_t(m_t))^{1/(1-\rho)} \\
\bar{\Lambda}_t^m &= ((1-\rho)\bar{\mathrm{v}}_t(m_t))^{-1+1/(1-\rho)}\,\bar{\mathrm{v}}_t^m(m_t)
\end{aligned}
$$

and an excellent approximation to the value function can be obtained by calculating the values of $\bar{\Lambda}$ at the same gridpoints used by the consumption function approximation, and interpolating among those points.

However, as with the consumption approximation, we can do even better if we realize that the $\bar{\Lambda}$ function for the optimist's problem is an upper bound for the Λ function in the presence of uncertainty, and the value function for the pessimist is a lower bound.

Analogously to (2), define an upper-case

$$\hat{\text{Ϙ}}_t(\mu_t) = \left(\frac{\bar{\Lambda}_t(\underline{m}_t + e^{\mu_t}) - \Lambda_t(\underline{m}_t + e^{\mu_t})}{\blacktriangle h_{t\rightarrow}\underline{\kappa}_t(\mathbb{C}_t^T)^{1/(1-\rho)}}\right) \tag{11}$$

{eq:Koppa}

with derivative (dropping arguments)

$$\hat{\text{Ϙ}}_t^{\mu} = (\blacktriangle h_{t\rightarrow}\underline{\kappa}_t(\mathbb{C}_t^T)^{1/(1-\rho)})^{-1}e^{\mu_t}\,(\bar{\Lambda}_t^m - \Lambda_t^m) \tag{12}$$

{eq:KoppaPrime}

and an upper-case version of the $\boldsymbol{\chi}$ equation in (3):

$$\begin{aligned} \hat{\mathrm{X}}_t(\mu_t) &= \log\left(\frac{1-\hat{\text{Ϙ}}_t(\mu_t)}{\hat{\text{Ϙ}}_t(\mu_t)}\right) \\ &= \log\left(1/\hat{\text{Ϙ}}_t(\mu_t) - 1\right) \end{aligned} \tag{13}$$

{eq:Chi}

with corresponding derivative

$$\hat{\mathrm{X}}_t^{\mu} = \left(\frac{-\hat{\text{Ϙ}}_t^{\mu}/\hat{\text{Ϙ}}_t^2}{1/\hat{\text{Ϙ}}_t - 1}\right) \tag{14}$$

and if we approximate these objects then invert them (as above with the $\hat{\text{ϙ}}$ and $\hat{\boldsymbol{\chi}}$ functions) we obtain a very high-quality approximation to our inverted value function at the same points for which we have our approximated value function:

$$\hat{\Lambda}_t = \bar{\Lambda}_t - \overbrace{\left(\frac{1}{1+\exp(\hat{\mathrm{X}}_t)}\right)}^{=\hat{\text{Ϙ}}_t}\blacktriangle h_{t\rightarrow}\underline{\kappa}_t(\mathbb{C}_t^T)^{1/(1-\rho)} \tag{15}$$

from which we obtain our approximation to the value function and its derivatives as

$$\begin{aligned} \hat{\mathrm{v}}_t &= \mathrm{u}(\hat{\Lambda}_t) \\ \hat{\mathrm{v}}_t^m &= \mathrm{u}^c(\hat{\Lambda}_t)\hat{\Lambda}^m \\ \hat{\mathrm{v}}_t^{mm} &= \mathrm{u}^{cc}(\hat{\Lambda}_t)(\hat{\Lambda}^m)^2 + \mathrm{u}^c(\hat{\Lambda}_t)\hat{\Lambda}^{mm}. \end{aligned} \tag{16}$$

Although a linear interpolation that matches the level of Λ at the gridpoints is simple, a Hermite interpolation that matches both the level and the derivative of the $\bar{\Lambda}_t$ function at the gridpoints has the considerable virtue that the $\bar{\mathrm{v}}_t$ derived from it numerically satisfies the envelope theorem at each of the gridpoints for which the problem has been solved.

If we use the double-derivative calculated above to produce a higher-order Hermite polynomial, our approximation will also match marginal propensity to consume at the gridpoints; this would guarantee that the consumption function generated from the value function would match both the level of consumption and the marginal propensity to consume at the gridpoints; the numerical differences between the newly constructed consumption function and the highly accurate one constructed earlier would be negligible within the grid.

0.4 Refinement: A Tighter Upper Bound

Carroll (2023b) derives an upper limit $\bar{\kappa}_t$ for the MPC as m_t approaches its lower bound. Using this fact plus the strict concavity of the consumption function yields the proposition that

$$\mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) < \bar{\kappa}_t \blacktriangle m_t. \tag{17}$$

The solution method described above does not guarantee that approximated consumption will respect this constraint between gridpoints, and a failure to respect the constraint can occasionally cause computational problems in solving or simulating the model. Here, we describe a method for constructing an approximation that always satisfies the constraint.

Defining $m_t^{\#}$ as the 'cusp' point where the two upper bounds intersect:

$$\begin{aligned} \left(\blacktriangle m_t^{\#} + \blacktriangle h_{t\rightarrow}\right) \underline{\kappa}_t &= \bar{\kappa}_t \blacktriangle m_t^{\#} \\ \blacktriangle m_t^{\#} &= \frac{\underline{\kappa}_t \blacktriangle h_{t\rightarrow}}{(1-\underline{\kappa}_t)\bar{\kappa}_t} \\ m_t^{\#} &= \frac{\underline{\kappa}_t h_{t\rightarrow} - \underline{h}_{t\rightarrow}}{(1-\underline{\kappa}_t)\bar{\kappa}_t}, \end{aligned}$$

we want to construct a consumption function for $m_t \in (\underline{m}_t, m_t^{\#}]$ that respects the tighter upper bound:

$$\begin{array}{rcl} \blacktriangle m_t \underline{\kappa}_t < & \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) < \bar{\kappa}_t \blacktriangle m_t & \\ \blacktriangle m_t(\bar{\kappa}_t - \underline{\kappa}_t) > & \bar{\kappa}_t \blacktriangle m_t - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t) & > 0 \\ 1 > & \left(\frac{\bar{\kappa}_t \blacktriangle m_t - \mathrm{c}_t(\underline{m}_t + \blacktriangle m_t)}{\blacktriangle m_t(\bar{\kappa}_t - \underline{\kappa}_t)}\right) & > 0. \end{array}$$

Again defining $\mu_t = \log \blacktriangle m_t$, the object in the middle of the inequality is

$$\begin{aligned} \check{\varphi}_t(\mu_t) &\equiv \frac{\bar{\kappa}_t - \mathrm{c}_t(\underline{m}_t + e^{\mu_t})e^{-\mu_t}}{\bar{\kappa}_t - \underline{\kappa}_t} \\ \check{\varphi}_t^{\mu}(\mu_t) &= \frac{\mathrm{c}_t(\underline{m}_t + e^{\mu_t})e^{-\mu_t} - \boldsymbol{\kappa}_t^m(\underline{m}_t + e^{\mu_t})}{\bar{\kappa}_t - \underline{\kappa}_t}. \end{aligned}$$

As m_t approaches $-\underline{m}_t$, $\check{\varphi}_t(\mu_t)$ converges to zero, while as m_t approaches $+\infty$, $\check{\varphi}_t(\mu_t)$ approaches 1.

As before, we can derive an approximated consumption function; call it $\check{\grave{\mathrm{c}}}_t$. This function will clearly do a better job approximating the consumption function for low values of m_t while the previous approximation will perform better for high values of m_t.

For middling values of m it is not clear which of these functions will perform better. However, an alternative is available which performs well. Define the highest gridpoint below $m_t^{\#}$ as $\bar{\check{m}}_t^{\#}$ and the lowest gridpoint above $m_t^{\#}$ as $\hat{\underline{m}}_t^{\#}$. Then there will be a unique interpolating polynomial that matches the level and slope of the consumption function at these two points. Call this function $\tilde{\mathrm{c}}_t(m)$.

Using indicator functions that are zero everywhere except for specified intervals,

$$\begin{aligned}
\mathbf{1}_{\text{Lo}}(m) &= 1 \text{ if } m \leq \bar{\check{m}}_t^{\#} \\
\mathbf{1}_{\text{Mid}}(m) &= 1 \text{ if } \qquad \bar{\check{m}}_t^{\#} < m < \underline{\hat{m}}_t^{\#} \\
\mathbf{1}_{\text{Hi}}(m) &= 1 \text{ if } \qquad\qquad\qquad \underline{\hat{m}}_t^{\#} \leq m
\end{aligned}$$

we can define a well-behaved approximating consumption function

$$\grave{\mathrm{c}}_t = \mathbf{1}_{\text{Lo}}\grave{\check{\mathrm{c}}}_t + \mathbf{1}_{\text{Mid}}\grave{\check{\mathrm{c}}}_t + \mathbf{1}_{\text{Hi}}\grave{\hat{\mathrm{c}}}_t. \tag{18}$$

This just says that, for each interval, we use the approximation that is most appropriate. The function is continuous and once-differentiable everywhere, and is therefore well behaved for computational purposes.

We now construct an upper-bound value function implied for a consumer whose spending behavior is consistent with the refined upper-bound consumption rule.

For $m_t \geq m_t^{\#}$, this consumption rule is the same as before, so the constructed upper-bound value function is also the same. However, for values $m_t < m_t^{\#}$ matters are slightly more complicated.

Start with the fact that at the cusp point,

$$\begin{aligned}
\bar{\mathrm{v}}_t(m_t^{\#}) &= \mathrm{u}(\bar{c}_t(m_t^{\#}))\mathbb{C}_t^T \\
&= \mathrm{u}(\blacktriangle m_t^{\#}\bar{\kappa}_t)\mathbb{C}_t^T.
\end{aligned}$$

But for *all* m_t,

$$\bar{\mathrm{v}}_t(m) = \mathrm{u}(\bar{c}_t(m)) + \bar{\mathrm{v}}_{t\rightarrow}(m - \bar{c}_t(m)),$$

and we assume that for the consumer below the cusp point consumption is given by $\bar{\kappa}\blacktriangle m_t$ so for $m_t < m_t^{\#}$

$$\bar{\mathrm{v}}_t(m) = \mathrm{u}(\bar{\kappa}_t\blacktriangle m) + \bar{\mathrm{v}}_{t\rightarrow}((1-\bar{\kappa}_t)\blacktriangle m),$$

which is easy to compute because $\bar{\mathrm{v}}_{t\rightarrow}(a_t) = \beta\bar{\mathrm{v}}_{t+1}(a_t\mathcal{R}+1)$ where $\bar{\mathrm{v}}_t$ is as defined above because a consumer who ends the current period with assets exceeding the lower bound will not expect to be constrained next period. (Recall again that we are merely constructing an object that is guaranteed to be an *upper bound* for the value that the 'realist' consumer will experience.) At the gridpoints defined by the solution of the consumption problem can then construct

$$\bar{\Lambda}_t(m) = ((1-\rho)\bar{\mathrm{v}}_t(m))^{1/(1-\rho)}$$

and its derivatives which yields the appropriate vector for constructing $\check{\mathrm{X}}$ and $\check{\mathrm{\Omega}}$. The rest of the procedure is analogous to that performed for the consumption rule and is thus omitted for brevity.

0.5 Extension: A Stochastic Interest Factor

Thus far we have assumed that the interest factor is constant at R. Extending the previous derivations to allow for a perfectly forecastable time-varying interest factor R_t would be trivial. Allowing for a stochastic interest factor is less trivial.

The easiest case is where the interest factor is i.i.d.,

$$\log \mathbf{R}_{t+n} \sim \mathcal{N}(\mathsf{r} + \phi - \sigma_{\mathbf{r}}^2/2, \sigma_{\mathbf{r}}^2) \;\forall\; n > 0 \tag{19}$$ {eq:distRisky}

where ϕ is the risk premium and the $\sigma_{\mathbf{r}}^2/2$ adjustment to the mean log return guarantees that an increase in $\sigma_{\mathbf{r}}^2$ constitutes a mean-preserving spread in the level of the return.

This case is reasonably straightforward because Merton (1969) and Samuelson (1969) showed that for a consumer without labor income (or with perfectly forecastable labor income) the consumption function is linear, with an infinite-horizon MPC[2]

$$\kappa = 1 - \left(\beta\, \mathbb{E}_{\leftarrow t}[\mathbf{R}_{t+1}^{1-\rho}]\right)^{1/\rho} \tag{20}$$ {eq:MPCExact}

and in this case the previous analysis applies once we substitute this MPC for the one that characterizes the perfect foresight problem without rate-of-return risk.

The more realistic case where the interest factor has some serial correlation is more complex. We consider the simplest case that captures the main features of empirical interest rate dynamics: An AR(1) process. Thus the specification is

$$\mathbf{r}_{t+1} - \mathbf{r} = (\mathbf{r}_t - \mathbf{r})\gamma + \epsilon_{t+1} \tag{21}$$

where $\mathbf{r}$ is the long-run mean log interest factor, $0 < \gamma < 1$ is the AR(1) serial correlation coefficient, and ϵ_{t+1} is the stochastic shock.

The consumer's problem in this case now has two state variables, m_t and $\mathbf{r}_t$, and is described by

$$\begin{aligned}
\mathrm{v}_t(m_t, \mathbf{r}_t) &= \max_{c_t}\; \mathrm{u}(c_t) + \mathbb{E}_{\leftarrow t}[\beta_{t+1}\mathrm{v}_{t+1}(m_{t+1}, \mathbf{r}_{t+1})] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
\mathbf{r}_{t+1} - \mathbf{r} &= (\mathbf{r}_t - \mathbf{r})\gamma + \epsilon_{t+1} \\
\mathbf{R}_{t+1} &= \exp(\mathbf{r}_{t+1}) \\
m_{t+1} &= \underbrace{(\mathbf{R}_{t+1}/\mathcal{G}_{t+1})}_{\equiv \mathcal{R}_{t+1}} a_t + \theta_{t+1}.
\end{aligned}$$

We approximate the AR(1) process by a Markov transition matrix using standard techniques. The stochastic interest factor is allowed to take on 11 values centered around the steady-state value $\mathbf{r}$. Given this Markov transition matrix, *conditional* on the Markov AR(1) state the consumption functions for the 'optimist' and the 'pessimist' will still be linear, with identical MPC's that are computed numerically. Given these MPC's, the (conditional) realist's consumption function can be computed for each Markov state, and the converged consumption rules constitute the solution contingent on the dynamics of the stochastic interest rate process.

In principle, this refinement should be combined with the previous one; further exposition of this combination is omitted here because no new insights spring from the combination of the two techniques.

[2] See `CRRA-RateRisk` for a derivation.

0.6 Imposing 'Artificial' Borrowing Constraints

Optimization problems often come with additional constraints that must be satisfied. Particularly common is an 'artificial' liquidity constraint that prevents the consumer's net worth from falling below some value, often zero.[3] The problem then becomes

$$\begin{aligned} \mathrm{v}_{t-1}(m_{t-1}) = \max_{c_{t-1}} \quad & \mathrm{u}(c_{t-1}) + \mathbb{E}_{t-1}[\beta \mathrm{v}_{\rightarrow(T)}(m_t)] \\ \text{s.t.} & \\ a_{t-1} &= m_{t-1} - c_{t-1} \\ m_t &= \mathcal{R}_t a_{t-1} + \theta_t \\ a_{t-1} &\geq 0. \end{aligned}$$

By definition, the constraint will bind if the unconstrained consumer would choose a level of spending that would violate the constraint. Here, that means that the constraint binds if the c_{t-1} that satisfies the unconstrained FOC

$$c_{t-1}^{-\rho} = \mathrm{v}^{a}_{(t-1)_\rightarrow}(m_{t-1} - c_{t-1}) \tag{22}$$ {eq:cUnc}

is greater than m_{t-1}. Call $\grave{\mathrm{c}}^{*}_{t-1}$ the approximated function returning the level of c_{t-1} that satisfies (22). Then the approximated constrained optimal consumption function will be

$$\grave{\mathrm{c}}_{t-1}(m_{t-1}) = \min[m_{t-1}, \grave{\mathrm{c}}^{*}_{t-1}(m_{t-1})]. \tag{23}$$ {eq:LiqCons}

The introduction of the constraint also introduces a sharp nonlinearity in all of the functions at the point where the constraint begins to bind. As a result, to get solutions that are anywhere close to numerically accurate it is useful to augment the grid of values of the state variable to include the exact value at which the constraint ceases to bind. Fortunately, this is easy to calculate. We know that when the constraint is binding the consumer is saving nothing, which yields marginal value of $\mathrm{v}^{a}_{(t-1)_\rightarrow}(0)$. Further, when the constraint is binding, $c_{t-1} = m_{t-1}$. Thus, the largest value of consumption for which the constraint is binding will be the point for which the marginal utility of consumption is exactly equal to the (expected, discounted) marginal value of saving 0. We know this because the marginal utility of consumption is a downward-sloping function and so if the consumer were to consume ϵ more, the marginal utility of that extra consumption would be *below* the (discounted, expected) marginal utility of saving, and thus the consumer would engage in positive saving and the constraint would no longer be binding. Thus the level of m_{t-1} at which the constraint stops binding is:[4]

$$\begin{aligned} \mathrm{u}^{c}(m_{t-1}) &= \mathrm{v}^{a}_{(t-1)_\rightarrow}(0) \\ m_{t-1} &= (\mathrm{v}^{a}_{(t-1)_\rightarrow}(0))^{(-1/\rho)} \\ &= \mathrm{c}_{(t-1)_\rightarrow}(0). \end{aligned}$$

[3] The word artificial is chosen only because of its clarity in distinguishing this from the case of the 'natural' borrowing constraint examined above; no derogation is intended – constraints of this kind certainly exist in the real world.

[4] The logic here repeats an insight from Deaton (1991).

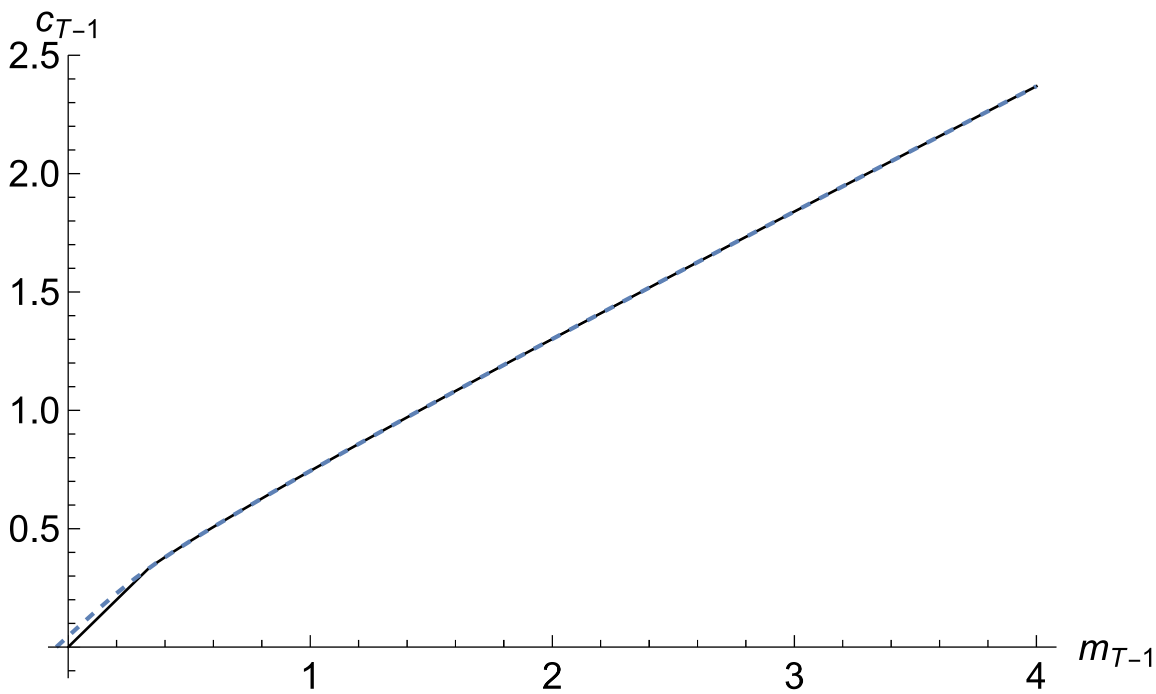

Figure 5 Constrained (solid) and Unconstrained (dashed) Consumption

{fig:cVScCon}

The constrained problem is solved in section "Artifical Borrowing Constraint" of the notebook, where the variable `constrained` is set to be a boolean type object. If the value of `constrained` is true, then the constraint is binding and their consumption behavior is computed to match (23). The resulting consumption rule is shown in Figure 5. For comparison purposes, the approximate consumption rule from Figure 5 is reproduced here as the solid line; this is accomplished by setting the boolean value of `constrained` to false.

The presence of the liquidity constraint requires three changes to the procedures outlined above:

1. We redefine $\underline{h}_{t\rightarrow}$, which now is the PDV of receiving $\theta_{t+1} = \underline{\theta}$ next period and $\theta_{t+n} = 0 \ \forall \ n > 1$ – that is, the pessimist believes he will receive nothing beyond period $t+1$

2. We augment the end-of-period `aVec` with zero and with a point with a small positive value so that the generated `mVec` will the binding point $m^{\#}$ and a point just above it (so that we can better capture the curvature around that point)

3. We redefine the optimal consumption rule as in equation (23). This ensures that the liquidity-constrained 'realist' will consume more than the redefined 'pessimist,' so that we will have ϱ still between 0 and 1 and the 'method of moderation' will proceed smoothly.

As expected, the liquidity constraint only causes a divergence between the two functions at the point where the optimal unconstrained consumption rule runs into the 45 degree line.

1 Recursion

{sec:recursion}

1.1 Theory

Before we solve for periods earlier than $T-1$, we assume for convenience that in each such period a liquidity constraint exists of the kind discussed above, preventing c from exceeding m. This simplifies things a bit because now we can always consider an `aVec` that starts with zero as its smallest element.

Recall now equations (21) and (12):

$$\begin{aligned} \mathrm{v}^{a}_{t\rightarrow}(a_t) &= \mathbb{E}_{\leftarrow t}[\beta \mathsf{R} \mathcal{G}^{-\rho}_{t+1} \mathrm{u}^{c}(\mathrm{c}_{t+1}(\mathcal{R}_{t+1} a_t + \theta_{t+1}))] \\ \mathrm{u}^{c}(c_t) &= \mathrm{v}^{a}_{t\rightarrow}(m_t - c_t). \end{aligned}$$

Assuming that the problem has been solved up to period $t+1$ (and thus assuming that we have an approximated $\grave{\mathrm{c}}_{t+1}(m_{t+1})$), our solution method essentially involves using these two equations in succession to work back progressively from period $T-1$ to the beginning of life. Stated generally, the method is as follows. (Here, we use the original, rather than the "refined," method for constructing consumption functions; the generalization of the algorithm below to use the refined method presents no difficulties.)

1. For the grid of values $a_{t,i}$ in `aVec_eee`, numerically calculate the values of $\mathrm{c}_{\bar{t}}(a_{t,i})$ and $\mathrm{c}^{a}_{\bar{t}}(a_{t,i})$,

$$\begin{aligned} \mathrm{c}_{\bar{t},i} &= \left(\mathrm{v}^{a}_{t\rightarrow}(a_{t,i})\right)^{-1/\rho}, \\ &= \left(\beta\, \mathbb{E}_{\leftarrow t}\left[\mathsf{R}\mathcal{G}^{-\rho}_{t+1}(\grave{\mathrm{c}}_{t+1}(\mathcal{R}_{t+1} a_{t,i} + \theta_{t+1}))^{-\rho}\right]\right)^{-1/\rho}, \qquad (24) \\ \mathrm{c}^{a}_{\bar{t},i} &= -(1/\rho)\left(\mathrm{v}^{a}_{t\rightarrow}(a_{t,i})\right)^{-1-1/\rho} \mathrm{v}^{aa}_{t\rightarrow}(a_{t,i}), \end{aligned}$$

{eq:vEndeq}

   generating vectors of values $\mathbf{c}_t$ and $\mathbf{c}^{a}_{\bar{t}}$.

2. Construct a corresponding vector of values of $\boldsymbol{m}_t = \boldsymbol{c}_t + \boldsymbol{a}_t$; similarly construct a corresponding list of MPC's $\boldsymbol{\kappa}_t$ using equation (9).

3. Construct a corresponding vector $\boldsymbol{\mu_t}$, the levels and first derivatives of $\boldsymbol{\varrho}_t$, and the levels and first derivatives of $\boldsymbol{\chi}_t$.

4. Construct an interpolating approximation $\grave{\chi}_t$ that smoothly matches both the level and the slope at those points.

5. If we are to approximate the value function, construct a corresponding list of values of $\boldsymbol{v}_t$, the levels and first derivatives of $\boldsymbol{\Omega}_t$, and the levels and first derivatives of $\hat{\mathbf{X}}_t$; and construct an interpolating approximation function $\hat{\mathrm{X}}_t$ that matches those points.

With $\grave{\chi}_t$ in hand, our approximate consumption function is computed directly from the appropriate substitutions in (4) and related equations. With this consumption rule in hand, we can continue the backwards recursion to period $t-1$ and so on back to the beginning of life.

Note that this loop does not contain an item for constructing $\hat{\mathrm{v}}_t^a(m_t)$. This is because with $\grave{\hat{\mathrm{c}}}_t(m_t)$ in hand, we simply *define* $\hat{\mathrm{v}}_t^m(m_t) = \mathrm{u}^c(\grave{\hat{\mathrm{c}}}_t(m_t))$ so there is no need to construct interpolating approximations - the function arises 'free' (or nearly so) from our constructed $\grave{\hat{\mathrm{c}}}_t(m_t)$ via the usual envelope result (cf. (7)).