

# 1 Notation

{sec:notation}

## 1.1 Periods, Stages, Steps

The problem specified above assumes that the agent has only one decision problem to solve in any period. In practice, it is increasingly common to model agents who have multiple choice stages per period; an agent's problem might have, say, a consumption decision (call it the  $c$  stage), a labor supply stage (call it  $\ell$ ) and a choice of what proportion  $\varsigma$  of their assets to invest in a risky asset (the portfolio-choice stage).

The modeler might well want to explore whether the order in which the stages are solved makes any difference, either to the substantive results or to aspects of the computational solution like speed and accuracy.

If, as in section 2, we hard-wire into the solution code for each stage an assumption that its successor stage will be something in particular (say, the consumption stage assumes that the portfolio choice is next), then if we want to change the order of the stages (say, labor supply after consumption, followed by portfolio choice), we will need to re-hard-wire each of the stages to know particular things about its new successor (for example, the specifics of the distribution of the rate of return on the risky asset would need to be known by whatever stage precedes the portfolio choice stage).

But one of the cardinal insights of Bellman's (1957, "Dynamic Programming") original work is that *everything that matters* for the solution to the current problem is encoded in a 'continuation-value function.'

Using Bellman's insight, we describe here a framework for isolating the stage problems within a period from each other, and the period from its successors in any future period; the advantage of this is that the isolated stage and period problems will then be 'modular': We can solve them in any order *without changing any code*. After considering the stage-order  $[\ell, c, \varsigma]$ , the modeler can costlessly reorder the stages to consider, say, the order  $[\ell, \varsigma, c]$ .<sup>1</sup>

## 1.2 Steps

The key is to distinguish, within each stage's Bellman problem, three steps:

1. **Arrival:** Incoming state variables (e.g.,  $k$ ) are known, but any shocks associated with the period have not been realized and decision(s) have not yet been made
2. **Decision:** All exogenous variables (like income shocks, rate of return shocks, and predictable income growth  $\mathcal{G}$ ) have been realized (so that, e.g.,  $m$ 's value is known) and the agent solves the optimization problem
3. **Continuation:** After all decisions have been made, their consequences are measured by evaluation of the continuing-value function at the values of the 'outgoing' state variables (sometimes called 'post-state' variables).

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<sup>1</sup>As long as the beginning-of-stage and end-of-stage value functions for the stages all depend on the same state variables; see the discussion in section ??.

When we want to refer to a specific toe in the stage we will do so by using an indicator which identifies that toe. Here we use the consumption stage problem described above to exemplify the usage:

Toe	Indicator	State	Usage	Explanation
Arrival	$\leftarrow$	$k$	$v_{\leftarrow}(k)$	value at entry to stage (before shocks)
Decision(s)	(blank)	$m$	$v(m)$	value of stage-decision (after shocks)
Continuation	$\rightarrow$	$a$	$v_{\rightarrow}(a)$	value at exit (after decision)

Notice that the value functions at different toes of the stage have distinct state variables. Only  $k$  is known at the beginning of the stage, and other variables take on their values with equations like  $b = k\mathcal{R}$  and  $m = b + \theta$ . We will refer to such within-the-stage creation of variables as evolutions.

### 1.3 Transitions

In the backward-induction world of Bellman solutions, to solve the problem of a particular period we must start with an end-of-period (continuation) value function, which we designate by explicitly including the period indicator in the subscript:

$$v_{t\rightarrow}(a) \mapsto \beta v_{\leftarrow(t+1)}(\overbrace{a}^{=k}), \quad (1) \quad \{\text{eq:trns-single-prd}\}$$

and we are not done solving the problem of the entire period until we have constructed a beginning-of-period value function  $v_{\leftarrow t}(k)$ .

Similarly, in order to solve the problem of any stage, we must endow it with an end-of-stage continuation-value function. For the last stage in a period the end-of-stage function is taken to be end-of-period value function:

$$v_{\rightarrow}(a) \mapsto v_{t\rightarrow}(a). \quad (2)$$

### 1.4 The Decision Problem in the New Notation

The Decision problem can now be written much more cleanly than in equation (10):

$$v(m) = \max_c u(c) + v_{\rightarrow}(\overbrace{m-c}^a) \quad (3) \quad \{\text{eq:vMid}\}$$

whose first order condition with respect to  $c$  is

$$u^c(c) = v_{\rightarrow}^a(m-c) \quad (4) \quad \{\text{eq:upEqbetaOp}\}$$

which is mathematically equivalent to the usual Euler equation for consumption. (We will reuse this formulation when we turn to section ??.)

Having defined these notational conventions, we are now ready to move to substance.