Structural Estimation of Life Cycle Models with Wealth in the Utility Function

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Abstract

Heterogeneous Agent Models (HAM) are a powerful tool for understanding the effects of monetary and fiscal policy on the economy. However, current state-of-the-art frameworks such as Heterogeneous Agent New Keynesian (HANK) models have limitations that hinder their ability to accurately replicate real-world economic phenomena. Specifically, HANK models struggle to account for the observed hoarding of wealth at the very top of the distribution and lack important life cycle properties such as time-varying preferences, mortality, and income risk. On the one hand, the inability to pin down wealth at the tail of the distribution has been a problem for HANK models precisely because it has implications for the transmission of monetary and fiscal policy. On the other hand, agents in HANK are generally conceived as perpetual youth with infinite horizons and without age-specific profiles of mortality and income risk. This is problematic as it ignores the effects of these policies on potentially more affected communities, such as young families with children or the low-wealth elderly. In this paper, I investigate the effects of both life cycle considerations as well as wealth in the utility on the structural estimation of HAMs. Structural estimation is the first step in evaluating the effect of monetary and fiscal policies in a HANK framework, and my hope is that this paper will lead to better models of the economy that can be used to inform policy.

Keywords structural estimation, life cycle, wealth in the utility

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me achieve my goals. All remaining errors are my own. The figures in this paper were generated using the Econ-ARK/HARK toolkit.

1 Introduction

1.1 HANK

Kaplan et al. [2018] Monetary policy according to HANK Acharya and Dogra [2020] Understanding HANK: Insights from a PRANK Ravn and Sterk [2020] Macroeconomic Fluctuations with HANK & SAM: An Analytical Approach Auclert et al. [2020] Micro Jumps, Macro Humps: Monetary Policy and Business Cycles in an Estimated HANK Model

1.2 Wealth Accumulation

Carroll [2000] Portfolios of the Rich Carroll [1998] Why do the Rich save so much? Mian et al. [2020] The Saving Glut of the Rich Dynan et al. [2004] Do the Rich Save More?

1.3 Life Cycle Models

Palumbo [1999] Uncertain Medical Expenses and Precautionary Saving Near the End of the Life Cycle Cagetti [2003] Wealth Accumulation Over the Life Cycle and Precautionary Savings Attanasio et al. [1999] Humps and Bumps in Lifetime Consumption Dynan et al. [2002] The Importance of Bequests and Life-Cycle Saving in Capital Accumulation: A New Answer Nardi et al. [2010] Why Do the Elderly Save? The Role of Medical Expenses

1.4 Related papers/tools

Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models Michaillat and Saez [2021] Resolving New Keynesian Anomalies with Wealth in the Utility Function

2 Life Cycle Incomplete Markets Models

An important extension to the Standard Incomplete Markets (SIM) model is the Life Cycle Incomplete Markets (LCIM) model as in Cagetti [2003], Gourinchas and Parker [2002], and Palumbo [1999], among others. The LCIM model is a natural extension to the SIM model that allows for age-specific profiles of preferences, mortality, and income risk.

2.1 The Baseline Model

The agent's objective is to maximize present discounted utility from consumption over the life cycle with a terminal period of T:

$$\mathrm{v}_t(\mathbf{p}_t, \mathbf{m}_t) = \max_{\{\mathrm{c}\}_t^T} \mathrm{u}(\mathbf{c}_t) + \mathbb{E}_t \left[\sum_{n=1}^{T-t} \beth^n \mathcal{L}_t^{t+n} \hat{\beta}_t^{t+n} \mathrm{u}(\mathbf{c}_{t+n}) \right] \tag{1}$$

where $\mathbf{p}_t$ is the permanent income level, $\mathbf{m}_t$ is total market resources, $\mathbf{c}_t$ is consumption, and

$$\begin{aligned} \beth &: \text{time-invariant 'pure' discount factor} \\ \mathcal{L}_t^{t+n} &: \text{probability to } \mathcal{L}\text{ive until age } t+n \text{ given alive at age } t \\ \hat{\beta}_t^{t+n} &: \text{age-varying discount factor between ages } t \text{ and } t+n. \end{aligned} \tag{2}$$

It will be convenient to work with the problem in permanent-income-normalized form as in Carroll [2004], which allows us to reduce a 2 dimensional problem of permanent income and wealth into a 1 dimensional problem of wealth normalized by permanent income. The recursive Bellman equation can be expressed as:

$$\begin{aligned} \mathrm{v}_t(m_t) &= \max_{c_t} \mathrm{u}(c_t) + \beth \mathcal{L}_{t+1} \hat{\beta}_{t+1} \mathbb{E}_t[(\mathbf{\Psi}_{t+1}\mathbf{\Phi}_{t+1})^{1-\rho} \mathrm{v}_{t+1}(m_{t+1})] \\ &\text{s.t.} \\ a_t &= m_t - c_t \\ m_{t+1} &= a_t \underbrace{\left(\frac{\mathsf{R}}{\mathbf{\Psi}_{t+1}\mathbf{\Phi}_{t+1}} \right)}_{\equiv \mathcal{R}_{t+1}} + \boldsymbol{\theta}_{t+1} \end{aligned} \tag{3}$$

where c, a, and m are consumption, assets, and market resources normalized by permanent income, respectively, v and u are now the normalized value and utility functions, and

$$\begin{aligned} \mathbf{\Psi}_{t+1} &: \text{mean-one shock to permanent income} \\ \mathbf{\Phi}_{t+1} &: \text{permanent income growth factor} \\ \boldsymbol{\theta}_{t+1} &: \text{transitory shock to permanent income} \\ \mathcal{R}_{t+1} &: \text{permanent income growth normalized return factor} \end{aligned} \tag{4}$$

with all other variables are defined as above. The transitory and permanent shocks to income are defined as:

$$\begin{aligned} \boldsymbol{\theta}_s &= \begin{cases} 0 & \text{with probability } \wp > 0 \\ \xi_s/\wp & \text{with probability } (1-\wp) \text{, where } \log \xi_s \sim \mathcal{N}(-\sigma^2_{[\xi,t]}/2, \sigma^2_{[\xi,t]}) \end{cases} \\ & \text{and } \log \mathbf{\Psi}_s \sim \mathcal{N}(-\sigma^2_{[\mathbf{\Psi},t]}/2, \sigma^2_{[\mathbf{\Psi},t]}). \end{aligned} \tag{5}$$

2.2 Wealth in the Utility Function

A simple extension to the Life Cycle Incomplete Markets (LCIM) model is to include wealth in the utility function. Carroll [1998] argues that models in which the only driver of wealth accumulation is consumption smoothing are not consistent with the saving behavior of the wealthiest households. Instead, they propose a model in which households derive utility from their level of wealth itself or they derive a flow of services from political power and social status, calling it the 'Capitalist Spirit' model. In turn, we can add this feature to the LCIM model by adding a utility function with consumption and wealth. We call this the Wealth in the Utility Function Incomplete Markets (WUFIM) model.

$$\begin{aligned} \mathrm{v}_t(m_t) &= \max_{c_t} \ \mathrm{u}(c_t, a_t) + \daleth\mathcal{L}_{t+1}\hat{\beta}_{t+1}\mathbb{E}_t[(\boldsymbol{\Psi}_{t+1}\boldsymbol{\Phi}_{t+1})^{1-\rho}\mathrm{v}_{t+1}(m_{t+1})] \\ &\text{s.t.} \\ a_t &= m_t - c_t \\ m_{t+1} &= a_t\mathcal{R}_{t+1} + \ \boldsymbol{\theta}_{t+1} \end{aligned} \tag{6}$$

Separable Utility Carroll [1998] presents extensive empirical and informal evidence for a LCIM model with wealth in the utility function. Specifically, the paper uses a utility that is separable in consumption and wealth:

$$\mathrm{u}(c_t, a_t) = \frac{c_t^{1-\rho}}{1-\rho} + \alpha_t \frac{(\mathrm{a}_t - \underline{a})^{1-\delta}}{1-\delta} \tag{7}$$

where α is the relative weight of the utility of wealth and δ is the relative risk aversion of wealth.

Non-separable Utility A different model that we will explore is one in which the utility function is non-separable in consumption and wealth; i.e. consumption and wealth are complimentary goods. In the case of the LCIM model, this dynamic complementarity drives the accumulation of wealth not only for the sake of wealth itself, but also because it increases the marginal utility of consumption.

$$\mathrm{u}(c_t, a_t) = \frac{(c_t^{1-\delta}(a_t - \underline{a})^{\delta})^{1-\rho}}{(1-\rho)} \tag{8}$$

3 Solution Methods

For a brief departure, let's consider how we solve these problems generally. Define the post-decision value function as:

$$\begin{aligned} \beta_{t+1}\mathrm{w}_t(a_t) &= \daleth\mathcal{L}_{t+1}\hat{\beta}_{t+1}\mathbb{E}_t[(\boldsymbol{\Psi}_{t+1}\boldsymbol{\Phi}_{t+1})^{1-\rho}\mathrm{v}_{t+1}(m_{t+1})] \\ &\text{s.t.} \\ m_{t+1} &= a_t\mathcal{R}_{t+1} + \ \boldsymbol{\theta}_{t+1} \end{aligned} \tag{9}$$

For our purposes, it will be useful to simplify the notation a bit by dropping time subscripts. The recursive problem can then be written as:

$$\begin{aligned} \mathrm{v}(m) &= \max_{c} \ \mathrm{u}(c, a) + \beta\mathrm{w}(a) \\ &\text{s.t.} \\ a &= m - c \end{aligned} \tag{10}$$

3.1 Endogenous Grid Method, Abridged

In the standard incomplete markets (SIM) model, the utility function is simply $\mathrm{u}(c)$ and the Euler equation is $\mathrm{u}'(c) = \beta \mathrm{w}'(a)$, which is a necessary and sufficient condition for an internal solution of the optimal choice of consumption. If $\mathrm{u}(c)$ is differentiable and its marginal utility is invertible, then the Euler equation can be inverted to obtain the optimal consumption function as $c(a) = \mathrm{u}'^{-1}(\beta \mathrm{w}'(a))$. Using an *exogenous* grid of post-decision savings $[\mathrm{a}]$, we can obtain an *endogenous* grid of market resources $[\mathrm{m}]$ by using the budget constraint $m([\mathrm{a}]) = [\mathrm{a}] + c([\mathrm{a}])$ such that this collection of grids satisfy the Euler equation. This is the endogenous grid method (EGM) of Carroll [2006].

In the presence of wealth in the utility function, the Euler equation is more complicated and may not be invertible in terms of optimal consumption. Consider the first order condition for an optimal combination of consumption and savings, denoted by *:

$$\mathrm{u}'_c(c^*, a^*) - \mathrm{u}'_a(c^*, a^*) = \beta \mathrm{w}'(a^*) \tag{11}$$

If the utility of consumption and wealth is additively separable, then the Euler equation can be written as $\mathrm{u}'_c(c) = \mathrm{u}'_a(a) + \beta \mathrm{w}'(a)$. This makes sense, as the agent will equalize the marginal utility of consumption with the marginal utility of wealth today plus the discounted marginal value of wealth tomorrow. In this case, the EGM is simple: we can invert the Euler equation to obtain the optimal consumption policy as $c(a) = \mathrm{u}'^{-1}_c\big(\mathrm{u}'_a(a) + \beta \mathrm{w}'(a)\big)$. We can proceed with EGM as usual, using the budget constraint to obtain the endogenous grid of market resources $m([\mathrm{a}]) = [\mathrm{a}] + c([\mathrm{a}])$.

3.2 Root Finding

When the utility of consumption and wealth is not additively separable, the Euler equation is not analytically invertible for the optimal consumption policy. The usual recourse is to use a root-finding algorithm to obtain the optimal consumption policy for each point on the grid of market resources, which turns out to be more efficient than grid search maximization.

Holding m constant, define a function f_m as the difference between the marginal utility of consumption and the marginal utility of wealth:

$$f_m(c) = \mathrm{u}'_c(c, m - c) - \mathrm{u}'_a(c, m - c) - \beta \mathrm{w}'(m - c) \tag{12}$$

The optimal consumption policy is the value of c that satisfies $f_m(c) = 0$. We can use a root-finding algorithm to obtain the optimal consumption policy for each point on the grid of market resources. Although this is more efficient than grid search maximization, it is still computationally expensive. Unlike the single-step EGM, root finding requires a number of iterations to find the optimal consumption policy, which makes it relatively slower. Nevertheless, we can use clever tricks to speed up the process. One such trick used in this paper is to use the optimal consumption policy from the previous iteration as the initial guess for the next iteration. This is possible because the optimal consumption policy is a continuous function of the grid of market resources and the optional decision from one period to the next is not too different. This is the method used in the code for this paper.

4 Quantitative Strategy

This section describes the quantitative strategy used for calibrating and estimating the Life Cycle Incomplete Markets model with and without Wealth in the Utility Function, following the works of Cagetti [2003], Palumbo [1999], Gourinchas and Parker [2002], and Sabelhaus and Song [2010], among others. The main objective is to find a set of parameters that can best match the empirical moments of some real-life data using simulation.

4.1 Calibration

The calibration of the Life Cycle Incomplete Markets model necessitates a richness not present in the SIM model precisely because we are interested in the heterogeneity of agents across different stages of the life cycle, such as the early working period, parenthood, saving for retirement, and retirement. To calibrate this model, we need to identify important patterns in preferences, mortality, and income risk across the life cycle. The first and perhaps most important departure from SIM is that life is finite and agents don't life forever; moreover, the terminal age is not certain as the probability of staying alive decreases with age. In this model, households start their life cycle at age $t = 25$ and live with certainty until retirement at age $t = 65$. After retirement, the probability of staying alive decreases with age, and the terminal age is set to $t = 91$. During their early adulthood, their utility of consumption might need to be adjusted by the arrival and subsequent departure of children. This is handled by a 'household-size-adjusted' discount factor that is greater than 1.0 in the presence of children. This is the rationale for parameters $\mathcal{L}_t$ and $\hat{\beta}_t$ in the model, whose values we take from Cagetti [2003] directly.

The unemployment probability is taken from Carroll et al. [1992] to be $\wp = 0.5$ which represents a long run equilibrium of 5% unemployment in the United States. The remaining life cycle attributes for the distribution of shocks to income ($\mathbf{\Phi}_t,\ \sigma_{[\Psi,t]},\ \sigma_{[\xi,t]}$) are taken from Sabelhaus and Song [2010]. In their paper, they analyze the variability of labor earnings growth rates between the 80's and 90's and find evidence for the "Great Moderation", a decline in variability of earnings across all age groups.

After careful calibration based on the Life Cycle Incomplete Markets literature, we can structurally estimate the remaining parameters $\beth$ and ρ to match specific empirical moments of the wealth distribution.

4.2 Estimation

Structural estimation consists of finding the set of parameters that, when used to solve and simulate the model, result in simulated moments that are as close as possible to the empirical moments observed in the data. For this exercise, we focus on matching the median of the wealth to permanent income ratio for 7 age groups starting from age 25-30 up to age 56-60. The data is aggregated from the waves of the Survey of Consumer Finances (SCF). Matching the median has been standard in the literature precisely because it

Table 1: LCIM Estimation Results

$\beth$	ρ
0.878	3.516
(0.0018)	(0.0266)

has been so difficult to match the mean of the wealth distribution given the high degree of wealth inequality in the United States. The Wealth in the Utility Function models however are constructed to better match the dispersion of wealth accumulation, and in future work we will attempt to match the mean of the wealth distribution as well.

Given an initial vector of parameters $\Theta_0 = \{\beth_0, \rho_0\}$, the first step in the estimation procedure is to solve for the steady state of the model. As this is a life cycle exercise, the strategy is to start from the terminal period and work backwards to the initial period. This is known as backward induction. The terminal period is characterized by simple decisions over consumption and bequest, as the agent is certain to die and has no continuation value and thus no use for savings. Having constructed the terminal policy functions and their corresponding value and marginal value, we can solve for the optimal policies in the second to last period using the methods described in the previous section. We can then continue this process until we arrive at the initial period. In the end, and unlike in the SIM model, we have a complete set of policy functions for consumption and saving for every age of the life cycle.

Having solved the steady state of the model for the given set of parameters, we can now use the optimal policy functions to generate simulated data of consumption and savings over the life cycle. We can then calculate the simulated moments of the wealth distribution at the 7 age groups. We can define the objective function as

$$g(\Theta) = \sum_{\tau=1}^{7} \omega_\tau |\varsigma^\tau - \mathbf{s}^\tau(\Theta)| \tag{13}$$

where ς^τ is the empirical moment of the wealth distribution at age τ, $\mathbf{s}^\tau(\Theta)$ is the simulated moment of the wealth distribution at age τ for a given set of parameters Θ, and ω_τ is the population weight for a particular age group in our data. The goal is thus to minimize the objective function by choice of Θ such that $\hat{\Theta} = \arg\min_\Theta g(\Theta)$. To find $\hat{\Theta}$, we use the Nelder-Mead algorithm which uses a simplex method and does not require derivatives of the objective function. This consists of trying a significant number of guesses for Θ, solving the model, and simulating moments which can be quite computationally intensive. Future work will focus on using more efficient methods such as those presented by Auclert et al. [2021], where the Jacobian (partial first derivatives) of the objective function is used to find the optimal parameters $\hat{\Theta}$ more efficiently and quickly.

Results for LCIM model We can see the estimated parameters for the LCIM model in Table 1. The estimated values for $\beth$ and ρ are 0.878 and 3.1516, respectively, with standard errors estimated via the bootstrap. Additionally, Figure 1 shows a contour plot of the objective function for the structural estimation exercise where the red star represents the estimated parameters. The contour plot shows that the objective function has a relatively flat region around the estimated parameters that extends toward higher values of ρ and lower values of $\beth$, showing the trade-offs between the estimation of these two parameters.

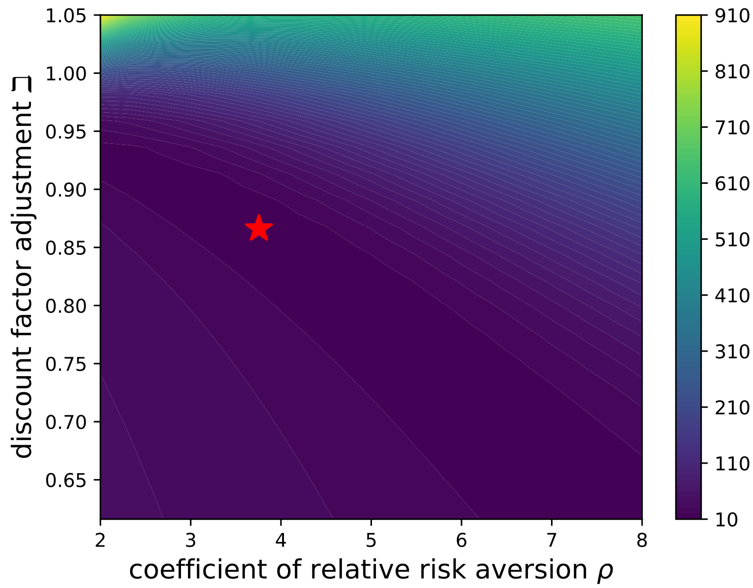

Figure 1: Contour plot of the objective function for the structural estimation of the Life Cycle Incomplete Markets model. The red dot represents the estimated parameters.

Results for WUFIM models

4.3 Sensitivity Analysis

Andrews et al. [2017] Measuring the Sensitivity of Parameter Estimates to Estimation Moments

For our sensitivity analysis, we use the methods introduced by Andrews et al. [2017].

5 Conclusion

6 References

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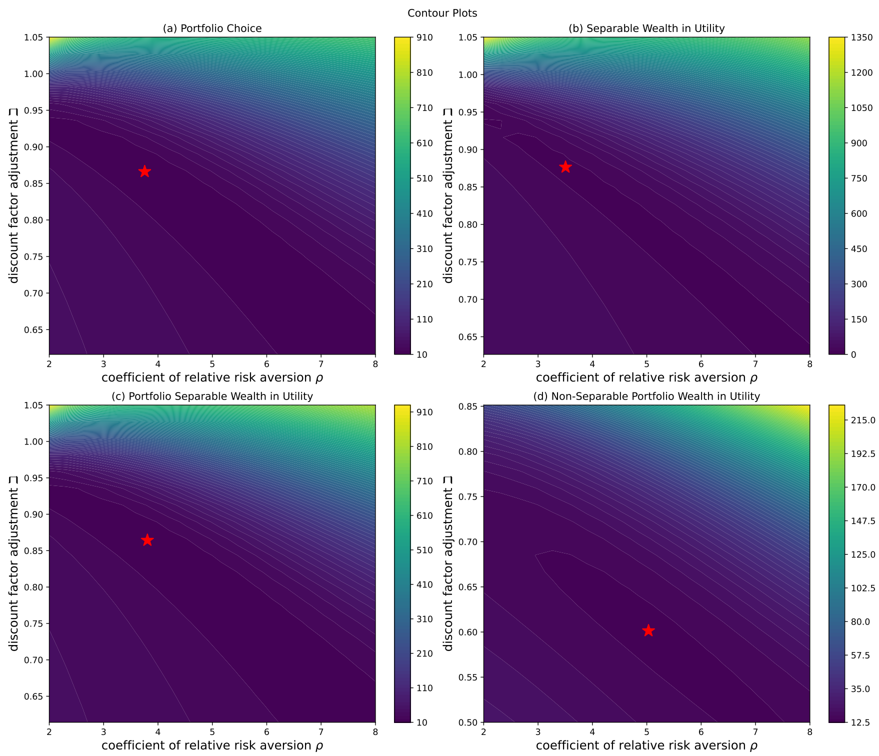

Figure 2: Contour plot of the objective function for the structural estimation of the Life Cycle Incomplete Markets model. The red dot represents the estimated parameters.

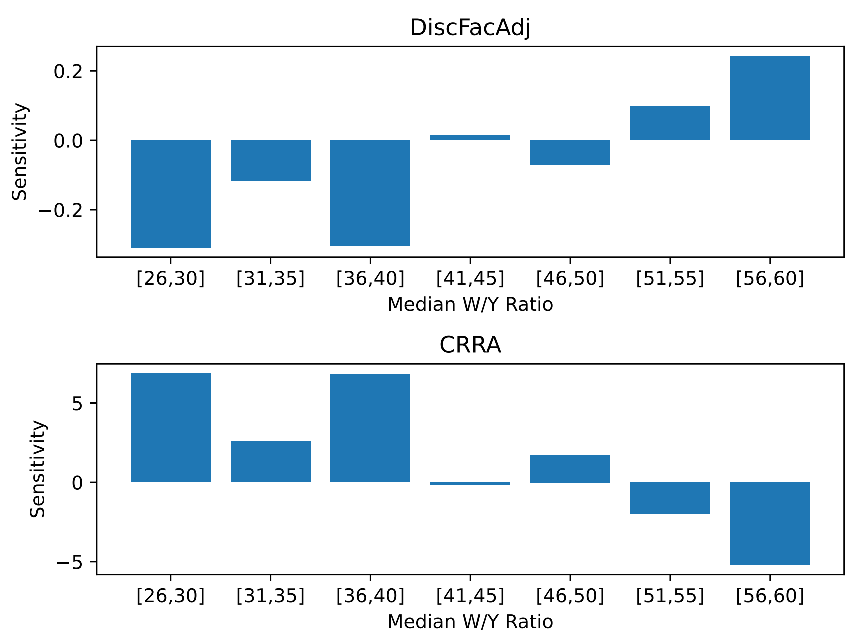

Figure 3: Sensitivity analysis of the structural estimation of the Life Cycle Incomplete Markets model. The red dot represents the estimated parameters.

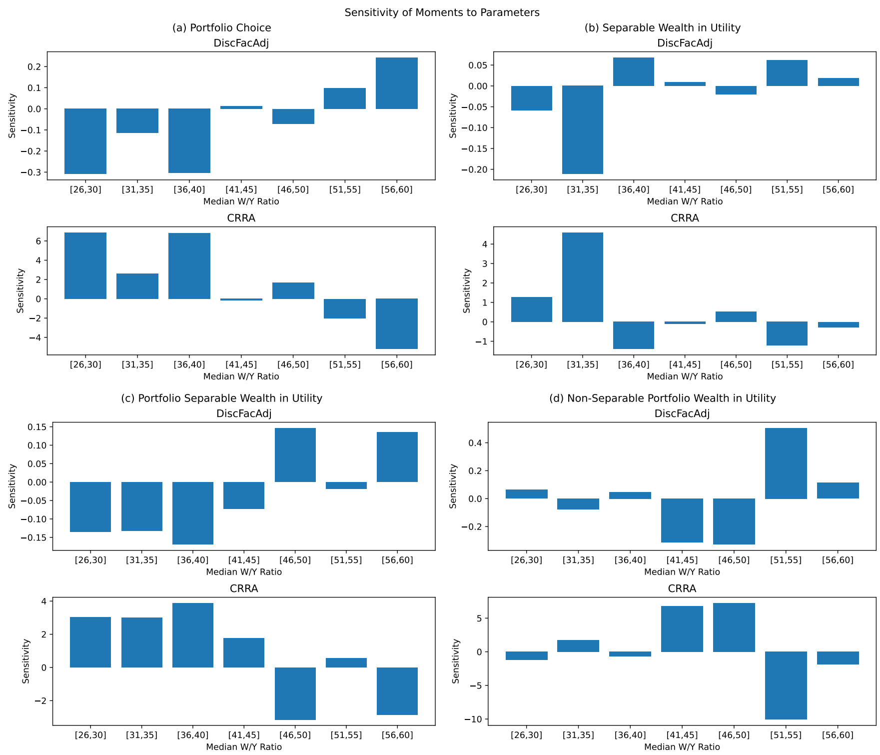

Figure 4: Sensitivity analysis of the structural estimation of the Life Cycle Incomplete Markets model. The red dot represents the estimated parameters.

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