
STRUCTURAL ESTIMATION OF LIFE CYCLE MODELS WITH WEALTH IN THE UTILITY FUNCTION

Alan Lujan¹

Ohio State University
Econ-ARK

Sunday 18th June, 2023

Abstract

Heterogeneous Agent Models (HAM) are a powerful tool for understanding the effects of monetary and fiscal policy on the economy. However, current state-of-the-art frameworks such as Heterogeneous Agent New Keynesian (HANK) models have limitations that hinder their ability to accurately replicate real-world economic phenomena. Specifically, HANK models struggle to account for the observed hoarding of wealth at the very top of the distribution and lack important life cycle properties such as time-varying mortality and income risk. On the one hand, the inability to pin down wealth at the tail of the distribution has been a problem for HANK models precisely because it has implications for the transmission of monetary and fiscal policy. On the other hand, agents in HANK are generally conceived as perpetual youth with infinite horizons and without age-specific profiles of mortality and income risk. This is problematic as it ignores the effects of these policies on potentially more affected communities, such as young families with children or the low-wealth elderly. In this paper, I investigate the effects of both life cycle considerations as well as wealth in the utility on the structural estimation of HAMs. Structural estimation is the first step in evaluating the effect of monetary and fiscal policies in a HANK framework, and my hope is that this paper will lead to better models of the economy that can be used to inform policy.

Keywords structural estimation, life cycle, wealth in the utility

I would like to thank my advisor, Chris Carroll, for his guidance and support throughout this project. His expertise and mentorship have been invaluable in shaping my work. Additionally, I would like to extend my appreciation to the members of the [Econ-ARK](#) team for fostering a dynamic and collaborative community that has greatly enriched my experience. Their contributions and feedback have been instrumental in helping me achieve my goals. All remaining errors are my own. The figures in this paper were generated using the [Econ-ARK/HARK](#) toolkit.

¹Correspondence to: alanlujan91@gmail.com

1 Introduction

2 Life Cycle Incomplete Markets Models

An important extension to the Standard Incomplete Markets (SIM) model is the Life Cycle Incomplete Markets (LCIM) model as in [Cagetti \[2003\]](#), [Gourinchas and Parker \[2002\]](#), and [Palumbo \[1999\]](#), among others. The LCIM model is a natural extension to the SIM model that allows for age-specific profiles of preferences, mortality, and income risk.

2.1 The Baseline Model

The agent's objective is to maximize present discounted utility from consumption over the life cycle with a terminal period of T :

$$v_t(\mathbf{p}_t, \mathbf{m}_t) = \max_{\{\mathbf{c}\}_t^T} u(\mathbf{c}_t) + \mathbb{E}_t \left[\sum_{n=1}^{T-t} \beta^n \mathcal{L}_t^{t+n} \hat{\beta}_t^{t+n} u(\mathbf{c}_{t+n}) \right] \quad (1)$$

where \mathbf{p}_t is the permanent income level, \mathbf{m}_t is total market resources, \mathbf{c}_t is consumption, and

$$\begin{aligned} \beta &: \text{time-invariant 'pure' discount factor} \\ \mathcal{L}_t^{t+n} &: \text{probability to Live until age } t+n \text{ given alive at age } t \\ \hat{\beta}_t^{t+n} &: \text{age-varying discount factor between ages } t \text{ and } t+n. \end{aligned} \quad (2)$$

It will be convenient to work with the problem in permanent-income-normalized form as in [Carroll \[2004\]](#), which allows us to reduce a 2 dimensional problem of permanent income and wealth into a 1 dimensional problem of wealth normalized by permanent income. The recursive Bellman equation can be expressed as:

$$\begin{aligned} v_t(m_t) &= \max_{c_t} u(c_t) + \beta \mathcal{L}_{t+1} \hat{\beta}_{t+1} \mathbb{E}_t[(\Psi_{t+1} \Phi_{t+1})^{1-\rho} v_{t+1}(m_{t+1})] \\ &\text{s.t.} \\ a_t &= m_t - c_t \\ m_{t+1} &= a_t \underbrace{\left(\frac{R}{\Psi_{t+1} \Phi_{t+1}} \right)}_{\equiv \mathcal{R}_{t+1}} + \theta_{t+1} \end{aligned} \quad (3)$$

where c , a , and m are consumption, assets, and market resources normalized by permanent income, respectively, v and u are now the normalized value and utility functions, and

$$\begin{aligned} \Psi_{t+1} &: \text{mean-one shock to permanent income} \\ \Phi_{t+1} &: \text{permanent income growth factor} \\ \theta_{t+1} &: \text{transitory shock to permanent income} \\ \mathcal{R}_{t+1} &: \text{permanent income growth normalized return factor} \end{aligned} \quad (4)$$

with all other variables are defined as above. The transitory and permanent shocks to income are defined as:

$$\theta_s = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \xi_s/\wp & \text{with probability } (1 - \wp), \text{ where } \log \xi_s \sim \mathcal{N}(-\sigma_{[\xi,t]}^2/2, \sigma_{[\xi,t]}^2) \end{cases} \quad (5)$$

$$\text{and } \log \Psi_s \sim \mathcal{N}(-\sigma_{[\Psi,t]}^2/2, \sigma_{[\Psi,t]}^2).$$

2.2 Wealth in the Utility Function

A simple extension to the Life Cycle Incomplete Markets (LCIM) model is to include wealth in the utility function. [Carroll \[1998\]](#) argues that models in which the only driver of wealth accumulation is consumption smoothing are not consistent with the saving behavior of the wealthiest households. Instead, they propose a model in which households derive utility from their level of wealth itself or they derive a flow of services from political power and social status, calling it the ‘Capitalist Spirit’ model. In turn, we can add this feature to the LCIM model by adding a utility function with consumption and wealth as follows:

$$\begin{aligned} v_t(m_t) &= \max_{c_t} u(c_t, a_t) + \beta_{t+1} \mathbb{E}_t[(\Psi_{t+1} \Phi_{t+1})^{1-\rho} v_{t+1}(m_{t+1})] \\ \text{s.t.} \\ a_t &= m_t - c_t \\ m_{t+1} &= a_t \mathcal{R}_{t+1} + \theta_{t+1} \end{aligned} \quad (6)$$

Separable Utility [Carroll \[1998\]](#) presents extensive empirical and informal evidence for a LCIM model with wealth in the utility function. Specifically, the paper uses a utility that is separable in consumption and wealth:

$$u(c_t, a_t) = \frac{c_t^{1-\rho}}{1-\rho} + \alpha_t \frac{(a_t - \underline{a})^{1-\delta}}{1-\delta} \quad (7)$$

where α is the relative weight of the utility of wealth and δ is the relative risk aversion of wealth.

Non-separable Utility A different model that we will explore is one in which the utility function is non-separable in consumption and wealth; i.e. consumption and wealth are complimentary goods. In the case of the LCIM model, this dynamic complementarity drives the accumulation of wealth not only for the sake of wealth itself, but also because it increases the marginal utility of consumption.

$$u(c_t, a_t) = \frac{(c_t^{1-\delta} (a_t - \underline{a})^\delta)^{1-\rho}}{(1-\rho)} \quad (8)$$

3 Solution Methods

For a brief departure, let’s consider how we solve these problems generally. Define the post-decision value function as:

$$\begin{aligned} \beta_{t+1} w_t(a_t) &= \beta_{t+1} \mathbb{E}_t[(\Psi_{t+1} \Phi_{t+1})^{1-\rho} v_{t+1}(m_{t+1})] \\ \text{s.t.} \\ m_{t+1} &= a_t \mathcal{R}_{t+1} + \theta_{t+1} \end{aligned} \quad (9)$$

For our purposes, it will be useful to simplify the notation a bit by dropping time subscripts. The recursive problem can then be written as:

$$\begin{aligned} v(m) &= \max_c u(c, a) + \beta w(a) \\ \text{s.t.} \\ a &= m - c \end{aligned} \tag{10}$$

3.1 Endogenous Grid Method, Abridged

In the standard incomplete markets (SIM) model, the utility function is simply $u(c)$ and the Euler equation is $u'(c) = \beta w'(a)$, which is a necessary and sufficient condition for an internal solution of the optimal choice of consumption. If $u(c)$ is differentiable and its marginal utility is invertible, then the Euler equation can be inverted to obtain the optimal consumption function as $c(a) = u'^{-1}(\beta w'(a))$. Using an *exogenous* grid of post-decision savings $[a]$, we can obtain an *endogenous* grid of market resources $[m]$ by using the budget constraint $m([a]) = [a] + c([a])$ such that this collection of grids satisfy the Euler equation. This is the endogenous grid method (EGM) of [Carroll \[2006\]](#).

In the presence of wealth in the utility function, the Euler equation is more complicated and may not be invertible in terms of optimal consumption. Consider the first order condition for an optimal combination of consumption and savings, denoted by $*$:

$$u'_c(c^*, a^*) - u'_a(c^*, a^*) = \beta w'(a^*) \tag{11}$$

If the utility of consumption and wealth is additively separable, then the Euler equation can be written as $u'_c(c) = u'_a(a) + \beta w'(a)$. This makes sense, as the agent will equalize the marginal utility of consumption with the marginal utility of wealth today plus the discounted marginal value of wealth tomorrow. In this case, the EGM is simple: we can invert the Euler equation to obtain the optimal consumption policy as $c(a) = u'^{-1}_c(u'_a(a) + \beta w'(a))$. We can proceed with EGM as usual, using the budget constraint to obtain the endogenous grid of market resources $m([a]) = [a] + c([a])$.

3.2 Generalized Endogenous Grid Method

When the utility of consumption and wealth is not additively separable, the Euler equation is not analytically invertible for the optimal consumption policy. The usual recourse is to use a root-finding algorithm to obtain the optimal consumption policy for each point on the grid of market resources, which turns out to be more efficient than grid search maximization. As far as I am aware, this is the first paper to avoid root-finding or grid search maximization by using an EGM-like approach to solve a problem with a non-invertible Euler equation.

Holding a constant, define a function f_a as the difference between the marginal utility of consumption and the marginal utility of wealth, and using an *exogenous* grid of $[c]$, label the difference as $[\chi_a]$:

$$f_a([c]) = u'_c([c], a) - u'_a([c], a) = [\chi_a] \tag{12}$$

So far, we haven't done much, as f_a is still not analytically invertible. However, as we have held a constant, this is now a single variable function with a single output. We can easily create a linear interpolator for the inverse of this function by reversing output with input appropriately. This is the key step that allows us to

avoid root-finding or grid search maximization. We will call this function \hat{f}_a^{-1} , as it is an approximation of the inverse of f_a which could still introduce some error¹.

$$\hat{f}_a^{-1}([\chi_a]) = [c] \quad (13)$$

We can now construct the function $g(a)$ as the composition of \hat{f}_a^{-1} and discounted marginal value of wealth $\beta w'$, which should provide an approximation of the optimal consumption policy for each point on the grid of post-decision savings:

$$g(a^*) = \hat{f}_{a^*}^{-1}(\beta w'(a^*)) = c^* \quad (14)$$

This completes the modified step in the Generalized Endogenous Grid Method (GEGM). If this looks familiar, it is because it exactly parallels the EGM method. In the SIM model, $f_a(c, a) = u'(c)$, which does not depend on a , so we can drop it. The inverse of this function is exactly $f^{-1}(\chi) = u'^{-1}(\chi)$, so we don't need an approximating interpolator. The composition of this function with the discounted marginal value of wealth exactly provides the consumption policy as $g(a) = u'^{-1}(\beta w'(a)) = c$. GEGM is a generalization of EGM to the case where the Euler equation is not analytically invertible.

An additional feature of GEGM is that the inverse interpolating function \hat{f}_a^{-1} only needs to be constructed once and can be used for all backward iterations, as long as the utility function isn't time-varying. This is because the inverse interpolating function only depends on the marginal utilities of consumption and wealth, whose parameters are constant in our specification. This also means that we can construct this interpolating function from the onset of our process to have arbitrary precision by optimally choosing the grid of $[c]$. Computationally, this adds just one additional step to the standard EGM at the beginning of the process, inheriting its substantial improvements in speed and accuracy relative to root finding and grid search maximization².

4 Quantitative Strategy

This section describes the quantitative strategy used for calibrating and estimating the Life Cycle Incomplete Markets model with and without Wealth in the Utility Function, following the works of [Cagetti \[2003\]](#), [Palumbo \[1999\]](#), [Gourinchas and Parker \[2002\]](#), and [Sabelhaus and Song \[2010\]](#), among others. The main objective is to find a set of parameters that can best match the empirical moments of some real-life data using simulation.

4.1 Calibration

The calibration of the Life Cycle Incomplete Markets model necessitates a richness not present in the SIM model precisely because we are interested in the heterogeneity of agents across different stages of the life

¹A main difference between EGM and GEGM is that EGM uses the exact inverse of the Euler equation, thereby giving the unique optimal consumption policy for each point on the grid of post-decision savings that exactly satisfies the Euler equation. GEGM uses an approximation of the inverse of the Euler equation, thereby giving an approximate optimal consumption policy for each point on the grid of market resources. With careful grid choice, the approximation error can be made arbitrarily small.

²If additional precision is needed, the resulting \hat{c} can be used as the initial guess for a root-finding algorithm with arbitrary precision, which should converge in few iterations due to the proximity of the initial guess to the true solution.

cycle, such as the early working period, parenthood, saving for retirement, and retirement. To calibrate this model, we need to identify important patterns in preferences, mortality, and income risk across the life cycle. The first and perhaps most important departure from SIM is that life is finite and agents don't live forever; moreover, the terminal age is not certain as the probability of staying alive decreases with age. In this model, households start their life cycle at age $t = 25$ and live with certainty until retirement at age $t = 65$. After retirement, the probability of staying alive decreases with age, and the terminal age is set to $t = 91$. During their early adulthood, their utility of consumption might need to be adjusted by the arrival and subsequent departure of children. This is handled by a 'household-size-adjusted' discount factor that is greater than 1.0 in the presence of children. This is the rationale for parameters \mathcal{L}_t and $\hat{\beta}_t$ in the model, whose values we take from [Cagetti \[2003\]](#) directly.

The unemployment probability is taken from [Carroll et al. \[1992\]](#) to be $\varphi = 0.5$ which represents a long run equilibrium of 5% unemployment in the United States. The remaining life cycle attributes for the distribution of shocks to income (Φ_t , $\sigma_{[\Psi,t]}$, $\sigma_{[\xi,t]}$) are taken from [Sabelhaus and Song \[2010\]](#). In their paper, they analyze the variability of labor earnings growth rates between the 80's and 90's and find evidence for the "Great Moderation", a decline in variability of earnings across all age groups.

After careful calibration based on the Life Cycle Incomplete Markets literature, we can structurally estimate the remaining parameters \mathfrak{I} and ρ to match specific empirical moments of the wealth distribution.

4.2 Estimation

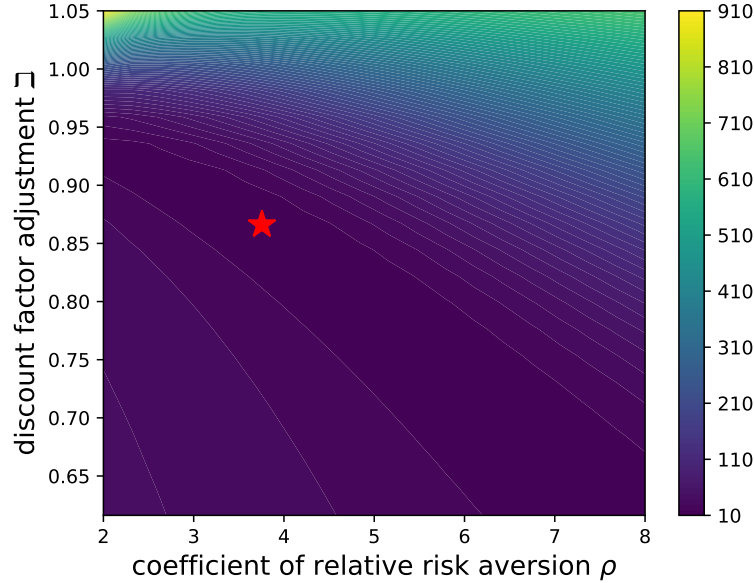


Figure 1: Contour plot of the objective function for the structural estimation of the Life Cycle Incomplete Markets model. The red dot represents the estimated parameters.

4.3 Sensitivity Analysis

For our sensitivity analysis, we use the methods introduced by [Andrews et al. \[2017\]](#).

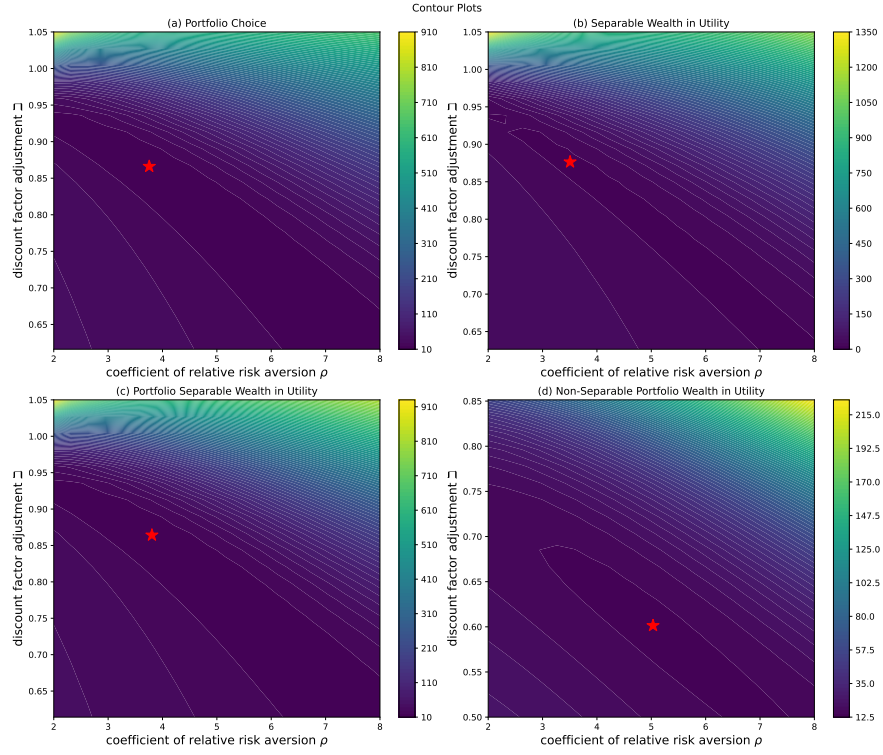


Figure 2: Contour plot of the objective function for the structural estimation of the Life Cycle Incomplete Markets model. The red dot represents the estimated parameters.

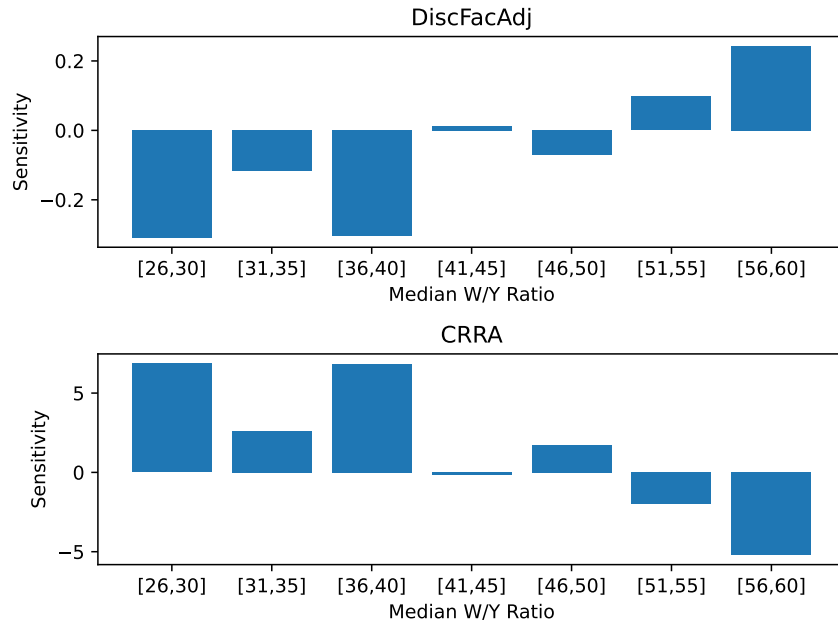


Figure 3: Sensitivity analysis of the structural estimation of the Life Cycle Incomplete Markets model. The red dot represents the estimated parameters.

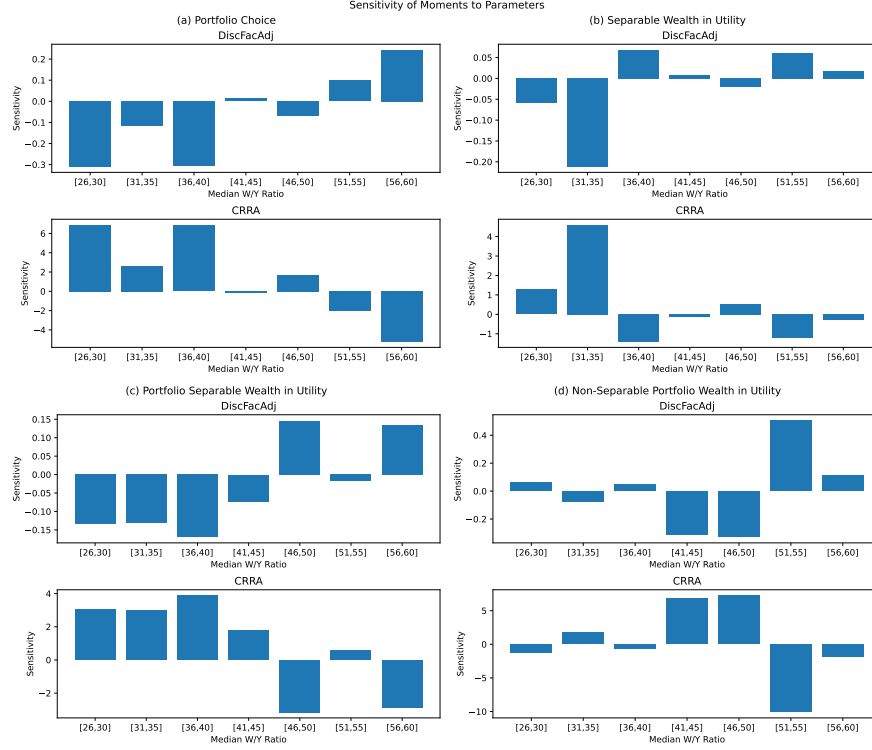


Figure 4: Sensitivity analysis of the structural estimation of the Life Cycle Incomplete Markets model. The red dot represents the estimated parameters.

5 Conclusion

6 References

Palumbo [1999] Carroll [2000] Carroll [1998] Michaillat and Saez [2021] Auclert et al. [2021] Mian et al. [2020] Kaplan et al. [2018] Auclert et al. [2020] Cagetti [2003] Andrews et al. [2017] Attanasio et al. [1999]

References

- Marco Cagetti. Wealth accumulation over the life cycle and precautionary savings. *Journal of Business & Economic Statistics*, 21(3):339–353, jul 2003. doi:[10.1198/073500103288619007](https://doi.org/10.1198/073500103288619007). URL <https://doi.org/10.1198/073500103288619007>.
- Pierre-Olivier Gourinchas and Jonathan A. Parker. Consumption over the life cycle. *Econometrica*, 70(1): 47–89, jan 2002. doi:[10.1111/1468-0262.00269](https://doi.org/10.1111/1468-0262.00269). URL <https://doi.org/10.1111/1468-0262.00269>.
- Michael G. Palumbo. Uncertain medical expenses and precautionary saving near the end of the life cycle. *Review of Economic Studies*, 66(2):395–421, apr 1999. doi:[10.1111/1467-937x.00092](https://doi.org/10.1111/1467-937x.00092). URL <https://doi.org/10.1111/1467-937x.00092>.
- Christopher Carroll. Theoretical foundations of buffer stock saving. Technical report, nov 2004. URL <https://doi.org/10.3386/2Fw10867>.
- Christopher Carroll. Why do the rich save so much? Technical report, may 1998. URL <https://doi.org/10.3386/2Fw6549>.

- Christopher D. Carroll. The method of endogenous gridpoints for solving dynamic stochastic optimization problems. *Economics Letters*, 91(3):312–320, jun 2006. doi:[10.1016/j.econlet.2005.09.013](https://doi.org/10.1016/j.econlet.2005.09.013). URL <https://doi.org/10.1016%2Fj.econlet.2005.09.013>.
- John Sabelhaus and Jae Song. The great moderation in micro labor earnings. *Journal of Monetary Economics*, 57(4):391–403, may 2010. doi:[10.1016/j.jmoneco.2010.04.003](https://doi.org/10.1016/j.jmoneco.2010.04.003). URL <https://doi.org/10.1016%2Fj.jmoneco.2010.04.003>.
- Christopher D. Carroll, Robert E. Hall, and Stephen P. Zeldes. The buffer-stock theory of saving: Some macroeconomic evidence. *Brookings Papers on Economic Activity*, 1992(2):61, 1992. doi:[10.2307/2534582](https://doi.org/10.2307/2534582). URL <https://doi.org/10.2307%2F2534582>.
- Isaiah Andrews, Matthew Gentzkow, and Jesse M. Shapiro. Measuring the sensitivity of parameter estimates to estimation moments*. *The Quarterly Journal of Economics*, 132(4) : 1553 – –1592, jun2017. doi : [10.1093/qje/qjx023](https://doi.org/10.1093/qje/qjx023). URL <https://doi.org/10.1093/qje/qjx023>.
- Christopher Carroll. Portfolios of the rich. Technical report, aug 2000. URL <https://doi.org/10.3386%2Fw7826>.
- Pascal Michaillat and Emmanuel Saez. Resolving new keynesian anomalies with wealth in the utility function. *The Review of Economics and Statistics*, 103(2):197–215, may 2021. doi:[10.1162/rest_a_00893](https://doi.org/10.1162/rest_a_00893). URL https://doi.org/10.1162%2Frest_a_00893.
- Adrien Auclert, Bence Bardóczy, Matthew Rognlie, and Ludwig Straub. Using the sequence-space jacobian to solve and estimate heterogeneous-agent models. *Econometrica*, 89(5):2375–2408, 2021. doi:[10.3982/ecta17434](https://doi.org/10.3982/ecta17434). URL <https://doi.org/10.3982%2Fecta17434>.
- Atif Mian, Ludwig Straub, and Amir Sufi. The saving glut of the rich. Technical report, apr 2020. URL <https://doi.org/10.3386%2Fw26941>.
- Greg Kaplan, Benjamin Moll, and Giovanni L. Violante. Monetary policy according to HANK. *American Economic Review*, 108(3):697–743, mar 2018. doi:[10.1257/aer.20160042](https://doi.org/10.1257/aer.20160042). URL <https://doi.org/10.1257%2Faer.20160042>.
- Adrien Auclert, Matthew Rognlie, and Ludwig Straub. Micro jumps, macro humps: Monetary policy and business cycles in an estimated HANK model. Technical report, jan 2020. URL <https://doi.org/10.3386%2Fw26647>.
- Orazio P. Attanasio, James Banks, Costas Meghir, and Guglielmo Weber. Humps and bumps in lifetime consumption. *Journal of Business & Economic Statistics*, 17(1):22–35, jan 1999. doi:[10.1080/07350015.1999.10524794](https://doi.org/10.1080/07350015.1999.10524794). URL <https://doi.org/10.1080%2F07350015.1999.10524794>.