Particle Swarm optimization

- Metaheuristic optimization technique to solve computationally hard optimization problems.
- It is motivated from foraging and social behavior of swarm.

swarm → group

social behavior of — flocking of birds. / school of fish

flock means — moves together in a crowd.

PSO is developed using two methodologies

- Artificial life / Nature life :- mimicking bird flocking, fish schooling and swarm theory.
- Evolutionary computation

Motivated from foraging and social behavior of swarm

- The swarm searches for food in a cooperative way
- Each member in the swarm learns from its experience and also from other members for changing the search pattern to locate the food.

other members — neighbours.

How PSO works ?

- PSO starts with initializing population randomly. } Swarm
- candidate solutions are assigned with randomized velocity to explore the search space.

Particle

Three distinct features of PSO

- Best fitness of each particle ($pbest_i$) : ~~the best~~ the best solution (fitness) achieved so far by particle i
- Best fitness of swarm ($gbest$) the best solution (fitness) achieved so far by any particle in the swarm
- velocity and position update of each particle for exploring and exploiting the search space to locate optimal solution

Position of particle (i) is updated as:

$$\boxed{x_i^{(t+1)} = \underbrace{x_i^{(t)}}_{\text{current position}} + \underbrace{v_i^{(t+1)}}_{\text{updated velocity}}}$$

velocity of particle (i) is updated as:

$$\boxed{v_i^{(t+1)} = \omega v_i^{(t)} + c_1 r_1 (pbest_i^{(t)} - x_i^{t}) + c_2 r_2 (gbest_i^{(t)} - x_i^{(t)})}$$

where,

i = i^{th} particle

t = generation counter

$v_i^{(0)}$ = set randomly (Initial velocity)

ω = adds to the inertia of the particle

r_1 and r_2 = Random number $\in [0, 1]$

$pbest_i^{(t)}$: local best of i^{th} particle

$gbest_i^{(t)}$: global best of swarm at t^{th} generation.

c_1 and c_2 are the acceleration coefficients of pbest and gbest

Velocity components

$$v_i^{(t+1)} = \underbrace{\omega\, v_i^{(t)}}_{①} + \underbrace{c_1 r_1 (pbest_i^{(t)} - x_i^t)}_{②} + \underbrace{c_2 r_2 (gbest^t - x_i^t)}_{③}$$

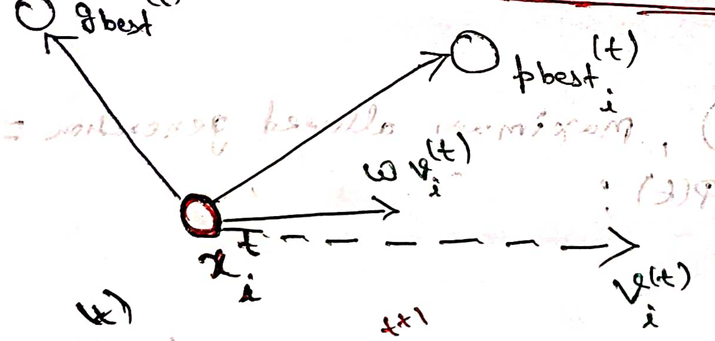
1) Momentum part : ωv_i^t
- inertia component (ω)
  - larger ω ⟹ facilitates global search
  - smaller ω ⟹ facilitates local search
- memory of previous flight direction — $\omega v_i \to v_i$ — control the extension of vector movement to particular direction
- prevent particle from drastically changing direction
- helps particle to chang direction gradually.

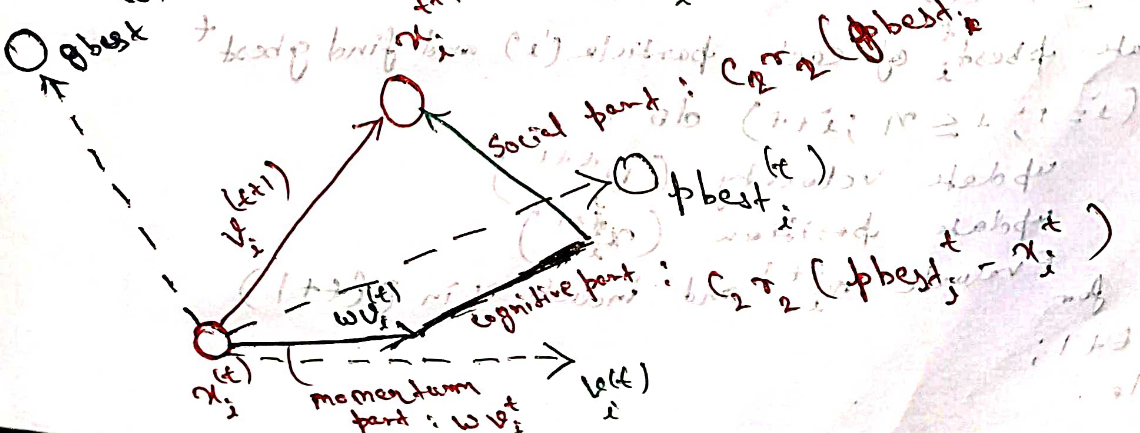
2) Cognitive part : $c_1 r_1 (pbest_i^{(t)} - x_i^t)$
- quantifies performance relative to past performance
- memory of previous best position
- nostalgia

3) Social part : $c_2 r_2 (gbest^t - x_i^t)$
- quantifies performance relative to neighbours
- envy

$c_1 > c_2$ ⟹ Facilitates Global Search
$c_2 > c_1$ ⟹ Facilitates Local Search

Geometrical Illustration of velocity Components





Local and Global Best Positions

N: $pbest_i^{(t)}$ is the personal best position of ith particle in/till t generation.

Assume minimization problem.

$$pbest_i^{(t+1)} = \begin{cases} x_i^{(t+1)} & \text{if } f(x_i^{(t+1)}) < f(pbest_i^t) \\ pbest_i^t & \text{otherwise} \end{cases}$$

fitness.

$gbest^{(t)}$ is the global best position in t generation which is calculated as

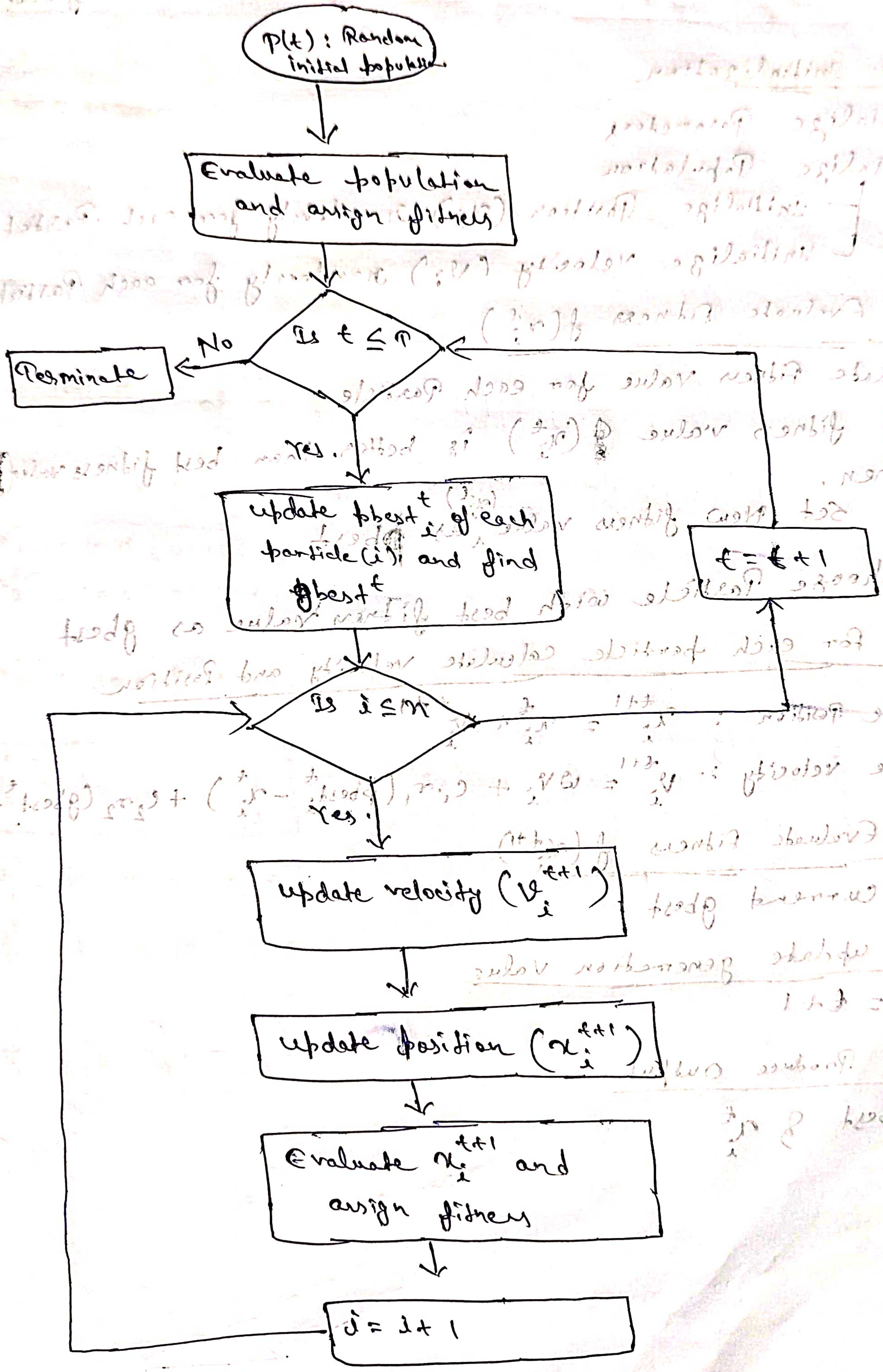
$$gbest^{(t)} \in \{pbest_1^t, pbest_2^t, \ldots\ldots pbest_n^t\} \mid f(gbest^{(t)}) = \min\{f(pbest_1^t), f(pbest_2^t) \ldots f(pbest_n^t)\}$$

where n = no. of the particles in swarm

Algorithm of PSO

1. Set $t = 1$ (Generation counter)
2. Initialize random swarm $P(t)$, maximum allowed generation $= T$
3. Evaluate fitness $P(t)$
4. while $t \leq T$ do.
5. update $pbest_i^t$ of each particle (i) and find $gbest^t$
6. for ($i = 1$; $i \leq n$; i++) do
7. update velocity (v_i^{t+1})
8. update position (x_i^{t+1})
9. Evaluate x_i^{t+1} and include it in $P(t+1)$
10. end for
13. $t = t+1$;
14. end while.

Flow chart of PSO



PSO Algorithm Steps

Aim:- follow the bird which is nearest to the food.

Step1:- Initialization

1) Initialize Parameters
2) Initialize Population
   - Initialize Position (x_i) randomly for each Particle
   - Initialize velocity (v_i) randomly for each Particle

Step2:- Evaluate fitness $f(x_i^t)$

1) Calculate fitness value for each Particle
2) If fitness value $f(x_i^t)$ is better than best fitness value (pbest) then.
   Set New fitness value (x_i^t) as pbest
3) choose Particle with best fitness value as gbest

Step3:- for each particle calculate velocity and Position

1) Particle Position : $x_i^{t+1} = x_i^t + v_i^{t+1}$
2) Particle velocity :- $v_i^{t+1} = \omega v_i + c_1 r_1 (pbest_i^t - x_i^t) + c_2 r_2 (gbest^t - x_i^t)$

Step4:- Evaluate fitness $f(x_i^{t+1})$

Find current gbest

Step5:- update generation value

$t = t+1$

Step6:- Produce output

gbest & x_i^t

Numerical Example of PSO

Objective function: Maximize $f(x) = 1 + 2x - x^2$

Let Control parameters be:

$w = 0.70$, $c_1 = 0.20$, $c_2 = 0.60$; $n = 5$ (Five Swarm Particle)

Random numbers used for updating velocity of particles be:

$r_1 = [0.4657, 0.8956, 0.3877, 0.4902, 0.5039]$

$r_2 = [0.5319, 0.8185, 0.8331, 0.7677, 0.1708]$

each one corresponding each particle

Initialization of Swarm Particles

$$\boxed{x_i^{t=0} = 10 * [r_{1,i} - 0.5]} \rightarrow \text{Initialize Position}$$

$x_1^{t=0} = 10 * [0.4656 - 0.5]$

$x_2^{t=0} = 10 \times [0.8956 - 0.5]$

$\vdots$

$x_5^{t=0} = 10 \times [0.5039 - 0.5]$.

$$\boxed{v_i^{t=0} = [r_{2,i} - 0.5]} \rightarrow \text{Initialize velocity}$$

$v_1^0 = [0.5319 - 0.5]$

$v_2^0 = [0.8185 - 0.5]$

$\vdots$

$v_5^0 = [0.1708 - 0.5]$

Numerical Example of PSO

objective function : Minimize

$$f(a_1, a_2, a_3) = 10 * (a_1 - 1)^2 + 20(a_2 - 2)^2 + 30(a_3 - 3)^2$$

Sol

Step 1 :- 1) Initialization of Parameters.

No. of Particles (Population size) : $N = 5$

Number of variables (parameters) : $m = 3$

Inertia weight : $\omega_{max} = 0.9$, $\omega_{min} = 0.4$

Acceleration factor : $C_1 = 2$, $C_2 = 2$;

Maximum Iteration size : $max_t = 50$

2) Initialize Position (x_i) randomly for each particle :

$$\text{Lower Bound} = \begin{array}{ccc} a_1 & a_2 & a_3 \\ [0, & 0, & 0] \end{array}$$

$$\text{upper Bound} = \begin{array}{ccc} a_1 & a_2 & a_3 \\ [10, & 10, & 10] \end{array}$$

Initial generation ($t = 0$)

$$i=1 \begin{cases} x^t_{1,a_1} = \text{LB of } a_1 + \text{rand}()\,[\text{UB of } a_1 - \text{LB of } a_1] \\ x^t_{i,a_2} = \text{LB of } a_2 + \text{rand}()\,[\text{UB of } a_2 - \text{LB of } a_2] \\ x_{i,a_3} = \text{LB of } a_3 + \text{rand}()\,[\text{UB of } a_3 - \text{LB of } a_3] \end{cases}$$

Initialization of first Particle Position with all their three parameter value.

Similarly Initialize second, third, fourth and fifth Particle position with all their parametric value.

	a_1	a_2	a_3
x_1^0	8	9	1
x_2^0	9	6	1
x_3^0	3	5	10
x_4^0	10	2	10
x_5^0	10	5	8

Initial Population.

3) Initialize Velocity (V_i) randomly for each particle

$$i=1 \begin{cases} V^t_{i,a_1} = 0.1 * x^0_{i,a_1} \\ V^t_{i,a_2} = 0.1 \times x^0_{i,a_2} \\ V^t_{i,a_3} = 0.1 \times x^0_{i,a_3} \end{cases}$$

:- Similarly Initialize second, third, fourth and fifth particle Velocity.

Initial velocity

	a_1	a_2	a_3
v_1^0	0.8	0.9	0.1
v_2^0	0.9	0.6	0.1
v_3^0	0.3	0.5	1
v_4^0	1	0.2	1
v_5^0	1	0.5	0.8

Initial velocity

Now In two ways we can proceed

1) Calculate the current position of each particle

$$\boxed{\text{Current position} = \text{Previous Position (Initial)} + \text{velocity}}$$

~~Updated~~ Initial Position ~~Population~~

	a_1	a_2	a_3
x_1^0	8 + 0.8 = 8.8	9 + 0.9 = 9.9	1 + 0.1 = 1.1
x_2^0	9 + 0.9 = 9.9	6 + 0.6 = 6.6	1 + 0.1 = 1.1
x_3^0	3.3	5.5	11
x_4^0	11	2.2	11
x_5^0	11	5.5	8.8

Proceed with updated Initial Population

2) Proceed with Initial population for first iteration

First Iteration $t=0$

Step 2 :- 1) Evaluate fitness $f(x_i^t)$

Calculate fitness value for each particle

$f(a_1, a_2, a_3) = 10(a_1-1)^2 + 20(a_2-2)^2 - 30(a_3-3)^2$

$f(x_1^0) = 10(8.8-1)^2 + 20(9.9-2)^2 + 30(1.1-3)^2$

$= 1.9649$

$f(x_2^0) = 1.3236$

$f(x_3^0) = 2.2179$

$f(x_4^0) = 2.9208$

$f(x_5^0) = 2.2542$

2) ~~choose~~ Set Particle with local best fitness value as pbest

$pbest_1^0 = f(x_1^0) = 1.9649 =$

$pbest_2^0 = f(x_2^0) = 1.3236 =$

$pbest_3^0 = f(x_3^0) = 2.2179 =$

$pbest_4^0 = f(x_4^0) = 2.9208 =$

$pbest_5^0 = f(x_5^0) = 2.2542 =$

a_1	a_2	a_3
8.8	9.9	1.1
9.9	6.6	1.1
3.3	5.5	11
11	2.2	11
11	5.5	8.8

3) Choose Particle with global best fitness value as gbest

$gbest^0 = f(x_2^0) = 1.3236 =$

a_1	a_2	a_3
3.3	6.6	1.1

~~gbest~~.

Step 3 :- for each particle calculate Velocity and Position

Particle Position :- $x_i^{t+1} = x_i^t + v_i^{t+1}$

Particle velocity. $v_i^{t+1} = \omega v_i^t + c_1 r_1 (pbest_i^t - x_i^t) + c_2 r_2 (gbest^t - x_i^t)$

update velocity for 1st particle

$v_{1,a_1}^{0+1} = 0.9 \times 0.8 + 2\, rand()(8.8-8.8) + 2\, rand()(9.9-8.8)$

$= 1.0667$

$v_{1,a_2}^{0+1} = 0.9 \times 0.9 + 2\, rand()(9.9-9.9) + 2\, rand()(6.6-9.9)$

$v_{1,a_3}^{0+1} = 0.9 \times 0.1 + 2\, rand()(1.1-1.1) + 2\, rand()(1.1-1.1)$

Similarly update velocity for 2nd particle, 3rd, 4th & 5th particle.

updated velocity for each particle

	a_1	a_2	a_3
v_1^1	1.0667	-4.4719	0.0900
v_2^1	0.810	0.540	0.090
v_3^1	7.488	2.572	-18.31
v_4^1	-0.1678	1.4286	1.4286
v_5^1	-1.2109	0.5286	23.66

Update 1st particle position

$$x_{1,a_1}^{0+1} = x_{1,a_1}^{0} + v_{1,a_1}^{0+1}$$
$$= 8.8 + 1.0667$$
$$x_{1,a_2}^{0+1} = x_{1,a_2}^{0} + v_{1,a_2}^{0+1}$$
$$= 9.9 - 4.4719$$
$$x_{1,a_3}^{0+1} = x_{1,a_3}^{0} + v_{1,a_3}^{0+1}$$
$$= 1.1 + 0.0900$$

⇒ Similarly update for 2nd particle, 3rd, 4th and 5th particle

updated position for each particle

	a_1	a_2	a_3
x_1^1	9.8667	5.4281	2
x_2^1	10.71	7.14	1.19
x_3^1	10.78	8.07	-7.317
x_4^1	10.83	3.628	12.42
x_5^1	9.78	6.028	-4.8635