2020/21 Final Exam

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1. Interpolation

(a) false, if you have 5 data points you can fit a unique quintic to the points. you can fit 2 3rd order (cubic) polynomials.

(b) Second order DD

$$FD_{2_i} = \frac{FD1_{i+1} - FD1_i}{x_{i+2} - x_i}$$

i	1	2	3	4	5
x	-2	-1	1	2	3
$f(x)$	-0.9	7.2	0.8	-3.1	0.1
FD1	8.1	-3.2	-3.9	3.2	
FD2	$-\frac{113}{30}$	$-\frac{7}{30}$	$\frac{71}{20}$		

$$FD1 = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$\frac{7.2-(-0.9)}{-1-(-2)} = \frac{81}{10} = 8.1$$

$$\frac{0.8-7.2}{1-(-1)} = -\frac{16}{5} = -3.2$$

$$\frac{-3.1-0.8}{2-1} = -\frac{39}{10} = -3.9$$

$$\frac{0.1-(-3.1)}{3-2} = \frac{16}{5} = 3.2$$

$$FD2 = \frac{FD1_{i+1} - FD1_i}{x_{i+2} - x_i}$$

$$\frac{-3.2-8.1}{1-(-2)} = -\frac{113}{30}$$

$$\frac{-3.9+3.2}{2-(-1)} = -\frac{7}{30}$$

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$$\frac{3.2+3.9}{3-1} = 3.55$$

$$f_2^{123} = -0.9 + (x+2)(8.1) + (x+2)(x+1)\left(\frac{-113}{30}\right)$$

$$y(0.5) = -0.9 + (0.5+2)(8.1) + (0.5+2)(0.5+1)\left(\frac{-113}{30}\right)$$

$$= -0.9 + \frac{81}{4} + \left(-\frac{113}{8}\right)$$

$$= \frac{209}{40}$$

This formula is from

$$f_n(x) = f(x_1) + (x-x_1)\,FD1 + (x-x_1)(x-x_2)\,FD2$$

(c) $$f_3 = \frac{209}{40} + (0.5+2)(0.5+1)(0.5-1)\,FD3$$

$$FD3_i = \frac{FD2_{i+1} - FD2_i}{x_{i+3} - x_i}$$

$$\frac{-\frac{70}{3} + \frac{113}{30}}{2-(-2)} = \frac{-587}{120}$$

$$f_3(0.5) = \frac{209}{40} + \left(-\frac{15}{8}\right)\left(-\frac{587}{120}\right) = \frac{4607}{320} = 14.397$$

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2. Root Finding

(a) The bracketing and bisection method will fail to converge if the root finding function has been called millions of times.

(b) The algorithm of newton Raphson method is

$$x_{n+1} = \frac{-f(x_n)}{f'(x_n)} + x_n$$

If we consider the newton Raphson one dimensional map, the fixed points will be the roots of ~~the~~ the function $f(x)$.

The algorithm will get caught in a period 2 orbit if the initial point x we give it is

$$x = F(F(x))$$

(c) $f(x) = \frac{1-2x}{x}$

$$u = 1-2x$$
$$v = x$$

$$\frac{du}{dx} = -2 \qquad \frac{dv}{dx} = 1$$

$$F(x) = x - \frac{f(x)}{f'(x)}$$

$$f'(x) = \frac{x(-2)-(1-2x)(1)}{x^2}$$

$$F(x) = x - \frac{\frac{1-2x}{x}}{\frac{-1}{x^2}}$$

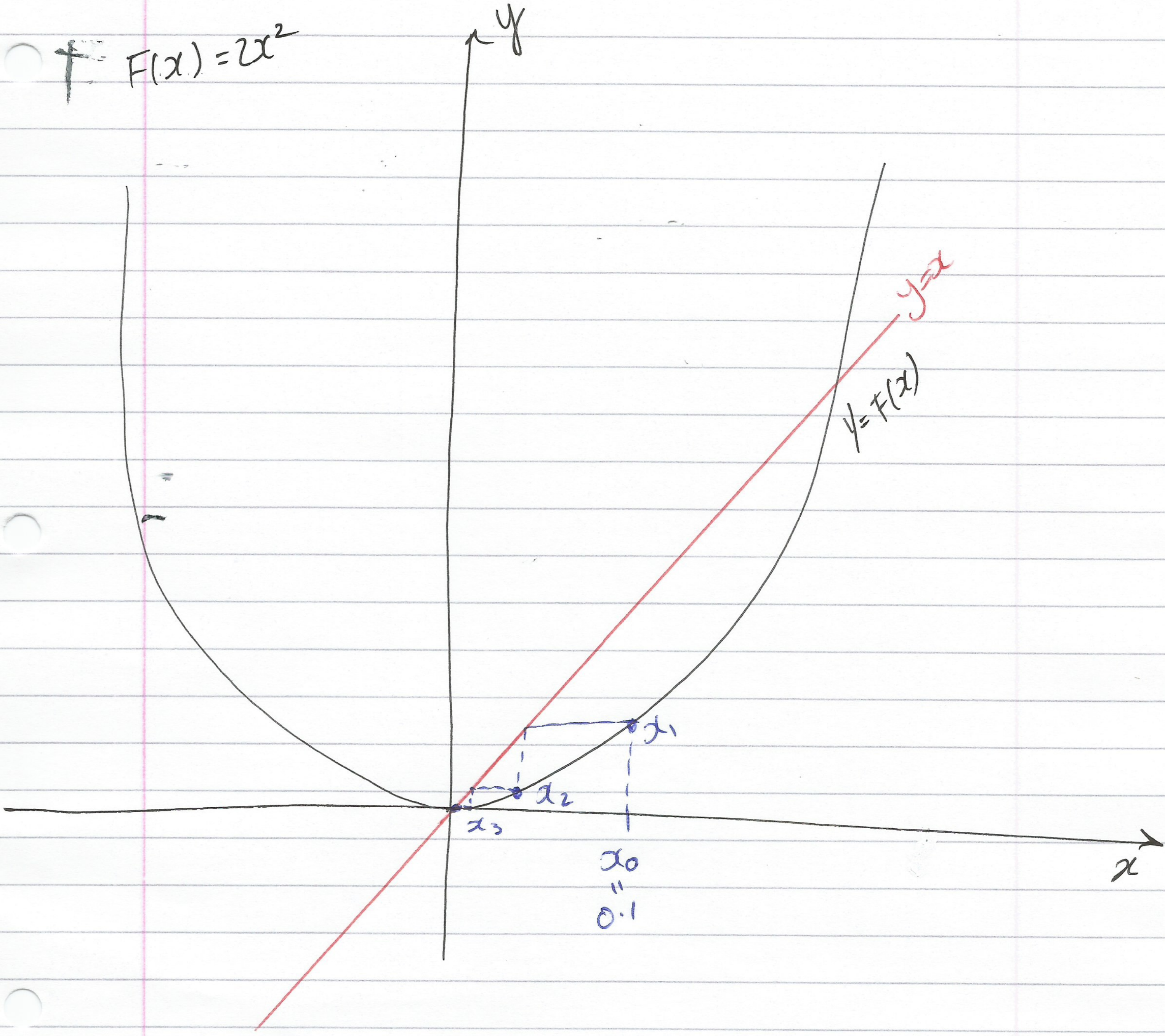
$$= \frac{-\cancel{2x} - 1 + \cancel{2x}}{x^2}$$

$$= \frac{-1}{x^2}$$

$$\frac{1-2x}{x} \div \left(\frac{-1}{x^2}\right) = \frac{1-2x}{\cancel{x}} \cdot \frac{x^2}{-1}$$

$$= \frac{1-2x(x)}{-1} = -x + 2x^2$$

$F(x) = 2x^2$



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(d) Secant method

$$f(x) = \sin(x) + \sqrt{x} - 2$$

$x_0 = 3.1$

$x_1 = 3.0$

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

Iteration 1:

$f(3.1) = \sin(3.1) + \sqrt{3.1} - 2 = -0.197737$

$f(3.0) = \sin(3.0) + \sqrt{3} - 2 = -0.126829$

$$x_{n+1} = 3.0 - \frac{(-0.126829)(-0.126829 + 0.197737)}{-0.126829 + 0.197737}$$

$x_2 = 3.127488$

Iteration 2:

$f(3.127488) = \sin(3.127488) + \sqrt{3.127488} - 2$

$= -0.2174254$

$$x_{n+1} = (3.127488) - \frac{(-0.2174254)(-0.2174254 + 0.126829)}{-0.2174254 + 0.126829}$$

$= 3.34491$

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3. Numerical Solutions to ODEs

(a) an IVP specifies all data at a single independant variables and BVP specifies data at multiple values of independant variables.

(b) $y'(x) = \sin(x)y^2 + x \qquad y(0) = -1$

$\Delta x = 0.1$

(i) Euler's method $y(0.3)$

$x_0 = 0 \qquad y_0 = -1$

$f_i = \sin(x_i)y_i^2 + x_i$

$f_0 = \sin(0)(-1)^2 + 0 = 0$

$x_1 = x_0 + \Delta x = 0 + 0.1 = 0.1$

$y_1 = y_0 + \Delta x f_0 = -1 + 0.1(0) = -1$

$f_1 = \sin(0.1)(-1)^2 + 0.1 = 0.1998334$

$x_2 = x_1 + \Delta x = 0.1 + 0.1 = 0.2$

$y_2 = -1 + 0.1(0.1998334) = -0.9800166$

$f_3 = \sin(0.2)(-0.9800166)^2 + 0.2$
$= 0.39080$

$x_3 = 0.3$

$y_3 = (-0.9800166) + 0.1(0.3908)$
$= -0.9409366$

$f_4 = \sin(0.3)(-0.9409366)^2 + 0.3$
$= 0.56164$

iii) initial values @ $x = 0.3$
$y =$

$y'(x) = \sin(x)y^2 + x$

$\frac{dy}{dx} =$

at $x = 0.3$ $\quad y(0.3) = 0.56164$

$$\int_0^{0.3} \sin(x)y^2 + x \; dx$$

Backwards Euler method

$x_{i+1} = x_i + \Delta x$

$y_{i+1} = y_i + \Delta x \, f(x_{i+1}, y_{i+1})$

use root finder ?

$$= \frac{0.1}{3}(8.72192)$$

$$= 0.291073$$

$$\int_{-0.1}^{0.5} \sin(2x) = -\frac{1}{2}\cos(2x)\Big|_{-0.1}^{0.5}$$

$$= \left(-\frac{1}{2}\cos(2(0.5))\right) - \left(-\frac{1}{2}\cos(2(-0.1))\right)$$

$$= -0.270151 + 0.490033$$
$$= 0.219882$$

$$0.219882 - 0.291073 = -0.071191$$

relative error is 0.071191

$\rightarrow$ 7.11%

(c)

(i) x_i Δx & f

$$\text{area} = \cancel{f(x_i)\Delta x + f(x_i+\Delta x)\cdot\Delta x}$$
$$= (\cancel{x_i} + \frac{\Delta x}{2} - \cancel{x_i})\cdot f(x_i) + (\cancel{x_i} + \Delta x - \cancel{x_i} + \frac{\Delta x}{2})\cdot f(x_i+\Delta x)$$

$$= f(x_i)\left(\frac{\Delta x}{2}\right) + f(x_i+\Delta x)\left(\Delta x + \frac{\Delta x}{2}\right)$$

(ii) midpoint integration

(iii) $\Delta x = 0.1$ $\int_0^{0.4} \sqrt{x}\, dx$

$$0 + \frac{0.1}{2} = \frac{0.1}{2}$$

$$I_1 = 0.1 \cdot f\left(0 + \frac{0.1}{2}\right)$$

$I_1 = 0.1\left(\sqrt{0.05}\right) = 0.02235$

$I_2 = 0.1 f\left(0.05 + \frac{0.1}{2}\right)$

$= 0.1 f(0.1) = 0.1\left(\sqrt{0.1}\right) = 0.03162$

$I_3 = 0.1 f\left(0.1 + \frac{0.1}{2}\right)$

$= 0.1 f(0.15) = 0.1\left(\sqrt{0.15}\right) = 0.03873$

$I_4 = 0.1 \; f(0.15 + 0.05) = 0.1 \; f(0.2)$

$= 0.1\left(\sqrt{0.2}\right) = \frac{\sqrt{5}}{50}$

integral $= 0.02235 + 0.03162 + 0.03873 + \frac{\sqrt{5}}{50}$

$= 0.13742$

4. Numerical Integration

(a)

(b) $\int_{-0.1}^{0.5} \sin(2x)\, dx$

~~$I_i = \frac{\Delta x}{3}[f(x_i$~~

$$\Delta x = \frac{0.5-(-0.1)}{6} = \frac{1}{10} = 0.1$$

x	y
-0.1	-0.1986
0	0
0.1	0.09983
0.2	0.19866
0.3	0.29552
0.4	0.38941
0.5	0.47942

$$I_{simps} = \frac{\Delta x}{3}\left(y_0 + 4\sum_{j:odd} y_j + 2\sum_{j:even} y_j + y_N\right)$$

$$I_{simps} = \frac{0.1}{3}\Big(-0.1986 + 4(0+0.19866+0.3841) + 2(0.09983 + 2.9552) + 0.47942\Big)$$

$$= \frac{0.1}{3}\Big(-0.1986 + 2.33104 + 6.11006 + 0.47942$$

f(i) $y(0.4)$ $\Delta x = 0.1$

$$\frac{dy}{dx} = x - y^2 \qquad y(0) = 1$$

$$k_1 = f(0,1) = 0 - 1^2 = -1$$

$$k_2 = f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{\Delta x}{2} k_1\right)$$

$$= f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}(-1)\right)$$

$$= f(0.05, 0.95)$$

$$= 0.05 - (0.95)^2 = -0.8525$$

$$k_3 = f(x_i + \Delta x, y_i + \Delta x(2k_2 - k_1))$$

$$= f(0.1, 1 + 0.1(2(-0.8525) + 1))$$

$$= f(0.1, 0.9295)$$

$$= 0.1 - (0.9295)^2 = -0.76397$$

$$y_{i+1} = 1 + \frac{0.1}{6}\left(-1 + 4(-0.8525) + (-0.76397)\right)$$

$$= 1 + \frac{0.1}{6}(-5.17517)$$

$$= 0.913747$$

† 5. Linear Algebra

(d) if a matrix system $Ax = b$ is said to be ill conditioned, it means a small change in b will cause a large change in x. This means matrix A is close to singular and has a large condition number.

(b)

$$\begin{bmatrix} 1.24 & -3.29 \\ -1.86 & 4.936 \end{bmatrix}$$

infinity norm: sum of absolute values of each row

$$\|A\|_\infty := \max_i \sum_{j=1} |a_{ij}|$$

$$|1.24| + |-3.29| = 4.53$$
$$|-1.86| + |4.936| = 6.796$$

$$\|A\|_\infty = 6.796$$

The condition number tells us if the matrix is close to singular or not. This will in turn let us know if we should expect a large error if we try to compute the inverse of A.

The condition number will also tell us if λ_{min} is small.

$$\det(A) = 1.24(4.936) - (-3.29)(-1.86)$$
$$= \frac{31}{250000}$$

$$\frac{1}{1.24\times10^{-3}}\begin{bmatrix} 4.936 & 3.29 \\ 1.86 & 1.24 \end{bmatrix} = A^{-1}$$

$$A^{-1} = \begin{bmatrix} 3980.65 & 2653.23 \\ 1500 & 1000 \end{bmatrix}$$

$$\|A^{-1}\|_\infty = 5633.88$$

$$K(A) = 5633.88 \times 6.796 = 38287.84848$$

The condition number is large thus the matrix is close to singular.

(c) $$\begin{bmatrix} 3 & 12 \\ 6 & 25 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\left.\begin{array}{cc} 3 & 12 \\ 6 & 25 \end{array}\right| \left.\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right| \begin{array}{c} 2 \\ 3 \end{array}$$

$R_1 \to \frac{1}{3}R_1$: $$\left.\begin{array}{cc} 1 & 4 \\ 6 & 25 \end{array}\right| \left.\begin{array}{cc} \frac{1}{3} & 0 \\ 0 & 1 \end{array}\right| \begin{array}{c} \frac{2}{3} \\ 3 \end{array}$$

$R_2 \to R_2 - 6R_1$: $$\left.\begin{array}{cc} 1 & 4 \\ 0 & 1 \end{array}\right| \left.\begin{array}{cc} \frac{1}{3} & 0 \\ -2 & 1 \end{array}\right| \begin{array}{c} \frac{2}{3} \\ -1 \end{array}$$

$R_1 \to R_1 - 4R_2$: $$\left.\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right| \left.\begin{array}{cc} \frac{25}{3} & -4 \\ -2 & 1 \end{array}\right| \begin{array}{c} \frac{14}{3} \\ -1 \end{array}$$

inverse matrix = $\begin{bmatrix} \frac{25}{3} & -4 \\ -2 & 1 \end{bmatrix}$ solution = $\begin{bmatrix} \frac{14}{3} \\ -1 \end{bmatrix}$