MAE 431

Heat Transfer Systems

DESIGN PROJECT

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California State University, Long Beach

Banh, Deedee

SID 9299

TABLE OF CONTENT

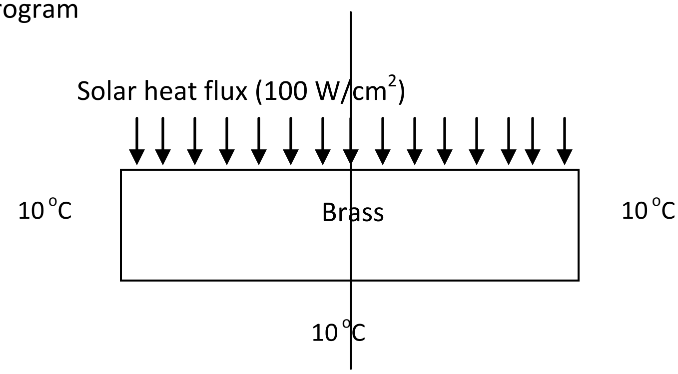
I. PROJECT STATEMENT:

Consider a long slab with a cross section of (4 x1 cm^2). The block is made of brass (k = 120 W/m.°C, ρ = 8,500 kg/m^3, C_p = 400 J/kg.K), and is initially at temperature of 10°C.
The top surface is suddenly exposed to a laser beam of 100 W/cm^2, all other surfaces are maintained at T = 10°C.

1. Find numerical solution of temperature distribution within the slab at time progresses. Determine how long it takes before steady-state solution is obtained.
2. Determine the maximum temperature in the slab, and where it is located. Plot isotherms at three different times between initial time and time it takes to reach steady-state solution. Please label or graphs, as without it, the graphs are purely worthless.
3. Find exact solution of steady-state temperature distribution within the slab.
4. Plot the temperature distribution along a line running through the center. Compare numerical with exact solution. What is the effect of mesh size, time step, on stability, and sensitivity of the solution?

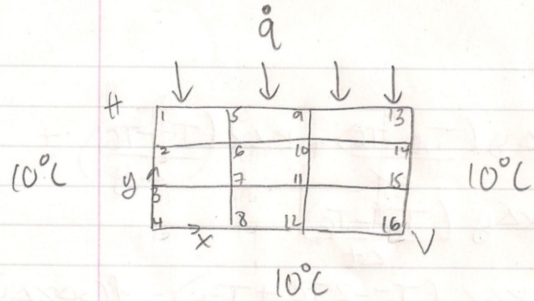
Note: You can use any programming medium (C, Matlab, Excel, etc.) that you like. You CANNOT use, however, any commercial software designed to do heat transfer analysis (such as NASTRAN). [You can use these software to validate your results, but not to substitute programming].

Attach the program



II. FORMULATION:

Formulation



$T(x,0) = T_\infty = 10°C$
$T(0,y) = T_\infty = 10°C$
$T(V,y) = T_\infty = 10°C$

$T(x,H)$ Top boundary

Nodes: $\dot{q}\Delta x + k\frac{\Delta y}{2}\frac{T_1-T_5}{\Delta x} + k\frac{\Delta y}{2}\frac{T_9-T_5}{\Delta x} + k\Delta x\frac{T_6-T_5}{\Delta y} =$

$\rho\Delta x\frac{\Delta y}{2}c_p\frac{(T_5^*-T_5)}{\Delta t}$

Mulitily by $\frac{2\Delta t}{\rho c_p\Delta x\Delta y}$

$$\frac{2\dot{q}\Delta t}{\rho c_p\Delta y} + \frac{k\Delta t}{\rho c_p(\Delta x)^2}(T_1-2T_5+T_9) + \frac{2k\Delta t}{\rho c_p(\Delta y)^2}(T_6-T_5) = T_5^*-T_5$$

$\alpha = k/\rho c_p$

$$T_5^* = \frac{2\dot{q}\Delta t}{\rho c_p\Delta y} + \frac{\alpha\Delta t}{(\Delta x)^2}(T_1-2T_5+T_9) + \frac{2\alpha\Delta t}{(\Delta y)^2}(T_6-T_5) = T_5^*-T_5$$

General equation for nodes exposed to flux →

$$T_{node}^{i+1} = \left(1-\frac{2\alpha\Delta t}{(\Delta x)^2}-\frac{2\alpha\Delta t}{(\Delta y)^2}\right)T_{node}^i + \frac{\alpha\Delta t}{(\Delta x)^2}\left(T_{left}^i + T_{right}^i\right) + \frac{2\alpha\Delta t}{(\Delta y)^2}T_{bottom}^i + \frac{2\dot{q}\Delta t}{\rho c_p\Delta y}$$

Internal node

Node 6 $K\Delta y\left(\frac{T_2-T_6}{\Delta x}\right)+K\Delta y\left(\frac{T_{10}-T_6}{\Delta x}\right)+K\Delta x\left(\frac{T_5-T_6}{\Delta y}\right)+$

$$K\Delta x\left(\frac{T_7-T_6}{\Delta y}\right)=\rho C_p\Delta x\Delta y\left(\frac{T_6^*-T_6}{\Delta t}\right)$$

$$\frac{K\Delta y}{\Delta x}\left(T_2-2T_6+T_{10}\right)+\frac{K\Delta x}{\Delta y}\left(T_5-2T_6+T_7\right)=\rho C_p\Delta x\Delta y\frac{(T_6^*-T_6)}{\Delta t}$$

multiply by $\frac{\Delta t}{\rho C_p\Delta x\Delta y}$

$$\frac{\alpha\Delta t}{(\Delta x)^2}\left(T_2-2T_6+T_{10}\right)+\frac{\alpha\Delta t}{(\Delta y)^2}\left(T_5-2T_6+T_7\right)=T_6^*-T_6$$

General equation for internal nodes →

$$T_{node}^{i+1}=\left(1-2\frac{\alpha\Delta t}{(\Delta x)^2}-2\frac{\alpha\Delta t}{(\Delta y)^2}\right)T_{node}^{i}+\frac{\alpha\Delta t}{(\Delta x)^2}\left(T_{left}^{i}+T_{right}^{i}\right)+\frac{\alpha\Delta t}{(\Delta y)^2}\left(T_{top}^{i}+T_{bottom}^{i}\right)$$

Stability Criterion:

$$1-\frac{2\alpha\Delta t}{(\Delta x)^2}-\frac{2\alpha\Delta t}{(\Delta y)^2}\geq 0$$

$$1\geq 2\alpha\left(\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}\right)$$

$$\Delta t\leq\frac{1}{2\alpha\left(\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}\right)}$$

III. VALIDATION

MATLAB Codes:

```
r=51; % Number of rows
c=4*r; % Number of columns
t_final=100;

%Given
q_dot=100*100^2; %[W/m^2]
k=120; % [W/m.C]
rho=8500; % [kg/m^3]
Cp=400; % [J/kg.K]
T_inf=10; % [C]

dx=(4/100)/(c-1); %[m]
dy=(1/100)/(r-1); %[m]
alpha=k/rho/Cp; %[m^2/K]
dt=1/(2*alpha*(1/dx^2+1/dy^2)); %[sec] Stability Criterion
beta=alpha*dt/dx^2;
gamma=alpha*dt/dy^2;

%Exact Solution
Te=zeros(r,1);
Te(:,1)=T_inf;
Tge=zeros(r,1);
Tge(:,1)=Te(:,1);
for y=0:dy:0.01
    ii=2:1:r-1;
    Tge(1,1)=q_dot*y/k+Te(1,1);
    Tge(ii,1)=Tge(1,1)-q_dot*(dy*(ii-1))/k;
end
ExactSolution_MaxTemp=Tge(1)

%Numerical Solution
for i=1:1:r
    j=1:1:c;
    T=zeros(r,c);
    T(1,:)=T_inf;
    T(i,:)=T_inf;
    T(:,1)=T_inf;
    T(:,j)=T_inf;
end
Tg=zeros(r,c);
Tg(1,:)=T(1,:);
Tg(i,:)=T(i,:);
Tg(:,1)=T(:,1);
Tg(:,j)=T(:,j);

for t=0:dt:t_final
    i=2:1:r-1;
    j=2:1:c-1;
    Tg(1,j)=(1-2*beta-2*gamma)*T(1,j)+beta*(T(1,j-
1)+T(1,j+1))+2*gamma*T(2,j)+2*q_dot*dt/(rho*Cp*dy); %Top boundary equation
    Tg(i,j)=(1-2*beta-2*gamma)*T(i,j)+beta*(T(i,j-
1)+T(i,j+1))+gamma*(T(i+1,j)+T(i-1,j)); %Internal nodes
    if (Tg-T)<1e-5
        break
```

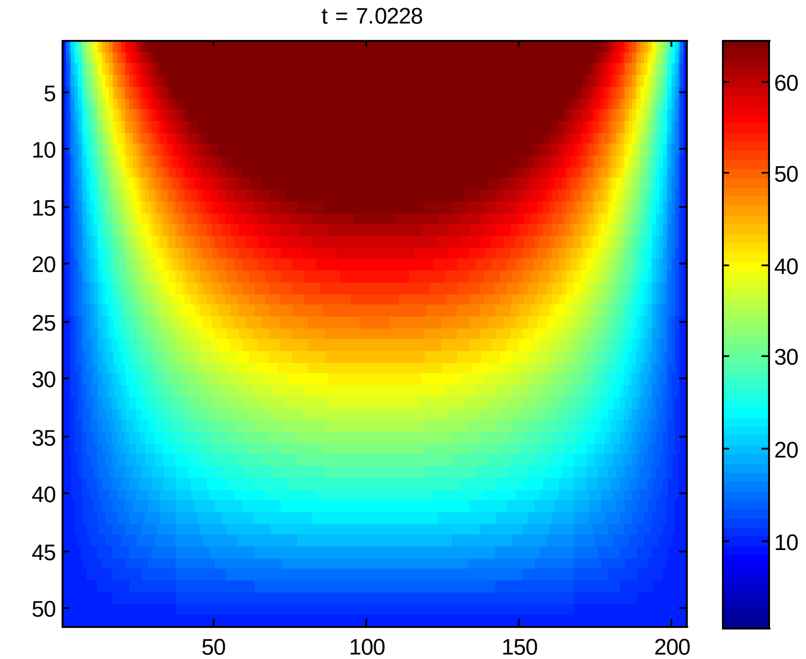
```
    end
    T=Tg;
    tf=0:dt:t;
    iter=length(tf);
    Tgmax(iter)=max(Tg(1,:));
end

MaxTemp=Tgmax(end)
FinalTime=t
percentoferror=(Tge(1)-Tgmax(end))/Tge(1)*100

figure(1)
x=0:dx:0.01;
plot(x,T(:,c/2),x,Tge)
legend('Numerical Method','Exact Solution')
xlabel('Distance from top wall to bottom wall')
ylabel('Temperature Distribution')
figure(2)
plot(tf,Tgmax)
xlabel('Time(sec)')
title('Temperature distribution along a line running through the center')
ylabel('Maximum Temperture')
title(['t = ' num2str(t)])
figure(3)
image(T)
title(['t = ' num2str(t)])
colorbar
```

Part (1)

<u>Numerical Method Results:</u>

For a distribution of 51 rows and 204 columns, the maximum temperature [°C] and the time [sec] to reach steady-state are:

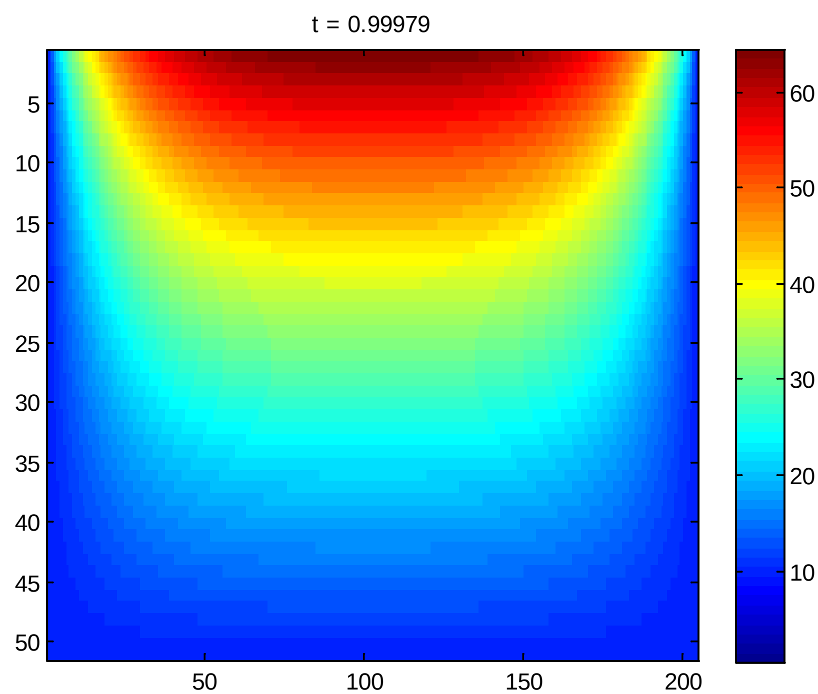
```
MaxTemp =

   87.4695


FinalTime =

    7.0228
```

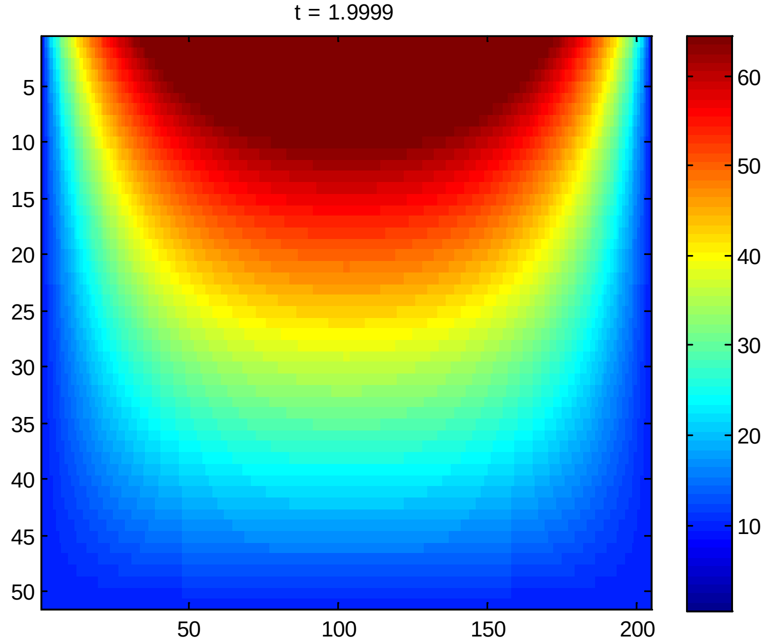


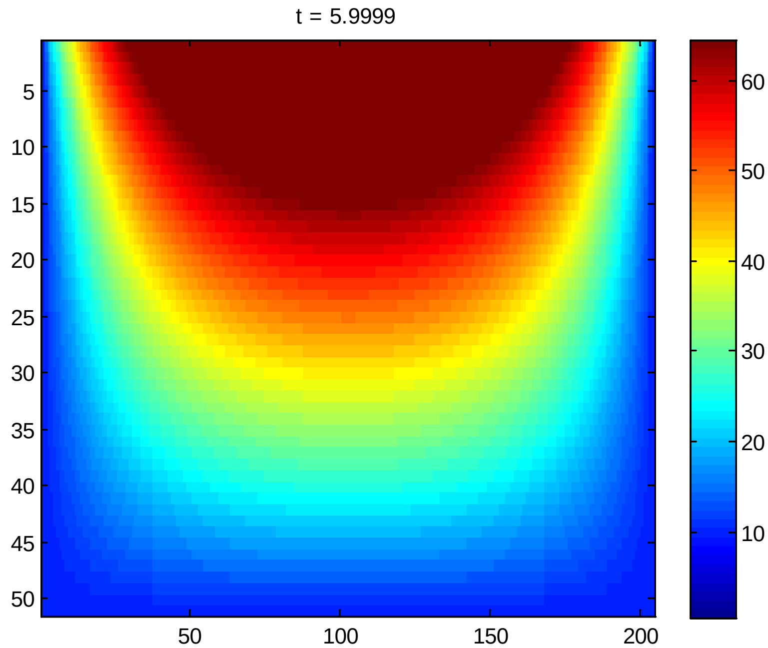
Part (2)

The maximum temperature occurs right at the centerline of the slab and at the top layer where the slab is exposed to the heat flux and. The maximum temperature is 87.4695°C.

Isotherms at three different times [sec] between the initial time and time it takes to reach steady-state solution:



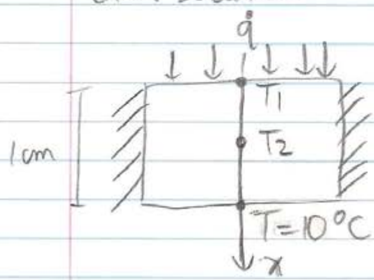

t = 1.9999
5
10
15
20
25
30
35
40
45
50
50
100
150
200
60
50
40
30
20
10


t = 5.9999
5
10
15
20
25
30
35
40
45
50
50
100
150
200
60
50
40
30
20
10

Part (3)

Exact Solution of steady state temperature distribution within the slab:

Exact Solution



$$\dot q = \frac{k\,dT}{dx} = \frac{k(T_1 - T)}{dx}$$

$$T_1 = \frac{\dot q\, dx}{k} + T$$

$$= \frac{100 \frac{W}{cm^2}(1\,cm)}{1.20\ W/cm°C} + 10°C$$

$$= 93.33°C$$

$$\dot q = \frac{k(T_1 - T_2)}{dx}$$

$$\dot q = \frac{k}{1/2\,cm}(93.33 - T_2) = 100\ W/cm^2$$

$$T_2 = -\frac{100(0.5)}{1.20} + 93.33 = 51.67°C$$

$$T(1,1) = \frac{\dot q(0.01)}{k} + T\_inf$$

$$T(2,1) = T(1,1) - \frac{\dot q(dx - x/2)}{k}$$

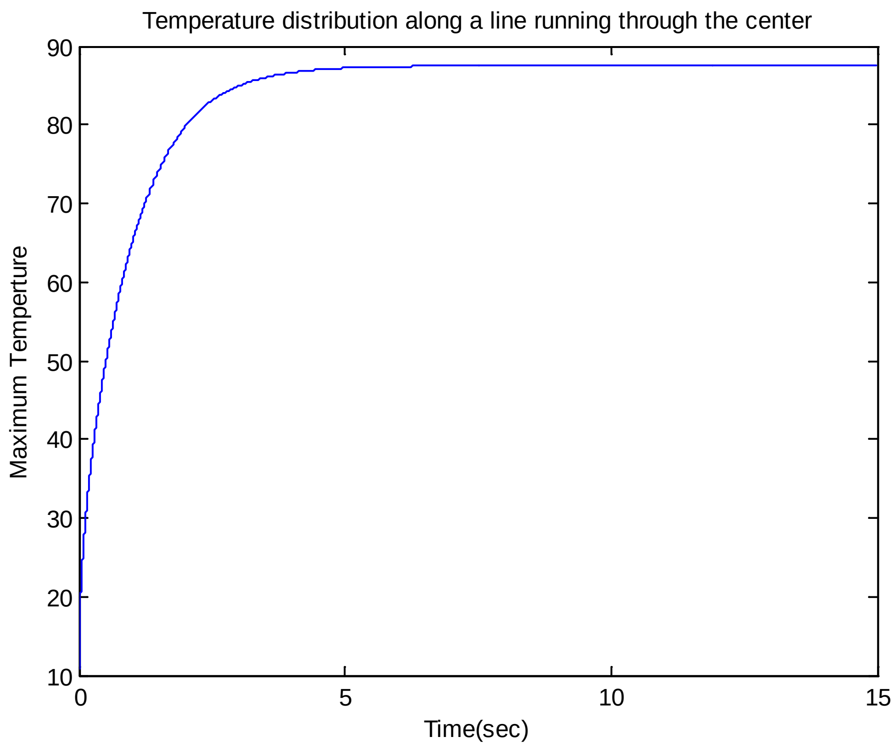
Exact Solution Results:

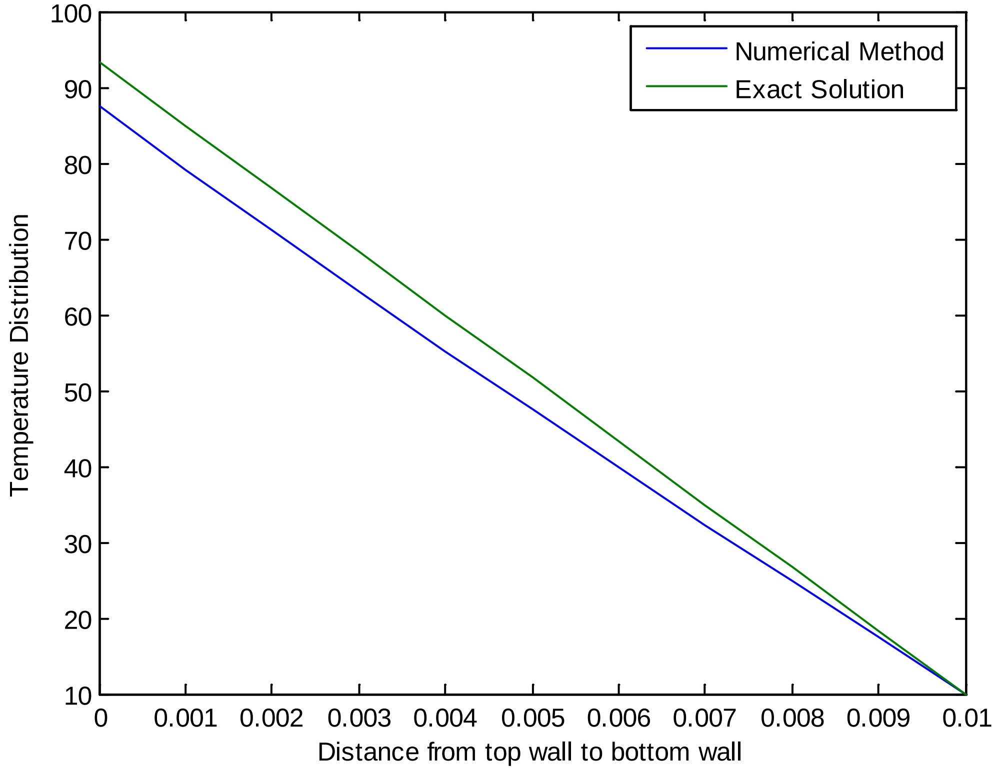
For a distribution of 51 rows, the maximum temperature [°C] and the time [sec] to reach steady-state are:

```
ExactSolution_MaxTemp =

   93.3333
```

Part (4)

The temperature distribution along a line running through the center:





The percent error between the steady-state result and the transient result is:

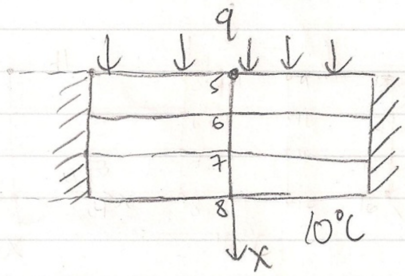
$$\%\,Error = \frac{93.3333 - 87.4695}{87.4695} * 100 = 10.49\%$$

The MATLAB code for the long slab is verified by insulating the left and right walls of the slab, making the problem one dimensional. For a distribution of 51 rows and 204 columns, the maximum temperature obtained using numerical method is 87.4695°C and the exact maximum temperature using one dimensional assumption is 92.1018°C. By insulating both right and left walls, all of the heat has nowhere to go but downward. But in reality the heat does escape to the sides of the walls and it is reasonable that the temperature at the center line is close but smaller than that from exact solutions. Small grid size will show better visual view of the heat conduction; however, since the slab is symmetrical at the centerline with no hole or stress concentration in the inner nodes, the grid size used in this project is sufficient enough to see the distribution of heat conduction.

To further verify that the code is indeed justifying the real result, a one dimensional numerical coding can be defined to compare to the exact solutions.

Validation

make the problem 1-D by insulating left and right walls.



$T(L) = T_\infty = 10°C$

Node 5:

$$\dot{q}\cancel{A} + k\cancel{A}\,\frac{T_6 - T_5}{\Delta x} = \rho \cancel{A} \Delta x\, C_p \frac{(T_5^* - T_5)}{\Delta t}$$

$$T_5^* = T_5 + \frac{\dot{q}\Delta t}{\rho C_p \Delta x} + \frac{k\Delta t}{\rho C_p (\Delta x)^2}(T_6 - T_5)$$

Top Node → $$T_{node}^{i+1} = \left(1 - \frac{\alpha\Delta t}{(\Delta x)^2}\right) T_{node}^{i} + \frac{\alpha \Delta t}{(\Delta x)^2} T_{bottom}^{i} + \frac{\dot{q}\Delta t}{\rho C_p \Delta x}$$

Node 6:

$$k\cancel{A}\frac{(T_5 - T_6)}{\Delta x} + k\cancel{A}\frac{(T_7 - T_6)}{\Delta x} = \rho\cancel{A}\Delta x\, C_p \frac{(T_6^* - T_6)}{\Delta t}$$

$$T_6^* = \frac{k\Delta t}{\rho C_p (\Delta x)^2}(T_5 - 2T_6 + T_7) + T_6$$

internal nodes: → $$T_{node}^{i+1} = \left(1 - \frac{2\alpha\Delta t}{(\Delta x)^2}\right) T_{node}^{i} + \frac{\alpha\Delta t}{(\Delta x)^2}(T_{top} + T_{bottom})$$

Stability Criterion

$$1 - \frac{2\alpha\Delta t}{(\Delta x)^2} \geq 0 \qquad \Delta t \leq \frac{1}{2\alpha/(\Delta x)^2} = \frac{(\Delta x)^2}{2\alpha}$$

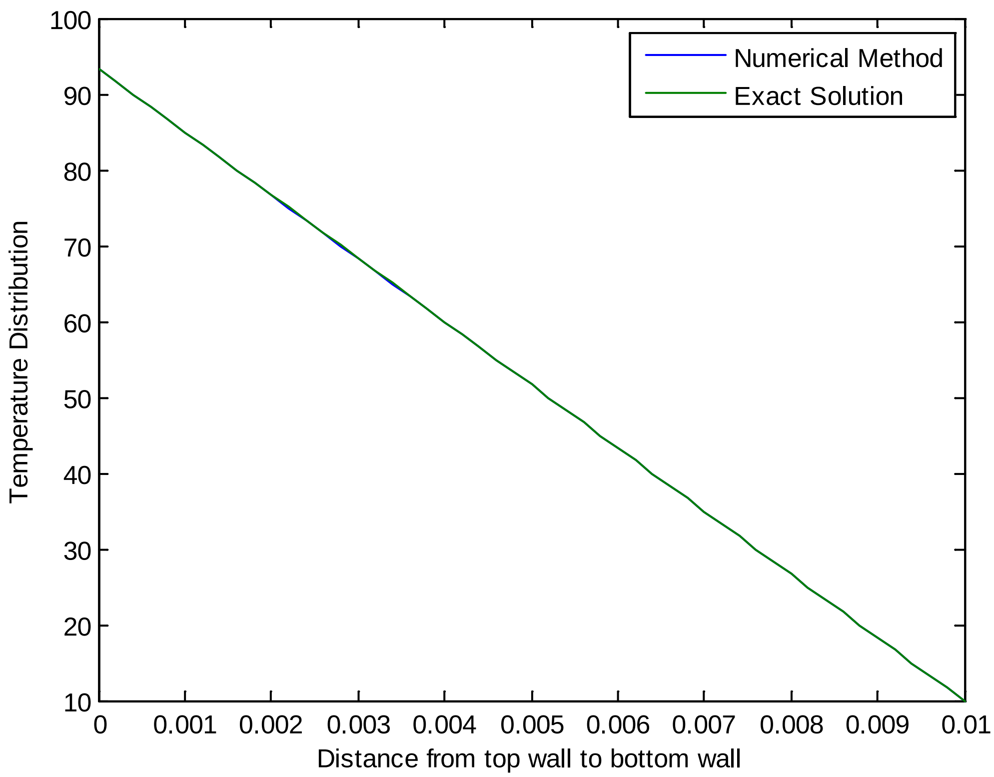
let $beta = \frac{\alpha\Delta t}{(\Delta x)^2}$

```
%1-D
%Given
q_dot=100*100^2; %[W/m^2]Heat Flux on top surface
k=120; % [W/m.C]
rho=8500; % [kg/m^3]
Cp=400; % [J/kg.K] Heat Capacity
T_inf=10; % [C]Temperature maintained at 10 degrees Celsius for left, right
and bottom surfaces.

r=51; % Number of rows

dy=(1/100)/(r-1); %[m] Mesh size x-direction
alpha=k/rho/Cp; %[m^2/K]
dt=dy^2/(2*alpha); %[sec] Time step is estimated using Stability Criterion
beta=alpha*dt/dy^2;

%Exact Solution
Te=zeros(r,1);
Te(:,1)=T_inf;
Tge=zeros(r,1);
Tge(:,1)=Te(:,1);
for y=0:dy:0.01
    ii=2:1:r-1;
    Tge(1,1)=q_dot*y/k+Te(1,1);
    Tge(ii,1)=Tge(1,1)-q_dot*(dy*(ii-1))/k;
end
ExactSolution_MaxTemp=Tge(1)

%Numerical Method
T=zeros(r,1);
T(:,1)=T_inf;
Tg=zeros(r,1);
Tg(:,1)=T(:,1);
t_final=100;
for t=0:dt:t_final
    i=2:1:r-1;
    Tg(1,1)=(1-beta)*T(1,1)+beta*T(2,1)+q_dot*dt/(rho*Cp*dy);
    Tg(i,1)=(1-2*beta)*T(i,1)+beta*(T(i-1,1)+T(i+1,1));
    if (Tg-T)<1e-5
        break
    end
    T=Tg;
    tf=0:dt:t;
    iter=length(tf);
    Tgmax(iter)=max(Tg(1,1));
end

NumericalMaxTemp=Tgmax(end)
FinalTime=t
percentoferror=(Tge(1)-Tgmax(end))/Tge(1)*100

%Comparing Numerical solutions to Exact Solutions
y=0:dy:0.01;
plot(y,T(:,1),y,Tge(:,1))
legend('Numerical Method','Exact Solution')
xlabel('Distance from top wall to bottom wall')
```

```
ylabel('Temperature Distribution')
```

Results:

ExactSolution_MaxTemp =
  93.3333
NumericalMaxTemp =
  93.3127
FinalTime =
  9.4888
percentoferror =
  0.0221



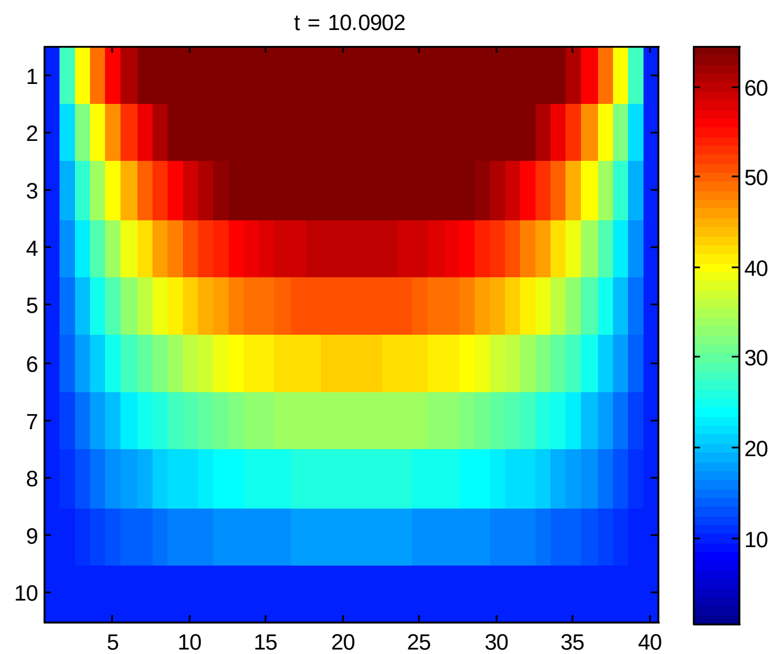
IV. SENSITIVITY ANALYSIS:

<u>Mesh size:</u>

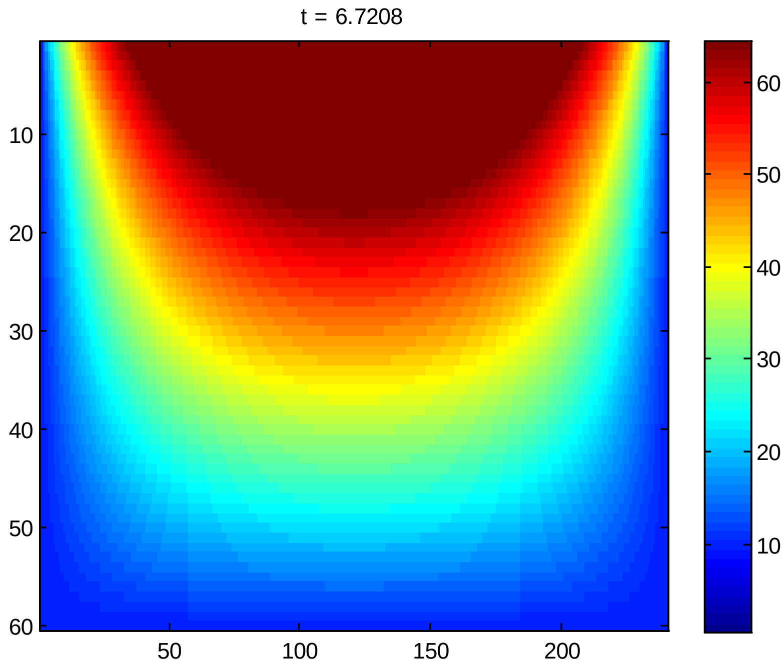
Delta x and delta y are to be determined depending on the shape of the slab. Since the slab is longer on the horizontal direction and the heat flux applies to the slab is from the top wall, it is easy to see that the temperatures on the vertical direction are changing more rapidly than the horizontal direction (heat is moving from top wall to bottom wall).

The mesh size is determined by how much the user wanted to see the distribution of heat conduction inside the slab. For smaller mesh size, the grids are much larger and the 'colors' are not blending well.

`For instance, r=10; c=4*r;`



`Versus when r=60; c=4*r;`



Looking at the isotherms, it is easy to see that the smaller the grids, the finer the picture and the faster the slab reaches steady-state solution since the temperature are closer to each other and at some time, when the slab reaches steady state, the iteration stops at difference of 1e-5, producing an isotherm of very close nodes with little difference between the maximum temperature of the node from one time to the next.

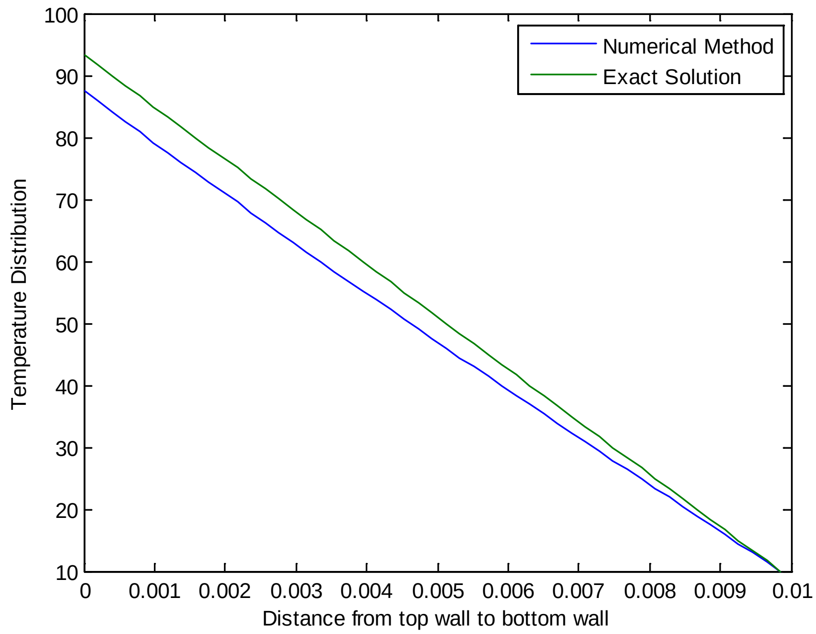
Time Step:

Using the Stability Criterion, the time step is calculated based on the mesh size of the subject.

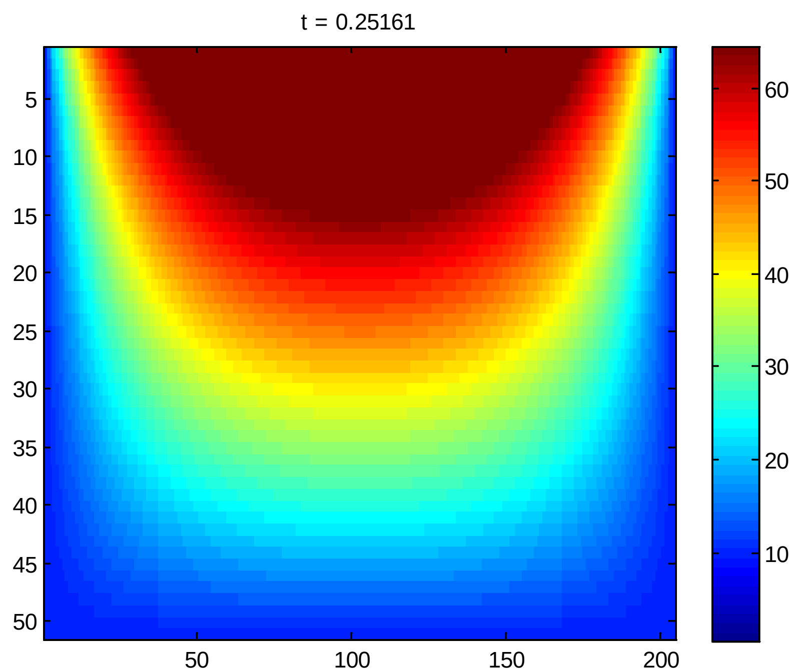
$$\Delta t \leq \frac{1}{2\alpha\left(\frac{1}{dx^2}+\frac{1}{dy^2}\right)}$$

For the slab with distribution of 51 rows and c=204 columns, dt= 2.7912e-004 sec.

When dt=1e-5, the temperature distribution is:

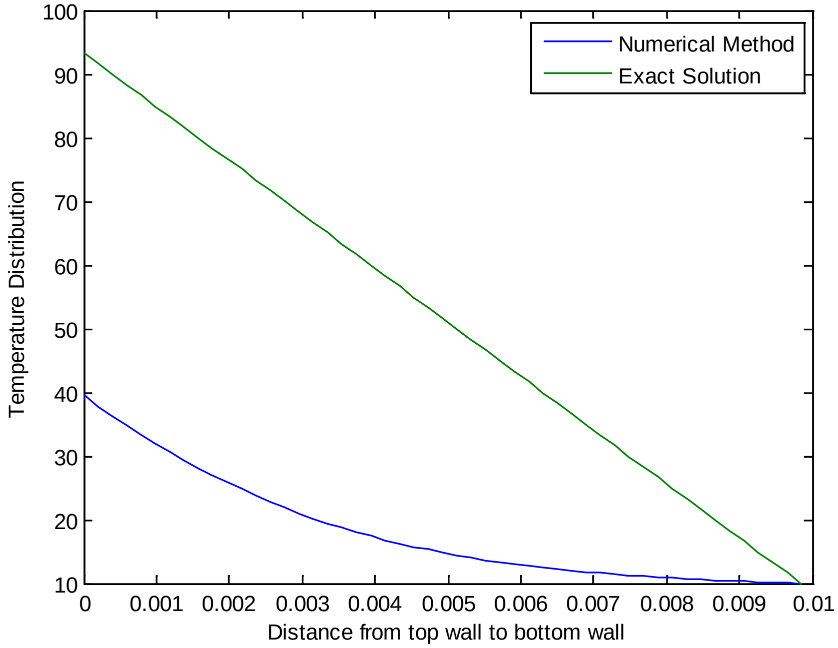


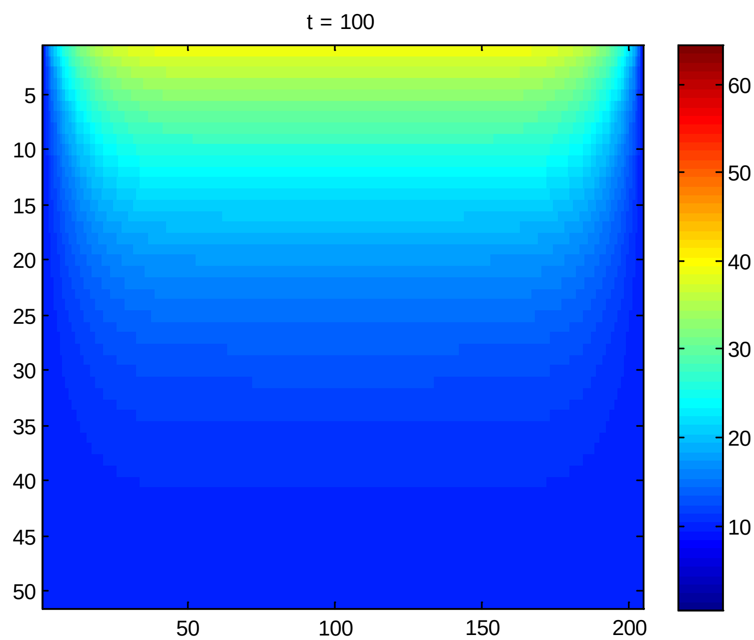
Numerical Method
Exact Solution
Temperature Distribution
Distance from top wall to bottom wall
100
90
80
70
60
50
40
30
20
10
0
0.001
0.002
0.003
0.004
0.005
0.006
0.007
0.008
0.009
0.01



t = 0.25161
5
10
15
20
25
30
35
40
45
50
50
100
150
200
60
50
40
30
20
10

For dt=0.1sec,





For the time step using the Stability Criterion requirement, the smaller dt the faster the slab reaches steady-state solution. For dt is greater than the stability requirement, the time step is not sufficiently small enough for the function to reach a steady-state; in fact, the function keeps going and reaches the maximum time by default (t_final=100 sec) with the percent of error of 57.6528% and yet did not reach a steady-state.

V. RESULTS:

The steady-state solution varies depending on the mesh size and essentially the time step defined for the problem. For small grid size, the steady-state is reached faster and has better visual results (finer resolution). The time step is defined by the stability criterion and any time step lower or equal to the smallest primary coefficient of all T_m^i in the T_m^{i-1} expressions are sufficient enough to avoid divergent oscillation in nodal temperatures. For a slab of wood, it is reasonable enough to choose dt to be exactly equal to the coefficient and a distribution of 51 rows and 204 columns since there are no stress concentration or asymmetry in the slab.