Problem 1. We have that $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|\cos(\theta)$ and $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos(\theta)$ where $\cos(\theta) = \dfrac{\langle x, y\rangle}{\|x\|\|y\|}$ which is proven in the textbook.

(i) $\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \frac{1}{4}(\|x\|^2+\|y\|^2+2\|x\|\|y\|\cos(\theta) - \|x\|^2 - \|y\|^2 + 2\|x\|\|y\|\cos(\theta)) = \frac{1}{4}(4\|x\|\|y\|\cos(\theta)) = \|x\|\|y\|\dfrac{\langle x, y\rangle}{\|x\|\|y\|} = \langle x, y\rangle$

(ii) $\frac{1}{2}(\|x+y\|^2 + \|x-y\|^2) = \frac{1}{2}(\|x\|^2+\|y\|^2+2\|x\|\|y\|\cos(\theta) + \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos(\theta)) = \frac{1}{2}(2(\|x\|^2 + \|y\|^2)) = \|x\|^2 + \|y\|^2$

Problem 2.

$$\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2) = \langle x, y\rangle + \frac{1}{4}(i\|x-iy\|^2 - i\|x+iy\|^2) =$$

$$= \langle x, y\rangle + \frac{1}{4}(-i(||x||^2 + \langle x, iy\rangle + \langle iy, x\rangle + ||y||^2 - ||x||^2 + \langle x, iy\rangle + \langle iy, x\rangle - ||y||^2)) =$$

$$= \langle x, y\rangle + \frac{i}{4}(2i\langle x, y\rangle - 2i\langle y, x\rangle) = \langle x, y\rangle$$

Problem 3.

(i) Let $f(x) = x$ and $g(x) = x^5$. Then

$$\theta = \cos^{-1}\Big(\frac{\langle f, g\rangle}{\|f\|\|g\|}\Big) = \cos^{-1}\Big(\frac{\int_0^1 x^6 dx}{\sqrt{\int_0^1 x^2 dx}\sqrt{\int_0^1 x^{10} dx}}\Big) = \cos^{-1}\Big(\frac{1/7}{\sqrt{1/3}\sqrt{1/11}}\Big) = 0.608$$

(ii) Let $f(x) = x^2$ and $g(x) = x^4$. Then

$$\theta = \cos^{-1}\Big(\frac{\int_0^1 x^6 dx}{\sqrt{\int_0^1 x^4 dx}\sqrt{\int_0^1 x^8 dx}}\Big) = \cos^{-1}\Big(\frac{1/7}{\sqrt{1/5}\sqrt{1/3}}\Big) = 0.984$$

Problem 8.

(i)

$$\langle\cos(t), \sin(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\sin(t)dt = \frac{1}{\pi} * 0 = 0$$

$$\langle\cos(t), \cos(2t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\cos(2t)dt = 0$$

$$\langle\cos(t), \sin(2t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\sin(2t)dt = 0$$

$$\langle\cos(t), \cos(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\cos(t)dt = 1$$

$$\langle \sin(t), \cos(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(t)\cos(2t)dt = 0$$
$$\langle \sin(t), \sin(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(t)\sin(2t)dt = 0$$
$$\langle \sin(t), \sin(t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(t)\sin(t)dt = 1$$
$$\langle \cos(2t), \sin(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(2t)\sin(2t)dt = 0$$
$$\langle \cos(2t), \cos(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(2t)\cos(2t)dt = 1$$
$$\langle \sin(2t), \sin(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(2t)\sin(2t)dt = 1$$

(ii)

$$\|t\| = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt} = \sqrt{\frac{2\pi^3}{3}}$$

(iii)

$$\begin{aligned}
&proj_X(\cos(3t)) = \langle \cos(t), \cos(3t)\rangle + \langle \sin(t), \cos(3t)\rangle + \langle \cos(2t), \cos(3t)\rangle + \langle \sin(2t), \cos(3t)\rangle = \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\cos(3t)dt + \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(t)\cos(3t)dt + \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2t)\cos(3t)dt + \\
&+ \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(2t)\cos(3t)dt = 0+0+0+0 = 0
\end{aligned}$$

(iv)

$$\begin{aligned}
&proj_X(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos(t)dt + \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(t)dt + \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos(2t)dt + \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(2t)dt \\
&= \frac{1}{\pi}(0 + 2\pi + 0 - \pi) = 1
\end{aligned}$$

Problem 9. Let $x = (x_1, x_2)$ and let $y = (y_1, y_2)$. Then $R_\theta x = (\cos(\theta)x_1 - \sin(\theta)x_2, \sin(\theta)x_1 + \cos(\theta)x_2)$ and $R_\theta y = (\cos(\theta)y_1 - \sin(\theta)y_2, \sin(\theta)y_1 + \cos(\theta)y_2)$. We now expand:

$$\begin{aligned}
\langle R_\theta x, R_\theta y\rangle =& (\cos(\theta)x_1 - \sin(\theta)x_2) * (\cos(\theta)y_1 - \sin(\theta)y_2) + \\
& (\sin(\theta)x_1 + \cos(\theta)x_2) * (\sin(\theta)y_1 + \cos(\theta)y_2) = \\
& \cos(\theta)^2 x_1y_1 - \sin(\theta)\cos(\theta)x_2y_1 - \cos(\theta)\sin(\theta)x_1y_2 + \sin(\theta)^2 x_2y_2 + \\
& \sin(\theta)^2 x_1y_1 + \cos(\theta)^2 x_2y_2 + \cos(\theta)\sin(\theta)x_1y_2 + \cos(\theta)\sin(\theta)x_2y_1 = \\
& (x_1y_1 + x_2y_2)(\cos(\theta)^2 + \sin(\theta)^2) = x_1y_1 + x_2y_2 = \langle x, y\rangle
\end{aligned}$$

Problem 10. Let $Q \in M_n(\mathbb{F})$ be an orthonormal matrix.

(i) ($\Rightarrow$) Let $Q \in M_n(\mathbb{F})$ be an orthonormal matrix. Then $\langle Qx, Qy\rangle = \langle x, y\rangle$ which expanding gives $\langle Qx, Qy\rangle = x^H Q^H Q y = x^H y$. This implies that $Q^H Q = I$. By Proposition 3.2.12, since Q is an orthonormal operator and $\mathbb{F}^n$ is finite dimensional, Q is invertible. Since inverses are unique, $Q^{-1} = Q^H$ so $Q^H Q = QQ^H = I$.
($\Leftarrow$) Let $Q^H Q = QQ^H = I$. Then $\langle Qx, Qy\rangle = x^H Q^H Q y = x^H I y = x^H y = \langle x, y\rangle$.

(ii) This follows directly from part i. $\|Qx\| = \sqrt{\langle Qx, Qx\rangle} = \sqrt{\langle x, x\rangle} = \|x\|$.

(iii) By part i we have $Q^{-1} = Q^H$. Then $\langle Q^H x, Q^H y\rangle = x^H QQ^H y = x^H I y = x^H y = \langle x, y\rangle$.

(iv) Consider $\langle Qe_i, Qe_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$ which is the Kronecker delta. Qe_i is column i of matrix Q, so the dot product of column i with itself is 1 and 0 when $i \neq j$, implying the columns of an orthonormal matrix are orthonormal.

(v) $\det(QQ^H) = \det(I) = 1$. Since $\det(Q) = \det(Q^H)$ we have $\det(Q)^2 = 1$ so $\det(Q) = 1$. The converse is not necessarily true. Consider:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix A is upper triangular and therefore has $\det(A) = 1$ but is not orthonormal because column 2 dot product itself is not equal to 1 which needs to be true by part (iv).

(vi) Let Q_1, Q_2 be orthonormal. Then $\langle Q_1Q_2x, Q_1Q_2y\rangle = x^H Q_2^H Q_1^H Q_1 Q_2 y = x^H Q_2^H I Q_2 y = x^H Q_2^H Q_2 y = x^H y = \langle x, y\rangle$ so the product Q_1Q_2 is orthonormal.

Problem 11. Apply the Gram-Schmidt orthonormalization process to a collection of linearly dependent vectors

Problem 16.

Problem 17. Let $A = \widehat{Q}\widehat{R}$ be reduced QR decomposition. Then the system $A^H A x = A^H b$ can be rewritten as

$$\begin{aligned} &(\widehat{Q}\widehat{R})^H(\widehat{Q}\widehat{R})x = (\widehat{Q}\widehat{R})^H b \\ &\Leftrightarrow \widehat{R}^H\widehat{Q}^H\widehat{Q}\widehat{R}x = \widehat{R}^H\widehat{Q}^H b \\ &\Leftrightarrow \widehat{R}^H\widehat{R}x = \widehat{R}^H\widehat{Q}^H b \\ &\Leftrightarrow \widehat{R}x = \widehat{Q}^H b \end{aligned}$$

Problem 23. We have that $\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$ so $\|x\| - \|y\| \leq \|x - y\|$. Similarly, $\|y\| - \|x\| \leq \|x - y\|$ since $\|x - y\| = \|y - x\|$. Putting these together implies that $|\|x\| - \|y\|| \leq \|x - y\|$ because if $\|x\| > \|y\|$ then $|\|x\| - \|y\|| = \|x\| - \|y\|$ and else $|\|x\| - \|y\|| = \|y\| - \|x\|$.

Problem 24.

(i)

(ii)

(iii)

Problem 26.

Problem 28.

Problem 29.

Problem 30.

Problem 37.

Problem 47. Let $P = A(A^H A)^{-1} A^H$.

(i) Since $(A^H A)^{-1} A^H A = I$ we have that

$$P^2 = PP = A(A^H A)^{-1} A^H A(A^H A)^{-1} A^H = A(A^H A)^{-1} A^H = P$$

(ii) We prove this by induction. We show the base case in part i. Now assume that $P^H = P$. Then $P^{H+1} = P^H P = PP = P^2 = P$.

(iii)