Problem 1. We have that $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|\cos(\theta)$ and $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos(\theta)$ where $\cos(\theta) = \dfrac{\langle x, y\rangle}{\|x\|\|y\|}$ which is proven in the textbook.

(i) $\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \frac{1}{4}(\|x\|^2 + \|y\|^2 + 2\|x\|\|y\|\cos(\theta) - \|x\|^2 - \|y\|^2 + 2\|x\|\|y\|\cos(\theta)) = \frac{1}{4}(4\|x\|\|y\|\cos(\theta)) = \|x\|\|y\|\dfrac{\langle x, y\rangle}{\|x\|\|y\|} = \langle x, y\rangle$

(ii) $\frac{1}{2}(\|x+y\|^2 + \|x-y\|^2) = \frac{1}{2}(\|x\|^2 + \|y\|^2 + 2\|x\|\|y\|\cos(\theta) + \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos(\theta)) = \frac{1}{2}(2(\|x\|^2 + \|y\|^2)) = \|x\|^2 + \|y\|^2$

Problem 2.

$$\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2) = \langle x, y\rangle + \frac{1}{4}(i\|x-iy\|^2 - i\|x+iy\|^2) =$$
$$= \langle x, y\rangle + \frac{1}{4}(-i(||x||^2 + \langle x, iy\rangle + \langle iy, x\rangle + ||y||^2 - ||x||^2 + \langle x, iy\rangle + \langle iy, x\rangle - ||y||^2)) =$$
$$= \langle x, y\rangle + \frac{i}{4}(2i\langle x, y\rangle - 2i\langle y, x\rangle) = \langle x, y\rangle$$

Problem 3.

(i) Let $f(x) = x$ and $g(x) = x^5$. Then

$$\theta = \cos^{-1}\big(\frac{\langle f, g\rangle}{\|f\|\|g\|}\big) = \cos^{-1}\big(\frac{\int_0^1 x^6 dx}{\sqrt{\int_0^1 x^2 dx}\sqrt{\int_0^1 x^{10} dx}}\big) = \cos^{-1}\big(\frac{1/7}{\sqrt{1/3}\sqrt{1/11}}\big) = 0.608$$

(ii) Let $f(x) = x^2$ and $g(x) = x^4$. Then

$$\theta = \cos^{-1}\big(\frac{\int_0^1 x^6 dx}{\sqrt{\int_0^1 x^4 dx}\sqrt{\int_0^1 x^8 dx}}\big) = \cos^{-1}\big(\frac{1/7}{\sqrt{1/5}\sqrt{1/3}}\big) = 0.984$$

Problem 8.

(i)

$$\langle \cos(t), \sin(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\sin(t)dt = \frac{1}{\pi} * 0 = 0$$
$$\langle \cos(t), \cos(2t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\cos(2t)dt = 0$$
$$\langle \cos(t), \sin(2t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\sin(2t)dt = 0$$
$$\langle \cos(t), \cos(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\cos(t)dt = 1$$

$$\langle \sin(t), \cos(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(t)\cos(2t)dt = 0$$
$$\langle \sin(t), \sin(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(t)\sin(2t)dt = 0$$
$$\langle \sin(t), \sin(t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(t)\sin(t)dt = 1$$
$$\langle \cos(2t), \sin(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(2t)\sin(2t)dt = 0$$
$$\langle \cos(2t), \cos(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(2t)\cos(2t)dt = 1$$
$$\langle \sin(2t), \sin(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(2t)\sin(2t)dt = 1$$

(ii)

$$\|t\| = \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} t^2 dt} = \sqrt{\frac{2\pi^3}{3}}$$

(iii)

$$\begin{aligned} &proj_X(\cos(3t)) = \langle \cos(t), \cos(3t) \rangle + \langle \sin(t), \cos(3t) \rangle + \langle \cos(2t), \cos(3t) \rangle + \langle \sin(2t), \cos(3t) \rangle = \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(t)\cos(3t)dt + \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(t)\cos(3t)dt + \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(2t)\cos(3t)dt + \\ &+ \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(2t)\cos(3t)dt = 0 + 0 + 0 + 0 = 0 \end{aligned}$$

(iv)

$$\begin{aligned} &proj_X(t) = \frac{1}{\pi} \int_{-\pi}^{\pi} t\cos(t)dt + \frac{1}{\pi} \int_{-\pi}^{\pi} t\sin(t)dt + \frac{1}{\pi} \int_{-\pi}^{\pi} t\cos(2t)dt + \frac{1}{\pi} \int_{-\pi}^{\pi} t\sin(2t)dt \\ &= \frac{1}{\pi}(0 + 2\pi + 0 - \pi) = 1 \end{aligned}$$

Problem 9. Let $x = (x_1, x_2)$ and let $y = (y_1, y_2)$. Then $R_\theta x = (\cos(\theta)x_1 - \sin(\theta)x_2, \sin(\theta)x_1 + \cos(\theta)x_2)$ and $R_\theta y = (\cos(\theta)y_1 - \sin(\theta)y_2, \sin(\theta)y_1 + \cos(\theta)y_2)$. We now expand:

$$\begin{aligned} \langle R_\theta x, R_\theta y \rangle = &(\cos(\theta)x_1 - \sin(\theta)x_2) * (\cos(\theta)y_1 - \sin(\theta)y_2) + \\ &(\sin(\theta)x_1 + \cos(\theta)x_2) * (\sin(\theta)y_1 + \cos(\theta)y_2) = \\ &\cos(\theta)^2 x_1 y_1 - \sin(\theta)\cos(\theta)x_2 y_1 - \cos(\theta)\sin(\theta)x_1 y_2 + \sin(\theta)^2 x_2 y_2 + \\ &\sin(\theta)^2 x_1 y_1 + \cos(\theta)^2 x_2 y_2 + \cos(\theta)\sin(\theta)x_1 y_2 + \cos(\theta)\sin(\theta)x_2 y_1 = \\ &(x_1 y_1 + x_2 y_2)(\cos(\theta)^2 + \sin(\theta)^2) = x_1 y_1 + x_2 y_2 = \langle x, y \rangle \end{aligned}$$

Problem 10. Let $Q \in M_n(\mathbb{F})$ be an orthonormal matrix.

(i) ($\Rightarrow$) Let $Q \in M_n(\mathbb{F})$ be an orthonormal matrix. Then $\langle Qx, Qy\rangle = \langle x, y\rangle$ which expanding gives $\langle Qx, Qy\rangle = x^H Q^H Q y = x^H y$. This implies that $Q^H Q = I$. By Proposition 3.2.12, since Q is an orthonormal operator and $\mathbb{F}^n$ is finite dimensional, Q is invertible. Since inverses are unique, $Q^{-1} = Q^H$ so $Q^H Q = QQ^H = I$.
($\Leftarrow$) Let $Q^H Q = QQ^H = I$. Then $\langle Qx, Qy\rangle = x^H Q^H Q y = x^H I y = x^H y = \langle x, y\rangle$.

(ii) This follows directly from part i. $\|Qx\| = \sqrt{\langle Qx, Qx\rangle} = \sqrt{\langle x, x\rangle} = \|x\|$.

(iii) By part i we have $Q^{-1} = Q^H$. Then $\langle Q^H x, Q^H y\rangle = x^H QQ^H y = x^H I y = x^H y = \langle x, y\rangle$.

(iv) Consider $\langle Qe_i, Qe_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$ which is the Kronecker delta. Qe_i is column i of matrix Q, so the dot product of column i with itself is 1 and 0 when $i \neq j$, implying the columns of an orthonormal matrix are orthonormal.

(v) $\det(QQ^H) = \det(I) = 1$. Since $\det(Q) = \det(Q^H)$ we have $\det(Q)^2 = 1$ so $\det(Q) = 1$. The converse is not necessarily true. Consider:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix A is upper triangular and therefore has $\det(A) = 1$ but is not orthonormal because column 2 dot product itself is not equal to 1 which needs to be true by part (iv).

(vi) Let Q_1, Q_2 be orthonormal. Then $\langle Q_1Q_2x, Q_1Q_2y\rangle = x^H Q_2^H Q_1^H Q_1 Q_2 y = x^H Q_2^H I Q_2 y = x^H Q_2^H Q_2 y = x^H y = \langle x, y\rangle$ so the product Q_1Q_2 is orthonormal.

Problem 11. Applying the Gram-Schmidt orthonormalization process to a collection of linearly dependent vectors we are ultimately forced to divide by zero when trying to form the orthonormal vector for the first dependent vector in the set. We can see this by setting WLOG the first dependent vector to be the second vector in the set. Then

$$q_1 = \frac{x_1}{\|x_1\|}$$

$$q_2 = \frac{x_2 - p_2}{\|x_2 - p_2\|}$$

where $p_2 = \langle x_2, q_1\rangle q_1$. However, assuming $x_2 = a_1x_1$, we have that

$$p_2 = \langle a_1x_1, \frac{x_1}{\|x_1\|}\rangle \frac{x_1}{\|x_1\|} = \langle \frac{x_1}{\|x_1\|}, \frac{x_1}{\|x_1\|}\rangle a_1x_1 = a_1x_1 = x_2$$

so $x_2 - p_2 = 0$.

Problem 16.

Problem 17. Let $A = \widehat{Q}\widehat{R}$ be reduced QR decomposition. Then the system $A^H Ax = A^H b$ can be rewritten as

$$\begin{aligned}&(\widehat{Q}\widehat{R})^H(\widehat{Q}\widehat{R})x = (\widehat{Q}\widehat{R})^H b\\ \Leftrightarrow\ &\widehat{R}^H\widehat{Q}^H\widehat{Q}\widehat{R}x = \widehat{R}^H\widehat{Q}^H b\\ \Leftrightarrow\ &\widehat{R}^H\widehat{R}x = \widehat{R}^H\widehat{Q}^H b\\ \Leftrightarrow\ &\widehat{R}x = \widehat{Q}^H b\end{aligned}$$

Problem 23. We have that $\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$ so $\|x\| - \|y\| \leq \|x - y\|$. Similarly, $\|y\| - \|x\| \leq \|x - y\|$ since $\|x - y\| = \|y - x\|$. Putting these together implies that $|\|x\| - \|y\|| \leq \|x - y\|$ because if $\|x\| > \|y\|$ then $|\|x\| - \|y\|| = \|x\| - \|y\|$ and else $|\|x\| - \|y\|| = \|y\| - \|x\|$.

Problem 24.

(i)

(ii)

(iii)

Problem 26.

Problem 28.

Problem 29.

Problem 30.

Problem 37.

Problem 47. Let $P = A(A^H A)^{-1}A^H$.

(i) Since $(A^H A)^{-1}A^H A = I$ we have that

$$P^2 = PP = A(A^H A)^{-1}A^H A(A^H A)^{-1}A^H = A(A^H A)^{-1}A^H = P$$

(ii) $P^H = (A(A^H A)^{-1}A^H)^H = (A^H)^H((A^H A)^{-1})^H A^H = A((A^H A)^H)^{-1}A^H = A(A^H A)^{-1}A^H = P$

(iii)