Problem 1. Let S be a nonempty subset of V. Consider $conv(S)$ which is the set of all elements such that $\lambda_1 x_1 + \cdots + \lambda_k x_k \in conv(S)$ where $x_i \in S, k \in \mathbb{N}$ and $\lambda_i \geq 0$ and $\lambda_1 + \cdots + \lambda_k = 1$. Consider then the case where $k = 2$. We then have that $\lambda_1 x + \lambda_2 y \in conv(S)$. Then by our assumptions, $\lambda_1 + \lambda_2 = 1$ so $\lambda \equiv \lambda_1 = 1 - \lambda_2$. Then for any $x, y \in S$ we have $\lambda x + (1-\lambda)y \in conv(S)$ so $conv(S)$ is convex.

Problem 2.

(i) Let $P = \{x \in V | \langle a, x\rangle = b\} \in V$ be a hyperplane where $a \in V, a \neq 0, b \in \mathbb{R}$. Let $x, y \in P$ and consider the point $\lambda x + (1-\lambda)y$. Then $\langle a, \lambda x + (1-\lambda)y\rangle = \lambda\langle a, x\rangle + (1-\lambda)\langle a, y\rangle = \lambda b + (1-\lambda)b = b$ so $\lambda x + (1-\lambda)y \in P$ and P is convex.

(ii) Let $H = \{x \in V | \langle a, x\rangle \leq b\} \in V$ be a half-space where $a \in V, a \neq 0, b \in \mathbb{R}$. Let $x, y \in H$ and consider the point $\lambda x + (1-\lambda)y$. Then $\langle a, \lambda x + (1-\lambda)y\rangle = \lambda\langle a, x\rangle + (1-\lambda)\langle a, y\rangle \leq \lambda b + (1-\lambda)b = b$ so $\lambda x + (1-\lambda)y \in H$ and H is convex.

Problem 4.

(i) $\|x-y\|^2 = \langle x-y, x-y\rangle = \langle (x-p)+(p-y), (x-p)+(p-y)\rangle = \|x-p\|^2 + \|p-y\|^2 + 2\langle x-p, p-y\rangle$

(ii) Let $\langle x-p, p-y\rangle \geq 0$. Then if $y \neq p$ we have by part 1 that $\|x-y\|^2 = \|x-p\|^2 + \|p-y\|^2 + 2\langle x-p, p-y\rangle$ where both the second and third terms on the right hand side are greater than 0 so the left hand side is greater than the right hand side and $\|x-y\|^2 > \|x-p\|^2$.

(iii) Let $z = \lambda y + (1-\lambda)p$ where $0 \geq \lambda \leq 1$. Then $\|x-z\|^2 = \|x - \lambda y - (1-\lambda)p\|^2 = \langle x - \lambda y - (1-\lambda)p, x - \lambda y - (1-\lambda)p\rangle = \langle (x-p) - \lambda(y-p), (x-p) - \lambda(y-p)\rangle = \|x-p\|^2 + 2\lambda\langle x-p, y-p\rangle + \lambda^2\|y-p\|^2$

(iv) Let p be the projection of point x onto C. Then by definition of the projection $\|x-p\|^2 \leq \|x-y\|^2,\ \forall y \in C$. We have from part (iii) that $\|x-z\|^2 - \|x-p\|^2 = 2\lambda\langle x-p, y-p\rangle + \lambda^2\|y-p\|^2$. Then by the definition of the projection $\|x-z\|^2 - \|x-p\|^2 \geq 0$ so $0 \leq 2\langle x-p, y-p\rangle + \lambda\|y-p\|^2$.

Problem 6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and consider the set $C = \{x \in \mathbb{R}^n | f(x) \leq c\} \subset \mathbb{R}^n$. Then for $x, y \in C$ and $0 \leq \lambda \leq 1$, since f is convex, we have that $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \leq \lambda c + (1-\lambda)c = c$. Therefore, $\lambda x + (1-\lambda)y \in C$ so C is convex.

Problem 7. Let C be a convex set and let $f_1, \cdots, f_k$ be convex functions where $f_i : C \to \mathbb{R}$ and $\lambda_1, \cdots, \lambda_k \in \mathbb{R}^+$. Then define the function f as $f(x) = \sum_{i=1}^{k} \lambda_i f_i(x)$. Consider the points

$x, y \in C$ and $0 \leq \lambda \leq 1$ so $\lambda x + (1-\lambda)y \in C$ since C is convex . Then

$$\begin{aligned}
f(\lambda x + (1-\lambda)y) &= \sum_{i=1}^{k} \lambda_i f_i(\lambda x + (1-\lambda)y) \\
&\leq \sum_{i=1}^{k} \lambda_i (\lambda f_i(x) + (1-\lambda) f_i(y)) \\
&= \sum_{i=1}^{k} \lambda_i \lambda f_i(x) + \sum_{i=1}^{k} \lambda_i (1-\lambda) f_i(y) \\
&= \lambda \sum_{i=1}^{k} \lambda_i f_i(x) + (1-\lambda) \sum_{i=1}^{k} \lambda_i f_i(y) \\
&= \lambda f(x) + (1-\lambda) f(y)
\end{aligned}$$

Problem 13. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and bounded above. Assume by contradiction that f is not constant. Then WLOG $\exists\, x, y, z, \in \mathbb{R}^n$ such that $x < y$ and $f(x) < f(y)$. Additionally, let $z \in (x, y)$. We have by convexity that $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y)$. If we let $\lambda = \dfrac{z-y}{y-x}$ then $\lambda x + (1-\lambda)y = z$. Rewriting the above expression we have that $f(z) \geq \lambda f(x) + (1-\lambda) f(y)$. If we let $y \to \infty$ then f violates the condition of being bounded above, and must therefore be a constant function.

Problem 20. Let $f : \mathbb{R}^n \to R$ be convex and let $-f$ also be convex. We have then that

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y)$$

$$-f(\lambda x + (1-\lambda)y) \geq -(\lambda f(x) + (1-\lambda) f(y))$$

These two conditions together imply that

$$f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y)$$

so f is linear and therefore affine.

Problem 21. Let x^* be the local minimizer for the problem

$$\begin{aligned}
\text{min} \quad & \phi \circ f(x) \\
\text{s.t} \quad & G(x) \leq 0 \\
& H(x) = 0
\end{aligned}$$