Problem Set #2 Part 1

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Problem 1 We first need to show that the set of complete and bounded functions $(\mathscr{C}, \|\cdot\|_{\sup})$ is complete. We now show that T is a contraction mapping on $\mathscr{C}$. We have that for a specific $y \in \mathbb{R}_+$:

$$
\begin{aligned}
&\big|Uw(y) - Uw'(y)\big| = \\
&= \big|u(\sigma(y)) + \beta \int w(f(y-\sigma(y))z)\phi(dz) - u(\sigma(y)) - \beta \int w'(f(y-\sigma(y))z)\phi(dz)\big| \\
&= \beta\big| \int [w(f(y-\sigma(y))z) - w'(f(y-\sigma(y))z)]\phi(dz)\big| \\
&\leq \beta \int \big|(w(f(y-\sigma(y))z) - w'(f(y-\sigma(y))z))\big|\phi(dz) \\
&\leq \beta \int \|w - w'\|_{\sup_{y\in\mathbb{R}_+}} \phi(dz) = \beta\|w - w'\|_{\sup_{y\in\mathbb{R}_+}}
\end{aligned}
$$

Taking the sup over $y \in \mathbb{R}_+$ gives us the desired result that

$$\|Uw - Uw'\|_{\sup_{y\in\mathbb{R}_+}} \leq \beta\|w - w'\|_{\sup_{y\in\mathbb{R}_+}}$$

This proves that U is a contraction mapping and there exists a unique fixed point solution. We now argue that the unique fixed point of U in $\mathscr{C}$ is v_σ. We know that v_σ is the expected lifetime utility given that policy $\sigma(y)$ is used. If $v'_\sigma(y')$ is the expected lifetime utility a period forward then $v'_\sigma(y') = v_\sigma(y')$ because otherwise the individual's preferences must have changed. It is therefore a fixed point of the equation.

Problem 2 Refer to the python notebook "DP_Part2".