Problem 1.

(3.6) Let $\{B_i\}_{i\in I}$ be a collection of elements in $\mathscr{F}$ such that $\{B_i\}_{i\in I}$ is a partition of Ω. Then for any $A \in \mathscr{F}$, since $B_i \cap B_j = \emptyset$ for all $i \neq j$, we have that $(A \cap B_i) \cap (A \cap B_j) = A \cap B_i \cap B_j = A \cap \emptyset = \emptyset$ for all $i \neq j$. Furthermore, $\cup_{i\in I}(A \cap B_i) = A$ because $\Omega = \cup_{i\in I} B_i$ and $A \cap \Omega = A$. Therefore, $\{(A \cap B_i)\}_{i\in I}$ forms a partition of A. It follows that $P(A) = P(\cup_{i\in I}(A \cap B_i)) = \sum_{i\in I} P(A \cap B_i)$.

(3.8) Let $\{E_1, E_2, ..., E_n\}$ be a collection of independent events. This implies that the compliments $\{E_1^c, E_2^c, ..., E_n^c\}$ are independent as well. Since the compliments are independent and by De Morgan's laws $\cup_{k=1}^n E_k = (\cap_{k=1}^n E_k^c)^c$ we have that

$$P(\cup_{k=1}^n E_k) = P((\cap_{k=1}^n E_k^c)^c) = 1 - P(\cap_{k=1}^n E_k^c) = 1 - \Pi_{k=1}^n P(E_k^c) = 1 - \Pi_{k=1}^n (1 - P(E_k))$$

(3.11)

$$P(s = \text{crime } | s \text{ tested } +) = \frac{P(s \text{ tested } + | s = \text{crime }) * P(s = \text{crime })}{P(s \text{ tested } +)} =$$

$$= \frac{P(s \text{ tested } + | s = \text{crime }) * P(s = \text{crime })}{P(s \text{ tested } + | s = \text{crime })P(s = \text{crime }) + P(s \text{ tested } + | s \neq \text{crime }) * P(s \neq \text{crime })}$$

We have that

$$P(s \text{ tested } + | s \neq \text{crime }) = \frac{1}{3000000}$$

$$P(s = \text{crime }) = \frac{1}{250000000}$$

$$P(s \neq \text{crime }) = 1 - \frac{1}{250000000}$$

$$P(s \text{ tested } + | s = \text{crime }) = 1$$

Therefore

$$P(s = \text{crime } | s \text{ tested } +) = .0118$$

(3.12) WLOG assume contestant chooses door 1. Denote the event that the car is behind door i for $i \in 1, 2, 3$ as C_1, C_2, C_3 where

$$P(C_i) = \frac{1}{3}, i \in 1, 2, 3$$

Denote the door Monty opens as D_1, D_2, D_3. Then the probability monty opens a door given that the contestant chooses door 1 and the car is behind door i is:

$$P(D_3|C_1) = 1/2$$

$$P(D_3|C_2) = 1$$

$$P(D_3|C_3) = 0$$

By Bayes' law we have that:

$$P(C_2|D_3) = \frac{P(D_3|C_2)P(C_2)}{P(D_3|C_1)P(C_1) + P(D_3|C_2)P(C_2) + P(D_3|C_3)P(C_3)} = \frac{2}{3}$$
$$P(C_1|D_3) = \frac{P(D_3|C_1)P(C_1)}{P(D_3|C_1)P(C_1) + P(D_3|C_2)P(C_2) + P(D_3|C_3)P(C_3)} = \frac{1}{3}$$

Therefore the probability of switching is higher. Similarly, if there are 10 doors and Monty opens 8, the probability of winning if the contestant switches is $\frac{9}{10}$ and the probability of winning if the contestant doesn't switch is $\frac{1}{10}$.

(3.16) Let X be a random variable with $\mathrm{E}[X] = \mu$ and X^2 be the random variable given by $X^2(\omega) = (X(\omega))^2$. Then

$$\mathrm{Var}[X] = \mathrm{E}[(X - \mathrm{E}[X])^2] = \mathrm{E}[X^2 - 2X\mathrm{E}[X] + \mathrm{E}[X]^2] = \mathrm{E}[X^2] - \mathrm{E}[2X\mathrm{E}[X]] + \mathrm{E}[\mathrm{E}[X]^2] =$$
$$= \mathrm{E}[X^2] - 2\mathrm{E}[X]\mathrm{E}[X] + \mathrm{E}[X]^2 = \mathrm{E}[X^2] - \mathrm{E}[X]^2 = \mathrm{E}[X^2] - \mu^2$$

(3.33) Let $B = B(n, p)$ be a binomial random variable for n trials with parameter p. Then since $\mu = pn$ and $\sigma^2 = n(p)(1 - p)$, for any $\epsilon > 0$ we have by Chebyshev's inequality that

$$P(|\frac{B}{n} - p| \geq \epsilon) = P(|B - pn| \geq n\epsilon) = P(|B - \mu| \geq n\epsilon) \leq \frac{p(1-p)n}{n^2\epsilon^2} = \frac{p(1-p)}{n\epsilon^2}$$

(3.36) The probability of each student enrolling is Bernoulli(.801) and the students are i.i.d., we have that if 6242 students are offered admission the expected number of students is 5000. Denote S_{6242} as the number of students enrolled given 6242 admissions, so by the Central Limit Theorem:

$$P(S_{6242} > 5500) = P(\frac{S_{6242} - 5000}{\sqrt{6242(.801)(.199)}} > \frac{5500 - 5000}{\sqrt{6242(.801)(.199)}}) =$$
$$= P(\frac{S_{6242} - 5000}{\sqrt{6242(.801)(.199)}} > 15.85) \approx 1 - \Phi(15.85) = 0$$

Problem 2.

(a) Events A, B, C must be pairwise independent, but not mutually independent. Consider four equally likely points $\Omega = \{1, 2, 3, 4\}$ and let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 1\}$. Then $P(A) = P(B) = P(C) = 1/4$ and $P(A \cap C) = P(\{1\}) = 1/4 = P(A)P(C)$. The other pairwise probabilities similarly follow because the sets have similar structures. However, $P(A \cap B \cap C) = P(\{\emptyset\}) = 0 \neq 1/8 = P(A)P(B)P(C)$.

(b) Now let events A, B, C must be mutually independent, but not pairwise independent in B and C. Consider eight equally likely points $\Omega = \{1, 2, ..., 8\}$ and let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{1, 3, 6, 7\}$. Then Then $P(A) = P(B) = P(C) = 1/2$ and $P(A \cap B \cap C) = P(\{3\}) = 1/8 = P(A)P(B)P(C)$. Additionally, $P(A \cap C) = P(\{1, 3\}) = 1/4 = P(A)P(C)$ and $P(A \cap B) = P(\{3, 4\}) = 1/4 = P(A)P(B)$ but $P(B \cap C) = P(\{\emptyset\}) = 0 \neq 1/4 = P(B)P(C)$.

Problem 3. In Benford's law, $P(d) = \log_{10}(1 + 1/d)$ for $d = 1, 2, ..., 9$. Then

$$\sum_{d=1}^{9} P(d) = \sum_{d=1}^{9} \log_{10}(1 + 1/d) = \log_{10}(\Pi_{d=1}^{9}(1 + 1/d)) = \log_{10}(2\frac{3}{2}\frac{4}{3}\frac{5}{4}...\frac{10}{9}) = \log_{10}(10) = 1$$

Problem 4.

(a)

$$\mathrm{E}[X] = \sum_{n=1}^{\infty} 2^n P(\text{first tail on flip } n) = \sum_{n=1}^{\infty} 2^n (\frac{1}{2})^n = \sum_{n=1}^{\infty} (\frac{2}{2})^n = \sum_{n=1}^{\infty} 1 = +\infty$$

(b)

$$\mathrm{E}[\ln X] = \sum_{n=1}^{\infty} \ln(2^n) P(\text{first tail on flip } n) = \sum_{n=1}^{\infty} n \ln(2) (\frac{1}{2})^n = \ln(2) \sum_{n=1}^{\infty} n (\frac{1}{2})^n =$$

$$= \ln(2) * \frac{\frac{1}{2}}{(1 - \frac{1}{2})^2} = 2\ln(2)$$

Problem 5. Let r be the risk free interest rate in both US and Switzerland. Then if the investors invest n in the assets of their respective countries, their returns will be $(1 + r)n$ since the interest rate is risk free. If they invest in a foreign country then

$$\mathrm{E}[\text{foreign investment}] = 0.5[1.25(1 + r)n + (1 + r)n/1.25] = (1 + r)n(1.025) > (1 + r)n$$

so the investors should invest in the asset of the foreign country.

Problem 6. Let $\Omega = [0, 1]$.

(a) Let X be the map from $\Omega = [0, 1]$ to the Pareto distribution on $\mathbb{R}$ where parameter $x_m = 1$. Then

$$f_X(x) = \begin{cases} \frac{\alpha x_m^\alpha}{x^{\alpha+1}} & \text{for } x \leq 0 \\ 0 & \text{for } x < x_m \end{cases}$$

When $\alpha \in (1, 2]$ the expectation is finite $\mathrm{E}[X] = \frac{\alpha x_m}{\alpha - 1}$ and the variance is infinite so $\mathrm{E}[X^2] = \infty$.

(b) Let X be a uniform distribution on $[7.5, 7.51]$ so $\mathrm{E}[X] > 7.5$ and let Y be the polynomial distribution on $[0, 10]$ with pdf $3/1000x^2$ so $\mathrm{E}[Y] = 7.5$. Since $P(Y > 7.51) > 0.57$ we have that $P(Y > X) > 1/2$.

(c) Let X be a uniform distribution on $[-3, 3]$; Y be a uniform distribution on $[-2, 2]$ and let Z be a uniform distribution on $[-1, 1]$. Then $\mathrm{E}[X] = \mathrm{E}[Y] = \mathrm{E}[Z] = 0$ and $P(X > Y)P(Y > Z)P(X > Z) > 0$ since each probability is greater than 0.

Problem 7. Let random variables X, Z be independent and $X \sim N(0, 1)$ and $P(Z = 1) = P(Z = -1) = \frac{1}{2}$. Let $Y = XZ$.

(a) True, $f_Y(x) = f_{XZ}(x) = \frac{1}{2}(f_X(x) + f_X(-x))) = \frac{1}{2}[\frac{1}{\sqrt{2\pi}}e^{-x^2/2} + \frac{1}{\sqrt{2\pi}}e^{-(-x)^2/2}] = \frac{1}{2}2\frac{1}{\sqrt{2\pi}}e^{-x^2/2} = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ so we have that $Y \sim N(0,1)$.

(b) True, $|Y| = |XZ| = |X|$ so $P(|X| = |Y|) = 1$.

(c) True, we easily show this by counter example: $P(Y > 0|X = 0) = 0 \neq P(Y > 0)$.

(d) True, $\text{Cov}(X,Y) = \text{E}[XY] - \text{E}[X]\text{E}[Y] = \text{E}[XXZ] - \text{E}[X]\text{E}[XZ] = \text{E}[X^2]\text{E}[Z] - 0*0 = \text{E}[X^2]*0 = 0$.

(e) False, we just showed using part(d) that it is possible for $\text{Cov}(X,Y) = 0$ even when X and Y are not independent.

Problem 8. Let the random variables $X_i, i = 1,2,...,n$ be i.i.d and $X_i \sim U[0,1]$. Let m and M be random variables such that $m = \min\{X_1, X_2, ..., X_n\}$ and $M = \max\{X_1, X_2, ..., X_n\}$. Then for $x \in [0,1]$ we have

$$P(M \leq x) = P(x \geq \max\{X_1, X_2, ..., X_n\}) = P(x \geq X_1, x \geq X_2, ..., x \geq X_n) =$$
$$= \Pi_{i=1}^n P(x \geq X_i) = (x)^n$$
$$P(m \leq x) = 1 - P(m \geq x) = 1 - \Pi_{i=1}^n P(X_i \geq x) = 1 - \Pi_{i=1}^n (1 - P(X_i \leq x)) =$$
$$= 1 - (1-x)^n$$

$$F(m) = \begin{cases} 1 & \text{for } x \leq 0 \\ 1-(1-x)^n & \text{for } x \in (0,1) \\ 0 & \text{for } x \geq 1 \end{cases} \qquad F(M) = \begin{cases} 0 & \text{for } x \leq 0 \\ (x)^n & \text{for } x \in (0,1) \\ 1 & \text{for } x \geq 1 \end{cases}$$

$$f_m(x) = n(1-x)^{n-1}$$
$$f_M(x) = n(x)^{n-1}$$

$$\text{E}[m] = \int_0^1 xn(1-x)^{n-1}dx = \frac{1}{n+1}$$
$$\text{E}[M] = \int_0^1 xn(x)^{n-1}dx = \frac{n}{n+1}$$

Problem 9.

(a) We know that $X_i \sim \text{Bernoulli}(.5)$ where the X_i are i.i.d with $\mu = 0.5$ and variance $p(1-p) = 0.25$ so

$$P(490 < S_1000 < 510) = P(\frac{-10}{\sqrt{1000*0.5*0.5}} < \frac{S_1000 - 500}{\sqrt{1000*0.5*0.5}}) < \frac{10}{\sqrt{1000*0.5*0.5}})$$
$$\approx 2\Phi(.632) - 1 = 0.47$$

(b) By the Weak Law of Large Numbers,

$$P(|\frac{S_{1000}}{1000} - \frac{1}{2}| \geq 0.005) \leq \frac{1}{4n * 0.005^2} \leq 0.01$$

Therefore we have that $n \geq 1000000$.

Problem 10. Let $\mathrm{E}[X] < 0$ and $\theta \neq 0$ such that $\mathrm{E}[e^{\theta X}] = 1$. Because $g(x) = e^x$ is a convex differentiable function, by Jensen's inequality, $\mathrm{E}[e^{\theta X}] \geq e^{\mathrm{E}[\theta X]} = e^{\theta \mathrm{E}[X]}$. Taking ln of both sides, we get that $0 = \ln(1) \geq \theta \mathrm{E}[X]$ and $\mathrm{E}[X] < 0$ and $\theta \neq 0$ implies that $0 > \theta \mathrm{E}[X]$ so $\theta > 0$.