Problem 2. Let $V = span(\{1, x, x^2\})$ and D is the derivative operator $D : V \to V$ such that $D[p](x) = p'(x)$. In the Chapter 3 exercises we showed that

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $\det(D - \lambda I) = (-\lambda)^3 = 0$. Therefore the eigenvalue is $\lambda = 0$ with algebraic multiplicity 3 and geometric multiplicity of 0 since all the eigenvectors are in the form $(a, 0, 0)$.

Problem 4. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(i) If $A = A^H$ then

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

Then $tr(A) = a+d$ and $\det(A) = ad-b^2$. A has real eigenvalues if $tr(A)^2 - 4\det(A) \geq 0$ where $tr(A)^2 - 4\det(A)$ is the term under the square root in the quadratic formula. We have that $tr(A)^2 - 4\det(A) = a^2 + d^2 + 2ad - 4ad + 4b^2 = (a-d)^2 + 4b^2 \geq 0$ so A has real eigenvalues.

(ii) If $A = -A^H$ then

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

Then $tr(A) = 0$ and $\det(A) = b^2$. We have that $tr(A)^2 - 4\det(A) = -4b^2 < 0$ so A has imaginary eigenvalues.

Problem 6. Let the matrix A be upper triangular. Then, since upper triangular matrices are closed under addition and subtraction, $A - \lambda I$ is also an upper triangular matrix with diagonal entires $a_{ii} - \lambda$. The determinant of an upper triangular matrix is the product of the diagonal entires so $p_A(\lambda) = \det(A - \lambda I) = \Pi_{i=1}^n (a_{ii} - \lambda) = 0$. Therefore the eigenvalues of A are the values of λ such that $a_{ii} - \lambda = 0 \forall i$ or equivalently $a_{ii} = \lambda \forall i$ where the a_{ii} are the diagonal entires of A.

Problem 8. Let $S = \{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$ and $V = span(\{S\})$.

(i) S is a basis for V if S spans V and is linearly independent. The first part follows from our assumptions. We now show the second.

$$0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin(x) \\ \cos(x) \\ \sin(2x) \\ \cos(2x) \end{bmatrix}$$

Since the $\det(I_{4\times 4}) = 1 \neq 0$ the set S is linearly independent. We also showed in problem 3.8 that the set is orthonormal and therefore linearly independent.

(ii) The derivatives of the basis are $\{\cos(x), -\sin(x), 2\cos(2x), -2\sin(2x)\}$ respectively. Then

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

(iii)

Problem 13.

Problem 15.

Problem 16.

Problem 18.

Problem 20.

Problem 24.

Problem 25.

Problem 27.

Problem 28.

Problem 31.

Problem 32.

Problem 33.

Problem 36.

Problem 38.