Problem Set #2 Part 1
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Problem 1 First checking for the spectral radius condition, the largest eigenvalue of matrix A has a value of 0.965538166352 < 1 so the spectral radius condition holds and the equation $x = Ax + b$ has a unique solution. Solving for x using both matrix algebra and successive approximations we get that:

$$x = \begin{bmatrix} -0.89552239 \\ 13.34328358 \\ 45.64179104 \end{bmatrix}$$

Problem 2 In a standard job search model, we have that $\bar{w}$ should satisfy

$$\bar{w} = c(1-\beta) + \beta \sum_{k=1}^{K} \max\{w_k,\, \bar{w}\}\; p_k$$

and that $T(x) : \mathbb{R}_+ \to \mathbb{R}_+$ where $T(x) = c(1-\beta) + \beta \sum_{k=1}^{K} \max\{w_k,\, x\}\; p_k$. First, we see that $\mathbb{R}_+$ is complete metric space because $\mathbb{R}$ is a complete metric space and $\mathbb{R}_+ \in \mathbb{R}$ is closed since it includes 0. Second, we need to show that T is a contraction mapping. Consider

$$\begin{aligned}
|T(x) - T(y)| &= \big|c(1-\beta) + \beta \sum_{k=1}^{K} \max\{w_k,\, x\}\; p_k - c(1-\beta) + \beta \sum_{k=1}^{K} \max\{w_k,\, y\}\; p_k\big| = \\
&= \beta\big|\sum_{k=1}^{K} \max\{w_k,\, x\}\, p_k - \sum_{k=1}^{K} \max\{w_k,\, y\}\; p_k\big| = \\
&= \beta\big|\sum_{k=1}^{K} (\max\{w_k,\, x\} - \max\{w_k,\, y\}\,) p_k\big| \\
&\leq \beta \sum_{k=1}^{K} \big|(\max\{w_k,\, x\} - \max\{w_k,\, y\}\,)\big| p_k \\
&\leq \beta \sum_{k=1}^{K} |x - y| p_k \\
&\leq \beta \sum_{k=1}^{K} \rho(x,y) p_k = \beta \rho(x,y)
\end{aligned}$$

And finally $\rho(T(x), T(y)) \leq \beta\rho(x,y)$ which proves that T is a contraction mapping and there exists a fixed point solution, or in other words, a $\bar{w}$ such that unemployed individuals offered a wage above $\bar{w}$ will choose to accept it, and will reject those below. They are indifferent if offered wage $\bar{w}$. Furthermore, $\bar{w}$ can be found by guessing an initial value and using successive approximations as in problem 1.

Problem 3 For any and all code refer to the python notebook "DP_part1". As unemployment benefits increase, $\bar{w}$ increases as well. Increased benefits increase the value of staying unemployed, so individuals are willing to wait for a higher wage offer. We can solve for $\frac{\partial \bar{w}}{\partial c}$ explicitly:

$$\frac{\partial \bar{w}}{\partial c} = \frac{1}{1 + \frac{\beta}{1-\beta}[1 - G(\bar{w})]} > 0$$

where G is the wage offer distribution. A graph is show below.