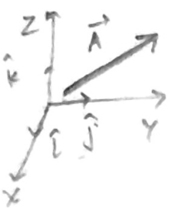
16th July 2020

MIT 18.02 Multivariable Calculus, Fall 2007

* **Vectors:**



- direction
- magnitude

$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \langle a_1, a_2, a_3 \rangle$

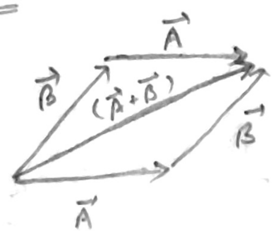
length $|\vec{A}|$ (a scalar), direction: dir $(\vec{A})$

$\vec{A} = \vec{PQ}$

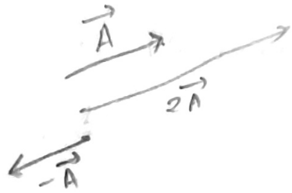
In general, $\vec{A} = \langle a_1, a_2, a_3 \rangle$

$$|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

* **Addition:** $\vec{A} + \vec{B} = \vec{B} + \vec{A} = \langle a_1+b_1,\ a_2+b_2,\ a_3+b_3 \rangle$



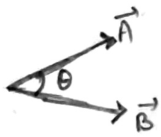
* **Multiplication by scalars:**
  ↳ scaled by the factor



* **Dot Product:**

Def: $\boxed{\vec{A}\cdot\vec{B} = \sum a_i b_i = a_1b_1 + a_2b_2 + a_3b_3}$ (is a scalar)

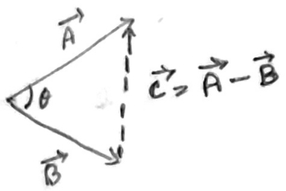
$\boxed{\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta}$



what does this defn mean?

① $\vec{A}\cdot\vec{A} = |\vec{A}|^2 = a_1^2 + a_2^2 + a_3^2$

②



$\vec{C} = \vec{A} - \vec{B}$

Law of cosines:

$|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - \underbrace{2|\vec{A}||\vec{B}|\cos\theta}$

$|\vec{C}|^2 = \vec{C}\cdot\vec{C} = (\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})$

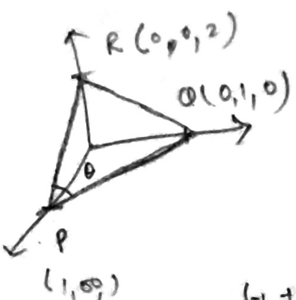
$= \vec{A}\cdot\vec{A} - \vec{A}\cdot\vec{B} - \vec{B}\cdot\vec{A} + \vec{B}\cdot\vec{B}$

$= |\vec{A}|^2 - \underbrace{2\vec{A}\cdot\vec{B}} + |\vec{B}|^2$

$\boxed{\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta}$ ✓

Applications: ① computing lengths & angles:

Ex.



$\vec{PQ}\cdot\vec{PR} = |\vec{PQ}|\,|\vec{PR}|\cos\theta$

$$\cos\theta = \frac{\vec{PQ}\cdot\vec{PR}}{|\vec{PQ}|\,|\vec{PR}|} = \frac{1}{\sqrt{2}\times\sqrt{5}}$$

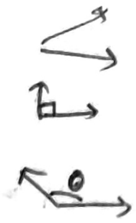
$$= \frac{\langle -1,1,0\rangle \langle -1,0,2\rangle}{\sqrt{2}\cdot\sqrt{5}} = \frac{1}{\sqrt{10}}$$

$(-1, +1, 0)$ $\quad X - 1 - 2$

$(-1, 0, +2)$

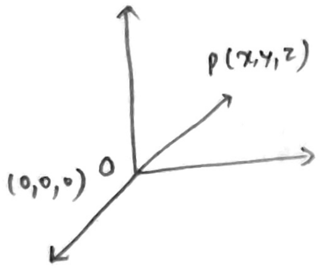
$\theta = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right) \approx 71.5°$

* sign of $\vec{A}\cdot\vec{B}$: >0 if $\theta < 90^\circ$
$=0$ if $\theta = 90^\circ$
<0 if $\theta > 90^\circ$



② **Detect orthogonality:**

Ex. $x + 2y + 3z = 0$ eqn of a plane.



$\vec{OP} = \langle x, y, z\rangle$

$\vec{A} = \langle 1, 2, 3\rangle$

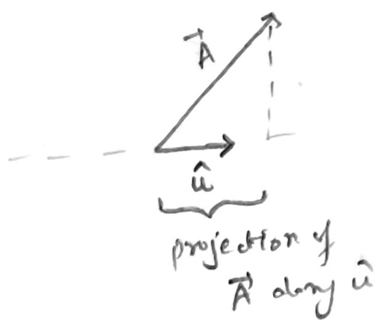
$\vec{OP}\cdot\vec{A} = 0 \iff \vec{OP} \perp \vec{A}$

Get: plane through O, perpendicular to $\vec{A}$

$\vec{A}\cdot\vec{B} = 0 \iff \cos\theta = 0 \iff \theta = 90^\circ$

$\iff \vec{A} \perp \vec{B}$

③ Components of $\vec{A}$ along direction $\hat{u}$ (a unit vector)



component of $\vec{A}$ along $\hat{u}$ $= |\vec{A}|\cos\theta$ → unit vector

$= |\vec{A}|\,|\hat{u}|\cos\theta$

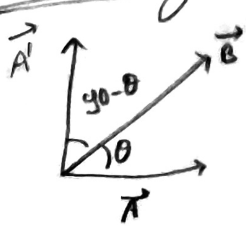
$= \vec{A}\cdot\hat{u}$

Area:

Area $= \frac{1}{2}\,|\vec{A}|\,|\vec{B}|\sin\theta$

We could: find $\cos\theta$, then solve $\sin^2\theta + \cos^2\theta = 1$.

Easier way:

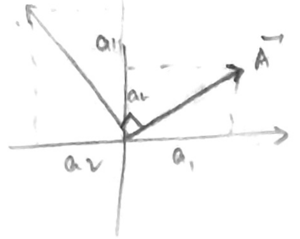


$\vec{A}' = \vec{A}$ rotated 90° $\Rightarrow \theta' = \frac{\pi}{2} - \theta$

$\cos(\theta') = \sin\theta$

$|\vec{A}|\,|\vec{B}|\sin\theta = |\vec{A}'|\,|\vec{B}|\cos\theta'$

$= \vec{A}'\cdot\vec{B}$



Rotating $\vec{A}$ by 90°

$\vec{A} = \langle a_1, a_2 \rangle$

$\vec{A}' = \langle -a_2, a_1 \rangle$

* $|\vec{A}||\vec{B}| \sin\theta = |\vec{A}'||\vec{B}| \cos\theta' = \vec{A}' \cdot \vec{B}$

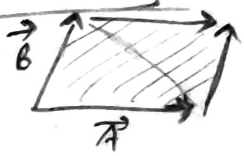
$= \langle -a_2, a_1 \rangle \cdot \langle b_1, b_2 \rangle$

$= a_1 b_2 - a_2 b_1$

$= \det(\vec{A}, \vec{B})$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$\boxed{\text{Determinant of } \vec{A} \text{ and } \vec{B} = \text{Area of the parallelogram } \vec{A}\vec{B}}$



$\pm$ area (▭) $= |\vec{A}||\vec{B}| \sin\theta = \det(\vec{A}, \vec{B})$

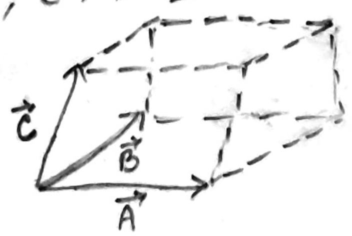
$\pm$ area (△) $= \frac{1}{2} |\vec{A}||\vec{B}| \sin\theta = \frac{1}{2} \det(\vec{A}, \vec{B})$

Determinant in space: 3 vectors A, B, C

$$\det(\vec{A}, \vec{B}, \vec{C}) = \begin{vmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Theorem: geometrically,

$\det(\vec{A}, \vec{B}, \vec{C}) = \pm$ volume of parallelepiped



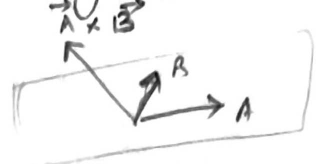
cross-product: of 2 vectors in 3-space.

Def: $\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$

is a vector

Theorem: • $|\vec{A}\times\vec{B}|$ = area of the parallelogram

• dir $(\vec{A}\times\vec{B})$ = $\perp$ to plane of the parallelogram with right hand rule



Another approach to find the volume:

Volume = area (base) . height $\qquad \hat{n} = \dfrac{\vec{B}\times\vec{C}}{|\vec{B}\times\vec{C}|}$

$= |\vec{B}\times\vec{C}| \times (\vec{A}\cdot\hat{n})$

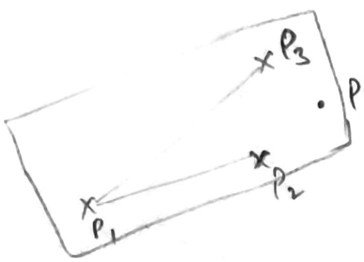
$= \cancel{|\vec{B}\times\vec{C}|} \times \vec{A}\cdot\dfrac{(\vec{B}\times\vec{C})}{\cancel{|\vec{B}\times\vec{C}|}}$

** $\boxed{\det(\vec{A},\vec{B},\vec{C}) = \vec{A}\cdot(\vec{B}\times\vec{C})}$

Lec 3 • $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ • $\vec{A}\times\vec{A} = 0$

Application: P_1, P_2, P_3 Equation of Plane containing $P_1\, P_2\, P_3$

= condition on (x,y,z) telling us whether P is in the plane or not.



$\det(\overrightarrow{P_1P}, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}) = 0$

Another solution: P is in the plane $\Leftrightarrow \overrightarrow{P_1P} \perp \vec{N}$ ← some vector $\perp$ plane (normal vector)

$\Leftrightarrow \boxed{\overrightarrow{P_1P}\cdot\vec{N} = 0}$

How to we find $\vec{N} \perp$ plane?:

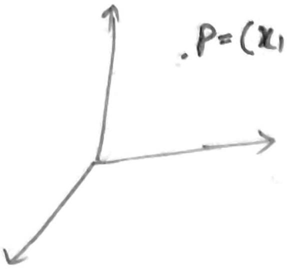
Answer: $\vec{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$

So: $\overrightarrow{P_1P}\cdot\vec{N} = \overrightarrow{P_1P}\cdot(\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}) = 0$ triple product = det

MATRICES

often: linear relations between variables

Ex. change of coordinates



$P=(x_1, x_2, x_3)$ (u_1, u_2, u_3)

$$\begin{cases} u_1 = 2x_1 + 3x_2 + 3x_3 \\ u_2 = 2x_1 + 4x_2 + 5x_3 \\ u_3 = x_1 + x_2 + 2x_3 \end{cases}$$

Express using matrix product:

$$\begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\boxed{A \cdot x = u}$$

Entries in matrix product: AX

dot-product between rows of A & column of X.

(A 3×3 matrix, x column vector $\Leftrightarrow$ 3×1 matrix)

Entries of AB

$$\underset{3\times4}{\overset{A}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ - & - & - & - \\ - & - & - & - \end{bmatrix}}} \underset{4\times2}{\overset{B}{\begin{bmatrix} \cdot & 0 \\ \cdot & 3 \\ \cdot & 0 \\ \cdot & 2 \end{bmatrix}}} = \underset{3\times2}{\overset{AB}{\begin{bmatrix} \cdot & 14 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}}}$$

For matrix multiplication: Width of A = Height of B

What AB represents: **do** transformation B, then transformation A.

$(AB)X = A(BX)$

Note: $AB \neq BA$

Identity matrix: $\boxed{IX = X}$

$$I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: in the plane, rotation by 90° counter-clockwise

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$R\hat{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \hat{j}$$

$$R\hat{j} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\hat{i}$$

$$R\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

* **Inverse matrix:**

Inverse of A: matrix M such that $AM = I$, $MA = I$

Need: A square matrix $n \times n$

$M = A^{-1}$

Solution of $AX = B$ is $X = A^{-1}B$

$AX = B \Rightarrow A^{-1}AX = A^{-1}B \Rightarrow \boxed{X = A^{-1}B}$

***

Formula: $\boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)}$ ← "adjoint"

How to find the adj (A)

Steps: (on a 3x3 example)

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}$$

① MINORS:

$\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} \to$
$$\begin{pmatrix} 3 & -1 & -2 \\ 3 & 1 & -1 \\ 3 & 4 & 2 \end{pmatrix}$$

② COFACTORS

flip signs in checkerboard

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix} \qquad \begin{pmatrix} 3 & 1 & -2 \\ -3 & 1 & 1 \\ 3 & -4 & 2 \end{pmatrix}$$

③ **Transpose:**

switch rows & columns

$$\begin{pmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{pmatrix} \Leftarrow \text{Adj}(A)$$

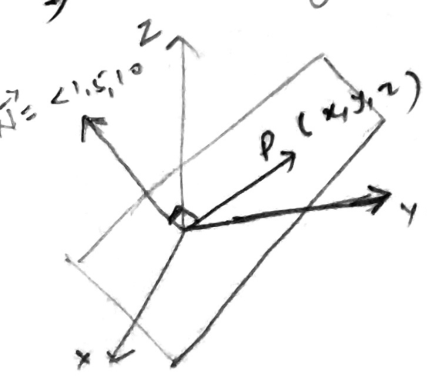
④ divide by determinant of (A).

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{pmatrix}$$

14:45, 16 July

Lec-4 **Equations of planes:** $ax + by + cz = d$ eqn of this form defines a plane

1) Plane through origin with normal vector $\vec{N} = \langle 1, 5, 10 \rangle$?

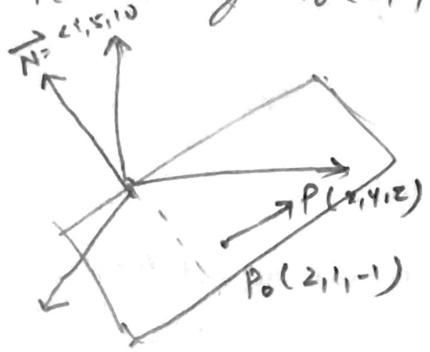


P is in a plane

$\Leftrightarrow \vec{OP} \cdot \vec{N} = 0$

$\Leftrightarrow x + 5y + 10z = 0$

② Plane through $P_0(2,1,-1)$ & $\perp \vec{N} = \langle 1,5,10 \rangle$.



P is in a plane
is $\vec{P_0P} \perp \vec{N}$

$$\vec{P_0P} \cdot \vec{N} = 0$$

$$\langle x-2, y-1, z+1 \rangle \cdot \langle 1,5,10 \rangle = 0$$

$$(x-2) + 5(y-1) + 10(z+1) = 0$$

$$\boxed{x + 5y + 10z = -3}$$

* In the equation $ax + by + cz = d$
  $\langle a,b,c \rangle$ = normal vector $\vec{N}$.

For example: get $\vec{N}$ by cross-product of 2 vectors in the plane

* 3x3 linear system?

example:
$$\begin{cases} x + \quad + z = 1 \\ x + y \quad = 2 \\ x + 2y + 3z = 3 \end{cases} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

3 planes.

⇒ 2 planes intersect in a line
→ the line $P_1 \cap P_2$ intersects P_3 in a point = the solution

Solution is given: $AX = B$ is given by:
$$\boxed{X = A^{-1}B}$$

Unless ---- the third plane P_3 is parallel to the line where P_1 & P_2 intersect

① line formed by $P_1 \cap P_2$ is parallel to the third plane & (not contained in it)
non-intersecting, hence no solution

② line formed by the intersection of $P_1 \cap P_2$ is contained in the plane P_3, then any point on that line is a solution. (infinitely many solution)

* $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

$\boxed{A \text{ is invertible} \Leftrightarrow \det(A) \neq 0,}$

Start with : Homogenous System Case :

$\boxed{AX = 0}$

e.g. $\begin{cases} x + z = 0 \\ x + y = 0 \\ x + 2y + 3z = 0 \end{cases}$

There's always an obvious solution $(0,0,0) \leftarrow$ TRIVIAL solution

(origin is solution because the 3 planes pass through the 0)

→ if $\det(A) \neq 0$: can invert A.

$AX = 0 \Rightarrow X = A^{-1}0 = 0$

no other solution..

→ if $\det(A) = 0$: $\Leftrightarrow \det(\vec{N_1}, \vec{N_2}, \vec{N_3}) = 0$

$\Leftrightarrow \vec{N_1}, \vec{N_2}, \vec{N_3}$ coplanar

Line through 0 perpendicular to plane of $\vec{N_1}, \vec{N_2}, \vec{N_3}$ is // to all 3 planes and in fact is contained in them.

$\Rightarrow \infty$ many solutions.

Ex: $\vec{N_1} \times \vec{N_2}$ is $\perp$ $\vec{N_1}$ & $\vec{N_2}$ -- but also to $\vec{N_3}$.

so, it's a non trivial solution.

General Case : $\boxed{AX = B}$

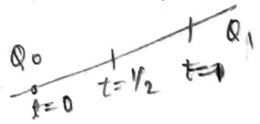
→ if $\det(A) \neq 0$ then unique solution $X = A^{-1}B$.

→ if $\det(A) = 0$ then either no sol or ∞ many solution.

$\boxed{22:42,\ 16\text{th May}}$ Lecture 5 multivariable calculus :

Equations of Lines : We have seen line = intersection of 2 planes. "parametric equation"

Another way: trajectory of a moving point :



Example : Line through :

$Q_0 = (-1, 2, 2)$

$Q_1 = (1, 3, -1)$

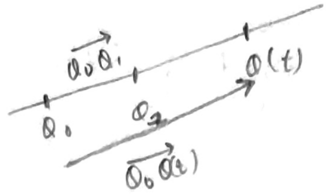
$Q(t)$ = moving point,

at $t = 0$ it's at Q_0. moves at constant speed on the line

what is the position at time t, $Q(t)$?

$$\overrightarrow{Q_0 Q(t)} = t\,\overrightarrow{Q_0 Q_1}$$

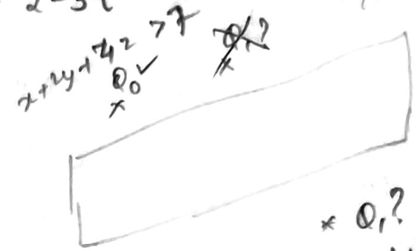
$$= t\langle 2,1,-3\rangle$$



$$Q(t) = x(t), y(t), z(t)$$

$$\begin{cases} x(t)+1 = t.2 \\ y(t)-2 = t \\ z(t)-2 = -3t \end{cases} \Rightarrow \begin{cases} x(t) = -1+2t \\ y(t) = 2+t \\ z(t) = 2-3t \end{cases} \Rightarrow \boxed{Q(t) = Q_0 + t\,\overrightarrow{Q_0Q_1}}$$

parametric equation



* Application :

→ intersection with a plane?

Consider the plane $x+2y+4z=7$

where does the line through Q_0, Q_1 intersect the plane?

$Q_0(-1,2,2)$ $\quad x+2y+4z = (-1)+2(2)+4(2) = 11 > 7$

not in the plane

$Q_1(1,3,-1)$ $\quad x+2y+4z = 1+2(3)+4(-1) = 3 < 7$

Q_0 & Q_1 are on opposite sides of the plane

→ what about $Q(t)$?

$$x(t)+2y(t)+4z(t)$$

$$= (-1+2t)+2(2+t)+4(2-3t)$$

$$= -8t+11 = 7$$

$Q(t)$ is in the plane when

$$-8t+11 = 7$$

$$\boxed{t = 1/2}$$

then $Q(1/2) = \left(0, \frac{5}{2}, \frac{1}{2}\right)$

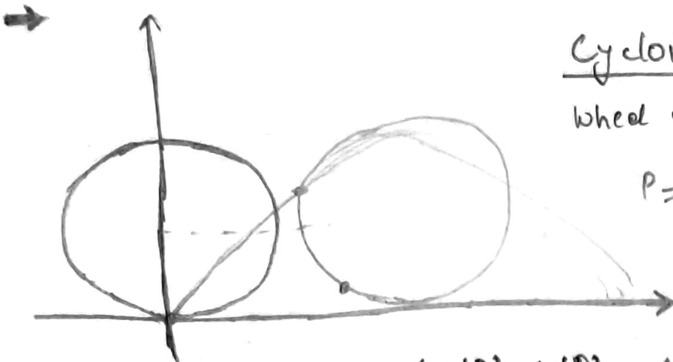
The line intersects the plane at

$$\left(0, \frac{5}{2}, \frac{1}{2}\right).$$

\# If the line is in the plane then plugging $(x(t), y(t), z(t))$ into eqn always satisfies the equation.

\# If the line is parallel to the plane, the $(x(t), y(t), z(t))$ never satisfy the eqn of the plane.

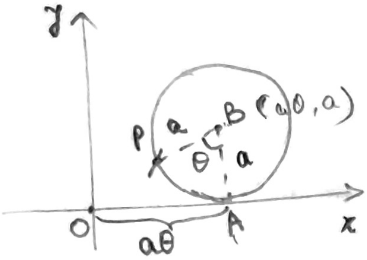
* More generally, we can use parametric equation for arbitrary motion in the plane or in space!



Cycloid:

Wheel of radius a rolling on floor = x-axis
P = point on rim of wheel starts at O.
What happens?

<u>Question</u>: position $(x(\theta), y(\theta))$ of the point P?
as a function of the angle by which the ~~angle~~ wheel has rotated.



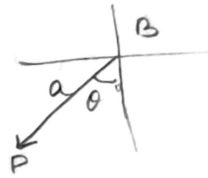
$$\vec{OP} = \vec{OA} + \vec{AB} + \vec{BP}$$

$$\vec{OA} = \langle \underset{\substack{\uparrow \\ \text{amount by which the wheel has moved}}}{a\theta}, 0 \rangle$$

$\vec{OA}$ = arc length from A to P.

$\vec{AB} = \langle 0, a \rangle$

$$\vec{BP} = \begin{cases} |\vec{BP}| = a \\ \text{angle } \theta \text{ with verticle} \end{cases}$$



$$\vec{BP} = \langle -a\sin\theta, -a\cos\theta \rangle$$

$$\vec{OP} = \langle \underbrace{a\theta - a\sin\theta}_{x(\theta)}, \underbrace{a - a\cos\theta}_{y(\theta)} \rangle$$

<u>Question</u>? What happens near the bottom point

<u>Answer</u>: take length unit = radius : a = 1

$$\begin{cases} x(\theta) = \theta - \sin\theta \\ y(\theta) = 1 - \cos\theta \end{cases} \qquad \begin{Bmatrix} \sin\theta \sim \theta \\ \cos\theta \sim 1 \end{Bmatrix} \text{ for small } \theta$$

<u>Taylor approximation</u>:

for t small:

$$f(t) \approx f(0) + t f'(0) + \frac{t^2}{2} f''(0) + \frac{t^3}{6} f'''(0) + \cdots$$

$$\sin\theta \approx \theta - \frac{\theta^3}{6} \qquad \cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$\begin{cases} x(\theta) = \theta - \left(\theta - \frac{\theta^3}{6}\right) \approx \frac{\theta^3}{6} \\ y(\theta) \approx 1 - \left(1 - \frac{\theta^2}{2}\right) \approx \frac{\theta^2}{2} \end{cases}$$

$$|x| \ll |y|$$

$$\frac{y}{x} = \frac{\theta^2/2}{\theta^3/6} = \frac{3}{\theta} \to \infty$$

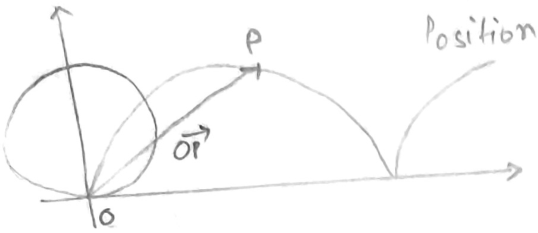
Slope at origin is ∞

00:63, 17 July

Lecture-5 18.02 Multivariable Calculus

Parametric equations

$(x(t), y(t), z(t))$ position of a moving point



Position vector $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Example: Cycloid (wheel radius 1, at unit speed)

$$r(t) = \langle t - \sin t, 1 - \cos t \rangle$$

Velocity vector: $\vec{v} = \frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$

Example: for cycloid:

$\vec{v} = \langle 1 - \cos t, \sin t \rangle$ (at $t=0$, $\vec{v} = 0$)

Speed: (scalar)

$$|\vec{v}| = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}$$

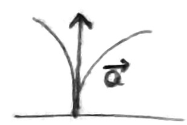
$$= \sqrt{2(1-\cos t)}$$

Acceleration: $\vec{a} = \frac{d\vec{v}}{dt}$

(a vector)

e.g. cycloid $\vec{a} = \langle \sin t, \cos t \rangle$

At $t=0$ $\vec{a} = \langle 0, 1 \rangle$

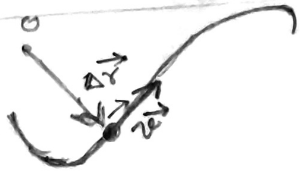


Arc length: s = distance travelled along trajectory

s vs t ? $\left|\frac{ds}{dt}\right| = \text{speed} = |\vec{v}|$

Example: length of an arch of cycloid is $\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$

Unit Tangent Vector: $\hat{T}$



$$\hat{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \times \frac{ds}{dt} \Rightarrow \hat{T}\frac{ds}{dt}$$

$\frac{d\vec{r}}{ds} = \hat{T}$, $\frac{ds}{dt} = |\vec{v}|$

Velocity has { direction: tangent to trajectory; length: speed $\frac{ds}{dt}$

$\Delta\vec{r} \approx \hat{T}\, \Delta s$

$\frac{\Delta\vec{r}}{\Delta t} \approx \hat{T}\, \frac{\Delta s}{\Delta t}$

$\frac{\Delta s}{\Delta t} \approx$ speed

Limit as $\Delta t \to 0$ gives

$$\frac{d\vec{r}}{dt} = \hat{T}\, \frac{ds}{dt}$$

Lecture 8: Multivariable Calculus

- Function of 1 variable: $f(x) = \sin x$
- Function of 2 variable: given $(x,y) \rightsquigarrow$ get a number $f(x,y)$

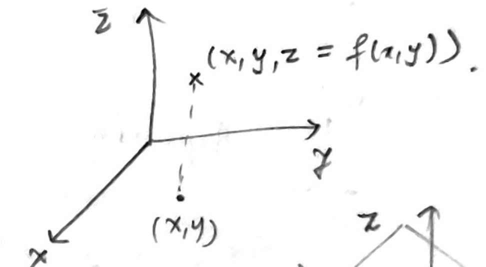
  Example: $f(x,y) = x^2 + y^2$

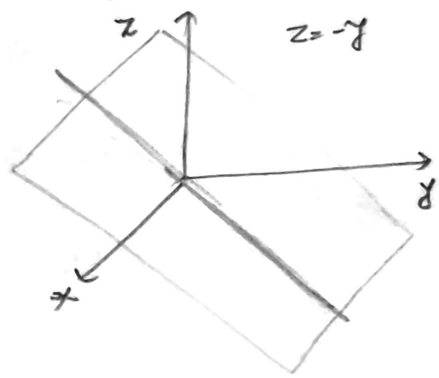
  $f(x,y) = \sqrt{y}$ only defined if $y \geq 0$

  $f(x,y) = \frac{1}{x+y}$ only defined if $x+y \neq 0$.

- or 3 or more parameters!
- For simplicity, focus mostly on 2 (or 3) variables.
- How to visualize f? $f = f^n$ of 2 variables

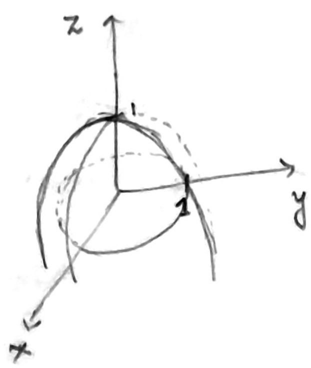
→ graph: $z = f(x,y)$





Example: $f(x,y) = -y \Rightarrow z = -y$.

Example: $f(x,y) = 1 - x^2 - y^2$



→ in yz plane

$x = 0, \quad z = 1 - y^2$

→ in xz plane

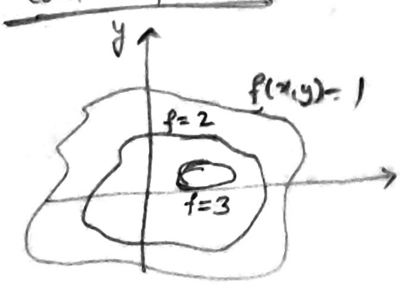
$y = 0, \quad z = 1 - x^2$

→ hit x-y plane, $z = 0$

$1 - x^2 - y^2 = 0$

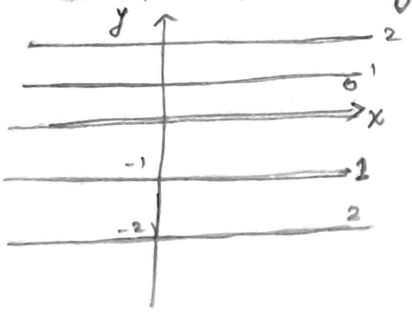
$x^2 + y^2 = 1$ unit circle

- Contour plot: shows all the points where $f(x,y) =$ some fixed constants, chosen at regular intervals.

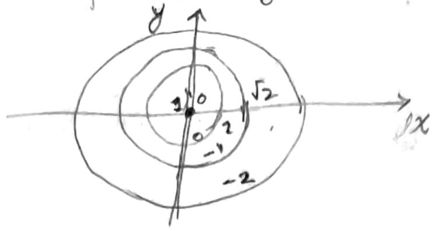


$\Longleftrightarrow$ we slice the graph by horizontal plane $z = c$

Example: $f(x,y) = -y$



$f(x) = 1 - x^2 - y^2 = 1 - (x^2 + y^2)$

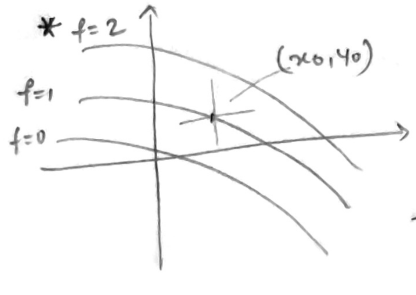


$f = 0 \Rightarrow x^2 + y^2 = 1$

$f = 1, \ x^2 + y^2 = 0$

$f = -1, \ x^2 + y^2 = 2$

$f =$



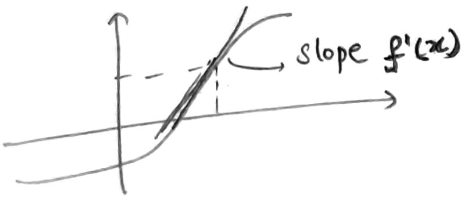
If $x\uparrow$ then $f(x,y)\uparrow$ If $x\downarrow$ then $f(x,y)\downarrow$

If $y\uparrow$ then $f(x,y)\uparrow$ If $y\downarrow$ then $f(x,y)\downarrow$

→ Partial derivatives:

function of 1 variable $f(x)$

$$\boxed{f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$



→ Approximation formula:

$\boxed{x_0 \longrightarrow f(x_0)}$

$$\boxed{f(x) \simeq f(x_0) + f'(x_0)(x - x_0)}$$

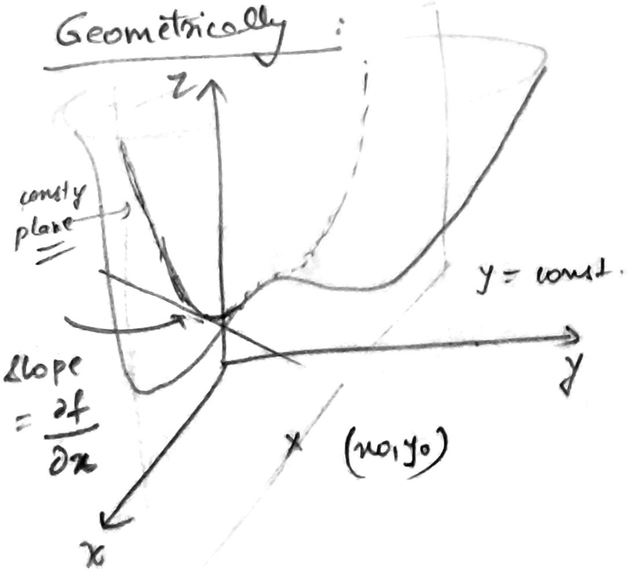
Question: do the same for $f(x,y)$?

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

"partial"

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

Geometrically:



* How to compute?

to find $\frac{\partial f}{\partial x} = f_x$?

treat y as constant & x as variable

Ex: $f(x) = x^3 y + y^2$

$$\frac{\partial f}{\partial x} = 3x^2 y + 0$$

$$\frac{\partial f}{\partial y} = x^3 + 2y$$

12:21, 17 July

Multivariable calculus : Lecture 9

Partial derivatives :

$f(x,y) \rightarrow \frac{\partial f}{\partial x} = f_x$ (vary x / y = const)

$\frac{\partial f}{\partial y} = f_y$ (vary y / x const)

Approximation formula :

→ If we change $x \sim x + \Delta x$

$y \sim y + \Delta y$

$z = f(x,y)$: then $\boxed{\Delta z \approx f_x \Delta x + f_y \Delta y}$

* Justif this formula : tangent plane to $z = f(x,y)$

Know : f_x & f_y are slopes of 2 tangent lines

If $\frac{\partial f}{\partial x}(x_0, y_0) = a \Rightarrow L_1 = \begin{cases} z = z_0 + a(x - x_0) \\ y = y_0 \end{cases}$

$\frac{\partial f}{\partial y}(x_0, y_0) = b \Rightarrow L_2 = \begin{cases} z = z_0 + b(y - y_0) \\ x = x_0 \end{cases}$

L_1 & L_2 are both tangent to the graph $z = f(x,y)$

Together they determine a plane.

$$z = z_0 + a(x - x_0) + b(y - y_0)$$

Approx. formula says : graph of f is close to its tangent plane.

→ Application of Partial derivatives

1) OPTIMIZATION PROBLEMS : ⟶ find min/max of a function $f(x,y)$

• At a local min/max, $\boxed{f_x = 0 \ \& \ f_y = 0.}$

⟺ tangent plane to the graph is horizontal !

Def^n : (x_0, y_0) is a critical point of f if $f_x(x_0, y_0) = 0$ & $f_y(x_0, y_0) = 0$

Example: $f(x,y) = x^2 - 2xy + 3y^2 + 2x - 2y$

$$\begin{cases} f_x = 2x - 2y + 2 = 0 \\ f_y = -2x + 6y - 2 = 0 \end{cases}$$

$4y = 0$, $2x + 2 = 0$

$\boxed{y = 0}$ $\boxed{x = -1}$

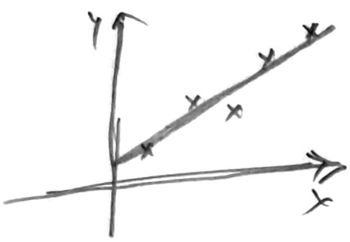
1 critical point: $(x,y) = (-1,0)$

→ Possibilities:
- local min
- local max
- saddle

$$f(x,y) = (x-y)^2 + 2y^2 + 2x - 2y$$
$$= \underbrace{(x-y+1)^2}_{+} - 1 + \underbrace{2y^2}_{\oplus} \geq -1 = f(-1,0)$$

So minimum

→ LEAST-SQUARED INTERPOLATION:



Given experimental data (x_i, y_i)

$(x_1,y_1)\ (x_2,y_2) \ldots (x_n,y_n)$

find the "best fit" line $y = \underline{\underline{a}}x + \underline{\underline{b}}$

Find the "best" a & b:

↳ minimizing the total square deviation.

deviation for each data point: $y_i - (ax_i + b)$

Minimize $D = \sum_{i=1}^{n} [y_i - (ax_i + b)]^2$

Want $\frac{\partial D}{\partial a} = 0 \Rightarrow \sum_{i=1}^{n} \cancel{2}[y_i - (ax_i+b)][-x_i] = 0$ — ①

$\frac{\partial D}{\partial b} = 0$ $\sum_{i=1}^{n} \cancel{2}[y_i - (ax_i+b)](-1) = 0$ — ②

$$\Leftrightarrow \begin{cases} \sum_{i=1}^{n} x_i^2 a + x_i b - x_i y_i = 0 \\ \sum_{i=1}^{n} x_i a + b - y_i = 0 \end{cases} \Leftrightarrow \begin{cases} \left(\sum_{i=1}^{n} x_i^2\right) a + \sum_{i=1}^{n} x_i b = \sum_{i=1}^{n} x_i y_i \\ \left(\sum_{i=1}^{n} x_i\right) a + nb = \sum_{i=1}^{n} y_i \end{cases}$$

→ Can show: it's a minimum.

⇝ 2×2 linear system.
Solve this for a & b.
(a, b)

Least-squares is more general.

- quadratic law: $y = ax^2 + bx + c$

$$D(a,b,c) = \sum_{i=1}^{n} \left(y_i - (ax_i^2 + bx_i + c)\right)^2$$

14:00, 17th July

Multivariable Calculus : Lecture 10

Recall : Critical points

of $f(x,y)$: where $f_x = 0$ & $f_y = 0$

Question : ① how do we decide between { – local min, – local max, – saddle ?

Question : ② how do we find global min / max ?

↳ these occur at a critical point or on boundary / at infinity (∞)

Second derivative test :

First consider $w = ax^2 + bxy + cy^2$ $(0,0)$ is a critical point

Example : $w = x^2 + 2xy + 3y^2 = (x+y)^2 + 2y^2 \geq 0$ $(0,0)$ min.

In general, if $a \neq 0$

$$w = a\left(x^2 + \frac{b}{a}xy\right) + cy^2$$

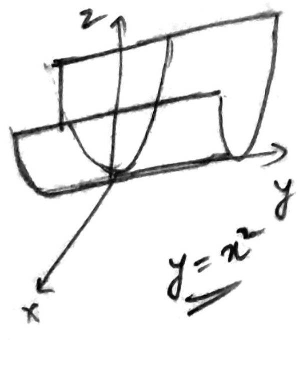
$$\Rightarrow w = a\left(x + \frac{b}{2a}y\right)^2 + \left(c - \frac{b^2}{4a}\right)y^2$$

$$\Rightarrow w = \frac{1}{4a}\left[4a^2\left(x + \frac{b}{2a}y\right)^2 + (4ac - b^2)y^2\right]$$ (sum/difference of 2 sq.)

3 cases : 1) $4ac - b^2 < 0$, one term ≥ 0, the other ≤ 0

$\Rightarrow$ saddle point.

2) $4ac - b^2 = 0$, degenerate critical point



③ $4ac - b^2 > 0$, $w = \frac{1}{4a}\underbrace{\left[+(\ )^2 + (\ )^2\right]}_{\geq 0}$

→ if $a > 0$, minimum

→ if $a < 0$, maximum.

* $b^2 - 4ac$ quadratic formula ??

$$w = ax^2 + bxy + cy^2 = \underbrace{y^2}_{\geq 0}\underbrace{\left[a\left(\frac{x}{y}\right)^2 + b\left(\frac{x}{y}\right) + c\right]}$$

if $b^2 - 4ac > 0 \Rightarrow$ has solutions.

then this takes + and - values.

$\Rightarrow$ w takes both +ve & -ve values.

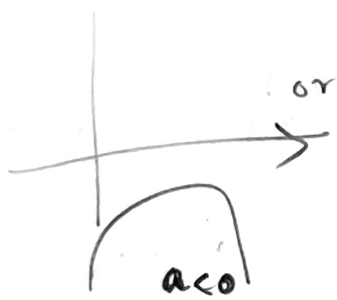
$\Rightarrow$ SADDLE point

If $b^2 - 4ac < 0$, $a\left(\frac{x}{y}\right)^2 + b\left(\frac{x}{y}\right) + c$

always $> 0 \rightarrow w \geq 0$, min

or always $< 0 \rightarrow w \leq 0$, max.

$a > 0$



$a < 0$

* In general, look at second derivatives !

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad f_{xy} = \frac{\partial f}{\partial x \partial y} = f_{yx} = \frac{\partial^2 f}{\partial y x} \qquad \frac{\partial^2 f}{dy^2} = f_{yy}$$

Second derivative test :

At a critical point (x_0, y_0) of f

let $A = f_{xx}(x_0, y_0)$ $\quad B = f_{xy}(x_0, y_0)$ $\quad C = f_{yy}(x_0, y_0)$

If $AC - B^2 > 0 \Rightarrow$ if $A > 0$, then it's a local minimum

$\Rightarrow$ if $A < 0$, then it's a local max.

If $AC - B^2 < 0 \Rightarrow$ Saddle point

If $AC - B^2 = 0 \Rightarrow$ can't conclude.

- Lets Verify in specific case $w = ax^2 + bxy + cy^2$?

$w_x = 2ax + by$ $\quad w_{xy} = b$ $\quad w_y = bx + 2cy$

$w_{xx} = 2a$ $\quad w_{yy} = 2c$

$A = 2a$ $\quad B = b$ $\quad C = 2c$ $\quad AC - B^2 = 4ac - b^2$

if $AC - B^2 < 0 \Rightarrow 4ac - b^2 < 0 \Rightarrow$ saddle point

if $AC - B^2 > 0 \Rightarrow 4ac - b^2 > 0 \Rightarrow$ if $a > 0 \rightarrow$ min
if $a < 0 \rightarrow$ max

if $AC - B^2 = 0 \Rightarrow 4ac - b^2 = 0 \Rightarrow$ can't conclude, degenerate critical point

* quadratic approximation:

$$\Delta f \approx f_x(x-x_0) + f_y(y-y_0) + \underbrace{\tfrac{1}{2} f_{xx}}_{\frac{1}{2}A = a}(x-x_0)^2 + \underbrace{f_{xy}}_{B = b}(x-x_0)(y-y_0) + \underbrace{f_{yy}}_{C = c}(y-y_0)^2$$

at critical point $f_x = 0$, $f_y = 0$.

→ So the general case reduces to the quadratic case.

→ In the degenerate case, what actually happens depends on the higher order derivates.

Example: $f(x,y) = x + y + \frac{1}{xy}$ $\quad xy > 0$ $\quad$ Min ? Max ?

Critical point $f_x = 1 - \frac{1}{x^2 y} = 0$ $\quad f_y = 1 - \frac{1}{xy^2} = 0$

$x^2 y = 1$ $\Rightarrow x = y$
$xy^2 = 1$ $\Rightarrow y^3 = 1 = y = 1, x = 1$

Only 1 solution
(1,1)
1 critical point

$f_{xx} = \frac{2}{x^3 y}$ (A) $\quad f_{xy} = \frac{1}{x^2 y^2}$ (B) $\quad f_{yy} = \frac{2}{xy^3}$ (C)

$AC - B^2 = \frac{2 \cdot 2 - 1}{x^4 y^4} = 3 > 0$,] local min.

Max, $f \rightarrow \infty$ when $x \rightarrow \infty$ or $y \rightarrow \infty$
or $x, y \rightarrow \infty$

5:30, 17 July — Multivariable Calculus - Lecture 11

* More tools to study functions.

→ **Differentiate :**

Implicit differentiation $\quad y = f(x)$ $\quad dy = f'(x)\,dx$

Example: $y = \sin^{-1}x$

$x = \sin y$

$dx = \cos y\, dy$

$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$ ✓

$\Rightarrow y = \sin^{-1}x \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Total differential : $f(x,y,z)$

$$df = f_x\,dx + f_y\,dy + f_z\,dz$$
$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

• Important : df is NOT $\underbrace{\Delta f}_{\text{number}}$

Can do : ① encode how changes in x, y, z affect f.

② placeholder for small variation $\Delta x, \Delta y, \Delta z$ to get an approx formula $\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$.

③ divide by something like dt to get ~~and~~ rate of change

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

where $x = x(t)$, $y = y(t)$, $z = z(t)$

⇑ Chain Rule

Why is this valid ?

1st attempt : $df = f_x\,dx + f_y\,dy + f_z\,dz$

$dx = x'(t)\,dt \quad dy = y'(t)\,dt \quad dz = z'(t)\,dt$

$df = f_x\, x'(t)\,dt + f_y\, y'(t)\,dt + f_z\, z'(t)\,dt$

Divide by dt ⇒ get chain rule.

Better : $\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$ (in time Δt)

$$\frac{\Delta f}{\Delta t} \approx f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + f_z \frac{\Delta z}{\Delta t}$$

when $\Delta t \to 0$ $\quad \frac{\Delta f}{\Delta t} \to \frac{df}{dt}, \quad \frac{\Delta x}{\Delta t} \to \frac{dx}{dt}, \ldots$

so $$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

Example: $w = x^2 y + z, \quad x = t, \; y = e^t, \quad z = \sin t$

Chain rule: $\dfrac{dw}{dt} = 2xy \dfrac{dx}{dt} + x^2 \dfrac{dy}{dt} + \dfrac{dz}{dt}$

$$\frac{dw}{dt} = 2te^t + t^2 e^t + \cos t$$

alternatively: $w = x^2 y + z = t^2 e^t + \sin t$

$w(t) = t^2 e^t + \sin t$

$$\frac{dw}{dt} = 2te^t + t^2 e^t + \cos t$$

Application: justify product rules
quotient

$f = uv \qquad u = u(t) \quad v = v(t)$

$$\frac{d(uv)}{dt} = f_u \frac{du}{dt} + f_v \frac{dv}{dt} = v\frac{du}{dt} + u\frac{dv}{dt}$$

$g = \dfrac{u}{v} \qquad u = u(t) \quad v = v(t)$

$$\frac{dg}{dt} = \frac{1}{v}\frac{du}{dt} + \left(\frac{-u}{v^2}\right)\frac{dv}{dt} = \frac{u'v - v'u}{v^2}$$

* Chain rule with more variable:

$w = f(x, y) \quad$ where $x = x(u, v), \; y = y(u, v)$

$= f(x(u,v), y(u,v))$

Question: $\dfrac{\partial w}{\partial u}, \dfrac{\partial w}{\partial v}$ in terms of $\dfrac{\partial w}{\partial x}, \dfrac{\partial w}{\partial y}, x_u, x_v, y_u, y_v$

$$dw = f_x\, dx + f_y\, dy$$

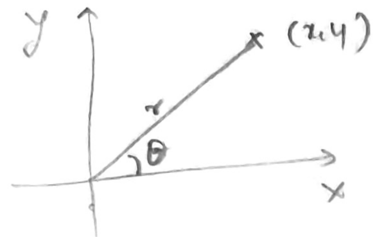
$$= f_x (x_u du + x_v dv) + f_y (y_u du + y_v dv)$$

$$= \underbrace{(f_x x_u + f_y y_u)}_{\frac{\partial f}{\partial u}} du + \underbrace{(f_y y_v + f_x x_v)}_{\frac{\partial f}{\partial v}} dv$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

Example: polar coordinates



$x = r\cos\theta$

$y = r\sin\theta$

$f = f(x,y)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$= f_x \cos\theta + f_y \sin\theta$$

$\boxed{18:00, 17 \text{ July}}$ Multivariable Calculus : Lecture 12

Recall : Chain Rule. $w = w(x,y,z)$

$x = x(t),\ y = y(t),\ z = z(t)$

$$\frac{dw}{dt} = w_x \frac{dx}{dt} + w_y \frac{dy}{dt} + w_z \frac{dz}{dt}$$

$$\boxed{\frac{dw}{dt} = \nabla w \cdot \frac{d\vec{r}}{dt}}$$

$$\boxed{\nabla w = \langle w_x, w_y, w_z \rangle}$$ Gradient of w at some point (x,y,z)

$$\frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

Theorem : $\nabla w \perp$ level surface $\{w = \text{constant}\}$

Example 1 : $w = a_1 x + a_2 y + a_3 z$

$\nabla w = \langle a_1, a_2, a_3 \rangle$ ✓

$= \frac{\partial w}{\partial x}\ \ \frac{\partial w}{\partial y}\ \ \frac{\partial w}{\partial z}$

level surface

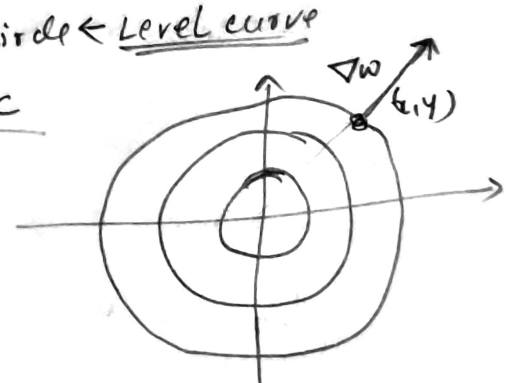
✓ $a_1 x + a_2 y + a_3 z = c$

plane with normal $\langle a_1, a_2, a_3 \rangle$

Example 2: $w = x^2 + y^2$

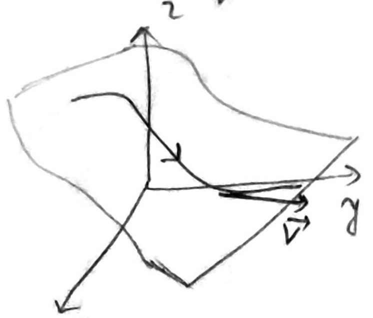
$w = c$ is a circle ← Level curve

$x^2 + y^2 = c$



Proof: Take a curve $\vec{r} = \vec{r}(t)$ that stays on the level $w = c$

velocity $\vec{v} = \frac{d\vec{r}}{dt}$ is tangent to the level $w = c$



By Chain Rule:

$$\frac{dw}{dt} = \nabla w \cdot \frac{d\vec{r}}{dt} = \nabla w \cdot \vec{v} = 0$$

since $w(t) = c$

So, $\nabla w \perp \vec{v}$. This is true for any motion on $w = c$

$\vec{v}$ can be any vector tanget to $w = c$

- Given any vector $\vec{v}$ tangent to level
  $$\nabla w \perp \vec{v}.$$
  So, $\nabla w \perp$ tanget to the level.

Exampl: find the tangent plane to surface $x^2+y^2-z^2=4$ at $(2,1,1)$?

Level set $w=4$, where $w=x^2+y^2-z^2$

gradient $\nabla w = \langle 2x, 2y, -2z\rangle$

$= \langle 4, 2, -2\rangle$

normal vector

equation is $4x+2y-2z=8$ ✓

Another way: $dw = 2x\,dx + 2y\,dy - 2z\,dz$

$= 4dx + 2dy - 2dz$

at $(2,1,1)$

$\Delta w \approx 4\Delta x + 2\Delta y - 2\Delta z$

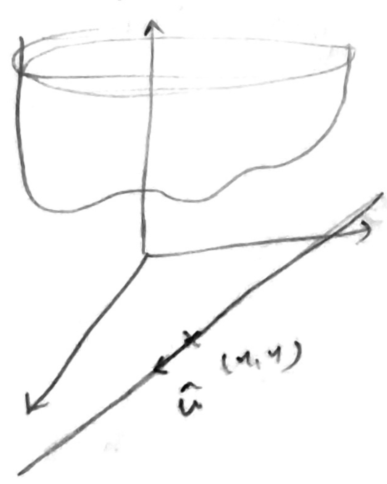
* So, $\Delta w = 0$ level. its tanget plane : $4\Delta x + 2\Delta y - 2\Delta z = 0$

$4(x-2)+2(y-1)-2(z-1)=0$ ✓

Directional derivatives.

$w = w(x,y) \rightsquigarrow$ know $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$ derivatives in direcⁿ of $\hat{i}$ or $\hat{j}$

what if we move in direction of $\hat{u}$ = unit vector?



straight line trajectory,

$\vec{r}(s)$, $\frac{d\vec{r}}{ds} = \hat{u}$

$\frac{\partial w}{\partial s} = ?$

If $\hat{u} = \langle a, b\rangle$

$$\begin{cases} x(s) = x_0 + as \\ y(s) = y_0 + bs \end{cases}$$

$\rightsquigarrow$ plug into w; $\frac{dw}{dw} = ?$

Defⁿ. $\boxed{\left.\frac{dw}{ds}\right|_{\hat{u}}}$

directional derivative in direcⁿ of $\hat{u}$

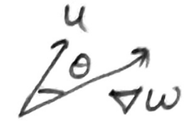
$\frac{dw}{ds}\Big|_{\hat{u}}$ = slope of slice of graph by a verticle plane $\parallel \hat{u}$.

Chain Rule : $\frac{dw}{ds} = \nabla w \cdot \frac{d\vec{r}}{ds} = \nabla w \cdot \hat{u}$

$$\boxed{\frac{dw}{ds}\Big|_{\hat{u}} = \nabla w \cdot \hat{u}}$$ component of gradient in direction of $\hat{u}$

Example: $\frac{dw}{ds}\Big|_{\hat{i}} = \nabla w \cdot \hat{i} = \frac{\partial w}{\partial x}$

Geometry : $\frac{dw}{ds}\Big|_{\hat{u}} = \nabla w \cdot \hat{u} = |\nabla w| \cancel{|\hat{u}|}^{1} \cos\theta$



→ maximal when $\cos\theta = 1 \Rightarrow \theta = 0°$, $\hat{u} = dir(\nabla w)$

So, direction of ∇w = dir of fastest increase of w.

$|\nabla w| = \frac{dw}{ds}\Big|_{\hat{u} = dir(\nabla w)}$

→ minimal $\frac{dw}{ds}\Big|_{\hat{u}}$ when $\cos\theta = -1$, $\theta = 180°$ $\hat{u}$ is in direc^n of $(-\nabla w)$

→ $\frac{dw}{ds}\Big|_{\hat{u}} = 0$ when $\cos\theta = 0$, $\theta = 90°$ $\hat{u} \perp \nabla w \Leftrightarrow \hat{u}$ tangent to level

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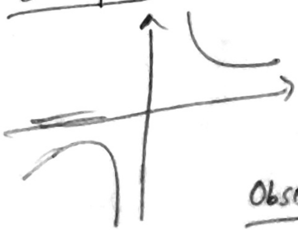
[00:16, 18th July] Multivariable Calculus : Lecture 13

* LAGRANGE MULTIPLIERS :

→ min/max a function $f(x,y,z)$ where x,y,z are not independent. $g(x,y,z) = c$

Example: point closest to the origin on hyperbola $xy = 3$



→ Minimize $f(x,y) = x^2+y^2$

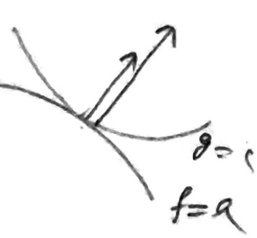
subject to the constraint $\underbrace{xy}_{g(x,y)} = 3$

Observe: ~~at~~ the minimum, level curve of F is tanget to hyperbola $g = 3$

level curve of f and g are tanget to each other!

→ How to find (x,y) where level curve of f and g are tanget to each other

when this happens, $\nabla f \parallel \nabla g$



so, $\boxed{\nabla f = \lambda \nabla g}$

λ → Lagrange multiplier

min/max.
2 variables x, y.
constraint: $g(x,y)=c$ → system of equations

$$\boxed{\nabla f = \lambda \nabla g} \quad \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases}$$

constraint $g = c$

$f = x^2 + y^2$

$g = xy$

$g = c$

$$\begin{cases} 2x = \lambda y \\ 2y = \lambda x \\ xy = 3 \end{cases} \qquad \left.\begin{matrix} 2x - \lambda y = 0 \\ \lambda x - 2y = 0 \\ xy = 3 \end{matrix}\right\} \quad \overset{M}{\begin{bmatrix} 2 & -\lambda \\ \lambda & -2 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Trivial solution $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ does not solve the constraint equation

Other solution exist only if $\det(M) = 0$.

$$\begin{vmatrix} 2 & -\lambda \\ \lambda & -2 \end{vmatrix} = -4 + \lambda^2 = 0 \qquad \lambda^2 = 4 \Rightarrow \boxed{\lambda = \pm 2}$$

$\lambda = 2 \Rightarrow x = y$, $x^2 = 3$

$(x, y) = (\sqrt{3}, \sqrt{3})\ (-\sqrt{3}, -\sqrt{3})$

$\lambda = -2$: $x = -y$, $-x^2 = 3$, no solution

→ **Why is this method valid?**

At constrainted min/max, in any direction along level $g=c$ the rate of change of f must be 0.

For any $\hat{u}$ tanget to $g=c$, we must have $\frac{df}{ds}\Big|_{\hat{u}} = 0$

$$\frac{df}{ds}\Big|_{\hat{u}} = \nabla f \cdot \hat{u} = 0$$

So any such $\hat{u}$ is $\perp \nabla f$, So $\nabla f \perp$ level set of g

Know that $\nabla g \perp$ level set of g

So $\nabla f \parallel \nabla g$

Warning: method doesn't tell whether a solution is min. or max!

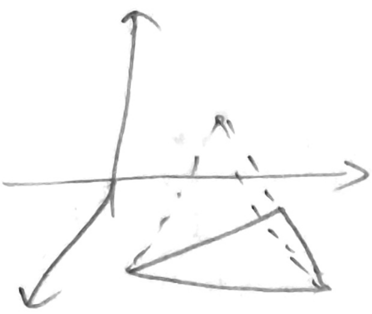
Can't use second derivatives.

* To find the min (or max), we compare values of f at various sol^{ns} to Lagrange equations.

Advanced Example :

Want to build a pyramid with given triangular base & given volume

minimize total surface area.



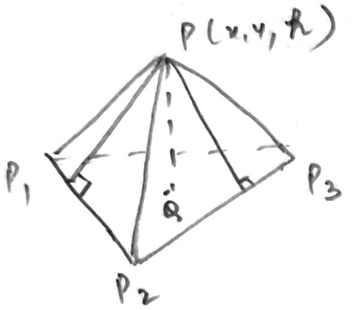
Vol $= \frac{1}{3}$ Area (base) . height

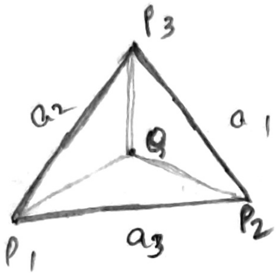
$\hookrightarrow$ fixed ! $= h$

Could do : $P_1(x_1, y_1, 0)$ $P_2(x_2, y_2, 0)$ $P_3(x_3, y_3, 0)$

$P(x, y, h)$ $\rightsquigarrow$ Area (x, y)

can't solve min





Let u_1, u_2, u_3 = dist from Q to sides

height of faces $= \sqrt{u_1^2 + h^2}$

side area $= \frac{1}{2} a_1 \sqrt{u_1^2 + h^2} + \frac{1}{2} a_2 \sqrt{u_2^2 + h^2} + \frac{1}{2} a_3 \sqrt{u_3^2 + h^2}$
(f)

Cut the base into 3 $\Rightarrow$ Area (base) $= \frac{1}{2} a_1 u_1 + \frac{1}{2} a_2 u_2 + \frac{1}{2} a_3 u_3$
(g)

$\nabla f = \lambda \nabla g$

$$\frac{\partial f}{\partial u_1} = \cancel{\tfrac{1}{2}} \cancel{a_1} \frac{u_1}{\sqrt{u_1^2 + h^2}} = \lambda \cancel{\tfrac{1}{2}} \cancel{a_1}$$

$$\frac{\partial f}{\partial u_2} = \cancel{\tfrac{1}{2}} \cancel{a_2} \frac{u_2}{\sqrt{u_2^2 + h^2}} = \lambda \cancel{\tfrac{1}{2}} \cancel{a_2}$$

$\boxed{u_1 = u_2 = u_3}$ $\Leftarrow$

$$\frac{\partial f}{\partial u_3} = \frac{u_3}{\sqrt{u_3^2 + h^2}} = \lambda$$

Q = incenter

02:40 , 18 July — Multivariable Calculus - Lecture 14

Non- independent variables :

$f(x, y, z)$ where $g(x, y, z) = c$

$\rightarrow$ If x, y, z are ~~both~~ related , $g(x, y, z) = c$ then $z = z(x, y)$

How to find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$?

Example : $\underbrace{x^2 + yz + z^3}_{g} = 8$ at $(2, 3, 1)$

Take differential $\Rightarrow$ $2x\,dx + z\,dy + (y\,dz + 3z^2)\,dz = 0$ because $g = 8$, $dg = 0$

at $(2, 3, 1)$ $4dx + dy + 6dz = 0$

If we view $z = z(x,y)$

$dz = -\frac{1}{6}(4dx + dy)$.

$\underbrace{\frac{\partial z}{\partial x} = -\frac{2}{3}}$ $\underbrace{\frac{\partial z}{\partial y} = -\frac{1}{6}}$

y constant, Set $dy = 0$ — x constant, set $dx = 0$

In general,

$g(x,y,z) = c$, then

$dg = g_x dx + g_y dy + g_z dz = 0$

Solve for dz:

$$dz = -\frac{g_x}{g_z}dx - \frac{g_y}{g_z}dy$$

So, for $\frac{dz}{dx}$: set y = constant $\Rightarrow dy = 0$

$$dz = -\frac{g_x}{g_z}dx \qquad \boxed{\frac{dz}{dx} = -\frac{g_x}{g_z}} \qquad \boxed{\frac{dz}{dy} = -\frac{g_y}{g_z}}$$

* Say $f(x,y) = x + y$ $\quad \frac{\partial f}{\partial x} = 1$

$x = u$
$y = u + v$; then $f(u,v) = x + y = 2u + v \Rightarrow \frac{\partial f}{\partial u} = 2$

So, $x = u$ but $\frac{\partial f}{\partial x} \neq \frac{\partial f}{\partial u}$

change $u = x$
Keep y constant

change $u = x$
keep $v = y - x$ constant.

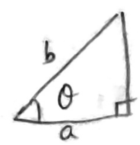
* Need clearer notation: $\left(\frac{\partial f}{\partial x}\right)_y$ = Keep y constant $\left(\frac{\partial f}{\partial y}\right)_x$ = Keep x constant

Example:

Area $= \frac{1}{2} ab \sin\theta$

$A = \frac{1}{2} ab \sin\theta$

Assume it's a right triangle



$\underbrace{a = b\cos\theta}_{\text{constraint}}$

Rate of change of A wrt θ?

① treat a, b, θ as independent $\frac{\partial A}{\partial \theta} = \left(\frac{\partial A}{\partial \theta}\right)_{a,b}$

Keep a, b fixed. $\frac{\partial A}{\partial \theta} = \frac{1}{2} ab \cos\theta$

2) Keep a constant, b will change : $b = b(a,\theta) = \left(b = \frac{a}{\cos\theta}\right)$

so that ~~so that~~ we keep a right angle:

$$\left(\frac{\partial A}{\partial \theta}\right)_a$$

3) Keep b costant, $a = a(b, \theta)$ $\left(\frac{\partial A}{\partial \theta}\right)_b$

Compute $\left(\frac{\partial A}{\partial \theta}\right)_a$?

Method 0: solve for b & substitute:

$a = b\cos\theta \quad b = \frac{a}{\cos\theta} \quad A = \frac{1}{2} ab \sin\theta = \frac{1}{2} a \cdot \frac{a}{\cos\theta} \sin\theta = \frac{a^2}{2} \frac{\sin\theta}{\cos\theta}$

$A = \frac{1}{2} a^2 \tan\theta$

$\left(\frac{\partial A}{\partial \theta}\right)_a = \frac{1}{2} a^2 \sec^2\theta$

2 systematic methods

1) differentials:
- keep a fixed $da = 0$
- contraint $a = b\cos\theta$

$da = \cos\theta \, db - b\sin\theta \, d\theta$

$0 = da = \cos\theta \, db - b\sin\theta \, d\theta$

so $\cos\theta \, db = b\sin\theta \, d\theta$

$db = b\tan\theta \, d\theta$

function: $A = \frac{1}{2} ab\sin\theta$

$dA = \frac{1}{2} b\sin\theta \, \overset{0}{da} + \frac{1}{2} a\sin\theta \, db + \frac{1}{2} ab\cos\theta \, d\theta$

so,

$dA = \frac{1}{2} a \sin\theta \, (b\tan\theta) \, d\theta + \frac{1}{2} ab\cos\theta \, d\theta$

$dA = \frac{1}{2} ab \underbrace{(\sin\theta\tan\theta + \cos\theta)}_{\sec\theta} d\theta \Rightarrow \left(\frac{\partial A}{\partial \theta}\right)_a = \frac{1}{2} ab\sec\theta$

Summary:
1) write dA in terms of $da, db, d\theta$
2) $a = $ const $\Rightarrow$ set $da = 0$
3) differentiat constraint $\Rightarrow$ solve for db in term of $d\theta$

plug into dA, get answer

2) Chain Rule: $\left(\frac{\partial}{\partial \theta}\right)_a$ in formula for A.

$\left(\frac{\partial A}{\partial \theta}\right)_a = A_\theta \left(\frac{\partial \theta}{\partial \theta}\right)_a + A_a \left(\frac{\partial a}{\partial \theta}\right)_a + A_b \left(\frac{\partial b}{\partial \theta}\right)_a$

$\left(\frac{\partial \theta}{\partial \theta}\right)_a \to 1$

$\left(\frac{\partial a}{\partial \theta}\right)_a \Rightarrow 0$ since $a = $ const

use the constraint

14:20, 21 July 2020

MIT- 18.02 Multivariate Calculus - Lecture 15

Summary: Topics :

① Functions of several variables - graphs & contour plots

② Partial derivatives $f_x = \frac{\partial f}{\partial x}$

Gradient vector : $\nabla f = \langle f_x, f_y, f_z \rangle$

Approximation formula = $\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \cdot \Delta z$

$= \nabla f \cdot \Delta \vec{r}$

Tangent plane approximation.

Tangent plane to a surface $f(x,y,z) = c$. can be found

Normal vector = ∇f

* Partial differential equation : (equations involving partial derivative of an unknown function).

eg: Heat equation : $\frac{\partial f}{\partial t} = k\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)$ $f(x,y,z,t)$ = temperature at (x,y,z) at time t

- Min/Max problems : critical point : when all partial derivative = 0

second derivatives test for a function of 2 variables to decide between min / max or saddle point

- Least-squared approximation
- Differentials: $df = f_x\, dx + f_y\, dy + f_z\, dz$

chain rule.

ex if $x = x(u,v)$ $\quad \frac{\partial f}{\partial u} = f_x \frac{\partial x}{\partial x} + \frac{\partial s}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$

$y = y(u,v)$

$z = z(u,v)$

- Non-independent variables : $g(x,y,z) = c$

→ min/max problems : Lagrange multipliers.

min/max of f subject to the constraint $g=c$.

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \end{cases} + \quad g = c$$

4 eqn

3 variable

→ constraint partial derivatives

$f(x,y,z)$ where $g(x,y,z)=c$

To find $\left(\frac{\partial f}{\partial z}\right)_y$ ← y is constant, z varies, $x = x(y,z)$

Rate of change of f w.r.t z?

Method 1 using differentials:

$$df = f_x\,dx + \cancel{f_y\,dy}^{\to 0} + f_z\,dz$$

y = constant $dy \to 0$

need: dx in terms of dz?

$$dg = g_x dx + \cancel{g_y dy}^{0} + g_z dz = 0$$

$(g=c)$

$$g_x\,dx + g_z\,dz = 0$$

$$\boxed{dx = -\frac{g_z}{g_x}\,dz} \Rightarrow \left(\frac{\partial x}{\partial z}\right)_y = \frac{-g_z}{g_x}$$

$$df = f_x \frac{-g_z}{g_x}\,dz + f_z\,dz$$

$$\boxed{df = \left(-f_x \frac{g_z}{g_x} + f_z\right) dz}$$

$$\left(-f_x \frac{g_z}{g_x} + f_z\right) = \left(\frac{\partial f}{\partial z}\right)_y$$

$$\boxed{\left(\frac{\partial f}{\partial z}\right)_y = -f_x \frac{g_z}{g_x} + f_z}$$

Method - 2

using chain rule:

$$\left(\frac{\partial f}{\partial z}\right)_y = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial z}\right)_y + \frac{\partial f}{\partial y}\cancel{\left(\frac{\partial y}{\partial z}\right)_y}^{\,0} + \frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial z}\right)_y^{=1}$$

to find this, use constraint

$$0 = \left(\frac{\partial g}{\partial z}\right)_y = \frac{\partial g}{\partial x}\left(\frac{\partial x}{\partial z}\right)_y + \frac{\partial g}{\partial y}\cancel{\left(\frac{\partial y}{\partial x}\right)_y}^{\,0} + \frac{\partial g}{\partial z}\left(\frac{\partial z}{\partial z}\right)_y^{=1}$$

$$0 = g_x\left(\frac{\partial x}{\partial z}\right)_y + g_z \Rightarrow \boxed{\left(\frac{\partial x}{\partial z}\right)_y = -\frac{g_z}{g_x}}$$

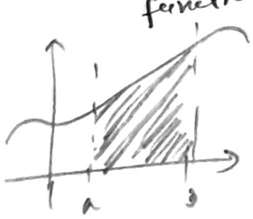
$$\boxed{\left(\frac{\partial f}{\partial z}\right)_y = f_x\left(-\frac{g_z}{g_x}\right) + f_z}$$

→ Directional Derivatives : $\boxed{\frac{df}{ds}\Big|_{\hat{u}} = \nabla f \cdot \hat{u}}$

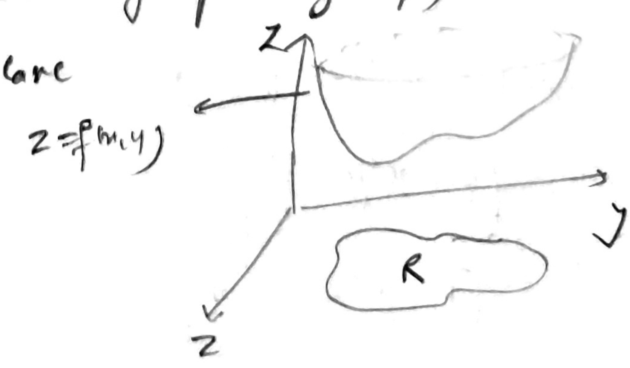
15:00, 21 July 2020 MIT - 18.02 - Multivariable Calculus [Lecture 16]

Double Integrals :

function of 1 variable : $\int_a^b f(x)\,dx$ = area below graph of f over $[a,b]$

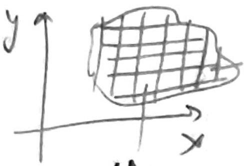


→ Double Integral = volume below graph $f(x,y)$ over a region R in x-y plane



$$= \iint_R f(x,y)\, \partial A$$

Definition : cut R into small pieces of area ΔA_i



ΔA_i

(x_i, y_i)

$$\left(\sum_i f(x_i, y_i)\, \Delta A_i \right)$$

Take the limit as $\Delta A_i = 0$ get $\iint$

→ To compute $\iint_R f(x,y)\,dA$: take slices.

Let $S(x)$ = area of slice by plane parallel // y-z plane

then volume $= \displaystyle\int_{x_{min}}^{x_{max}} S(x)\, dx$

For a given x, $\quad S(x) = \displaystyle\int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\, dy$

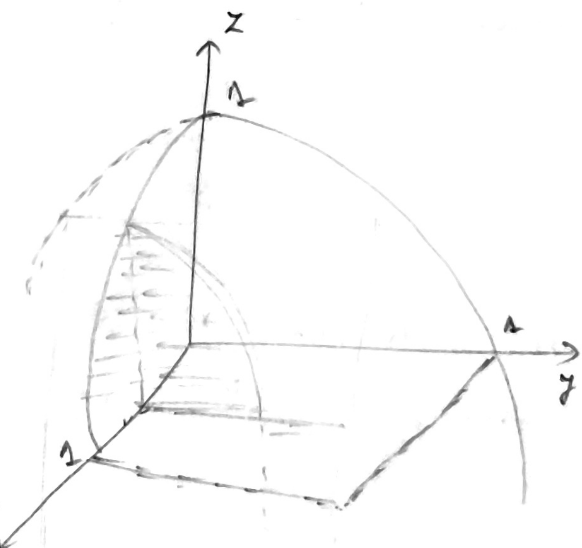
$$\Rightarrow \quad \iint_R f(x,y)\, dA = \int_{x_{max}}^{x_{min}} \left[\int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\, dy \right] dx$$

ITERATED INTEGRAL

Example: 1

$z = 1 - x^2 - y^2$

region : $0 \le x \le 1$
$\quad 0 \le y \le 1$



$$= \int_0^1 \int_0^1 (1 - x^2 - y^2)\, dy\, dx$$

1) Inner integral : $\displaystyle\int_0^1 (1 - x^2 - y^2)\, dy$

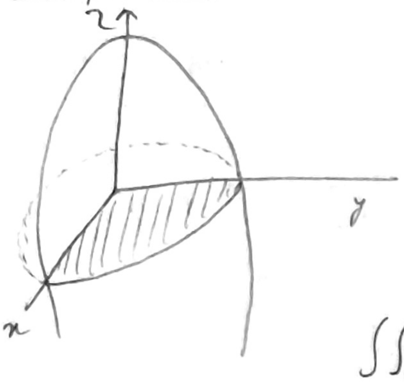
$$= \left[y - x^2 y - \frac{y^3}{3} \right]_0^1$$

$$= \left[1 - x^2 - \frac{1}{3} - 0 \right] = \frac{2}{3} - x^2$$

2) Outer Integral : $\displaystyle\int_0^1 \left(\frac{2}{3} - x^2 \right) dx = \left[\frac{2x}{3} - \frac{x^3}{3} \right]_0^1 = \frac{1}{3}$

$*\ dA = dy\,dx$ because $\Delta A = \Delta x\,\Delta y$ for small rectangles

Example 2 : Same function



R = quarter-disk
$x^2+y \le 1$
$x \ge 0$ & $y \ge 0$

For given x, the range of y is from $y=0$ to $\sqrt{1-x^2}$

$$\iint_R 1-x^2-y^2\,dA = \int_0^1 \int_0^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy\,dx$$

$$= \int_0^1 \left[(1-x^2)y - \frac{y^3}{3}\right]_0^{\sqrt{1-x^2}} dx$$

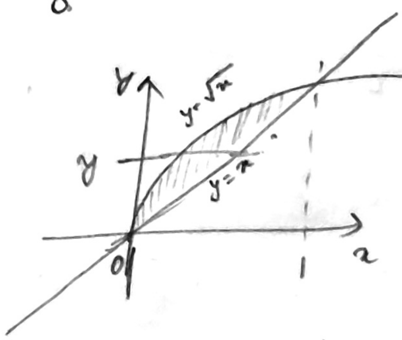
$$= \int_0^1 \left[(1-x^2)\sqrt{1-x^2} - \frac{(\sqrt{1-x^2})^3}{3}\right] dx$$

$$= \int_0^1 \left[(1-x^2)^{3/2} - \frac{(1-x^2)^{3/2}}{3}\right] dx$$

$$= \frac{2}{3} \int_0^1 (1-x^2)^{3/2}\,dx$$

$$= \frac{\pi}{8}$$

Example 3 : $\displaystyle\int_0^1 \int_x^{\sqrt{x}} \frac{e^y}{y}\,dy\,dx$



We have to reverse the order of integral for easier calculation

$$= \int_0^1 \int_{y^2}^{y} \frac{e^y}{y}\,dx\,dy = \int_0^1 e^y - y e^y\,dy = \left[-ye^y + 2e^y\right]_0^1$$

$$= e - 2.$$

=

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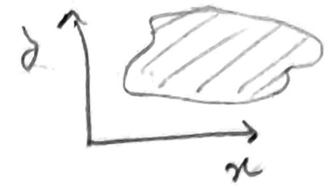
→ Double Integrals

$$\iint_R f(x,y)\,dA = \iint f(x,y)\,dy\,dx$$



example:

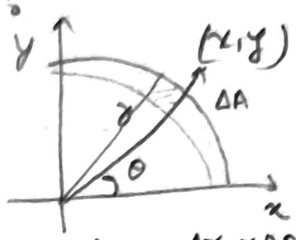
$$\iint (1-x^2-y^2)\,dA \quad \text{in polar coordinates?}$$

$x^2+y^2 \le 1$

$x, y \ge 0.$

$x = r\cos\theta$

$y = r\sin\theta$



$\Delta A \approx \Delta r \, r \Delta\theta$

$\boxed{dA = r\,dr\,d\theta}$

in polar coordinates

$$\int_0^{\pi/2}\int_0^1 f(r,\theta)\, r\,dr\,d\theta$$

$$f = 1-x^2-y^2 = 1-(x^2+y^2) = 1-r^2$$

$$\int_0^{\pi/2}\int_0^1 (1-r^2)\, r\,dr\,d\theta = \int_0^{\pi/2} \left(\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4}\,d\theta = \frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{8} \checkmark$$

Applications :

1) Find the area of region R.

$$\text{Area}(R) = \iint_R 1\,dA$$

2) Find the average value of f in Region(R).

$$\text{Average of } f = \bar{f} = \frac{1}{\text{Area}(R)}\iint_R f\,dA = \frac{1}{\text{Area}(R)}\iint_R f\,dA$$

Weighted Average of f: $\frac{1}{\text{Mass}(R)} \iint_R f\,\delta\,dA$

with desity δ

$\left\{ \text{Mass} = \iint_R \delta\,dA \right\}$

* Center of mass of a (planar) objects (with density δ)

$(\bar{x}, \bar{y})$ $\quad \bar{x} = \frac{1}{\text{Mass}} \iint x\,\delta\,dA$

$\bar{y} = \frac{1}{\text{Mass}} \iint y\,\delta\,dA$