14-07-2020

Coursera:

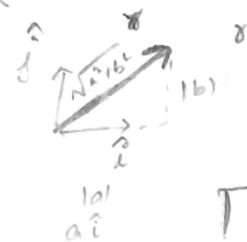
Mathematics for Machine Learning: Linear Algebra

$$2a + 3b = 8$$
$$10a + b = 13$$
$$\Rightarrow \begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \end{bmatrix}$$

Matrix   vector

linear algebra problem

* Modulus & inner product:

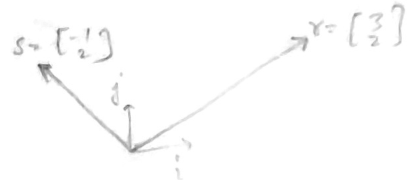


$r = a\hat{i} + b\hat{j} = \begin{bmatrix} a \\ b \end{bmatrix}$

$r = \begin{vmatrix} a \\ b \end{vmatrix}$

$$\boxed{|r| = \sqrt{a^2+b^2}}$$

$s = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad r = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$



$\boxed{r \cdot s = r_i \cdot s_i + r_j s_j}$

$= 3 \times -1 + 2 \times 2$

$r \cdot s = 1$

Properties of **Inner product** (dot product):

- commutative $\Rightarrow \boxed{r \cdot s = s \cdot r}$
- distributive over addition $\Rightarrow \boxed{r \cdot (s+t) = r \cdot s + r \cdot t}$
- associative over scalar multiplication $\Rightarrow \boxed{r \cdot (as) = a(r \cdot s)}$ (scalar, scalar)

Relationship between Modulus & innerproduct:

$$r \cdot r = r_i \cdot r_i + r_j \cdot r_j = r_i^2 + r_j^2 = |r|^2$$

$$\boxed{r \cdot r = |r|^2}$$

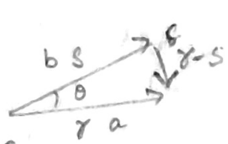
$$\boxed{r \cdot s = |r||s| \cos\theta}$$

θ → angle between the vectors

Cosine & dot product:

cosine rule:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



$$|r-s|^2 = |r|^2 + |s|^2 - 2|r||s|\cos\theta$$

$$(r-s)\cdot(r-s) = |r|^2 + |s|^2 - 2|r||s|\cos\theta$$

$$|r|^2 + |s|^2 - 2r\cdot s = |r|^2 + |s|^2 - 2|r||s|\cos\theta$$

**

$$\boxed{r \cdot s = |r||s|\cos\theta}$$

$\theta = 0 \Rightarrow r \cdot s = |r||s|$, $\theta = 90^\circ \Rightarrow r \cdot s = 0$

$\theta = 180^\circ \Rightarrow r \cdot s = -|r||s|$

* Projection

$$r \cdot s = |r| \underbrace{|s| \cos\theta}_{\text{projection of } s \text{ on } r}$$

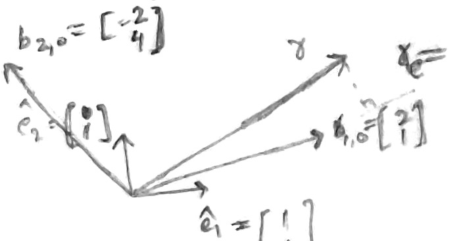
$$= |r| * \text{projection}$$

$$\boxed{\text{scalar projection} = |s|\cos\theta = \frac{r \cdot s}{|r|}}$$

$$\boxed{\text{vector projection } r = \frac{r \cdot s}{|r|\,|r|}\, r = \frac{r \cdot s}{r \cdot r}\, r = (r \cdot s)\, \frac{r}{r \cdot r}}$$

unit vector in r

* Changing the reference frame:



$$r_e = 3\hat{e}_1 + 4\hat{e}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad r_b = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

Basis vectors (explained on next page)

$$\cos\theta = \frac{b_1 \cdot b_2}{|b_1|\,|b_2|} = \frac{-4+4}{} = 0$$

$$\theta = 90^\circ$$

scalar projection of r on b_1 $\rightarrow$ $\frac{r_e \cdot b_1}{|b_1|^2} = \frac{3 \times 2 + 4 \times 1}{2^2 + 1^2} = \frac{10}{5} = 2$

$$\frac{r_e \cdot b_1}{|b_1|^2}\, b_1 = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

scalar projection of r on b_2 $\quad \frac{r_e \cdot b_2}{|b_2|^2} = \frac{-6+16}{4+16} = \frac{10}{20} = \frac{1}{2}$

$$\frac{r_e \cdot b_2}{|b_2|^2}\, b_2 = \frac{1}{2} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

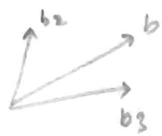
$$r_b = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix} = 2b_1 + \frac{1}{2} b_2$$

Basis, vector space & linear independence:

⇒ Basis is a set of n vectors that:

* are not linear combinations of each other (linearly independent)
* span the space.
* The space is then n-dimensional

$\boxed{\text{Linearly independent}}$ vectors ⇒ impossible to represent a vector as combination of other



$b_3 \neq a_1 b_1 + a_2 b_2$

↑ linearly independent (impossible to find a_1 & a_2)

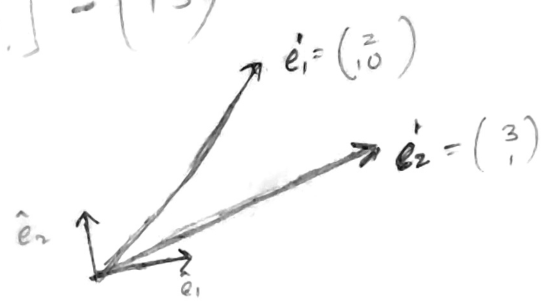
* Matrices, vectors & solving simultaneous equation problems

$2a + 3b = 8$
$10a + 1b = 13$

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

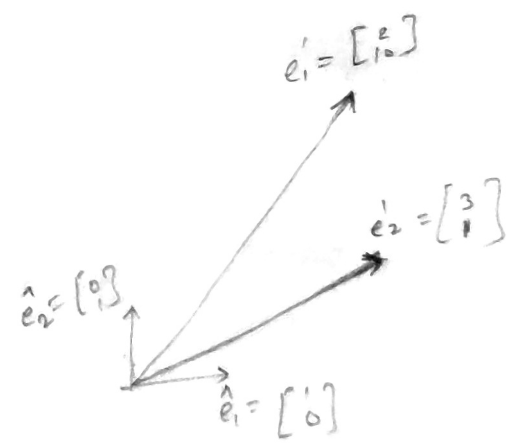
$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



* How matrices transform space

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$



$$\boxed{A r = r'}$$

$$A(nr) = nr'$$
$$A(r+s) = Ar + As$$
$$A(n\hat{e}_1 + m\hat{e}_2) = nA\hat{e}_1 + mA\hat{e}_2$$
$$= n\hat{e}_1' + m\hat{e}_2'$$

$$\begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{pmatrix} 12 \\ 32 \end{pmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix} \left(3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 3\left(\begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 2\left(\begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

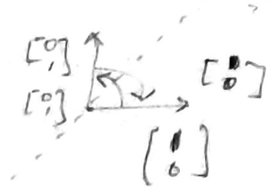
$$= 3\begin{bmatrix} 2 \\ 10 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 32 \end{bmatrix}$$

Type of Matrix Transformation:

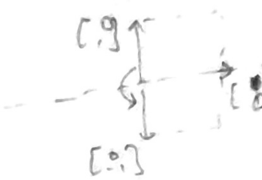
$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$$

I
(Identity Matrix)

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}y\\x\end{pmatrix}$$

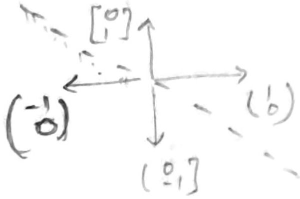


$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\-y\end{pmatrix}$$

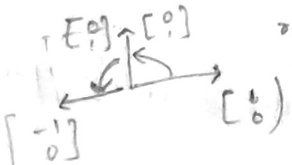


mirror on x-axis

$$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-y\\-x\end{pmatrix}$$

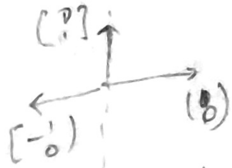


$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-y\\x\end{pmatrix}$$



rotation 90° anticlock

$$\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-x\\y\end{pmatrix}$$



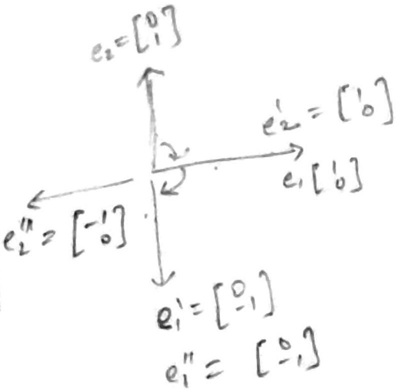
mirror transfor in y-axis

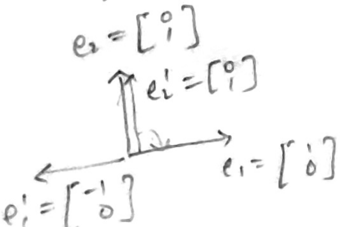
General expression for rotation in 2D $= \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$

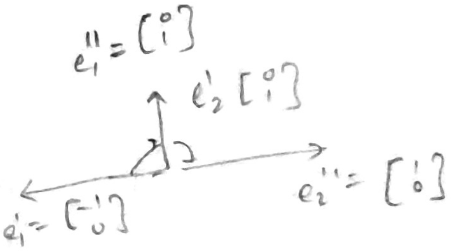
Composition or combination of matrix transformation:

$A_2(A_1 r)$ — first transformation, second transformation

$A_1=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ $A_2=\begin{bmatrix}-1&0\\0&1\end{bmatrix}$







$$A_2A_1=\begin{bmatrix}0&-1\\-1&0\end{bmatrix}$$

$$A_2A_1=\begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}=\begin{bmatrix}0&-1\\-1&0\end{bmatrix}$$

$$A_1A_2=\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}-1&0\\0&1\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

* Matrices multiplications is <u>not commutative</u>. $A_2A_1 \neq A_1A_2$
* $A_3(A_2A_1)=(A_3A_2)A_1$ they are <u>associative</u>.

Gaussian elimination :

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

$A \quad \gamma \quad = \quad S$

$\boxed{A^{-1} A = I}$ **

Inverse　　Identity

$A\gamma = S$

$A^{-1} A\gamma = A^{-1}S$

$I\gamma = A^{-1}S$

$\boxed{\gamma = A^{-1}S}$

example

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ 21 \\ 13 \end{pmatrix}$$

substract row1 of row2

substract row1 of row3

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ 6 \\ -2 \end{pmatrix}$$

multiple row3 ×(-1)

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ 6 \\ 2 \end{pmatrix}$$

elimation

back substitution

row1 − 3 row3

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \\ 2 \end{pmatrix}$$

row1 − row2

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$$

I

Identity

Gaussian elimation to finding inverse matrix :

$\boxed{A^{-1} A = I}$

$A\gamma = S$

$A^{-1} A\gamma = A^{-1}S$

$I\gamma = A^{-1}S$

$\boxed{\gamma = A^{-1}S}$

inverse of A

$A \quad B = I \quad \{B = A^{-1}\}$

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = I$$

$$\begin{pmatrix}1&1&3\\1&2&4\\1&1&2\end{pmatrix}\begin{pmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&3\\0&1&1\\0&0&+1\end{pmatrix}B=\begin{pmatrix}1&0&0\\-1&1&0\\+1&0&-1\end{pmatrix}$$

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}B=\begin{pmatrix}-2&0&3\\-2&1&1\\1&0&-1\end{pmatrix}$$

$$\boxed{IB=B}$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}B=\begin{pmatrix}0&-1&2\\-2&1&1\\1&0&-1\end{pmatrix}$$

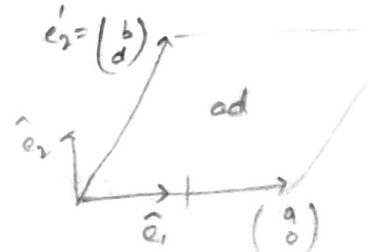
$$IB=\begin{pmatrix}0&-1&2\\-2&1&1\\1&0&-1\end{pmatrix}$$

Determinants & inverses :

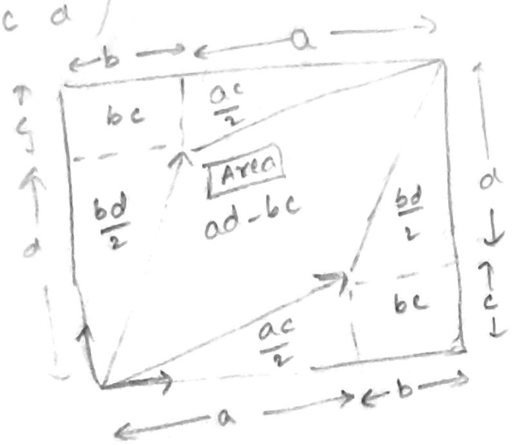
$\begin{pmatrix}a&0\\0&d\end{pmatrix}$ $e_2'\begin{pmatrix}0\\d\end{pmatrix}$

determinant =

$\begin{pmatrix}a&b\\0&d\end{pmatrix}$



$$A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$



Area = $(a+b)(c+d) - ac - bd - 2bc$

$$\boxed{|A| = ad-bc}$$

Determinant

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}=\begin{pmatrix}ad-bc&0\\0&ad-bc\end{pmatrix}$$

$$\frac{1}{ad-bc}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

A

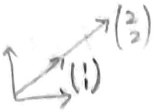
A'

$$\boxed{A^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}}$$

$AA' = I$

Look up QR decomposition.

$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$



$|A| = 0$

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 2 & 3 & 7 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 17 \\ 29 \end{pmatrix}$$

row 3 = row 1 + row 2

col 3 = 2 × col 1 + col 2

row 2 − row 1

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \\ 0 \end{pmatrix}$$

$0c = 0$

* Inverse does not exist for linearly dependent matrices.

Introduct: Einstein summation convention & the symmetry of the dot product:

$$\underset{A}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}} \underset{B}{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & & & \\ b_{n1} & & & b_{nn} \end{pmatrix}} = \underset{AB}{\begin{pmatrix} & & \\ & a_{ik} & \\ & & \end{pmatrix}}$$

row column

$(ab)_{23} = a_{21}b_{13} + a_{22}b_{23} + \dots + a_{2n}b_{n3}$

summation convention

$$\boxed{ab_{ik} = \sum_j a_{ij} b_{jk} = a_{ij} b_{jk}}$$

$2\times3,\ 3\times2$

$a_{13} = a_{11} \cdot b_{13} + a_{12} b_{23} + a_{13} b_{33}$

$$2\overset{3}{\begin{pmatrix} \cdot & \cdot \\ & \end{pmatrix}} \; 3\overset{4}{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ & & & \end{pmatrix}} = 2\overset{4}{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}}$$

$2 \times \boxed{3 \quad 3} \times 4 \rightarrow 2 \times 4$

same

Resulting matrix dimension

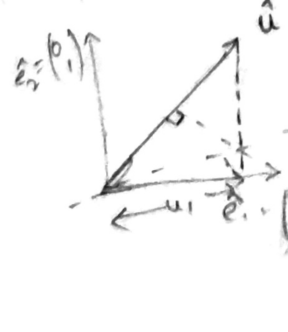
$u \quad v$

$\begin{pmatrix} u_i \end{pmatrix} \quad \begin{pmatrix} v_i \end{pmatrix}$

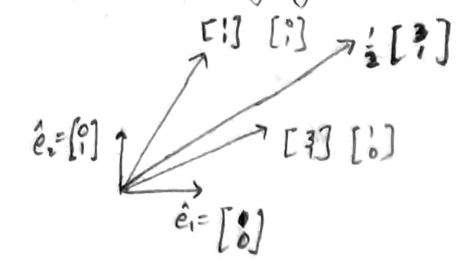
$\underline{u_i v_i}$

$[u_1\ u_2\ \dots\ u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

$\hat{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$



Matrices changing basis.



Basis vector $= [\,^3_1]\,[\,^1_1]$ in my frame $[\,^1_0][\,^0_1]$

Transformation matrix $= \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ ✓

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \text{ in my frame}$$

⇓ in basis vector $[\,^3_1][\,^1_1]$ → in basis vector $[\,^1_0][\,^0_1]$

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

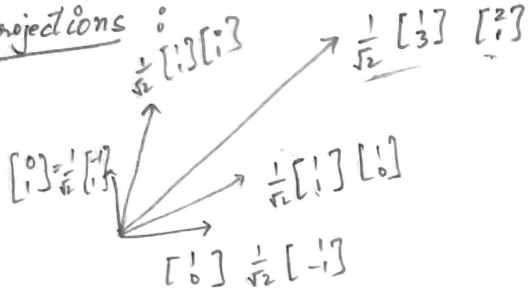
Bear's basis in my frame — Bear's vector — my vector

$B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ $B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$ my basis in Bear's world

$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$ ✓

Projections :



$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \times 4 = 2$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2} \times 2 = 1$$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Only happens becoz they are orthogonal.

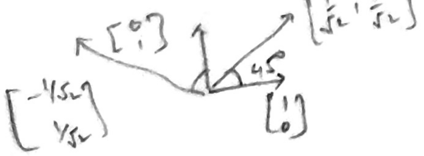
Doing a transformation in changed basis :

B

$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$

$$\left[B^{-1} \left(\underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}}_{\text{rotated } 45^\circ \text{ in my word.}} \right) \right]$$

rotated 45° in bear's world.



Transform for 45° rotation in my world $= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$B^{-1} R B$ = Rotation in bear's system (R_B)

$$\boxed{R_B = B^{-1} R B}$$

R_B: Rotation in bear's world

B^{-1}: inverse of B

R: Rotation in my world

B: Basis of bear in my world

Orthogonal Matrices :

Transpose : $\boxed{A^T_{ij} = A_{ji}}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$A_{n\times n}$ = orthogonal matrix

$$\begin{pmatrix} \begin{pmatrix} a_1 \end{pmatrix} & \begin{pmatrix} a_2 \end{pmatrix} & \cdots & \begin{pmatrix} a_n \end{pmatrix} \end{pmatrix}$$

where $a_i \cdot a_j = 0 \quad i \neq j$; $a_i \cdot a_j = 1 \quad i = j$ } orthonormal basis set

$$A^T = \begin{pmatrix} (- - - a_1 &) \\ (\quad a_2 &) \\ \vdots & \\ (\quad a_n &) \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I$$

Identity

$\boxed{A^T A = A^{-1} A = I}$

only for orthogonal matrix

Orthogonal Matrix basis set $\boxed{A^T = A^{-1}}$

The Gram-Schmidt process : (to create orthogonal matrices)

$v = \{v_1, v_2 \ldots v_n\}$ set of vectors such that they span the space

linear independent

$v_1 \rightarrow \boxed{e_1 = \frac{v_1}{|v_1|}}$

$v_2 = (v_2 \cdot e_1)\, e_1 + u_2$

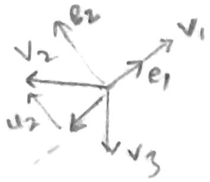
$\boxed{u_2 = v_2 - (v_2 \cdot e_1)\, e_1}$

$\boxed{\frac{u_2}{|u_2|} = e_2}$

$u_3 = v_3 - (v_3 \cdot e_1)\, e_1 - (v_3 \cdot e_2)\, e_2$

$\boxed{e_3 = \frac{u_3}{|u_3|}}$

normal to the plane e_1, e_2



Example : Reflecting in a Plane :

$$\underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{v_1} \quad \underbrace{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}}_{v_2} \quad \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}_{v_3}$$

plane of the mirror

$$e_1 = \frac{v_1}{|v_1|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad e_2 = \frac{u_2}{|u_2|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$u_2 = v_2 - (v_2 \cdot e_1)\, e_1$$

$$= \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right] \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\cancel{3}} \times \cancel{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

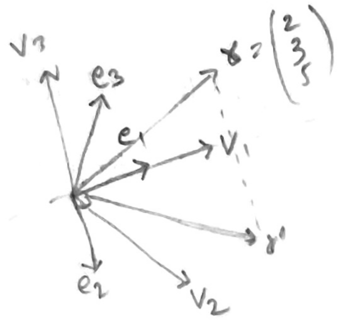
$$u_3 = v_3 - (v_3 \cdot e_1)e_1 - (v_3 \cdot e_2)e_2$$

$$= \begin{pmatrix}3\\1\\-1\end{pmatrix} - \begin{pmatrix}3\\1\\-1\end{pmatrix}\cdot\frac{1}{\sqrt3}\begin{pmatrix}1\\1\\1\end{pmatrix}\frac{1}{\sqrt3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}3\\1\\-1\end{pmatrix}\cdot\frac{1}{\sqrt2}\begin{pmatrix}1\\-1\\0\end{pmatrix}\frac{1}{\sqrt2}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$= \begin{pmatrix}3\\1\\-1\end{pmatrix} - \frac{1}{3}\times 3\times\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{2}\times 2\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\-2\end{pmatrix} \checkmark$$

$$e_3 = \frac{u_3}{|u_3|} = \frac{1}{\sqrt6}\begin{pmatrix}1\\1\\-2\end{pmatrix}$$



$$E = \left(\begin{pmatrix}e_1\end{pmatrix}\begin{pmatrix}e_2\end{pmatrix}\begin{pmatrix}e_3\end{pmatrix}\right)$$

$$= \left(\frac{1}{\sqrt3}\begin{pmatrix}1\\1\\1\end{pmatrix}\frac{1}{\sqrt2}\begin{pmatrix}1\\-1\\0\end{pmatrix}\frac{1}{\sqrt6}\begin{pmatrix}1\\1\\-2\end{pmatrix}\right)$$

$$T_E = \begin{array}{c} \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix} \\ \underbrace{e_1 \quad e_2} \quad e_3 \end{array}$$

$E^T = E^{-1}$

since we created ~~an~~ orthogonal matrix

$$\begin{tikzcd}
\boxed{r} \arrow[r, "\text{hard}"] \arrow[d, "E^{-1}"'] & \boxed{r'} \\
\boxed{r_E} \arrow[r, "T_E"'] & \boxed{r'_E} \arrow[u, "E"']
\end{tikzcd}$$

$$\boxed{r' = E T_E E^{-1} r}$$

my frame

inplane frame

Reflection in plane

back to my plane with reflection

$$T_E E^T = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt3}(1\ 1\ 1)\\ \frac{1}{\sqrt2}(1\ -1\ 0)\\ \frac{1}{\sqrt6}(1\ 1\ -2)\end{pmatrix}$$

$$E T_E E^T = \frac{1}{3}\begin{pmatrix}2&-1&2\\-1&2&2\\2&2&-1\end{pmatrix} = T$$

$$r' = Tr = T\begin{pmatrix}2\\3\\5\end{pmatrix} = \frac{1}{3}\begin{pmatrix}11\\14\\5\end{pmatrix}$$

Eigenvalues & Eigenvectors :

- Eigenvectors are those which lie along the same span both before & after applying a linear transform to a space.
- Eigenvalues are the amount that each of those vectors has been stretched ~~or~~ in the process.

⇒ Calculating Eigenvectors :

$Ax = \lambda x$

A → n dimensional transform

x → n dimensional vector

$(A - \lambda I)x = 0$

$\boxed{\det(A - \lambda I) = 0}$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \det\left(\begin{pmatrix} a & b \\ c & a \end{pmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$

$\lambda^2 - (a+d)\lambda + ad - bc = 0$

example ①

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda) = 0$

$\boxed{\lambda = 1, 2}$

$(A - \lambda I)x = 0$

@ $\lambda = 1$ $\quad \begin{pmatrix} 1-1 & 0 \\ 0 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = 0$

@ $\lambda = 2$ $\quad \begin{pmatrix} 1-2 & 0 \\ 0 & 2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ 0 \end{pmatrix} = 0$

e_2, e_1

@ $\lambda = 1$ $\quad x = \begin{pmatrix} t \\ 0 \end{pmatrix}$ $\qquad$ @ $\lambda = 2$ $\quad x = \begin{pmatrix} 0 \\ t \end{pmatrix}$

② $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad \lambda^2 + 1 = 0$ ⇒ no solution (real)

No eigen vector exists for this transform

rotation 90° anti clock

* Diagonal Matrix : if all the element except those at the diagonal are zero, we call the matrix a diagonal matrix.

$T = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ $\qquad T^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$

$$C = \begin{pmatrix} x_1 & x_2 & x_3 \\ \vdots & \vdots & \vdots \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$T = CDC^{-1}$$

$$T^2 = CD\underbrace{C^{-1}C}_{I}DC^{-1} = CDDC^{-1} = CD^2C^{-1}$$

$$T^n = CD^nC^{-1}$$

Example: $T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ @ $\lambda = 1$ $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, @ $\lambda = 2$ $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 0+2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$(Tx) \Rightarrow \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}}_{(T)}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1+1 \\ 0+2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$(T^2x) \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\underbrace{\begin{pmatrix} 0 \\ 2 \end{pmatrix}}_{(Tx)} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \checkmark$$

$$\Rightarrow (T^2x) \Rightarrow \underbrace{\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}}_{T^2}\underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \checkmark$$

$$T^2 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$C^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

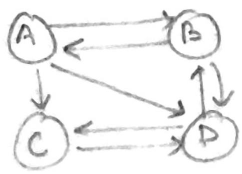
$$T^2 = CD^2C^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^2\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$$

Introduction to Page Rank : (Google it)



$$L_A = (0, 1/3, 1/3, 1/3)$$

$$L_B = (1/2, 0, 0, 1/2)$$

$$L_C = (0, 0, 0, 1)$$

$$L_D = (0, 1/2, 1/2, 0)$$

$$L = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{pmatrix}$$

$$r_A = \sum_{j=1}^{n} L_{Aj} r_j \Rightarrow r^{i+1} = L r^i \qquad r = \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}$$

$$r^{i+1} = d(Lr^i) + \frac{1-d}{n}$$