CCA with Ledoit-Wolf

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∇_{11}, ∇_{22} and ∇_{12} remains the same, since

$$\nabla = \frac{\partial f}{\partial \hat{\Sigma}}$$

and we haven't changed how we get from estimated covariance to trace norm of T.

However, now that

$$\hat{\Sigma}_{11} = (1-s)\Sigma_{11} + s\mu I$$

where $\Sigma_{11} = H_1 H_1^T/m$ is the sample covariance matrix, $\mu = \mathrm{tr}(\Sigma_{11})/D$. s is the shrinkage factor estimated from Σ_{11}. We should note that mathematically, s, μ both depend on Σ_{11}. However, I would like to treat s as an external variable unrelated to Σ_{11}, at least for now.

First consider $a \neq b$. This one is easy since $\hat{\Sigma}^{11}_{ab} = (1-s)\Sigma^{11}_{ab}$.

$$\frac{\partial \hat{\Sigma}^{11}_{ab}}{\partial \Sigma^{11}_{cd}} = \left\{\begin{matrix} 1-s, & \text{for } a=c \;\&\; b=d \\ 0, & \text{otherwise} \end{matrix}\right\} = (1-s)\cdot\delta_{ac}\cdot\delta_{bd}$$

Then consider $a = b$. Now $\hat{\Sigma}^{11}_{ab} = (1-s)\Sigma^{11}_{aa} + s\mu = (1-s)\Sigma^{11}_{aa} + \frac{s}{D}\sum_{i=1}^{d}\Sigma^{11}_{ii}$

$$\frac{\partial \hat{\Sigma}^{11}_{aa}}{\partial \Sigma^{11}_{cd}} = \left\{\begin{matrix} (1-s)+s/D, & \text{for } a=c \;\&\; c=d \\ s/D, & \text{for } c=d \\ 0, & \text{otherwise} \end{matrix}\right\} = (1-s)\cdot\delta_{ac}\cdot\delta_{cd} + \frac{s}{D}\cdot\delta_{cd}$$

In the DeepCCA paper, both $(\nabla_{11})_{ab} = \frac{\partial f}{\partial \hat{\Sigma}^{11}_{ab}}$ and $\frac{\partial \Sigma^{11}_{cd}}{\partial H^1_{ij}}$ has been given. $\frac{\partial \hat{\Sigma}^{11}_{ab}}{\partial \Sigma^{11}_{cd}}$ will be the final piece.

Then

$$\begin{aligned} \frac{\partial f}{\partial H^1_{ij}} &= \sum_{ab}\nabla^{11}_{ab}\frac{\hat{\Sigma}^{11}_{ab}}{H^1_{ij}} + \sum_{ab}\nabla^{12}_{ab}\frac{\partial \hat{\Sigma}^{12}_{ab}}{\partial H^1_{ij}} \\ &= \sum_{ab}\nabla^{11}_{ab}\sum_{cd}\frac{\partial \hat{\Sigma}^{11}_{ab}}{\partial \Sigma^{11}_{cd}}\frac{\partial \Sigma^{11}_{cd}}{\partial H^1_{ij}} + \sum_{ab}\nabla^{12}_{ab}\frac{\partial \hat{\Sigma}^{12}_{ab}}{\partial H^1_{ij}} \end{aligned}$$

The second term should stay the same, since we haven't changed the way to estimate cross-covariance. So let's look at the first term only.

$$\begin{aligned}
&\sum_{ab} \nabla^{11}_{ab} \sum_{cd} \frac{\partial \hat{\Sigma}^{11}_{ab}}{\partial \Sigma^{11}_{cd}} \frac{\partial \Sigma^{11}_{cd}}{\partial H^1_{ij}} \\
&= \sum_{ab, a\neq b} \nabla^{11}_{ab} \sum_{cd} \frac{\partial \hat{\Sigma}^{11}_{ab}}{\partial \Sigma^{11}_{cd}} \frac{\partial \Sigma^{11}_{cd}}{\partial H^1_{ij}} + \sum_{a} \nabla^{11}_{aa} \sum_{cd} \frac{\partial \hat{\Sigma}^{11}_{aa}}{\partial \Sigma^{11}_{cd}} \frac{\partial \Sigma^{11}_{cd}}{\partial H^1_{ij}}
\end{aligned}$$

Really the first part is easy.

$$\sum_{ab, a\neq b} \nabla^{11}_{ab} \sum_{cd} \frac{\partial \hat{\Sigma}^{11}_{ab}}{\partial \Sigma^{11}_{cd}} \frac{\partial \Sigma^{11}_{cd}}{\partial H^1_{ij}} = (1-s) \sum_{ab, a\neq b} \nabla^{11}_{ab} \frac{\partial \Sigma^{11}_{ab}}{\partial H^1_{ij}}$$

The second part is a little tougher, but not too bad

$$\begin{aligned}
\sum_{a} \nabla^{11}_{aa} \sum_{cd} \frac{\partial \hat{\Sigma}^{11}_{aa}}{\partial \Sigma^{11}_{cd}} \frac{\partial \Sigma^{11}_{cd}}{\partial H^1_{ij}} &= \sum_{a} \nabla^{11}_{aa} \left(\sum_{cd, c\neq d} \frac{\partial \hat{\Sigma}^{11}_{aa}}{\partial \Sigma^{11}_{cd}} \frac{\partial \Sigma^{11}_{cd}}{\partial H^1_{ij}} + \sum_{c} \frac{\partial \hat{\Sigma}^{11}_{aa}}{\partial \Sigma^{11}_{cc}} \frac{\partial \Sigma^{11}_{cc}}{\partial H^1_{ij}} \right) \\
&= \sum_{a} \nabla^{11}_{aa} \left(0 + \sum_{c} \frac{\partial \hat{\Sigma}^{11}_{aa}}{\partial \Sigma^{11}_{cc}} \frac{\partial \Sigma^{11}_{cc}}{\partial H^1_{ij}} \right) \\
&= \sum_{a} \nabla^{11}_{aa} \sum_{c} \frac{\partial \hat{\Sigma}^{11}_{aa}}{\partial \Sigma^{11}_{cc}} \frac{\partial \Sigma^{11}_{cc}}{\partial H^1_{ij}} \\
&= \sum_{a} \nabla^{11}_{aa} \left(\frac{s}{D} \sum_{c} \frac{\partial \Sigma^{11}_{cc}}{\partial H^1_{ij}} + (1-s) \frac{\partial \Sigma^{11}_{aa}}{\partial H^1_{ij}} \right) \\
&= \frac{s}{D} \left(\sum_{a} \nabla^{11}_{aa} \sum_{c} \frac{\partial \Sigma^{11}_{cc}}{\partial H^1_{ij}} \right) + (1-s) \left(\sum_{a} \nabla^{11}_{aa} \frac{\partial \Sigma^{11}_{aa}}{\partial H^1_{ij}} \right) \\
&= \frac{s}{D} \left(\sum_{a} \nabla^{11}_{aa} \sum_{c} \frac{\partial \Sigma^{11}_{cc}}{\partial H^1_{ij}} \right) + (1-s) \left(\sum_{a} \nabla^{11}_{aa} \frac{\partial \Sigma^{11}_{aa}}{\partial H^1_{ij}} \right)
\end{aligned}$$

Put the two parts together, we get

$$\sum_{ab} \nabla^{11}_{ab} \sum_{cd} \frac{\partial \hat{\Sigma}^{11}_{ab}}{\partial \Sigma^{11}_{cd}} \frac{\partial \Sigma^{11}_{cd}}{\partial H^1_{ij}} = (1-s) \sum_{ab} \nabla^{11}_{ab} \frac{\partial \Sigma^{11}_{ab}}{\partial H^1_{ij}} + \frac{s}{D} \cdot \sum_{a} \nabla^{11}_{aa} \sum_{c} \frac{\partial \Sigma^{11}_{cc}}{\partial H^1_{ij}}$$

Put the first term and the second term together, we get

$$\frac{\partial f}{\partial H^1_{ij}} = (1-s) \sum_{ab} \nabla^{11}_{ab} \frac{\partial \Sigma^{11}_{ab}}{\partial H^1_{ij}} + \frac{s}{D} \cdot \sum_{a} \nabla^{11}_{aa} \sum_{c} \frac{\partial \Sigma^{11}_{cc}}{\partial H^1_{ij}} + \sum_{ab} \nabla^{12}_{ab} \frac{\partial \hat{\Sigma}^{12}_{ab}}{\partial H^1_{ij}}$$

Since

$$\frac{\partial \Sigma^{11}_{ab}}{\partial H^1_{ij}} = \frac{1}{m} (\delta_{ai} \bar{H}^1_{bj} + \delta_{bi} \bar{H}^1_{aj})$$

We know,

$$\begin{aligned}
\frac{s}{D} \cdot \sum_a \nabla^{11}_{aa} \sum_c \frac{\partial \Sigma^{11}_{cc}}{\partial H^1_{ij}} &= \frac{s}{D} \cdot \frac{2}{m} \sum_a \nabla^{11}_{aa} \bar{H}^1_{ij} \\
&= \frac{s}{D} \cdot \frac{2}{m} \bar{H}^1_{ij} \mathrm{tr}(\nabla^{11})
\end{aligned}$$

As a result,

$$\frac{\partial f}{\partial H^1_{ij}} = \frac{1}{m}((1-s)(\nabla_{11}\bar{H}_1)_{ij} + (1-s)(\nabla'_{11}\bar{H}_1)_{ij} + (\nabla_{12}\bar{H}_2)_{ij} + 2s \cdot \frac{\mathrm{tr}(\nabla_{11})}{D} \bar{H}^1_{ij})$$

$$\begin{aligned}
\frac{\partial f}{\partial H_1} &= \frac{1}{m}(2(1-s)\nabla_{11}\bar{H}_1 + 2s\frac{\mathrm{tr}(\nabla_{11})}{D}\bar{H}_1 + \nabla_{12}\bar{H}_2) \\
&= \frac{1}{m}(2\nabla_{11}\bar{H}_1 + \nabla_{12}\bar{H}_2) + \frac{1}{m}(2s(\frac{\mathrm{tr}(\nabla_{11})}{D}I - \nabla_{11})\bar{H}_1)
\end{aligned}$$