Indian Institute of Technology Roorkee
Spring Semester 2023-24
MAI-102 (Mathematics II)
Assignment 6

Topics: Bivariate random variables: Joint, marginal, and conditional distributions, statistical independence. Distributions of functions of random variables. Correlation and regression.

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(1) Suppose 2 balls are drawn without replacement from an urn containing 3 balls numbered 1, 2, and 3. Let X be the number on the first ball drawn and Y be the larger of the two numbers drawn. Find
(a) the joint pmf of X and Y, (b) the conditional pmf of X given $Y = 3$,
(c) $Cov(X, Y)$.

(2) If X and Y are two independent random variables having the same geometric distribution with the parameter p, find $P(X = Y)$.

(3) Let X and Y have the joint density $f_{X,Y}(x, y) = \begin{cases} c(1 - x - y), & x > 0, y > 0, x + y < 1 \\ 0, & \text{elsewhere}, \end{cases}$.
Then find
(a) the constant c, and the marginal densities of X and Y,
(b) the conditional density of X given $Y = y$, and $E(X \mid Y = \frac{1}{2})$,
(c) the correlation coefficient of X and Y.

(4) Let X and Y be two discrete random variables with the joint pmf $f_{X,Y}(x, y) = \frac{x+2y}{24}$, $(x, y) = (0, 1), (0, 2), (1, 1), (1, 2), (2, 1), (2, 2)$, zero elsewhere. Find the conditional mean and conditional variance of X given $Y = 2$. Also determine the correlation coefficient of X and Y.

(5) Let two random variables X, Y have the joint density $f_{X,Y}(x, y) = \begin{cases} e^{-(x+y)}, & x > 0, y > 0, \\ 0, & \text{elsewhere} \end{cases}$.
Then find (a) $P(1 < X + Y < 2)$ (b) $P(X < Y | X < 2Y)$.

(6) Let two random variables X, Y have the joint density $f_{X,Y}(x, y) = \begin{cases} \frac{1}{2}xy, & 0 < y < x < 2, \\ 0, & \text{elsewhere} \end{cases}$.
(a) Find $P(X \leq \frac{3}{2} | Y = 1)$, and the conditional variance of X given $Y = 1$.
(b) Are X, Y independent? Explain.

(7) Let X and Y be two discrete random variables such that the pmf of X is $f_X(x) = \frac{x}{3}$, $x = 1, 2$, and the conditional distribution of Y, given $X = x$, is a binomial distribution with parameters x and $\frac{1}{2}$. Find (a) the joint pmf of X and Y (b) $E(Y)$.

(8) Three points X_1, X_2 and X_3 are selected at random on a line segment of length L. What is the probability that X_2 lies between X_1 and X_3?

(9) Suppose that the joint cdf of two random variables X and Y is

$$F_{X,Y}(x, y) = \begin{cases} (1 - e^{-ax})y^2, & 0 \leq x < \infty,\ 0 \leq y < 1, \\ 1 - e^{-ax}, & 0 \leq x < \infty,\ 1 \leq y < \infty, \\ 0, & \text{elsewhere}, \end{cases}$$

where $a > 0$ is a constant. Show that X and Y are independent.

(10) Let X and Y be random variables such that $Var(X) = 4$, $Var(Y) = 2$, and $Var(X+2Y) = 15$. Determine the correlation coefficient of X and Y.

(11) Prove that
(a) sum of r independent geometric random variables each with parameter p is a negative binomial random variable with the parameters (r, p).
(b) the sum of two independent binomial random variables with the parameters (n_1, p) and $n_2, p)$ is a binomial random variable with the parameters $(n_1 + n_2, p)$.

(c) sum of two independent Poisson random variables with the parameters λ_1 and λ_2 is a Poisson random variable with the parameter $\lambda_1 + \lambda_2$.
(d) sum of r independent exponential random variables each with the parameter β is a gamma random variable with the parameters (r, β).
(e) sum of two independent gamma random variables with the parameters (α_1, β) and (α_2, β) is a gamma random variable with the parameters $(\alpha_1 + \alpha_2, \beta)$.
(f) sum of two independent χ^2-random variables with n_1 and n_2 degrees of freedom is a χ^2-random variable with $n_1 + n_2$ degrees of freedom.
(g) any linear combination of two independent normal random variables is again a normal random variable.

(12) If the pdf of X is $f(x) = 2xe^{-x^2}$, $x > 0$, zero elsewhere, determine the pdf of $Y = X^4$.

(13) Let $f(x) = \frac{4-x}{12}$, $-1 < x < 3$, zero elsewhere, be the pdf of X. Find the pdf of $Y = X^2$ using (i) distribution function technique (ii) transformation technique.

(14) (a) Let X_1 and X_2 be two independent random variables each following an exponential distribution with the parameter $\beta = 1$. Find the pdf of $X_1 + X_2$.

(b) Let X and Y be two independent standard normal random variables. Find the pdf of $Z = X^2 + Y^2$.

(15) Let X_1, X_2 and X_3 be three independent random variables each having the pdf $f(x) = 5x^4$, $0 < x < 1$, zero elsewhere. Let Y be the largest of X_1, X_2 and X_3. Find the pdf of Y.

(16) Find the coefficient of correlation between X and Y from the following data table:

X	5.5	6.5	7.5	8.5	9.5	10.5	11.4
Y	10.9	7.8	8.3	6.2	7.1	5.3	4.8

(17) While calculating the coefficient of correlation between two variables x and y, the following results were obtained: $n = 25$, $\sum x = 125$, $\sum y = 100$, $\sum x^2 = 650$, $\sum y^2 = 460$, $\sum xy = 508$. It was however later discovered at the time of checking that two pairs of observations (x, y) were copied $(6, 14)$ and $(8, 6)$, while the correct values were $(8, 12)$ and $(6, 8)$ respectively. Determine the correct value of the coefficient of correlation.

(18) Let X and Y be two random variables with variances σ_x^2 and σ_y^2 respectively and r is the correlation coefficient between them. If $U = X + kY$ and $V = X + \frac{\sigma_x}{\sigma_y} Y$, find the value of k such that U and V are uncorrelated.

(19) If X and Y are random variables each with variance 1 and

$$r(aX + bY, bX + aY) = \frac{1 + 2ab}{a^2 + b^2},$$

then find $r(X, Y)$, the correlation between X and Y.

(20) Obtain the equations of two lines of regression for the following data: Also, obtain the

X	65	66	67	67	68	69	70	72
Y	67	68	65	68	72	72	69	71

estimate of X for $Y = 70$.

(21) In a partially destroyed record of an analysis of correlation data, only the following results were legible:
Variance of $X = 9$. Regression equation: $8X - 10Y + 66 = 0$ and $40X - 18Y = 214$.
Find the
(a) The mean values of X and Y.
(b) The coefficient of correlation between X and Y.

(c) The standard deviation of Y.

(22) Find two regression equations when it is given that

$$\overline{X} = 68.2,\ \overline{Y} = 9.9,\ \frac{\sigma_Y}{\sigma_X} = 0.44 \ \text{ and } \ r = 0.7.$$

(23) For 50 students of a class, the regression equation of marks in Statistics (X) on the marks in Mathematics (Y) is $3Y - 5X + 108 = 0$. The mean marks of mathematics is 44 and the variance of marks in statistics is $\frac{9}{16}^{\text{th}}$ of the variance of marks in mathematics. Find the mean marks in statistics and the co-efficient of correlation between the marks in the two subjects.

(24) Can $Y = 5 + 2.8X$ and $X = 3 - 0.5Y$ be the estimated regression equations of Y on X and X on Y respectively? Justify your answer.

ANSWERS

(1) (a) $f(1,2) = f(1,3) = f(2,2) = f(2,3) = \frac{1}{6}$, $f(3,2) = 0$, $f((3,3) = \frac{1}{3}$
(b) $f_{X|Y}(1|3) = f_{X|Y}(2|3) = \frac{1}{4}$, $f_{X|Y}(3|3) = \frac{1}{2}$
(c) $\frac{1}{6}$

(2) $\frac{p}{2-p}$

(3) (a) $c = 6$, marginal densities: $f_X(x) = 3(1-x)^2, 0 < x < 1$, zero elsewhere; $f_Y(y) = 3(1-y)^2$, $0 < y < 1$, zero elsewhere
(b) $f_{X|Y}(x|y) = \frac{2(1-x-y)}{(1-y)^2}, 0 < x < 1-y$, zero elsewhere; $E(X|\frac{1}{2}) = \frac{1}{6}$
(c) $\rho_{X,Y} = -\frac{1}{3}$

(4) $E(X|Y=2) = \frac{17}{15}$, $Var(X|Y=2) = \frac{146}{225}$; $\rho_{X,Y} = -\frac{1}{\sqrt{345}}$

(5) (a) $2e^{-1} - 3e^{-2}$
(b) $\frac{3}{4}$

(6) (a) $\frac{5}{12}$; conditional variance of X given $Y = 1$ is $\frac{13}{162}$.
(b) No

(7) (a) Joint pmf $f(x,y) = \binom{x}{y}\left(\frac{1}{2}\right)^x \frac{x}{3}$, $x = 1,2;\ 0 \le y \le x$, zero elsewhere.
(b) $\frac{5}{6}$

(8) $\frac{1}{3}$

(10) $\rho_{X,Y} = \frac{3}{8\sqrt{2}}$

(12) $f(y) = \frac{1}{2\sqrt{y}}\, e^{-\sqrt{y}}, \quad y > 0.$

(13)

$$f(y) = \begin{cases} \frac{1}{3\sqrt{y}}, & 0 < y < 1, \\ \frac{1}{6\sqrt{y}} - \frac{1}{24}, & 1 \le y < 9, \\ 0, & \text{elsewhere .} \end{cases}$$

(14) (a) $f(y) = ye^{-y}, \quad y > 0$, zero elsewhere, where $Y = X_1 + X_2$.

(b) $f(z) = \frac{1}{2}e^{-\frac{z}{2}}, \quad z > 0$, zero elsewhere.

(15) $f(y) = 15y^{14}, \quad 0 < y < 1.$
(16) -0.915
(17) 0.67
(18) $k = \begin{cases} \text{any real number when } r = -1, \\ -\frac{\sigma_x}{\sigma_y} \text{ when } r \ne -1. \end{cases}$
(19) $\frac{a^2+b^2}{(a^2-b^2)^2-2ab}$
(20) $y = 0.667x + 23.667$, $x = 0.545y + 30.364$, 68.514
(21) (a) 13, 17
(b) +0.6
(c) 4
(22) Y on X: $y = 0.308x - 11.1065$, X on Y: $x = 1.591y + 52.4491$.
(23) $\overline{X} = 48$ and $r = 0.8$
(24) No