Indian Institute of Technology Roorkee
Spring Semester 2023-24
MAI-102 (Mathematics II)
Assignment 7

Topics: ***Sampling Distributions:*** *Random sampling and sampling distributions, central limit theorem.* ***Estimation:*** *Point estimation, unbiased estimators, maximum likelihood estimation. Interval estimation, interval estimation of mean, variance and proportion for normal populations.*

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(1) (a) Let $\overline{X}_n$ be the sample mean of a random sample of size n from rectangular (or uniform) distribution on $[0,1]$. Let $U_n = \sqrt{n}\left(\overline{X}_n - \frac{1}{2}\right)$. Show that $F(u) = \lim_{n\to\infty} P(U_n < u)$ exists and determine it.

(b) Let $X_1, X_2, \ldots, X_n, \ldots$ be independent and identically distributed Bernoulli variates such that:

$$P(X_k = 1) = p,\ P(X_k = 0) = q = (1-p),\ k = 1, 2, \ldots, n, \ldots.$$

Examine whether the sequence $\left\{\sum_{i=1}^{n} X_i\right\}$ follows central limit theorem.

(2) In a communication system each data packet consists of 1000 bits. Due to the noise each bit may be received in error with probability 0.1. It is assumed bit errors occur independently. Find the probability that there are more than 120 errors in a certain data packet.

(3) A bank teller serves customers standing in the queue one by one. Suppose that the service time X_i for customer i has mean $E(X_i) = 2$ (minutes) and $\mathrm{Var}(X_i) = 1$. We assume that service times for different bank customers are independent. Let Y be the total time the bank teller spends serving 50 customers. Find $P(90 < Y < 110)$.

(4) Let n numbers $X_1, X_2, \ldots, X_n$ in decimal form, be each approximated by the closest integer. If X_i is the i^{th} true number and Y_i is the nearest integer, then $U_i = X_i - Y_i$ is the error made by the rounding process. Suppose that $U_1, U_2, \ldots, U_n$ are independent and each is uniform on $(-0.5, 0.5)$.

(a) What is the probability that the true sum is within 'a' units of the approximated sum?

(b) If $n = 300$ terms are added, find 'a' so that we are 95% sure that the approximation is within 'a' units of the true sum.

(5) (a) Suppose that a sample of $n = 1,600$ tires of the same type are obtained at random from an ongoing production process in which 8% of all such tires produced are defective. What is the probability that in such a sample not more than 150 tires will be defective? (use the normal approximation to the binomial distribution with continuity correction.)

(b) Based on past experience, 7% of all luncheon vouchers are in error. If a random sample of 400 vouchers is selected, what is the approximate probability that (use the normal approximation to the binomial distribution with continuity correction)

(i) exactly 25 are in error?
(ii) fewer than 25 are in error?
(iii) between 20 and 25 (inclusive) are in error?

(6) (a) A radioactive element disintegrates such that it follows a Poisson distribution. If the mean number of particles (α) emitted is recorded in a 1 second interval as 69, then using the normal approximation to the Poisson distribution (continuity correction), evaluate the probability of:

(i) Less than 60 particles are emitted in 1 second.
(ii) Between 65 and 75 particles inclusive are emitted in 1 second.

(b) The annual number of earthquakes registering at least 2.5 on the Richter Scale and having an epicenter within 40 miles of downtown Memphis follows a Poisson distribution with mean 6.5. What is the probability that at least 9 such earthquakes will strike next year?

(i) Use the Poisson distribution to calculate the exact probability.
(ii) Use the normal approximation to the Poisson distribution with continuity correction.

(7) (a) If X is a binomial distribution with parameters n and θ, show that $n \cdot \frac{X}{n} \cdot \left(1 - \frac{X}{n}\right)$ is not an unbiased estimator of the variance of X.
(b) Let $\overline{X}$ be the mean of a random sample $X_1, X_2, \ldots, X_n$ from a distribution with variance σ^2. Show that $T = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is not an unbiased estimator of σ^2.

(8) (a) Find the maximum likelihood estimate(MLE) of the parameter p for $Binomial(n, p)$ population. Check whether the estimate is unbiased or not.
(b) Find the MLE of the parameter λ for $Poisson(\lambda)$ population.
(c) Find the MLE of the parameters μ and σ for $Normal(\mu, \sigma)$ population.
(d) A population has a density function given by

$$f(x) = \frac{x^{p-1}e^{-x/\theta}}{\theta^p\Gamma(p)}, \quad 0 < x < \infty, \; p \text{ is given and } \theta \text{ is a parameter.}$$

Find the MLE of θ. Show that the estimate is consistent and unbiased.
(e) Estimate the parameter μ given by the distribution

$$f(x) = \frac{1}{1+\mu}\left(\frac{\mu}{1+\mu}\right)^{x_i}, \quad x_i = 0, 1, 2, \ldots; \; \mu > 0,$$

by the method of maximum likelihood and show that the estimate is consistent and unbiased.
(f) Estimate the parameter θ of a continuous population having the density function $(1+\theta)x^\theta$, $(0 < x < 1)$ by the method of maximum likelihood.
(g) Find the MLE of the parameter α of a continuous population, having density function $\frac{2(\alpha - x)}{\alpha^2}$, $(0 < x < \alpha)$, for a sample of unit mass and check the unbiasedness of the estimate.
(h) Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution with probability density function

$$f(x, \theta) = \begin{cases} \frac{1}{\theta}, & 0 < x < \infty, \; \theta > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Obtain the maximum likelihood estimator for θ.

(9) (a) Let $\overline{X}$ denotes the mean of a random sample of size n from a distribution that has mean μ and variance $\sigma^2 = 20$. Find n with 90% confidence that the random interval $(\overline{X} - 2, \overline{X} + 2)$ includes μ.
(b) A random sample of size 100 is taken from a population with $\sigma = 0.5$ gram. Given that the sample mean is 0.75, construct a 95% confidence interval for the population mean μ.
(c) A normal distribution has mean 7 and standard deviation 2. How large a sample should be taken to have a 95% confidence interval for μ of length 0.6.
(d) A random sample of size 80 is taken from a population with sample variance $S^2 = 30.85$. Given that the sample mean is 18.85, construct a 99% confidence interval for the population mean μ.

(10) A signal having value μ is transmitted from location A. The value received at location B is normally distributed with unknown mean μ and variance σ^2. A particular value is

transmitted 9 times. Find the 95% confidence interval for the transmitted data μ, when the successive values received are $5, 8.5, 12, 15, 7, 9, 7.5, 6.5$ and 10.5.

(11) A standardized procedure is expected to produce washers with very small deviation in their thickness. Suppose that 10 such washers were chosen and measured. The thickness of these washers were found to be (in inches) as follows:

$$0.123, 0.124, 0.126, 0.120, 0.130, 0.133, 0.125, 0.128, 0.124, 0.126.$$

What is the 90% confidence interval for the standard deviation σ of the thickness of a washer produced by this procedure?

(12) An optical firm purchases glass for making lenses. Assume that the refractive index of 20 pieces of glass have a variance of 1.20×10^{-4}. Construct a 95% confidence interval for σ^2, the variance of the refractive index of all such pieces of glass.

(13) The following are weights (in decagrams) of 10 packages of grass seed distributed by a certain company:

$$46.4, 46.1, 45.8, 47.0, 46.1, 45.9, 45.8, 46.9, 45.2 \text{ and } 46.0.$$

Find a 95% confidence interval for the standard deviation σ of all such packages of grass seed distributed by this company.

(14) A random sample size 100 is taken from a population with $\sigma = 5.1$. Given that the sample mean is 21.6. What can be asserted with probability 0.95 about the maximum size of the error for the sample mean?

(15) Let two independent random samples, each of size 10, from two normal distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ yield $\overline{x}_1 = 4.8, s_1^2 = 8.64, \overline{x}_2 = 5.6, s_2^2 = 7.88$. Find a 95% confidence interval for $\mu_1 - \mu_2$.

(16) A study of two kinds of photocopying equipment shows that 60 failures of the first kind of equipment took on the average 80.7 minutes to repair with a standard deviation of 19.4 minutes, whereas 50 failures of the second kind of equipment took on the average 88.1 minutes to repair with a standard deviation of 18.8 minutes. Find a 99% confidence interval for the difference between the true average amounts of time it takes to repair failures of the two kinds of photocopying equipment.

(17) In a sample survey it was found that among 100 people surveyed only 18 see a particular television program. Construct a 98% confidence interval for the corresponding true proportion.

(18) Suppose that 175 heads and 225 tails resulted from 400 tosses of a coin. Find a 90% confidence interval for the probability of a head.

ANSWERS

(1) (a) $\Phi(\sqrt{12}u)$, where $\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\left(-\frac{1}{2}x^2\right) dx$
(2) 0.0175
(4) (a) Required probability

$$p = P\left(\left|\sum_{i=1}^{n}(X_i - Y_i)\right| \leq a\right) = P\left(-a \leq \sum_{i=1}^{n} U_i \leq a\right) = 2\Phi(a\sqrt{12/n}) - 1.$$

(b) $p = 0.95 \Rightarrow \Phi(a\sqrt{12/300}) = 0.975 \Rightarrow a = 9.8$
(5) (a) 0.9808
(b) (i) 0.0670
(ii) 0.2451
(iii) 0.2646
(6) (a) (i) 0.1271
(ii) 0.4877
(b) (i) 0.208
(ii) 0.218
(8) (a) $\widehat{p} = \frac{\overline{\mathrm{x}}}{N}$, unbiased.
(8) (b) $\widehat{\lambda} = \overline{\mathrm{x}}$.
(8) (c) $\widehat{\mu} = \overline{\mathrm{x}}$, $\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(\mathrm{x}_i - \overline{\mathrm{x}})^2$.
(8) (d) $\widehat{\theta} = \frac{\overline{\mathrm{x}}}{p}$.
(8) (e) $\widehat{\mu} = \overline{\mathrm{x}}$.
(8) (f) $\widehat{\theta} = -1 - \frac{n}{\ln(x_1 x_2 \ldots x_n)}$.
(8) (g) $\widehat{\alpha} = 2\mathrm{x}_1$, where x_1 is a random sample of unit size.
(8) (h) $\widehat{\theta} = X_{(n)}$ (the largest sample observation).
(9) (a) $n \approx 14$.
(9) (b) $(0.652, 0.848)$.
(9) (c) $n \approx 171$.
(9) (d) $(16.92, 20.68)$.
(10) $(6.631, 11.369)$.
(11) $(2.696 \times 10^{-3}, 6.072 \times 10^{-3})$.
(12) $(0.694 \times 10^{-4}, 2.56 \times 10^{-4})$.
(13) $(0.368, 0.976)$.
(14) 0.9996.
(15) $(-3.825, 2.225)$.
(16) $(-16.799, 1.999)$.
(17) $(0.0906, 0.2694)$.
(18) $(0.3967, 0.4783)$.