Indian Institute of Technology Roorkee
Spring Semester 2023-24
MAI-102 (Mathematics II)
Assignment 3

(*Topics: Inner-product spaces, Gram-Schmidt process, orthonormal basis; spectral theorem for real symmetric matrices*)

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(1) (a) (i) In $\mathbb{R}^2$, let $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2)$. Determine whether $\langle , \rangle$ is a real inner product for $\mathbb{R}^2$ if $\langle \cdot, \cdot \rangle$ be defined by

$$\langle \alpha, \beta \rangle = 2a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2.$$

(ii) Use Gram-Schmidt process to obtain an orthonormal basis from the basis set $\{(1, -1),\ (1, 0)\}$ of $\mathbb{R}^2$ with the above inner product.

(b) (i) In $\mathbb{R}^3$, let $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3)$. Determine whether $\langle \cdot, \cdot \rangle$ is a real inner product for $\mathbb{R}^3$ if $\langle , \rangle$ be defined by

$$\langle \alpha, \beta \rangle = a_1b_1 + (a_2 + a_3)(b_2 + b_3) + a_2b_2.$$

(ii) Use Gram-Schmidt process to obtain an orthonormal basis from the basis set $\{(1, 0, 1),\ (1, 1, 1),\ (1, 3, 4)\}$ of $\mathbb{R}^2$ with the above inner product.

(2) Provide reasons why each of the following is not an inner product on the given vector spaces.
(a) $\langle (a, b), (c, d) \rangle = ac - bd$ on $\mathbb{R}^2$.
(b) $\langle A, B \rangle = \mathrm{Tr}(A + B)$ on $M_{2\times 2}(\mathbb{R})$.
(c) $\langle f, g \rangle = \int_0^1 f'(x)g(x)dx$ on $\mathcal{P}(\mathbb{R})$ where $'$ denotes differentiation.
(d) $\langle f, g \rangle = \int_0^{1/2} f(x)g(x)dx$ on $C([0, 1])$ over $\mathbb{R}$.

(3) Prove that for all α, β in a real vector space V,
(a) $\langle \alpha, \beta \rangle = 0$ if and only if $\|\alpha + \beta\| = \|\alpha - \beta\|$,
(b) $\langle \alpha + \beta,\ \alpha - \beta \rangle = 0$ if and only if $\|\alpha\| = \|\beta\|$.

(4) Let V be an inner product space.
(a) Prove that $\|x \pm y\|^2 = \|x\|^2 \pm \mathrm{Re}(\langle x, y \rangle) + \|y\|^2$, for all $x, y \in V$, where $\mathrm{Re}(\langle x, y \rangle)$ denotes the real part of the complex number $\langle x, y \rangle$.
(b) Suppose that x and y are orthogonal vectors in V. Prove that

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Deduce Pythagorean theorem in $\mathbb{R}^2$.
(c) Prove the parallelogram law on an inner product space V; that is, show that

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad \text{for all} \quad x, y \in V.$$

What does this equation state about parallelograms in $\mathbb{R}^2$?
(d) If V is an inner product space over $\mathbb{R}$, prove the following polar identity:

$$\langle x, y \rangle = \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2.$$

(e) If V is a complex inner product space, the show that

$$\langle x, y \rangle = \frac{1}{4}\left[\|x + y\|^2 - \|x - y\|^2 + \|u + \iota v\|^2 \iota - \|x - \iota y\|^2 \iota\right].$$

(5) Let V be an inner product space over $\mathbb{F}$.
(a) (Cauchy-Schwarz inequality) Prove that $|\langle x, y \rangle| \leq \|x\|\|y\|$ for all $x, y \in V$.

(b) Prove that $|\langle x, y\rangle| = \|x\|\|y\|$ if and only if one of the vectors x or y is a multiple of the other.

(6) In $C([0,1])$, over $\mathbb{R}$, show that $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$ defines a norm. Let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g\rangle$, $\|f\|$, $\|g\|$ and $\|f+g\|$. Then verify both the Cauchy-Schwarz inequality and the triangle inequality.

(7) (a) Let $V = M_{n\times n}(\mathbb{C})$, and define $\langle A, B\rangle = \mathrm{Tr}(B^*A)$ for $A, B \in V$ (Frobenius inner product), where B^* denotes the conjugated transpose of B. Verify that $\langle\cdot,\cdot\rangle$ defines an inner product in V.

(b) Use the Frobenius inner product to compute $\|A\|$, $\|B\|$ and $\langle A, B\rangle$ for

$$A = \begin{pmatrix} 1 & 2+\iota \\ 3 & \iota \end{pmatrix}, \quad B = \begin{pmatrix} 1+\iota & 0 \\ \iota & -\iota \end{pmatrix}.$$

(c) In $\mathbb{C}^2$, show that $\langle x, y\rangle = xAy^*$ is an inner product where $A = \begin{pmatrix} 1 & \iota \\ -\iota & 2 \end{pmatrix}$. Compute $\langle x, y\rangle$ for $x = (1-\iota, 2+3\iota)$ and $y = (2+\iota, 2-3\iota)$.

(8) Suppose $u, v \in V$ are such that $\|u\| = 3$, $\|u+v\| = 4$, $\|u-v\| = 6$. What number does $\|v\|$ equal?

(9) (a) Let T be a linear operator on an inner product space V, and suppose that

$$\|Tx\| = \|x\| \quad \text{for all} \quad x.$$

Prove that T is one-to-one.

(b) Let V be a vector space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and let W be an inner product space over $\mathbb{F}$ with inner product $\langle\cdot,\cdot\rangle$. If $T : V \to W$ is linear, prove that $(x, y) := \langle Tx, Ty\rangle$ defines an inner product on V if and only if T is one-to-one.

(10) (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping such that

$$\ker(T) = \{(x, y, z) \in \mathbb{R}^3 : \ 2x + 3y - z = 0\}.$$

(i) Use Gram-Schmidt process to obtain an orthogonal basis of $\ker(T)$ with the standard inner product.

(ii) Use rank-nullity theorem to find $\dim(\mathcal{L}(V, \mathrm{range}(T)))$.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping such that

$$\mathrm{range}(T) = \{(x, y, z) \in \mathbb{R}^3 : \ 2x + 3y - z = 0\}.$$

Use Gram-Schmidt process to obtain an orthogonal basis of range T with the standard inner product.

(11) Find an orthonormal basis of the a) row space and b) column space of the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

(12) Apply Gram-Schmidt process to convert the given basis B of the Euclidean space $\mathbb{R}^4$ with standard inner product into an orthonormal basis.

$$\{(1,1,0,1), \ (1,-2,0,0), \ (1,0,-1,2), \ (0,0,0,1)\}.$$

(13) In each part, apply the Gram- Schmidt process to the given subset S of the inner product space V to obtain an orthogonal basis for $\mathrm{span}(S)$. Then normalize the vectors in this basis to obtain an orthonormal basis β for $\mathrm{span}(S)$.

(a) $V = \mathbb{R}^3$, $S = \{(1,1,1),(0,1,1),(0,0,1)\}$.
(b) $V = \mathcal{P}_2(\mathbb{R})$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$, $S = \{1, x, x^2\}$.
(c) $V = \mathbb{R}^4$, $S = \{(1,0,1,0),(1,1,1,1),(0,1,2,1)\}$.
(d) $V = M_{2\times 2}(\mathbb{R})$, $S = \left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right\}$.

(14) Verify spectral theorem for the following real symmetric matrices.
(a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
(b) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$
(c) $\begin{pmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix}$

(15) (a) If A is orthogonally diagonalizable, then show that $A^\top = A$.
(b) If A is an $m \times n$ real matrix, then show that $A^\top A$ is diagonalizable.

ANSWERS

(1): (a) $\{(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}),\ (0,1)\}$
(b) $\{\frac{1}{\sqrt{3}}(1,0,1),\ \frac{1}{\sqrt{6}}(-1,3,-1),\ \frac{1}{\sqrt{2}}(-1,-1,1)\}$
(6): $\langle f, g\rangle = 1$, $\|f\| = \frac{\sqrt{3}}{3}$, $\|g\| = \sqrt{\frac{e^2-1}{2}}$ **and** $\|f+g\| = \sqrt{\frac{11+3e^2}{6}}$
(8): $\sqrt{17}$.
(4): (a) **(i)** $\mathbf{ker}(T) = \mathrm{span}\{(1,0,2),\ (-\frac{6}{5}, 1, \frac{3}{5})\}$.
(ii) 3.
(b) $\mathbf{range}(T) = \mathrm{span}\{(1,0,2),\ (-\frac{6}{5}, 1, \frac{3}{5})\}$.
(11): $\{\frac{1}{\sqrt{2}}(1,0,0,1),\ \frac{1}{\sqrt{6}}(1,2,0,-1),\ \frac{1}{2\sqrt{3}}(-1,1,3,1)\}$, $\{(1,0,0),\ (0,1,0),\ (0,0,1)\}$
(12): $\{\frac{1}{\sqrt{3}}(1,1,0,1),\ \frac{1}{\sqrt{42}}(4,-5,0,1),\ \frac{1}{\sqrt{105}}(4,2,7,-6),\ \frac{1}{\sqrt{30}}(-2,-1,4,3)\}$
(13): (a) $\left\{\frac{\sqrt{3}}{3}(1,1,1), \frac{\sqrt{6}}{6}(-2,1,1), \frac{\sqrt{2}}{2}(0,-1,1)\right\}$.
(b) $\{1, 2\sqrt{3}(x-\frac{1}{2}), \sqrt{5}(x^2 - x + \frac{1}{6})\}$.
(c) $\left\{\frac{1}{\sqrt{2}}(1,0,1,0), \frac{1}{\sqrt{2}}(0,1,0,1), \frac{1}{\sqrt{2}}(-1,0,1,0)\right\}$.
(d) $\left\{ \frac{1}{6}\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \frac{1}{6\sqrt{2}}\begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}, \frac{1}{9\sqrt{2}}\begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix} \right\}$