Indian Institute of Technology Roorkee
Spring Semester 2023-24
MAI-102 (Mathematics II)
Assignment 1

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(1) Determine whether the following set of vector are vector spaces or not over $\mathbb{R}$, with respect to given addition '+' and scalar multiplication '.':

(a) The set
$$\{(a, b) \in \mathbb{R}^2 : b = 5a + 1\},$$
with respect to usual addition and multiplication.

(b) The set $M_n(\mathbb{R})$ of all $n \times n$ real matrices over $\mathbb{R}$ with respect to usual addition and multiplication.

(c) The set $\mathbb{R}^2$ over $\mathbb{R}$ with respect to the addition and scalar multiplication defined by
$$\lambda \cdot (a, b) := (\lambda a, 0)$$
for $(a, b) \in \mathbb{R}^2, \lambda \in \mathbb{R}$.

(d) The set $C[a, b]$ of all real valued continuous functions defined on the interval $[a, b]$, with respect to the following addition and scalar multiplication: $(f + g)(t) := f(t) + g(t)$ and $(\lambda \cdot f)(t) := \lambda f(t)$ for $f, g \in C[a, b], t \in [a, b], \lambda \in \mathbb{R}$.

(e) The set of all positive real numbers over $\mathbb{R}$, with respect to the following addition and scalar multiplication: $x + y := xy$ and $\lambda \cdot x := x^\lambda$ for $x, y \in \mathbb{R}, \lambda \in \mathbb{R}$.

(f) The set P_n of all real polynomials of order less than or, equal to n, with respect to usual addition and scalar multiplication.

(2) Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define
$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2)\ .$$
Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

(3) Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.
(a) $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$
(b) $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$
(c) $W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$
(d) $W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

(4) Describe the smallest subspace of the space $M_2(\mathbb{R})$ containing $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Justify your answer.

(5) Determine whether or not the set $\{f \in C[0, 1] : \int_0^1 f(x)dx = 0\}$ is a subspace of $C[0, 1]$.

(6) Show that the subspaces of $\mathbb{R}^3$ are precisely $\{0\}, \mathbb{R}^3$, all lines in $\mathbb{R}^3$ through the origin, and all planes in $\mathbb{R}^3$ through the origin.

(7) If U_1, U_2 are two subspaces of a vector space V then $U_1 + U_2$ is the direct sum $U_1 \oplus U_2$ of U_1, U_2 if every element u of $U_1 + U_2$ can be uniquely written as $u = u_1 + u_2$ where $u_1 \in U_1$ and $u_2 \in U_2$.

(a) Prove or give a counterexample: if U_1, U_2, W are subspaces of a vector space V such that $V = U_1 \oplus W = U_2 \oplus W$ then $U_1 = U_2$.
(b) Let U_e denote the set of real-valued even functions on $\mathbb{R}$ and let U_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$, where $\mathbb{R}^{\mathbb{R}}$ denotes the space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(8) For each of the following lists of vectors in $\mathbb{R}^3$, determine whether the first vector can be expressed as a linear combination of the other two.
(a) $(-2, 0, 3), (1, 3, 0), (2, 4, -1)$
(b) $(3, 4, 1), (1, -2, 1), (-2, -1, 1)$
(c) $(5, 1, -5), (1, -2, -3), (-2, 3, -4)$

(9) Determine whether the vectors $v_1 = (1, -1, 4), v_2 = (-2, 1, 3)$ and $v_3 = (4, -3, 5)$ span $\mathbb{R}^3$.

(10) Verify that $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ span $M_2(\mathbb{R})$.

(11) Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all 2×2 symmetric matrices.

(12) Determine whether the following sets are linearly dependent or linearly independent.
(a) $\left\{ \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -8 \\ -9 & 10 \end{pmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$.
(b) $\{x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\}$ in $P_3(\mathbb{R})$.

(13) Determine the subspace of the vector space P_2 spanned by $p_1(x) = 1 + 3x, p_2(x) = x + x^2$, and decide whether $\{p_1, p_2\}$ is a spanning set for P_2.

(14) Show that the set of solutions to the systems of linear equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

is a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

(15) Determine whether the given set of vectors in $\mathbb{R}^n$ is linearly dependent or linearly independent:
(a) $\{(1, 2, 3), (1, 0, 1), (1, -1, 5)\}$
(b) $\{(1, -1, 3, -1), (1, -1, 4, 2), (1, -1, 5, 7)\}$

(16) Let V be a vector space over F. A subset $\mathcal{B}$ of V is called a basis if it is linearly independent and it spans V over $\mathbb{F}$. Prove that a set $\{v_1, v_2, \ldots, v_n\}$ of vectors in a vector space V is a basis if and only if every $v \in V$ can be written uniquely as $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ where $c_i \in \mathbb{F}$ for all i.

(17) Determine a basis for the subspace of $\mathbb{R}^n$ spanned by the given set of vectors:

(a) $\{(1, 3, 3), (-3, -9, -9), (1, 5, -1), (2, 7, 4), (1, 4, 1)\}$

(b) $\{(1, 1, -1, 2), (2, 1, 3, -4), (1, 2, -6, 10)\}$

(18) Show that the set vectors $\{f_1, f_2, f_3\}$ is linearly independent where $f_1(x) = 2x - 3$, $f_2(x) = x^2 + 1$, $f_3(x) = 2x^2 - x$. Complete the set to form a basis for P_3, the set of all polynomials of degree no more than 3.

(19) (a) Find a basis and the dimension of the subspace

$$W = \{(a + b + 2c, 2a + 2b + 4c + d, b + c + d, 3a + 3c + d) : a, b, c, d \in \mathbb{R}\}.$$

(b) Let u, v, and w be distinct vectors of a vector space V. Show that if $\{u, v, w\}$ is a basis of V, then $\{u + v + w, v + w, w\}$ is also a basis of V.

(c) Find a basis of the space $U = \{p \in P_4(\mathbb{R}) : p(6) = 0\}$.

(20) Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional.

ANSWERS

(1) (a) No (b) Yes (c) No (d) Yes (e) Yes

(2) No

(3) (a) Yes (b) Yes (c) No (d) No

(4) $\left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$

(5) Yes

(6)

(7) (a) $V = \mathbb{R}^2, W = \{(x, 0) : x \in \mathbb{R}\}, U_1 = \{(0, y) : y \in \mathbb{R}\}, U_2 = \{(x, y) \in \mathbb{R}^2 : x = y\}$ then $V = U_1 \oplus W = U_2 \oplus W$ but $U_1 \neq U_2$.

(8) (a) Yes (b) No (c) No

(9) No

(10) Yes

(11)

(12) (a) Linearly dependent (b) Linearly independent

(13) Not same

(14) $\{(1, 1, 1)\}$

(15) (a) Linearly dependent (b) Linearly independent

(16)

(17) (a) $\{(1, 3, 3), (1, 5, -1)\}$ (answers may vary)
(b) $\{(1, 1, -1, 2), (2, 1, 3, -4)\}$ (answers may vary)

(18) $\{f_1, f_2, f_3, f_4\}$ where $f_4(x) = x^3$ (answers may vary)

(19) (a) $\{(1, 2, 0, 3), (1, 2, 1, 0), (0, 1, 1, 1)\}$ and dimension is 3
(b)
(c) $\{(x - 6), x(x - 6), x^2(x - 6), x^3(x - 6)\}$.

(20)