Indian Institute of Technology Roorkee
Spring Semester 2023-24
MAI-102 (Mathematics II)
Assignment 5

Topics: (Discrete): Binomial, Poisson, Negative binomial, and Geometric distributions. (Continuous): Uniform, Exponential, Gamma, and Normal distributions.

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(1) (a) Let X be a random variable having a binomial distribution with parameters $n = 25$ and $p = 0.2$. Evaluate $P[X < \mu - 2\sigma]$.

(b) Using Chebyshev's inequality, show that the probability that in 900 flips of a balanced coin the head will occur between 360 and 540 times is at least $\frac{35}{36}$.

(c) Suppose an unbiased die is cast at random seven times independently. What is the probability that each side appears at least once given that side 1 appears exactly twice?

(d) Let X denote the number of successes throughout n independent repetitions of a random experiment having probability of success $\frac{1}{4}$. Determine the smallest value of n so that probability of at least one success is at least 0.80.

(2) If X is a random variable such that $E[X] = 10$ and $\sigma_X = 3$, can X have a negative binomial distribution $NB(r; p)$ in which X represents the number of failures preceding the rth success? Explain.

(3) If the probability is 0.75 that any person will believe a rumour, find the probabilities that
(i) the fifth person to hear it is the first to believe it,
(ii) the eighth person to hear the rumour will be the fifth to believe it,
(iii) at least 4 persons do not believe the rumour before the tenth person believes it.

(4) The probability of a successful rocket launching equals 0.8. Suppose that launching attempts are made until 3 successful launchings have occurred.
(a) Find the probability that (i) exactly 6 attempts will be required (ii) fewer than 6 attempts will be required.
(b) Suppose that each launching costs Rs. 5 lakh. In addition, a launching failure results in an additional cost of Rs. 50 thousand. Evaluate the expected cost for launchings (for attempts until 3 successful launchings have occurred).

(5) Let X have a geometric distribution, where X represents the number of failures before the first success. Show that $P(X \geq k + j | X \geq k) = P(X \geq j)$ (the memoryless property of X), where k, j are non-negative integers.

(6) (a) If a random variable X has a Poisson distribution such that $P(X = 1) = P(X = 2)$, find (i) $P(X = 4)$, (ii) $P(X \geq 3)$.

(b) A merchant has found that the number of items of brand A that he can sell in a day is a Poisson random variable with mean 4. What is the expected number of days out of 25 that the merchant will sell no items of brand A?

(7) There are 270 typographical errors in a book of 675 pages, and these errors are randomly distributed throughout the book. Find the probabilities that (i) a randomly chosen page will have more than two errors (ii) exactly three of the five randomly chosen pages will be free of errors (iii) five randomly chosen pages will have less than three errors altogether.

(8) Suppose that a container contains $10,000$ particles. The probability that such a particle escapes from the container is equal to 0.0004. What is the probability that more than 5 such escapes occur?

(9) A radioactive source is observed during 7 time intervals of 10 seconds each. If the number of particles emitted during each time interval follows a Poisson probability law with a rate of 0.5 particles per second, what is the probability that there are at least two time intervals during each of which no more than 2 particles are emitted.

(10) (a) If the mgf of a random variable X is $\left(\frac{2}{3}+\frac{1}{3}e^t\right)^9$, determine $P\left(|X-\mu|<2\sigma\right)$.

(b) If the mgf of a random variable X is $e^{100(e^t-1)}$, find a lower bound on $P(75<X<125)$ using Chebyshev's inequality.

(11) Find the moment generating function of the random variable X having the pdf

$$f_X(x)=\begin{cases}\frac{1}{3}, & -1<x<2,\\ 0, & \text{elsewhere.}\end{cases}$$

Find its mean and variance.

(12) Suppose that a certain type of electronic component has an exponential distribution with a mean life of 500 hours. If X denotes the life of this component (or the time to failure of this component), then what is the probability that the component will last an additional 600 hours, given that it has operated for 300 hours?

(13) Two programmers started working on their computer programs at 9:00 am. The time taken to finish the program by the first programmer is uniformly distributed with parameters $a=1$ hr, $b=4$ hrs. The time taken by the second programmer follows a gamma distribution with the parameters $\alpha=5, \beta=1$ hrs^{-1}. Which programmer has a better chance to finish before 11:00 am?

(14) In studies of anticancer drugs it was found that if mice are injected with cancer cells, the survival time can be modeled with the exponential distribution. Without treatment the expected survival time was 10 hrs. What is the probability that

(a) A randomly selected mouse will survive at least 8 hrs? At most 12 hrs? Between 8 and 12 hrs?

(b) The survival time of a mouse exceeds the mean value by more than 2 standard deviations?

(15) (a) Let X have an exponential distribution. If x, y are positive real numbers, show that

$$P(X>x+y|X>x)=P(X>y).$$

Thus the exponential distribution has the memoryless property.

(b) Let X be a random variable such that $E(X^m)=(m+1)!2^m$, $m=1,2,3,\ldots$. Determine the mgf and the distribution of X.

(c) Let X have a gamma distribution with pdf $f_X(x)=\dfrac{1}{\beta^2}xe^{-x/\beta}$, $0<x<\infty$, zero elsewhere. If $X=2$ is the unique mode of the distribution, find the parameter β and the distribution.

(16) Find the uniform distribution of the continuous type on the interval (b,c) that has the same mean and the same variance as those of a χ^2 distribution with 8 degrees of freedom. That is, find b and c.

(17) (a) If $X \sim N(\mu, \sigma^2)$, then find b so that $P[-b < (X-\mu)/\sigma < b] = 0.90$, by using table.
(b) Let $X \sim N(\mu, \sigma^2)$ so that $P(X < 89) = 0.90$ and $P(X < 94) = 0.95$. Find μ and σ^2.
(c) Show that the constant c can be selected so that $f(x) = c2^{-x^2}$, $-\infty < x < \infty$, satisfies the conditions of a normal pdf.
(d) Evaluate $\int_2^3 \exp[-2(x-3)^2]dx$
(e) If e^{3t+8t^2} is the mgf of the random variable X, then find the distribution of X and determine $P(-1 < X < 9)$.
(f) If $X \sim N(\mu, \sigma^2)$, then show that $E(|X-\mu|) = \sigma\sqrt{2/\pi}$.
(g) If $X \sim N(1, 4)$, where $\sigma^2 = 4$, compute the probability $P(1 < X^2 < 9)$.

(18) Based on extensive data from an urban freeway near Toronto, Canada, "it is assumed that free speeds can best be represented by a normal distribution". The mean and standard deviation reported were 119 km/hr and 13.1 km/hr, respectively.
(a) What is the probability that the speed of a randomly selected vehicle is between 100 and 120 km/hr?
(b) What speed characterizes the fastest 10% of all speeds?
(c) The posted speed limit was 100 km/hr. What percentage of vehicles was traveling at speeds exceeding this posted limit?
(d) If five vehicles are randomly and independently selected, what is the probability that at least one is not exceeding the posted speed limit?
(e) What is the probability that the speed of a randomly selected vehicle exceeds 70 miles/hr?

(19) The plasma cholesterol level (mg/dL) for patients with no prior evidence of heart disease who experience chest pain is normally distributed with mean 200 and standard deviation 35. Consider randomly selecting an individual of this type. What is the probability that the plasma cholesterol level
(a) is at most 250?
(b) is between 300 and 400?
(c) differs from the mean by at least 1.5 standard deviations?

(20) There are two machines available for cutting corks intended for use in badminton shuttlecocks. The first produces corks with diameters that are normally distributed with mean 3 cm and standard deviation 0.1 cm. The second machine produces corks with diameters that have a normal distribution with mean 3.04 cm and standard deviation 0.02 cm. Acceptable corks have diameters between 2.9 and 3.1 cm. Which machine is more likely to produce an acceptable cork?

ANSWERS

(1) (a) $(0.8)^{25}$ (1) (b) $\frac{35}{36}$ (1) (c) $\frac{24}{625}$ (1) (d) 6

(2) No (3) (i) $(0.25)^4(0.75)$ (ii) $35(0.75)^5(0.25)^3$ (iii) $\sum_{r=1}^{6}\binom{9}{r-1}(0.75)^r(0.25)^{10-r}$

(4) (a) (i) 0.041 (ii) 0.94208 (b) 19.125 lakh

(6) (a) (i) $\frac{2}{3}e^{-2}$ (ii) $1-5e^{-2}$ (6) (b) 0.458 (approx.)

(7) (i) $1-(1.48)e^{-0.4}$ (ii) $10e^{-1.2}(1-e^{-0.4})^2$ (iii) $5e^{-2}$

(8) 0.215 (9) $1-2(0.875)^7$ (approx.)

(10) (a) $\sum_{x=1}^{5}\binom{9}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{9-x}$ (10) (b) 0.84

(11) $M_X(t)=\begin{cases}\frac{e^{2t}-e^{-t}}{3t}, & t\neq 0\\ 1, & t=0,\end{cases}$ $\mu=\frac{1}{2}$, $\sigma^2=\frac{3}{4}$.

(12) $e^{-6/5}$.

(13) The first programmer has a better chance to finish before 11 : 00 am.

(14) (a) 0.4493, 0.6988 and 0.1481.
(b) 0.0498.

(15) (b) $M_X(t)=(1-2t)^{-2}$ and $X\sim\chi^2(4)$.
(c) $\beta=2$ and $X\sim\chi^2(4)$.

(16) $b=8-4\sqrt{3}$ and $c=8+4\sqrt{3}$.

(17) (a) 1.645.
(b) 71.3, 189.7.
(c) $c=1/[\sqrt{2\pi}\sqrt{1/(2\ln 2)}]$.
(d) $\sqrt{\frac{\pi}{2}}\left[\frac{1}{2}-\Phi(-2)\right]=0.598$.
(e) $X\sim N(3,16)$ and 0.774.
(f)
(g) 0.477.

(18) (a) 0.4584.
(b) 135.8 km/hr.
(c) 92.65%.
(d) 0.3173.
(e) 0.6844.

(19) (a) 0.9236.
(b) 0.0021.
(c) 0.1336.

(20) Second machine.