Comparison between the Extended Kalman Filter (EKF) and the Unscented Kalman Filter (UKF) for state estimation of pendulum systems

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Abstract

This paper compares two variants of the Kalman filter, the extended Kalman filter (EKF) and the unscented Kalman filter (UKF). The EKF is an early extension of the Kalman filter that uses linearization to be able to estimate nonlinear dynamical systems with the Gaussian distributions. This approach is desirable because of its computational efficiency but comes at the cost of some risks, like divergence and inaccuracy of estimates for high nonlinearities. The UKF was developed to counteract those shortcomings by using a set of points around the current estimate to compute a new estimate instead of linearizing around the current estimate. The two filters are compared using the single and double pendulum in different configurations and the performance is analyzed by the mean squared error and the uncertainty of the estimates. It is shown that the UKF generally estimates the states of the pendulums better, however the EKF still gets good estimates even when the system is expected to be more difficult to estimate.

Introduction

State estimation has been a problem for many applications such as localization, tracking, simultaneous localization and mapping (SLAM) and many more. One of the most popular and widely used family of filters for state estimation are Gaussian filters. The most well known filter of this family is the Kalman filter which was developed in the 1950s. However, the standard Kalman filter can only be applied to linear dynamical systems since it approximates the state with the first and second moments of a Gaussian distribution [6]. Since most real-world applications have some non-linearities, this is a problem. This is why the extended Kalman filter (EKF) was developed in the early 1960s by NASA for the estimation of the position and velocity of a spaceship [5]. In short, the EKF uses the linearized system and measurement equations to compute the covariance matrix and the Kalman gain. This can lead to some problems, such as divergence or inaccurate estimations depending on the degree of the non-linearities and the degree of the uncertainty in the belief [6]. Despite these problems, the EKF is widely used in common estimation problems such as navigation or tracking of various vehicles [7]. Still, there are some alternative methods to better handle nonlinear estimation tasks, such as the unscented Kalman filter and the iterative extended Kalman filter [1]. In this paper, our goal is to compare the EKF and the UKF for estimating the states of a pendulum and a double pendulum, two nonlinear systems, to show that the UKF does indeed deliver the better estimate and to compare other properties such as the stability of the estimate and the uncertainty. The results indicate that the UKF estimate is overall more stable and in most cases slightly better, although it can be argued about since the margin is quite small.

Related Work

The extended Kalman filter was developed by the NASA Ames to estimate the position and velocity of rockets for guidance purposes [5]. The reason for not using the already existing particle filter was for computational purposes since the estimator had to be integrated into the on-board computer. Even though it is widely used, the EKF suffers from some problems like divergence or inaccuracy for higly nonlinear systems. The predecessor of the unscented Kalman filter (UKF), an alternative approach to estimate the state of nonlinear dynamical systems, was introduced by the authors of [4]. Instead of linearly approximating the system like the extended Kalman filter, the UKF propagates a set of points around the current mean through the system equations to generate a new mean by taking a weighted average of the new points. Similarly, the covariance is computed by the weighted difference between the predicted mean and each of the propagated points. This procedure avoids the computation of Jacobians completely and therefore leads to a more accurate estimate up to the fourth order as opposed to the EKF which correctly predicts nonlinear systems up to the second order [4]. The filter depends on how the so-called sigma points, that are propagated through the system to generate a new mean and covariance, are chosen. The unscented Kalman filter generates a minimum number of sigma points as described in [3] and transforms them using the unscented transform presented in [2]. The authors of [1] compared the extended Kalman filter witht he iterated EKF and a linear regression Kalman filter (LRKF), where the UKF implementation by Julier was used as the LRKF. It was found that the UKF was better at handling nonlinear process functions because it takes an area around the interest point into account, not just the point itself and it does not need the Jacobian, which can be difficult or impossible to obtain.

Methodology

This section will go over the methodology and implementation used to evaluate the behaviour of the extended Kalman filter compared to the unscented Kalman filter. The systems used for the comparison of the filters are the single pendulum as well as the double pendulum as the systems are quite straightforward to implement and contain some nonlinearities. The single pendulum was mainly used for feasibility and testing reasons, although it gives some interesting results during the experiments. The main part of the experiments was conducted on the double pendulum which contains some higher nonlinearities and should be more difficult to estimate due to its chaotic nature. The motion of the pendulum was simulated in the ideal case, i.e. without loss of energy using the 4th order Runge-Kutta method with time step $dt = 0.1s$. The Runge-Kutta method is a numerical solver for differential equations. The state vectors of the the single pendulum consists of the angle θ with respect to the vertical axis and the angular velocity ω of the mass at the end of the pendulum. The state vector of the double pendulum contains of the angles of the upper and lower rods θ_1 and θ_2 with respect to the vertical axis as well the angular velocities ω_1 and ω_2. The gravitational acceleration is g and the lengths and masses of the pendulums are noted as L_i and m_i for $i = 1, 2$, respectively. This results in the nonlinear motion models in equations 1 and 2 . The measurement model, however, is linear and only consists of the reading of the angle(s). Since there was no noise added to the measurements, they are simultaneously the ground truth of the pendulum angle.

Differential equations for single pendulum:

$$\begin{aligned} \theta(k+1) &= \theta(k) + \omega(k)dt \\ \omega(k+1) &= \omega(k) - gsin(\theta(k))/L \end{aligned} \tag{1}$$

Differential equations for double pendulum:

$$\begin{aligned} \theta_1(k+1) &= \theta_1(k) + \omega_1(k)dt \\ \theta_2(k+1) &= \theta_2(k) + \omega_2(k)dt \\ \omega_1(k+1) &= \omega_1(k) \\ &\quad + \frac{g(2m_1 m_2)sin(\theta_1) - m_2 gsin(\theta_1 - 2\theta_2)}{L_1(2m_1 + m_2 - m_2 cos(2\theta_1 - 2\theta_2))} \\ &\quad - \frac{2sin(\theta_1 - \theta_2)m_2(\omega_2^2 L_2 + \omega_1^2 L_1 cos(\theta_1 - \theta_2))}{L_1(2m_1 + m_2 - m_2 cos(2\theta_1 - 2\theta_2))} dt \\ \omega_2(k+1) &= \omega_2(k) \\ &\quad + \frac{2sin(\theta_1 - \theta_2)(\omega_1^2 L_1(m_1 + m_2)}{L_2(2m_1 + m_2 - m_2 cos(2\theta_1 - 2\theta_2))} \\ &\quad + \frac{2sin(\theta_1 - \theta_2)(g(m_1 + m_2)cos(\theta_1))}{L_2(2m_1 + m_2 - m_2 cos(2\theta_1 - 2\theta_2))} \\ &\quad + \frac{2sin(\theta_1 - \theta_2)(\omega_2^2 L_2 m_2 cos(\theta_1 - \theta_2))}{L_2(2m_1 + m_2 - m_2 cos(2\theta_1 - 2\theta_2))} dt \end{aligned} \tag{2}$$

For the estimation of the states, the EKF and UKF algorithms calculate the prediction and update for each time step dt in the simulation. The extended Kalman filter calculates the Jacobian of the state transition model G and calculates the prediction by propagating the previous estimate $\bar{\mu}$ through the nonlinear motion model $g(\cdot)$ with some additional white noise to increase the nonlinear behaviour and test the robustness of the filter. The prediction of the uncertainty $\bar{\Sigma}$ is obtained by multiplying it with the Jacobian from both sides and adding the process noise R. For the update step, the Jacobian of the measurement model H is needed, however, it is identical to the measurement model matrix since it is linear. The updated estimate μ is obtained by adding the innovation which is weighted by the Kalman gain K.

Equations for prediction step of the EKF:

$$\bar{\mu}_k = g(\mu_{k-1}, m, L, dt) + \mathcal{N}(0, 0.1) \tag{3}$$

$$\bar{\Sigma}_k = G^T \Sigma G + R \tag{4}$$

Equations for the update step of the EKF:

$$K = \bar{\Sigma}_k H^T / (H \bar{\Sigma}_k H^T + Q) \tag{5}$$

$$\mu_k = \bar{\mu}_k + K(z - H\bar{\mu}_k) \tag{6}$$

$$\Sigma_k = (I - KH)\bar{\Sigma}_k \tag{7}$$

The unscented Kalman filter first computes the sigma points χ around the previous estimation as well as the weight vectors W_m and W_c for the calculating the new mean and covariance respectively. The spread of the sigma points and the weights are determined by a scaling factor λ which is determined by two adjustable parameters α and κ and the dimension of the state vector n. The scaling factor is denoted as

$$\lambda = \alpha^2(n + \kappa) - n. \tag{8}$$

The sigma points are found using

$$\chi_k = \mu_{k-1} \pm \sqrt{(n + \lambda)\Sigma_{k-1}}. \tag{9}$$

The weights are given as follows, where β is an additional factor for the first weight of the covariance:

$$W_{m,1} = \lambda/(n + \lambda), \tag{10}$$

$$W_{c,1} = \lambda/(n + \lambda) + (1 - \alpha^2 + \beta), \tag{11}$$

$$W_{m,i} = W_{c,i} = 1/(2((n + \lambda)), \tag{12}$$

for $i = 2, ..., 2n + 1$.

The sigma points are then propagated through the nonlinear motion model $g(\cdot)$ with some additional white noise to increase the nonlinear behaviour and their weighted sum is used to get the predicted mean. The predicted covariance is obtained by the weighted sum of the differences between each sigma point and the predicted mean.

Equations for prediction step of the UKF:

$$\bar{\chi}_i = g(\chi_i, m, L, dt) + \mathcal{N}(0, 0.1) \tag{13}$$

$$\bar{\mu}_k = \sum_i W_{m,i} \bar{\chi}_i \tag{14}$$

$$\bar{\Sigma}_k = \sum_i (\bar{\chi}_i - \bar{\mu}_k) W_{c,i} (\bar{\chi}_i - \bar{\mu}_k)^T \tag{15}$$

In the update step, the measurements are incorporated similarly to the extended Kalman filter, however, the Kalman gain K is computed using the measurement covariance Σ_{yy}

and the cross-covariance between state and measurement Σ_{xy}.

Equations for the update step of the UKF:

$$\bar{Z}_i = C\bar{\chi}_i \tag{16}$$

$$\hat{Z} = \sum_i W_{m,i}\bar{Z}_i \tag{17}$$

$$\Sigma_{xy} = \sum_i (\bar{\chi}_i - \bar{\mu}_k)W_{c,i}(\bar{Z}_i - \hat{Z}) \tag{18}$$

$$\Sigma_{yy} = \sum_i (\bar{Z}_i - \hat{Z})W_{c,i}(\bar{Z}_i - \hat{Z}) \tag{19}$$

$$K = \Sigma_{xy}/\Sigma_{yy} \tag{20}$$

$$\mu_k = \bar{\mu}_k + K(z - \hat{Z}) \tag{21}$$

$$\Sigma_k = \bar{\Sigma}_k - K\Sigma_{yy}K^T \tag{22}$$

It is also worth mentioning that the implementation of the UKF only required minor changes going from the single pendulum to the double pendulum while the EKF required a recalculation of the Jacobian which was much more complex for the double pendulum than for the single pendulum.

Experimental results

For the experiments we mainly changed the magnitude of the measurement and process noises and their ratio, as well as the starting angle of the pendulum. The results were evaluated by looking at the plots of the estimated angles versus the ground truth, as well as the mean squared error and the covariance of the angle estimate. All of the tests were performed with the same initial conditions and the same measurement and process noises. The parameters used for the UKF algorithm were obtained experimentally and are $\alpha = 1$, $\kappa = 0$ and $\beta = 2$. Since the mean squared error of the estimates of the angular velocities did not bring any additional insight, this measure will not be further discussed.

Single Pendulum

The simulations for the single pendulum overall show that the behaviour of the two filters is quite similar for all configurations. It can be pointed out that the uncertainty of the EKF estimate is generally lower than that of the UKF estimate, and the error of the UKF estimate is generally smaller than the error of the EKF estimate, however, only by an arguably insignificant margin. First, we'll have a look at the case of low measurement and process noises with a small starting angle, i.e. $\theta_0 = \pi/4$, in figure 1. The figure shows the similar behaviour of the two filters which is to be expected because the single pendulum does not have high order non-linearities, especially not in the case of a lower starting angle where the linear approximation of the extended Kalman filter is quite accurate. However, the error of the UKF estimate is a bit lower which might be due to the fact that the noise parameters used are not optimal for the EKF. For starting angles near $0rad$, the two filters become nearly identical, even with the given noise parameters. The figure 1 also shows the different uncertainties between the two filters. It is expected that the uncertainty of the EKF is lower

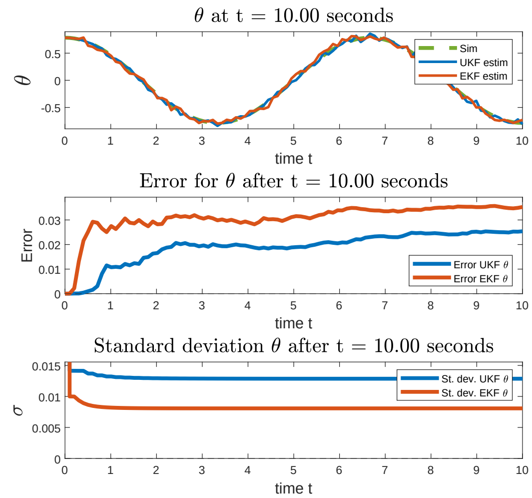

Figure 1: Position, error and covariance plots for the UKF and EKF estimates of the single pendulum with starting angle $\theta_0 = \pi/4$ and low process and measurement noise.

since it is known to become overconfident in its belief and thus underestimate the covariance. Before analysing the next case, it should be mentioned that the changing the measurement and process noise by the same factor resulted in the same results except that the standard deviation of the angle estimate is scaled by the square root of the scaling factor, which makes sense because it won't change how much the measurements influence the estimate in the update step.

Next, we'll have a look at the different noise ratios. For simplicity, we will define ratio_1 to correspond to high process noise vs low measurement noise and ratio_2 to correspond to low process noise vs high measurement noise. For ratio_1, see figure 2, the error for both the UKF and the EKF estimate is much lower than for the case with low process and measurement noise, see figure 1. In this case, the two filters both estimate the system almost perfectly. Regarding the uncertainty, the standard deviation of the EKF estimate is about the same as with the low noise parameters while the uncertainty of the UKF estimate has increased. It can also be noted that the standard deviation of both estimates converges almost instantly. Figure 2 only shows the results for the starting angle $\theta_0 = \pi/2$, however, it is worth mentioning that the results for the smaller starting angle look very similar.

The results for ratio_2 can be seen in figure 3. In contrast to ratio_1 the mean squared error is higher than the error of the estimates with low process and measurement noise for both filters. The errors are very similar in the beginning but the error of the EKF estimate increases after $8s$ and seems to keep increasing after the simulation time, while the UKF error remains quite stable. The uncertainty of the UKF estimate decreased compared to the case with ratio_1 while the opposite is true for the EKF estimate. This results in the fact that the standard deviation of both estimates converge to the same value which is quite interesting. Furthermore, the standard deviations converge much slower than for the case with

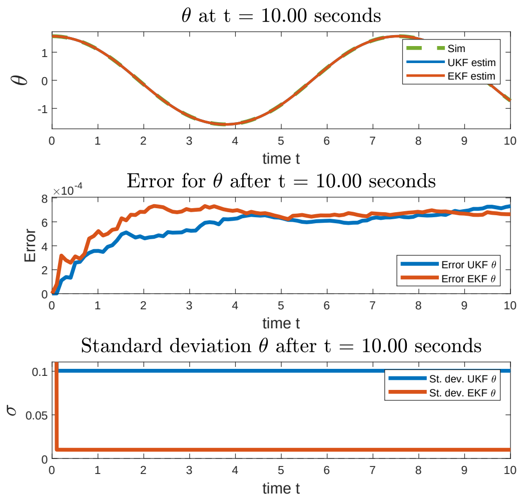

Figure 2: Position, error and covariance plots for the UKF and EKF estimates of the single pendulum with starting angle $\theta_0 = \pi/2$ and high process noise vs low measurement noise (ratio_1).

ratio_1 or even low process and measurement noise.

Double Pendulum

For the double pendulum the results are not quite as they were expected, since the UKF only performed marginally better than the EKF. However, it can be said for sure that the UKF is more stable and robust to changes in the measurement vs process noise ratio. The reason why the difference in performance is so small is not entirely clear but will be discussed in this section. The experiments were also conducted with two different starting configurations,

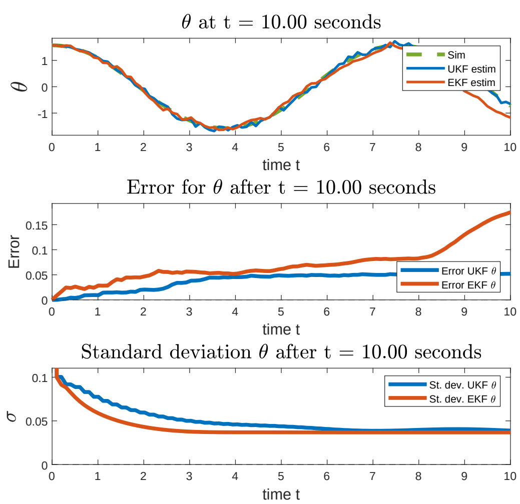

Figure 3: Position, error and covariance plots for the UKF and EKF estimates of the single pendulum with starting angle $\theta_0 = \pi/2$ and low process noise vs high measurement noise (ratio_2).

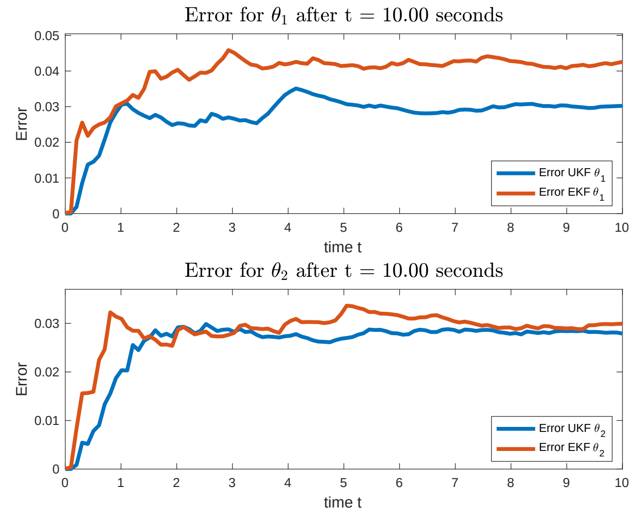

Figure 4: Error plots for the UKF and EKF estimates of the double pendulum with starting angles $\theta_{1,0} = \theta_{2,0} = \pi/4$ and low process and measurement noise.

$\theta_{1,0} = \theta_{2,0} = \pi/4$ and $\theta_{1,0} = \theta_{2,0} = \pi/2$. Additionally, the measurement and process noises and their ratio were altered. Similarly to the single pendulum, the performance of the filters is analyzed by their mean squared error and covariance of the angle estimates.

First we will look at the different measurement and process noises for the starting angle $\theta_{1,0} = \theta_{2,0} = \pi/4$. The first scenario will be for equally low measurement and process noise. As can be seen in figure 4, the mean squared error for the UKF estimate of the angle θ_1 is much lower than for the EKF estimate. For the second angle, there is barely any difference between the errors of both algorithms. The results of the estimte of θ_1 look similar to the ones for the single pendulum with the same initial conditions. The reason for that could be that the system is not chaotic or nonlinear enough with the given starting angle.

The figure 5 shows the standard deviations of the estimates. As expected, the UKF estimate has a slightly higher standard deviation, however, it is also more stable since there are minor fluctuations in the standard deviation of the EKF estimate. Compared to the single pendulum, the standard deviation of the UKF estimate is very similar, but the standard deviation of the EKF estimate seems to be a bit unstable which makes sense considering the higher complexity of the double pendulum system.

Next, the ratio between the measurement and process noise is altered using starting angles $\theta_{1,0} = \theta_{2,0} = \pi/4$. For the case of high process noise and low measurement noise (ratio_1), see figure 6, the behaviour of the standard deviations is similar to the single pendulum with the same initial conditions. Both the UKF and the EKF estimate are very stable and converge almost immediately and the difference in uncertainty is very high, as the EKF is more confident than the UKF. For the higher starting angle, the standard deviations of the estimates also look the same. Similarly to the single pendulum, the mean squared errors of the estimates also have a lower magnitude in this scenario than with the noise ratio 1.

For the case of low process and high measurement noise (ratio_2) seen in figure 7, the EKF estimate is very unstable and the UKF estimate has some small fluctuations but is stable compared to the EKF. This was observed for both the lower and the higher initial angle and shows that the UKF is pretty robust to changes of the noise parameters compared to the EKF. This behaviour was not observed for the single pendulum, so the added complexity of the system might cause the estimate to become very unstable. Additionally, the mean squared error of the EKF estimate for this noise ratio steadily increases for both the single and the double pendulum and seems to keep increasing after the recorded time while the error of the UKF estimate seems to flatten out.

Lastly, the results for the double pendulum starting at $\theta_{1,0} = \theta_{2,0} = \pi/2$ with high process and measurement noise, seen in figure 8, will be analysed. Even though the noise was high, there was no big difference between the performance of the EKF and the UKF regarding the mean squared error for the angle of the upper rod. The error of the second angle starts off bad for the EKF estimate but it seems to converge towards the error of the UKF estimate which is quite steady. The bigger difference in performance can be seen in the plots of the standard deviation of the estimates in figure 9. Here, the effect that was seen for the lower starting angle with low process and measurement noise in figure 5 is amplified, as the standard deviation of the EKF estimates varies quite a bit. In contrast to the noise ratio_2 though, the value stays below the standard deviation of the UKF estimates. This means that having high measurement noise leads to some fluctuation in the uncertainty of the EKF estimate.

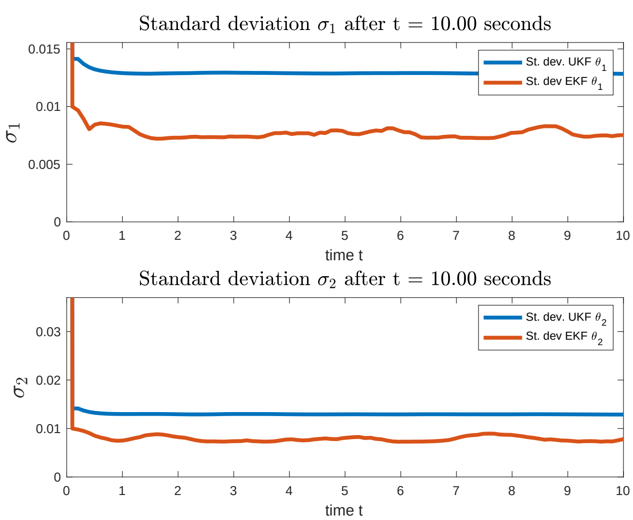

Figure 5: Standard deviation plots for the UKF and EKF estimates of the double pendulum with starting angles $\theta_{1,0} = \theta_{2,0} = \pi/4$ and low process and measurement noise.

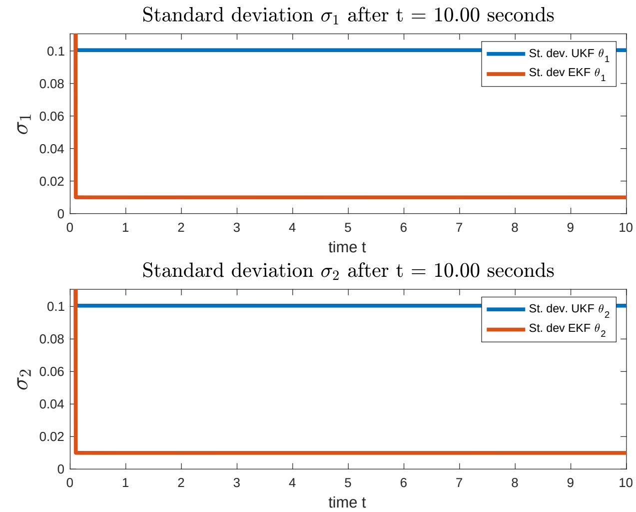

Figure 6: Standard deviation plots for the UKF and EKF estimates of the double pendulum with starting angles $\theta_{1,0} = \theta_{2,0} = \pi/4$ and high process vs low measurement noise (ratio_1).

Discussion

In general, the hypothesis that the unscented Kalman filter estimates nonlinear systems better than the extended Kalman filter is met, as its mean squared error is always smaller or approximately equal to the mean squared error of the EKF estimate. For the single pendulum it was expected that the EKF estimate would be close to the UKF estimate since the system is not highly nonlinear and can be well approximated by linearization. For the double pendulum, however, it was expected that the UKF estimate would be much

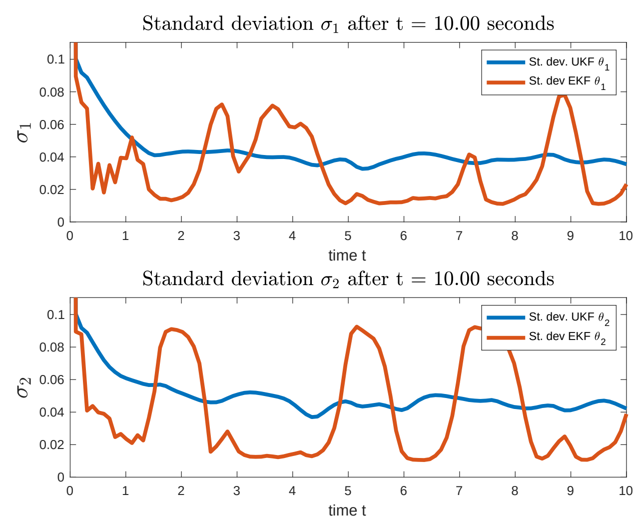

Figure 7: Standard deviation plots for the UKF and EKF estimates of the double pendulum with starting angles $\theta_{1,0} = \theta_{2,0} = \pi/4$ and low process vs high measurement noise (ratio_2).

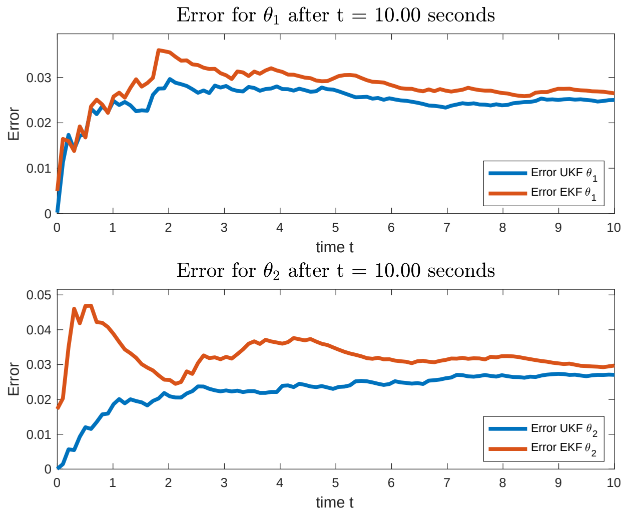

Figure 8: Error plots for the UKF and EKF estimates of the double pendulum with starting angles $\theta_{1,0} = \theta_{2,0} = \pi/2$ and high process and measurement noise.

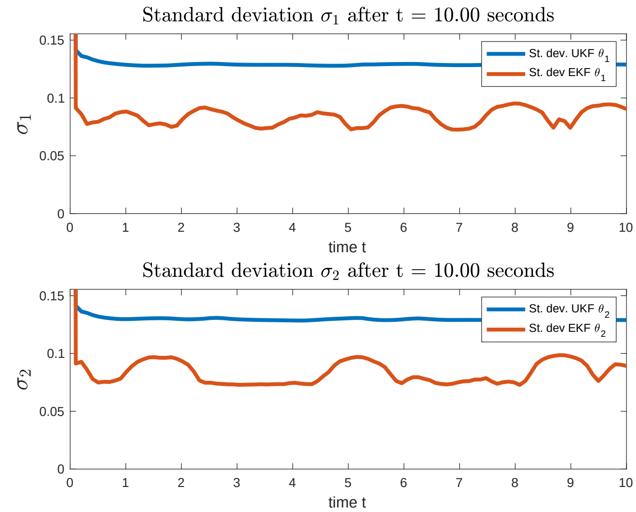

Figure 9: Standard deviation plots for the UKF and EKF estimates of the double pendulum with starting angles $\theta_{1,0} = \theta_{2,0} = \pi/2$ and high process and measurement noise.

better than the EKF because of the chaotic behaviour of the double pendulum. It is hard to pinpoint why the two estimates were so close, even in the more nonlinear system. A possible explanation could be that comparing the two filters with the same noise ratios might not be the best way, because both filters react differently to noise, so one filter could perform best for a certain noise ratio while the other one shows best performance for another noise ratio. This means that the best configuration of the UKF should be much better than the best configuration of the EKF. Another interesting observation was that the EKF was especially affected by changes of the noise ratio and it performed best for the noise ratio that represented the noise ratio of the simulated system the best. In the simulation the sensor was modelled without any noise and the state transition model was modelled with white noise as described in *Methodology* which corresponds to the noise ratio_1 where the EKF showed its best results. For the opposite noise configuration, the EKF performed the worst, which then makes sense.

Conclusion

This paper compares the performance of the extended Kalman filter and the unscented Kalman filter in the task of estimating the state of a single and a double pendulum. The two filters were tested by running different configurations of the single and double pendulum and estimating the state composed of the angles and the angular velocities of the pendulums. The performance was evaluated using the mean squared error of the estimates and the uncertainty of the guesses. The hypothesis that the unscented Kalman filter performs better under nonlinear systems, especially if the nonliniearties are higher was partially confirmed since the UKF estimated the system states of the different pendulum configurations more accurately, however, in some cases the EKF estimate was still very good. The reason for that might be that the systems were not nonlinear enough. Additionally, it was found that the EKF tends to be very reactive to changes on the process and measurement noise ratio while the UKF showed robustness against these changes and consistently estimated the states of the pendulums well. Since the filters were only compared with the same noise configurations, it would be interesting to individually optimize the parameters for each filter and then compare their best version side by side. Additionally, since the systems didn't seem to be nonlinear enough,it would be interesting to see a comparison for highly nonlinear systems.

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