# Asymptotic Notation, Review of Functions & Summations

# Asymptotic Complexity

- Running time of an algorithm as a function of input size $n$ **for large $n$**.
- Expressed using only the **highest-order term** in the expression for the exact running time.
  - Instead of exact running time, say $\Theta(n^2)$.
- Describes behavior of function in the limit.
- Written using ***Asymptotic Notation***.

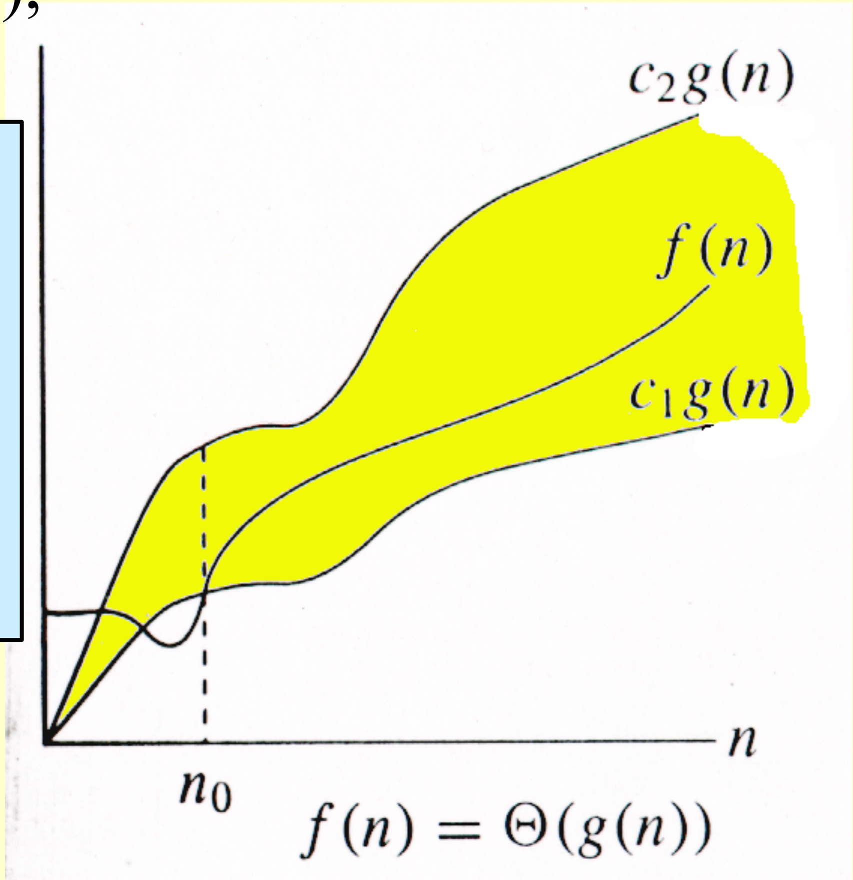
# Asymptotic Notation

- **$\Theta$, *O*, $\Omega$, *o*, $\omega$**
- Defined for functions over the natural numbers.
  - **Ex:** $f(n) = \Theta(n^2)$.
  - Describes how $f(n)$ grows in comparison to $n^2$.
- Define a ***set*** of functions; in practice used to compare two function sizes.
- The notations describe different rate-of-growth relations between the defining function and the defined set of functions.

# $\Theta$-notation

For function $g(n)$, we define $\Theta(g(n))$, big-Theta of $n$, as the set:

$\Theta(g(n)) = \{f(n)$ :
$\exists$ **positive constants $c_1$, $c_2$, and $n_0$,**
**such that $\forall n \geq n_0$,**
**we have $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$**
$\}$




***Intuitively***: Set of all functions that have the same *rate of growth* as $g(n)$.

**$g(n)$ is an *asymptotically tight bound* for $f(n)$.**

# $\Theta$-notation

For function $g(n)$, we define $\Theta(g(n))$, big-Theta of $n$, as the set:

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$\exists$ **positive constants** $c_1, c_2$, **and** $n_0$,
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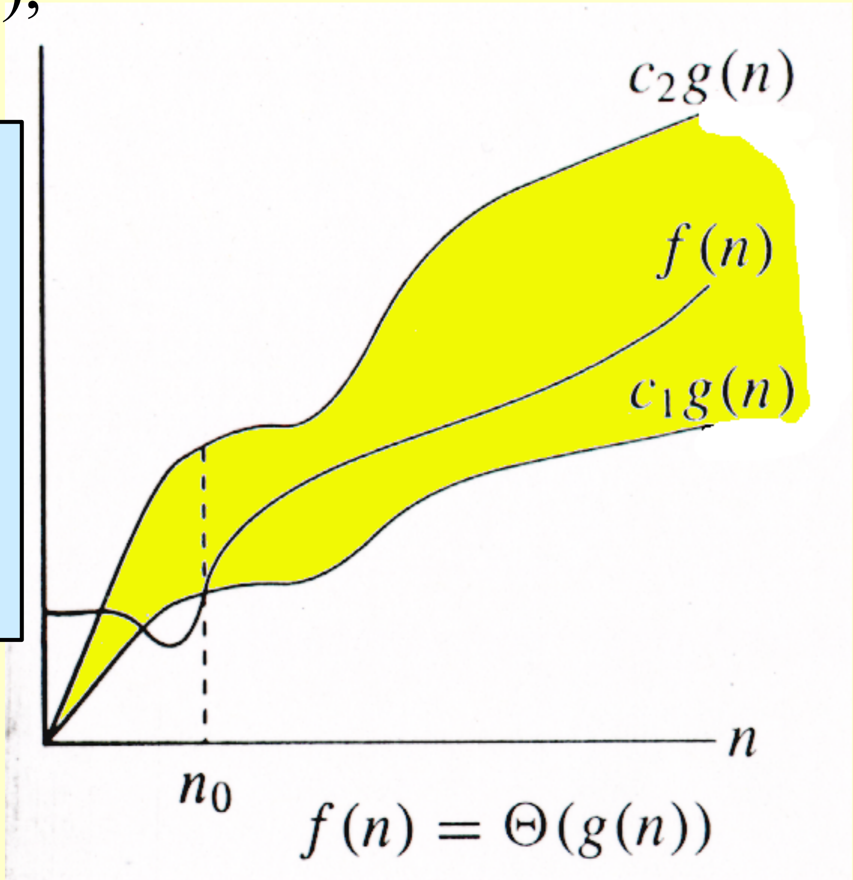



Technically, $f(n) \in \Theta(g(n))$.
Older usage, $f(n) = \Theta(g(n))$.
I'll accept either…

***f*(*n*) and *g*(*n*) are nonnegative, for large *n*.**

# Example

$\Theta(g(n)) = \{f(n) :$ **∃ positive constants $c_1$, $c_2$, and $n_0$, such that** $\forall n \geq n_0, \quad 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$

- $10n^2 - 3n = \Theta(n^2)$
- What constants for $n_0$, $c_1$, and $c_2$ will work?
- Make $c_1$ a little smaller than the leading coefficient, and $c_2$ a little bigger.
- ***To compare orders of growth, look at the leading term.***
- Exercise: Prove that $n^2/2-3n= \Theta(n^2)$

# Example

$\Theta(g(n)) = \{f(n) :$ **$\exists$ positive constants $c_1$, $c_2$, and $n_0$,**
**such that $\forall n \geq n_0$,** $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$

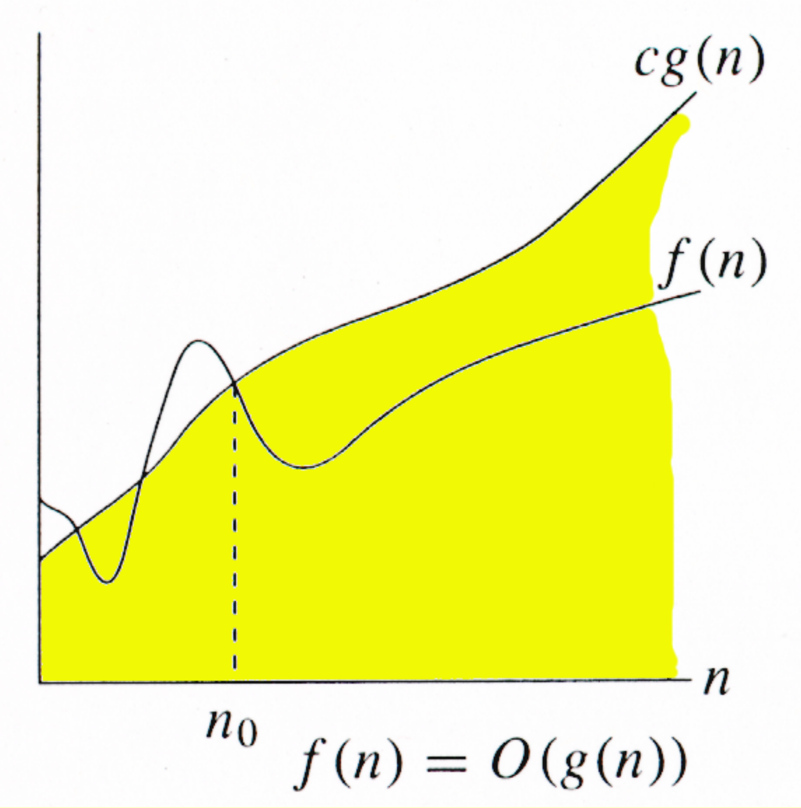
- Is $3n^3 \in \Theta(n^4)$ ??
- How about $2^{2n} \in \Theta(2^n)$??

# $O$-notation

For function $g(n)$, we define $O(g(n))$, big-O of $n$, as the set:

$O(g(n)) = \{f(n) :$
$\exists$ **positive constants** $c$ **and** $n_0$, **such that** $\forall n \geq n_0$,

**we have** $0 \leq f(n) \leq cg(n) \}$




***Intuitively***: Set of all functions whose *rate of growth* is the same as or lower than that of $g(n)$.

**$g(n)$ is an *asymptotic upper bound* for $f(n)$.**

$f(n) = \Theta(g(n)) \Rightarrow f(n) = O(g(n))$.

$\Theta(g(n)) \subset O(g(n))$.

# Examples

$O(g(n)) = \{f(n) :$ $\exists$ **positive constants** $c$ **and** $n_0$**, such that** $\forall n \geq n_0$**, we have** $0 \leq f(n) \leq cg(n)$ $\}$

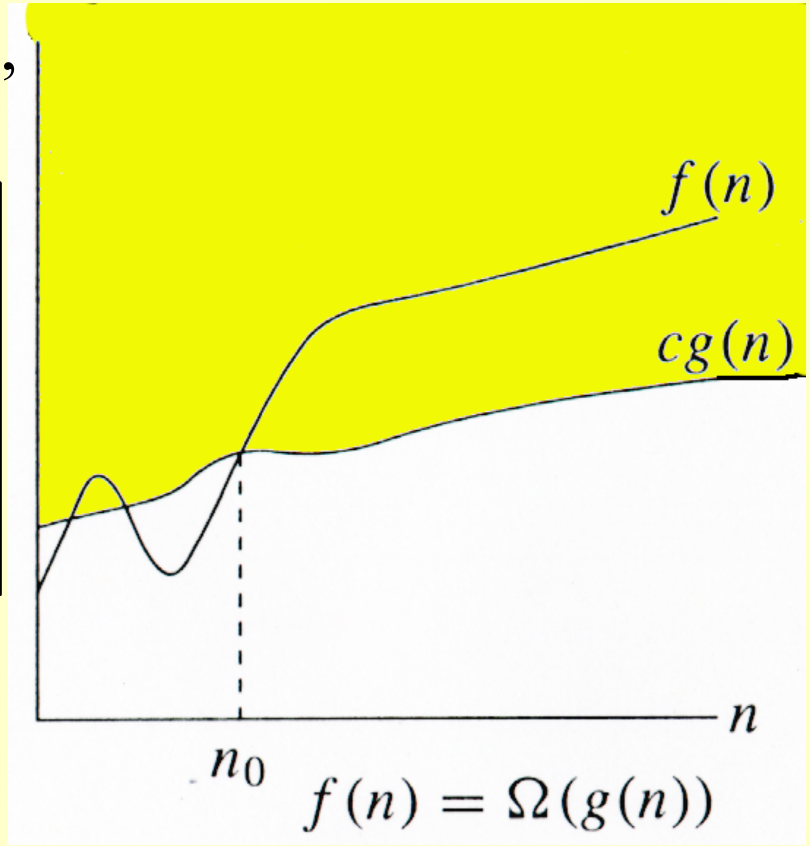
- Any linear *function* $an + b$ is in $O(n^2)$. **How?**
- Show that $3n^3 = O(n^4)$ for appropriate $c$ and $n_0$.

# $\Omega$ -notation

For function $g(n)$, we define $\Omega(g(n))$, big-Omega of $n$, as the set:

$\Omega(g(n)) = \{f(n) :$
$\exists$ positive constants $c$ and $n_0$, such that $\forall n \geq n_0$,
we have $0 \leq cg(n) \leq f(n)\}$

***Intuitively***: Set of all functions whose *rate of growth* is the same as or higher than that of $g(n)$.




**$g(n)$ is an *asymptotic lower bound* for $f(n)$.**

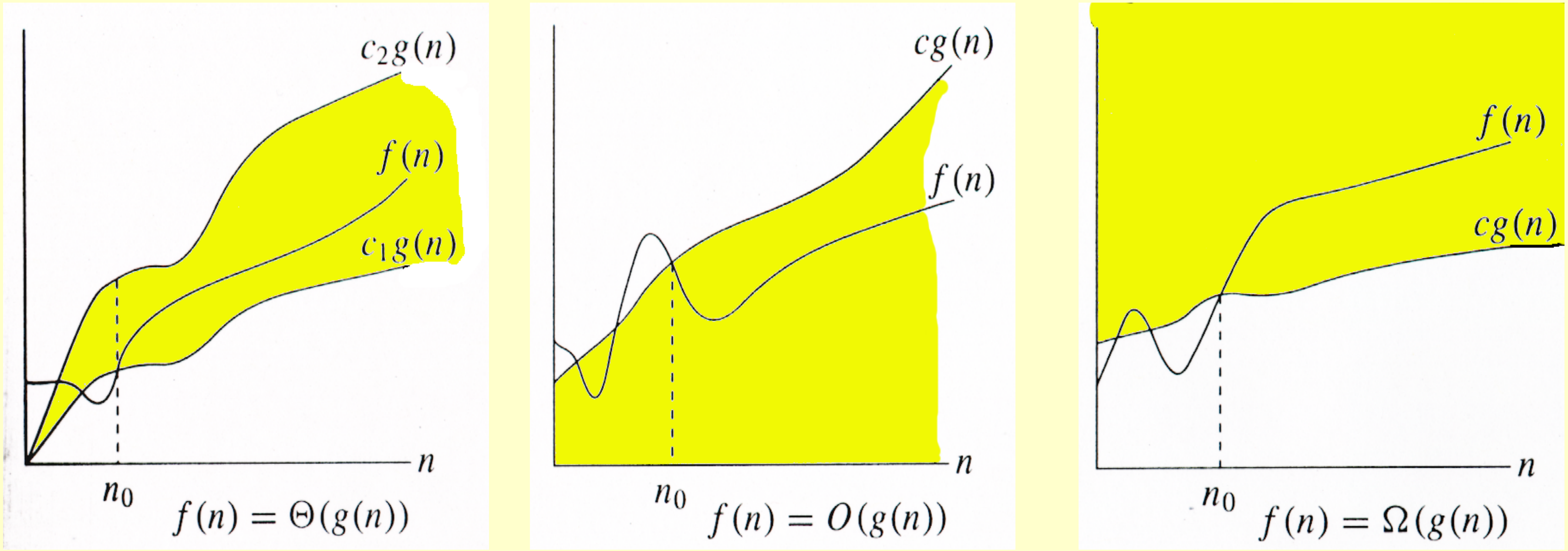
$f(n) = \Theta(g(n)) \Rightarrow f(n) = \Omega(g(n))$.
$\Theta(g(n)) \subset \Omega(g(n))$.

# Example

$\Omega(g(n)) = \{f(n) :$ $\exists$ positive constants $c$ and $n_0$, such that $\forall n \geq n_0$, we have $0 \leq cg(n) \leq f(n)\}$

- $\sqrt{n} = \Omega(\lg n)$. Choose $c$ and $n_0$.

# Relations Between Θ, *O*, Ω



# Relations Between $\Theta$, $\Omega$, $O$

**Theorem :** For any two functions $g(n)$ and $f(n)$,

$\boldsymbol{f(n) = \Theta(g(n))}$ iff

$\boldsymbol{f(n) = O(g(n))}$ **and** $\boldsymbol{f(n) = \Omega(g(n))}$.

- I.e., $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$
- In practice, asymptotically tight bounds are obtained from asymptotic upper and lower bounds.

# Running Times

- "Running time is $O(f(n))$" $\Rightarrow$ Worst case is $O(f(n))$
- $O(f(n))$ bound on the worst-case running time $\Rightarrow$ $O(f(n))$ bound on the running time of every input.
- $\Theta(f(n))$ bound on the worst-case running time $\not\Rightarrow$ $\Theta(f(n))$ bound on the running time of every input.
- "Running time is $\Omega(f(n))$" $\Rightarrow$ Best case is $\Omega(f(n))$
- Can still say "Worst-case running time is $\Omega(f(n))$"
  - Means worst-case running time is given by some unspecified function $g(n) \in \Omega(f(n))$.

# Example

- ***Insertion sort*** takes $\Theta(n^2)$ in the worst case, so sorting (as a *problem*) is $O(n^2)$. **Why?**

- Any sort algorithm must look at each item, so sorting is $\Omega(n)$.

- In fact, using (e.g.) merge sort, sorting is $\Theta(n \lg n)$ in the worst case.
  - Later, we will prove that we cannot hope that any comparison sort to do better in the worst case.

# Asymptotic Notation in Equations

- Can use asymptotic notation in equations to replace expressions containing lower-order terms.
- For example,
  $4n^3 + 3n^2 + 2n + 1 = 4n^3 + 3n^2 + \Theta(n)$
  $= 4n^3 + \Theta(n^2) = \Theta(n^3)$. **How to interpret?**
- In equations, $\Theta(f(n))$ always stands for an ***anonymous function*** $g(n) \in \Theta(f(n))$
  - In the example above, $\Theta(n^2)$ stands for $3n^2 + 2n + 1$.

# $o$-notation

For a given function $g(n)$, the set little-$o$:

$$o(g(n)) = \{f(n) : \forall\, c > 0, \exists\, n_0 > 0 \text{ such that } \forall\, n \geq n_0, \text{ we have } 0 \leq f(n) < cg(n)\}.$$

$f(n)$ becomes insignificant relative to $g(n)$ as $n$ approaches infinity:

$$\lim_{n \to \infty} [f(n) / g(n)] = 0$$

$g(n)$ is an ***upper bound*** for $f(n)$ that is not asymptotically tight.

Observe the difference in this definition from previous ones. **Why?**

# $\omega$ -notation

For a given function $g(n)$, the set little-omega:

$\omega(g(n))$ = $\{f(n)$: $\forall\, c > 0$, $\exists\, n_0 > 0$ such that $\forall\, n \geq n_0$, we have $0 \leq cg(n) < f(n)\}$.

$f(n)$ becomes arbitrarily large relative to $g(n)$ as $n$ approaches infinity:

$$\lim_{n \to \infty} [f(n) / g(n)] = \infty.$$

$g(n)$ is a ***lower bound*** for $f(n)$ that is not asymptotically tight.

# Comparison of Functions

$$f \leftrightarrow g \approx a \leftrightarrow b$$

$$f(n) = O(g(n)) \approx a \leq b$$

$$f(n) = \Omega(g(n)) \approx a \geq b$$

$$f(n) = \Theta(g(n)) \approx a = b$$

$$f(n) = o(g(n)) \approx a < b$$

$$f(n) = \omega(g(n)) \approx a > b$$

# Limits

- $\lim_{n\to\infty} [f(n) / g(n)] = 0 \Rightarrow f(n) \in \mathrm{o}(g(n))$
- $\lim_{n\to\infty} [f(n) / g(n)] < \infty \Rightarrow f(n) \in \mathrm{O}(g(n))$
- $0 < \lim_{n\to\infty} [f(n) / g(n)] < \infty \Rightarrow f(n) \in \Theta(g(n))$
- $0 < \lim_{n\to\infty} [f(n) / g(n)] \Rightarrow f(n) \in \Omega(g(n))$
- $\lim_{n\to\infty} [f(n) / g(n)] = \infty \Rightarrow f(n) \in \omega(g(n))$
- $\lim_{n\to\infty} [f(n) / g(n)]$ undefined $\Rightarrow$ can't say

# Properties

- **Transitivity**

$$f(n) = \Theta(g(n)) \ \& \ g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$$
$$f(n) = O(g(n)) \ \& \ g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$$
$$f(n) = \Omega(g(n)) \ \& \ g(n) = \Omega(h(n)) \Rightarrow f(n) = \Omega(h(n))$$
$$f(n) = o\,(g(n)) \ \& \ g(n) = o\,(h(n)) \Rightarrow f(n) = o\,(h(n))$$
$$f(n) = \omega(g(n)) \ \& \ g(n) = \omega(h(n)) \Rightarrow f(n) = \omega(h(n))$$

- **Reflexivity**

$$f(n) = \Theta(f(n))$$
$$f(n) = O(f(n))$$
$$f(n) = \Omega(f(n))$$

# Properties

- **Symmetry**

$f(n) = \Theta(g(n))$ *iff* $g(n) = \Theta(f(n))$

- **Complementarity**

$f(n) = O(g(n))$ *iff* $g(n) = \Omega(f(n))$

$f(n) = o(g(n))$ *iff* $g(n) = \omega((f(n))$

# Common Functions

# Monotonicity

- $f(n)$ is
  - **monotonically increasing** if $m \le n \Rightarrow f(m) \le f(n)$.
  - **monotonically decreasing** if $m \ge n \Rightarrow f(m) \ge f(n)$.
  - **strictly increasing** if $m < n \Rightarrow f(m) < f(n)$.
  - **strictly decreasing** if $m > n \Rightarrow f(m) > f(n)$.

# Exponentials

- **Useful Identities:**

$$a^{-1} = \frac{1}{a}$$

$$(a^m)^n = a^{mn}$$

$$a^m a^n = a^{m+n}$$

- **Exponentials and polynomials**

$$\lim_{n\to\infty} \frac{n^b}{a^n} = 0$$

$$\Rightarrow n^b = o(a^n)$$

# Logarithms

$x = \log_b a$ is the exponent for $a = b^x$.

Natural log: $\ln a = \log_e a$

Binary log: $\lg a = \log_2 a$

$\lg^2 a = (\lg a)^2$

$\lg \lg a = \lg (\lg a)$

$$a = b^{\log_b a}$$

$$\log_c(ab) = \log_c a + \log_c b$$

$$\log_b a^n = n \log_b a$$

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$\log_b(1/a) = -\log_b a$$

$$\log_b a = \frac{1}{\log_a b}$$

$$a^{\log_b c} = c^{\log_b a}$$

# Logarithms and exponentials – Bases

- If the base of a logarithm is changed from one constant to another, the value is altered by a constant factor.
  - **Ex:** $\log_{10} n * \mathbf{\log_2 10} = \log_2 n$.
  - Base of logarithm is not an issue in asymptotic notation.
- Exponentials with different bases differ by a exponential factor (not a constant factor).
  - **Ex:** $2^n = \mathbf{(2/3)^n} * 3^n$.

# Polylogarithms

- **For $a \geq 0$, $b > 0$,** $\lim_{n\to\infty} ( \lg^a n \,/\, n^b ) = 0$, so $\lg^a n = o(n^b)$, and $n^b = \omega(\lg^a n)$
  - Prove using L'Hopital's rule repeatedly

- $\lg(n!) = \Theta(n \lg n)$
  - Prove using Stirling's approximation (in the text) for $\lg(n!)$.

# Exercise

Express functions in A in asymptotic notation using functions in B.

| A | B | |
|---|---|---|
| $5n^2 + 100n$ | $3n^2 + 2$ | $A \in \Theta(B)$ |

$A \in \Theta(n^2),\ n^2 \in \Theta(B) \Rightarrow A \in \Theta(B)$

| | | |
|---|---|---|
| $\log_3(n^2)$ | $\log_2(n^3)$ | $A \in \Theta(B)$ |

$\log_b a = \log_c a / \log_c b$; $A = 2\lg n / \lg 3$, $B = 3\lg n$, $A/B = 2/(3\lg 3)$

| | | |
|---|---|---|
| $n^{\lg 4}$ | $3^{\lg n}$ | $A \in \omega(B)$ |

$a^{\log b} = b^{\log a}$; $B = 3^{\lg n} = n^{\lg 3}$; $A/B = n^{\lg(4/3)} \to \infty$ as $n \to \infty$

| | | |
|---|---|---|
| $\lg^2 n$ | $n^{1/2}$ | $A \in o(B)$ |

$\lim_{n\to\infty} ( \lg^a n / n^b ) = 0$ (here $a = 2$ and $b = 1/2$) $\Rightarrow A \in o(B)$

# Summations – Review

# Review on Summations

- Why do we need summation formulas?

  **For computing the running times of iterative constructs** (loops). (CLRS – Appendix A)

  **Example:** Maximum Subvector

  Given an array $A[1\ldots n]$ of numeric values (can be positive, zero, and negative) determine the subvector $A[i\ldots j]$ ($1 \le i \le j \le n$) whose sum of elements is maximum over all subvectors.

| | | | |
|---|---|---|---|
| 1 | -2 | 2 | 2 |

# Review on Summations

```
MaxSubvector(A, n)
        maxsum ← 0;
        for i ← 1 to n
           do for j = i to n
                    sum ← 0
                    for k ← i to j
                        do sum += A[k]
                    maxsum ← max(sum, maxsum)
        return maxsum
```

- $T(n) = \sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} 1$
- NOTE: This is not a simplified solution. What *is* the final answer?

# Review on Summations

- **Constant Series:** For integers $a$ and $b$, $a \le b$,

$$\sum_{i=a}^{b} 1 = b - a + 1$$

- **Linear Series (Arithmetic Series):** For $n \ge 0$,

$$\sum_{i=1}^{n} i = 1 + 2 + \boxed{?} + n = \frac{n(n+1)}{2}$$

- **Quadratic Series:** For $n \ge 0$,

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \boxed{?} + n^2 = \frac{n(n+1)(2n+1)}{6}$$

# Review on Summations

- **Cubic Series:** For $n \geq 0$,

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \boxed{?} + n^3 = \frac{n^2(n+1)^2}{4}$$

- **Geometric Series:** For real $x \neq 1$,

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \boxed{?} + x^n = \frac{x^{n+1}-1}{x-1}$$

For $|x| < 1$,

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

# Review on Summations

- **Linear-Geometric Series:** For $n \geq 0$, real $c \neq 1$,

$$\sum_{i=1}^{n} ic^i = c + 2c^2 + \boxed{?} + nc^n = \frac{-(n+1)c^{n+1} + nc^{n+2} + c}{(c-1)^2}$$

- **Harmonic Series:** $n$th harmonic number, $n \in \mathrm{I}^+$,

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \boxed{?} + \frac{1}{n}$$

$$= \sum_{k=1}^{n} \frac{1}{k} = \ln(n) + O(1)$$

# Review on Summations

- **Telescoping Series:**

$$\sum_{k=1}^{n} a_k - a_{k-1} = a_n - a_0$$

- **Differentiating Series:** For $|x| < 1$,

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

# Review on Summations

- **Approximation by integrals:**
  - For monotonically increasing $f(n)$

$$\int_{m-1}^{n} f(x)dx \le \sum_{k=m}^{n} f(k) \le \int_{m}^{n+1} f(x)dx$$

  - For monotonically decreasing $f(n)$

$$\int_{m}^{n+1} f(x)dx \le \sum_{k=m}^{n} f(k) \le \int_{m-1}^{n} f(x)dx$$

- **How?**

# Review on Summations

- ***n*th harmonic number**

$$\sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} \frac{dx}{x} = \ln(n+1)$$

$$\sum_{k=2}^{n} \frac{1}{k} \leq \int_{1}^{n} \frac{dx}{x} = \ln n$$

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{k} \leq \ln n + 1$$

# Reading Assignment

- Chapter 4 of CLRS.