VECTOR SPACES

Vector space: Let F be a field, V is non empty set then the set V is said to be vector space over the field F, If the following axioms are satisfied for every α, β, γ belons to V and for every A, B belongs to F

1) V is an abelian group under addition
   a. Closure law: α, β $\in V \rightarrow$ α+ β $\in V$
   b. Commutative law: α+ β = β + α ∀ α, β $\in V$
   c. Associative law: (α + β) + γ = α + (β+ γ) ∀ α, β, γ $\in V$
   d. Identity law : α+0 = 0+ α = α, ∀ α $\in V$
   e. Inverse law : α+(- α) = 0 = (-α)+ α , ∀ -α $\in V$
2) a.(α+ β)= a.α+ a.β
3) (a+b). α= a. α + b. α
4) (a.b). α =a.(b. α)
5) 1. α = α = α.1 , ∀ α, β $\in V$, a, b, 1 ∈ F

1 . Show that the set of all 2X 2 matrix with real elements is a vector space over the field of real numbers

Ans: Let A, B, C be 2X2 matrix are belongs to V

$A = [a_{ij}]_{2X2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = [b_{ij}]_{2X2}$ and $C = [c_{ij}]_{2X2}$

a. V is abelian
   i) Closure law : $A = [a_{ij}]_{2X2}$, , $B = [b_{ij}]_{2X2} \in V$
      A + B = $[a_{ij}]_{2X2} + [b_{ij}]_{2X2} \in V$
   ii) Commutative law : A + B = B + A
      A + B = $[a_{ij}]_{2X2} + [b_{ij}]_{2X2}$ = $[a_{ij} + b_{ij}]_{2X2} \in V$
      B + A = $[b_{ij} + a_{ij}]_{2X2} \in V$
   iii) Associative law : A, B, C $\in V$
      (A+B)+C = A+(B+C)
      (A+B)+C = $([a_{ij}]_{2X2} + [b_{ij}]_{2X2}) + [c_{ij}]_{2X2}$ == $[a_{ij} + b_{ij} + c_{ij}]_{2X2}$
      A+(B+C) = $[a_{ij}]_{2X2} + \left([b_{ij}]_{2X2} + [c_{ij}]_{2X2}\right) = [a_{ij} + b_{ij} + c_{ij}]_{2X2}$
   iv) Identity law: A + O = 0 +A
      A + O = $[a_{ij}]_{2X2} + [0]_{2X2} = [a_{ij}]_{2X2}$
      0 +A = $[0]_{2X2} + [a_{ij}]_{2X2} = [a_{ij}]_{2X2}$, A$\in V$
   v) Inverse law: A== $[a_{ij}]_{2X2}$, -A = - $[a_{ij}]_{2X2}$
      A + (-A) = (-A) +A = $[0]_{2X2}$

b. Let $\alpha \in F$, A, B $\in V$, $\alpha(A+B)= \alpha([a_{ij}]_{2X2} + [b_{ij}]_{2X2}) = \alpha[a_{ij}]_{2X2} + \alpha[b_{ij}]_{2X2}$
$= (\alpha a_{ij} + \alpha b_{ij})_{2X2}$
$\alpha(A+B)= \alpha A+\alpha B$

c. $\alpha, \beta \in F$, A $\in V$
$(\alpha+\beta)A = (\alpha+\beta)\ [a_{ij}]_{2X2} = \alpha\ [a_{ij}]_{2X2} + \beta\ [a_{ij}]_{2X2} = (\alpha A+\beta A) \in V$

d. $(\alpha\beta)A = (\alpha\beta)\ [a_{ij}]_{2X2} = \alpha\left(\beta\ [a_{ij}]_{2X2}\right) = \alpha(\beta A) \in V$

e. $I \in V = [1]_{2X2}$, A $\in V$
$I.A=[1]_{2X2}[a_{ij}]_{2X2} = [a_{ij}]_{2X2}[1]_{2X2}$

2. Show that the polynomial of degree at most 3 with real coefficients is a vector space over the field of real numbers

Ans: Let P(x) be the polynomial of degree at most 3, $P(x) \in V$

$A(x), B(x), C(x) \in V$

Let $A(x) = a_0+a_1x+a_2x^2+a_3x^3$, $B(x) = b_0+b_1x+b_2x^2+b_3x^3$, $C(x) = c_0+c_1x+c_2x^2+c_3x^3$

i) $A +B = a_0+ a_1x +a_2x^2+a_3x^3 + b_0+b_1x+b_2x^2+b_3x^3$
$= (a_0+ b_0)+(a_1 + b_1)x+(a_2+ b_2) x^2 +(a_3 + b_3) x^3 \in V$

ii) $A +B = B+A$,
$(a_0+ b_0)+(a_1 + b_1)x+(a_2+ b_2) x^2 +(a_3 + b_3) x^3= (b_0 +a_0)+(b_1+a_1)x+(b_2+a_2) x^2 +(b_3+a_3) x^3 \in V$

iii) $(A+B)+C = A+(B+C)$
$=(a_0+ b_0+c_0)+(a_1 + b_1+c_1)x+(a_2+ b_2+c_2) x^2 +(a_3 + b_3+c_3) x^3 \in V$

iv) $A + 0 = 0 +A =A$
$= (a_0+a_1x+a_2x^2+a_3x^3) + 0 =0 +(a_0+a_1x+a_2x^2+a_3x^3) = (a_0+a_1x+a_2x^2+a_3x^3) \in V$

v) $A+ (-A) = (-A) +A = 0$
$(a_0+a_1x+a_2x^2+a_3x^3) -(a_0+a_1x+a_2x^2+a_3x^3) = 0 \in V$

b. $\alpha(A+B) = \alpha.A + \alpha.B$

$= \alpha(a_0+a_1x+a_2x^2+a_3x^3) + \alpha(b_0+b_1x+b_2x^2+b_3x^3) \in V$

c. $(\alpha+\beta)A = (\alpha+\beta)\ (a_0+a_1x+a_2x^2+a_3x^3)$

$= \alpha(a_0+a_1x+a_2x^2+a_3x^3) + \beta(a_0+a_1x+a_2x^2+a_3x^3)$

$= \alpha A+\beta A \in V$

d. $(\alpha\beta)A = \alpha(\beta A)$

$= \alpha(\beta(a_0+a_1x+a_2x^2+a_3x^3)) \in V$

e. $I \in V$, $A \in V$, $I.A = 1.\ (a_0+a_1x+a_2x^2+a_3x^3)$

$= (a_0+a_1x+a_2x^2+a_3x^3)$

Sub space: A non-empty subset W of a vector space V over a field F is called a subspace of V if W is itself a vector space over F under the same operation of addition and scalar multiplication as defined in V

A non-empty subset W of a vector space V over a field F is a subspace V over a Field F is a subspace of V if and only if

i) $\forall\ \alpha,\ \beta \in W, \alpha+\beta \in W$
ii) $c\in F,\ \ \alpha \in W$ such that $c.\alpha\in W$

1) Let V = R^3 the vector space of all ordered triplets of real number over the field of real numbers show that the subset W = { (x, 0, 0)/ x $\in R$} is a subspace of R^3

Ans: Let $\alpha = (x_1 ,0,0)$, $\beta = (x_2, 0, 0)$

i) $\alpha+ \beta=(x_1 ,0,0) +(x_2, 0, 0)$

$= (x_1+x_2, 0, 0) \in W$

ii) Let $c\in R, \alpha \in W,$

$\alpha = (x_1 ,0, 0)$

$c\ \alpha = c(x_1 ,0, 0)$

$=(cx_1,0, 0) \in W$

W is a sub space of V(R)

2) Prove that the set W = {(x, y, z)/ (x-3y+4z=0} of a vector space V_3(R) is subspace of V_3(R)

Ans: Let $\alpha = \{(x_1 , y_1 , z_1)/ (x_1 -3y_1 +4z_1 =0\}$

and $\beta=\{(x_2 , y_2 , z_2)/ (x_2 -3y_2 +4z_2 =0\}$

$\alpha+ \beta=(x_1 , y_1 , z_1) +(x_2 , y_2 , z_2)$

$= (x_1+ x_2, y_1+ y_2, z_1+ z_2)$

$= (x_1+ x_2)-3(y_1+ y_2)+4(z_1+ z_2)$

$= (x_1 -3y_1 +4z_1) +(x_2 -3y_2 +4z_2)$

$= 0+0 =0$

$c \in R, \alpha \in W$

$\alpha = (x_1 , y_1 , z_1),$

$c\, \alpha = c(x_1 , y_1 , z_1)$

$= c\,(x_1 -3y_1 +4z_1)$

$= c.0$

$= 0$

$c\, \alpha \in W$

W is a subspace of vector space V

Linear combination and Linear span of a set

Let V be a vector space over a field F and $\alpha_1, \alpha_2, \alpha_3,$ --------------- α_n be any n vectors of V then the vectors of the form $c_1\alpha_2 +c_2\alpha_2+$--------------- $+ c_n\alpha_n$ where $c_1, c_2, c_3,$ --------------- c_n belongs to F is called linear combination of the vector $\alpha_1, \alpha_2, \alpha_3,$ --------------- α_n

Linear span of set

Let S be a non-empty subset of a vector space V the set of all linear combination of finite numbers of elements of S is called linear span of S denoted by L(S)

1) Express the vector (3,5,2) is a linear combination of the vectors (1,1,0),(2,3,0), (0,0,1) of $V_3(R)$

Ans: $(3,5,2) = c_1 (1,1,0) + c_2 (2,3,0)+ c_3 (0,0,1)$

$= (c_1 +2 c_2 ,\ c_1 +3 c_2 ,\ c_3)$

$3 = c_1 +2 c_2$

$5 = c_1 +3 c_2$

$2 = c_3$

Solving the above equations we get $c_1 = -1$, $c_2 = 2$, $c_3 = 2$

$(3,5,2) = -1\,(1,1,0) + 2\,(2,3,0) + 2\,(0,0,1)$

2) Let S = { (1, -3, 2), (2,4,1), (1,1,1) } be a subset of $V_3(R)$ Show that the vector (3, -7, 6) is in linear span of S

Ans: $(3, -7, 6) = c_1\,(1, -3, 2) + c_2\,(2,4,1) + c_3\,(1,1,1)$

$3 = c_1 + 2c_2 + c_3$

$-7 = -3c_1 + 4c_2 + c_3$

$6 = 2c_1 + c_2 + c_3$

$AX = B$

$$A:B = \begin{bmatrix} 1 & 2 & 1; & 3 \\ -3 & 4 & 1; & -7 \\ 2 & 1 & 1; & 6 \end{bmatrix}$$

$R_2 = R_2 + 3R_1$, $R_3 = R_3 - 3/2\, R_2$

$$= \begin{bmatrix} 1 & 2 & 1; & 3 \\ 0 & 10 & 4; & 2 \\ 0 & -3 & -1; & 0 \end{bmatrix} \quad R_2 = R_2/2$$

$$= \begin{bmatrix} 1 & 2 & 1; & 3 \\ 0 & 5 & 2; & 1 \\ 0 & -3 & -1; & 0 \end{bmatrix} \quad R3 = R3 + 3/5\, R2$$

$$= \begin{bmatrix} 1 & 2 & 1; & 3 \\ 0 & 5 & 2; & 1 \\ 0 & 0 & 1/5; & 3/5 \end{bmatrix}$$

Rank of A = Rank of [A:B]=3

$1/5\, c_3 = 3/5$, $c_3 = 3$

$5c_2 + 2c_3 = 1$, $c_2 = -1$

$C_1 + 2c_2 + c_3 = 3$, $c_1 = 2$

3)Show that the vector (2, -5, 3) is not in L(s) where S= {(1,-3,2), (2,-4,-1), (1,-5,7)}

$(2, -5, 3) = c_1 \ (1,-3,2)+ c_2 \ (2,-4,-1) + c_3 \ (1,-5,7)$

$2 = c_1 +2 c_2+c_3$

$-5 = -3 c_1 -4 c_2 - 5c_3$

$3 = 2 c_1 - c_2+ 7c_3$

$$A:B = \begin{bmatrix} 1 & 2 & 1; & 2 \\ 3 & 4 & 5; & 5 \\ 2 & -1 & 7; & 3 \end{bmatrix}$$

$R_2= R_2-3R_1, \quad R_3= R_3- 2 R_1$

$$= \begin{bmatrix} 1 & 2 & 1; & 2 \\ 0 & -2 & 2; & -1 \\ 0 & -5 & 5; & -1 \end{bmatrix} \quad R_2 = R_2/2 \ , \ R_3 = R_3/5$$

$$= \begin{bmatrix} 1 & 2 & 1\,; & 2 \\ 0 & -1 & 1; & -1/2 \\ 0 & -1 & 1; & -1/5 \end{bmatrix} \quad R_3= R_3- R_2$$

$$= \begin{bmatrix} 1 & 2 & 1\,; & 2 \\ 0 & -1 & 1; & -1/2 \\ 0 & 0 & 0; & 3/10 \end{bmatrix}$$

Rank of A =2, Rank of [A:B]=3

Rank of A $\neq$ Rank of [A:B]

S does not span of a set

<u>Linearly Dependence :</u>

A set { $\alpha_1, \alpha_2, \alpha_3,$ --------------- α_n} of a vector of a vector space V(F) is said to be Linearly dependent if there exists a scalars $c_1, c_2, c_3,$ --------------- $c_n \in F$ not all zeros such that $c_1\alpha_2 + c_2\alpha_2+$--------------- $+ c_n\alpha_n = 0$

OR

det(S) = 0

Linearly Independence :

A set { α_1, α_2, α_3, --------------- α_n} of a vector of a vector space V(F) is said to be Linearly independent if there exists a scalars c_1, c_2, c_3, --------------- $c_n \in F$ such that $c_1\alpha_2 + c_2\alpha_2 +$--------------- + $c_n\alpha_n = 0$ when $c_1=0$, $c_2 = 0$, $c_3= 0$, --------------- $c_n = 0$

OR

$\det(S) \neq 0$

1) Show that the vectors (1,2,3), (3,-2,1), (1,-6,-5) are linearly dependent

Ans: $S = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 1 & -6 & -5 \end{bmatrix}$

$|S| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 1 & -6 & -5 \end{vmatrix}$

$= 1(10 + 6) - 3(-10 + 18) + 1\,(2+6)$

$= 0$

S is L.D

2) Show that the vectors (1,2,-3,4), (3,-1,2,1) (1,-5,8,-7) are linearly dependent

Ans: $S = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 2 & 1 \\ 1 & -5 & 8 & -7 \end{bmatrix}$ R2 = R2-3R1, R3 = R3 – R1

$= \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 11 & -11 \\ 0 & -7 & 11 & -11 \end{bmatrix}$ R3 = R3 – R2 ,

$= \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 11 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The maximum number of non zero rows is 2 which is less than the numbers of given vectors

Therefore the given vectors are L.D

3) Show that the set S = {(1,1,2,4), (2,-1,-5,2), (1,-1,-4,0), (2,1,1,6)} are L.D

Ans: $|S|=\begin{vmatrix}1 & 1 & 2 & 4\\ 2 & -1 & -5 & 2\\ 1 & -1 & -4 & 0\\ 2 & 1 & 1 & 6\end{vmatrix}$

$$= 1\begin{vmatrix}-1 & -5 & 2\\ -1 & -4 & 0\\ 1 & 1 & 6\end{vmatrix} - 1\begin{vmatrix}2 & -5 & 2\\ 1 & -4 & 0\\ 2 & 1 & 6\end{vmatrix} + 2\begin{vmatrix}2 & -1 & 2\\ 1 & -1 & 0\\ 2 & 1 & 6\end{vmatrix} - 4\begin{vmatrix}2 & -1 & -5\\ 1 & -1 & -4\\ 2 & 1 & 1\end{vmatrix}$$

= 1(0) -1(0)+2(0)-4(0) = 0

S is L.D

4) Find the value of K for which the vectors (1, -2, k),(2,-1,5), (3,-5,7k) are L.D

Ans: the set A is L.D then Det(A) = 0

$$\begin{vmatrix}1 & -2 & k\\ 2 & -1 & 5\\ 3 & -5 & 7k\end{vmatrix} = 0$$

1(-7k+25)+2(14k-15)+k(-10+3) = 0

-7k+25+28k-30-7k = 0

14k-5 = 0

14k = 5

k= 5/14

Basis and Dimension:

Basis: A subset B of a vector space V(F) is called a Basis of V if

i) B is L.I
ii) B spans V, i.e. L[B] = V

Finite Dimension: A vector space V[F] is said to be finite dimensional space if it as finite basis

1) Determine whether the set (1,2,1), (3,4,-7) and (3,1,5) is a basis of V_3(R)

Ans: $|S| = \begin{vmatrix}1 & 2 & 1\\ 3 & 4 & -7\\ 3 & 1 & 5\end{vmatrix}$

= 1(20 +7) – 2(15 +21) + 1 (3-12)

= -54 $\neq$ 0

S is L.I

and satisfies the linear combination

S spans V

S is a Basis

2) Determine whether the set S = { (1,2,3), (3,1,0), (-2, 1, 3)} is a basis determine the dimension and the basis of the subspace spanned by S

Ans: $|S| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 1 & 3 \end{vmatrix}$

= 1(3-0) – 2(9-0) +3(3+2) = 0

S is L.D

S is not a Basis

$S = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 1 & 3 \end{bmatrix}$ R2 = R2 – 3 R1, R3 = R3 +2 R1

$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -9 \\ 0 & -5 & 9 \end{bmatrix}$ R3 = R3 +R2

$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -9 \\ 0 & 0 & 0 \end{bmatrix}$

Dimension (S) = 2 (number of Non zero rows)

3) Show that the set B = {(1,1,0), (1,0,1), (0,1,1)} is a basis of the vector space V_3(R)

Ans: $|B| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$

= 1(0 – 1) -1(1-0) +0(1-0) = -2 $\neq$ 0

B is L.I

$(x_1, x_2, x_3) = c_1 (1,1,0) + c_2(1,0,1) + c_3(0,1,1)$

$= (c_1+c_2, c_1+c_3, c_2+c_3)$

$c_1+c_2= x_1,$

$c_1+c_3= x_2$

$c_2+c_3 = x_3$

solving the above equations, we get

$c_2 = (x_1-x_2 +x_3)/2$

$c_3= x_3-c_2= x_3 – (x_1-x_2 +x_3)/2$

$c_1=x_2-c_3 = x_2 – (x_3 – (x_1-x_2 +x_3)/2)$

$(x_1, x_2, x_3) = (x_1-x_2 +x_3)/2\ (1,1,0) + (x_3 – (x_1-x_2 +x_3)/2)\ (1,0,1) + (x_2 – (x_3 – (x_1-x_2 +x_3)/2)$

$)\ (0,1,1)$

It satisfies the linear combination

B spans V

B is a basis of V

4) Find the dimension and the basis of the subspace spanned by the vectors (2,4,2), (1,-1,0), (1,2.1) and (0,3,1) in V_3(R)

Ans: $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix}$, R1 = R1 /2, R3 = 2R3

$= \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 2 & 4 & 2 \\ 0 & 3 & 1 \end{bmatrix}$, R3 = R3 – 2 R1, R2 = R2 – R1

$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix}$, R4 = R4 +R2

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A is subspace of S which spans V_3(R)

A = { (1,2.1), (0,-3, -1)}

Dim(A) = 2

Linear transformation:

Def: Let U and V be two vector spaces over the field F the mapping T: U→V is said to be linear transformation if

T(α+β) = T(α) + T(β) , for all α, β $\in U$

T(c. α) = c T(α) for all c $\in F$, α $\in U$

1) If f: V_3(R) → V_2(R) is defined by f(x,y,z) = (x+y, y+z), show that f is a linear transformation

Ans: Let $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$

$F(\alpha+\beta) = F(x_1 + x_2 , y_1 + y_2, z_1 + z_2)$

$= (x_1 + x_2 + y_1 + y_2, y_1 + y_2 + z_1 + z_2)$

$= ((x_1 + y_1)+(x_2 + y_2), (y_1 + z_1)+(y_2 + z_2))$

$= ((x_1 + y_1), (y_1 + z_1)) + (x_2 + y_2), (y_2 + z_2))$

$= F(x_1, y_1, z_1) + F(x_2, y_2, z_2)$

$=F(\alpha) + F(\beta)$

$F(c.\ \alpha) = F(cx_1, cy_1, cz_1)$

$= (cx_1 + cy_1, cy_1 + cz_1)$

$= c(x_1 + y_1, y_1 + z_1)$

$= c\ F(x_1, y_1, z_1)$

$= c\,F(\alpha)$

F is a Linear transformation

2) Find the matrix of the linear transformation T: $V_2(R) \to V_3(R)$ defined by T(x,y) = (x+y, x, 3x-y) w.r.t $B_1 = \{(1,1), (3,1)\}$ and $B_2 = \{(1,1,1), (1,1,0), (1,0,0)\}$

Ans: $B_1 = \{(1,1), (3,1)\}$

$B_2 = \{(1,1,1), (1,1,0), (1,0,0)\}$

$T(x,y) = (x+y, x, 3x-y)$

$T(1,1) = (1+1, 1, 3(1)-1) = (2, 1, 2)$

$(2, 1, 2) = c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$

$(2, 1, 2) = (c_1+c_2+c_3, c_1+c_2, c_1)$

$2 = c_1+c_2+c_3$

$1 = c_1+c_2$

$2 = c_1$

Solving above equation we get $c_1 = 2$, $c_2 = -1$, $c_3 = 1$

$T(x,y) = (x+y, x, 3x-y)$

$T(3,1) = (3+1, 3, 3(3)-1) = (4, 3, 8)$

$(4, 3, 8) = c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$

$(4, 3, 8) = (c_1+c_2+c_3, c_1+c_2, c_1)$

$4 = c_1+c_2+c_3$

$3 = c_1+c_2$

$8 = c_1$

Solving above equation we get $c_1 = 8$, $c_2 = -5$, $c_3 = 1$

Matrix form of the linear transformation is

$A_T = \begin{bmatrix} 2 & 8 \\ -1 & -5 \\ 1 & 1 \end{bmatrix}$

3) Find the matrix of the linear transformation $T: V_3(R) \rightarrow V_2(R)$ defined by $T(x,y,z) = (x+y, y+z)$ w.r.t $B_1 = \{(1,1,1), (1,0,0), (1,1,0)\}$ and $B_2 = \{(1,0), (0,1)\}$

Ans: $B_1 = \{(1,1,1), (1,0,0), (1,1,0)\}$

$T(x,y,z) = (x+y, y+z)$

$T(1,1,1) = (2, 2)$

$(2, 2) = c_1(1, 0) + c_2(0, 1)$

$= (1c_1, 0) + (0, 1c_2)$

$(2, 2) = (c_1, c_2)$

$c_1 = 2, \; c_2 = 2$

$(2, 2) = 2(1, 0) + 2(0, 1)$

$T(x,y,z) = (x+y, y+z)$

$T(1,0,0) = (1, 0)$

$(1, 0) = c_1(1, 0) + c_2(0, 1)$

$(1, 0) = (c_1, c_2)$

$c_1 = 1, \; c_2 = 0$

$T(x,y,z) = (x+y, y+z)$

$T(1,1,0) = (2, 0)$

$(2, 0) = c_1(1, 0) + c_2(0, 1)$

$(2, 0) = (c_1, c_2)$

$c_1 = 2, \; c_2 = 0$

$A_T = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

Rank of the Linear transformation

Let T: V →W be a linear transformation the dimension of the range space R(T) is called rank of the linear transformation is denoted by r(T)

Column space and Null space

Consider the M x N matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad c_1=\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad c_2=\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \text{-------} \quad c_n=\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

[c_1, c_2, c_3, ------------------ c_n] is called column vector of A is called column space of A

The solution of the system of homogeneous linear equation AX = 0 is called Null space of A

1) Find the rank of the linear transformation defined by T(x,y,z) = (x+y, x-y, 2x+z)

Ans: the standard basis of V_3(R) are {(1,0,0), (0,1,0),(0,0,1)}

T(x,y,z) = (x+y, x-y, 2x+z)

T(1,0,0) = (1,1,2)

T(0,1,0) = (1,-1,0)

T(0,0,1) = (0,0,1)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Det(A) = -2 ≠ 0

A is L. I , it is a basis of R(T)

$\alpha = x_1\,\alpha_1 + x_2\,\alpha_2 + x_3\,\alpha_3$

$= x_1(1,1,2) + x_2(1,-1,0) + x_3(0,0,1)$

$= (x_1+x_2,\ x_1-x_2,\ 2x_1+x_3)$

$R(T) = (x_1+x_2, x_1-x_2, 2x_1+x_3)$

$r(T)=3$

2) Find the rank of the linear transformation defined by $T(x,y,z) = (y-x, y-z)$

Ans: the standard basis of $V_3(R)$ are {(1,0,0), (0,1,0),(0,0,1)}

$T(1,0,0) = (-1,0)$

$T(0,1,0) = (1,1)$

$T(0,0,1) = (0,-1)$

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \quad R2 = R2+R1$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \quad R3 = R3+R2$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Dim[R(T)] = 2 = rank of T