MODULE-I

Random variable: A real variable associated with every outcome of a random experiment is called random variable. Therefore a random variable is a function $X: S \to R$.

Discrete Random Variable: If a random variable X takes finite or countable number of values then X is called discrete random variable.

Continuous Random Variable: If a random variable X takes uncountable number of values then X is called continuous random variable.

Probability function of discrete Random Variable: $f(x)$ is called probability function for a discrete random variable X, If (i) $f(x) \geq 0$ and (ii) $\sum_x f(x) = 1$ iii) $P(X = x) = f(x)$.

Collection $[x, f(x)]$ is called **discrete probability distribution of X**.

Example: A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Solution: Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school. Then x may takes the value 0, 1, or 2.

$$f(0) = P(X = 0) = \frac{\binom{3}{0}\binom{5}{2}}{\binom{8}{2}} = \frac{10}{28},$$

$$f(1) = P(X = 1) = \frac{\binom{3}{1}\binom{5}{1}}{\binom{8}{2}} = \frac{15}{28}$$

$$f(2) = P(X = 2) = \frac{\binom{3}{2}\binom{5}{0}}{\binom{8}{2}} = \frac{3}{28}$$

Thus the probability distribution of X is

x	0	1	2
$f(x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

Note:

1. Mean$= \mu = \sum_x x f(x)$.
2. Variance $= V = \sum_x x^2 f(x) - \mu^2$.
3. Standard deviation$= \sigma = \sqrt{V}$.
4. Cumulative distributive function: $F(x) = P(X \leq x)$

   $\therefore$ $F(x) = \sum_{t \leq x} f(t)$. (c.d.f. is defined for all real values of x)
5. Expectation of $h(X) = E[h(X)] = \sum_x h(x) f(x)$.

Therefore $\mu = E[X]$ and $V = E[X^2] - \{E[X]\}^2$.

Examples:

1. The probability density function of a discrete random variable X is given below:

x	0	1	2	3	4	5	6
$f(x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find (i) k; (ii) $F(4)$; (iii) $P(X \leq 5)$; (iv) $P(2 \leq X < 5)$; (v) $E(X)$ and (vi) $Var\,(X)$.

Solution: (i) Since $\sum_{x\in X} f(x) = 1 \quad \Longrightarrow \quad 49k = 1\ , \quad \therefore \quad k = \frac{1}{49}$.

(ii) $F(4) = \sum_{x\in X} P(x \leq 4) = 25k = \frac{25}{49}$.

(iii) $P(X \leq 5) = 36k = \frac{36}{49}$.

(iv) $P(2 \leq X < 5) = f(2) + f(3) + f(4) = 21k = \frac{21}{49}$.

(v) $E(X) = \mu = \sum xf(x) = 203k = \frac{29}{7} = 4.1429$.

(vi) $V(X) = \sum_{x\in X} x^2 f(x) - \mu^2 = \frac{973}{49} - \left(\frac{29}{7}\right)^2 = \frac{132}{49} = 2.6939.$

2. A random variable X has the following probability distribution.

x	-2	-1	0	1	2	3
$f(x)$	0.1	k	0.2	$2k$	0.3	k

Find (i) k; (ii) $F(2)$; (iii)$P(-2 < X < 2)$; (iv) $P(-1 < X \leq 2)$; (v) $E(X)$ and (vi) $Var\,(X)$.

Solution: (i) Since $\sum_{x\in X} P(x) = 1 \quad \Longrightarrow \quad 4k + 0.6 = 1\ , \quad \therefore \quad k = 0.1$.

Therefore probability distribution is

x	-2	-1	0	1	2	3
$f(x)$	0.1	0.1	0.2	0.2	0.3	0.1

(ii) $F(2) = \sum_{x\in X} P(x \leq 2) = 0.9$.

(iii) $P(-2 < X < 2) = f(-1) + f(0)fP(1) = 0.5$.

(iv) $P(-1 < X \leq 2) = f(0) + f(1) + f(2) = 0.7$.

(v) $E(X) = \mu = \sum xf(x) = -0.2 - 0.1 + 0 + 0.2 + 0.6 + 0.3 = 0.8$.

(vi) $V(X) = \sum_{x\in X} x^2 f(x) - \mu^2 = 0.4 + 0.1 + 0 + 0.2 + 1.2 + 0.9 - 0.8^2 = 2.16$.

Continuous Random Variable: A real valued function $f(x)$ is called probability density function (p.d.f.) for a continuous random variable X, If (i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x)dx = 1$ and iii) $P(a < X < b) = \int_a^b f(x)\,dx$.

Cumulative distribution function (c.d.f.):

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt \ . \quad \text{Therefore p.d.f.} \quad f(x) = \frac{d}{dx}[F(x)] \ .$$

Note: 1. Mean= $\mu = \int_{-\infty}^{\infty} xf(x)dx$.

2. Variance $= V = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$.

3. Standard deviation= $\sigma = \sqrt{V}$

4. Expectation of $h(x) = E[h(x)] = \int_{-\infty}^{\infty} h(x)f(x)dx$.

5. $P(a < X < b) = \int_a^b f(x)dx = F(b) - F(a)$.

1. A random variable X has density function $f(x) = \begin{cases} kx^2, & 0 \leq x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$.

Evaluate k , and find (i) $P(x \leq 1)$ (ii) $P(1 \leq x \leq 2)$ (iii) $P(x \leq 3)$ (iv) $P(x > 1)$ (v) $P(x > 2)$.

Solution: Since $\int_{-\infty}^{\infty} f(x)dx = 1 \implies \int_0^3 kx^2 dx = 1 \implies \frac{kx^3}{3}\Big|_0^3 = 1 \implies k = \frac{1}{9}$.

(i) $P(x \leq 1) = \int_{-\infty}^{1} f(x)dx = \frac{1}{9}\int_0^1 x^2 dx = \frac{x^3}{27}\Big|_0^1 = \frac{1}{27}$.

(ii) $P(1 \leq x \leq 2) = \frac{x^3}{27}\Big|_1^2 = \frac{7}{27}$.

(iii) $P(x \leq 3) = \frac{x^3}{27}\Big|_0^3 = 1$.

(iv) $P(x > 1) = \frac{x^3}{27}\Big|_1^3 = \frac{26}{27}$.

(v) $P(x > 2) = \frac{x^3}{27}\Big|_2^3 = \frac{19}{27}$.

2. Suppose that the error in the reaction, for a controlled laboratory experiment is a continuous random

variable having the p.d.f $\quad f(x) = \begin{cases} \frac{x^2}{3}, & -1 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$

Find (i) c.d.f. $F(x)$ and (ii) use it to evaluate $P\ (0 < X \leq 1)$.

Solution: If $x < -1$, then c.d.f. $F(x) = 0$.

If $-1 \leq x < 2$, then $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-1}^{x} \frac{x^2}{3} dx = \frac{x^3}{9}\Big|_{-1}^{x} = \frac{x^3+1}{9}$.

And if $2 \leq x$ then $F(x) = 1$.

Therefore $F(x) = \begin{cases} 0, & for\ \ x \leq -1 \\ \frac{x^3+1}{9}, & for\ \ -1 < x < 2 \\ 1, & for\ \ 2 \leq x \end{cases}$.

And $P\ (0 < X \leq 1) = F(1) - F(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$.

3. The trouble shooting of an I.C. is a R.V. X whose cumulative distribution function is given by

$$F(x) = \begin{cases} 0, & \text{for } x \leq 3 \\ 1 - \frac{9}{x^2}, & \text{for } x > 3 \end{cases}$$

If X denotes the number of years, find the probability that the I.C. will work properly

(a) less than 8 years, (b) beyond 8 years, (c) from 5 to 7 years, (d) from 2 to 5 years.

Solution: (a) $P(X < 8) = F(8) = 1 - \frac{9}{64} = \frac{55}{64}$.

(b) $P(X \geq 8) = 1 - F(8) = \frac{9}{64}$.

(c) $P(5 \leq X \leq 7) = F(7) - F(5) = \left[1 - \frac{9}{49}\right] - \left[1 - \frac{9}{25}\right] = \frac{216}{1225} = 0.1763$.

(d) $P(2 \leq X \leq 5) = F(5) - F(2) = \left[1 - \frac{9}{25}\right] - 0 = \frac{16}{25} = 0.64$.

Bernoulli's trail: A trial having only two outcomes say success and failure is called Bernoulli's trail.

Let p be the probability of success and q be the probability of failure in each trail. Then $p + q = 1$.

Binomial Distribution: If a Bernoulli's trail is conducted n times,

Probability of x successes in n trail is given by

$$b(x; n, p) = P(X = x) = f(x) = \binom{n}{x} p^x q^{n-x}.$$

The distribution $[x,\ \ b(x; n, p)\ \ ]$ is called binomial distribution.

Mean and variance of the binomial distribution:

Mean$= \mu = \sum_{x \in X} x f(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$

$$= \sum_{x=0}^{n} x . \frac{n}{x} . \binom{n-1}{x-1} p^x q^{n-x} \qquad \because \binom{n}{x} = \frac{n}{x} . \binom{n-1}{x-1}$$

$$= np\sum_{x=1}^{n}\binom{n-1}{x-1}p^{x-1}q^{(n-1)-(x-1)}. \qquad \text{by Binomial theorem}$$

$$= np(p+q)^{n-1} = np\,. \qquad \because\ p+q=1.$$

Variance $= V = \sum_{x\in X} x^2 f(x) - \mu^2 = \sum_{x\in X}[x(x-1)+x]f(x) - \mu^2$

$$= \sum_{x\in X}[x(x-1)]f(x) + \mu - \mu^2$$

$$= \sum_{x=0}^{n}[x(x-1)]\binom{n}{x}p^x q^{n-x} + np - n^2p^2$$

$$= \sum_{x=0}^{n}[x(x-1)]\frac{n(n-1)}{x(x-1)}.\binom{n-2}{x-2}p^x q^{n-x} + np - n^2p^2 \qquad \because\ {}^{n}C_x = \frac{n(n-1)}{x(x-1)}.\binom{n-2}{x-2}$$

$$= n(n-1)p^2\sum_{x=2}^{n}\binom{n-2}{x-2}p^{x-2}q^{(n-2)-(x-2)} + np - n^2p^2$$

$$= n(n-1)p^2(p+q)^{n-2} + np - n^2p^2 = n^2p^2 - np^2 + np - n^2p^2$$

$$= -np^2 + np = np(1-p) = npq. \qquad \because\ q = 1-p.$$

∴ **Standard deviation** $= \sigma = \sqrt{V} = \sqrt{npq}\,.$

Problems:

1. It is known that among the 10 telephone lines available in an office, the chance that any telephone is busy at an instant of time is 0.2. Find the probability that (i) exactly 3 lines are busy, (ii) What is the most probable number of busy lines and compute its probability, and (iii) What is the probability that all the telephones are busy?

   Solution: Let p be the probability that any telephone is busy at an instant of time.

   Then, $p = 0.2,\ q = 0.8$. Since 10 telephone lines available in an office, $n = 10$.

   Therefore probability that x telephone lines are busy at an instant of time is

   $f(x) = {}^{10}C_x 0.2^x 0.8^{(10-x)}$.

   (i) The probability that exactly 3 lines are busy $P(X=3) = f(3) = {}^{10}C_3 0.2^3 0.8^7 = 0.2013$.

   (ii) The most probable number of busy lines is $\mu = np = 2$.

   And $f(2) = {}^{10}C_2 0.2^2 0.8^8 = 0.3020$

   (iii) The probability that all the telephones are busy$= f(10) = {}^{10}C_{10} 0.2^{10} = 0.000000102 \approx 0$.

2. The chance that a bomb dropped from an airplane will strike a target is 0.4. 6 bombs are dropped from the airplane. Find the probability that (i) exactly 2 bombs strike the target? (ii) At least 1 strikes the target. (iii) None of the bombs hits the target?

   Solution: Clearly $p = 0.4,\ q = 0.6\ .\ n = 6.$

   Therefore probability that x bombs strike the target is $f(x) = {}^{6}C_x 0.4^x 0.6^{(6-x)}$.

   (i) The probability that exactly 2 bombs strike the target $P(X=2) = f(2) = {}^{6}C_2 0.4^2 0.6^4 = 0.3110$.

(ii) The probability that at least one bomb strike the target= $P(X \geq 1)$.

$$P(X \geq 1) = 1 - f(0) = 1 - {}^6C_0 0.6^6 = 0.9533.$$

(iii) The probability that none of the bombs strike the target= $f(0) = {}^6C_0 0.6^6 = 0.0467$.

3. If 10% of the rivets produced by a machine are defective, find the probability that out of 12 randomly chosen rivets (i) exactly 2 will be defective (ii) at least 2 will be defective (iii) none will be defective.

Solution: Let p be the probability that a rivet is defective. Then, $p = 0.1,\ q = 0.9$. $n = 12$.

Therefore probability that x rivets are defective is $f(x) = {}^{12}C_x 0.1^x 0.9^{(12-x)}$.

(i) The probability that exactly 2 will be defective= $f(2) = {}^{12}C_2 0.1^2 0.9^{10} = 0.2301$.

(ii) The probability that at least 2 will be defective= $P(X \geq 2)$

$$P(X \geq 2) = 1 - [f(0) + f(1)] = 1 - [0.2824 + 0.3766] = 0.3410.$$

(iii) The probability that none will be defective= $f(0) = 0.2824$.

4. A die is thrown 8 times. Find the probability that 3 falls,

(i) Exactly 2 times (ii) at least once (iii) at the most 7 times.

Solution: In each throw chance of 3 falls is 1 out of 6. Therefore $p = \frac{1}{6},\ q = \frac{5}{6},\ n = 8.$

The probability that 3 falls x times in 8 throw is $f(x) = {}^8C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{(8-x)} = \frac{{}^8C_x 5^{(8-x)}}{6^8}$.

(i) The probability that 3 falls exactly 2 times= $f(2) = \frac{{}^8C_2 5^6}{6^8} = 0.2605.$

(ii) The probability that 3 falls at least once= $P(X \geq 1) = 1 - f(0) = 1 - \frac{{}^8C_0 5^8}{6^8} = 0.7674.$

(iii) The probability that 3 falls at the most 7 times= $P(X \leq 7) = 1 - f(8)$

$$= 1 - \frac{{}^8C_8}{6^8} = 0.999999405 \approx 1.$$

5. Fit a binomial distribution for the following data and obtain the expected frequencies.

x	0	1	2	3	4	5	6	7	8	9	10
f	6	20	28	12	8	6	0	0	0	0	0

Solution: $\mu = \overline{x} = \frac{\sum f \times x}{\sum f} = \frac{20+2\times28+3\times12++4\times8+5\times6}{6+20+28+12+8+6} = \frac{174}{80} = 2.175.$

In binomial distribution $\mu = np = 2.175$, Since x takes the values 0 to 10 , $n = 10.$

And hence $p = 0.2175,\quad q = 0.7825.\quad f(x) = {}^{10}C_x 0.2175^x 0.7825^{(10-x)}$.

Therefore binomial distribution and the expected frequencies are given below.

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	0.0861	0.2392	0.2992	0.2218	0.1079	0.0360	0.0083	0.0013	0.0001	0	0
f	7	19	24	18	9	3	0	0	0	0	0

Poisson distribution: The limiting case of the binomial distribution by making n very large and p very small, and keeping np fixed ($np = m$) is Poisson distribution.

Probability function is $p(x;m) = f(x) = \dfrac{m^x e^{-m}}{x!}$.

Proof: In binomial distribution, if $n \to \infty$, $p \to 0$ and $np = m$

$$f(x) = \binom{n}{x} p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots\cdots(n-x+1)p^x q^{n-x}}{x!}$$

$$= \frac{(np)^x\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\cdots\left(1-\frac{x-1}{n}\right)q^{n-x}}{x!}$$

$$= \frac{m^x q^{n-x}}{x!} = \frac{m^x e^{-m}}{x!}\ .$$

$$\because \quad q^{n-x} = (1-p)^{n-x} = \left(1-\frac{m}{n}\right)^{n-x} = \left\{\left(1+-\frac{m}{n}\right)^{-\frac{n}{m}}\right\}^{-\frac{m}{n}(n-x)} = e^{-m}\,, \text{ as } n \to \infty.$$

Mean and variance of the Poisson distribution:

Mean$= \mu = \sum_{x\in X} x f(x) = \sum_{x=0}^{\infty} x \frac{m^x e^{-m}}{x!}$

$$= me^{-m} \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

$$= me^{-m}\left\{1 + m + \frac{m^2}{2!} + \cdots\cdots\right\} = me^{-m}e^{m} = m.$$

Variance $= V = \sum_{x\in X} x^2 f(x) - \mu^2 = \sum_{x\in X}[x(x-1)+x]f(x) - \mu^2$

$$= \sum_{x\in X}[x(x-1)]f(x) + \mu - \mu^2$$

$$= \sum_{x=0}^{\infty}[x(x-1)]\frac{m^x e^{-m}}{x!} + m - m^2$$

$$= m^2 e^{-m} \sum_{x=2}^{\infty} \frac{m^{x-2}}{(x-2)!} + m - m^2$$

$$= m^2 e^{-m}\left\{1 + m + \frac{m^2}{2!} + \cdots\cdots\right\} + m - m^2 = m^2 e^{-m}e^{m} + m - m^2 = m.$$

$\therefore$ **Standard deviation** $= \sigma = \sqrt{V} = \sqrt{m}$.

And hence mean and variance of a Poisson distribution are same.

1. The probability that an individual suffers a bad reaction from a certain injection is 0.001. Determine the probability that out of 2000 individuals (i) exactly 3 (ii) more than 2 will suffer a bad reaction.

Solution: Given that $p = 0.001$, $n = 2000$. Since p is very small, consider Poisson distribution,

$$m = np = 2. \qquad \therefore\ f(x) = \frac{2^x e^{-2}}{x!}\ .$$

(i) The probability that out of 2000 individuals exactly 3 will suffer a bad reaction$= P(X = 3)$

$$= f(3) = \frac{2^3 e^{-2}}{3!} = 0.1804.$$

(ii) The probability that out of 2000 individuals more than 2 will suffer a bad reaction$= P(X \geq 3)$

$$P(X \geq 3) = 1 - [f(0) + f(1) + f(2)] = 1 - [e^{-2} + 2e^{-2} + 2e^{-2}] = 1 - 5e^{-2} = 0.3233.$$

2. It is known that the chance of an error in the transmission of a message through a communication channel is 0.002. 2000 messages are sent through the channel; find the probability that at least 3 messages will be received incorrectly.

Solution: Given that $p = 0.002$, $n = 2000$. Since p is very small, consider Poisson distribution,

$$m = np = 4. \qquad \therefore\ f(x) = \frac{4^x e^{-4}}{x!}\ .$$

The probability that at least 3 messages will be received incorrectly $= P(X \geq 3)$

$$P(X \geq 3) = 1 - [f(0) + f(1) + f(2)] = 1 - [e^{-4} + 4e^{-4} + 8e^{-4}] = 1 - 13e^{-4} = 0.7619.$$

3. A car hire –firm has two cars which it hires out on a day to day basis. The number of demands for a car is known to be Poisson distribution with mean 1.5. Find the probability of day on which (i) There is no demand for the car and (ii) The demand is rejected.

Solution: Given that, $m = 1.5$. $\quad \therefore\ f(x) = \frac{1.5^x e^{-1.5}}{x!}\ .$

(i) The probability of day on which there is no demand for the car $= P(0) = e^{-1.5} = 0.2231.$

(ii) The probability of day on which the demand is rejected$= P(X \geq 3)$

$$P(X \geq 3) = 1 - [f(0) + f(1) + f(2)] = 1 - [e^{-1.5} + 1.5e^{-1.5} + 1.125e^{-1.5}] = 0.1912\ .$$

4. Fit a Poisson distribution for the following data and calculate the theoretical frequencies.

x	0	1	2	3	4
f	122	60	15	2	1

Solution : $\quad \mu = \overline{x} = \frac{\sum fx}{\sum f} = \frac{100}{200} = 0.5.$

In Poisson distribution $\mu = m = 0.5$, $P(x) = \frac{0.5^x e^{-0.5}}{x!}\ .$

The Poisson distribution and the theoretical frequencies are given below.

x	0	1	2	3	4
$P(x)$	0.6065	0.3033	0.0758	0.0126	0.0016
f	121	61	15	3	0

Exponential distribution: Probability density function is $f(x) = \begin{cases} \propto e^{-\propto x}, & for\ \ x \geq 0 \\ 0, & for\ \ x < 0 \end{cases}$.

Mean and variance of the Exponential distribution:

Mean= $\mu = \int_{-\infty}^{\infty} xf(x)dx = \propto \int_0^{\infty} xe^{-\propto x}dx$

$$= \propto \left[\frac{xe^{-\propto x}}{-\propto} - \frac{e^{-\propto x}}{\propto^2}\right]_0^{\infty} = \propto \left[0 + \frac{1}{\propto^2}\right] = \frac{\mathbf{1}}{\propto} .$$

Variance $= V = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 \ = \propto \int_0^{\infty} x^2 e^{-\propto x}dx - \frac{1}{\propto^2}$

$$= \propto \left[\frac{x^2e^{-\propto x}}{-\propto} - \frac{2xe^{-\propto x}}{\propto^2} + \frac{2e^{-\propto x}}{-\propto^3}\right]_0^{\infty} - \frac{1}{\propto^2} = \propto \left[0 + \frac{2}{\propto^3}\right] - \frac{1}{\propto^2} = \frac{\mathbf{1}}{\propto^{\mathbf{2}}} .$$

Standard deviation= $\sigma = \sqrt{V} = \frac{\mathbf{1}}{\propto}$**.**

Hence, mean and standard deviation of an exponential distribution are same.

Problems:

1. The duration of telephone conservation has been found to have an exponential distribution with mean 2 minutes. Find the probabilities that the conservation may last (i) more than 3 minutes, (ii) less than 4 minutes and (iii) between 3 and 5 minutes.

Solution: In exponential distribution $\mu = \frac{1}{\propto} = 2, \qquad \therefore \propto = \frac{1}{2}$

Probability density function is $f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}}, & for\ \ x \geq 0 \\ 0, & for\ \ x < 0 \end{cases}$.

(i) The probability that the conservation may last for more than 3 minutes $= P(X > 3)$

$$P(X > 3) = \int_3^{\infty} f(x)dx = -e^{-\frac{x}{2}}\Big|_3^{\infty} = e^{-\frac{3}{2}} = 0.2231.$$

(ii) The probability that the conservation may last for less than 4 minutes= $P(X < 4)$

$$P(X < 4) = \int_{-\infty}^{4} f(x)dx = \int_0^4 \frac{1}{2}e^{-\frac{x}{2}}dx = -e^{-\frac{x}{2}}\Big|_0^4 = 1 - e^{-2} = 0.8647.$$

(ii) The probability that the conservation may last between 3 and 5 minutes= $P(3 < X < 5)$

$$P(3 < X < 5) = \int_3^5 f(x)dx = -e^{-\frac{x}{2}}\Big|_3^5 = e^{-\frac{3}{2}} - e^{-\frac{5}{2}} = 0.1410.$$

2. In a town, the duration of a rain is exponentially distributed with mean equal to 5 minutes. What is the probability that (i) the rain will last not more than 10 minutes (ii) between 4 and 7 minutes and (iii) between 5 and 8 minutes?

Solution: In exponential distribution $\mu = \frac{1}{\alpha} = 5$, $\therefore \alpha = \frac{1}{5}$

Probability density function is $f(x) = \begin{cases} \frac{1}{5}e^{-\frac{x}{5}}, & for\ x \geq 0 \\ 0, & for\ x < 0 \end{cases}$.

(i) The probability that the rain will last not more than 10 minutes $= P(X \leq 10)$

$$P(X \leq 10) = \int_0^{10} f(x)dx = -e^{-\frac{x}{5}}\Big|_0^{10} = 1 - e^{-2} = 0.8647.$$

(ii) The probability the rain will last between 4 and 7 minutes $= P(4 \leq X \leq 7)$

$$P(4 \leq X \leq 7) = \int_4^7 f(x)dx = -e^{-\frac{x}{5}}\Big|_4^7 = e^{-\frac{4}{5}} - e^{-\frac{7}{5}} = 0.2027.$$

(iii) The probability the rain will last between 5 and 8 minutes $= P(5 \leq X \leq 8)$

$$P(4 \leq X \leq 7) = \int_5^8 f(x)dx = -e^{-\frac{x}{5}}\Big|_5^8 = e^{-1} - e^{-\frac{8}{5}} = 0.1660.$$

Normal distribution (**Gaussian distribution N (μ, σ^2)):** A continuous random variable X from $-\infty$ to ∞ is said to have normal distribution with parameters, μ and σ^2, if its p.d.f. is

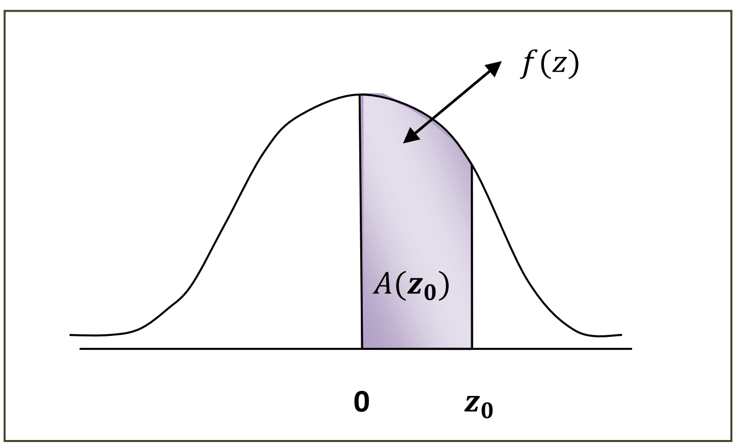
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \text{ for } -\infty < x < \infty, \qquad \text{where } -\infty < \mu < \infty, \text{ and } \sigma > 0.$$

Clearly mean$= E(X) = \mu,\ \ and\ \ variance = V(X) = \sigma^2$.

If $\mu = 0\ \ and\ \sigma = 1$ then the normal distribution is called standard normal distribution.

Standard normal variable: Probability density function is $f(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$.

Area under the standard normal curve



$$P(0 \leq z \leq z_0) = A(z_0) = \frac{1}{\sqrt{2\pi}}\int_0^{z_0} e^{-\frac{z^2}{2}} dz$$

Let x be a normal variable with mean μ and standard deviation σ , then $z = \frac{x-\mu}{\sigma}$.

Problems:

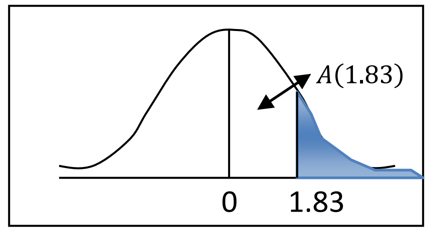
1. In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and standard deviation of 60 hours. Estimate the number of bulbs likely to burn for (a) more than 2150 hours, (b) less than 1940 hours and (c) more than 1920 hours and but less than 2160 hours.

Solution: Let x be the life of a bulb in hours. Given that $\mu = 2040$ hours, $\sigma = 60$ hours .

If z is the standard normal variant, then $z = \frac{x-2040}{60}$.

(a) Probability that a bulb burn for more than 2150 hours= $P(x > 2150)$

$$P(x > 2150) = P(z > 1.83)$$
$$= 0.5 - A(1.83)$$
$$= 0.5 - 0.4664 = 0.0336$$

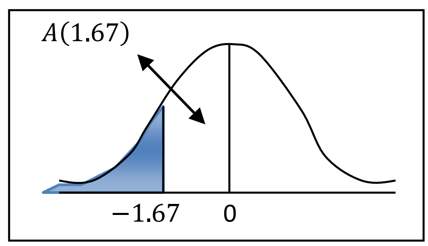


$$0.0336 \times 2000 = 67.2 \approx 67$$

Therefore out of 2000 bulbs 67 bulbs are likely to burn for more than 2150 hours.

(b) Probability that a bulb burn for less than 1940 hours= $P(x < 1940)$

$$P(x < 1940) = P(z < -1.67)$$
$$= 0.5 - A(1.67)$$
$$= 0.5 - 0.4525 = 0.0475$$

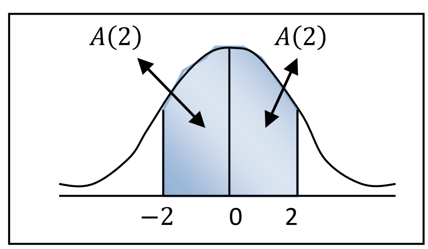


$$0.0475 \times 2000 = 95.$$

Therefore out of 2000 bulbs 95 bulbs are likely to burn for less than 1940 hours.

(c) Probability that a bulb burn for more than 1920 hours and but less than 2160 hours

$$= P(1920 < x < 2160)$$
$$= P(-2 < z < 2)$$
$$= 2A(2)$$
$$= 2 \times 0.4772 = 0.9544$$



$$0.9544 \times 2000 = 1908.8 \approx 1909.$$

Therefore out of 2000 bulbs 1909 bulbs are likely to burn between 1920 hours and 2160 hours.

2. An analog signal received at a detector (measured in micro volts) may be modeled as a Gaussian random variable N (200, 256) at a fixed point in time. What is the probability that the signal will exceed 240 micro volts? What is the probability that the signal is less than 240 micro volts, given that it is larger than 210 micro volts? (N (200, 256) means $\mu = 200$, $\sigma^2 = 256$.

Solution: Let x be the signal voltage. Given that $\mu = 200$, $\sigma = 16$.

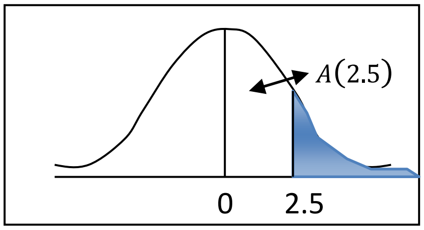
If z is the standard normal variant, then $z = \frac{x-200}{16}$.

(a) Probability that the signal will exceed 240 micro volts $= P(x > 240)$

$= P(z > 2.5)$

$= 0.5 - A(2.5)$

$= 0.5 - 0.4938 = 0.0062.$

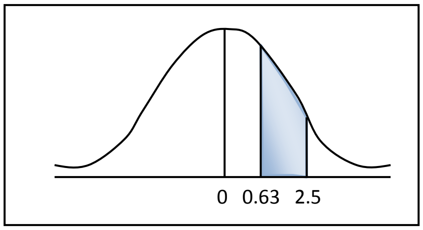


(b) Probability that the signal is between 210 micro volts and 240 micro volts $= P(210 < x < 240)$

$= P(0.63 < z < 2.5)$

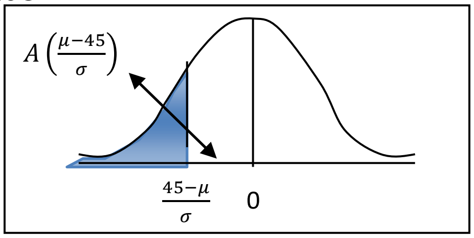
$= A(2.5) - A(0.63)$

$= 0.4938 - 0.2357 = 0.2581.$



3. In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and Standard deviation given that $A(0.5) = 0.19$, and $A(1.4) = 0.42$.

Solution: Given that $P(x < 45) = 0.31$ and $P(x > 64) = 0.08$



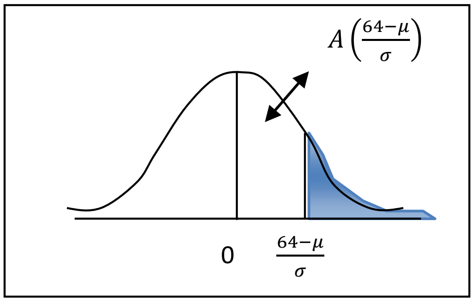
$\Longrightarrow\ P\left(z < \frac{45-\mu}{\sigma}\right) = 0.31$ and $P\left(z > \frac{64-\mu}{\sigma}\right) = 0.08$

$\Longrightarrow\ 0.5 - A\left(\frac{\mu-45}{\sigma}\right) = 0.31$ and $0.5 - A\left(\frac{64-\mu}{\sigma}\right) = 0.08$

$\Longrightarrow\ A\left(\frac{\mu-45}{\sigma}\right) = 0.19$ and $A\left(\frac{64-\mu}{\sigma}\right) = 0.42$

$\therefore\ \frac{\mu-45}{\sigma} = 0.5$ *and* $\frac{64-\mu}{\sigma} = 1.4$

On solving we get, $\mu = 50$, *and* $\sigma = 10$.



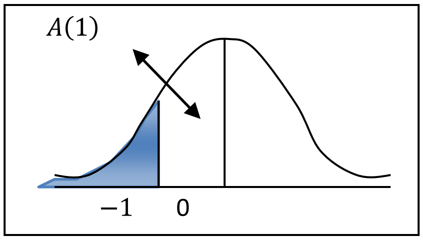
4) The marks of 1000 students in an examination follow a normal distribution with mean 70 and standard deviation 5. Find the number of students whose marks will be i) less than 65 ii) more than 75 iii) between 65 and 75 given that $A(1) = 0.3413$.

Solution: Let x be the marks of a student. Given that $\mu = 70, \quad \sigma = 5$.

If z is the standard normal variant, then $z = \frac{x-70}{5}$.

(i) Probability that a student with marks less than 65= $P(x < 65)$

$$P(x < 65) = P(z < -1)$$
$$= 0.5 - A(1)$$
$$= 0.5 - 0.3413 = 0.1687$$

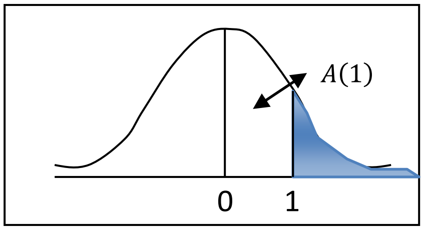


$$0.1687 \times 1000 \approx 169.$$

Therefore 169 students are with marks less than 65.

(ii) Probability that a student with marks more than 75= $P(x > 75)$

$$P(x > 75) = P(z > 1)$$
$$= 0.5 - A(1)$$
$$= 0.5 - 0.3413 = 0.1687$$



$$0.1687 \times 1000 \approx 169.$$

Therefore 169 students are with marks more than 75.

(iii) Probability that a student with marks between 65 and 75

$$= P(65 < x < 75)$$
$$= P(-1 < z < 1) = 2A(1)$$
$$= 2 \times 0.3413 = 0.6826$$

$$0.6826 \times 1000 \approx 683.$$

683 students are with marks between 65 and 75.