MODULE-2

Joint probability distribution: Joint Probability distribution for two discrete random variables, expectation, covariance and correlation. Markov Chain: Introduction to Stochastic Process, Probability Vectors, Stochastic matrices, Regular stochastic matrices, Markov chains, higher transition probabilities, Stationary distribution of Regular Markov chains and absorbing states.

Joint probability distribution: $P(x,\ y)$ is called joint Probability function for two discrete random variables X and Y If i) $P(x,\ y) \geq 0$ for all x, y ii) $\sum_x \sum_y P(x,\ y) = 1$.

Then $[(x,\ y),\ \ P(x,\ y)]$ is called joint probability distribution.

Marginal distribution of X : $[x,\ \ f(x)]$ where $f(x) = \sum_y P(x,\ y)$.

Marginal distribution of Y : $[y,\ \ g(y)]$ where $g(y) = \sum_x P(x,\ y)$.

$$E(X) = \mu_X = \sum xf(x), \qquad E(Y) = \mu_Y = \sum yg(y).$$

$$V(X) = \sum x^2 f(x) - (\mu_X)^2, \qquad V(Y) = \sum y^2 f(y) - (\mu_Y)^2.$$

$$\sigma_X = \sqrt{V(X)} \qquad \sigma_Y = \sqrt{V(Y)}$$

$$E(XY) = \sum_x \sum_y xyP(x,\ y)$$

$$Cov\ (X,\ Y) = E(XY) - E(X)E(Y)$$

Correlation $\rho(X,\ Y) = \frac{Cov\ (X,Y)}{\sigma_X \sigma_Y}$.

If $P(x,\ y) = f(x)g(y)$ for all $x,\ y$ then X and Y are independent.

1. A fair coin is tossed 3 times. Let X denote 0 or 1 according as a head or tail occurs on the first toss. Let Y denote the number of heads which occur. Find the joint distribution and marginal distribution of X and Y . Also find $Cov\ (X,\ Y)$.

Solution:

S	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
x	0	0	0	0	1	1	1	1
y	3	2	2	1	2	1	1	0

Joint distribution is

$X \backslash Y$	0	1	2	3	Sum
0	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{1}{2}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{1}{2}$
Sum	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	

Marginal distribution of X

x	0	1
$f(x)$	$\frac{1}{2}$	$\frac{1}{2}$

Marginal distribution of Y

y	0	1	2	3
$g(y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$E(XY) = \sum_x \sum_y xyP(x,\ y) = \frac{2}{8} + \frac{2}{8} = \frac{1}{2}$,

$E(X) = \sum xf(x) = \frac{1}{2}$,

$E(Y) = \sum yg(y) = \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{3}{2}$.

$Cov\ (X,\ Y) = E(XY) - E(X)E(Y) = \frac{1}{2} - \frac{3}{4} = -\frac{1}{4}$.

2. The joint probability distribution for two random variables X and Y as follows

Y \ X	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Determine i) marginal distribution of X and Y ii) $Cov\ (X,\ Y)$ iii) Correlation of X and Y

Solution: Given that

Y \ X	-2	-1	4	5	Sum
1	0.1	0.2	0	0.3	0.6
2	0.2	0.1	0.1	0	0.4
Sum	0.3	0.3	0.1	0.3	

Marginal distribution of X

x	-2	-1	4	5
$f(x)$	0.3	0.3	0.1	0.3

$E(X) = \mu_X = \sum xf(x) = 1,$

$V(X) = \sum x^2 f(x) - (\mu_X)^2 = 9.6,$

Marginal distribution of Y

y	1	2
$g(y)$	0.6	0.4

$E(Y) = \mu_Y = \sum yg(y) = 1.4.$

$V(Y) = \sum y^2 f(y) - (\mu_Y)^2 = 0.24.$

$\sigma_X = \sqrt{V(X)} = 3.098$, $\sigma_Y = \sqrt{V(Y)} = 0.4899$

$E(XY) = \sum_x \sum_y xyP(x, y) = 0.9$

$Cov\ (X, Y) = E(XY) - E(X)E(Y) = -0.5$

Correlation $\rho(X, Y) = \frac{Cov\ (X,Y)}{\sigma_X \sigma_Y} = -0.3294.$

3. The distribution of two independent variables X and Y are as follows

x	0	1	2
$f(x)$	0.3	0.3	0.4

y	1	2
$g(y)$	0.3	0.7

Determine the joint distribution, verify that $Cov\ (X, Y) = 0$.

Solution: Since X and Y are independent, $P(x, y) = f(x)g(y)$ for all x, y .

The joint distribution is

Y \ X	0	1	2	Sum
1	0.09	0.09	0.12	0.3
2	0.21	0.21	0.28	0.7
Sum	0.3	0.3	0.4	

$E(X) = \mu_X = \sum xf(x) = 1.1$, $E(Y) = \mu_Y = \sum yg(y) = 1.7.$

$E(XY) = \sum_x \sum_y xyP(x, y) = 1.87$

$Cov\ (X, Y) = E(XY) - E(X)E(Y) = 1.87 - 1.1 \times 1.7 = 0$.

Exercise:

1. If X and Y are two independent variables, X take the values 1 and 2 with probabilities 0.7, 0.3 and Y take the values $-2, 5, 8$ with probabilities $0.3, 0.5, 0.2$ respectively. Find the joint probability distribution of X and Y and $Cov(x, y)$.

2. The joint distribution of two discrete random variables X and Y is given by $P(x, y) = k(2x + y)$, where x and y are integers such that $0 \le x \le 2$, $0 \le y \le 3$.
Determine i) k ii) marginal distribution of X and Y . iii) Check whether X and Y are independent?

3. Two pens are selected at random from a box that contains 3 blue pens, 2 red pens and 3 green pens. If X is the number of blue pens selected, and Y is the number of red pens selected, then find i) Joint probability distribution ii) $P(x + y \le 1)$.

Stochastic process:

Stochastic Process: The process $\{t,\ x_t\}$ is called stochastic Process. Where t is the parameter and x_t are the values of the random variables called states.

1. Discrete state discrete parameter process.

Example: Number of telephone calls in different days in a telephone booth.

2. Discrete state continuous parameter process.

Example: Number of telephone calls in different time intervals in a telephone booth.

3. Continuous state discrete parameter process.

Example: Average duration of a telephone calls in different days in a telephone booth.

4. Continuous state continuous parameter process.

Example: Average duration of telephone calls in different time intervals in a telephone booth.

Probability vector: A vector $v = (v_1,\ v_2,\ v_3, \cdots\cdots\ v_n)$ is said to be probability vector if

i) $v_i \geq 0$ for all i and ii) $\sum_{i=1}^{n} v_i = 1$.

Examples: $(\ 0.6,\ 0.4),\ (0.1,\ 0.3,\ 0.4,\ 0.2),\ (1,\ 0,\ 0),\ (\frac{1}{2},\ 0,\ \frac{1}{2})$.

Stochastic matrices: A square matrix P is said to be stochastic matrices if each row of P is a probability vector.

Examples: $\begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0 & 1 & 0 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$, $\begin{bmatrix} 0.3 & 0.7 \\ 1 & 0 \end{bmatrix}$.

Regular stochastic matrices: A stochastic matrices P is said to be regular if all the entries of P^k are nonzero for some positive integer k.

Fixed point Or unique fixed probability vector: Let P be regular stochastic matrices, and v be a probability vector such that $vP = v$, then v is called unique fixed probability vector for P.

Marko-chain: Marko-chain is a discrete state discrete parameter process in which state space is finite and the probability of any state depends at the most predecessor state.

Transition probability matrix of a Marko-chain is $P = \langle p_{ij} \rangle$,
Where p_{ij} is the probability of the transition of ith state to jth state.

Transient state of Marko-chain: A state a_i is said to be transient if the chain may be in the state a_i in some step but after the finite number of steps chain never comes back to the state a_i .

Example: In modeling a computer program as a Marko-chain by considering every step as a state, except the last step every step is a transient state.

Absorbing state: A state a_i is said to be absorbing if the chain once it reaches the state a_i then it remains in the same state.

Example: Consider the transition probability matrix of a Marko-chain with state space (a_1, a_2, a_3)

$$P = \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0 & 1 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}.$$ Clearly the state a_2 is absorbing.

Recurrent state: A state a_i is said to be recurrent (or periodic) if the chain return to the state a_i from the state a_i in finite number of steps with probability 1. Minimum number of steps required to return is called period.

Higher transition probability: Let P be the transition probability matrix of a Marko-chain, and v_0 be the initial probability vector of the chain.

Then after the first step the probability vector is $v_1 = v_0 P$,

after the second step the probability vector is $v_2 = v_1 P = v_0 P^2$,

after the third step the probability vector is $v_3 = v_2 P = v_0 P^3$, and so on.

A Marko-chain is irreducible iff the transition probability matrix is regular.

Problems:

1. Show that $P = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$ is regular stochastic matrix and find the associated unique fixed probability vector.

Solution: Since $P^2 = \begin{bmatrix} 1/6 & 1/2 & 1/3 \\ 1/12 & 23/36 & 5/18 \\ 1/9 & 5/9 & 1/3 \end{bmatrix}$,

all the entries of P^2 are nonzero, hence P is regular.

Let $v = (x,\ y,\ z)$ be the fixed probability vector,

then $vP = v$, that is

$$(x,\ y,\ z)\begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = (x,\ y,\ z)$$

$$\Rightarrow \begin{matrix} x+y+z=1 \\ \frac{y}{6} = x \\ x + \frac{y}{2} + \frac{2z}{3} = y \end{matrix} \quad \Rightarrow \begin{matrix} x+y+z=1 \\ -x + \frac{y}{6} = 0 \\ x - \frac{y}{2} + \frac{2z}{3} = 0 \end{matrix} \quad \Rightarrow \begin{matrix} x = 0.1 \\ y = 0.6 \\ z = 0.3 \end{matrix}.$$

$\therefore v = (0.1,\ 0.6,\ 0.3)$.

2. Three boys A, B, C are throwing a ball to each other. A always throws ball to B, B always throws the ball to C, but C is just as likely to throw the ball to B as to A. If C was the first person to throw the ball, find the probability that after three throws i) A has the ball, ii) B has the ball, iii) C has the ball.

Solution: Let a, b, c indicate the states ball is with A, B or C respectively.

Transition probability matrix is $P = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$.

Since C was the first person to throw the ball, $v_0 = (0,\ 0,\ 1)$

Then after the first throw the probability vector is $v_1 = v_0 P = (1/2,\ 1/2,\ 0)$,

after the second throw the probability vector is $v_2 = v_1 P = (0,\ 1/2,\ 1/2)$,

after the third throw the probability vector is $v_3 = v_2 P = (1/4,\ 1/4,\ 1/2)$.

Therefore after third throw probabilities that A has the ball, B has the ball, and C has the ball are respectively $1/4,\ 1/4,\ 1/2$.

3. Show that $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is regular.

Solution: $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$,

$$A^3 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \\ 0.25 & 0.5 & 0.25 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 0.25 & 0.25 & 0.5 \\ 0.25 & 0.5 & 0.25 \\ 0.125 & 0.375 & 0.5 \end{bmatrix}.$$

Since the entries of A^5 are all nonzero's, therefore A is regular.

4. A student's study habits are as follows. If he studies one night, he is 70% sure not to study the next night. On the other hand if he does not study one night, he is 60% sure not to study the next night. In long run how often does he study?

Solution: Let S denote he study, N denote he does not study.

Transition probability matrix is $P = \begin{array}{c c} & \begin{array}{cc} S & N \end{array} \\ \begin{array}{c} S \\ N \end{array} & \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$.

Let $v = (x,\ y)$ be the fixed probability vector, then $vP = v$, that is

$$(x,\ y)\begin{bmatrix}0.3 & 0.7\\0.4 & 0.6\end{bmatrix} = (x,\ y) \ \Longrightarrow \ \begin{matrix}x + y = 1\\0.3x + 0.4y = x\end{matrix}$$

$$\Longrightarrow \ \begin{matrix}x + y = 1\\-0.7x + 0.4y = 0\end{matrix} \ \Longrightarrow \ \begin{matrix}x = \frac{4}{11}\\y = \frac{7}{11}\end{matrix}$$

In long run he studies with the probability $\frac{4}{11}$.

Exercise:

1. Show that following stochastic matrices are not regular.

i. $\begin{bmatrix}1/2 & 1/4 & 1/4\\0 & 1 & 0\\1/2 & 0 & 1/2\end{bmatrix}$ ii. $\begin{bmatrix}1/2 & 1/2 & 0\\1/2 & 1/2 & 0\\1/4 & 1/4 & 1/2\end{bmatrix}$

2. Every year, a man trades his car for a new car. If he has a Maruti, he trades it for a Ford, If he has a Ford, he trades it for Santro. However, if he has a Santro, he is just as likely to trade it for a new Santro as to trade it for a Maruti or a Ford. In 2014 he bought his first car, which was a Santro.

Find the probability that he has a i) 2015 Santro, ii) 2016 Maruti , iii) 2017 Ford.

3. There are 2 white marbles in box A and 3 red marbles in box B. At each step of the process a marble is selected from each box and the two marbles selected are interchanged. Let the state a_i of the system is number of i red marbles in box A. a) Find the transition probability matrix. b) What is the probability that there are 2 red marbles in box A after 3 steps? c) In long run what is the probability that there are 2 red marbles in box A?