EE 604
Digital Image Processing

IIT KANPUR
Indian Institute of Technology, Kanpur

Tanaya Guha
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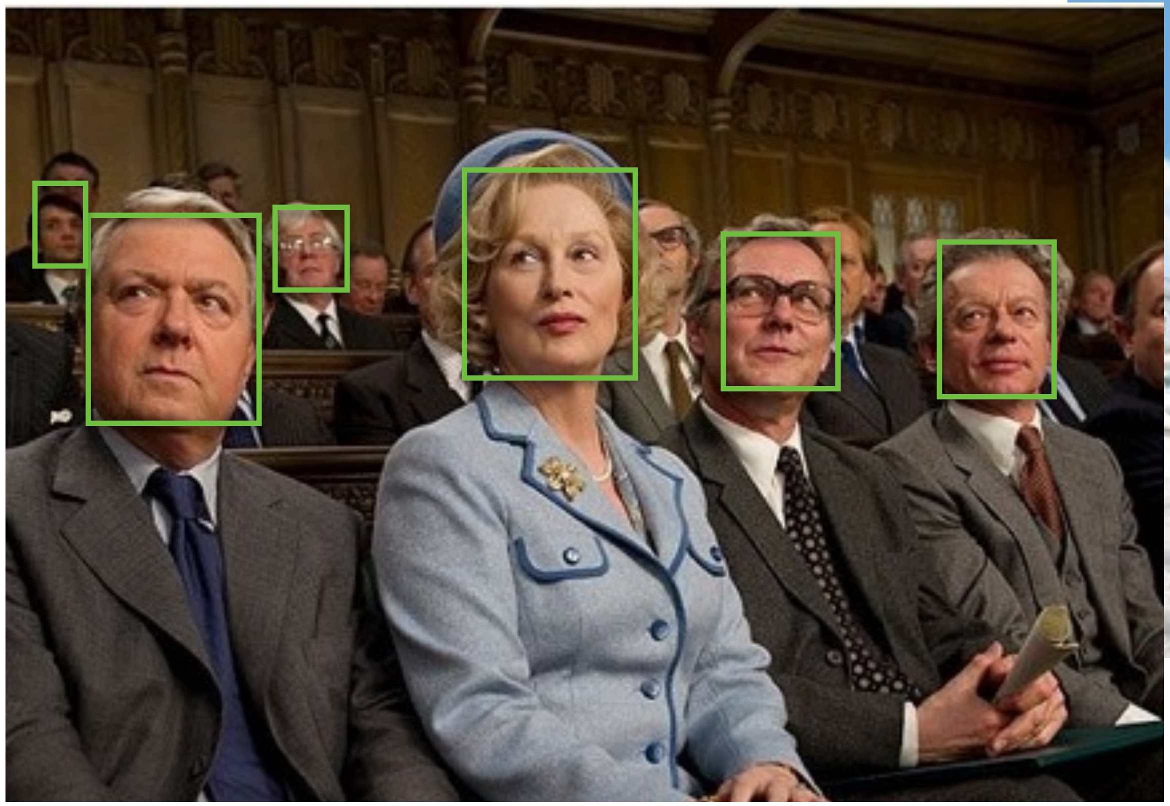
Multiscale Image Analysis

The multiscale concept

- Images contain useful information at different scales

- Analyzing information at any one scale will not be effective



The multiscale concept

- How to analyze image in multiple scales?
    - Vary the window size
    - Alternatively, vary the image size, keeping window size the same.
- Larger objects can be examined at low resolution
- Smaller objects need to be examined at higher resolution

Multiscale analysis

- Various ways of multiscale analysis
    - Image pyramid
    - Subband coding
    - **<u>Wavelet decomposition</u>**

Wavelet decomposition

Fourier transform:

$$X(\omega) = \int_{-\infty}^{-\infty} x(t) e^{-j\omega t} dt$$

Basis function

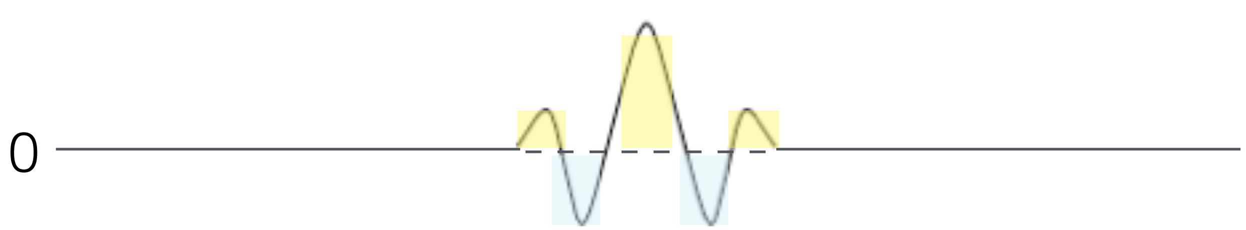
Wavelet transform:

$$X(a,b) = \int_{-\infty}^{-\infty} x(t) \psi^*_{a,b}(t) dt$$

Wavelet decomposition

- Linear decomposition of a signal over a set of basis functions
    - basis functions are called **wavelets**
- Wavelet has compact support (unlike Fourier basis)

- Wavelet function has zero mean



sum( ) = sum( )

Wavelet decomposition

Fourier transform:

$$X(\omega) = \int_{-\infty}^{-\infty} x(t) e^{-j\omega t} dt$$

frequency

Wavelet transform:

$$X(a, b) = \int_{-\infty}^{-\infty} x(t) \psi^*_{a,b}(t) dt$$

scale, translation

Wavelet decomposition

The entire set of wavelet basis can be formed by translating and dilating a prototype wavelet (**mother wavelet**)

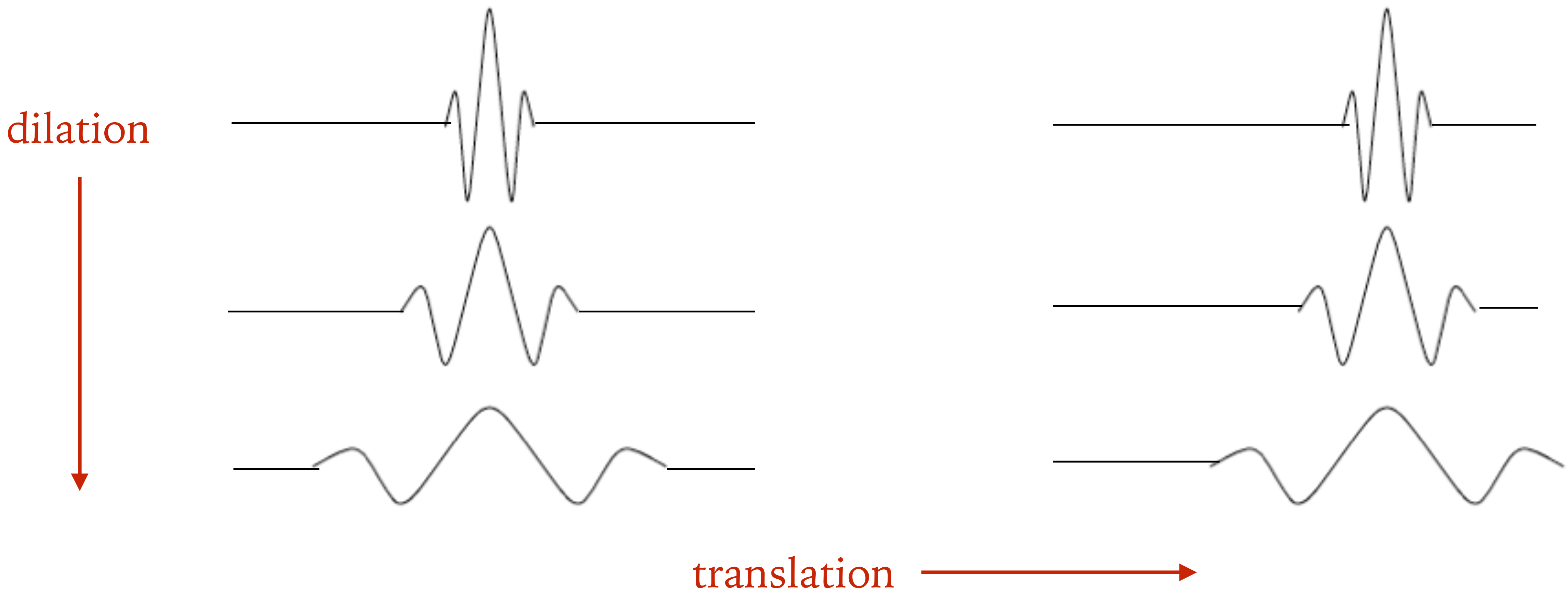
$$\psi_{a,b}(t) = 2^{-a/2}\psi\left(\frac{t-b}{2^a}\right)$$

translation

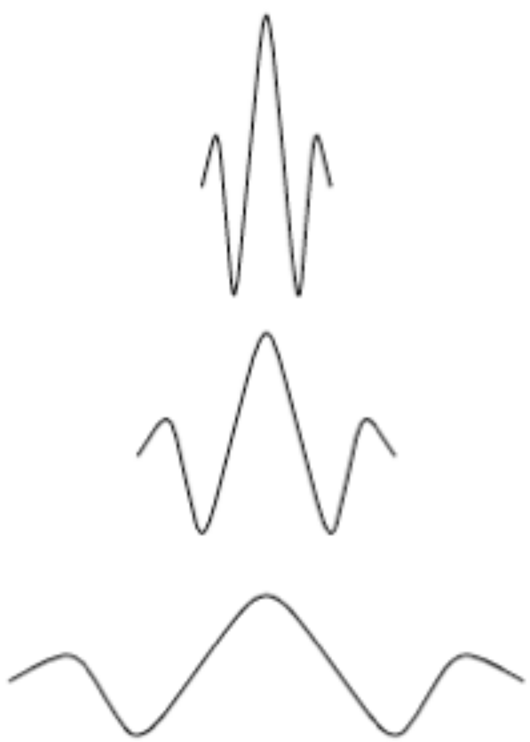
normalize

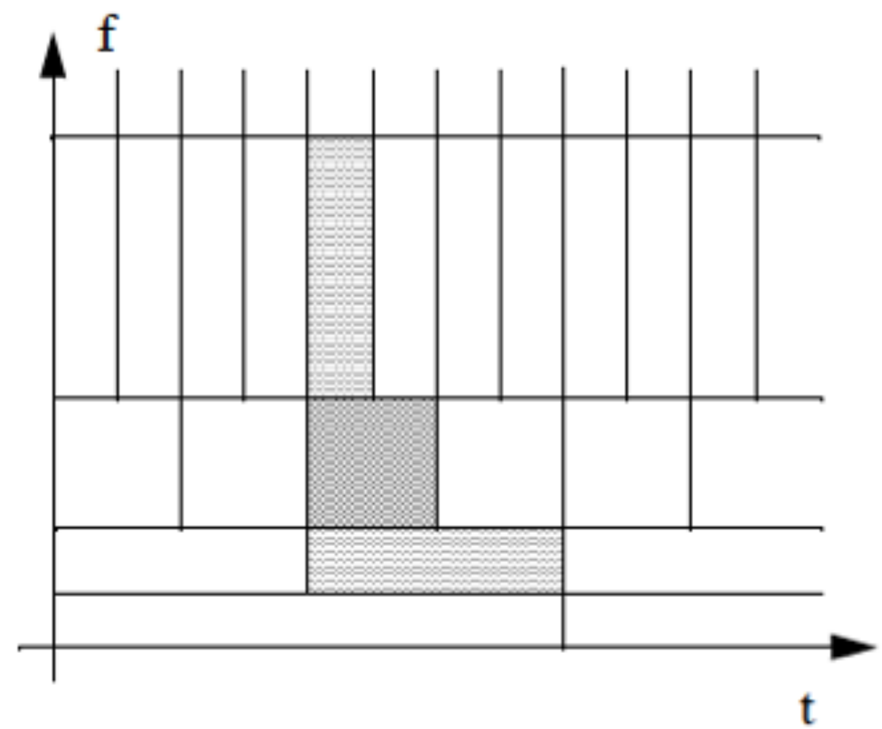
dilation

$a, b \in \mathbb{Z}$



Wavelet decomposition



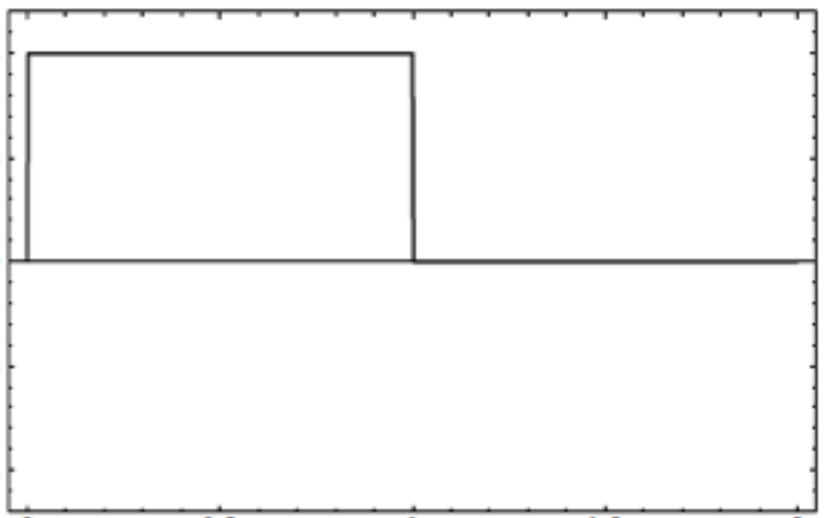


Time-frequency plane

Vettereli and Kovacevic, 2007

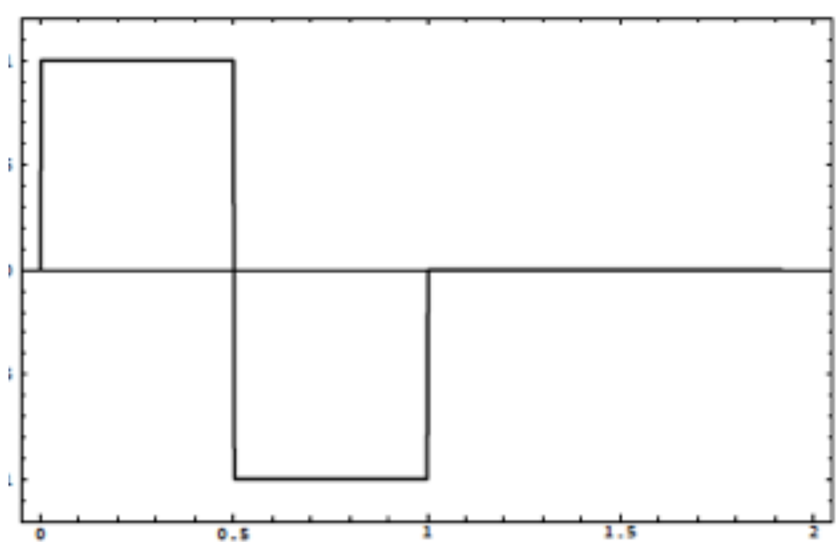
Haar wavelet

Haar scaling function



$$\varphi(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

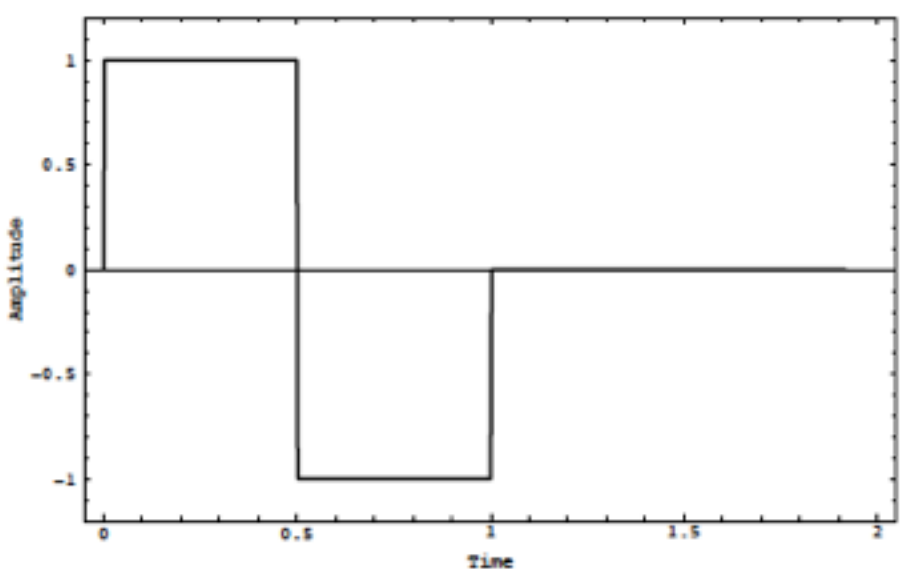
Haar (mother) wavelet

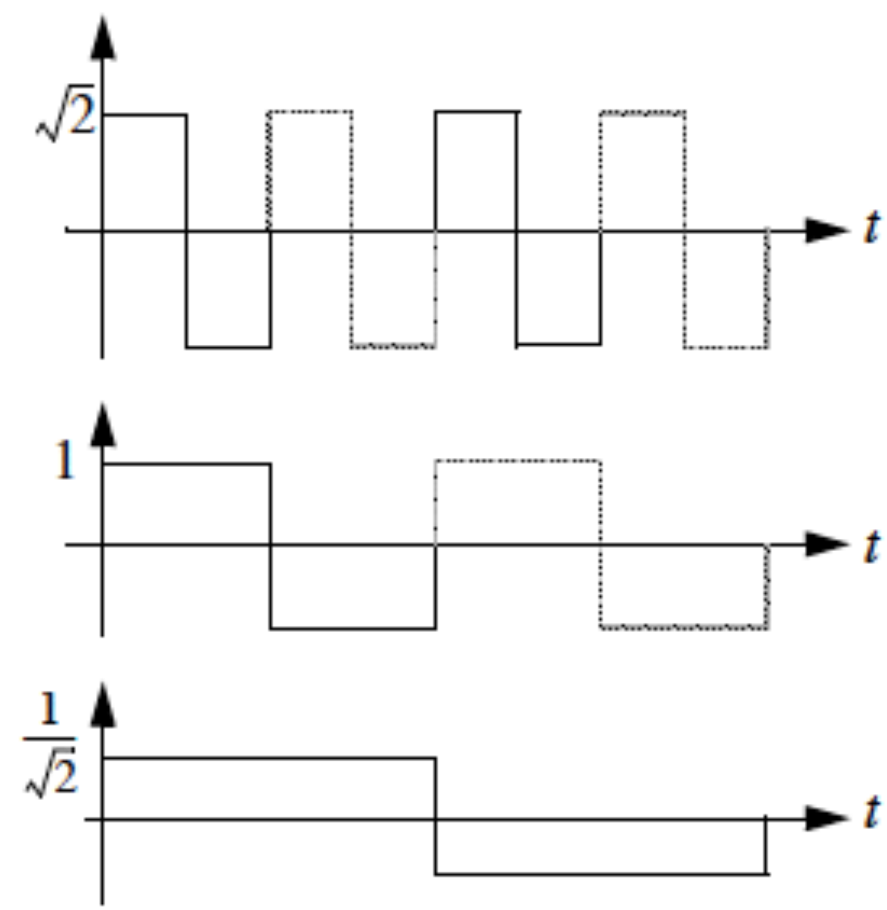


$$\psi(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2} \\ -1 & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

Haar wavelet

Haar wavelet





Haar matrix

$$\begin{array}{rc}
a=0 & \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \\
a=1 & \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix} \\
a=2 & \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{bmatrix} \\
\text{scaling function:} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}
\end{array} = \mathbf{H}^T$$

Haar transform

$$\underset{\substack{\uparrow \\ \text{signal}}}{\mathbf{f}} = \mathbf{H}\underset{\substack{\uparrow \\ \text{coefficients}}}{\mathbf{w}} \qquad\qquad \mathbf{w} = \mathbf{H}^{-1}\mathbf{f} = \mathbf{H}^{T}\mathbf{f}$$

Example:

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\mathbf{H}$ is column normalized for orthonormality.

Haar transform & filterbank

Discrete Haar wavelet:

$$\varphi_{2k}[n] = \begin{cases} \frac{1}{\sqrt{2}} & n = 2k,\ 2k+1, \\ 0 & \text{otherwise}, \end{cases} \qquad \varphi_{2k+1}[n] = \begin{cases} \frac{1}{\sqrt{2}} & n = 2k, \\ -\frac{1}{\sqrt{2}} & n = 2k+1 \\ 0 & \text{otherwise}. \end{cases}$$

Haar expansion:

$$X[2k] = \langle \varphi_{2k}, x \rangle = \frac{1}{\sqrt{2}}(x[2k] + x[2k+1])$$

$$X[2k+1] = \langle \varphi_{2k+1}, x \rangle = \frac{1}{\sqrt{2}}(x[2k] - x[2k+1])$$

$$x[n] = \sum_{k \in \mathbb{Z}} X[k]\ \varphi_k[n]$$

Vettereli and Kovacevic, 2007 (chap 3)

Haar transform & filterbank

$$h_0[n] = \begin{cases} \frac{1}{\sqrt{2}} & n = -1,\ 0 \\ 0 & \text{otherwise.} \end{cases} \qquad h_1[n] = \begin{cases} \frac{1}{\sqrt{2}} & n = 0, \\ -\frac{1}{\sqrt{2}} & n = -1, \\ 0 & \text{otherwise} \end{cases}$$

$$h_0[n] * x[n] \mid_{n=2k} = \sum_{l \in \mathbb{Z}} h_0[2k-l]\ x[l] = \frac{1}{\sqrt{2}} x[2k] + \frac{1}{\sqrt{2}} x[2k+1] = X[2k]$$

(the Haar inner product)

$$\begin{aligned} h_1[n] * x[n] \mid_{n=2k} &= \sum_{l \in \mathbb{Z}} h_1[2k-l]\ x[l] \\ &= \frac{1}{\sqrt{2}} x[2k] - \frac{1}{\sqrt{2}} x[2k+1] = X[2k+1] \end{aligned}$$

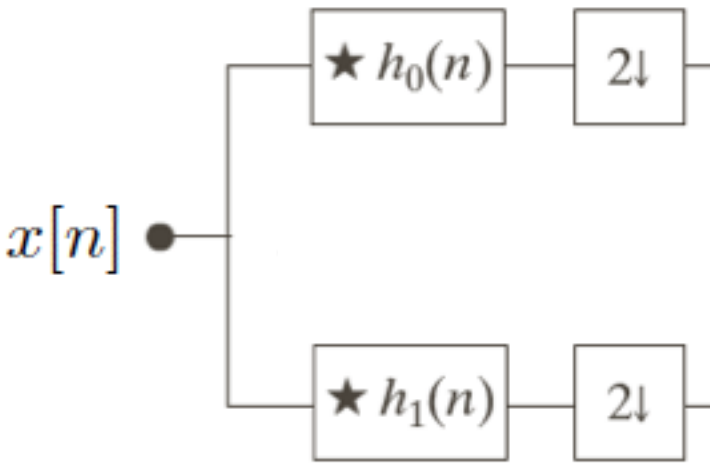
(the Haar inner product)

So convolution and downsampling can provide the same Haar inner products.

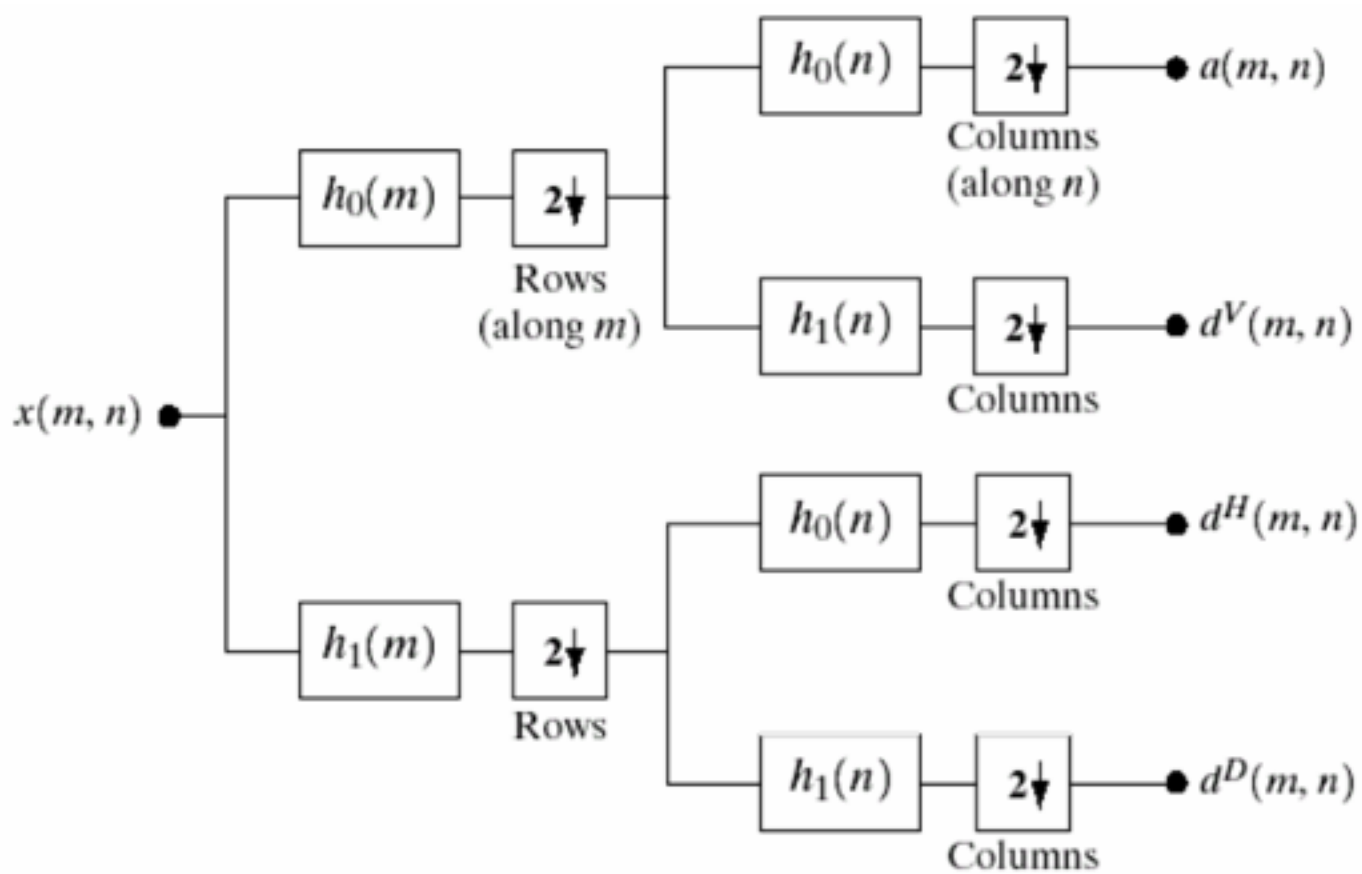
This is **filterbank**! So, Haar wavelet can be implemented as filter banks.

Vettereli and Kovacevic, 2007 (chap 3)

Haar transform & filterbank

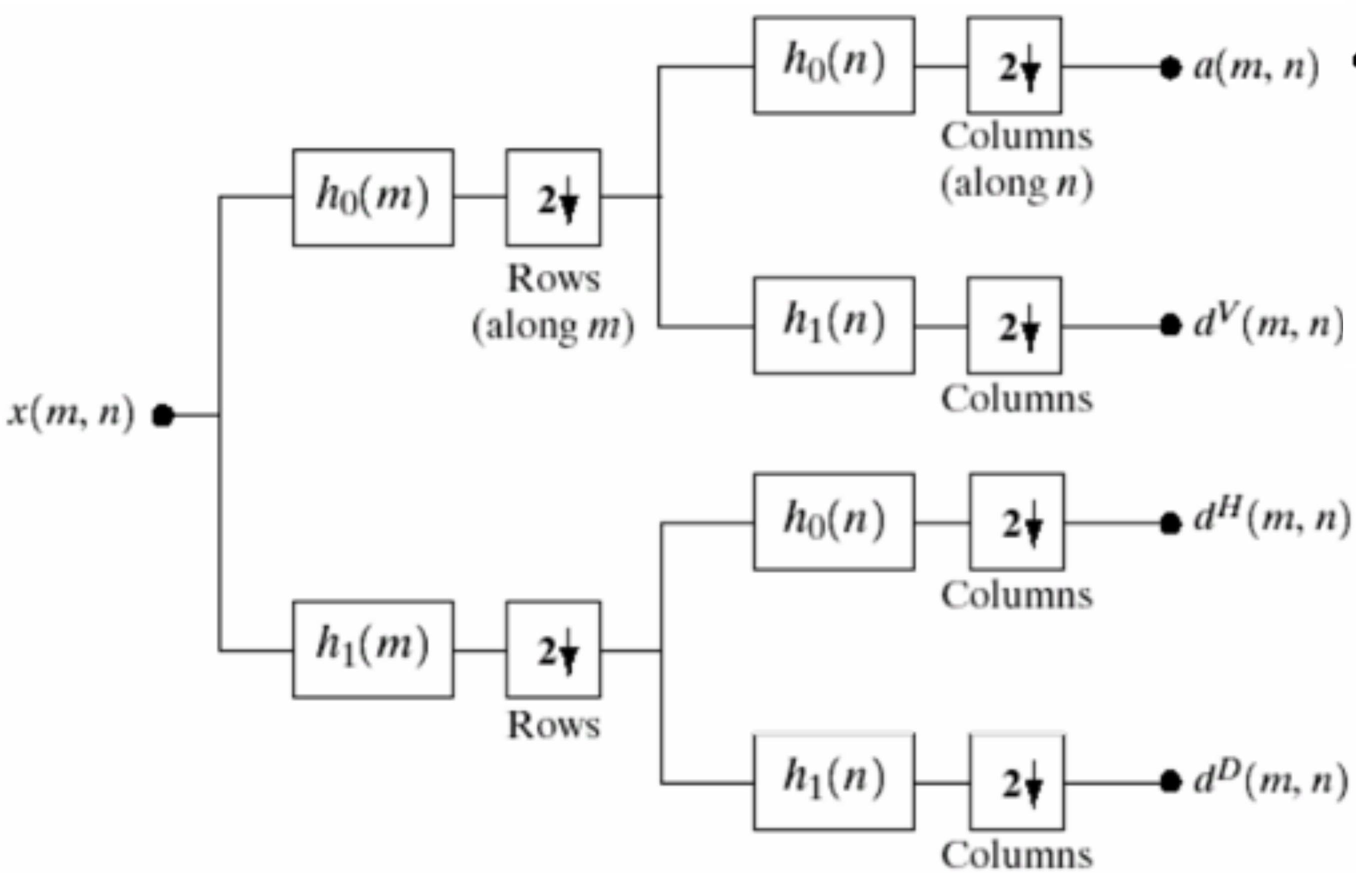


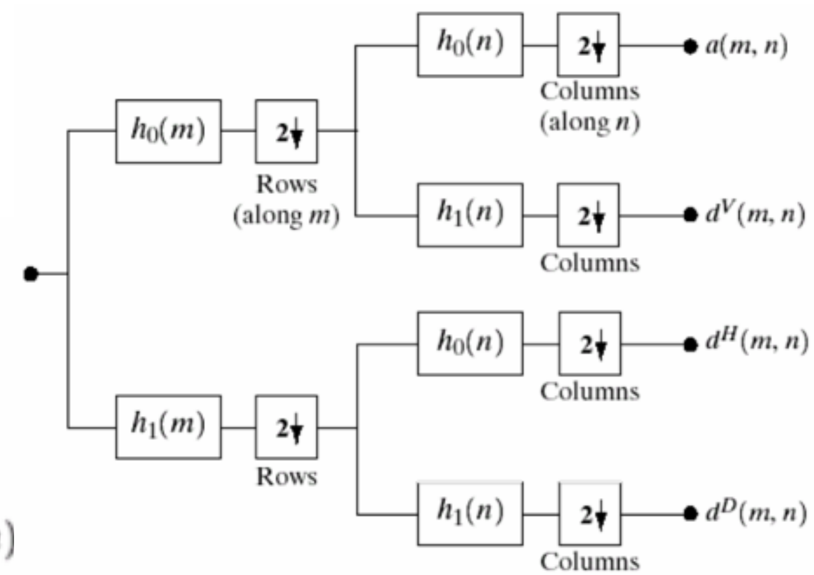
Single scale (1D)



Single scale (2D)

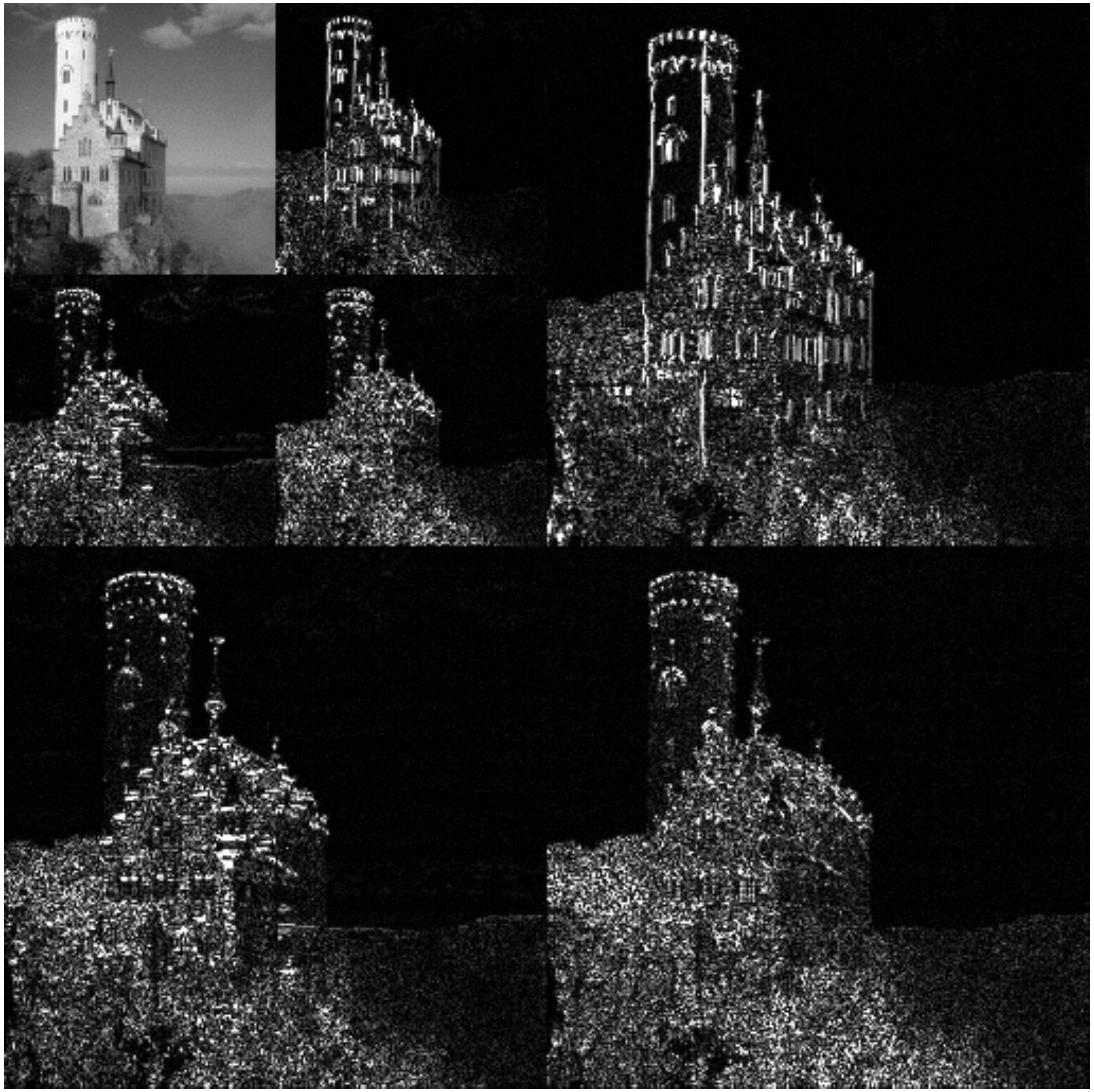
Haar wavelet as filterbank





Multiscale (2D)

Wavelet decomposition



2-level Wavelet decomposition

image source: wikipedia