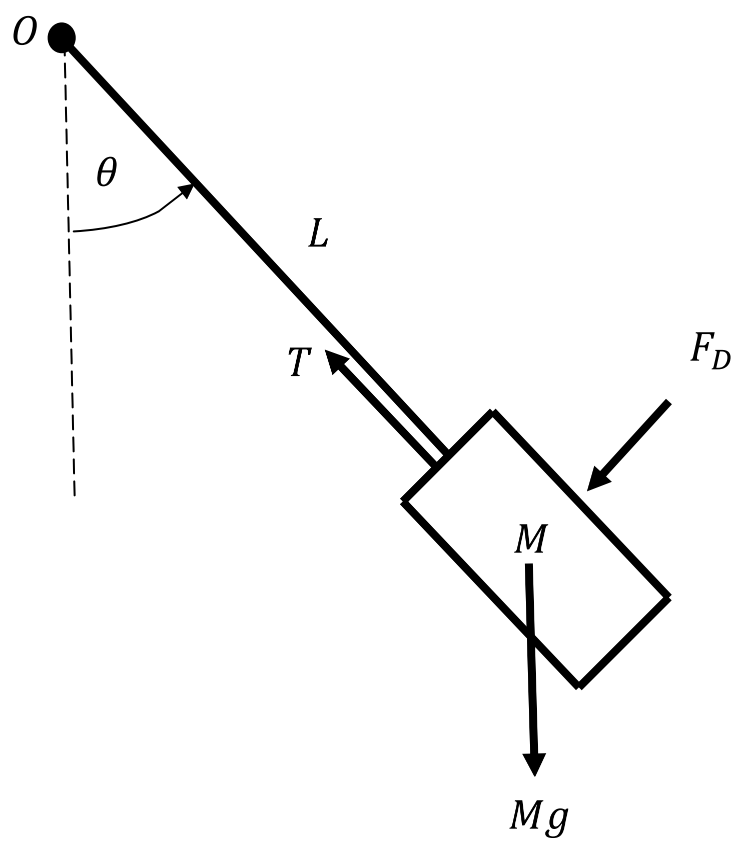
Modeling of a Simple Pendulum



Simple Linear Model:

- The simplest analytical model for a "standard" pendulum can be derived by assuming no drag force and small variations in the angle of the pendulum. Assume that the entire mass of the pendulum is concentrated in the bob and that the length from the pivot point to the center of gravity of the bob is given by L.
- Find the sum of the moments around the point O taking the counterclockwise direction as positive. Note that the tension, T, in the thin rod does not contribute to the moment about the point O.

$$\sum M_O = 0$$

$$-L(Mg)sin(\theta) - I\alpha = 0$$

- The mass moment of inertia of the pendulum, I, is given by:

$$I = M \cdot L^2$$

- The angular acceleration of the pendulum, α, is the second derivative of the angle θ.

$$\alpha = \ddot{\theta}$$

- The differential equation governing the dynamics of the pendulum becomes:

$$-L(Mg)sin(\theta) - ML^2\ddot{\theta} = 0$$

- Dividing through by $(M \cdot L^2)$ and rearranging terms yields the following second order differential equation:

$$\ddot{\theta} + \frac{g}{L}sin(\theta) = 0$$

- Although this equation appears simple enough, it is nonlinear in nature due to the presence of a transcendental function. This non-linear equation can, however, be linearized by assuming small variations in the angle of the pendulum, θ.
- For small angles (expressed in radians), $sin(\theta)$ is approximately equal to θ. For example:

$$for\ \theta = 22.5° = 0.3927\ radians, \quad sin(\theta) = 0.3827$$

- Therefore for small variations in angle, the linear differential equation describing the dynamics of a simple pendulum can be approximated by:

$$\ddot{\theta} + \frac{g}{L}\theta = 0$$

- This equation can be solved analytically by considering the basic properties of Laplace transforms. Recall that the second derivative of a function can be expressed in the Laplace domain as follows:

$$F''^{(t)} = s^2F(s) - sf(0) - f'(0)$$

- The differential equation describing the dynamics of the pendulum becomes:

$$s^2\theta - s\theta_o - \dot{\theta}_0 + \frac{g}{L}\theta = 0$$

- Assume that the pendulum is released from some initial angle, θ_o, no initial velocity ($\dot{\theta}_0 = 0$).

$$s^2\theta - s\theta_o + \frac{g}{L}\theta = 0$$

- Solving for θ yields:

$$\theta\left(s^2 + \frac{g}{L}\right) = s\theta_o$$

$$\theta = \frac{s\theta_o}{\left(s^2 + \frac{g}{L}\right)}$$

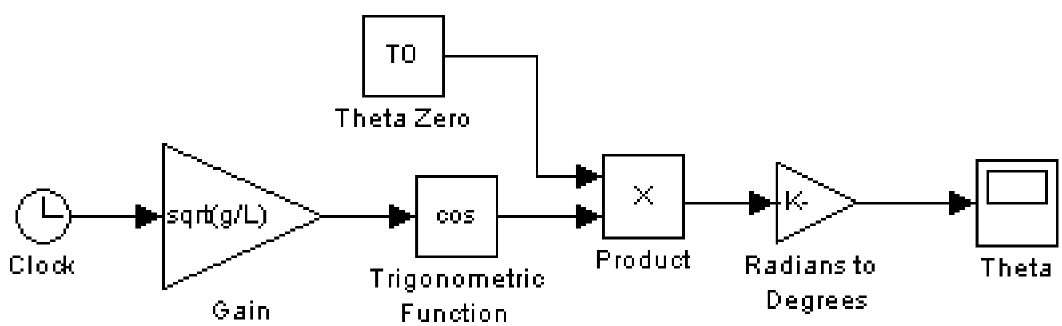
- This can easily be put into the form of the following basic Laplace transform:

$$\mathcal{L}^{-1}\left\{K\frac{s}{s^2 + \omega^2}\right\} = Kcos(\omega t)u(t)$$

- The angle of the pendulum, θ, as a function of time based on a simple, linear model becomes:

$$\theta = \theta_o cos\left(\sqrt{\frac{g}{L}}t\right)u(t)$$

- Although it is not necessary to use Matlab-Simulink to visualize this function, for comparison purposes the SImulink model becomes:



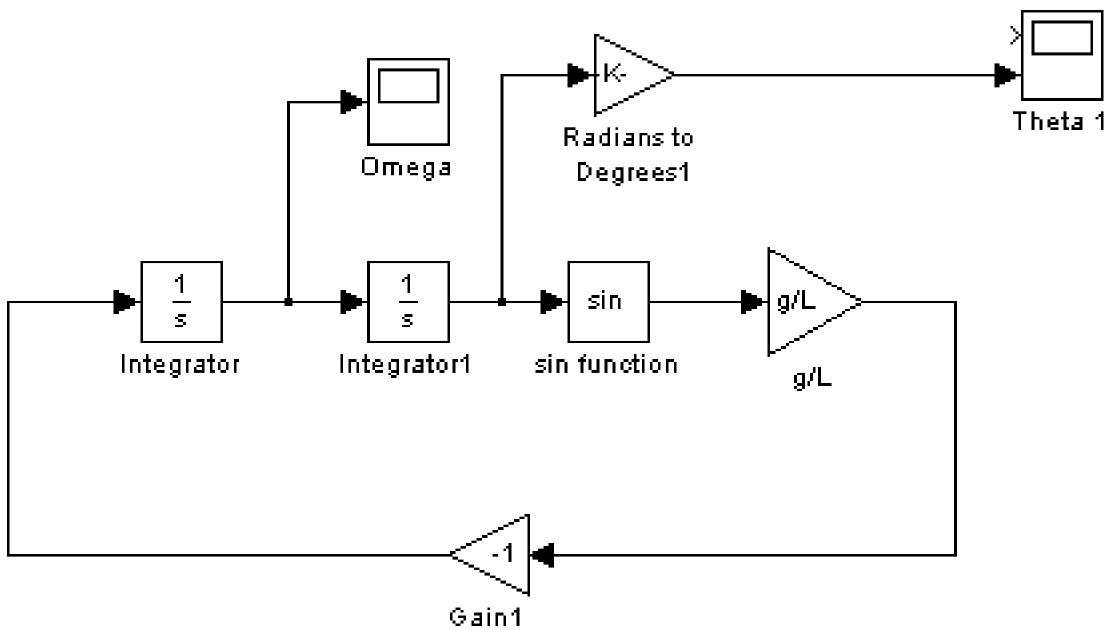
Simple Non-Linear Model:

- Although finding an analytical solution to the simple non-linear model is not evident, it can be solved numerically with the aid of Matlab-Simulink. Start by isolating the highest order derivative from the linear model:

$$\ddot{\theta} + \frac{g}{L} sin(\theta) = 0$$

$$\ddot{\theta} = -\frac{g}{L} sin(\theta)$$

- The Matlab-Simulink model that solves this equation is given by:



Non-Linear Model Including a Constant Aerodynamic Drag Force:

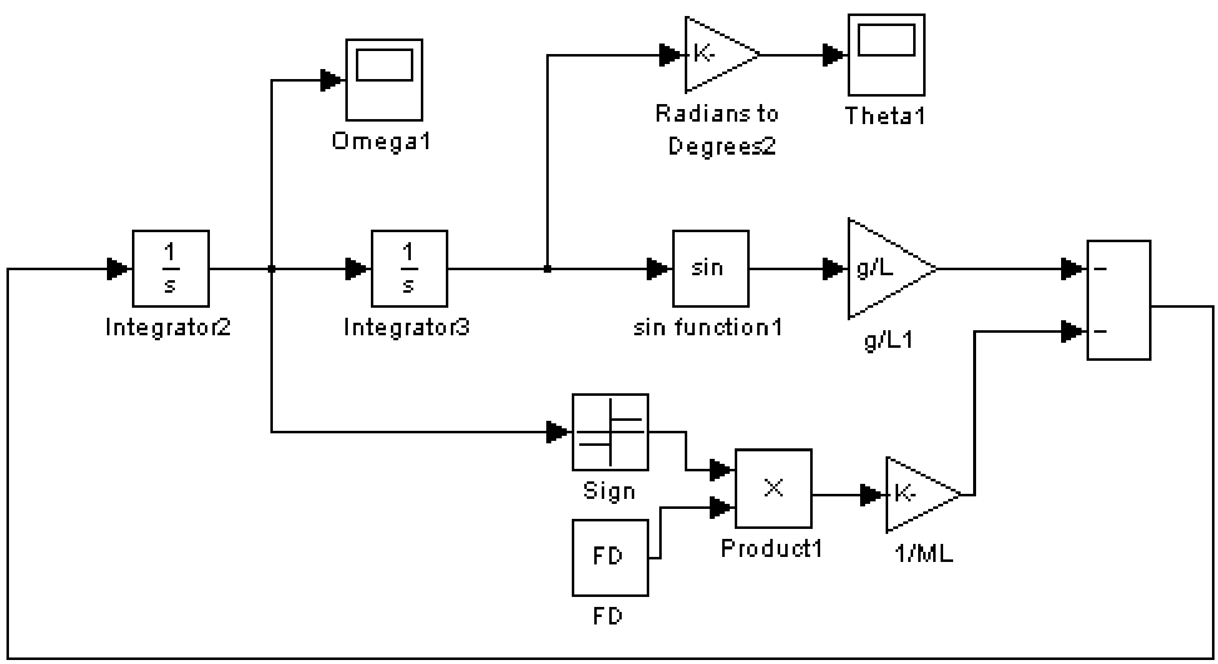
- Assume that the aerodynamic drag acts only on the bob; i.e., the drag on the thin rod is negligible. The drag force, F_D, imposes an additional moment about the point O.
- The differential equation governing the dynamics of the pendulum becomes:

$$-L(Mg)sin(\theta) - ML^2\ddot{\theta} - LF_D = 0$$

- Isolating the highest order derivative of θ yields:

$$\ddot{\theta} = -\frac{g}{L}sin(\theta) - \frac{1}{ML}F_D$$

- The Matlab-Simulink model that solves this equation is given by:



- Note that the drag force must change sign based on the angular velocity of the pendulum. In other words, although the drag force is taken as constant, it must always act opposite to the direction of motion.

Non-Linear Model Including Aerodynamic Drag Force that is a function of Velocity:

- The aerodynamic drag for incompressible flow over any body is given by:

$$F_D = \frac{1}{2} \cdot C_D \cdot A \cdot \rho \cdot V^2$$

- where V is the velocity of the incompressible fluid, ρ is the density of the fluid, A is the frontal area (note: not the *surface* area), and C_D is the drag coefficient.
- The drag coefficient for a cylindrical body is a function of the Reynolds number as shown in the figure on the following page.

- Recall that the Reynolds number is given by:

$$RE = \frac{\rho \cdot V \cdot D}{\mu}$$

- where D is the diameter of the cylinder and μ is the viscosity of the incompressible fluid (in this case air).

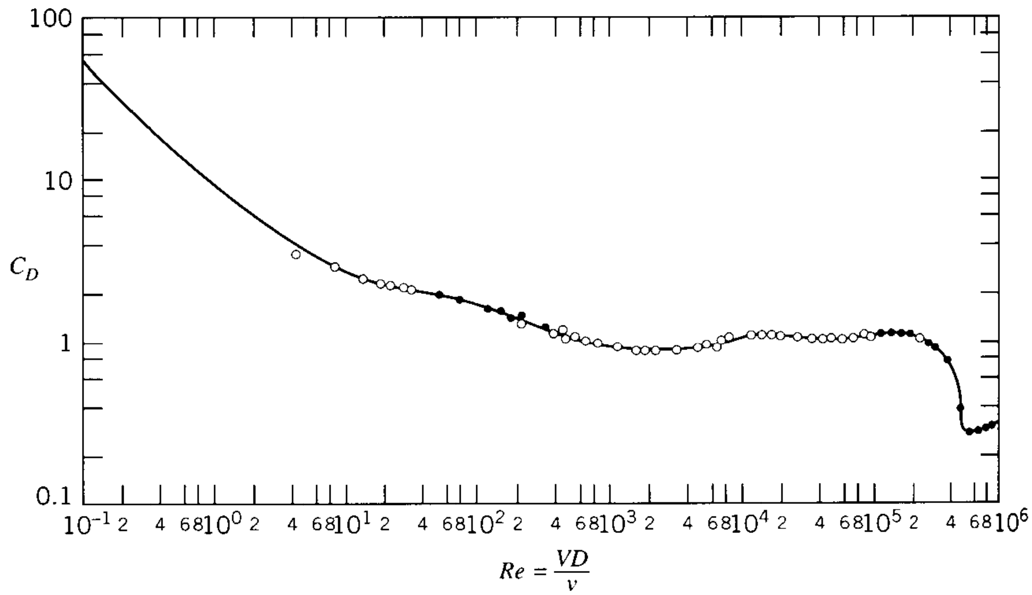
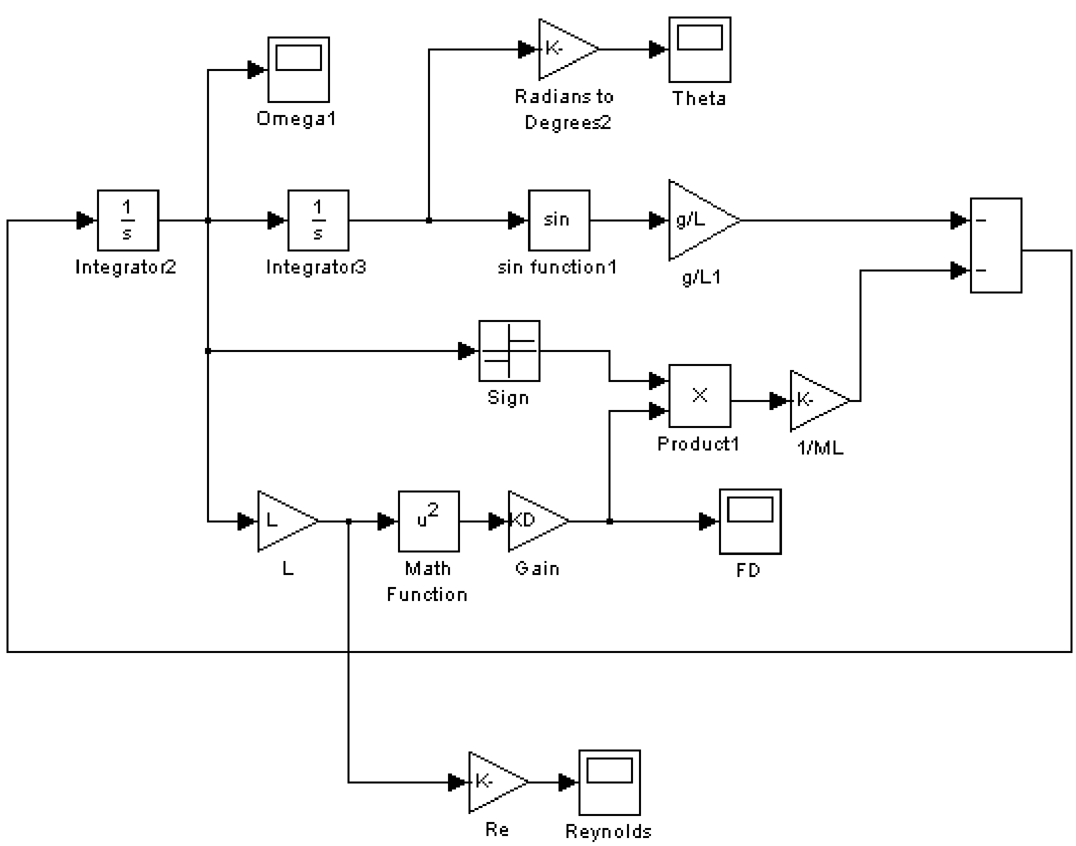


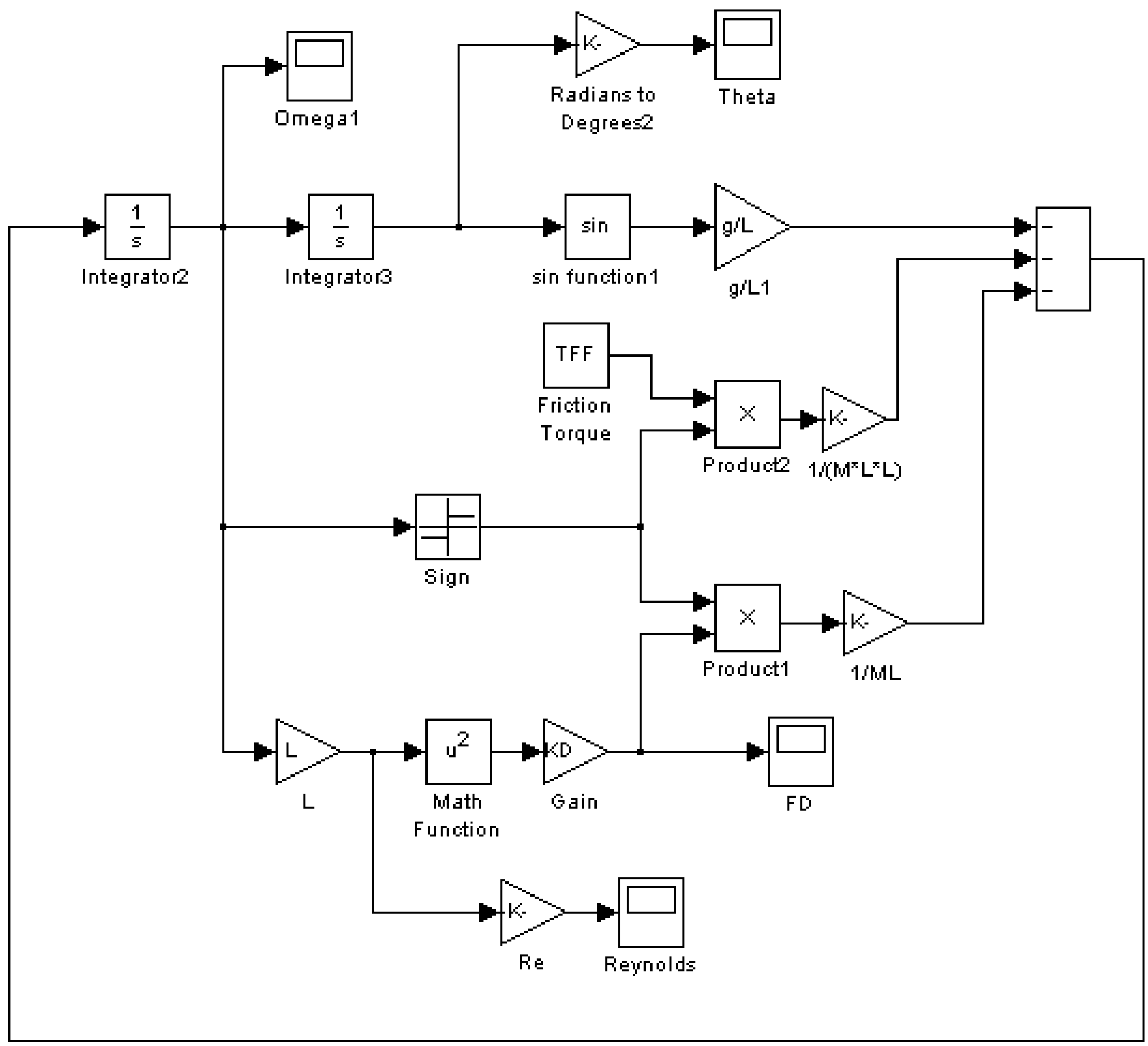
Fig. 9.13 Drag coefficient for a smooth circular cylinder as a function of Reynolds number [3].

- The Matlab-Simulink model for the pendulum with variable aerodynamic drag is given by:

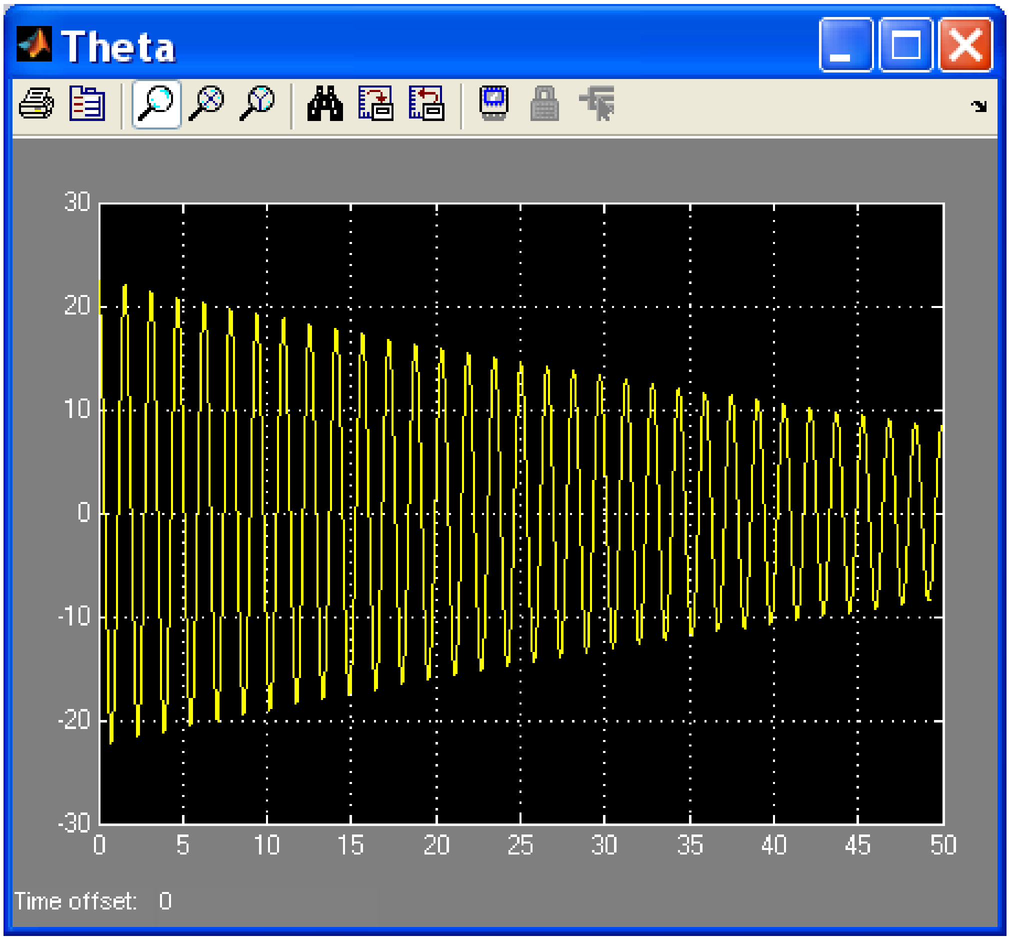


Non-Linear Model Including Variable Aerodynamic Drag and Friction at the Pivot:

- There could potentially be a significant amount of friction at the pivot point.
- The Matlab-Simulink model including aerodynamic drag and Coulomb friction at the pivot point is given by:



- The response of the system based on an initial pendulum angle of 22.5 degrees is illustrated below.



- The following table summarizes the parameters used in the simulation:

```
T0 = 22.5*pi/180;    %% Initial pendulum angle (radians)
L = 0.6;             %% Length of rod (distance to CG) (m)
g = 9.81;            %% Acceleration due to gravity (m/s^2)

M = 0.06;            %% Mass of bob (kg)
FD = 0.0007;         %% Drag Force (N)

DB = 0.0333;         %% Diameter of bob (m)
HB = 0.0508;         %% Height of bob (m)
A = DB*HB;           %% Frontal area of bob (m^3)

RHO = 1.23;          %% Density of air (kg/m^2)
MEW = 1.79E-5;       %% Viscosity of air (kg/(m s))
CD = 1;              %% Drag Coefficient

KD = 0.5*CD*A*RHO;   %% Drag constant (kg/m)
TFF = 0.0005;        %% Friction torque at pivot (N-m)
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