Lab Session 4

MA-423 : Matrix Computation   July-November, 2019   S. Bora

The purpose of the following exercises is to illustrate the immense difficulties in handling polynomials in finite precision computation.

1. Write a MATLAB function program $\texttt{y} = \texttt{Horner}(\texttt{p}, \texttt{x})$ which uses Horner's rule (see `Note 1` below) to evaluate a polynomial $p(z) = p_1 z^n + p_2 z^{n-1} + \cdots + p_n z + p_{n+1}$ at $z = x$ by taking $x \in \mathbb{R}$ and a vector p with $p(i) = p_i, i = 1 : n + 1$ as input. Design your program in such a way that if the user passes x as a vector (say in $\mathbb{R}^m$), then the output is a vector $y \in \mathbb{R}^m$ satisfying $y(i) = p(x(i)), i = 1 : m$.

2. Write a MATLAB program $\texttt{x} = \texttt{bisect}[\texttt{p}, \texttt{x}_0, \texttt{x}_1, \texttt{tol}]$ which finds a root of a polynomial $p(x) = \Sigma_{k=0}^{m} a_k x^k$ in an interval $[x_0, x_1]$ upto a given tolerance *tol* using the Bisection method (see `Note 2` below). You should use the function `Horner` to evaluate the polynomial $p(x)$ at a point x.

   Now apply your algorithm to find a root of the polynomial

   $$p(x) = (x-2)^9 = x^9-18x^8+144x^7-672x^6+2016x^5-4032x^4+5376x^3-4608x^2+2304x-512.$$

   in different intervals $[x_0, x_1]$ that lie within $[1.95, 2.05]$. What do you observe?

3. This exercise should help you to understand the observations made in the previous exercise.

   Evaluate the polynomial p introduced in the previous exercise at 151 equidistant points (use `linspace` command) in the interval [1.93, 2.08] using the program $\texttt{y} = \texttt{Horner}(\texttt{p}, \texttt{x})$. Then evaluate p for the same points by directly using the formula $p(x) = (x - 2)^9$.

   [If $\texttt{x} = \texttt{linspace}(1.93, 2.08, 151)$, then a vector $\texttt{z}$ for which $z(i) = p(x_i)$, $i = 1 : 151$ is obtained via the command $\texttt{z} = (\texttt{x} - 2).\hat{}\,9$.]

   Plot the graphs for both the procedures in the same figure. (Use the `hold on` command and different colors to distinguish between the plots).

   Do the plots differ from one another? If yes, can you think of possible reasons? Explain the results of the previous exercise in the light of the difference in the plots.

Note 1 *(***Horner's method***)* `Given` $p(z) = \Sigma_{k=0}^{n} p_{n-k+1} z^k$, `the Horner's method uses the fact that`

$$p_1 x^n + p_2 x^{n-1} + \cdots + p_n x + p_{n+1} = p_{n+1} + x(p_n + \cdots + x(p_3 + x(p_2 + p_1 x)))$$

`to evaluate` $p(z)$ `at` $z = x$.

Note 2 *(***Bisection method:***)* `The traditional bisection method is a tool to find a root of a polynomial` $p(x)$ `in an interval which makes repeated use of the fact that given a continuous function` f `on an interval` $[a, b]$, `such that` $f(a)$ `and` $f(b)$ `are of opposite sign, there exists at least one` $c \in (a, b)$ `such that` $f(c) = 0$.

Choosing an initial interval say $[x_0, x_1]$ satisfying $p(x_0)p(x_1) < 0$, the first step is to evaluate $p(x)$ at the midpoint $x_{mid} = (x_0+x_1)/2$ (hence the name 'bisection') and check the signs of $p(x_0), p(x_{mid})$ and $p(x_1)$. If $p(x_0)$ and $p(x_{mid})$ are of opposite sign, then the procedure is repeated for the interval $[x_0, x_{mid}]$. Otherwise it is repeated for $[x_{mid}, x_1]$. The method is necessarily *iterative* (i.e. it does not terminate after a finite number of steps). In practice, it is terminated upon finding a number that lies in an interval $[x_r - tol, x_r + tol]$ where x_r is an exact root of $p(x)$ in the initial interval $[x_0, x_1]$ and tol is a tolerance value provided by the user.(This is achieved by writing a **while** statement which is executed as long as the length of the interval to be bisected at any given step is greater than $2\,tol$.)