SPR Lab 3

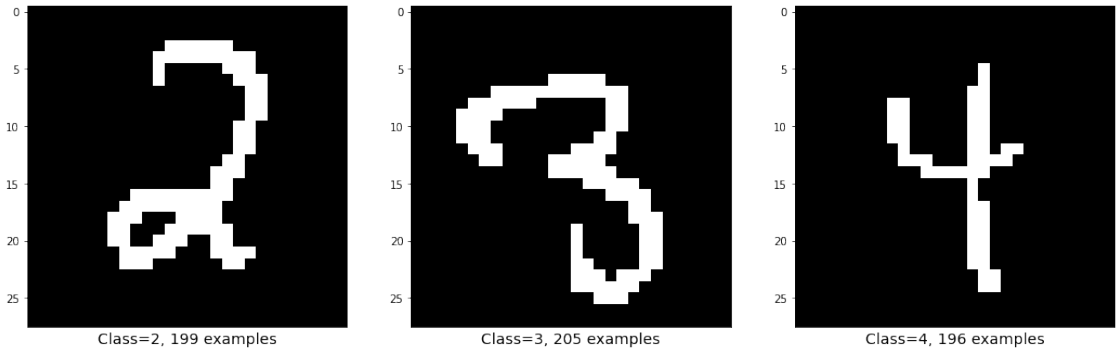
March 3, 2021

1 Multivariate Bernoulli Distribution

Run EM and K-Means algorithm on same dataset and compare the results. ## Reading MNIST dataset

[1]:
```
import numpy as np
import torch
import pytorch_lightning as pl
from sklearn.model_selection import train_test_split
import matplotlib.pyplot as plt
import scipy
import seaborn as sns
np.random.seed(42)

class Config:
    N = 600
    K = 3
    D = 784
    iterations = 50

def load_mnist():
    mnist = pd.read_csv('mnist_train.csv').values
    X_train, y_train = mnist[:, 1:], mnist[:, 0]

    # take only digits 2, 3 and 4
    take = (y_train >= 2) & (y_train <= 4)
    X_train, y_train = X_train[take], y_train[take]

    # making arrays have only zeros or ones
    X_train = ((X_train/255.0) > .5).astype(int)

    # make sure you have a total of 600 examples
    train_size = Config.N / y_train.shape[0]
    X_train, _, y_train, _ = train_test_split(X_train, y_train,␣
↪train_size=train_size, stratify=y_train, random_state=42)

    # show one example from each class
    fig, axs = plt.subplots(1, 3, figsize=(18, 6))
```

```
    for i, c in enumerate([2, 3, 4]):
        img = X_train[y_train == c][0].reshape((28, 28))
        t = (y_train==c).sum()
        axs[i].imshow(img, cmap='CMRmap')
        axs[i].set(xticks = [])
        axs[i].set_xlabel(f"Class={c}, {t} examples", fontsize=14)
    plt.show()
    return X_train, y_train

X_train, y_train = load_mnist()
```



1.1 EM algorithm for Multivariate Bernoulli Mixture

One iteration of EM updates π_k and μ_k as follows

$P(x|\mu_k) = \prod_{i=1}^{D} \mu_{ki}^{x_i}(1-\mu_{ki})^{(1-x_i)}$

$\gamma_{nk} = \frac{\pi_k P(x_n|\mu_k)}{\sum_{j=1}^{K} \pi_j P(x_n|\mu_j)}$

$N_k = \sum_{n=1}^{N} \gamma_{nk}$

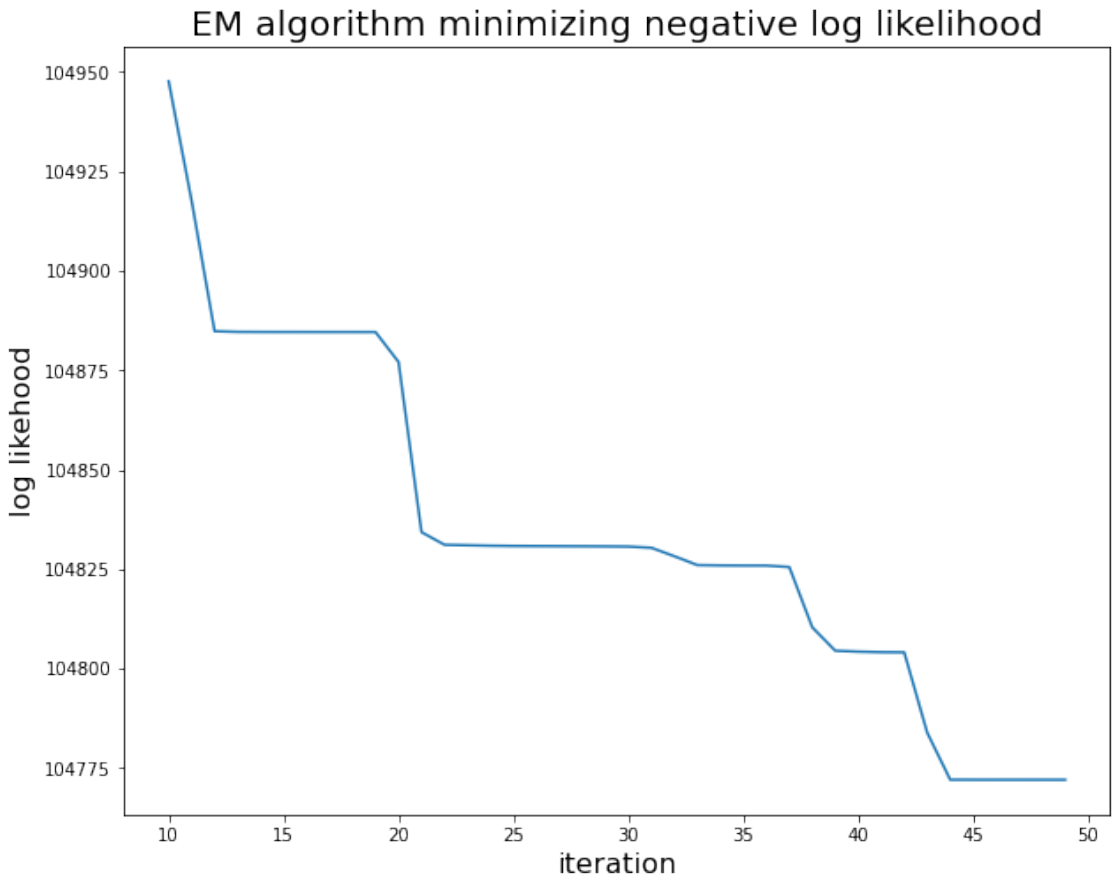
$\mu_k = \frac{\sum_{n=1}^{N} \gamma_{nk} x_n}{N_k}$

[2]:
```
class BernoulliMixtureModel:
    def __init__(self):
        self.pi = np.ones((Config.K,1)) / Config.K  # pi_k
        self.mu = np.random.uniform(low=.25, high=.75, size=(Config.K, Config.
↪D)) # mu_kj
        self.mu /= self.mu.sum(axis=1, keepdims=True) # sum mu_kj over j should␣
↪be 1

    # helpers
    def prob(self, X_nd, k): # examples, k -> [P(example | k)]
        mu_1d = self.mu[k:k+1]
        p_nd = mu_1d**X_nd * (1- mu_1d)**(1-X_nd) #(1, d)*(n, d)
```

```
        p_n = np.prod(p_nd, axis=1) # product over dimensions, axis=1
        return p_n

    def getGamma(self, X_nd):
        g_kn = np.array([
            self.pi[k, 0] * self.prob(X_nd, k)
            for k in range(Config.K)
        ]) # (k n)

        denom = g_kn.sum(axis=0, keepdims=True)
        denom[denom == 0.0] = 1
        g_kn /= denom # sum over clusters (k) must be one

        return g_kn, denom

    def oneIteration(self, X_nd):
        """Performs one iteration of the EM algorithm for bernoulli mix model
        Args
            * X_nd is an np.array with n examples, each of d dimensions
        Function
            * updates Params.mu and Params.pi
        """
        g_kn, denom = self.getGamma(X_nd)

        # N_k is gamma summed over n
        N_k = g_kn.sum(axis=1, keepdims=True) # shape (k,)

        # new mu is examples averaged over gamma
        mu_kd = np.matmul(g_kn, X_nd)
        self.mu = mu_kd / N_k
        self.pi = N_k / Config.N

        # return log likelihood
        return - np.log(denom).sum()

    def fit(self, X_train):
        return [
            self.oneIteration(X_train)
            for i in range(Config.iterations)
        ]
```

[3]:
```
model = BernoulliMixtureModel()
neg_log_likelihoods = model.fit(X_train)
fig, ax = plt.subplots(figsize=(10, 8))
ax.plot(range(10, Config.iterations), neg_log_likelihoods[10:])
ax.set_ylabel('log likehood', fontsize=16)
ax.set_xlabel('iteration', fontsize=16)
```

```
ax.set_title('EM algorithm minimizing negative log likelihood', fontsize=20);
```

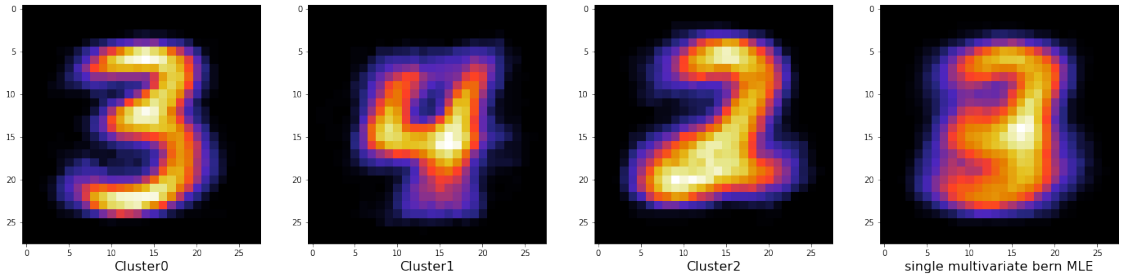


1.2 Most likely estimate for Single Multivariate Bernoulli

$Z = [z_i] \in R^d$

$Z_{mle} = \frac{\sum_{n=1}^{N} X_i}{N}$

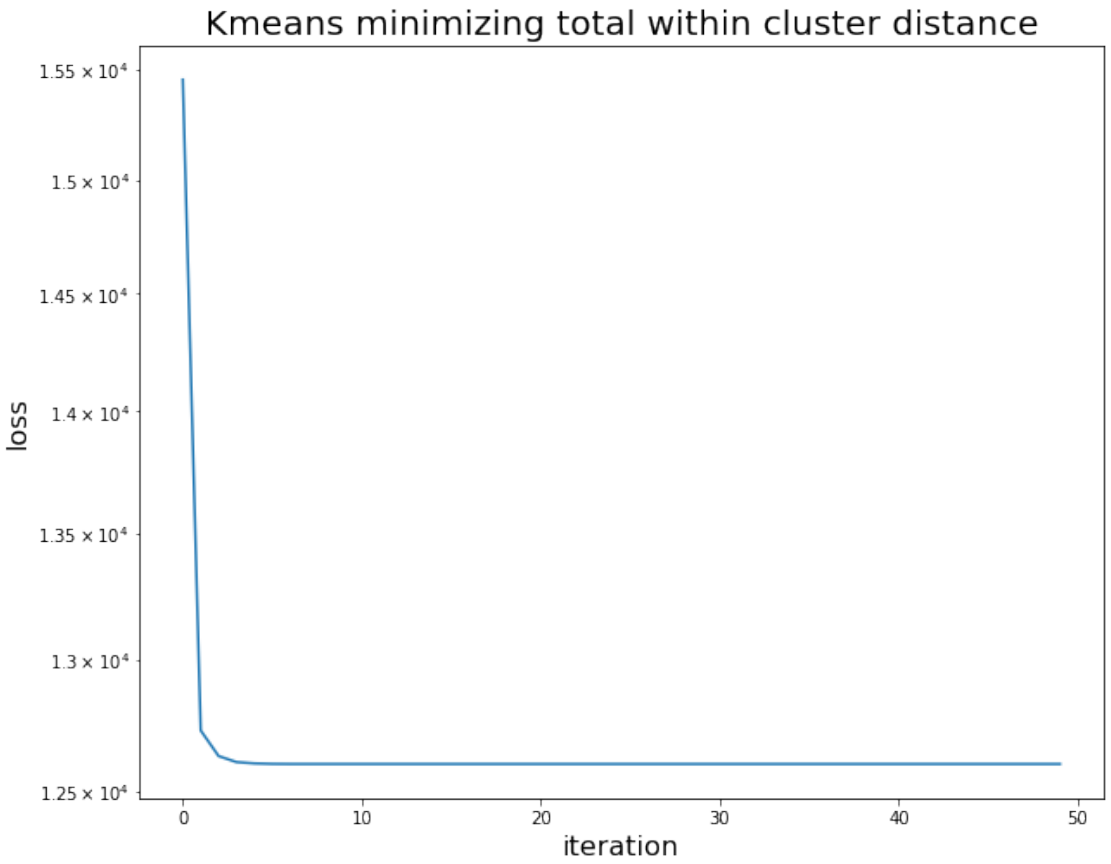
[4]:
```
# display EM algo mus
images = (model.mu * 255).reshape((Config.K, 28, 28)).astype(np.uint8)
fig, axs = plt.subplots(1, 4, figsize=(24, 6))
for k in range(Config.K):
    axs[k].imshow(images[k], cmap='CMRmap')
    axs[k].set_xlabel(f'Cluster{k}', fontsize=16)

img = (X_train.mean(axis=0) * 255).reshape((28, 28)).astype(np.uint8)
axs[Config.K].imshow(img, cmap='CMRmap')
axs[Config.K].set_xlabel('single multivariate bern MLE', fontsize=16)
plt.show()
```

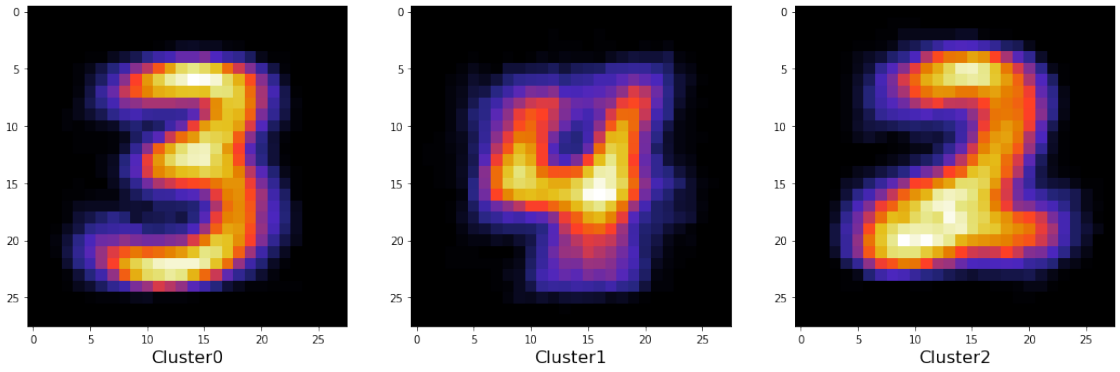


1.3 K-means clustering for MNIST

[5]:
```python
class Kmeans:
    # standardize data
    def standardize(self, X_train):
        sigma = X_train.std(axis=0, keepdims=True)
        sigma[sigma == 0] = 1
        X_stan = X_train  / sigma
        return X_stan, sigma

    def fit(self, X_train):
        # init means as K random points from data
        X_stan, sigma = self.standardize(X_train)
        idx  = np.random.choice(np.arange(Config.N), Config.K)
        self.mu = X_stan[idx] * 1.0

        # init assign
        z_n  = np.random.choice(np.arange(Config.K), Config.N)

        losses = []
        for i in range(Config.iterations):
            # calc pairwise dists (ND - K1D -> KN shapes for ref)
            d_kn = np.linalg.norm(X_stan - self.mu[:,None,:], axis=-1)
            # assign cluster based on proximity
            z_n  = d_kn.argmin(axis=0) # N
            losses.append(d_kn.min(axis=0).sum())

            # update cluster means
            z_kn = np.eye(3)[:, z_n] # 3 N
            z_kn/= z_kn.sum(axis=1, keepdims=True) # normalize
            self.mu = z_kn @ X_stan

        self.mu = self.mu * sigma
        return losses

    def predict(self, X):
```

```
        d_kn = np.linalg.norm(X - self.mu[:,None,:], axis=-1)
        # assign cluster based on proximity
        z_n  = d_kn.argmin(axis=0) # N
        return z_n

    def plot(self, losses, display_means=True):
        fig, ax = plt.subplots(figsize=(10, 8))
        ax.plot(range(Config.iterations), losses)
        ax.set_ylabel('loss', fontsize=16)
        ax.set_xlabel('iteration', fontsize=16)
        ax.set_title('Kmeans minimizing total within cluster distance',␣
↪fontsize=20)
        plt.yscale('log')
        plt.show()

        # display the k means
        if display_means:
            images = (model.mu * 255).reshape((Config.K, 28, 28)).astype(np.
↪uint8)
            fig, axs = plt.subplots(1, 3, figsize=(18, 6))
            for k in range(Config.K):
                axs[k].imshow(images[k], cmap='CMRmap')
                axs[k].set_xlabel(f'Cluster{k}', fontsize=16)
            plt.show()

model = Kmeans()
losses = model.fit(X_train)
model.plot(losses)
```



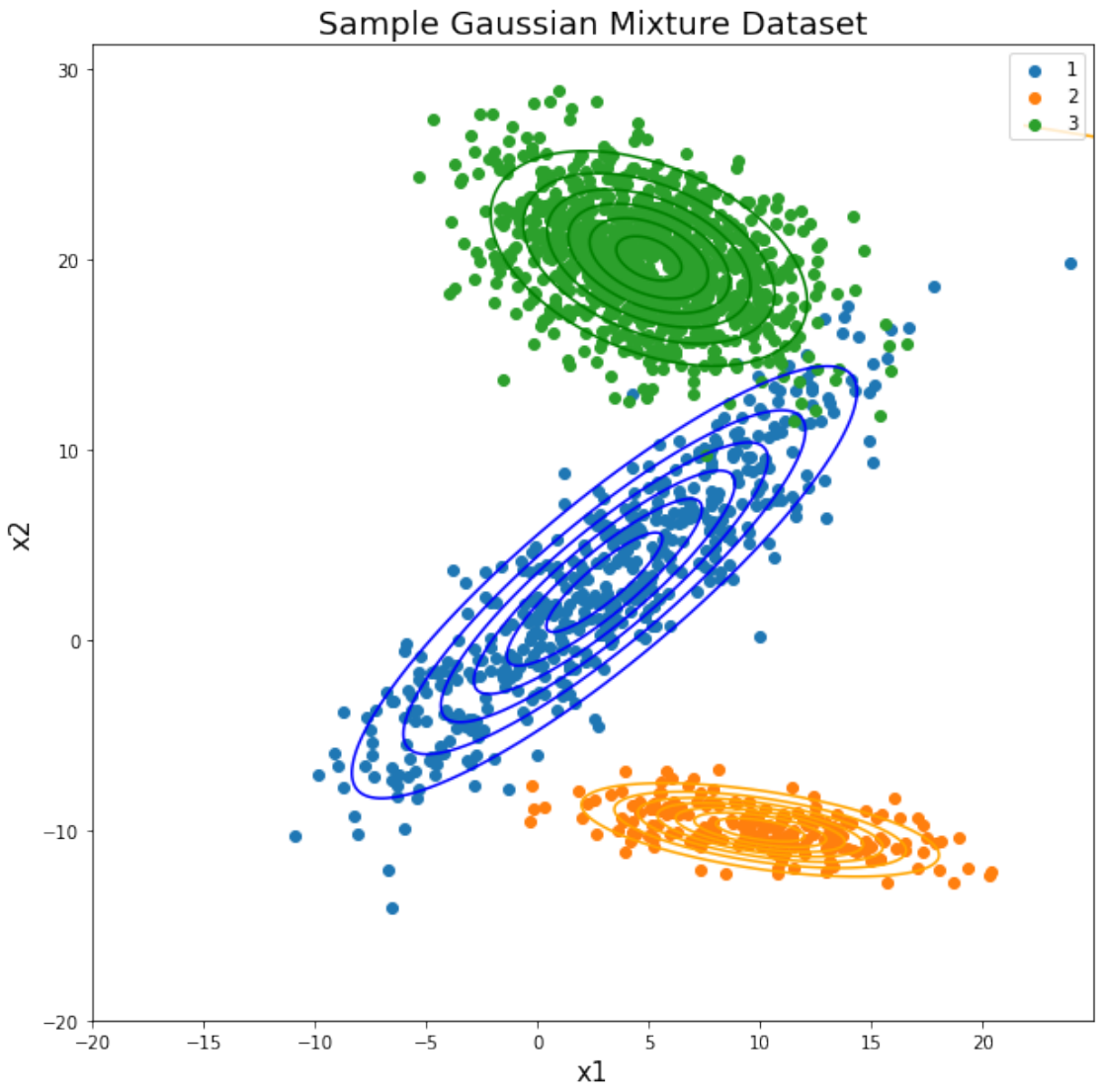


2 GMM on custom data

Sample data from univariate and multivariate Gaussian Mixture Models and verify whether EM algorithm is able to retrieve the mixture densities in each of the case.

[6]:
```
# Config has hardcoded values and are not meant to be changed at all for this␣
 ↪notebook
class Config:
    N = 1700
    K = 3 # theser
    D = 2 # dont change this we need 2d for interpretting plots
    iterations = 75
    eps = 1e-3

class Actual:
    mu = np.array([[3,3], [10, -10], [5, 20]])
    cov_norot = [
        [[4, 0], [0, 64]],
        [[16, 0], [0, 1]],
        [[16, 0], [0, 6]]
    ]
    cov = []
    rots = [45, 10, 30]
    sizes = [500, 200, 1000]

def rotate2D(theta):
    c, s = np.cos(theta), np.sin(theta)
    return np.array([[c, -s],[s, c]])

def sampleMultiGaussian(mu, cov, theta, size):
    R = rotate2D(theta)
    Actual.cov.append(R.T @ cov @ R)
    return np.random.multivariate_normal(mu, Actual.cov[-1], size=size)

def sampleDistriubtion():
    fig, ax = plt.subplots(figsize=(10, 10))
    X_train = []
    y_train = []

    x_grid = np.arange(-20, 25, .02)
    y_grid = np.arange(-20, 27, .02)
    x_grid, y_grid = np.meshgrid(x_grid, y_grid)
    coordinates = np.array([x_grid.ravel(), y_grid.ravel()]).T
    colors = ['blue', 'orange', 'green']

    for i in range(Config.K):
        samples = sampleMultiGaussian(Actual.mu[i], Actual.cov_norot[i],
                    theta=np.radians(Actual.rots[i]), size=Actual.sizes[i])
        labels  = np.ones(Actual.sizes[i]) * i
        X_train.append(samples)
        y_train.append(labels)
        ax.scatter(samples[:,0], samples[:, 1], label=str(i+1))
```

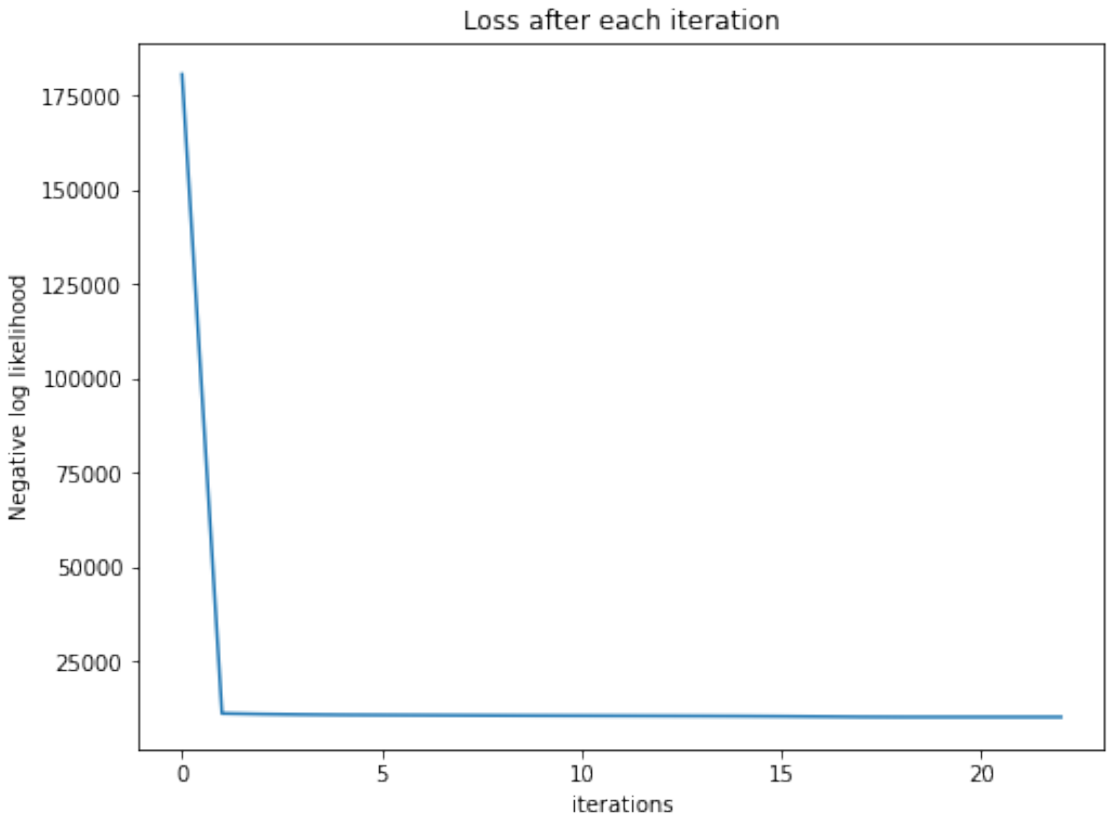
```
        z_grid = scipy.stats.multivariate_normal(Actual.mu[i], Actual.cov[i]).
↪pdf(coordinates).reshape(x_grid.shape)
        ax.contour(x_grid, y_grid, z_grid, colors=colors[i])

    ax.set_xlabel('x1', fontsize=15)
    ax.set_ylabel('x2', fontsize=15)

    ax.set_title('Sample Gaussian Mixture Dataset', fontsize=18)
    ax.legend()

    return np.concatenate(X_train, axis=0), np.concatenate(y_train, axis=0)

X_train, y_train = sampleDistriubtion()
```



[7]:
```
class GaussianMixtureModel:
    def fit(self, X_train):
        self.pi = np.ones((Config.K,1)) / Config.K
↪   # pi_k
        idx = np.random.choice(np.arange(Config.N), Config.K)
        self.mu = X_train[idx] * 1.0 # choose K samples randomly from dist
        self.cov = np.array([np.eye(Config.D) for i in range(Config.K)]) # init
↪K covs to identity

        nll = []
        for i in range(Config.iterations):
            loss = self.em_one_iteration(X_train)
            nll.append(loss)
```

```
            if len(nll) > 1 and np.abs(loss - nll[-2]) < Config.eps:
                break
        return nll

    # helpers
    def prob(self, X_nd, k): # (examples, k) -> [P(example1 | k),␣
↪P(example2|k), ..]
        mu_1d = self.mu[k:k+1]
        cov_dd = self.cov[k]
        z_nd = X_nd - mu_1d
        A_nd = (z_nd @ np.linalg.inv(cov_dd))
        z_n  = (A_nd * z_nd).sum(axis=1)
        p_n  = (2*np.pi) ** (- Config.D / 2) * np.linalg.det(cov_dd) ** -.5 *␣
↪np.exp(-.5 * z_n)
        return p_n

    def getGamma(self, X_nd):
        g_kn = np.array([
            self.pi[k, 0] * self.prob(X_nd, k)
            for k in range(Config.K)
        ]) # (k n)
        denom = g_kn.sum(axis=0, keepdims=True)
        denom[denom == 0.0] = 1
        g_kn /= denom # sum over clusters (k) must be one

        return g_kn, denom

    def em_one_iteration(self, X_nd):
        """Performs one iteration of the EM algorithm for gaussian mixture model
        Args
            * X_nd is an np.array with n examples, each of d dimensions
        Function
            * updates Params.mu, Params.cov and Params.pi
        """
        g_kn, denom = self.getGamma(X_nd)

        # N_k is gamma summed over n
        N_k1 = g_kn.sum(axis=1, keepdims=True) # shape (k,1)

        # new mu is examples averaged over gamma
        self.mu = np.matmul(g_kn, X_nd) / N_k1
        self.pi = N_k1 / Config.N

        # new mean is used to calculate new cov
        # calculate cov as E[x-mu] but over dist of reponse coeff
        for k in range(Config.K):
            g_n1 = g_kn[k:k+1].T
```

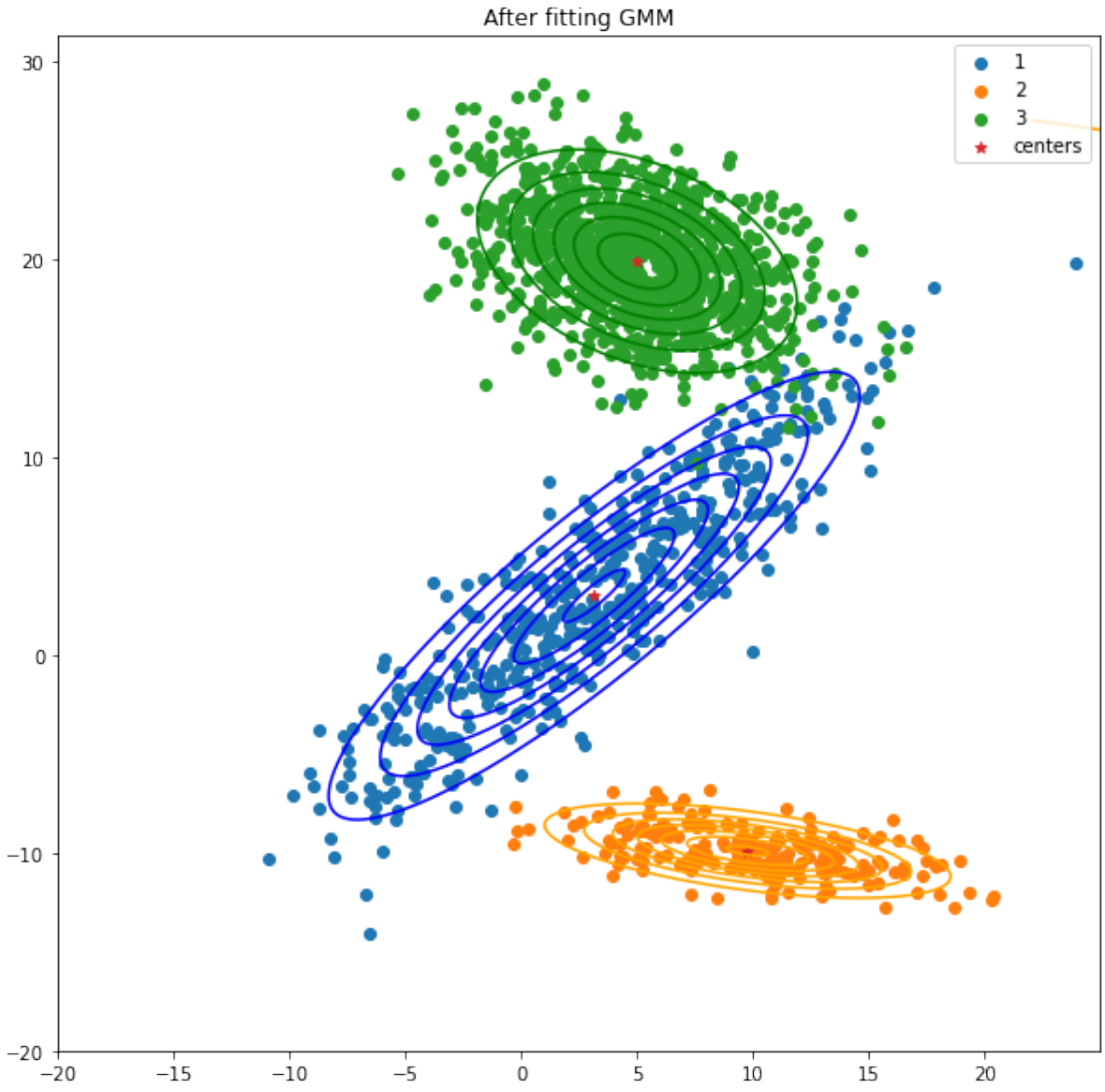
```
            z_nd = X_nd - self.mu[k:k+1]
            self.cov[k] = ((g_n1 * z_nd).T @ z_nd) / N_k1[k]

        # return log likelihood
        neg_log_likelihood = - np.log(denom).sum()
        return neg_log_likelihood

model = GaussianMixtureModel()
nll = model.fit(X_train)
fig, ax = plt.subplots(figsize=(8, 6))
plt.plot(nll)
plt.xlabel('iterations')
plt.ylabel('Negative log likelihood')
plt.title('Loss after each iteration')
plt.show()

print("Model means:")
print(model.mu)
print()
print("Actual means:")
print(Actual.mu)
```



```
Model means:
[[ 5.00015899 19.8556918 ]
 [ 9.76671785 -9.90188685]
 [ 3.14197768  2.97072897]]

Actual means:
[[  3   3]
 [ 10 -10]
 [  5  20]]
```

2.1 Plotting learned distributions

[8]:
```
fig, ax = plt.subplots(figsize=(10, 10))
parts = np.cumsum([0] + Actual.sizes)

x_grid = np.arange(-20, 25, .02)
y_grid = np.arange(-20, 27, .02)
x_grid, y_grid = np.meshgrid(x_grid, y_grid)
coordinates = np.array([x_grid.ravel(), y_grid.ravel()]).T
colors = ['green', 'orange', 'blue']

for k in range(Config.K):
    x = X_train[parts[k]:parts[k+1]]
    ax.scatter(x[:,0], x[:,1], label=str(k+1))
    z_grid = scipy.stats.multivariate_normal(model.mu[k], model.cov[k]).
 ↪pdf(coordinates).reshape(x_grid.shape)
    ax.contour(x_grid, y_grid, z_grid, colors=colors[k])

ax.scatter(model.mu[:,0], model.mu[:,1], marker='*', label='centers')
ax.set_title('After fitting GMM')
plt.legend();
```

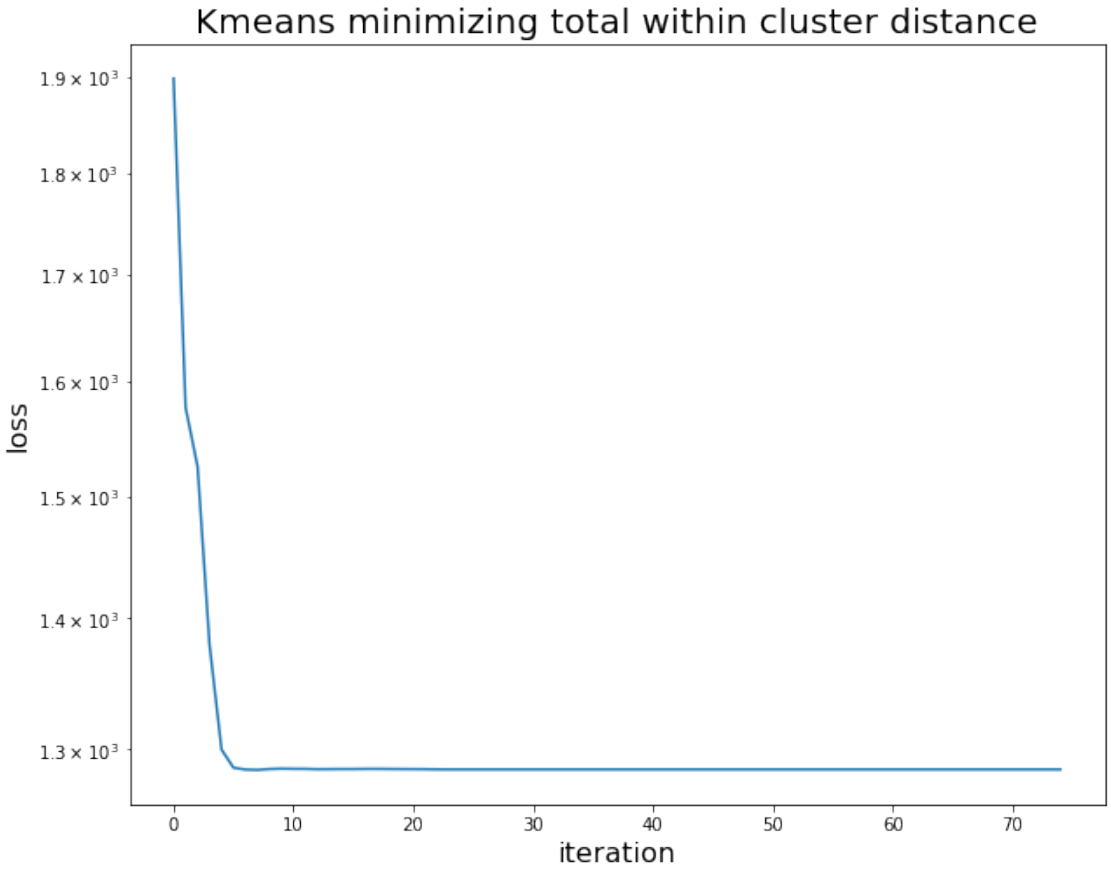


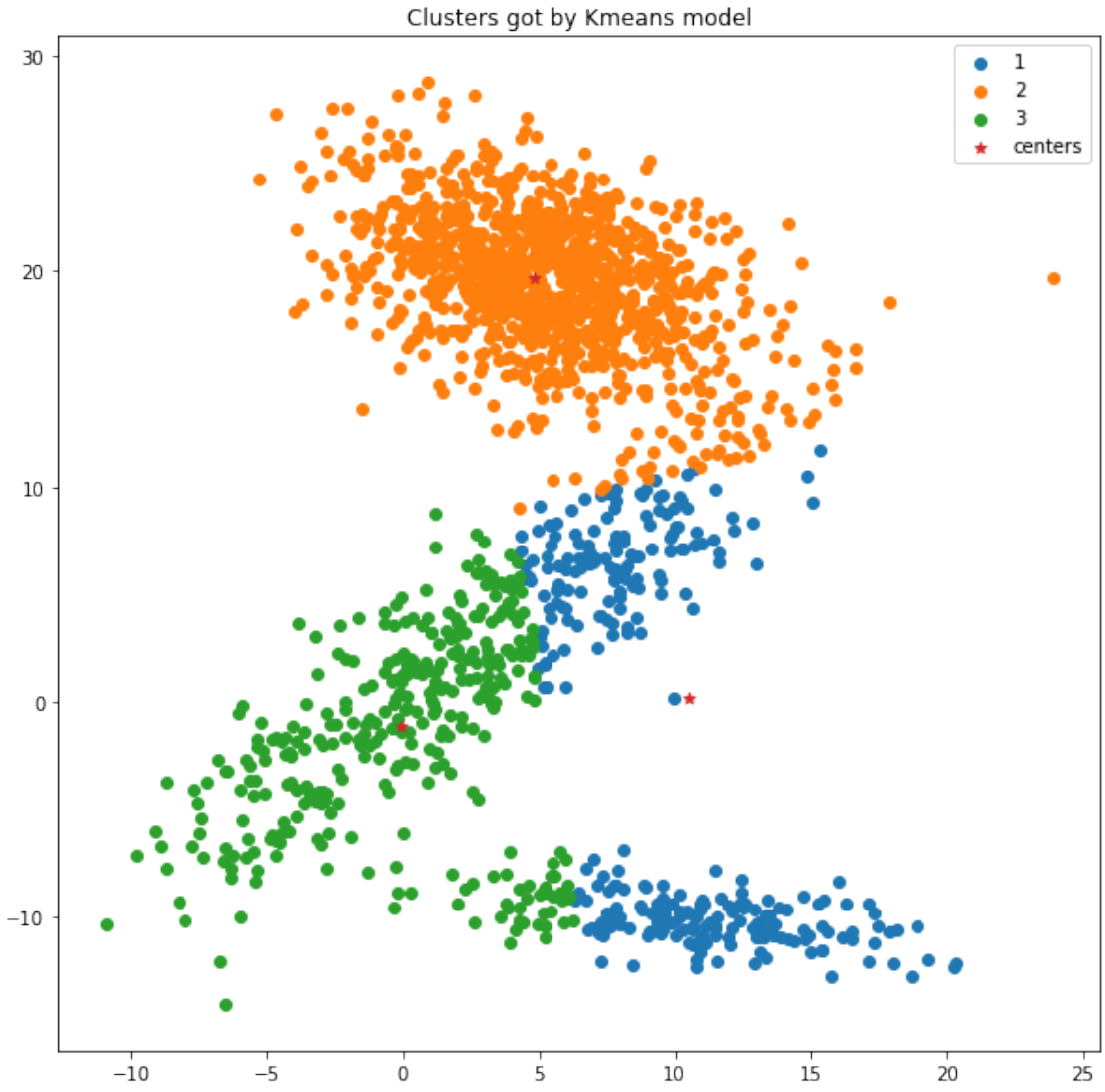
3 Parzen Window and K-means on the same custom data

Run K-Means and Parzen Window/Kernel density estimation on dataset generated in question 2 and compare the results with EM algorithm. Try different window sizes and different kernels for Kernel density estimation. Compare it with histogram based density estimation.

[9]:
```
model = Kmeans()
losses = model.fit(X_train)
model.plot(losses, display_means=False)
labels = model.predict(X_train)

# plotting clusters got by model
fig, ax = plt.subplots(figsize=(10, 10))
```

```
colors = ['blue', 'green', 'orange']

for k in range(Config.K):
    x = X_train[labels == k]
    ax.scatter(x[:,0], x[:,1], label=str(k+1))

ax.scatter(model.mu[:,0], model.mu[:,1], marker='*', label='centers')
ax.set_title('Clusters got by Kmeans model')
plt.legend();
```

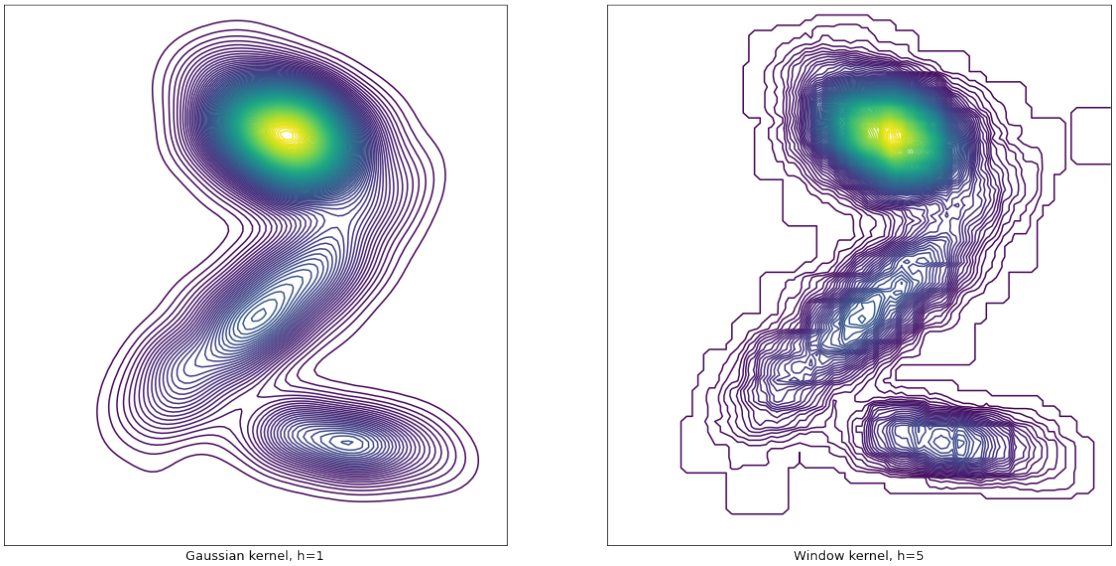




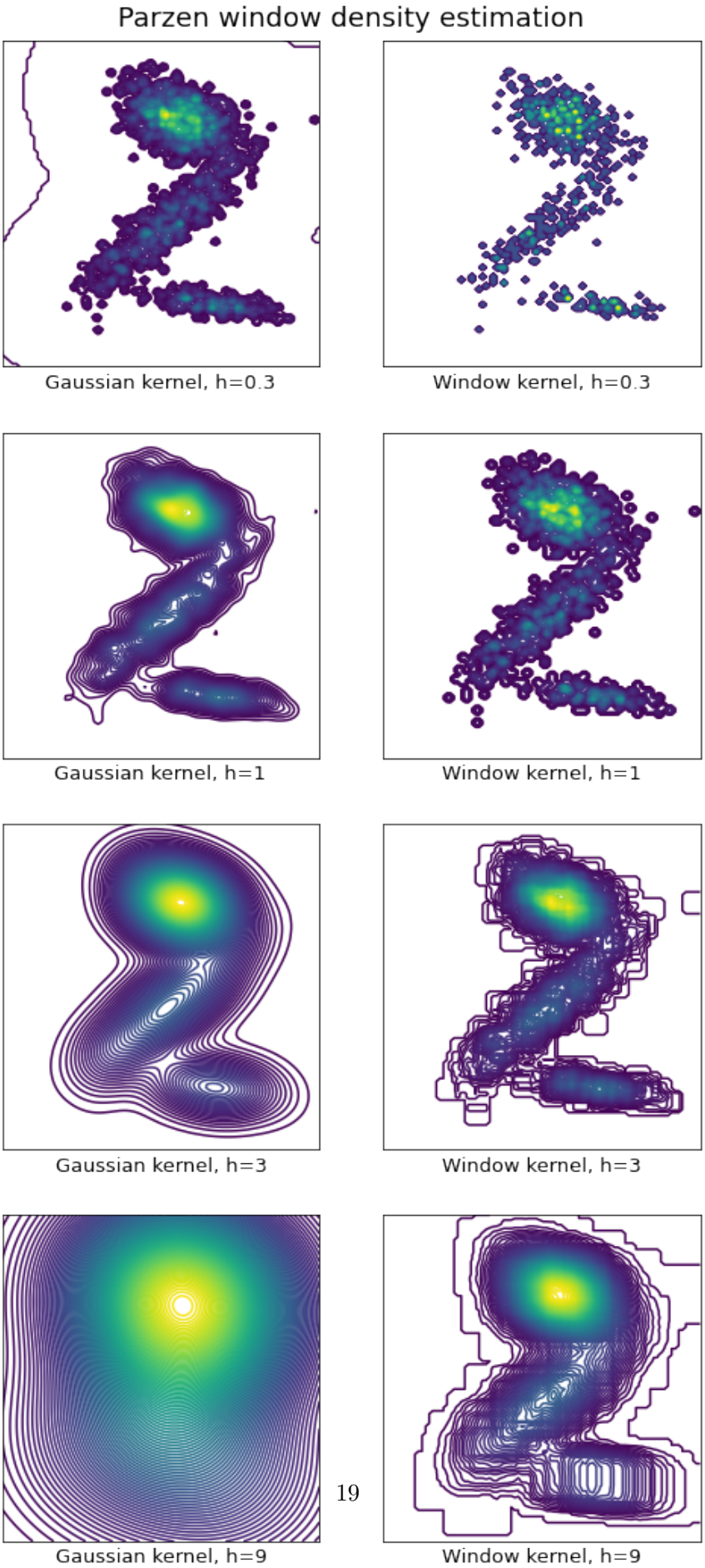
[10]:
```
def gaussianKernel(z_mtd, h): # z = (x - xi) / h of shape (m, t, d) m train ex, t for predict arg, d = Config.d
    z_mt = np.linalg.norm(z_mtd, axis=2)
    phi_mt = (2*np.pi) ** (Config.D / 2) * np.exp(- z_mt**2 / 2)
    phi_t  = phi_mt.mean(axis=0) / h ** Config.D
    return phi_t

def windowKernel(z_mtd, h):
    phi_mt = np.prod(z_mtd**2 <= .25, axis=2) # reduce d
    phi_t  = phi_mt.mean(axis=0) / h ** Config.D  # reduce m
    return phi_t

class ParzenWindow:
```

```
    def __init__(self, X_train, h, kernel='gaussian'):
        self.X = X_train
        self.h = h
        if kernel.lower() == 'gaussian':
            self.kernel = gaussianKernel
        else:
            self.kernel = windowKernel

    def predict(self, X_t):
        z_mtd = (X_t - self.X[:,None,:]) / self.h
        phi_t = self.kernel(z_mtd, self.h)
        return phi_t
```

[11]:
```
x_grid = np.arange(-20, 25, .5)
y_grid = np.arange(-20, 33, .5)
x_grid, y_grid = np.meshgrid(x_grid, y_grid)
coordinates = np.array([x_grid.ravel(), y_grid.ravel()]).T

model_gk = ParzenWindow(X_train, 2, 'gaussian')
density_g = model_gk.predict(coordinates).reshape(x_grid.shape)

model_wk = ParzenWindow(X_train, 5, 'window')
density_w = model_wk.predict(coordinates).reshape(x_grid.shape)

fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(20, 10))
fig.suptitle('Parzen window density estimation', fontsize=18)

ax1.contour(x_grid, y_grid, density_g, levels=np.linspace(density_g.min(),␣
 ↪density_g.max(), num=100))
ax1.set_xlabel('Gaussian kernel, h=1', fontsize=13)
ax1.set(xticks=[], yticks=[])
ax2.contour(x_grid, y_grid, density_w, levels=np.linspace(density_w.min(),␣
 ↪density_w.max(), num=100))
ax2.set_xlabel('Window kernel, h=5', fontsize=13)
ax2.set(xticks=[], yticks=[]);
```



[12]:
```
h_list = [0.3, 1, 3, 9]

fig, axs = plt.subplots(len(h_list), 2, figsize=(8, 16),␣
 ↪gridspec_kw=dict(left=0.1, right=0.9, bottom=0.1, top=0.96))
plt.subplots_adjust(top=0.8)
fig.suptitle('Parzen window density estimation', fontsize=18)

for i, h in enumerate(h_list):
    model_gk = ParzenWindow(X_train, h, 'gaussian')
    density_g = model_gk.predict(coordinates).reshape(x_grid.shape)

    model_wk = ParzenWindow(X_train, h, 'window')
    density_w = model_wk.predict(coordinates).reshape(x_grid.shape)

    ax1, ax2 = axs[i,0], axs[i,1]
    ax1.contour(x_grid, y_grid, density_g, levels=np.linspace(density_g.min(),␣
 ↪density_g.max(), num=100))
    ax1.set_xlabel(f'Gaussian kernel, h={h}', fontsize=13)
    ax1.set(xticks=[], yticks=[])
    ax2.contour(x_grid, y_grid, density_w, levels=np.linspace(density_w.min(),␣
 ↪density_w.max(), num=100))
    ax2.set_xlabel(f'Window kernel, h={h}', fontsize=13)
    ax2.set(xticks=[], yticks=[]);
plt.show()
```


Parzen window density estimation
Gaussian kernel, h=0.3
Window kernel, h=0.3
Gaussian kernel, h=1
Window kernel, h=1
Gaussian kernel, h=3
Window kernel, h=3
Gaussian kernel, h=9
Window kernel, h=9

- As the window size increases, the density function becomes smoother and smoother.
- Small window sizes => density function changes over small changes in position
- Large window sizes => even if we add points in a small region, a larger surrounding will have its density function value changed because of the large window size. The contour plot is a graphical representation of the same.

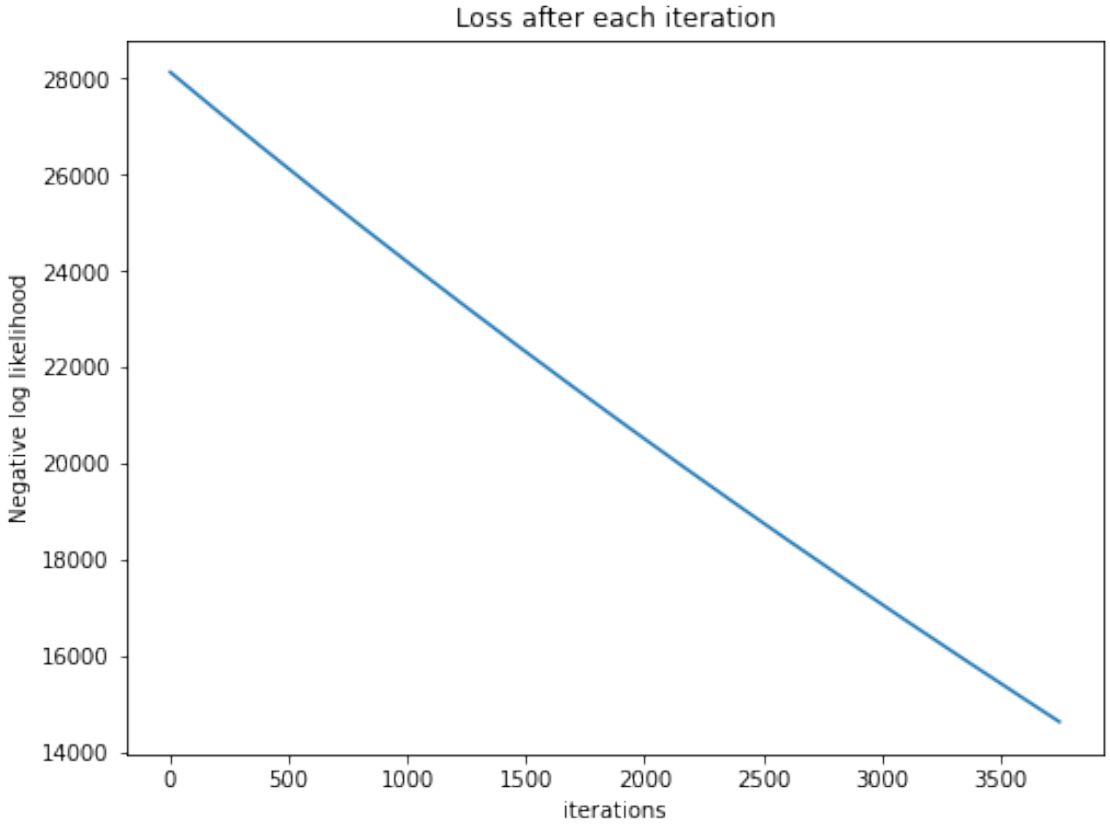
3.1 GD GMM

- $\pi_k \in [0,1]$ and $\sum \pi_k = 1$ then we declare a unconstrained variable ξ_k such that $\pi = softmax(\xi)$
- The covariance matrices must be positive semidefinite. so we declare unconstrained matrices M_k such that $\sum_k^{-1} = M_k M_k^T$

[13]:
```
np.random.seed(42)
class GaussianMixtureModelGrad:
    def fit(self, X_train):
        self.si = np.zeros((Config.K,1))                                    # pi_k
        # pi is softmax(si)

        idx = np.random.choice(np.arange(Config.N), Config.K)
        self.mu = X_train[idx] * 1.0 # choose K samples randomly from dist
#       #  correct mu
#         self.mu = np.array([[3,3], [10, -10], [5, 20]]).astype(np.float32)
        self.precision = np.array([np.random.randn(Config.D, Config.D) for i in␣
 ↪range(Config.K)]) # init K covs to identity
        self.eta = 2e-3/Config.N

        nll = []
        for i in range(Config.iterations*50):
            loss = self.gd_one_iteration(X_train)
            nll.append(loss)
        return nll

    # helpers
    def prob(self, X_nd, k): # (examples, k) -> [P(example1 | k),␣
 ↪P(example2|k), ..]
        mu_1d = self.mu[k:k+1]
#         inv_cov_dd = self.precision[k] @ self.precision[k].T
        inv_cov_dd = np.linalg.inv(Actual.cov[k])
        z_nd = X_nd - mu_1d
        A_nd = z_nd @ inv_cov_dd
        z_n  = (A_nd * z_nd).sum(axis=1)
        p_n  = (2*np.pi) ** (- Config.D / 2) * np.linalg.det(inv_cov_dd) ** .5␣
 ↪* np.exp(-.5 * z_n)
        return p_n
```

```
    def getGamma(self, X_nd):
        g_kn = np.array([
            np.exp(self.si[k, 0]) * self.prob(X_nd, k)
            for k in range(Config.K)
        ]) # (k n)
        denom = g_kn.sum(axis=0, keepdims=True)
        denom[denom == 0.0] = 1
        g_kn /= denom # sum over clusters (k) must be one

        return g_kn, denom

    def gd_one_iteration(self, X_nd):
        """Performs one iteration of the gradient descent for gaussian mixture␣
↪model
        Args
            * X_nd is an np.array with n examples, each of d dimensions
        Function
            * updates Params.mu, Params.cov and Params.pi
        """
        g_kn, denom = self.getGamma(X_nd)

        # N_k is gamma summed over n
        N_k1 = g_kn.sum(axis=1, keepdims=True) # shape (k,1)

        # new mu is examples averaged over gamma
        for k in range(Config.K):
            z_nd = X_nd - self.mu[k:k+1]
            g_n1 = g_kn[k:k+1].T
#             inv_cov = self.precision[k] @ self.precision[k].T
            inv_cov = np.linalg.inv(Actual.cov[k])
            delta_mu = g_n1.T @ (inv_cov @ z_nd.T).T

            # delta_cov = .5 * (inv_cov @ ((g_n1 * z_nd).T @ z_nd) @ inv_cov -␣
↪N_k1[k] * inv_cov)
#             delta_precision = np.linalg.inv(self.precision[k].T) * N_k1[k] -␣
↪self.precision[k] @ ((g_n1 * z_nd).T @ z_nd)

#             self.precision[k] += self.eta * delta_precision
            self.mu[k:k+1]  += self.eta * delta_mu

        delta_si = N_k1
        self.si += self.eta * delta_si

        # return log likelihood
        neg_log_likelihood = - np.log(denom).sum()
        return neg_log_likelihood
```

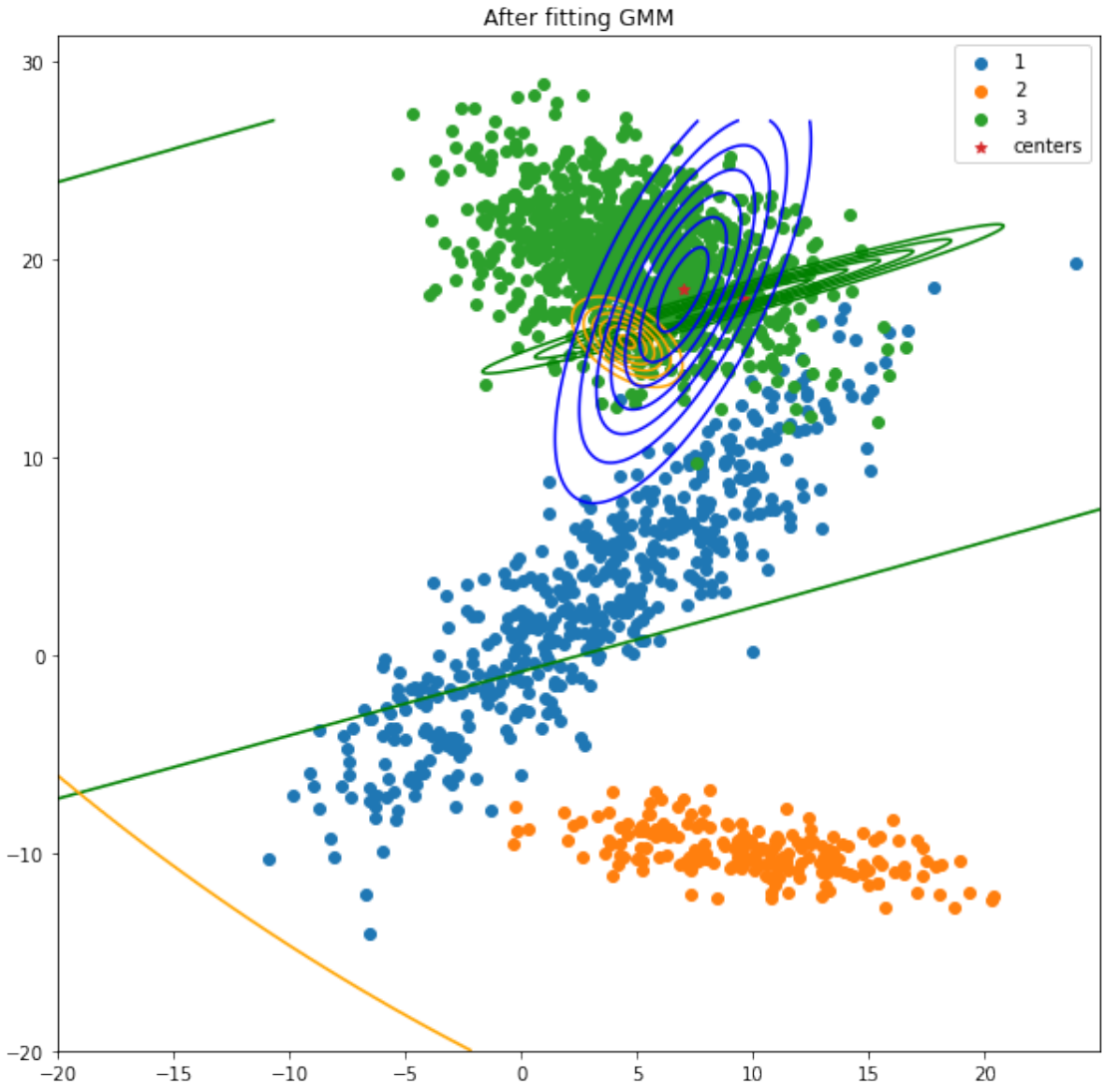
```
model = GaussianMixtureModelGrad()
nll = model.fit(X_train)
fig, ax = plt.subplots(figsize=(8, 6))
plt.plot(nll)
plt.xlabel('iterations')
plt.ylabel('Negative log likelihood')
plt.title('Loss after each iteration')
plt.show()

print("Model means:")
print(model.mu)
print()
print("Actual means:")
print(Actual.mu)

print(model.si)
```




```
Model means:
[[ 9.56550783 17.95945598]
 [ 4.56785114 15.78477678]
```

```
 [ 6.97719713 18.4234213 ]]

Actual means:
[[  3   3]
 [ 10 -10]
 [  5  20]]
[[2.89094286]
 [0.40089697]
 [4.20816017]]
```

[14]:
```
fig, ax = plt.subplots(figsize=(10, 10))
parts = np.cumsum([0] + Actual.sizes)

x_grid = np.arange(-20, 25, .02)
y_grid = np.arange(-20, 27, .02)
x_grid, y_grid = np.meshgrid(x_grid, y_grid)
coordinates = np.array([x_grid.ravel(), y_grid.ravel()]).T
colors = ['green', 'orange', 'blue']

for k in range(Config.K):
    x = X_train[parts[k]:parts[k+1]]
    ax.scatter(x[:,0], x[:,1], label=str(k+1))
    cov = np.linalg.inv(model.precision[k] @ model.precision[k].T)
    z_grid = scipy.stats.multivariate_normal(model.mu[k], cov).pdf(coordinates).
 ↪reshape(x_grid.shape)
    ax.contour(x_grid, y_grid, z_grid, colors=colors[k])

ax.scatter(model.mu[:,0], model.mu[:,1], marker='*', label='centers')
ax.set_title('After fitting GMM')
plt.legend();
```



[15]:
```
nll[-10:]
```

[15]:
```
[14645.892619211156,
 14642.69083918832,
 14639.489220479289,
 14636.287763098833,
 14633.086467061868,
 14629.885332383456,
 14626.684359078783,
 14623.483547163189,
 14620.282896652141,
 14617.082407561255]
```

[16]:
```
Actual.cov
```

[16]:
```
[array([[34., 30.],
        [30., 34.]]),
 array([[15.54769466, -2.56515107],
        [-2.56515107,  1.45230534]]),
 array([[13.5       , -4.33012702],
        [-4.33012702,  8.5       ]])]
```