Q) ## Formulation

Decision variables:

x_1 & x_2 = number of dairy cows & laying hens, respectively.

x_3, x_4 and x_5 = acreage of paddy crop, bajra crop & jowar crop, respectively

x_6 and x_7 = extra man-hours utilized in Sept - May and June - Aug, respectively.

The LP model:

Maximize (net annual cash income)

$$Z = 3,500 x_1 + 200 x_2 + 1,200 x_3 + 800 x_4 + 850 x_5 + 2 x_6 + 3 x_7$$

Subject to:

$$100 x_1 + 0.6 x_2 + 40 x_3 + 20 x_4 + 25 x_5 + x_6 \leq 3,500$$

(Man-hours for Sept - May)

$$50 x_1 + 0.4 x_2 + 50 x_3 + 35 x_4 + 40 x_5 + x_7 \leq 4,000$$

(Man-hours for June - Aug)

$$1.5 x_1 + x_3 + x_4 + x_5 \leq 100$$

(Land availability)

$x_1 \leq 32$ (accommodation restriction for dairy cows)

$x_2 \leq 4,000$ (accommodation restriction for laying hens)

$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$ (non-negativity constraints).

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Q) **Sensitivity Analysis**

(a) Let the unit profit of product A be C_1.

$$\therefore \bar{C}_2 = 3 - (5 \;\; C_1)\begin{pmatrix} 1/5 \\ 7/5 \end{pmatrix} = 3 - \left(1 + \frac{7C_1}{5}\right)$$

$$\bar{C}_4 = 0 - (5 \;\; C_1)\begin{pmatrix} 2/5 \\ -1/5 \end{pmatrix} = -\left(2 - \frac{C_1}{5}\right)$$

$$\bar{C}_5 = 0 - (5 \;\; C_1)\begin{pmatrix} -1/5 \\ 3/5 \end{pmatrix} = -\left(-1 + \frac{3C_1}{5}\right)$$

In order that the current solution remains optimal,

$$3 - \left(1 + \frac{7C_1}{5}\right) \leq 0 \Rightarrow 2 - \frac{7C_1}{5} \leq 0 \Rightarrow C_1 \geq \frac{10}{7}$$

$$-\left(2 - \frac{C_1}{5}\right) \leq 0 \Rightarrow -2 + \frac{C_1}{5} \leq 0 \Rightarrow C_1 \leq 10$$

$$-\left(-1 + \frac{3C_1}{5}\right) \leq 0 \Rightarrow 1 - \frac{3C_1}{5} \leq 0 \Rightarrow C_1 \geq 5/3$$

$$\therefore \boxed{5/3 \leq C_1 \leq 10}$$

When $C_1 = 12$

C_B	Basis \ C_j	12	3	5	0	0	Const	M.R
		X_1	X_2	X_3	X_4	X_5		
5	X_3	0	1/5	1	2/5	-1/5	6/5	3 ←
12	X_1	1	7/5	0	-1/5	3/5	27/5	∞
$\bar{C}$ Row		0	-74/5	0	2/5 ↑	-31/5	Z-354/5	
0	X_4	0	1/2	5/2	1	-1/2	3	
12	X_1	1	3/2	1/2	0	1/2	6	
$\bar{C}$ Row		0	-15	-1	0	-6	Z = 72	

$\therefore$ optimal value of obj. $\boxed{f^n = 72}$

(b) Let b_1 denote the man-hour availability.

For the current basis to remain optimal,

$$B^{-1}b \geqslant 0 \quad \Rightarrow \begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{bmatrix} \begin{bmatrix} b_1 \\ 12 \end{bmatrix} \geqslant 0$$

$$\Rightarrow \begin{bmatrix} 2b_1/5 - 12/5 \\ -b_1/5 + 36/5 \end{bmatrix} \geqslant 0$$

$$\Rightarrow \quad 2b_1/5 \geqslant 12/5 \Rightarrow b_1 \geqslant 6$$

$$\& \quad -b_1/5 \geqslant -36/5 \Rightarrow b_1 \leqslant 36$$

Thus, the current basis remains optimal as long as $\boxed{6 \leqslant C_1 \leqslant 36}$

If $b_1 = 40$, RHS $= B^{-1}b = \begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{bmatrix} \begin{bmatrix} 40 \\ 12 \end{bmatrix} = \begin{bmatrix} 68/5 \\ -4/5 \end{bmatrix}$

Thus, the final table becomes primal infeasible.

C_B	Basis \ C_j	4	3	5	0	0	Const
		X_1	X_2	X_3	X_4	X_5	
5	X_3	0	1/5	1	2/5	−1/5	68/5
4	X_1	1	7/5	0	−1/5	3/5	−4/5 ←
$\bar{C}$ Row		0	−18/5	0	−6/5 ↑	−7/5	$Z = 324/5$
5	X_3	2	3	1	0	1	12
0	X_4	−5	−7	0	1	−3	4
$\bar{C}$ Row		−6	−12	0	0	−5	$Z = 60$

Thus, the optimal solution is $X_3 = 12$, $X_4 = 4$ and $Z = 60$

(c) Shadow prices of the resources are the simplex multipliers of the final table.

$$\therefore \quad \pi = C_B B^{-1}$$

$$= (5 \quad 4)\begin{pmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{pmatrix} = \boxed{(6/5,\ 7/5)}$$

$\therefore$ The shadow price of 'man-hour' is $6/5$ and that of 'raw materials' is $7/5$

(d) Let X_6 be the number of new product produced.

$$\therefore \quad \bar{C}_6 = C_6 - \pi P_6 = 6 - (6/5 \quad 7/5)\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$= 6 - 27/5 = 3/5$$

Thus, the change is beneficial.

$$\bar{P}_6 = B^{-1} P_6 = \begin{pmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1/5 \\ 8/5 \end{pmatrix}$$

C_B	Basis \ C_j	4	3	5	0	0	6	Const	M.R
		X_1	X_2	X_3	X_4	X_5	X_6		
5	X_3	0	$1/5$	1	$2/5$	$-1/5$	$-1/5$	$6/5$	∞
4	X_1	1	$7/5$	0	$-1/5$	$3/5$	$8/5$	$27/5$	$27/8 \leftarrow$
$\bar{C}$ Row		0	$-18/5$	0	$-6/5$	$-7/5$	$3/5$ $\uparrow$	$Z = 138/5$	
5	X_3	$1/8$	$3/8$	1	$3/8$	$-1/8$	0	$15/8$	
6	X_6	$5/8$	$7/8$	0	$-1/8$	$3/8$	1	$27/8$	
$\bar{C}$ Row		$-3/8$	$-33/8$	0	$-9/8$	$-13/8$	0	$Z = 237/8$	

The new product-mix is

$$\boxed{X_3 = 15/8,\ X_6 = 27/8 \text{ and } Z = 237/8}$$

(e) If the supervision constraint $x_1 + x_2 + x_3 \leq 6$ is added to the original problem, the current solution is no longer optimal, as $27/5 + 0 + 6/5 = 33/5 > 6$

$\therefore$ We introduce the constraint $x_1 + x_2 + x_3 + x_6 = 6$ in the final table where x_6 is a slack variable.

Not in canonical. So multiply 1st & 2nd row by -1 & add to 3rd row.

C_B	C_j / Basis	4	3	5	0	0	0	Const
		x_1	x_2	x_3	x_4	x_5	x_6	
5	x_3	0	$1/5$	1	$2/5$	$-1/5$	0	$6/5$
4	x_1	1	$7/5$	0	$-1/5$	$3/5$	0	$27/5$
0	x_6	1	1	1	0	0	1	6
$\bar{C}$ Row		0	$-18/5$	0	$-6/5$	$-7/5$	0	$Z = 138/5$
5	x_3	0	$1/5$	1	$2/5$	$-1/5$	0	$6/5$
4	x_1	1	$7/5$	0	$-1/5$	$3/5$	0	$27/5$
0	x_6	0	$-3/5$	0	$-1/5$	$-2/5$	1	$-3/5$
$\bar{C}$ Row		0	$-18/5$	0	$-6/5$	$-7/5$	0	$Z = 138/5$
5	x_3	0	$1/2$	1	$1/2$	0	$-1/2$	$3/2$
4	x_1	1	$1/2$	0	$-1/2$	0	$3/2$	$9/2$
0	x_5	0	$3/2$	0	$1/2$	1	$-5/2$	$3/2$
$\bar{C}$ Row		0	$-3/2$	0	$-1/2$	0	$-7/2$	$Z = 51/2$

Primal infeasible. So solve by dual-simplex method.

$$\boxed{\text{Thus, the new optimal solution is } x_1 = 9/2,\ x_3 = 3/2,\ x_5 = 3/2 \ \& \ Z = 51/2}$$

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Q) **Revised Simplex Method**

Converting the problem to standard form:

$$\text{Max } Z = 2X_1 + X_2$$

$$\text{S.T.} \quad 3X_1 + 4X_2 + X_3 = 6$$

$$6X_1 + X_2 + X_4 = 3$$

$$X_1, X_2, X_3, X_4 \geq 0 \quad (X_3 \,\&\, X_4 \rightarrow \text{slack variables})$$

$$P_1 = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \quad P_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad P_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad P_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad b = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

Basis	B^{-1}		Const	Var to enter	Pivot Col	M.R
X_3	1	0	6	X_1	3	$6/3=2$
X_4	0	1	3		6 ←	$3/6=1/2$
X_3	1	$-1/2$	$9/2$	X_2	$7/2$ ←	$9/7$
X_1	0	$1/6$	$1/2$		$1/6$	3
X_2	$2/7$	$-1/7$	$9/7$			
X_1	$-1/21$	$4/21$	$2/7$			

$$\pi = C_B B^{-1} = (0\ 0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (0\ 0)$$

$$\bar{C}_1 = 2 - (0\ 0)\begin{pmatrix} 3 \\ 6 \end{pmatrix} = 2 \leftarrow$$

$$\bar{C}_2 = 1 - (0\ 0)\begin{pmatrix} 4 \\ 1 \end{pmatrix} = 1$$

$$\bar{P}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$\pi = (0\ 2)\begin{pmatrix} 1 & -1/2 \\ 0 & 1/6 \end{pmatrix} = (0\ 1/3)$$

$$\bar{C}_2 = 1 - (0\ 1/3)\begin{pmatrix} 4 \\ 1 \end{pmatrix} = 2/3$$

$$\bar{C}_4 = 0 - (0\ 1/3)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1/3$$

$$\bar{P}_2 = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/6 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 7/2 \\ 1/6 \end{pmatrix}$$

$$\pi = (1\ 2)\begin{pmatrix} 2/7 & -1/7 \\ -1/21 & 4/21 \end{pmatrix} = \left(\frac{4}{21}\ \frac{5}{21}\right)$$

$$\bar{C}_3 = 0 - \left(\frac{4}{21}\ \frac{5}{21}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -4/21$$

$$\bar{C}_4 = 0 - \left(\frac{4}{21}\ \frac{5}{21}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -5/21$$

Thus, the optimal solution is

$$X_1 = 2/7, \quad X_2 = 9/7$$

$$Z_{max} = 2 \times \frac{2}{7} + 1 \times \frac{9}{7} = \frac{13}{7}$$

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Q) **Transportation Problem:** (Balanced transportation problem as $120 + 80 + 80 = 150 + 80 + 50 = 280$)

IBFS using N-W corner rule:

	D_1	D_2	D_3	
S_1	120	0	0	~~120~~ 0
S_2	30	50	0	~~80~~ ~~50~~ 0
S_3	0	30	50	~~80~~ ~~50~~ 0
	~~150~~ ~~30~~ 0	~~80~~ ~~30~~ 0	~~50~~ 0	

$$\text{Cost} = 120 \times 8 + 30 \times 15 + 50 \times 10 + 30 \times 9 + 50 \times 10 = 2680$$

Optimal solution using u-v method:

	$v_1 = 8$	$v_2 = 3$	$v_3 = 4$	
$u_1 = 0$	(120) $\boxed{8}$	$\boxed{2}$ $\boxed{5}$	$\boxed{2}$ $\boxed{6}$	120
$u_2 = 7$	(30) $-\theta$ $\boxed{15}$	(50) $+\theta$ $\boxed{10}$	$\boxed{1}$ $\boxed{12}$	80
$u_3 = 6$	θ $\boxed{-11}$ $\boxed{3}$	(30) $-\theta$ $\boxed{9}$	(50) $\boxed{10}$	80
	150	80	50	

$\theta = 30$

$$\text{Cost} = 120 \times 8 + 80 \times 10 + 30 \times 3 + 0 \times 9 + 50 \times 10 = 2350$$

A case of degeneracy has occurred with one basic variable as 0.

	$v_1 = 8$	$v_2 - 14$	$v_3 = 15$
$u_1 = 0$	(120) $-\theta$ $\boxed{8}$	θ $\boxed{-9}$ $\boxed{5}$	$\boxed{-9}$ $\boxed{6}$
$u_2 = -4$	$\boxed{11}$ $\boxed{15}$	(80) $\boxed{10}$	$\boxed{1}$ $\boxed{12}$
$u_3 = -5$	(30) $+\theta$ $\boxed{3}$	(0) $-\theta$ $\boxed{9}$	(50) $\boxed{10}$

If this cell is chosen, solution will be obtained in the next step.

$\theta = 0$

$$\text{Cost} = 120 \times 8 + 0 \times 5 + 80 \times 10 + 30 \times 3 + 50 \times 10 = 2350$$

	$v_1 = 8$	$v_2 = 5$	$v_3 = 15$
$u_1 = 0$	120 $-\theta$ $\boxed{8}$	0 $+\theta$ $\boxed{5}$	$\boxed{9}$ $\boxed{6}$
$u_2 = 5$	$\boxed{2}$ $\boxed{15}$	80 $-\theta$ $\boxed{10}$	$\boxed{-8}$ θ $\boxed{12}$
$u_3 = -5$	30 $+\theta$ $\boxed{3}$	$\boxed{9}$ $\boxed{9}$	50 $-\theta$ $\boxed{10}$

$\theta = 50$

Cost $= 70 \times 8 + 50 \times 5$
$+ 30 \times 10 + 50 \times 12$
$+ \cancel{30}\ 80 \times 3$
$= 1950$

	$v_1 = 8$	$v_2 = 5$	$v_3 = 7$
$u_1 = 0$	70 $\boxed{8}$	50 $-\theta$ $\boxed{5}$	$\boxed{-1}$ θ $\boxed{6}$
$u_2 = 5$	$\boxed{2}$ $\boxed{15}$	30 $+\theta$ $\boxed{10}$	50 $-\theta$ $\boxed{12}$
$u_3 = -5$	80 $\boxed{3}$	$\boxed{9}$ $\boxed{9}$	$\boxed{8}$ $\boxed{10}$

$\theta = 50$

Cost $= 70 \times 8 + 0 \times 5$
$+ 50 \times 6 + 80 \times 10$
$+ 80 \times 3$
$= 1900$

	$v_1 = 8$	$v_2 = 5$	$v_3 = 6$
$u_1 = 0$	70 $\boxed{8}$	0 $\boxed{5}$	50 $\boxed{6}$
$u_2 = 5$	$\boxed{2}$ $\boxed{15}$	80 $\boxed{10}$	$\boxed{1}$ $\boxed{12}$
$u_3 = -5$	80 $\boxed{3}$	$\boxed{9}$ $\boxed{9}$	$\boxed{9}$ $\boxed{10}$

Since all C_{ij} Δ are positive, optimal solⁿ has been obtained.

Total transportation cost = $\boxed{\text{Rs } 1900}$

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