Proofs
A Long-Form Mathematics Textbook
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Chapter 1

Intuitive Proofs

Principle 1.1 (The Pigeonhole Principle). *If $kn+1$ objects are placed into n boxes, then at least one box has at least $k+1$ objects.*

Proposition 1.1. *Given any 101 integers frrom $1, 2, 3, ..., 200$, at least one of these numbers will divide another.*

Scratch Work. Since there are 101 items, we can consider the pigeon hole principle with $k = 1$ and $n = 100$.

Let us consider the following boxes. Create a box for each of the odd numbers $1, 3, 5, ..., 199$ and for any number x if x is of the form $x = 2^k \cdot m$, where m is odd and $k >= 0$, we can put x in the box m.

There are 100 odd numbers in the set so we have 100 boxes. And any two numbers in a box only differ by 2^k for some k. Thus, for any two numbers in one box, the smaller number divides the larger one.

For any odd number larger than 101, it will be the only number in that box.

Proof. For each number n from the set $1, 2, 3, ..., 200$, write it in the form of $n = 2^k \cdot m$ where $k >= 0$ and m is an odd number.

Now, create a box for each odd number from 1 to 199. There will be 100 such boxes. For each of the given 101 integers,

If $n = 2^k \cdot m$ then put n in the box numbered m.

Since 101 integers are placed in 100 boxes, there must be at least one box with more than 1 integer by 1.1.

Suppose the box m contains two numbers of the form $n_1 = 2^k \cdot m$ and $n_2 = 2^l \cdot m$ where without loss of generality $k > l$. Then we can show that

$$\frac{n_1}{n_2} = \frac{2^k \cdot m}{2^l \cdot m} = 2^{k-l}$$

Here, 2^{k-l} is an integer since $k > l$, thus, n_2 divides n_1.

Thus, proved. $\square$

Proposition 1.2. *Suppose G is a graph with $n \geq 2$ vertices. Then, G contains two vertices which have the same degree.*

Proof Idea. The possible degrees of a vertex is any number between 0 and $n-1$. Thus, there are n boxes for each possible value for the degree of a vertex and n vertices.

We can show that at least one box must be empty. Therefore, we need to put n vertices in $n-1$ boxes and by The Pigeonhole Principe (1.1), there must be at least two vertices in the same box, i.e., have the same degree.

We can show that both box 0 and box $n-1$ cannot have a vertex because if vertex v_1 is in box $n-1$ then it has an edge connecting it to every other vertex.

Thus, every other vertex has an edge connecting it to v_1 which implies that every other vertex has at least a degree of 1 and box 0 must be empty.

If there is no vertex in box $n-1$ then we have box $n-1$ that is empty.

Thus, at least one box is empty in both scenarios.

Proof. Let G be a graph with $n \geq 2$ vertices. Create boxes numbered from 0 to $n-1$.

Now, for each vertex, let us say it's degree is d, then put that vertex in box d. Let us take box 0 and $n-1$. Both of these boxes are either empty or have some vertex in them.

<u>Case 1</u>. Box $n-1$ is empty.
Since box $n-1$ is empty, we have n vertices being placed into $n-1$ boxes. Therefore, by The Pigeonhole Principle (1.1), there are at least one box with at least two vertices.

Thus, there are at least two vertices with the same degrees.

<u>Case 2</u>. Box $n-1$ is not empty.
The vertex in box $n-1$ must have a degree of $n-1$ which implies it has an edge connecting to $n-1$ vertices.
Therefore, all n vertices have at least one edge connecting them to another edge and all n vertices have a degree of at least 1.

This implies that box 0 must be empty since all vertices have a degree of at least 1.

Since box 0 is empty, there are n vertices placed into $n-1$ boxes.

Therefore, by The Pigeonhole Principle (1.1), there are at least two vertices in the same box and have the same degree.

Thus, proved.

□

Proposition 1.3. *If you draw five points on the surface of an orange in marker, then there is always a way to cut the orange in half so that four points (or some part of each of those points) all lie on one of the halves.*

Scratch Work. There are two subtle statements in the propsition. First it asserts that "always a way to cut the orange in half so that...". If doesn't assert that *any* such cut has this property.

Second, it is important that we say "or some part of each of those points". When you use a marker to make the points, the points are big enough that when you slice through any point, part of the point appears on *both* halves.

Classical Geometry Theorem. Given any two points on the sphere, there is a great circle that passes through those two points.

Proof. Take 2 out of 5 given points. By Classical Geometry Theorem, there is a great circle passing through these points. Thus, this great circle divides that sphere in two halves.

The remaining three points are placed among these two halves. Thus, by The Pigeonhole Principle (1.1), there are at least two points on one of the havles.

Adding the two initially chosen points to both halves, we have one half with atleast four points.

Hence, proved. □

Exercises

Problem 1.1. *Read* The Secret of Raising Smart Kids *by Carol Dweck and write a few paragraphs about what you learned and how it may help you be successful in proof-based math class.*

Solution. Not Interested.

Problem 1.2. *Explain the error in the following "proof" that* $2 = 1$.
Let $x = y$. *Then,*

$$x^2 = xy \tag{1.1}$$
$$x^2 - y^2 = xy - y^2 \tag{1.2}$$
$$(x + y)(x - y) = y(x - y) \tag{1.3}$$
$$x + y = y \tag{1.4}$$
$$2y = y \tag{1.5}$$
$$2 = 1 \tag{1.6}$$

Solution. Since $x = y$, $x - y = 0$ and therefore, we cannot divided by $x - y$ in step 3 to get $x + y = y$ from $(x + y)(x - y) = y(x - y)$. Thus, solved.

Problem 1.3. *Suppose that* m *and* n *are positive odd integers. Using* 2×1 *dominos,*

(a) Does there exist a perfect cover of the $m \times n$ *chessboard?*

(b) If I remove 1 square from the $m \times n$ *chessboard, will it have a perfect cover?*

Solution (a)**.** In this case, there are $m \times n$ cells on the board which is an odd number. Since each domino covers only 2 cells, the total number of cells covered will always be even.

Hence, no perfect cover exists.

Scratch Work (b)**.** Let us take 3×3 chessboard. There are 9 cells on the board. Without loss of generality, let us say there are 4 white cells and 5 black cells.

Since a domino always covers 1 white and 1 black cell, the number of white and black cell must be equal for a perfect cover.

Let us remove a black cell from the above chessboard. Now there are 4 white cells and 4 black cells.

Checking all 5 black squares for removal, we find that we have a cover in every case.

Solution (b)**.** Let us assume that the board has x white cells and $x + 1$ black cells. Note: If it is not the case, we can always swap the colors and have the same setup.

Since each domino must cover exactly 1 white and 1 black cell, we must remove a black cell to have a perfect cover.

In this scenario, all the corners will have black cells since there are more black cells than white.

Now, the question is, whether we can remove any black cell.

Lemma 1.1. *For every chessboard of size $m \times n$, there exists a cover if either m or n is even.*

Proof. Let us assume that m is even. We can always turn the board if n is even.

For every column, we have an even number of cells in that column as m is even. Hence, we can cover that column with dominos.

Hence, proved. □

Let us say we removed a black cell from row r. Now, there are two cases:

Case 1. r is odd.

In this case, we can divide the remaining chessboard into $(r - 1) \times n$ and $(m - r) \times n$ and cover them by Lemma 1.1.

Note: In case $r = 1$ or $r = m$, we only have one remaining part. The second part is empty and thus, requires no cover.

Since the corners are black, the left most cell of every odd row must be black because the colors are alternating. That is, all the cells in first column and rows $1, 3, 5, ..., m$ must be black.

Since r is odd, the left most cell in it must be black. Thus, the columns containing black cells in row r are odd, i.e., cells in columns $1, 3, 5, ..., n$ and row r are black.

Thus, if we remove any black cell from row r we will have divided the row into two even sized pieces, which can be covered by the dominos by Lemma 1.1.

Case 2. r is even.

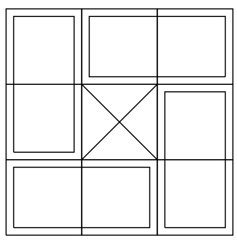
In this case, we can take rows $r-1, r, r+1$ and divide the remaining chess board in $(r-2) \times n$ and $(m-r-1) \times n$ and cover them by Lemma 1.1.

Since r is even, all the cells in row r and columns $2, 4, 8, ..., n-1$ are black.

Let us say we remove the cell in column c. Now, we can take column $c-1, c$ and $c+1$, and divide the rest of cells into chess boards of sizes $(c-2) \times 3$ and $(n-c-1) \times 3$. Since c is even, therefore, $c-2$ and $n-c-1$ are even as well.

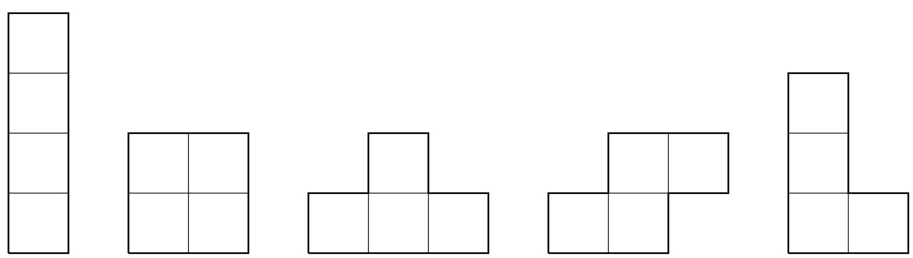
Thus, we can cover these boards using Lemma 1.1.

Now, for the remainig 3×3 board without its center, we can cover it like this:



Hence, proved.

Problem 1.4. *The game **Tetris** is played with five different shapes – the five shapes that can be obtained by piecing together four squares.*



For the questions below, we also allow these pieces to be "flipped over".

(a) Is it possible to perfectly cover a 4×5 chessboard using each of these shapes exactly once? Prove that it is impossible, or show by example that it is possible.

(b) Is it possible to perfectly cover an 8×5 chessboard using each of these shapes exactly twice? Prove that it is impossible, or show by example that it is possible.

Scratch Work. Let's color the chessboard in black and white. Here, we can see that all the shapes will cover 2 black cells and 2 white cells except the third shape.

The third shape will cover either 3 black and 1 white cell or 1 black and 3 white cells.

Therefore, if we use each shape exactly once, we will get either get a total of 11 black and 9 white cells or 9 black and 11 white cells.

Solution (a)**.** Let's assume that it is possible to cover a 4×5 chessboard using these shapes exactly once.

The chess board has exactly 10 black and 10 white cells in it. Each shape will take up exactly 2 white and 2 black cells except the third shape.

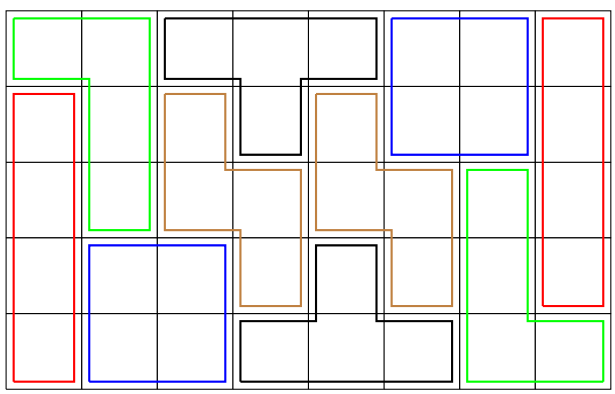
The third shape will either take up 3 black and 1 white cell or 3 white or 1 black cell. This is because all adjacent cells must be different color so if the center of the third shape is white, all the rest 3 cells of that shape must be black and vice versa.

Let's place each shape one by one.
After placing the first shape, we will have 8 black and 8 white cells.
After placing the second shape, we will have 6 black and 6 white cells.
After placing the fourth shape, we will have 4 black and 4 white cells.
After placing the fifth shape, we will have 2 black and 2 white cells.

Now, we don't have enough white or black cells to place the third shape. This is a contradiction. Therefore, it is impossible to cover a 4×5 chessboard using each of these shapes exactly once.

Hence, proved.

Solution (b)**.** Giving an example:



Hence, proved.

Problem 1.5. *If I remove two squares of different colors from an 8×8 chessboard, must the result have a perfect square?*

Solution. TODO

Problem 1.6. *If I remove four squares – two white, two black – from an 8×8 chessboard, must the result have a perfect cover?*

→ *If you believe a perfect cover exists, justify why.*

→ *If you belive a perfect cover does not need to exist, give an example of four squares that you could remove for which the result does not have a perfect cover.*

Solution. TODO

Problem 1.7. *In chess, a **knight** is a piece that can move two squares vertically and one square horizontally, or two squares horizontall and one square vertically.*

*A **knight** can legally move to any square provided there is not another piece on that same square.*

(a) Suppose there is a knight on every square of a 7×7 chessboard. Is it possible for every one of those knights to simultaneously make a legal move?

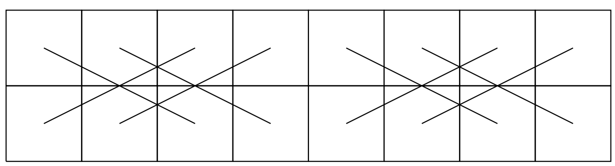
(b) Suppose there is a knight on every square of a 8×8 chessboard. Is it possible for every one of those knights to simultaneously make a legal move?

Solution (a)**.** Let us color the chessboard such that there are 24 white squares and 25 black squares without loss of generality.

In one move, a **knight** on a white square moves to a black square and vice-verse.

Since there are more black squares than white squares, we cannot move all the knight simultaneuosly such that all of them occupy different squares after the move by Principle 1.1.

Solution (b)**.** In the first two rows of the 8×8 chessboard, there are 8 white squares and 8 black squares. We can pair them up like so:



This pattern can be repeat by every two rows of the board. And the knights in these places can swap positions.

Hence, proved.

Problem 1.8. *Prove that if one chooses $n+1$ numbers from $\{1, 2, 3, ..., 2n\}$, it is guaranteed that two of the numbers that they choose are consecutive.*

Also, before the proof, write 2 example for $n = 3$, $n = 4$ and $n = 5$.

Scratch Work. For $n = 3$, we can choose 4 numbers from $\{1, 2, 3, 4, 5, 6\}$. Let them be $1, 3, 5, 6$. Here, 5 and 6 are consecutive.

For $n = 4$, we can choose 5 numbers from $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Let them be $1, 3, 5, 7, 8$. Here, 7 and 8 are consecutive.

For $n = 5$, we can choose 6 numbers from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let them be $1, 3, 5, 7, 9, 10$. Here, 9 and 10 are consecutive.

Solution. Let us define the n boxes numbered 1 to n. For each selected x, if $x = 2k - 1$ or $x = 2k$, put it in the box k.

Thus, box k will only contain two numbers: $2 \cdot k - 1$ and $2 \cdot k$. Both these numbers are consecutive.

Since there are $n + 1$ selected numbers atleast two numbers must be in the same box by Principle 1.1 which implies that they are consecutive.

Hence, proved.

Problem 1.9. *Assume that n is a positive integer. Prove that if one selects any $n+1$ numbers from the set $\{1, 2, 3, ..., 2n\}$, then two of the selected numbers will sum to $2n+1$.*

Solution. Let us define n boxes numbered 1 to n such that box i contains the numbers i and $2n+1-i$.

Thus, we will get boxes with numbers $(1, 2n), (2, 2n-1), ...(n, n+1)$. Note that the numbers in a box add up to $2n+1$.

Now, since there are $n+1$ selected numbers atleast two numbers must be in the same box by Principle 1.1 which implies that they add up two $2n+1$.

Problem 1.10. *Explain in your own words what the general pigeonhole princple says.*

Solution. If there are n objects that are placed into m boxes then there is atleast one box with atleast $\lfloor \frac{n}{m} \rfloor$ items in it.

Problem 1.11. *Prove that there are atleast two U.S. residents that have the same weight when rounded to the nearest* ***millionth*** *of a pound.*

Solution. A quick google search tells us that there are only 3.2 million people over 300 pounds in the U.S. and the population of the U.S. is 340 million people.

Thus, there are more than 330 million people that weigh between 0 and 300 pounds.

Let us create box for each weight with a millionth of a pound of precision. This will give us 300 million boxes each denoting a weight with a precision of a millionth of a pound.

Since there are more than 300 million people in the U.S. who weigh between 0 and 300 pounds, by Principle 1.1, we can conclude that there are atleast two people with the exact same weight when rounded to a millionth of a pound.

Problem 1.12. *Determine whetehr or not the pigeonhole principle guarantees that two students at your school have the exact three leter initials.*

Solution. My school had 1000 students in each year so a total of 4000 students.

There are $26 \cdot 26 \cdot 26 = 17576$ unique three letter initials.

Therefore, the pigeonhole principle doesn't guarantee that two students at my school have the same three letter initial.

Problem 1.13. *Find your own real-world example of the pigeonhole principle.*

Solution. There are $10,000$ engineers at my workplace. But there are only 366 days in the year.

Therefore, atleast $\lfloor 10000/366 \rfloor = 27$ employees have the exact same joining anniversary.

Definition. Two integers m and n are said to be *relatively prime* if there is no integer larger than 1 which divides both m and n.

This definition will be used in the following exercise.

Problem 1.14. *Prove that if one chooses* 31 *numbers from the set* $\{1, 2, 3, ..., 60\}$*, that two of the numbers must be relatively prime.*

Solution. We can use the same method as last one. We can create 30 boxes where each box k will contain the numbers $2k - 1$ and $2k$. Thus, we will get boxes that contain the numbers $(1, 2), (3, 4), (5, 6)..., (59, 60)$.

Here, it is obvious that both numbers in a box are relatively prime.

Thus, putting 31 selected numbers in these boxes, we will get atleast one box which has atleast two numbers.

Therefore, there are two numbers that are relatively prime.

Hence, proved.

Problem 1.15. *Assume that* n *is a positive integer. Prove that if one chooses* $n+1$ *distince odd integers from* $\{1, 2, 3, ..., 3n\}$*, then atleast one of these numbers will divide another.*

Also, before your proof, check all possible selection of 4 *odd numbers from* $\{1, 2, 3, ..., 9\}$*, and for each selection locate two of the numbers for which one divdes the other.*

Scratch Work. The following are the selections:

- $\{1, 3, 5, 7\}$: 1 divides 3.
- $\{1, 3, 5, 9\}$: 1 divides 3.
- $\{1, 3, 7, 9\}$: 1 divides 3.
- $\{1, 5, 7, 9\}$: 1 divides 5.
- $\{3, 5, 7, 9\}$: 3 divides 9.

Since there are $\lfloor \frac{3n+1}{2} \rfloor$ odd numbers in the given setand we are choosing $n + 1$, we need to put these numbers in n boxes.

Let us take the boxes as follows: $(1,), (3, 9), (5, 15), ...$

Solution.