Pattern avoidance

Yajit Jain, Deepak Narayanan, and Leon Zhang

November 19, 2014

Permutations

A permutation of a finite set $\{1, \cdots, n\}$ is some *ordering* of the elements.

Permutations

A permutation of a finite set $\{1, \cdots, n\}$ is some *ordering* of the elements.

51243 is a permutation of $\{1, 2, 3, 4, 5\}$.

Permutations

A permutation of a finite set $\{1, \cdots, n\}$ is some *ordering* of the elements.

51243 is a permutation of $\{1, 2, 3, 4, 5\}$.

S_n is the set of permutations on $\{1, \cdots, n\}$.

Permutations

A permutation of a finite set $\{1, \cdots, n\}$ is some *ordering* of the elements.

51243 is a permutation of $\{1, 2, 3, 4, 5\}$.

S_n is the set of permutations on $\{1, \cdots, n\}$.

$$51243 \in S_5$$

Permutations

A permutation of a finite set $\{1, \cdots, n\}$ is some *ordering* of the elements.

51243 is a permutation of $\{1, 2, 3, 4, 5\}$.

S_n is the set of permutations on $\{1, \cdots, n\}$.

$$51243 \in S_5$$

$\#S_n = n!$

Includes and Avoids

5 1 2 4 3

Includes and Avoids

5 1 2 4 3

includes $\left\{ \begin{array}{l} 123 \\ \\ \\ \end{array} \right.$

Includes and Avoids

$$5\ 1\ 2\ 4\ 3 \quad \text{includes} \quad \left\{ \begin{array}{l} 123 \\ 132 \\ \\ \\ \end{array} \right.$$

Includes and Avoids

$$\underset{\uparrow}{5}\ 1\ \underset{\uparrow}{2}\ \underset{\uparrow}{4}\ 3 \quad \text{includes} \quad \left\{\begin{array}{l} 123 \\ 132 \\ 312 \\ \\ \\ \end{array}\right.$$

Includes and Avoids

$$5\ 1\ 2\ 4\ 3 \quad \text{includes} \quad \left\{ \begin{array}{l} 123 \\ 132 \\ 312 \\ 321 \end{array} \right.$$

$\uparrow \qquad\qquad\qquad \uparrow \quad \uparrow$

Includes and Avoids

$$\text{includes} \left\{ \begin{array}{l} 123 \\ 132 \\ 312 \\ 321 \\ 4123 \end{array} \right.$$

$$\begin{array}{ccccc} 5 & 1 & 2 & 4 & 3 \\ \uparrow & \uparrow & \uparrow & \uparrow & \end{array}$$

Includes and Avoids

$$5\ 1\ 2\ 4\ 3 \quad \text{includes} \quad \left\{ \begin{array}{l} 123 \\ 132 \\ 312 \\ 321 \\ 4123 \end{array} \right.$$

Includes and Avoids

$$\text{includes} \begin{cases} 123 \\ 132 \\ 312 \\ 321 \\ 4123 \end{cases}$$

5 1 2 4 3

Includes and Avoids

5 1 2 4 3

includes $\left\{ \begin{array}{l} 123 \\ 132 \\ 312 \\ 321 \\ 4123 \end{array} \right.$

avoids $\left\{ \begin{array}{l} 213 \\ \\ \end{array} \right.$

Includes and Avoids

5 1 2 4 3

includes $\begin{cases} 123 \\ 132 \\ 312 \\ 321 \\ 4123 \end{cases}$

---

avoids $\begin{cases} 213 \\ 231 \end{cases}$

Includes and Avoids

5 1 2 4 3

includes $\begin{cases} 123 \\ 132 \\ 312 \\ 321 \\ 4123 \end{cases}$

---

avoids $\begin{cases} 213 \\ 231 \\ 1234 \end{cases}$

Questions

Let $\pi = 312 \in S_3$.

- Question: How many permutations avoid π?

Questions

Let $\pi = 312 \in S_3$.

- Question: How many permutations avoid π?

(a lot)

Questions

Let $\pi = 312 \in S_3$.

- Question: How many permutations avoid π?
- Better Question: How many permutations in S_n avoid π?

Let's count!

- How many permutations in S_1 avoid $\pi = 312$?

Let's count!

- How many permutations in S_1 avoid $\pi = 312$? 1

Let's count!

- How many permutations in S_1 avoid $\pi = 312$? 1
- How many permutations in S_2 avoid $\pi = 312$?

Let's count!

- How many permutations in S_1 avoid $\pi = 312$? 1
- How many permutations in S_2 avoid $\pi = 312$? 2

Let's count!

- How many permutations in S_1 avoid $\pi = 312$? 1
- How many permutations in S_2 avoid $\pi = 312$? 2
- How many permutations in S_3 avoid $\pi = 312$?

Let's count!

- How many permutations in S_1 avoid $\pi = 312$? 1
- How many permutations in S_2 avoid $\pi = 312$? 2
- How many permutations in S_3 avoid $\pi = 312$? 5

Let's count!

- How many permutations in S_1 avoid $\pi = 312$? 1
- How many permutations in S_2 avoid $\pi = 312$? 2
- How many permutations in S_3 avoid $\pi = 312$? 5
- How many permutations in S_4 avoid $\pi = 312$?

Let's count!

- How many permutations in S_1 avoid $\pi = 312$? 1
- How many permutations in S_2 avoid $\pi = 312$? 2
- How many permutations in S_3 avoid $\pi = 312$? 5
- How many permutations in S_4 avoid $\pi = 312$? ??????

Permutations in S_4 that avoid $\pi = 312$?

How many permutations in S_4 avoid $\pi = 312$?

Permutations in S_4 that avoid $\pi = 312$?

How many permutations in S_4 avoid $\pi = 312$?

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Permutations in S_4 that avoid $\pi = 312$?

How many permutations in S_4 avoid $\pi = 312$?

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Permutations in S_4 that avoid $\pi = 312$?

How many permutations in S_4 avoid $\pi = 312$?

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Permutations in S_4 that avoid $\pi = 312$?

How many permutations in S_4 avoid $\pi = 312$?

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Permutations in S_4 that avoid $\pi = 312$?

How many permutations in S_4 avoid $\pi = 312$?

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Permutations in S_4 that avoid $\pi = 312$?

How many permutations in S_4 avoid $\pi = 312$?

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Permutations in S_4 that avoid $\pi = 312$?

How many permutations in S_4 avoid $\pi = 312$? 14

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Definition

Let $a_n(\pi)$ be the number of permutations in S_n that avoid π.

Definition

Let $a_n(\pi)$ be the number of permutations in S_n that avoid π.

We want to compute the sequences $(a_n(\pi))$ for some $\pi \in S_k$.

Definition

Let $a_n(\pi)$ be the number of permutations in S_n that avoid π.

We want to compute the sequences $(a_n(\pi))$ for some $\pi \in S_k$.

Example: $(a_n(312)) = 1, 2, 5, 14,$

Definition

Let $a_n(\pi)$ be the number of permutations in S_n that avoid π.

We want to compute the sequences $(a_n(\pi))$ for some $\pi \in S_k$.

Example: $(a_n(312)) = 1, 2, 5, 14,$ 42, 132, 429,...

Main Result

Theorem

For $\pi \in S_3$, $(a_n(\pi))$ is equal to the Catalan numbers:

$$(a_n(\pi)) = 1, 2, 5, 14, 42, 132, 429...$$

What if we take $\pi \in S_4$?

$$(a_n(2314)) := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

What if we take $\pi \in S_4$?

$$(a_n(2314)) := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

$$(a_n(1243)) := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots$$

What if we take $\pi \in S_4$?

$$(a_n(2314)) := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

$$(a_n(1243)) := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots$$

$$(a_n(4231)) := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots$$

What if we take $\pi \in S_4$?

$$(a_n(2314)) := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

$$(a_n(1243)) := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots$$

$$(a_n(4231)) := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots$$

????

Variations

What if we count the number of permutations that avoid π in a subset of S_n?

$$T_{2m}$$

The Catalan numbers

Definition

The Catalan numbers are the sequence of positive integers C_i defined as follows,

$$C_0 = 1,\ C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \text{ for } n \geq 0$$

Avoidance of 312

Let's first look at some examples of permutations that don't avoid 312!

Avoidance of 312

Let's first look at some examples of permutations that don't avoid 312!

Example

1 2 6 5 3 4

Avoidance of 312

Let's first look at some examples of permutations that don't avoid 312!

Example

$1 \quad 2 \quad 6 \quad 5 \quad 3 \quad 4 \implies$ 126534 does not avoid 312

Avoidance of 312

Let's first look at some examples of permutations that don't avoid 312!

Example

$1\quad 2\quad \color{red}{6}\quad 5\quad \color{red}{3}\quad \color{red}{4} \quad \Longrightarrow$ 126534 does not avoid 312

Example

$1\quad 5\quad 6\quad 3\quad 2\quad 4$

Avoidance of 312

Let's first look at some examples of permutations that don't avoid 312!

Example

$1 \quad 2 \quad 6 \quad 5 \quad 3 \quad 4 \implies$ 126534 does not avoid 312

Example

$1 \quad 5 \quad 6 \quad 3 \quad 2 \quad 4 \implies$ 156324 does not avoid 312

Avoidance of 312

How about some permutations that do avoid 312?

Avoidance of 312

How about some permutations that do avoid 312?

Example

$1 \quad 2 \quad 3 \quad 6 \quad 5 \quad 4 \quad \Longrightarrow$ 123654 avoids 312

Avoidance of 312

How about some permutations that do avoid 312?

Example

$1 \quad 2 \quad 3 \quad 6 \quad 5 \quad 4 \quad \Longrightarrow$ 123654 avoids 312

Example

$2 \quad 1 \quad 4 \quad 5 \quad 6 \quad 3 \quad \Longrightarrow$ 214563 avoids 312

Avoidance of 312

Do the permutations that avoid 312 **have any special properties?**

Avoidance of 312

Do the permutations that avoid 312 have any special properties?

$$1 \quad 2 \quad 3 \quad 6 \quad 5 \quad 4$$

Avoidance of 312

Do the permutations that avoid 312 have any special properties?

$$\underbrace{1 \quad 2 \quad 3}_{} \quad 6 \quad 5 \quad 4$$

All < 4, and avoid 312

Avoidance of 312

Do the permutations that avoid 312 have any special properties?

$$\underbrace{1 \quad 2 \quad 3}_{} \quad \underbrace{6 \quad 5}_{} \quad \color{blue}{4}$$

All $<$ 4, and avoid 312

All $>$ 4, and avoid 312

Avoidance of 312

Do the permutations that avoid 312 have any special properties?

$$\underbrace{1 \quad 2 \quad 3}_{\text{All } < 4\text{, and avoid } 312} \quad \underbrace{6 \quad 5}_{\text{All } > 4\text{, and avoid } 312} \quad {\color{blue}4}$$

2 1 4 5 6 3

Avoidance of 312

Do the permutations that avoid 312 **have any special properties?**

$$\underbrace{1 \quad 2 \quad 3}_{\text{All} < 4\text{, and avoid } 312} \quad \underbrace{6 \quad 5}_{\text{All} > 4\text{, and avoid } 312} \quad \color{blue}{4}$$

$$\underbrace{2 \quad 1}_{\text{All} < 3\text{, and avoid } 312} \quad 4 \quad 5 \quad 6 \quad \color{blue}{3}$$

Avoidance of 312

Do the permutations that avoid 312 have any special properties?

$$\underbrace{1 \quad 2 \quad 3} \quad \underbrace{6 \quad 5} \quad {\color{blue}4}$$

All <4, and avoid 312 All >4, and avoid 312

$$\underbrace{2 \quad 1} \quad \underbrace{4 \quad 5 \quad 6} \quad {\color{blue}3}$$

All <3, and avoid 312 All >3, and avoid 312

Avoidance of 312

What happens with permutations that don't have this property?

Avoidance of 312

What happens with permutations that don't have this property?

1 2 6 5 3 4

Avoidance of 312

What happens with permutations that don't have this property?

1 2 6 5 3 4

All numbers < 4 in blue All numbers > 4 in green

Avoidance of 312

What happens with permutations that don't have this property?

1 2 6 5 3 4

All numbers < 4 in blue All numbers > 4 in green

$\Downarrow$

1 2 6 5 3 4

Avoidance of 312

What happens with permutations that don't have this property?

1 2 6 5 3 4

All numbers <4 in blue   All numbers >4 in green

$$\Downarrow$$

1 2 6 5 3 4

Doesn't avoid 312 anymore!

Avoidance of 312

Lemma

The permutations in S_{k+1} ending in i that avoid 312 are precisely those of the form,

$$\pi_1 \pi_2 i$$

where π_1 and π_2 are permutations of $\{1, 2, \ldots, i-1\}$ and $\{i+1, \ldots, k+1\}$ that avoid 312.

Avoidance of 312

Theorem

The n^{th} term of the sequence $a_n(312)$ is equal to C_n, the n^{th} Catalan number, for $n > 0$.

Avoidance of 312

Proof.

- Assume that for all i from 1 to k, the number of permutations of S_i that avoid 312 is C_i.

Avoidance of 312

Proof.

- Assume that for all i from 1 to k, the number of permutations of S_i that avoid 312 is C_i.
- It follows from the above lemma that the total number of permutations π avoiding 312 and ending in i is

$$C_{i-1} \cdot C_{k-i+1}$$

Avoidance of 312

Proof.

- Assume that for all i from 1 to k, the number of permutations of S_i that avoid 312 is C_i.
- It follows from the above lemma that the total number of permutations π avoiding 312 and ending in i is

$$C_{i-1} \cdot C_{k-i+1}$$

- Summing over all possible values of i, the total number of permutations of $\{1, 2, ..., k+1\}$ that avoid 312 is equal to,

$$\sum_{i=1}^{k+1} C_{i-1} \cdot C_{k-i+1} = \sum_{i=0}^{k} C_i \cdot C_{k-i}$$

Avoidance of 312

Proof.

- Assume that for all i from 1 to k, the number of permutations of S_i that avoid 312 is C_i.
- It follows from the above lemma that the total number of permutations π avoiding 312 and ending in i is

$$C_{i-1} \cdot C_{k-i+1}$$

- Summing over all possible values of i, the total number of permutations of $\{1, 2, ..., k+1\}$ that avoid 312 is equal to,

$$\sum_{i=1}^{k+1} C_{i-1} \cdot C_{k-i+1} = \sum_{i=0}^{k} C_i \cdot C_{k-i} = C_{k+1}$$

The Reversing Lemma

Definition (Reversing)

We define the *reverse* of a permutation $b_1 \cdots b_n$ to be the permutation $b_n \cdots b_1$. The reversing operator is denoted by $\mathcal{R}$.

The Reversing Lemma

Definition (Reversing)

We define the *reverse* of a permutation $b_1 \cdots b_n$ to be the permutation $b_n \cdots b_1$. The reversing operator is denoted by $\mathcal{R}$.

Example

$\mathcal{R}(1324) = 4231.$

Example

$\mathcal{R}(1243) = 3421.$

The Reversing Lemma

Definition (Reversing)

We define the *reverse* of a permutation $b_1 \cdots b_n$ to be the permutation $b_n \cdots b_1$. The reversing operator is denoted by $\mathcal{R}$.

Example

$\mathcal{R}(1324) = 4231$.

Example

$\mathcal{R}(1243) = 3421$.

Note that 1243 including 132 implies that $\mathcal{R}(1243) = 3421$ includes $\mathcal{R}(132) = 231$.

The Reversing Lemma

Lemma (Reversing Lemma)

The permutation σ avoids the permutation π iff $\mathcal{R}(\sigma)$ avoids $\mathcal{R}(\pi)$.

Corollary

For a permutation π, $a_n(\pi) = a_n(\mathcal{R}(\pi))$.

The Flipping Lemma

Definition (Flipping)

We define the *flip* of a sequence b as the sequence c with the same elements as b, but with

- the largest element swapped with the smallest element
- the second largest element swapped with the second smallest element

and so on.
The flipping operator is denoted by $\mathcal{F}$.

The Flipping Lemma

Example

$$1 \quad 2 \quad 3 \quad 4 \qquad \Longrightarrow \qquad 4 \quad 3 \quad 2 \quad 1$$

$$\mathcal{F}(1234) = 4321$$

The Flipping Lemma

Example

$$1 \quad 2 \quad 3 \quad 4 \quad \Longrightarrow \quad 4 \quad 3 \quad 2 \quad 1$$

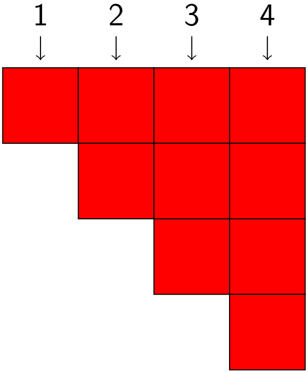
$$\mathcal{F}(1234) = 4321$$

Example

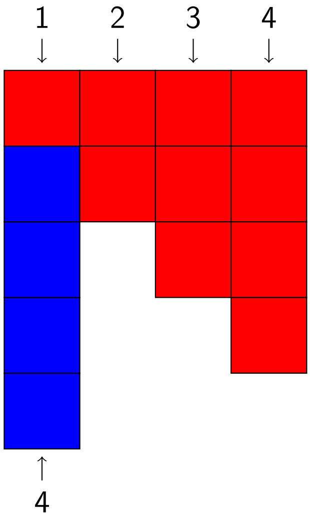
$$1 \quad 2 \quad 4 \quad 3 \quad \Longrightarrow \quad 4 \quad 3 \quad 1 \quad 2$$

$$\mathcal{F}(1243) = 4312$$

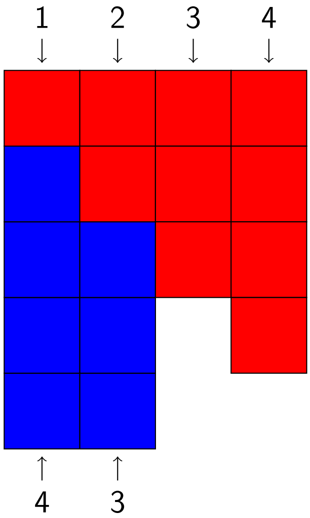
The Flipping Lemma



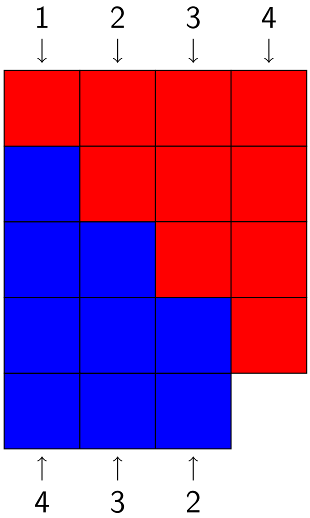
The Flipping Lemma



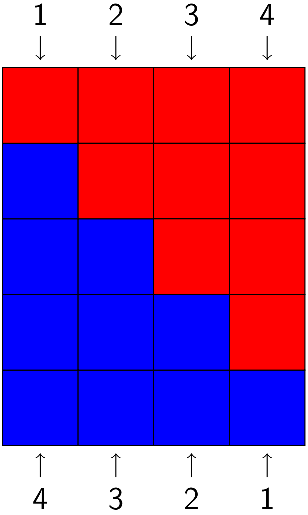
The Flipping Lemma



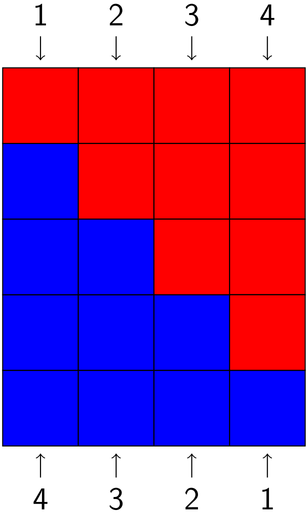
The Flipping Lemma



The Flipping Lemma

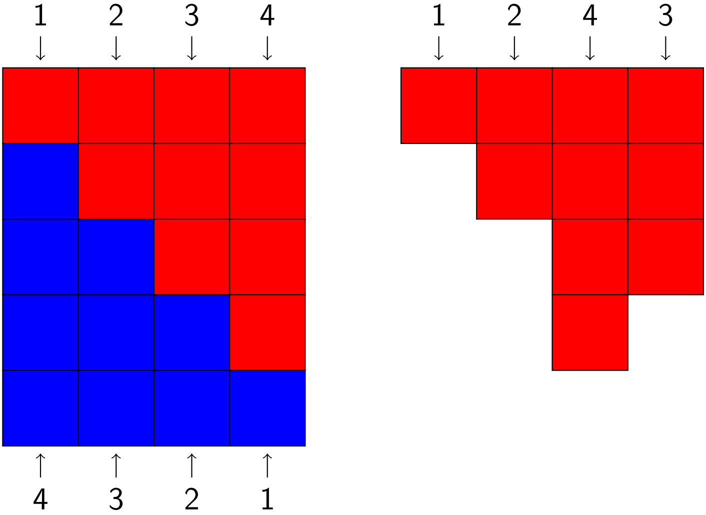


The Flipping Lemma



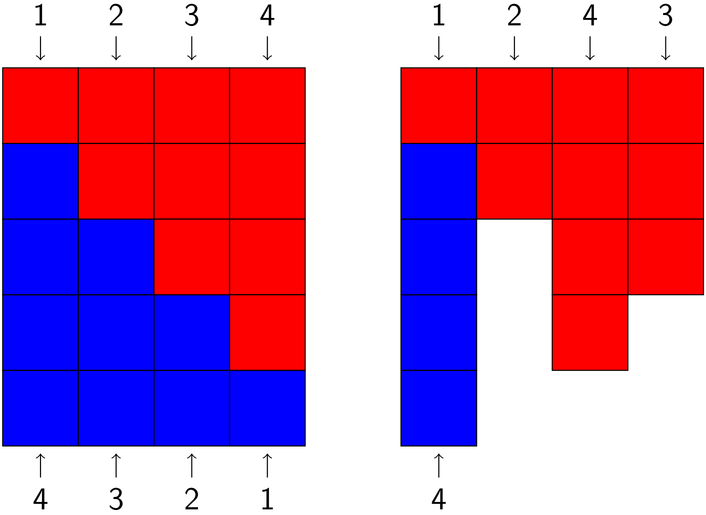
$$\mathcal{F}(1234) = 4321$$

The Flipping Lemma



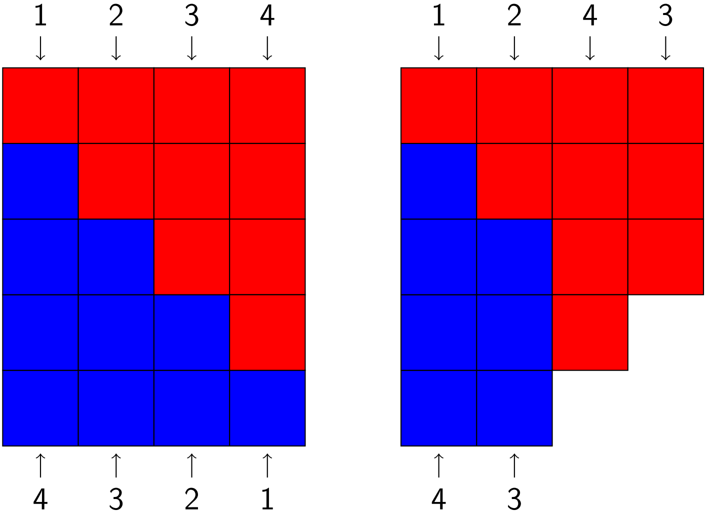
$$\mathcal{F}(1234) = 4321$$

The Flipping Lemma



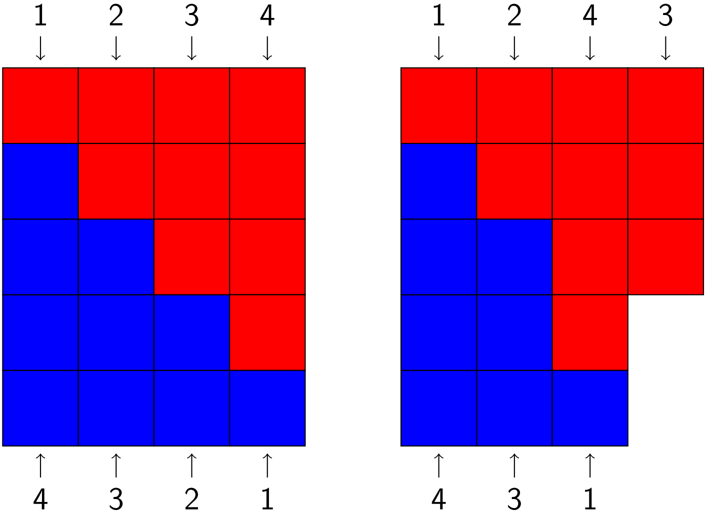
$$\mathcal{F}(1234) = 4321$$

The Flipping Lemma



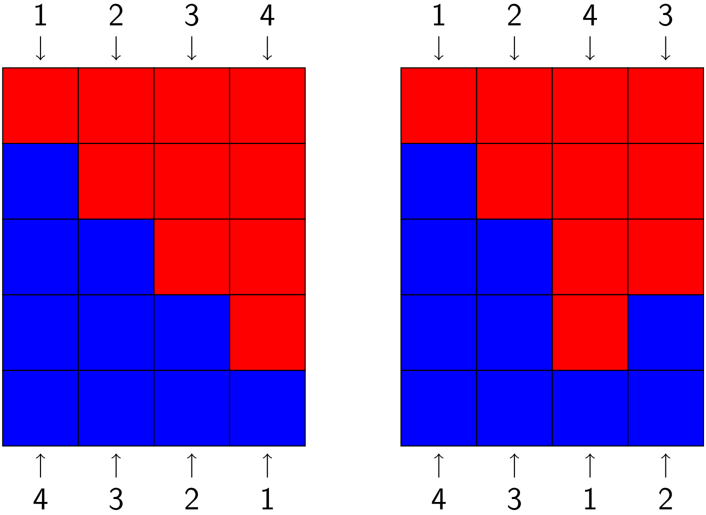
$$\mathcal{F}(1234) = 4321$$

The Flipping Lemma



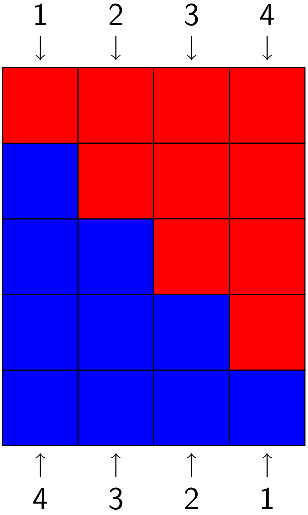
$$\mathcal{F}(1234) = 4321$$

The Flipping Lemma

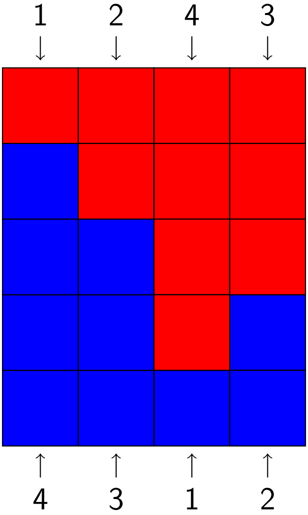


$$\mathcal{F}(1234) = 4321$$

The Flipping Lemma



$\mathcal{F}(1234) = 4321$



$\mathcal{F}(1243) = 4312$

The Flipping Lemma

Lemma (Flipping Lemma)

The permutation σ avoids the permutation π iff $\mathcal{F}(\sigma)$ avoids $\mathcal{F}(\pi)$.

Corollary

For a permutation π, $a_n(\pi) = a_n(\mathcal{F}(\pi))$.

Avoidance of other permutations in S_3

-
$$213 = \mathcal{R}(312)$$
$$132 = \mathcal{F}(312)$$
$$231 = \mathcal{R}(132)$$

Avoidance of other permutations in S_3

- $$\left.\begin{array}{l} 213 = \mathcal{R}(312) \\ 132 = \mathcal{F}(312) \\ 231 = \mathcal{R}(132) \end{array}\right\} \Rightarrow \begin{array}{l} (a_n(213)),\ (a_n(132)) \\ \text{and } (a_n(231)) \text{ are all the} \\ \text{sequence of Catalan numbers} \end{array}$$

Avoidance of other permutations in S_3

- $$\left.\begin{array}{l} 213 = \mathcal{R}(312) \\ 132 = \mathcal{F}(312) \\ 231 = \mathcal{R}(132) \end{array}\right\} \Rightarrow \begin{array}{l} (a_n(213)),\ (a_n(132)) \\ \text{and } (a_n(231)) \text{ are all the} \\ \text{sequence of Catalan numbers} \end{array}$$

- However, it is much harder to prove that the sequences $(a_n(123))$ and $(a_n(321))$ are the sequence of Catalan numbers.

Three sequences

$$A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

$$B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots$$

$$C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots$$

Permutations in S_4

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Permutations in S_4

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Flipping and reversing buckets

$$\{1243, 4312, 2134, 3421\}, \{2413, 3142\},$$
$$\{1432, 4123, 2341, 3214\}, \{1234, 4321\},$$
$$\{4132, 1423, 2314, 3241\}, \{2143, 3412\},$$
$$\{4213, 1342, 3124, 2431\}, \{4231, 1324\}.$$

Decomposition

$$A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

$$B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots$$

$$C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots$$

Decomposition

$$A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

$$B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots$$

$$C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots$$

B	A	C
1243, 4312, 2134, 3421	4132, 1423, 2314, 3241	4231, 1324
1432, 4123, 2341, 3214,	4213, 1342, 3124, 2431	
2143, 3412	2413, 3142	
1234, 4321		

Decomposition

$$A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

$$B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots$$

$$C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots$$

B	A	C
1243, 4312, 2134, 3421	4132, 1423, 2314, 3241	4231, 1324
1432, 4123, 2341, 3214,	4213, 1342, 3124, 2431	
2143, 3412	2413, 3142	
1234, 4321		

???

Cycle notation

Take $\sigma = 45213$. We can think of it as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

Cycle Notation

Take $\sigma = 45213$. We can think of it as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

<u>Cycle Notation</u>

$$\sigma = (14)(253)$$

Cycle Notation

Take $\sigma = 45213$. We can think of it as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

<u>Cycle Notation</u>

$$\sigma = (14)(253)$$

Cycle Notation

Take $\sigma = 45213$. We can think of it as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \color{red}{\downarrow} & \color{green}{\downarrow} & \color{green}{\downarrow} & \color{red}{\downarrow} & \color{green}{\downarrow} \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

<u>Cycle Notation</u>

$$\sigma = \color{red}{(14)}\color{green}{(253)}$$

Buckets in cycle notation

(34), (1423), (12), (1324)	(24), (1432), (13), (1234)
(142), (243), (123), (134)	(143), (234), (132), (124)
(1)(2)(3)(4), (14)(23)	(12)(34), (13)(24)
(14), (23)	(1243), (1342)

Cycle decomposition

$$A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

$$B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots$$

$$C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots$$

Cycle decomposition

$$A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

$$B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots$$

$$C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots$$

B	A	C
(34), (1423), (12), (1324)	(142), (243), (123), (134)	(14), (23)
(24), (1432), (13), (1234)	(143), (234), (132), (124)	
(12)(34), (13)(24)	(1243), (1342)	
(1)(2)(3)(4), (14)(23)		

Conjectures

Conjectures.

- There are three possible sequences for $(a_n(\pi))$.

Conjectures

Conjectures.

- There are three possible sequences for $(a_n(\pi))$.
- Two $\mathcal{F}\&\mathcal{R}$ buckets that "look the same" generate the same sequence.

Other observations

$$A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

$$B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots$$

$$C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots$$

Other observations

$$A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

$$B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots$$

$$C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots$$

- Significantly fewer sequences than expected.

Other observations

$$A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

$$B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots$$

$$C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots$$

- Significantly fewer sequences than expected.
- Uneven distribution of permutations generating the sequences.

Other observations

$$A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

$$B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots$$

$$C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots$$

- Significantly fewer sequences than expected.
- Uneven distribution of permutations generating the sequences.
- Different growth rates?

Avoidance in T_{2m}

Definition

Let m be a positive integer. The set T_{2m} is defined as all permutations in S_{2m} such that:

- the odd numbers appear in increasing order,
- each even number $2i$ appears to the right of $2i - 1$.

Example

The set S_2 is $\{12, 21\}$. The set T_2 is just $\{12\}$.

Permutations in T_4

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Permutations in T_4

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Permutations in T_4

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Problem Statement

Definition

Given a permutation $\pi \in S_k$, define $b_m(\pi)$ as

$$b_m(\pi) = \#\{\sigma \in T_{2m} \mid \sigma \text{ avoids } \pi\}.$$

Problem

Let $\pi \in S_3$, and let m be an arbitrary positive integer. What is $b_m(\pi)$?

Values of $b_m(\pi)$

π	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
123	1	0	0	0	0
132	1	1	1	1	1
213	1	2	4	8	16
231	1	2	4	8	16
312	1	3	12	55	273
321	1	3	12	55	273

Computing $b_m(123)$

π	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
123	1	0	0	0	0
132	1	1	1	1	1
213	1	2	4	8	16
231	1	2	4	8	16
312	1	3	12	55	273
321	1	3	12	55	273

Pick any $\sigma \in T_{2m}$ with $m \geq 2$. Then:

- 3 comes after 1,
- 4 comes after 3,
- So 134 is a subsequence of σ.

Computing $b_m(132)$

π	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
123	1	0	0	0	0
132	1	1	1	1	1
213	1	2	4	8	16
231	1	2	4	8	16
312	1	3	12	55	273
321	1	3	12	55	273

Let $m \geq 2$, and pick any $\sigma \in T_{2m}$ avoiding 132. Then:

- 1 first,
- An even integer $2i$ must come before $2i+1$,
- So $\sigma = 1234\ldots(2m)$.

On $b_m(213)$ and $b_m(231)$

π	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
123	1	0	0	0	0
132	1	1	1	1	1
213	1	2	4	8	16
231	1	2	4	8	16
312	1	3	12	55	273
321	1	3	12	55	273

Theorem

$b_m(213) = 2^{m-1}$, *and* $b_m(231) = 2^{m-1}$.

On $b_m(312)$ and $b_m(321)$

π	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
123	1	0	0	0	0
132	1	1	1	1	1
213	1	2	4	8	16
231	1	2	4	8	16
312	1	3	12	55	273
321	1	3	12	55	273

Theorem

$b_m(312) = b_m(321)$.

On $b_m(312)$ and $b_m(321)$

π	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
123	1	0	0	0	0
132	1	1	1	1	1
213	1	2	4	8	16
231	1	2	4	8	16
312	1	3	12	55	273
321	1	3	12	55	273

Recall.
Let $\sigma \in S_3$. Then $a_n(\sigma) = C_n = \binom{2n}{n} \cdot \frac{1}{n+1}$.

Conjecture.
$b_m(312) = \binom{3m}{m} \cdot \frac{1}{2m+1}$.

Thank you!

Questions?