Pattern avoidance

An explanation and proof

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November 19, 2014

A permutation of a finite set $\{1, \cdots, n\}$ is some *ordering* of the elements.

54123 is a permutation of $\{1, 2, 3, 4, 5\}$.

S_n is the set of permutations on $\{1, \cdots, n\}$.

$$54123 \in S_5$$

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$$54123 \in S_5$$

54123

includes $\begin{cases} 123 \\ 312 \\ 4312 \end{cases}$

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avoids $\begin{cases} 132 \\ 312 \\ 213 \\ 231 \end{cases}$

Let $\pi = 312 \in S_3$.

- Question: How many permutations avoid π? (a lot)
- Better Question: How many permutations in S_n avoid π?

Let's count!

- How many permutations in S_1 avoid π? 1
- How many permutations in S_2 avoid π? 2
- How many permutations in S_3 avoid π? 5
- How many permutations in S_4 avoid π? ??????

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Permutations in S_4 that avoid $\pi = 312$?

How many permutations in S_4 avoid π? 14

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Definition

Let $a_n(\pi)$ be the number of permutations in S_n that avoid π.

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Example: $(a_n(312)) = 1, 2, 5, 14,$ 42,132,429,...

Main Result

Theorem

For $\pi \in S_3$, $(a_n(\pi))$ is equal to the Catalan numbers:

$$(a_n(\pi)) = 1, 2, 5, 14, 42, 132, 429...$$

What if we take $\pi \in S_4$?

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots \\ \end{cases}$$

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Avoidance of 312

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Example

1 2 6 5 3 4

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$$1 \quad 2 \quad \color{red}{6} \quad 5 \quad \color{red}{3} \quad \color{red}{4} \quad \Longrightarrow \text{126534 does not avoid 312}$$

Avoidance of 312

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$1 \quad 2 \quad \color{red}{6} \quad 5 \quad \color{red}{3} \quad \color{red}{4} \quad \Longrightarrow$ 126534 does not avoid 312

Example

$1 \quad 5 \quad 6 \quad 3 \quad 2 \quad 4$

Avoidance of 312

Let's first look at some examples of permutations that don't avoid 312!

Example

$$1 \quad 2 \quad \textcolor{red}{6} \quad 5 \quad \textcolor{red}{3} \quad \textcolor{red}{4} \implies 126534 \text{ does not avoid } 312$$

Example

$$1 \quad \textcolor{red}{5} \quad 6 \quad \textcolor{red}{3} \quad 2 \quad \textcolor{red}{4} \implies 156324 \text{ does not avoid } 312$$

Avoidance of 312

How about some permutations that do avoid 312?

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Example

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$1 \quad 2 \quad 3 \quad 6 \quad 5 \quad 4 \quad \Longrightarrow$ 123654 avoids 312

Avoidance of 312

How about some permutations that do avoid 312?

Example

$1 \quad 2 \quad 3 \quad 6 \quad 5 \quad 4 \quad \Longrightarrow$ 123654 avoids 312

Example

$2 \quad 1 \quad 4 \quad 5 \quad 6 \quad 3$

Avoidance of 312

How about some permutations that do avoid 312?

Example

1 2 3 6 5 4 $\Longrightarrow$ 123654 avoids 312

Example

2 1 4 5 6 3 $\Longrightarrow$ 214563 avoids 312

Avoidance of 312

Do the permutations that avoid 312 have any special properties?

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$$\underbrace{1 \quad 2 \quad 3}_{\text{All } < 4\text{, and avoid } 312} \quad 6 \quad 5 \quad {\color{blue}4}$$

Avoidance of 312

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$$\underbrace{1 \quad 2 \quad 3}_{\text{All } < 4\text{, and avoid } 312} \quad \underbrace{6 \quad 5}_{\text{All } > 4\text{, and avoid } 312} \quad {\color{blue}4}$$

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2 1 4 5 6 3

Avoidance of 312

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$$\underbrace{1 \quad 2 \quad 3}_{\text{All } < 4 \text{, and avoid } 312} \quad \underbrace{6 \quad 5}_{\text{All } > 4 \text{, and avoid } 312} \quad \color{blue}{4}$$

$$\underbrace{2 \quad 1}_{\text{All } < 3 \text{, and avoid } 312} \quad 4 \quad 5 \quad 6 \quad \color{blue}{3}$$

Avoidance of 312

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$$\underbrace{1\quad 2\quad 3}_{\text{All} < 4\text{, and avoid } 312}\quad \underbrace{6\quad 5}_{\text{All} > 4\text{, and avoid } 312}\quad {\color{blue}4}$$

$$\underbrace{2\quad 1}_{\text{All} < 3\text{, and avoid } 312}\quad \underbrace{4\quad 5\quad 6}_{\text{All} > 3\text{, and avoid } 312}\quad {\color{blue}3}$$

Avoidance of 312

What happens with permutations that don't have this property?

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What happens with permutations that don't have this property?

1 2 6 5 3 4

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All numbers < 4 in blue

All numbers > 4 in green

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All numbers <4 in blue   All numbers >4 in green

$$\Downarrow$$

1 2 6 5 3 4

Avoidance of 312

What happens with permutations that don't have this property?

1 2 6 5 3 4

All numbers < 4 in blue   All numbers > 4 in green

$$\Downarrow$$

1 2 6 5 3 4

Doesn't avoid 312 anymore!

Avoidance of 312

Lemma

The permutations of $\{1, 2, \ldots, k, k+1\}$ *ending in* i *that avoid the pattern* 312 *are precisely those of the form,*

$$\pi_1 \pi_2 i$$

the concatenation of π_1, π_2, *and* i, *where* π_1 *is a permutation of* $\{1, 2, \ldots, i-1\}$ *that avoids the pattern* 312 *and* π_2 *is a permutation of* $\{i+1, \ldots, k+1\}$ *that avoids the pattern* 312.

Avoidance of 312

Definition

The Catalan numbers are the sequence of positive integers C_i defined as follows,

$$C_0 = 1, \; C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \text{ for } n \geq 0$$

Avoidance of 312

Theorem

The n^{th} term of the sequence $a_n(312)$ is equal to C_n, the n^{th} Catalan number, for $n > 0$.

Avoidance of 312

Proof.

Assume that for all i from 1 to k, the number of permutations of $\{1, 2, ..., i\}$ that avoid 312 is C_i.
It follows from the above lemma that the total number of permutations π avoiding 312 and ending in i is

$$C_{i-1} \cdot C_{k-i+1}$$

Summing over all possible values of i, the total number of permutations of $\{1, 2, ..., k+1\}$ that avoid 312 is equal to,

$$\sum_{i=1}^{k+1} C_{i-1} \cdot C_{k-i+1} = \sum_{i=0}^{k} C_i \cdot C_{k-i} = C_{k+1}$$

The Reversing Lemma

Definition (Reversing)

We define the *reverse* of a permutation $b_1 \cdots b_n$ to be the permutation $b_n \cdots b_1$. The reversing operator is denoted by $\mathcal{R}$.

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Example

$\mathcal{R}(1324) = 4231.$

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Example

$\mathcal{R}(1324) = 4231.$

Example

$\mathcal{R}(1243) = 3421.$

The Reversing Lemma

Lemma (Reversing Lemma)

The permutation σ avoids the permutation π iff $\mathcal{R}(\sigma)$ avoids $\mathcal{R}(\pi)$.

Corollary

For a permutation π, $a_n(\pi) = a_n(\mathcal{R}(\pi))$.

The Flipping Lemma

Definition (Flipping)

We define the *flip* of a sequence b as the sequence c with the same elements as b, but with the largest element swapped with the smallest element, the second largest element swapped with the second smallest element, etc. The flipping operator is denoted by $\mathcal{F}$.

The Flipping Lemma

Example

$$1 \quad 2 \quad 3 \quad 4 \quad \Longrightarrow \quad 4 \quad 3 \quad 2 \quad 1$$

$$\mathcal{F}(1234) = 4321$$

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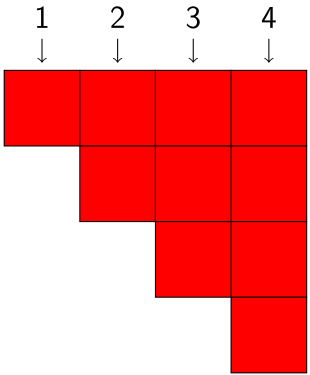
Example

$$1 \quad 2 \quad 4 \quad 3 \quad \Longrightarrow \quad 4 \quad 3 \quad 1 \quad 2$$

$$\mathcal{F}(1243) = 4312$$

The Flipping Lemma



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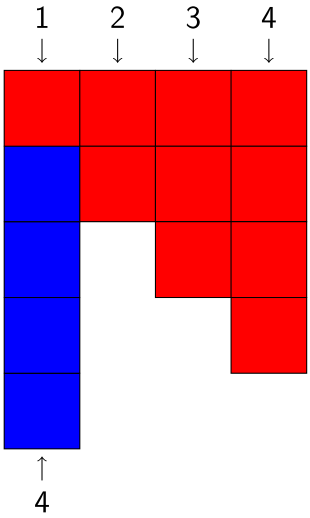

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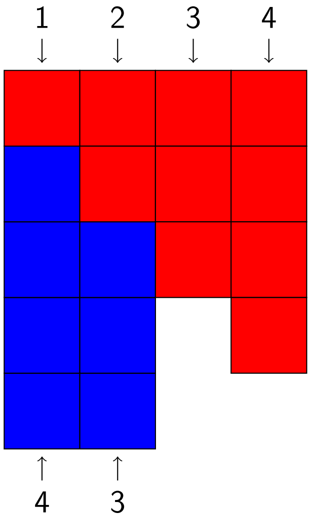

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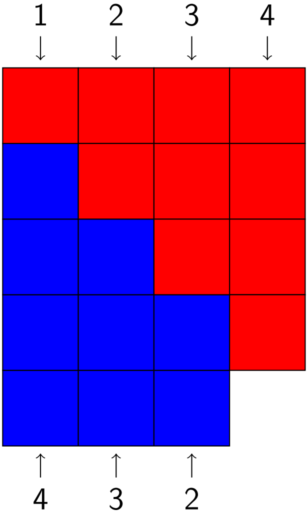

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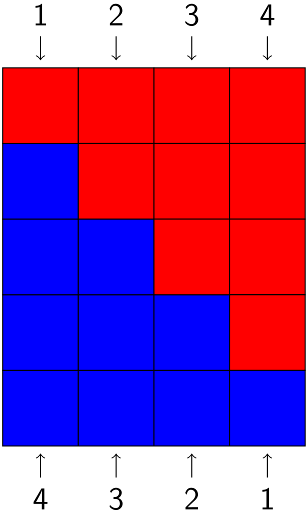

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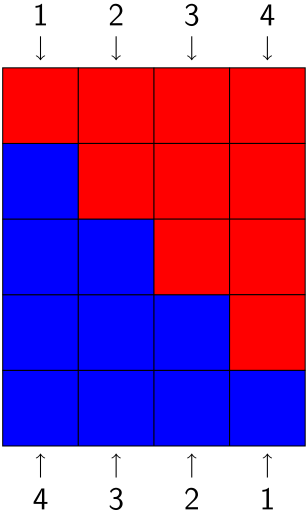

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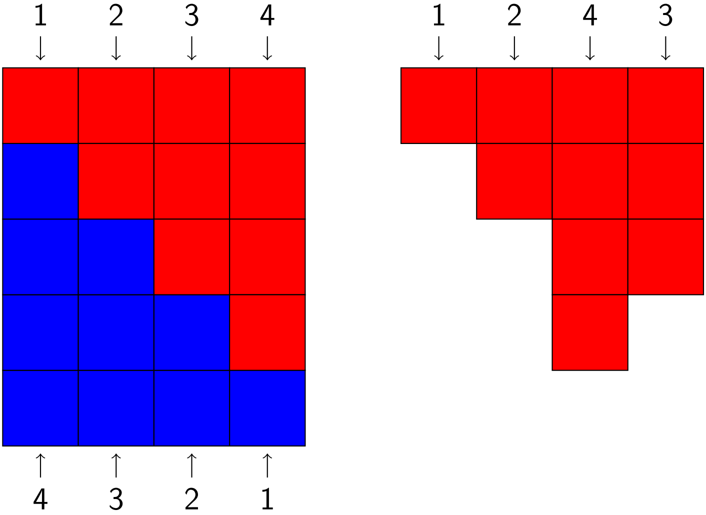

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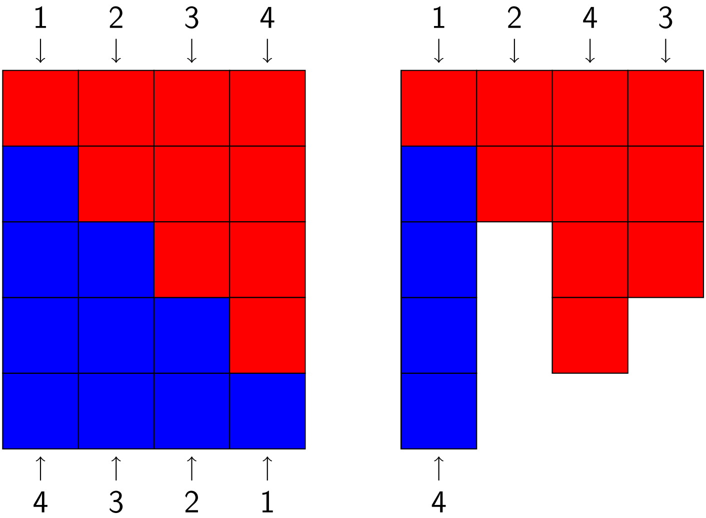

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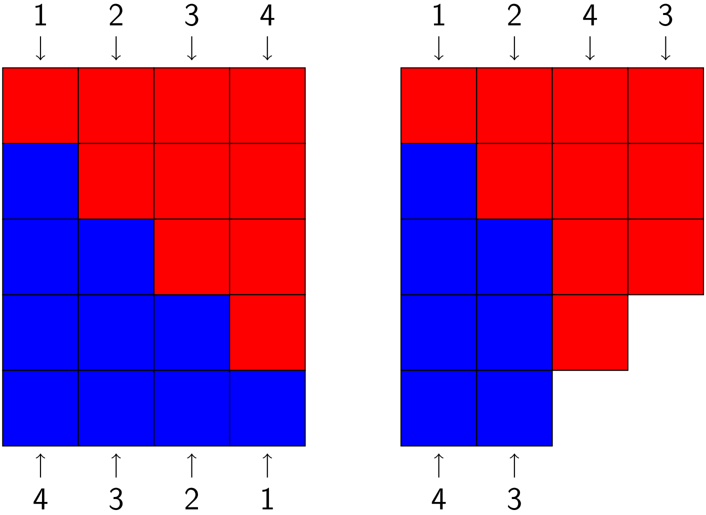

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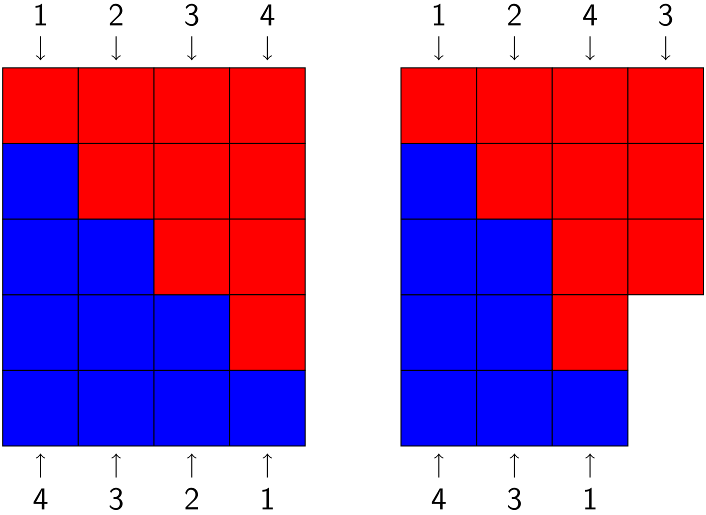

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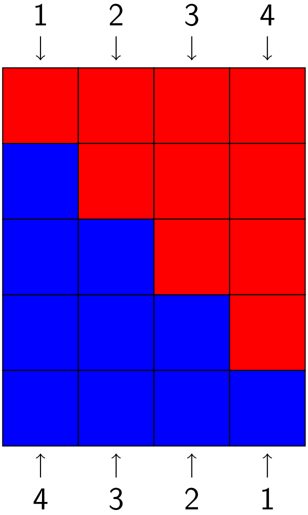

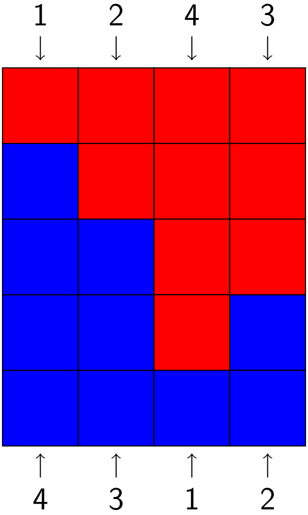

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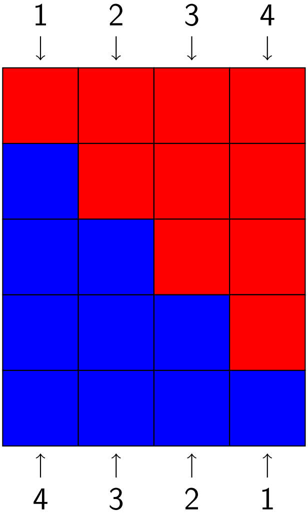

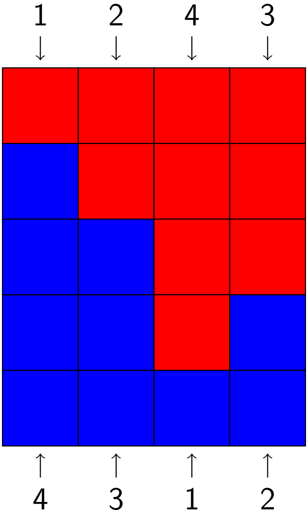

$\mathcal{F}(1234) = 4321$ $\qquad$ $\mathcal{F}(1243) = 4312$

The Flipping Lemma

Lemma (Flipping Lemma)

The permutation σ avoids the permutation π iff $\mathcal{F}(\sigma)$ avoids $\mathcal{F}(\pi)$.

Corollary

For a permutation π, $a_n(\pi) = a_n(\mathcal{F}(\pi))$.

Avoidance of other permutations in S_3

- From the Flipping Lemma and Reversing Lemmas, the sequences $(a_n(213))$, $(a_n(132))$ and $(a_n(231))$ are the sequence of Catalan numbers as well.
- However, it is much harder to prove that the sequences $(a_n(123))$ and $(a_n(321))$ are the sequence of Catalan numbers.

Recall

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots \\ C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots \end{cases}$$

Permutations in S_4

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2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Flipping and reversing buckets

$$\{1243, 4312, 2134, 3421\}, \{2413, 3142\},$$
$$\{1432, 4123, 2341, 3214\}, \{1234, 4321\},$$
$$\{4132, 1423, 2314, 3241\}, \{2143, 3412\},$$
$$\{4213, 1342, 3124, 2431\}, \{4231, 1324\}$$

Decomposition

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots \\ C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots \end{cases}$$

B	A	C
1234, 4321	4132, 1423	4231, 1324
1243, 4312	4213, 1342	
1432, 4123	2431, 3124	
2134, 3421	2413, 3142	
2143, 3412	2314, 3241	
2341, 3214		

Cycle notation

Take $\sigma \in S_5$ with

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

One-line Notation	Cycle Notation
$\sigma = 45213$	$\sigma = (14)(253)$

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Cycle decomposition

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B	A	C
(1)(2)(3)(4),(14)(23)	(243),(142)	(23),(14)
(34),(1423)	(234),(143)	
(24),(1432)	(124),(132)	
(12),(1324)	(123),(134)	
(12)(34),(13)(24)	(1243),(1342)	
(1234),(13)		

Avoidance in T_n

Definition

Let m be a positive integer. The set T_{2m} is defined as all permutations in S_{2m} such that:

- the odd numbers appear in increasing order,
- each even number $2i$ appears to the right of $2i-1$.

Example

The set $T_2 \subset S_2$ consists of the single permutation 12. The other permutation in S_2, 21, is not in T_2.

Permutations in T_4

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Problem Statement

Definition

Given a permutation $\pi \in S_k$, we define $t_m(\pi)$ as

$$t_m(\pi) = \#\{\sigma \in T_{2m} \mid \sigma \text{ avoids } \pi\}.$$

Problem

Let $\pi \in S_3$, and m an arbitrary positive integer. Compute $t_m(\pi)$.

We can run code to compute $t_m(\pi)$ for small m and for each $\pi \in S_3$. We get

π	$m = 2$	$m = 4$	$m = 6$	$m = 8$	$m = 10$
123	1	0	0	0	0
132	1	1	1	1	1
213	1	2	4	8	16
231	1	2	4	8	16
312	1	3	12	55	273
321	1	3	12	55	273

- Easy to see that $t_n(123) = 0$ when $n \geq 2$
  - The subsequence 134 is always present in a permutation $\sigma \in t_n(123)$.
- Also easy to see that $t_n(132) = 1$.
  - The permutation $123\ldots(2n)$ is the only permutation in T_n that avoids 132.