18.821 Project 2

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1 Terms in the sequence $s_n(312)$

We claim that the sequence of numbers $s_n(312)$ is in fact the sequence of Catalan numbers. We state this result formally as the following theorem,

Theorem 1. *The total number of permutations of* $\{1, 2, 3, ..., n\}$ *that avoid the order* 312 *as a subsequence is* C_n *where* C_n *is the* n^{th} *Catalan number.*

Before proving the theorem, we state and prove the following lemma, that will be used in our proof of the theorem.

Lemma 2. *All permutations of* $\{1, 2, ..., k, k+1\}$ *ending in* i *that avoid the order* 312 *as a sub-sequence must be of the form,*

$$\pi_1 \pi_2 i$$

where π_1 *is a permutation of* A *that avoids the order* 312 *as a sub-sequence and* π_2 *is a permutation of* B *that avoids the order* 312 *as a sub-sequence.*

Proof. It is clear that the permutations π_1 and π_2 themselves must avoid the order 312 as a subsequence.

Now, we need to prove that the ordering between π_1 and π_2 is necessary. We proceed with a proof by contradiction.

For the sake of contradiction, let us assume that there exists some permutation π of $\{1, 2, ..., k, k+1\}$ that ends with value i such that some integer $x < i$ is to the right of some integer $y > i$. Clearly, this permutation is not of the form described above. It is also easy to see that π does **not** avoid the order 312 since the triple (y, x, i) satisfies the condition $y > x > i$ and hence is in the order 312.

□

With this lemma proven, we move on to the proof of our theorem.

Proof. The inductive hypothesis holds for our base case of $\{1\}$, since the only permutation of $\{1\}$ avoids 312.

Now, we need to prove the inductive case. Let us first assume that for all i from 1 to k, the number of permutations of $\{1, 2, ..., i\}$ that avoid the order 312 as a subsequence is C_i.

Now, we want to prove the inductive hypothesis for $\{1, 2, ..., k, k+1\}$ as well, that is the number of permutations of $\{1, 2, ..., k, k+1\}$ that avoid the order 312 as a subsequence is C_{k+1}.

We count the number of permutations of $\{1, 2, ..., k, k+1\}$ that avoid 312 by enumerating through all possible values of the last term of a valid permutation. If the last term of the permutation is i (where $i \in \{1, 2, ..., k, k+1\}$), then let us define the subsets A and B of the set $\{1, 2, ..., k+1\} \setminus \{i\}$ as the set of integers less than i and the set of integers greater than i respectively. It is clear from the definition of A and B that A and B are disjoint from each other.

Now, from the above lemma, we know that all permutations of $\{1, 2, ..., k, k+1\}$ ending in i that avoid 312 must be of the form,

$$\pi = \pi_1 \pi_2 i$$

where π_1 is a permutation of A that avoids the order 312 as a sub-sequence and π_2 is a permutation of B that avoids the order 312 as a sub-sequence. It is clear that the above permutation contains all integers between 1 and $k+1$, from the definitions of the subsets A and B, which implies that $\pi_1 \pi_2 i$ is permutation of the set $\{1, 2, ..., k, k+1\}$.

Now, the total number of permutations π is,

$$n_{\pi_1} \cdot n_{\pi_2} = C_{i-1} \cdot C_{k-i+1}$$

since the total number of valid permuations π_1 is simply going to be C_{i-1} (total number of valid permutations of length $i-1$ that avoid the order 312 as a sub-sequence is C_{i-1}, and correspondingly $n_{\pi_2} = C_{k-1+1}$)

Now, summing over all valid i, we see that the total number of permutations of $\{1, 2, ..., k+1\}$ that avoid 312 is equal to,

$$\sum_{i=1}^{k+1} C_{i-1} \cdot C_{k-i+1} = \sum_{i=0}^{k} C_i \cdot C_{k-i}$$

which is in fact C_{k+1}, and we are done.

$\square$