Pattern avoidance

An explanation and proof

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November 14, 2014

Plan of action

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Definition, Introduction and Motivation

Lemma (Reversing Lemma)

The permutation σ avoids the permutation π if and only if $\mathcal{R}(\sigma)$ avoids $\mathcal{R}(\pi)$.

Proof.

Need to only prove one direction (why?).
Suppose that $\mathcal{R}(\sigma)$ does not avoid $\mathcal{R}(\pi)$ if σ avoids π.
Then if $\mathcal{R}(\sigma)$ is reversed again again, the subsequence of $\mathcal{R}(\sigma)$ that follows the same pattern as $\mathcal{R}(\pi)$ will reverse to a subsequence that follows the same pattern as π – a contradiction.
Therefore $\mathcal{R}(\sigma)$ avoids $\mathcal{R}(\pi)$.

□

Corollary

For a permutation π, $s_n(\pi) = s_n(\mathcal{R}(\pi))$.

Definition

We define the *flip* of a sequence a as the sequence b with the same elements as a, but with the largest element swapped with the smallest element, the second largest element swapped with the second smallest element, etc. The flipping operator will be denoted by $\mathcal{F}$.

Example

$\mathcal{F}(1324) = 4231$.

Example

$\mathcal{F}(1243) = 4312$.

Lemma (Flipping Lemma)

The permutation σ avoids the permutation π if and only if $\mathcal{F}(\sigma)$ avoids $\mathcal{F}(\pi)$.

Proof.

Need to only prove one direction (why?).
Suppose $\mathcal{F}(\sigma)$ does not avoid $\mathcal{F}(\pi)$ if σ avoids π. (Here, $\sigma \in S_n$ and $\pi \in S_k$)
Let $\mathcal{F}(\sigma) = a_1 \cdots a_n$ and let $\mathcal{F}(\pi) = b_1 \cdots b_k$. Then there is some subsequence $a_{i_1} \cdots a_{i_k}$ so that $a_{i_c} < a_{i_d}$ if and only if $b_c < b_d$.
But then, $n + 1 - a_{i_c} > n + 1 - a_{i_d}$ if and only if $k + 1 - b_c > k + 1 - b_d$. This implies that σ does not avoid π, a contradiction (why?). $\square$

Corollary

For a permutation π, $s_n(\pi) = s_n(\mathcal{F}(\pi))$.

Lemma

The permutations of $\{1, 2, \ldots, k, k+1\}$ *ending in* i *that avoid the pattern* 312 *are precisely those of the form,*

$$\pi_1 \pi_2 i$$

the concatenation of π_1, π_2*, and* i*, where* π_1 *is a permutation of* $\{1, 2, \ldots, i-1\}$ *that avoids the pattern* 312 *and* π_2 *is a permutation of* $\{i+1, \ldots, k+1\}$ *that avoids the pattern* 312*.*

Proof.

For the sake of contradiction, assume that there exists some permutation π of $\{1, 2, ..., k, k+1\}$ that avoids 312, ending with value i such that some integer $x < i$ is to the right of some integer $y > i$. Clearly, π does not avoid 312. (why?)
It follows that for any permutation $\pi \in S_{k+1}$ avoiding 312 and ending in i, all $x < i$ must be to the left of all $y > i$. Hence, π can be written as $\pi_1\pi_2 i$ where π_1 is a permutation of $\{1, 2, \ldots, i-1\}$ and π_2 is a permutation of $\{i+1, i+2, \ldots, k+1\}$. Furthermore, it is clear that any subsequences of the permutation π must avoid 312 if the entire permutation π is to avoid 312 as well; this implies that the permutations π_1 and π_2 must avoid 312 as well. $\square$

Definition

The Catalan numbers are the sequence of positive integers C_i defined as follows,

$$C_0 = 1,\ C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \text{ for } n \geq 0$$

Theorem

$s_n(312), s_n(132), s_n(213)$, and $s_n(231)$ are equal to C_n, the n^{th} Catalan number.

Proof.

Assume that for all i from 1 to k, the number of permutations of $\{1, 2, ..., i\}$ that avoid the order 312 as a subsequence is C_i.
Want to prove that the number of permutations of $\{1, 2, ..., k, k+1\}$ that avoid the order 312 as a subsequence is C_{k+1}.
The number of permutations of $\{1, 2, ..., k, k+1\}$ that avoid 312 can be counted by enumerating through all possible values of the last term of a valid permutation.
From the above lemma, all permutations of $\{1, 2, ..., k, k+1\}$ ending in i that avoid 312 are precisely those of the form

$$\pi = \pi_1 \pi_2 i$$

where π_1 is a permutation of $\{1, 2, \ldots, i-1\}$ that avoids 312 and π_2 is a permutation of $\{i+1, i+2, \ldots, k+1\}$ that avoids 312.

Proof (contd.)

It follows that the total number of permutations π avoiding 312 and ending in i is

$$C_{i-1} \cdot C_{k-i+1}$$

(by our induction hypothesis)
Now, summing over all possible values of i, the total number of permutations of $\{1, 2, ..., k+1\}$ that avoid 312 is equal to,

$$\sum_{i=1}^{k+1} C_{i-1} \cdot C_{k-i+1} = \sum_{i=0}^{k} C_i \cdot C_{k-i}$$

which equals C_{k+1}
By the flipping lemma and reversing lemma, $s_n(132)$, $s_n(213)$ and $s_n(231)$ are the sequence of Catalan numbers as well. $\square$

Avoidance in T_n