Pattern avoidance

An explanation and proof

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November 14, 2014

Plan of action

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Definition, Introduction and Motivation

Lemma

The permutations of $\{1, 2, \ldots, k, k+1\}$ *ending in* i *that avoid the pattern* 312 *are precisely those of the form,*

$$\pi_1 \pi_2 i$$

the concatenation of π_1, π_2, *and* i, *where* π_1 *is a permutation of* $\{1, 2, \ldots, i-1\}$ *that avoids the pattern* 312 *and* π_2 *is a permutation of* $\{i+1, \ldots, k+1\}$ *that avoids the pattern* 312.

Proof.

For the sake of contradiction, assume that there exists some permutation π of $\{1, 2, ..., k, k+1\}$ that avoids 312, ending with value i such that some integer $x < i$ is to the right of some integer $y > i$. Clearly, π does not avoid 312. (why?)

It follows that for any permutation $\pi \in S_{k+1}$ avoiding 312 and ending in i, all $x < i$ must be to the left of all $y > i$. Hence, π can be written as $\pi_1\pi_2 i$ where π_1 is a permutation of $\{1, 2, \ldots, i-1\}$ and π_2 is a permutation of $\{i+1, i+2, \ldots, k+1\}$. Furthermore, it is clear that any subsequences of the permutation π must avoid 312 if the entire permutation π is to avoid 312 as well; this implies that the permutations π_1 and π_2 must avoid 312 as well. $\square$

Definition

The Catalan numbers are the sequence of positive integers C_i defined as follows,

$$C_0 = 1,\ C_{n+1} = \sum_{i=1}^{n} C_i C_{n-i} \text{ for } n \geq 0$$

Theorem

$s_n(312), s_n(132), s_n(213)$, and $s_n(231)$ are equal to C_n, the n^{th} Catalan number.

Proof.

Assume that for all i from 1 to k, the number of permutations of $\{1, 2, ..., i\}$ that avoid the order 312 as a subsequence is C_i.
Want to prove that the number of permutations of $\{1, 2, ..., k, k+1\}$ that avoid the order 312 as a subsequence is C_{k+1}.
The number of permutations of $\{1, 2, ..., k, k+1\}$ that avoid 312 can be counted by enumerating through all possible values of the last term of a valid permutation.
From the above lemma, all permutations of $\{1, 2, ..., k, k+1\}$ ending in i that avoid 312 are precisely those of the form

$$\pi = \pi_1 \pi_2 i$$

where π_1 is a permutation of $\{1, 2, \ldots, i-1\}$ that avoids 312 and π_2 is a permutation of $\{i+1, i+2, \ldots, k+1\}$ that avoids 312.

Proof (contd.)

It follows that the total number of permutations π avoiding 312 and ending in i is

$$C_{i-1} \cdot C_{k-i+1}$$

(by our induction hypothesis)
Now, summing over all possible values of i, the total number of permutations of $\{1, 2, ..., k+1\}$ that avoid 312 is equal to,

$$\sum_{i=1}^{k+1} C_{i-1} \cdot C_{k-i+1} = \sum_{i=0}^{k} C_i \cdot C_{k-i}$$

which equals C_{k+1}
By the flipping lemma and reversing lemma, $s_n(132)$, $s_n(213)$ and $s_n(231)$ are the sequence of Catalan numbers as well. $\square$

Avoidance in T_n