

18.821 Project 2

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October 27, 2014

1 Introduction

In this paper we will consider investigate pattern avoidance in permutations of finite sets of positive integers.

Definition 1. Two finite sequences a_1, \dots, a_k and b_1, \dots, b_k have the same relative order if the i^{th} largest entry of each unordered set $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ appears in the same position in $a_1 \dots a_k$ and $b_1 \dots b_k$ for each $1 \leq i \leq k$.

Definition 2. A finite sequence of numbers $a_1 a_2 \dots a_n$ avoids a sequence $b_1 \dots b_k$ with $n \geq k$ if no subsequence $a_{i_1} a_{i_2} \dots a_{i_k}$ of $a_1 a_2 \dots a_n$ has its terms in the same relative order as $b_1 \dots b_k$.

Example 1. The sequence 15432 avoids 123 while the sequence 13425 does not. The latter example fails because 134 appears in 15342 with the same relative order as 123.

Remark 1. Let S_n be the set of permutations of $\{1, \dots, n\}$. There are two conventions commonly used to write elements of S_n . The first is *one-line* notation where permutations in S_n are written explicitly, e.g. 43215 is a permutation of $\{1, 2, 3, 4, 5\}$ in S_5 .

The second is *cycle notation*, where a cycle $(a_1 \dots a_n) \in S_n$ with $1 \leq a_i \leq n$ acts on a sequence by sending $a_1 \mapsto a_2, a_2 \mapsto a_3, \dots, a_n \mapsto a_1$. For example, the cycle (14)(23) sends 12345 to 43215. Unless otherwise stated, we will refer to permutations in one-line notation.

Definition 3. When $n \geq k$, a permutation $\sigma \in S_n$ avoids a permutation $\pi \in S_k$ if π and σ avoid each other when written as sequences in one-line notation.

Definition 4. For a permutation π , we write $s_n(\pi)$ for the number of permutations in S_n that avoid π .

In this paper we concern ourselves with the sequences $s_n(\pi)$ for various π . In section 2 we will consider sequence $s_n(\pi)$ for $\pi \in S_3$, and in section 3 we will consider sequence $s_n(\pi)$ for $\pi \in S_4$. In section 4 we consider a variation on the avoidance problem that considers the problem of avoiding sequences in a subset T_n of S_n .

1.1 Generating Trees

Before continuing we introduce generating trees, a tool we can use to study pattern avoidance. The generating tree for the sequence $s_n(\pi)$ is the infinite tree whose vertices are elements of S_n that avoid π . To construct the tree, we define the children of an arbitrary node. Let $\omega \in S_k$ be a node of the generating tree of $s_n(\pi)$. Then the children of ω are elements of S_{k+1} that avoid π and that can be obtained by inserting the integer $k+1$ between integers in ω . Figure 1 is an example of the generating tree of $s_n(312)$.

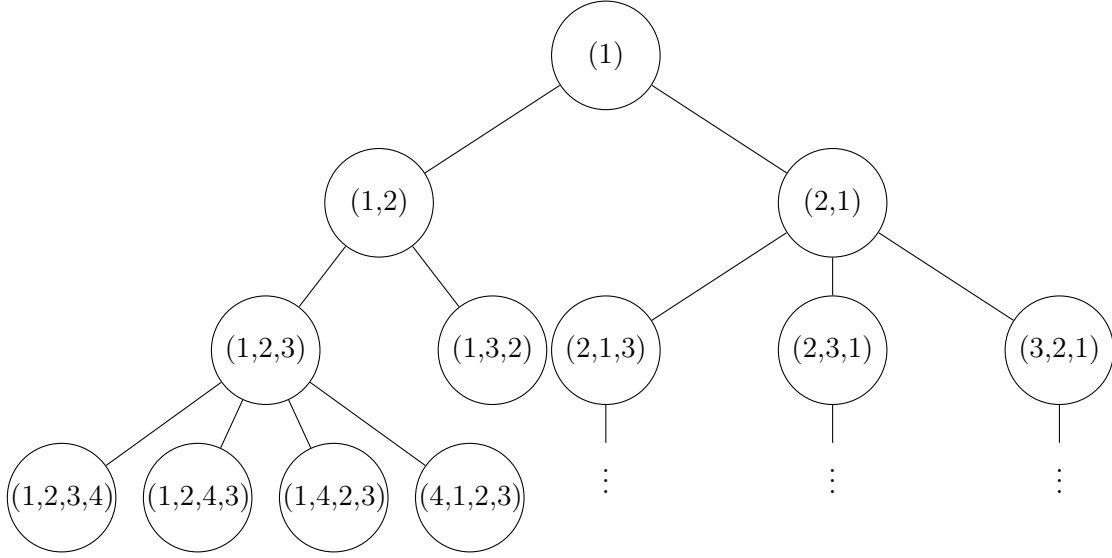


Figure 1: Generating tree of the sequence $s_n(312)$

1.2 Flipping

2 Terms in the sequence $s_n(312)$

We claim that the sequence of numbers $s_n(312)$ is in fact the sequence of Catalan numbers. We state this result formally as the following theorem,

Theorem 5. *The total number of permutations of $\{1, 2, 3, \dots, n\}$ that avoid the order 312 as a subsequence is C_n where C_n is the n^{th} Catalan number.*

Before proving the theorem, we state and prove the following lemma, that will be used in our proof of the theorem.

Lemma 6. *All permutations of $\{1, 2, \dots, k, k+1\}$ ending in i that avoid the order 312 as a sub-sequence must be of the form,*

$$\pi_1 \pi_2 i$$

where π_1 is a permutation of $\{1, 2, \dots, (i-1)\}$ that avoids the order 312 as a sub-sequence and π_2 is a permutation of $\{(i+1), \dots, (k+1)\}$ that avoids the order 312 as a sub-sequence.

Proof. It is clear that any subsequences of the permutation $\pi = \pi_1\pi_2i$ must avoid 312 if the entire permutation π is to avoid 312 as well; this implies that the permutations π_1 and π_2 must avoid 312 as well.

We proceed with a proof by contradiction.

Before proceeding, we define the sets A and B to be $\{1, 2, \dots, (i-1)\}$ and $\{(i+1), (i+2), \dots, (k+1)\}$ respectively. For the sake of contradiction, let us assume that there exists some permutation π of $\{1, 2, \dots, k, k+1\}$ that ends with value i such that some integer $x < i$ (that is, $x \in A$) is to the right of some integer $y > i$ ($y \in B$). Clearly, this permutation is not of the form described above. It is also easy to see that π does *not* avoid the order 312 since the triple (y, x, i) satisfies the condition $y > x > i$ and is in the order 312.

From this we conclude that only permutations of the form described above can avoid 312. □

With this lemma proven, we move on to the proof of our theorem.

Proof. The inductive hypothesis holds for our base case of $\{1\}$, since the only permutation of $\{1\}$ trivially avoids 312.

Now, we need to prove the inductive case. Let us first assume that for all i from 1 to k , the number of permutations of $\{1, 2, \dots, i\}$ that avoid the order 312 as a subsequence is C_i .

Now, we want to prove the inductive hypothesis for $\{1, 2, \dots, k, k+1\}$ as well, that is the number of permutations of $\{1, 2, \dots, k, k+1\}$ that avoid the order 312 as a subsequence is C_{k+1} .

We count the number of permutations of $\{1, 2, \dots, k, k+1\}$ that avoid 312 by enumerating through all possible values of the last term of a valid permutation. If the last term of the permutation is i (where $i \in \{1, 2, \dots, k, k+1\}$), then let us define the subsets A and B of the set $\{1, 2, \dots, k+1\} \setminus \{i\}$ as the set of integers less than i and the set of integers greater than i respectively. It is clear from the definition of A and B that A and B are disjoint from each other.

Now, from the above lemma, we know that all permutations of $\{1, 2, \dots, k, k+1\}$ ending in i that avoid 312 must be of the form,

$$\pi = \pi_1\pi_2i$$

where π_1 is a permutation of A that avoids the order 312 as a sub-sequence and π_2 is a permutation of B that avoids the order 312 as a sub-sequence. It is clear that the above permutation contains all integers between 1 and $k+1$, from the definitions of the subsets A and B , which implies that $\pi_1\pi_2i$ is permutation of the set $\{1, 2, \dots, k, k+1\}$.

Now, the total number of permutations π is,

$$n_{\pi_1} \cdot n_{\pi_2} = C_{i-1} \cdot C_{k-i+1}$$

since the total number of valid permutations π_1 is simply going to be C_{i-1} (total number of valid permutations of length $i-1$ that avoid the order 312 as a sub-sequence is C_{i-1} ; similarly $n_{\pi_2} = C_{k-i+1}$)

Now, summing over all possible values of i , we see that the total number of permutations of $\{1, 2, \dots, k+1\}$ that avoid 312 is equal to,

$$\sum_{i=1}^{k+1} C_{i-1} \cdot C_{k-i+1} = \sum_{i=0}^k C_i \cdot C_{k-i}$$

which is in fact C_{k+1} , and we are done. □

3 Conjectures on $s_n(\pi)$ for $\pi \in S_4$

In this section we will present our findings and conjectures about sequences $s_n(\pi)$ for $\pi \in S_4$. We computed values up to s_{10} . It appears that there are three sequences that appear, and we have sorted the permutations according to which sequence they produce and paired them with their flips.

We define the following sequences based on the first 8 terms:

$$A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \dots$$

$$B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \dots$$

$$C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \dots$$

The table below lists the elements of S_4 generate sequences A, B, C for the first 8 terms.

B	A	C
1234, 4321	4132, 1423	4231, 1324
1243, 4312	4213, 1342	
1432, 4123	2431, 3124	
2134, 3421	2413, 3142	
2143, 3412	2314, 3241	
2341, 3214		

There is no guarantee that these sequences will not diverge and produce more than three distinct sequences, however for now we hypothesize that A , B , and C are the only sequences that arise for $s_n(\pi)$ with $\pi \in S_4$. Assuming this hypothesis, if we write each length four permutation in cycle notation, a pattern begins to emerge.

B	A	C
(1),(14)(23)	(243),(142)	(23),(14)
(34),(1423)	(234),(143)	
(24),(1432)	(124),(132)	
(12),(1324)	(123),(134)	
(12)(34),(13)(24)	(1243),(1342)	
(1234),(13)		

So we conjecture that sequence A is characterized by elements of S_4 that are three cycles or four cycles when written in cycle notation. Sequence B seems to be characterized by either a pair of disjoint two cycles paired with a flip that is a 2-cycle, or a single 4-cycle paired with a flip that is a 2-cycle. Sequence C appears to be characterized by a 2-cycle paired with a 2-cycle as a flip.

4 Avoidance of permutations of T_n

We now impose further restrictions on the set of permutations S_n . This new set of permutations, T_n is defined follows.

Definition 7. For even n , T_n is defined as the set of all permutations $\sigma \in S_n$ for which $1, 3, 5, \dots, 2n-1$ appear in increasing order, and $2i$ always appears to the right of $2i-1$.

Since T_n is only defined for even n , henceforth we shall refer to T_n as T_{2m} where m is any integer greater than or equal to 0.

Given the above definition of T_{2m} , we prove the following result about the cardinality of T_{2m} .

Theorem 8. $|T_{2m}| = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2m-1)$

Proof. Observe that since $1, 3, 5, \dots, 2m-1$ must appear in increasing order in T_{2m} , our problem is now reduced to determining the relative order of $2, 4, 6, \dots, 2m$ with respect to each other, as well as with respect to $1, 3, 5, \dots, 2m-1$.

Let us first try to insert $2m$ into the sequence $1, 3, 5, \dots, 2m-1$. Clearly, the way T_{2m} is defined, $2m$ can be inserted into only 1 slot, the one following $2m-1$. (Here we define a slot as a gap between two existing elements of the sequence, or the gap that follows the last element of the sequence or the gap that precedes the first element of the sequence)

Now, let us try to insert $(2m-2)$ into the sequence. $(2m-2)$ can be inserted into 3 slots in the sequence – the one between $2m-3$ and $2m-1$, the one between $2m-1$ and $2m$ or the one following $2m$.

Now, if we try to insert $(2m-4)$ into this incompletely formed sequence, we will see that there are 5 possible locations into which this number can be inserted. Continuing this for all even numbers upto 2, we observe that the total number of ways such a sequence can be created is equal to $1 \cdot 3 \cdot 5 \cdot \dots \cdot (2m-1)$, as desired. \square