

Pattern avoidance

An explanation and proof

Yajit Jain, Deepak Narayanan and Leon Zhang

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A permutation of a finite set $\{1, \dots, n\}$ is some *ordering* of the elements.

54123 is a permutation of $\{1, 2, 3, 4, 5\}$.

S_n is the set of permutations on $\{1, \dots, n\}$.

$$54123 \in S_5$$

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54123

includes

$$\left\{ \begin{array}{l} 123 \\ 312 \\ 4312 \end{array} \right.$$

avoids

$$\left\{ \begin{array}{l} 132 \\ 312 \\ 213 \\ 231 \end{array} \right.$$

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Let $\pi = 312 \in S_3$.

- Question: How many permutations avoid π ? (a lot)
- Better Question: How many permutations in S_n avoid π ?

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- How many permutations in S_1 avoid π ? 1
- How many permutations in S_2 avoid π ? 2
- How many permutations in S_3 avoid π ? 5
- How many permutations in S_4 avoid π ? ??????

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Permutations in S_4 that avoid $\pi = 312$?

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How many permutations in S_4 avoid π ? 14

1234 1243 1324 1342 1423 1432

2134 2143 2314 2341 2413 2431

3124 3142 3214 3241 3412 3421

4123 4132 4213 4231 4312 4321

Definition

Let $a_n(\pi)$ be the number of permutations in S_n that avoid π .

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We want to compute the sequences $(a_n(\pi))$ for some $\pi \in S_k$.

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Example: $(a_n(312)) = 1, 2, 5, 14,$

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Definition

Let $a_n(\pi)$ be the number of permutations in S_n that avoid π .

We want to compute the sequences $(a_n(\pi))$ for some $\pi \in S_k$.

Example: $(a_n(312)) = 1, 2, 5, 14, 42, 132, 429, \dots$

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Theorem

For $\pi \in S_3$, $(a_n(\pi))$ is equal to the Catalan numbers:

$$(a_n(\pi)) = 1, 2, 5, 14, 42, 132, 429 \dots$$

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$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \dots \end{cases}$$

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$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \dots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \dots \end{cases}$$

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Let's first look at some examples of permutations that don't avoid 312!

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Example

1 2 6 5 3 4

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Example

1 2 6 5 3 4 \implies 126534 does not avoid 312

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Example

1 2 6 5 3 4 \implies 126534 does not avoid 312

Example

1 5 6 3 2 4

Let's first look at some examples of permutations that don't avoid 312!

Example

1 2 **6** 5 **3** **4** \implies 126534 does not avoid 312

Example

1 **5** 6 **3** 2 **4** \implies 156324 does not avoid 312

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How about some permutations that do avoid 312?

How about some permutations that do avoid 312?

Example

1 2 3 6 5 4

How about some permutations that do avoid 312?

Example

1 2 3 6 5 4 \implies 123654 avoids 312

How about some permutations that do avoid 312?

Example

1 2 3 6 5 4 \implies 123654 avoids 312

Example

2 1 4 5 6 3

How about some permutations that do avoid 312?

Example

1 2 3 6 5 4 \implies 123654 avoids 312

Example

2 1 4 5 6 3 \implies 214563 avoids 312

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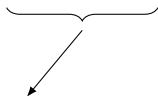
Do the permutations that avoid 312 have any special properties?

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1 2 3 6 5 4

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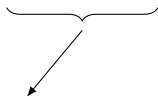
1 2 3 6 5 4



All < 4 , and avoid 312

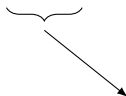
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1 2 3



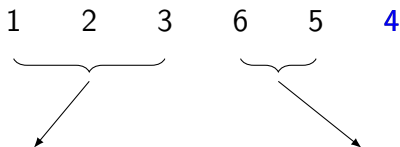
All < 4 , and avoid 312

6 5 4



All > 4 , and avoid 312

Do the permutations that avoid 312 have any special properties?



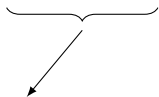
All < 4 , and avoid 312

All > 4 , and avoid 312

2 1 4 5 6 3

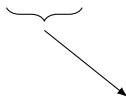
Do the permutations that avoid 312 have any special properties?

1 2 3



All < 4 , and avoid 312

6 5 4



All > 4 , and avoid 312

2 1 4



All < 3 , and avoid 312

5 6 3

Do the permutations that avoid 312 have any special properties?

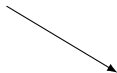
1 2 3 6 5 4



All < 4 , and avoid 312

All > 4 , and avoid 312

2 1 4 5 6 3



All < 3 , and avoid 312

All > 3 , and avoid 312

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What happens with permutations that don't have this property?

What happens with permutations that don't have this property?

1 2 6 5 3 4

What happens with permutations that don't have this property?

1 2 6 5 3 4

All numbers < 4 in blue

All numbers > 4 in green

What happens with permutations that don't have this property?

1 2 6 5 3 4

All numbers < 4 in blue

All numbers > 4 in green



1 2 6 5 3 4

What happens with permutations that don't have this property?

1 2 6 5 3 4

All numbers < 4 in blue

All numbers > 4 in green



1 2 6 5 3 4

Doesn't avoid 312 anymore!

Lemma

The permutations of $\{1, 2, \dots, k, k + 1\}$ ending in i that avoid the pattern 312 are precisely those of the form,

$$\pi_1 \pi_2 i$$

the concatenation of π_1, π_2 , and i , where π_1 is a permutation of $\{1, 2, \dots, i - 1\}$ that avoids the pattern 312 and π_2 is a permutation of $\{i + 1, \dots, k + 1\}$ that avoids the pattern 312.

Definition

The Catalan numbers are the sequence of positive integers C_i defined as follows,

$$C_0 = 1, \quad C_{n+1} = \sum_{i=0}^n C_i C_{n-i} \text{ for } n \geq 0$$

Theorem

The n^{th} term of the sequence $a_n(312)$ is equal to C_n , the n^{th} Catalan number, for $n > 0$.

Proof.

Assume that for all i from 1 to k , the number of permutations of $\{1, 2, \dots, i\}$ that avoid 312 is C_i .

It follows from the above lemma that the total number of permutations π avoiding 312 and ending in i is

$$C_{i-1} \cdot C_{k-i+1}$$

Summing over all possible values of i , the total number of permutations of $\{1, 2, \dots, k+1\}$ that avoid 312 is equal to,

$$\sum_{i=1}^{k+1} C_{i-1} \cdot C_{k-i+1} = \sum_{i=0}^k C_i \cdot C_{k-i} = C_{k+1}$$

The Reversing Lemma

Definition (Reversing)

We define the *reverse* of a permutation $b_1 \cdots b_n$ to be the permutation $b_n \cdots b_1$. The reversing operator is denoted by \mathcal{R} .

The Reversing Lemma

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Example

$$\mathcal{R}(1324) = 4231.$$

The Reversing Lemma

Definition (Reversing)

We define the *reverse* of a permutation $b_1 \cdots b_n$ to be the permutation $b_n \cdots b_1$. The reversing operator is denoted by \mathcal{R} .

Example

$$\mathcal{R}(1324) = 4231.$$

Example

$$\mathcal{R}(1243) = 3421.$$

The Reversing Lemma

Lemma (Reversing Lemma)

The permutation σ avoids the permutation π iff $\mathcal{R}(\sigma)$ avoids $\mathcal{R}(\pi)$.

Corollary

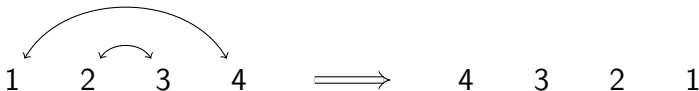
For a permutation π , $a_n(\pi) = a_n(\mathcal{R}(\pi))$.

Definition (Flipping)

We define the *flip* of a sequence b as the sequence c with the same elements as b , but with the largest element swapped with the smallest element, the second largest element swapped with the second smallest element, etc. The flipping operator is denoted by \mathcal{F} .

The Flipping Lemma

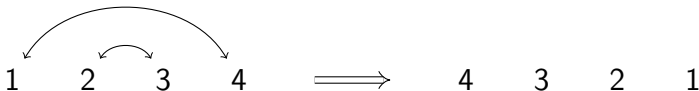
Example



$$\mathcal{F}(1234) = 4321$$

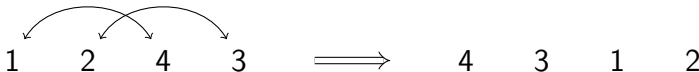
The Flipping Lemma

Example



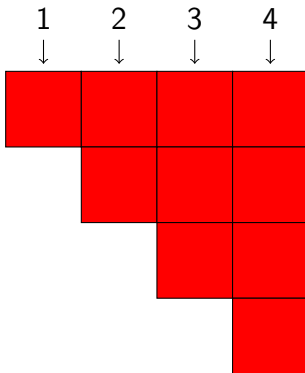
$$\mathcal{F}(1234) = 4321$$

Example

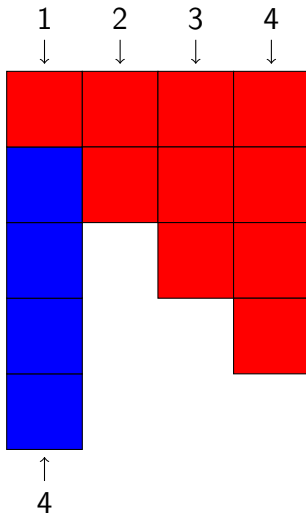


$$\mathcal{F}(1243) = 4312$$

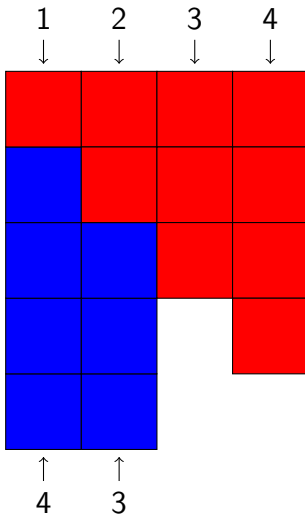
The Flipping Lemma



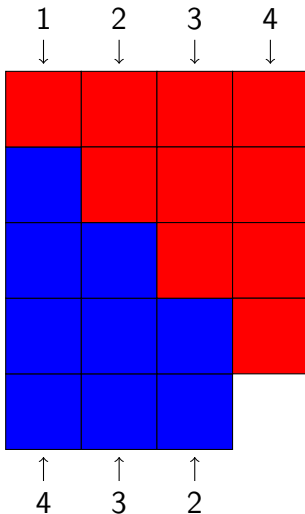
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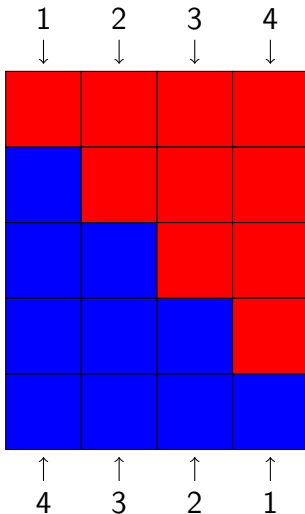
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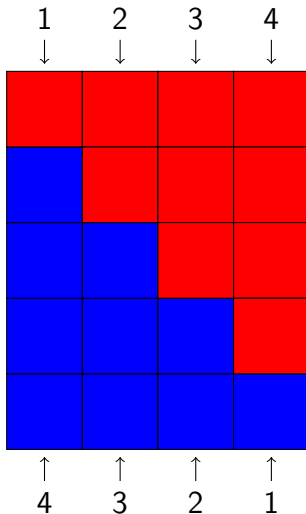
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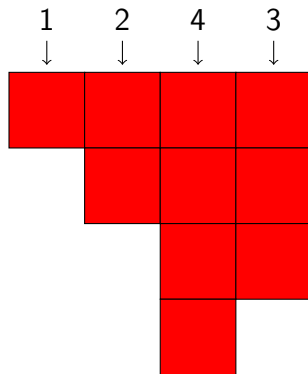
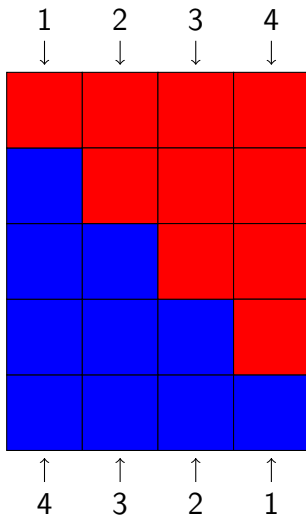


The Flipping Lemma



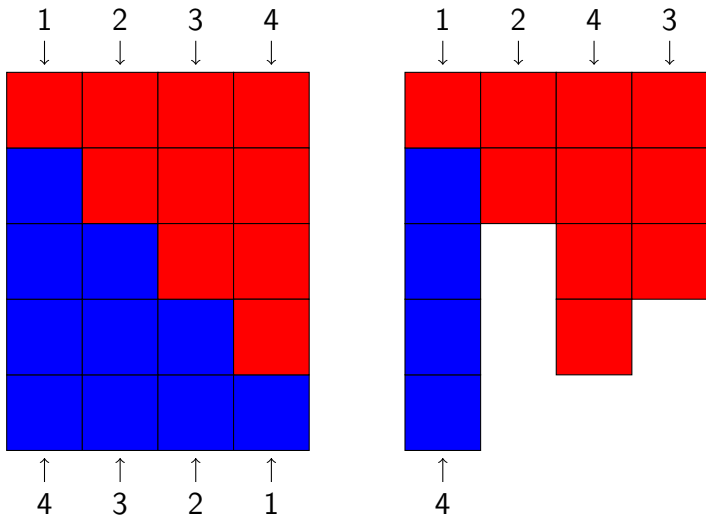
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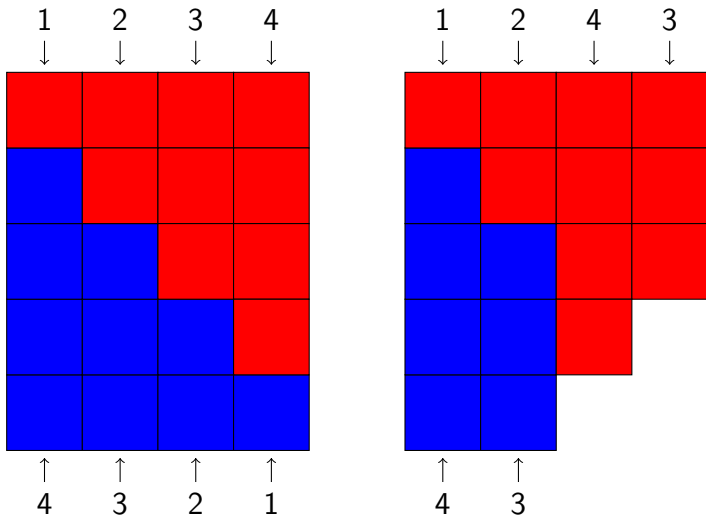
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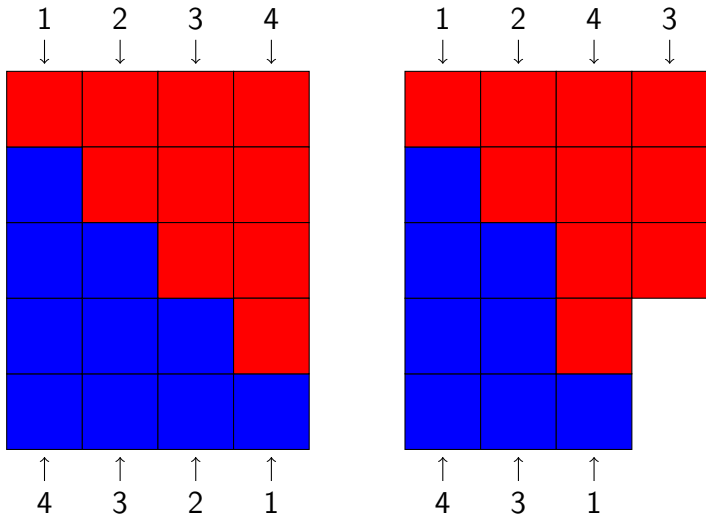
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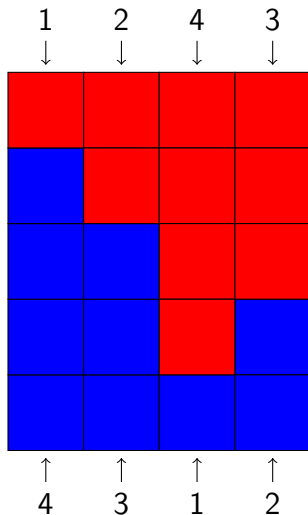
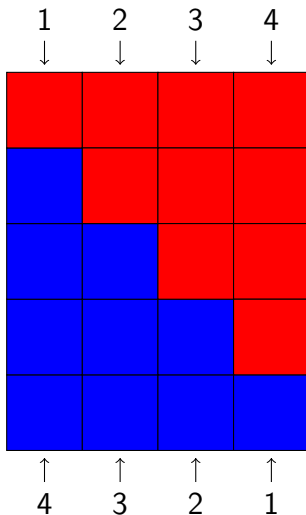
$$\mathcal{F}(1234) = 4321$$

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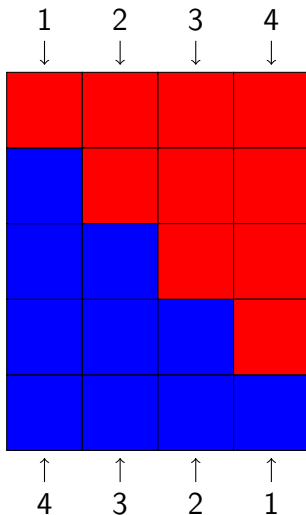
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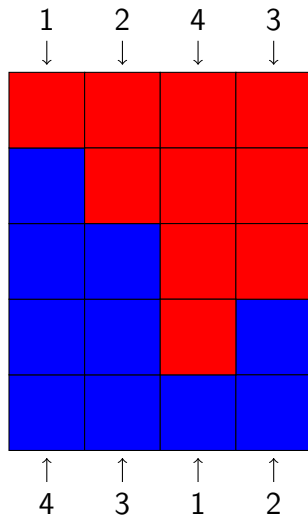


$$\mathcal{F}(1234) = 4321$$

The Flipping Lemma



$$\mathcal{F}(1234) = 4321$$



$$\mathcal{F}(1243) = 4312$$

The Flipping Lemma

Lemma (Flipping Lemma)

The permutation σ avoids the permutation π iff $\mathcal{F}(\sigma)$ avoids $\mathcal{F}(\pi)$.

Corollary

For a permutation π , $a_n(\pi) = a_n(\mathcal{F}(\pi))$.

Avoidance of other permutations in S_3

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- From the Flipping Lemma and Reversing Lemmas, the sequences $(a_n(213))$, $(a_n(132))$ and $(a_n(231))$ are the sequence of Catalan numbers as well.
- However, it is much harder to prove that the sequences $(a_n(123))$ and $(a_n(321))$ are the sequence of Catalan numbers.

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \dots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \dots \\ C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \dots \end{cases}$$

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1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

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$$\begin{aligned} &\{1243, 4312, 2134, 3421\}, \{2413, 3142\}, \\ &\{1432, 4123, 2341, 3214\}, \{1234, 4321\}, \\ &\{4132, 1423, 2314, 3241\}, \{2143, 3412\}, \\ &\{4213, 1342, 3124, 2431\}, \{4231, 1324\} \end{aligned}$$

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \dots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \dots \\ C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \dots \end{cases}$$

B	A	C
1234, 4321	4132, 1423	4231, 1324
1243, 4312	4213, 1342	
1432, 4123	2431, 3124	
2134, 3421	2413, 3142	
2143, 3412	2314, 3241	
2341, 3214		

Take $\sigma \in S_5$ with

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

One-line Notation

$$\sigma = 45213$$

Cycle Notation

$$\sigma = (14)(253)$$

Take $\sigma \in S_5$ with

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

One-line Notation

$$\sigma = 45213$$

Cycle Notation

$$\sigma = (14)(253)$$

Take $\sigma \in S_5$ with

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

One-line Notation

$$\sigma = 45213$$

Cycle Notation

$$\sigma = (14)(253)$$

Take $\sigma \in S_5$ with

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

One-line Notation

$$\sigma = 45213$$

Cycle Notation

$$\sigma = (14)(253)$$

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \dots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \dots \\ C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \dots \end{cases}$$

B	A	C
(1)(2)(3)(4), (14)(23)	(243), (142)	(23), (14)
(34), (1423)	(234), (143)	
(24), (1432)	(124), (132)	
(12), (1324)	(123), (134)	
(12)(34), (13)(24)	(1243), (1342)	
(1234), (13)		

Definition

Let m be a positive integer. The set T_{2m} is defined as all permutations in S_{2m} such that:

- the odd numbers appear in increasing order,
- each even number $2i$ appears to the right of $2i - 1$.

Example

The set $T_2 \subset S_2$ consists of the single permutation 12.
The other permutation in S_2 , 21, is not in T_2 .

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Introduction

Avoidance in S_n

Avoidance of 312

The Reversing
Lemma

The Flipping Lemma

Avoidance of other
permutations in S_3 Conjectures on S_4 Avoidance in T_n

1234	1243	1324	1342	1423	1432
2134	2143	2314	2341	2413	2431
3124	3142	3214	3241	3412	3421
4123	4132	4213	4231	4312	4321

Definition

Given a permutation $\pi \in S_k$, we define $t_m(\pi)$ as

$$t_m(\pi) = \#\{\sigma \in T_{2m} \mid \sigma \text{ avoids } \pi\}.$$

Problem

*Let $\pi \in S_3$, and m an arbitrary positive integer.
Compute $t_m(\pi)$.*

We can run code to compute $t_m(\pi)$ for small m and for each $\pi \in S_3$. We get

π	$m = 2$	$m = 4$	$m = 6$	$m = 8$	$m = 10$
123	1	0	0	0	0
132	1	1	1	1	1
213	1	2	4	8	16
231	1	2	4	8	16
312	1	3	12	55	273
321	1	3	12	55	273

- Easy to see that $t_n(123) = 0$ when $n \geq 2$
 - The subsequence 134 is always present in a permutation $\sigma \in t_n(123)$.
- Also easy to see that $t_n(132) = 1$.
 - The permutation $123 \dots (2n)$ is the only permutation in T_n that avoids 132.