

18.821 Project 2

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1 Terms in the sequence $s_n(312)$

We claim that the sequence of numbers $s_n(312)$ is in fact the sequence of Catalan numbers. We state this result formally as the following theorem,

Theorem 1. *The total number of permutations of $\{1, 2, 3, \dots, n\}$ that avoid the order 312 as a subsequence is C_n where C_n is the n^{th} Catalan number.*

Before proving the theorem, we state and prove the following lemma, that will be used in our proof of the theorem.

Lemma 2. *All permutations of $\{1, 2, \dots, k, k+1\}$ ending in i that avoid the order 312 as a sub-sequence must be of the form,*

$$\pi_1 \pi_2 i$$

where π_1 is a permutation of $\{1, 2, \dots, (i-1)\}$ that avoids the order 312 as a sub-sequence and π_2 is a permutation of $\{(i+1), \dots, (k+1)\}$ that avoids the order 312 as a sub-sequence.

Proof. It is clear that any subsequences of the permutation $\pi = \pi_1 \pi_2 i$ must avoid 312 if the entire permutation π is to avoid 312 as well; this implies that the permutations π_1 and π_2 must avoid 312 as well.

We proceed with a proof by contradiction.

Before proceeding, we define the sets A and B to be $\{1, 2, \dots, (i-1)\}$ and $\{(i+1), (i+2), \dots, (k+1)\}$ respectively. For the sake of contradiction, let us assume that there exists some permutation π of $\{1, 2, \dots, k, k+1\}$ that ends with value i such that some integer $x < i$ (that is, $x \in A$) is to the right of some integer $y > i$ ($y \in B$). Clearly, this permutation is not of the form described above. It is also easy to see that π does *not* avoid the order 312 since the triple (y, x, i) satisfies the condition $y > x > i$ and is in the order 312.

From this we conclude that only permutations of the form described above can avoid 312.

□

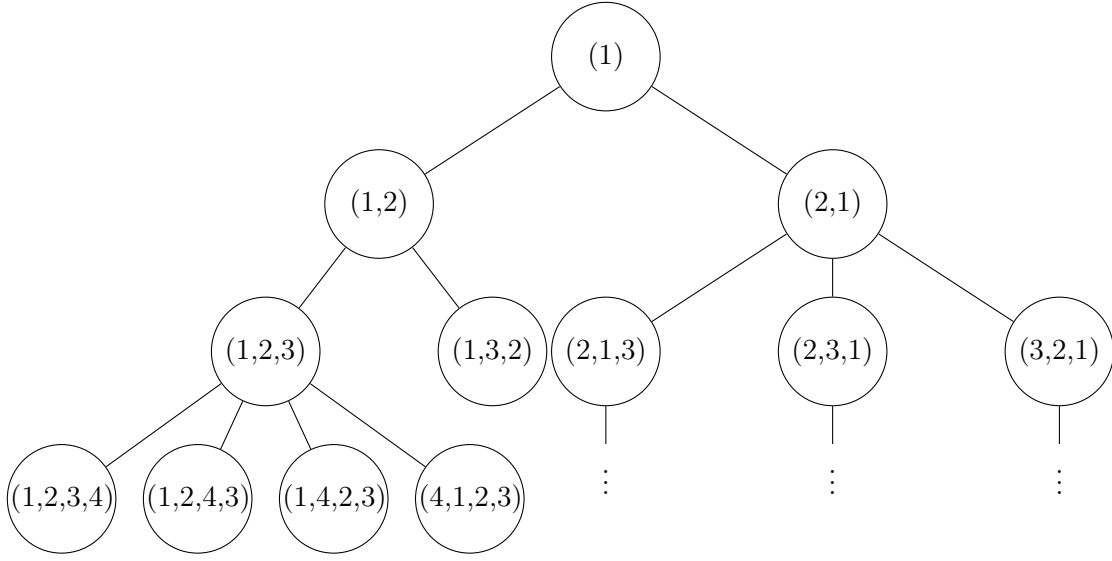


Figure 1: Generating tree of the sequence $s_n(312)$

With this lemma proven, we move on to the proof of our theorem.

Proof. The inductive hypothesis holds for our base case of $\{1\}$, since the only permutation of $\{1\}$ trivially avoids 312.

Now, we need to prove the inductive case. Let us first assume that for all i from 1 to k , the number of permutations of $\{1, 2, \dots, i\}$ that avoid the order 312 as a subsequence is C_i .

Now, we want to prove the inductive hypothesis for $\{1, 2, \dots, k, k+1\}$ as well, that is the number of permutations of $\{1, 2, \dots, k, k+1\}$ that avoid the order 312 as a subsequence is C_{k+1} .

We count the number of permutations of $\{1, 2, \dots, k, k+1\}$ that avoid 312 by enumerating through all possible values of the last term of a valid permutation. If the last term of the permutation is i (where $i \in \{1, 2, \dots, k, k+1\}$), then let us define the subsets A and B of the set $\{1, 2, \dots, k+1\} \setminus \{i\}$ as the set of integers less than i and the set of integers greater than i respectively. It is clear from the definition of A and B that A and B are disjoint from each other.

Now, from the above lemma, we know that all permutations of $\{1, 2, \dots, k, k+1\}$ ending in i that avoid 312 must be of the form,

$$\pi = \pi_1 \pi_2 i$$

where π_1 is a permutation of A that avoids the order 312 as a sub-sequence and π_2 is a permutation of B that avoids the order 312 as a sub-sequence. It is clear that the above permutation contains all integers between 1 and $k+1$, from the definitions of the subsets A and B , which implies that $\pi_1 \pi_2 i$ is permutation of the set $\{1, 2, \dots, k, k+1\}$.

Now, the total number of permutations π is,

$$n_{\pi_1} \cdot n_{\pi_2} = C_{i-1} \cdot C_{k-i+1}$$

since the total number of valid permutations π_1 is simply going to be C_{i-1} (total number of valid permutations of length $i-1$ that avoid the order 312 as a sub-sequence is C_{i-1} ; similarly $n_{\pi_2} = C_{k-i+1}$)

Now, summing over all possible values of i , we see that the total number of permutations of $\{1, 2, \dots, k+1\}$ that avoid 312 is equal to,

$$\sum_{i=1}^{k+1} C_{i-1} \cdot C_{k-i+1} = \sum_{i=0}^k C_i \cdot C_{k-i}$$

which is in fact C_{k+1} , and we are done. □

2 Avoidance of permutations of T_n

We now impose further restrictions on the set of permutations S_n . This new set of permutations, T_n is defined follows.

Definition 3. For even n , T_n is defined as the set of all permutations $\sigma \in S_n$ for which $1, 3, 5, \dots, 2n-1$ appear in increasing order, and $2i$ always appears to the right of $2i-1$.

Since T_n is only defined for even n , henceforth we shall refer to T_n as T_{2m} where m is any integer greater than or equal to 0.

Given the above definition of T_{2m} , we prove the following result about the cardinality of T_{2m} .

Theorem 4. $|T_{2m}| = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2m-1)$

Proof. Observe that since $1, 3, 5, \dots, 2m-1$ must appear in increasing order in T_{2m} , our problem is now reduced to determining the relative order of $2, 4, 6, \dots, 2m$ with respect to each other, as well as with respect to $1, 3, 5, \dots, 2m-1$.

Let us first try to insert $2m$ into the sequence $1, 3, 5, \dots, 2m-1$. Clearly, the way T_{2m} is defined, $2m$ can be inserted into only 1 slot, the one following $2m-1$. (Here we define a slot as a gap between two existing elements of the sequence, or the gap that follows the last element of the sequence or the gap that precedes the first element of the sequence)

Now, let us try to insert $(2m-2)$ into the sequence. $(2m-2)$ can be inserted into 3 slots in the sequence – the one between $2m-3$ and $2m-1$, the one between $2m-1$ and $2m$ or the one following $2m$.

Now, if we try to insert $(2m-4)$ into this incompletely formed sequence, we will see that there are 5 possible locations into which this number can be inserted. Continuing this for all even numbers unto 2, we observe that the total number of ways such a sequence can be created is equal to $1 \cdot 3 \cdot 5 \cdot \dots \cdot (2m-1)$, as desired. □