Pattern avoidance

An explanation and proof

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Plan of action

1. Definition, Introduction and Motivation
2. Avoidance in S_n
3. Avoidance in T_n

Definition, Introduction and Motivation

Lemma (Reversing Lemma)

The permutation σ avoids the permutation π iff $\mathcal{R}(\sigma)$ avoids $\mathcal{R}(\pi)$.

Proof.

Need to only prove one direction (why?).
Suppose that $\mathcal{R}(\sigma)$ does not avoid $\mathcal{R}(\pi)$ if σ avoids π.
Reversing $\mathcal{R}(\sigma)$ again produces a contradiction (why?), which implies that $\mathcal{R}(\sigma)$ avoids $\mathcal{R}(\pi)$ if σ avoids π. $\square$

Corollary

For a permutation π, $s_n(\pi) = s_n(\mathcal{R}(\pi))$.

Definition (Flip of a sequence)

We define the *flip* of a sequence a as the sequence b with the same elements as a, but with the largest element swapped with the smallest element, the second largest element swapped with the second smallest element, etc. The flipping operator is denoted by $\mathcal{F}$.

Example

$\mathcal{F}(1324) = 4231.$

Example

$\mathcal{F}(1243) = 4312.$

Lemma (Flipping Lemma)

The permutation σ avoids the permutation π iff $\mathcal{F}(\sigma)$ avoids $\mathcal{F}(\pi)$.

Proof.

As before, need to only prove one direction.
Suppose $\mathcal{F}(\sigma)$ does not avoid $\mathcal{F}(\pi)$ if σ avoids π. (Here, $\sigma \in S_n$ and $\pi \in S_k$)
Let $\mathcal{F}(\sigma) = a_1 \cdots a_n$ and let $\mathcal{F}(\pi) = b_1 \cdots b_k$. Then there is some subsequence $a_{i_1} \cdots a_{i_k}$ such that $a_{i_c} < a_{i_d}$ iff $b_c < b_d$.
But then, $n + 1 - a_{i_c} > n + 1 - a_{i_d}$ iff $k + 1 - b_c > k + 1 - b_d$.
This implies that σ does not avoid π, a contradiction (why?). □

Corollary

For a permutation π, $s_n(\pi) = s_n(\mathcal{F}(\pi))$.

Lemma

The permutations of $\{1, 2, \ldots, k, k+1\}$ *ending in* i *that avoid the pattern* 312 *are precisely those of the form,*

$$\pi_1 \pi_2 i$$

the concatenation of π_1, π_2, *and* i, *where* π_1 *is a permutation of* $\{1, 2, \ldots, i-1\}$ *that avoids the pattern* 312 *and* π_2 *is a permutation of* $\{i+1, \ldots, k+1\}$ *that avoids the pattern* 312.

Proof.

Assume that $\exists$ some permutation π of $\{1, 2, ..., k, k+1\}$ that avoids 312, ending with value i such that some integer $x < i$ is to the right of some integer $y > i$. Clearly, π does not avoid 312. (why?)

It follows that for any permutation π avoiding 312 and ending in i, all $x < i$ must be to the left of all $y > i$. Hence, π can be written as

$$\pi_1 \pi_2 i$$

where π_1 is a permutation of $\{1, 2, \ldots, i-1\}$ and π_2 is a permutation of $\{i+1, i+2, \ldots, k+1\}$.

Furthermore, it is clear that the permutations π_1 and π_2 must avoid 312 as well. (why?) $\square$

Definition

The Catalan numbers are the sequence of positive integers C_i defined as follows,

$$C_0 = 1,\ C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \text{ for } n \geq 0$$

Theorem

$s_n(312), s_n(132), s_n(213)$, and $s_n(231)$ are equal to C_n, the n^{th} Catalan number.

Proof.

Assume that for all i from 1 to k, the number of permutations of $\{1, 2, ..., i\}$ that avoid the order 312 as a subsequence is C_i.
The number of permutations of $\{1, 2, ..., k, k+1\}$ that avoid 312 can be counted by enumerating through all possible values of the last term of a valid permutation.
From the above lemma, all permutations of $\{1, 2, ..., k, k+1\}$ ending in i that avoid 312 are precisely those of the form

$$\pi = \pi_1 \pi_2 i$$

where π_1 is a permutation of $\{1, 2, \ldots, i-1\}$ that avoids 312 and π_2 is a permutation of $\{i+1, i+2, \ldots, k+1\}$ that avoids 312.

Proof (contd.)

It follows that the total number of permutations π avoiding 312 and ending in i is

$$C_{i-1} \cdot C_{k-i+1}$$

Summing over all possible values of i, the total number of permutations of $\{1, 2, ..., k+1\}$ that avoid 312 is equal to,

$$\sum_{i=1}^{k+1} C_{i-1} \cdot C_{k-i+1} = \sum_{i=0}^{k} C_i \cdot C_{k-i} = C_{k+1}$$

By the flipping lemma and reversing lemma, $s_n(132)$, $s_n(213)$ and $s_n(231)$ are the sequence of Catalan numbers as well. □

Avoidance in T_n

Definition

Let $n = 2m$. Then

$$T_n = \{\sigma \in S_n \mid 1, 3, 5, \ldots, 2n-1 \text{ appear in increasing order, and } 2i \text{ is always to the right of 2i-1}\}.$$

Example

The set $T_2 \subset S_2$ consists of the single permutation 12.

Example

The set $T_4 \subset S_4$ consists of the permutations $1234, 1324$, and 1342.

Example

The set $T_6 \subset S_6$ consists of the following permutations:

$$T_n = \{123564, 123456, 123546, 132564, 132456, 132546, 135264, 134256, 135246, 135624, 134526, 135426, 135642, 134562, 135462\}.$$

Theorem

Recall that $n = 2m$. The set T_n has size

$$\#T_n = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot 2m - 1.$$

Proof.

1. $\#T_n$ given by the number of valid ways to insert $2, 4, \ldots, 2m$ into the sequence $1, 3, \ldots, 2m - 1$.
2. Insert in reverse order: first $2m$, then $2m - 2$, and so on.
3. Only one way to insert $2m$. Then three ways to insert $2m - 2$. Then five ways to insert $2m - 4$. Repeat, until there are $2m - 1$ ways to insert 2.

□

Lemma

Let $\sigma \in T_n$, and let $a, b \in \{1, 2, \ldots, n\}$ with a to the left of b. If $a > b$, then b is even.

Proof.

Two possibilities for a:

- a is odd. Then b is not odd, as odd integers must appear in increasing order.
- a is even. Then
  - $a - 1$ is to the left of a, so to the left of b
  - Since $a - 1 \geq b$, and $a - 1 \neq b$, we have $a - 1 > b$.

  and we have reduced to the previous case.

□

Problem Statement

Definition

Given a permutation $\pi \in S_k$, we define $t_n(\pi)$ as

$$t_n(\pi) = \#\{\sigma \in T_n \mid \sigma \text{ avoids } \pi\}.$$

Problem

Let $\pi \in S_3$, and n an arbitrary positive integer. Compute $t_n(\pi)$.

We can run code to compute $t_n(\pi)$ for small n and for each $\pi \in S_3$. We get

π	$n = 2$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
123	1	0	0	0	0
132	1	1	1	1	1
213	1	2	4	8	16
231	1	2	4	8	16
312	1	3	12	55	273
321	1	3	12	55	273

- Easy to see that $t_n(123) = 0$ when $n \geq 2$
    - The subsequence 134 is always present in a permutation $\sigma \in t_n(123)$.
- Also easy to see that $t_n(132) = 1$.
    - The permutation $123 \dots (2n)$ is the only permutation in T_n that avoids 132.
- What about the other permutations in S_3?

Theorem

The total number of permutations in T_n avoiding 231, $t_n(231)$, is equal to $2^{\frac{n}{2}-1}$.

Proof outline.

Fix an arbitrary $\sigma \in T_n$. Consider an arbitrary $2i \in \{1, 2, \ldots, n\}$, and define

$$A = \{a \in \{1, 2, \ldots, n\} \mid a > 2i \text{ and } a \text{ to the left of 2i}\}.$$

Proof steps:

- The set A contains no even numbers.
- The set A contains no odd numbers greater than $2i + 1$.
- Any integer to the left of $2i$ must be less than $2i$, unless it is $2i + 1$.

Proof outline (Continued).

- Use induction to show that either
  - $2i$ is in the $2i$th position, with $2i+1$ in the $2i+1$st position, or
  - $2i+1$ is in the $2i$th position, with $2i$ in the $2i+1$st position.
- Conclude that there are $2^{\frac{n}{2}-1}$ options for σ.

$\square$

Other results and conjectures

Theorem

The total number of permutations in T_n avoiding 213, $t_n(213)$, is equal to $2^{\frac{n}{2}-1}$.

Theorem

There exists a bijection between the set of permutations in T_n avoiding 312 and the set of permutations in T_n avoiding 321. As a consequence, $t_n(312) = t_n(321)$.

Conjecture.

The total number of permutations in T_n avoiding 312 or 321, $t_n(312)$ and $t_n(321)$ respectively, is given by the equation $\binom{3n/2}{n/2}/(n+1)$.