

Pattern avoidance

An explanation and proof

Yajit Jain, Deepak Narayanan and Leon Zhang

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Plan of action

- ➊ Definition, Introduction and Motivation
- ➋ Avoidance in S_n
- ➌ Avoidance in T_n

Definition, Introduction and Motivation

Lemma (Reversing Lemma)

The permutation σ avoids the permutation π if and only if $\mathcal{R}(\sigma)$ avoids $\mathcal{R}(\pi)$.

Proof.

Need to only prove one direction (why?).

Suppose that $\mathcal{R}(\sigma)$ does not avoid $\mathcal{R}(\pi)$ if σ avoids π .

Then if $\mathcal{R}(\sigma)$ is reversed again again, the subsequence of $\mathcal{R}(\sigma)$ that follows the same pattern as $\mathcal{R}(\pi)$ will reverse to a subsequence that follows the same pattern as π – a contradiction.

Therefore $\mathcal{R}(\sigma)$ avoids $\mathcal{R}(\pi)$.



Corollary

For a permutation π , $s_n(\pi) = s_n(\mathcal{R}(\pi))$.

Definition (Flip of a sequence)

We define the *flip* of a sequence a as the sequence b with the same elements as a , but with the largest element swapped with the smallest element, the second largest element swapped with the second smallest element, etc. The flipping operator will be denoted by \mathcal{F} .

Example

$$\mathcal{F}(1324) = 4231.$$

Example

$$\mathcal{F}(1243) = 4312.$$

Lemma (Flipping Lemma)

The permutation σ avoids the permutation π if and only if $\mathcal{F}(\sigma)$ avoids $\mathcal{F}(\pi)$.

Proof.

Need to only prove one direction (why?).

Suppose $\mathcal{F}(\sigma)$ does not avoid $\mathcal{F}(\pi)$ if σ avoids π . (Here, $\sigma \in S_n$ and $\pi \in S_k$)

Let $\mathcal{F}(\sigma) = a_1 \cdots a_n$ and let $\mathcal{F}(\pi) = b_1 \cdots b_k$. Then there is some subsequence $a_{i_1} \cdots a_{i_k}$ so that $a_{i_c} < a_{i_d}$ if and only if $b_c < b_d$.

But then, $n + 1 - a_{i_c} > n + 1 - a_{i_d}$ if and only if

$k + 1 - b_c > k + 1 - b_d$. This implies that σ does not avoid π , a contradiction (why?). □

Corollary

For a permutation π , $s_n(\pi) = s_n(\mathcal{F}(\pi))$.

Lemma

The permutations of $\{1, 2, \dots, k, k + 1\}$ ending in i that avoid the pattern 312 are precisely those of the form,

$$\pi_1 \pi_2 i$$

the concatenation of π_1, π_2 , and i , where π_1 is a permutation of $\{1, 2, \dots, i - 1\}$ that avoids the pattern 312 and π_2 is a permutation of $\{i + 1, \dots, k + 1\}$ that avoids the pattern 312.

Proof.

For the sake of contradiction, assume that there exists some permutation π of $\{1, 2, \dots, k, k+1\}$ that avoids 312, ending with value i such that some integer $x < i$ is to the right of some integer $y > i$. Clearly, π does not avoid 312. (why?)

It follows that for any permutation $\pi \in S_{k+1}$ avoiding 312 and ending in i , all $x < i$ must be to the left of all $y > i$. Hence, π can be written as $\pi_1 \pi_2 i$ where π_1 is a permutation of $\{1, 2, \dots, i-1\}$ and π_2 is a permutation of $\{i+1, i+2, \dots, k+1\}$. Furthermore, it is clear that any subsequences of the permutation π must avoid 312 if the entire permutation π is to avoid 312 as well; this implies that the permutations π_1 and π_2 must avoid 312 as well. \square

Definition

The Catalan numbers are the sequence of positive integers C_i defined as follows,

$$C_0 = 1, C_{n+1} = \sum_{i=0}^n C_i C_{n-i} \text{ for } n \geq 0$$

Theorem

$s_n(312)$, $s_n(132)$, $s_n(213)$, and $s_n(231)$ are equal to C_n , the n^{th} Catalan number.

Proof.

Assume that for all i from 1 to k , the number of permutations of $\{1, 2, \dots, i\}$ that avoid the order 312 as a subsequence is C_i .

The number of permutations of $\{1, 2, \dots, k, k+1\}$ that avoid 312 can be counted by enumerating through all possible values of the last term of a valid permutation.

From the above lemma, all permutations of $\{1, 2, \dots, k, k+1\}$ ending in i that avoid 312 are precisely those of the form

$$\pi = \pi_1 \pi_2 i$$

where π_1 is a permutation of $\{1, 2, \dots, i-1\}$ that avoids 312 and π_2 is a permutation of $\{i+1, i+2, \dots, k+1\}$ that avoids 312.

Proof (contd.)

It follows that the total number of permutations π avoiding 312 and ending in i is

$$C_{i-1} \cdot C_{k-i+1}$$

Now, summing over all possible values of i , the total number of permutations of $\{1, 2, \dots, k+1\}$ that avoid 312 is equal to,

$$\sum_{i=1}^{k+1} C_{i-1} \cdot C_{k-i+1} = \sum_{i=0}^k C_i \cdot C_{k-i} = C_{k+1}$$

By the flipping lemma and reversing lemma, $s_n(132)$, $s_n(213)$ and $s_n(231)$ are the sequence of Catalan numbers as well. \square

Avoidance in T_n