18.821 Project 2

Yajit Jain, Deepak Narayanan, Leon Zhang

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1 Introduction

In this paper we will consider investigate pattern avoidance in permutations of finite sets of positive integers.

Definition 1. *Two finite sequences $a_1, \cdots a_k$ and $b_1, \cdots b_k$ have the same relative order if the i^{th} largest entry of each unordered set $\{a_1, \cdots, a_n\}$ and $\{b_1, \cdots, b_n\}$ appears in the same position in $a_1 \cdots a_k$ and $b_1 \cdots b_k$ for each $1 \leq i \leq k$.*

Definition 2. *A finite sequence of numbers $a_1 a_2 \cdots a_n$ avoids a sequence $b_1 \cdots b_k$ with $n \geq k$ if no subsequence $a_{i_1} a_{i_2} \cdots a_{i_k}$ of $a_1 a_2 \cdots a_n$ has its terms in the same relative order as $b_1 \cdots b_k$.*

Example 1. The sequence 15432 avoids 123 while the sequence 13425 does not. The latter example fails because 134 appears in 15342 with the same relative order as 123.

Remark 1. Let S_n be the set of permutations of $\{1, \cdots, n\}$. There are two conventions commonly used to write elements of S_n. The first is *one-line* notation where permutations in S_n are written explicitly, e.g. 43215 is a permutation of $\{1, 2, 3, 4, 5\}$ in S_5.

The second is *cycle notation*, where a cycle $(a_1 \cdots a_n) \in S_n$ with $1 \leq a_i \leq n$ acts on a sequence by sending $a_1 \mapsto a_2$, $a_2 \mapsto a_3$, $\cdots$, $a_n \mapsto a_1$. For example, the cycle (14)(23) sends 12345 to 43215. Unless otherwise stated, we will refer to permutations in one-line notation.

Definition 3. *When $n \geq k$, a permutation $\sigma \in S_n$ avoids a permutation $\pi \in S_k$ if pi and σ avoid each other when written as sequences in one-line notation.*

Definition 4. *For a permutation π, we write $s_n(\pi$ for the number of permutations in S_n that avoid π.*

In this paper we concern ourselves with the sequences $s_n(\pi)$ for various π. In section 2 we will consider sequence $s_n(\pi)$ for $\pi \in S_3$, and in section 4 we will consider sequence $s_n(\pi)$ for $\pi \in S_4$. In section 5 we consider a variation on the avoidance problem that considers the problem of avoiding sequences in a subset T_n of S_n.

1.1 Generating Trees

Before continuing we introduce generating trees, a tool we can use to study pattern avoidance. The generating tree for the sequence $s_n(\pi)$ is the infinite tree whose vertices are elements of S_n that avoid π. To construct the tree, we define the children of an arbitrary node. Let $\omega \in S_k$ be a node of the generating tree of $s_n(\pi)$. Then the children of ω are elements of S_{k+1} that avoid π and that can be obtained by inserting the integer $k+1$ between integers in ω. Figure 1 is an example of the generating tree of $s_n(312)$.

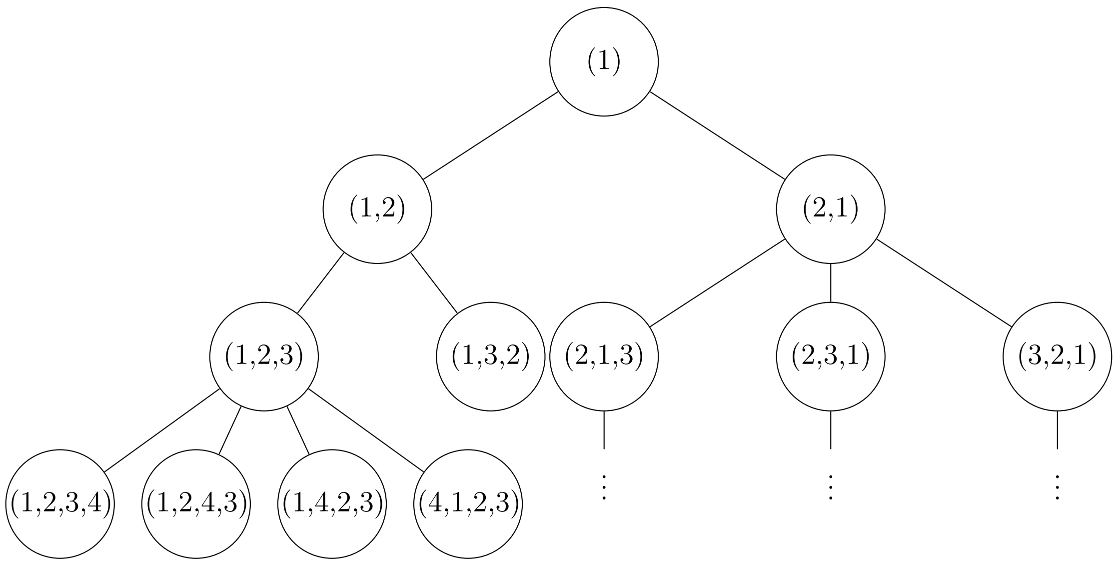


Figure 1: Generating tree of the sequence $s_n(312)$

2 Terms in the sequence $s_n(312)$

We claim that the sequence of numbers $s_n(312)$ is in fact the sequence of Catalan numbers. We state this result formally as the following theorem,

Theorem 5. *The total number of permutations of* $\{1, 2, 3, ..., n\}$ *that avoid the order* 312 *as a subsequence is* C_n *where* C_n *is the* n^{th} *Catalan number.*

Before proving the theorem, we state and prove the following lemma, that will be used in our proof of the theorem.

Lemma 6. *All permutations of* $\{1, 2, \ldots, k, k+1\}$ *ending in* i *that avoid the order* 312 *as a sub-sequence must be of the form,*

$$\pi_1 \pi_2 i$$

where π_1 *is a permutation of* $\{1, 2, \ldots, (i-1)\}$ *that avoids the order* 312 *as a sub-sequence and* π_2 *is a permutation of* $\{(i+1), \ldots, (k+1)\}$ *that avoids the order* 312 *as a sub-sequence.*

Proof. It is clear that any subsequences of the permutation $\pi = \pi_1\pi_2 i$ must avoid 312 if the entire permutation π is to avoid 312 as well; this implies that the permutations π_1 and π_2 must avoid 312 as well.

We proceed with a proof by contradiction.

Before proceeding, we define the sets A and B to be $\{1, 2, \ldots, (i-1)\}$ and $\{(i+1), (i+2), \ldots, (k+1)\}$ respectively. For the sake of contradiction, let us assume that there exists some permutation π of $\{1, 2, ..., k, k+1\}$ that ends with value i such that some integer $x < i$ (that is, $x \in A$) is to the right of some integer $y > i$ ($y \in B$). Clearly, this permutation is not of the form described above. It is also easy to see that π does *not* avoid the order 312 since the triple (y, x, i) satisfies the condition $y > x > i$ and is in the order 312.

From this we conclude that only permutations of the form described above can avoid 312.

$\square$

With this lemma proven, we move on to the proof of our theorem.

Proof. The inductive hypothesis holds for our base case of $\{1\}$, since the only permutation of $\{1\}$ trivially avoids 312.

Now, we need to prove the inductive case. Let us first assume that for all i from 1 to k, the number of permutations of $\{1, 2, ..., i\}$ that avoid the order 312 as a subsequence is C_i.

Now, we want to prove the inductive hypothesis for $\{1, 2, ..., k, k+1\}$ as well, that is the number of permutations of $\{1, 2, ..., k, k+1\}$ that avoid the order 312 as a subsequence is C_{k+1}.

We count the number of permutations of $\{1, 2, ..., k, k+1\}$ that avoid 312 by enumerating through all possible values of the last term of a valid permutation. If the last term of the permutation is i (where $i \in \{1, 2, ..., k, k+1\}$), then let us define the subsets A and B of the set $\{1, 2, ..., k+1\} \setminus \{i\}$ as the set of integers less than i and the set of integers greater than i respectively. It is clear from the definition of A and B that A and B are disjoint from each other.

Now, from the above lemma, we know that all permutations of $\{1, 2, ..., k, k+1\}$ ending in i that avoid 312 must be of the form,

$$\pi = \pi_1\pi_2 i$$

where π_1 is a permutation of A that avoids the order 312 as a sub-sequence and π_2 is a permutation of B that avoids the order 312 as a sub-sequence. It is clear that the above permutation contains all integers between 1 and $k+1$, from the definitions of the subsets A and B, which implies that $\pi_1\pi_2 i$ is permutation of the set $\{1, 2, ..., k, k+1\}$.

Now, the total number of permutations π is,

$$n_{\pi_1} \cdot n_{\pi_2} = C_{i-1} \cdot C_{k-i+1}$$

since the total number of valid permuations π_1 is simply going to be C_{i-1} (total number of valid permutations of length $i-1$ that avoid the order 312 as a sub-sequence is C_{i-1}; similarly $n_{\pi_2} = C_{k-1+1}$)

Now, summing over all possible values of i, we see that the total number of permutations of $\{1, 2, ..., k+1\}$ that avoid 312 is equal to,

$$\sum_{i=1}^{k+1} C_{i-1} \cdot C_{k-i+1} = \sum_{i=0}^{k} C_i \cdot C_{k-i}$$

which is in fact C_{k+1}, and we are done.

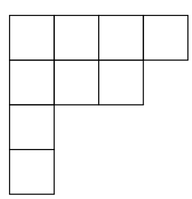
$\square$

3 Terms in the sequence $s_n(321)$

Unfortunately, there is no clear way to reformulate Lemma 2 for the case of avoiding the permutation 321 or 123. As a consequence, the proof in the previous section simply does not hold. Fortunately, however, we still have a closed form expression for $s_n(321)$ (recall that $s_n(321) = s_n(123)$ by our flipping principle). Surprisingly, in fact, the sequence remains the same: $s_n(321) = C_n$, the n^{th} Catalan number. In order to prove this result, however, we will need the machinery of Young tableaux. What follows is a brief exposition of the key results, adapted from Stanley's *Algebraic Combinatorics.*

Definition 7. *A Young diagram of a partition $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of the integer $\sum_{i=1}^{n} \lambda_i$ is a left-justified array of squares, with λ_i squares in the ith row.*

Example 2. The Young diagram of $(4, 3, 1, 1)$ looks like:



Definition 8. *A standard Young tableau (or SYT) consists of the Young diagram D of some partition λ of an integer n, together with the numbers $1, 2, \ldots, n$ inserted into the squares of D, so that each number appears exactly once, and every row and column is* increasing. *We call λ the* shape *of the SYT.*

Example 3. There are five SYT of the shape $(2, 2, 1)$. They are given by

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Definition 9. *Let u be a square of the Young diagram of the partition λ. Then the hook $H(u)$ is the set of all squares directly to the right of u or directly below u, including u itself. The size of $H(u)$ is called the hook length of u, and is denoted $h(u)$.*

Example 4. In the Young diagram of the partition $(4, 2, 2)$ below, each square u contains its hook length $h(u)$.

6	5	2	1
3	2		
2	1		

Theorem 10. *Let λ be a partition of n. Then f^λ, the number of SYT of shape λ, is given by*

$$f^\lambda = \frac{n!}{\prod_{u\in\lambda} h(u)},$$

where the notation $u \in \lambda$ means that u ranges over all squares of the Young diagram of λ.

The theorem above is called the *hook-length formula.*

Example 5. The diagram above for the hook lengths of $\lambda = (4, 2, 2)$ tells us that the number of SYT of shape λ is given by

$$\frac{8!}{6 \cdot 5 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 2 \cdot 1} = 56.$$

It is an easy application of the hook-length formula to prove the following theorem:

Theorem 11. *The number of SYT of shape (n, n) is given by C_n, the n^{th} Catalan number.*

We need only one more result before we are ready to prove our central result.

Theorem 12. *There exists an algorithm (called the* RSK algorithm*) which defines a bijection between the set of permutations in $S(n)$ avoiding a decreasing subsequence of length 3 and the set of pairs of size-n SYT of the same shape and at most two rows.*

Using all this machinery and all these results, we are finally able to prove the closed-form expression for $s_n(321)$.

Theorem 13. *The total number of permutations of $\{1, 2, 3, \dots, n\}$ that avoid the order 321 as a subsequence is C_n, where C_n is the n^{th} Catalan number.*

Proof. By the first part of Theorem 12, there are $s_n(321)$ elements in the set A of pairs of size-n SYT of the same shape and at most two rows. By Theorem 11, the set B_n of SYT of shape (n, n) has its size given by C_n. We will construct a bijection between the sets A_n and B_n: because $|A_n| = s_n(321)$ and $|B_n| = C_n$, it will therefore follow that $s_n(321) = C_n$, as desired.

Given any pair of size n Young tableaux of the same shape and at most two parts, take the second tableau and invert its numbers: that is, send i to $2n + 1 - i$ for all i. Now rotate the Young tableau and 'fit' it into the first tableau. We demonstrate this process for a pair of Young tableaux with $n = 6$: we begin with

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2}\end{array}.$$

Inverting the numbers of the second tableau gives us

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 12 & 10 & 9 & 8 \\ \hline 11 & 7 \\ \cline{1-2}\end{array}.$$

Finally, rotating the second Young tableau and fitting it into the first gives us

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 7 & 11 \\ \hline 5 & 6 & 8 & 9 & 10 & 12 \\ \hline\end{array},$$

a valid Young tableau of shape $(6, 6)$.

Beginning with a pair of size n Young tableaux of shape λ and at most two parts, it is easy to see that this process always yields a valid Young tableau of shape (n, n). Indeed, we can also define the inverse: given any Young tableau for the partition (n, n), we can find the shape formed by all numbers greater than n: split it off from the original tableau, rotate it, and invert the numbers to get a pair of size n Young tableau of the same shape and at most two parts. These processes can be easily checked to be inverses: hence the two sets are of the same size, and the proof follows. $\square$

4 Conjectures on $s_n(\pi)$ for $\pi \in S_4$

5 Avoidance of permutations of T_n

We now impose further restrictions on the set of permutations S_n. This new set of permutations, T_n is defined follows.

Definition 14. *For even n, T_n is defined as the set of all permutations $\sigma \in S_n$ for which $1, 3, 5, \ldots, 2n-1$ appear in increasing order, and $2i$ always appears to the right of $2i - 1$.*

Since T_n is only defined for even n, henceforth we shall refer to T_n as T_{2m} where m is any integer greater than or equal to 0.

Given the above definition of T_{2m}, we prove the following result about the cardinality of T_{2m}.

Theorem 15. $|T_{2m}| = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2m-1)$

Proof. Observe that since $1, 3, 5, \ldots, 2m-1$ must appear in increasing order in T_{2m}, our problem is now reduced to determining the relative order of $2, 4, 6, \ldots, 2m$ with respect to each other, as well as with respect to $1, 3, 5, \ldots, 2m-1$.

Let us first try to insert $2m$ into the sequence $1, 3, 5, \ldots, 2m-1$. Clearly, the way T_{2m} is defined, $2m$ can be inserted into only 1 slot, the one following $2m-1$. (Here we define a slot as a gap between two existing elements of the sequence, or the gap that follows the last element of the sequence or the gap that precedes the first element of the sequence)

Now, let us try to insert $(2m-2)$ into the sequence. $(2m-2)$ can be inserted into 3 slots in the sequence – the one between $2m-3$ and $2m-1$, the one between $2m-1$ and $2m$ or the one following $2m$.

Now, if we try to insert $(2m-4)$ into this incompletely formed sequence, we will see that there are 5 possible locations into which this number can be inserted. Continuing this for all even numbers upto 2, we observe that the total number of ways such a sequence can be created is equal to $1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2m-1)$, as desired. $\square$