# Pattern avoidance

## An explanation and proof

Yajit Jain, Deepak Narayanan and Leon Zhang

November 19, 2014

A permutation of a finite set $\{1, \cdots, n\}$ is some *ordering* of the elements.

A permutation of a finite set $\{1, \cdots, n\}$ is some *ordering* of the elements.

54123 is a permutation of $\{1, 2, 3, 4, 5\}$.

A permutation of a finite set $\{1, \cdots, n\}$ is some *ordering* of the elements.

54123 is a permutation of $\{1, 2, 3, 4, 5\}$.

$S_n$ is the set of permutations on $\{1, \cdots, n\}$.

A permutation of a finite set $\{1, \cdots, n\}$ is some *ordering* of the elements.

$$54123 \text{ is a permutation of } \{1, 2, 3, 4, 5\}.$$

$S_n$ is the set of permutations on $\{1, \cdots, n\}$.

$$54123 \in S_5$$

54123

includes $\begin{cases} 123 \\ 312 \\ 4312 \end{cases}$

---

avoids $\begin{cases} 132 \\ 312 \\ 213 \\ 231 \end{cases}$

Let $\pi = 312 \in S_3$.

- Question: How many permutations avoid $\pi$? (a lot)
- Better Question: How many permutations in $S_n$ avoid $\pi$?

# Let's count!

- How many permutations in $S_1$ avoid $\pi$?

# Let's count!

- How many permutations in $S_1$ avoid $\pi$? 1

# Let's count!

- How many permutations in $S_1$ avoid $\pi$? 1
- How many permutations in $S_2$ avoid $\pi$?

# Let's count!

- How many permutations in $S_1$ avoid $\pi$? 1
- How many permutations in $S_2$ avoid $\pi$? 2

# Let's count!

- How many permutations in $S_1$ avoid $\pi$? 1
- How many permutations in $S_2$ avoid $\pi$? 2
- How many permutations in $S_3$ avoid $\pi$?

# Let's count!

- How many permutations in $S_1$ avoid $\pi$? 1
- How many permutations in $S_2$ avoid $\pi$? 2
- How many permutations in $S_3$ avoid $\pi$? 5

# Let's count!

- How many permutations in $S_1$ avoid $\pi$? 1
- How many permutations in $S_2$ avoid $\pi$? 2
- How many permutations in $S_3$ avoid $\pi$? 5
- How many permutations in $S_4$ avoid $\pi$?

# Let's count!

- How many permutations in $S_1$ avoid $\pi$? 1
- How many permutations in $S_2$ avoid $\pi$? 2
- How many permutations in $S_3$ avoid $\pi$? 5
- How many permutations in $S_4$ avoid $\pi$? ??????

# Permutations in $S_4$ that avoid $\pi = 312$?

How many permutations in $S_4$ avoid $\pi$?

# Permutations in $S_4$ that avoid $\pi = 312$?

How many permutations in $S_4$ avoid $\pi$?

| | | | | | |
|---|---|---|---|---|---|
| 1234 | 1243 | 1324 | 1342 | 1423 | 1432 |
| 2134 | 2143 | 2314 | 2341 | 2413 | 2431 |
| 3124 | 3142 | 3214 | 3241 | 3412 | 3421 |
| 4123 | 4132 | 4213 | 4231 | 4312 | 4321 |

# Permutations in $S_4$ that avoid $\pi = 312$?

How many permutations in $S_4$ avoid $\pi$? 14

| | | | | | |
|---|---|---|---|---|---|
| 1234 | 1243 | 1324 | 1342 | 1423 | 1432 |
| 2134 | 2143 | 2314 | 2341 | 2413 | 2431 |
| 3124 | 3142 | 3214 | 3241 | 3412 | 3421 |
| 4123 | 4132 | 4213 | 4231 | 4312 | 4321 |

## Definition

Let $a_n(\pi)$ be the number of permutations in $S_n$ that avoid $\pi$.

## Definition

Let $a_n(\pi)$ be the number of permutations in $S_n$ that avoid $\pi$.

We want to compute the sequences $(a_n(\pi))$ for some $\pi \in S_k$.

## Definition

Let $a_n(\pi)$ be the number of permutations in $S_n$ that avoid $\pi$.

We want to compute the sequences $(a_n(\pi))$ for some $\pi \in S_k$.

Example: $(a_n(312)) = 1, 2, 5, 14,$

## Definition

Let $a_n(\pi)$ be the number of permutations in $S_n$ that avoid $\pi$.

We want to compute the sequences $(a_n(\pi))$ for some $\pi \in S_k$.

Example: $(a_n(312)) = 1, 2, 5, 14,$ 42,132,429,...

# Main Result

## Theorem

*For $\pi \in S_3$, $(a_n(\pi))$ is equal to the Catalan numbers:*

$$(a_n(\pi)) = 1, 2, 5, 14, 42, 132, 429...$$

# What if we take $\pi \in S_4$?

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots \\ \end{cases}$$

# What if we take $\pi \in S_4$?

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots \\ \end{cases}$$

# What if we take $\pi \in S_4$?

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots \\ C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots \end{cases}$$

# What if we take $\pi \in S_4$?

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots \\ C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots \end{cases}$$

????

# Avoidance of 312

Let's first look at some examples of permutations that don't avoid 312!

# Avoidance of 312

Let's first look at some examples of permutations that don't avoid 312!

## Example

1 2 6 5 3 4

# Avoidance of 312

Let's first look at some examples of permutations that don't avoid 312!

## Example

$$1 \quad 2 \quad \color{red}{6} \quad 5 \quad \color{red}{3} \quad \color{red}{4} \quad \Longrightarrow \text{126534 does not avoid 312}$$

# Avoidance of 312

Let's first look at some examples of permutations that don't avoid 312!

## Example

$1 \quad 2 \quad 6 \quad 5 \quad 3 \quad 4 \implies$ 126534 does not avoid 312

## Example

$1 \quad 5 \quad 6 \quad 3 \quad 2 \quad 4$

# Avoidance of 312

Let's first look at some examples of permutations that don't avoid 312!

## Example

$1\ 2\ 6\ 5\ 3\ 4 \implies$ 126534 does not avoid 312

## Example

$1\ 5\ 6\ 3\ 2\ 4 \implies$ 156324 does not avoid 312

# Avoidance of 312

How about some permutations that do avoid 312?

# Avoidance of 312

How about some permutations that do avoid 312?

## Example

1 2 3 6 5 4

# Avoidance of 312

How about some permutations that do avoid 312?

## Example

$$1 \quad 2 \quad 3 \quad 6 \quad 5 \quad 4 \quad \Longrightarrow 123654 \text{ avoids } 312$$

# Avoidance of 312

How about some permutations that do avoid 312?

## Example

$1 \quad 2 \quad 3 \quad 6 \quad 5 \quad 4 \quad \Longrightarrow$ 123654 avoids 312

## Example

$2 \quad 1 \quad 4 \quad 5 \quad 6 \quad 3$

# Avoidance of 312

How about some permutations that do avoid 312?

## Example

$1 \quad 2 \quad 3 \quad 6 \quad 5 \quad 4 \quad \Longrightarrow$ 123654 avoids 312

## Example

$2 \quad 1 \quad 4 \quad 5 \quad 6 \quad 3 \quad \Longrightarrow$ 214563 avoids 312

# Avoidance of 312

**Do the permutations that avoid** 312 **have any special properties?**

# Avoidance of 312

**Do the permutations that avoid 312 have any special properties?**

1 2 3 6 5 4

# Avoidance of 312

**Do the permutations that avoid 312 have any special properties?**

$$\underbrace{1 \quad 2 \quad 3}_{\text{All } < 4\text{, and avoid } 312} \quad 6 \quad 5 \quad 4$$

# Avoidance of 312

**Do the permutations that avoid 312 have any special properties?**

$$\underbrace{1 \quad 2 \quad 3}_{\text{All } < 4\text{, and avoid } 312} \quad \underbrace{6 \quad 5}_{\text{All } > 4\text{, and avoid } 312} \quad 4$$

# Avoidance of 312

**Do the permutations that avoid 312 have any special properties?**

$$1 \quad 2 \quad 3 \quad 6 \quad 5 \quad 4$$

All $<4$, and avoid 312 — All $>4$, and avoid 312

$$2 \quad 1 \quad 4 \quad 5 \quad 6 \quad 3$$

# Avoidance of 312

**Do the permutations that avoid 312 have any special properties?**

$$\underbrace{1 \quad 2 \quad 3}_{\text{All} < 4\text{, and avoid } 312} \quad \underbrace{6 \quad 5}_{\text{All} > 4\text{, and avoid } 312} \quad 4$$

$$\underbrace{2 \quad 1}_{\text{All} < 3\text{, and avoid } 312} \quad 4 \quad 5 \quad 6 \quad 3$$

# Avoidance of 312

**Do the permutations that avoid 312 have any special properties?**

$$\underbrace{1 \quad 2 \quad 3}_{\text{All} < 4\text{, and avoid } 312} \quad \underbrace{6 \quad 5}_{\text{All} > 4\text{, and avoid } 312} \quad {\color{blue}4}$$

$$\underbrace{2 \quad 1}_{\text{All} < 3\text{, and avoid } 312} \quad \underbrace{4 \quad 5 \quad 6}_{\text{All} > 3\text{, and avoid } 312} \quad {\color{blue}3}$$

# Avoidance of 312

**What happens with permutations that don't have this property?**

# Avoidance of 312

**What happens with permutations that don't have this property?**

1 2 6 5 3 4

# Avoidance of 312

**What happens with permutations that don't have this property?**

1 2 6 5 3 4

All numbers $< 4$ in blue

All numbers $> 4$ in green

# Avoidance of 312

**What happens with permutations that don't have this property?**

$$1 \quad 2 \quad 6 \quad 5 \quad 3 \quad 4$$

All numbers $< 4$ in blue    All numbers $> 4$ in green

$$\Downarrow$$

$$1 \quad 2 \quad 6 \quad 5 \quad 3 \quad 4$$

# Avoidance of 312

**What happens with permutations that don't have this property?**

1 2 6 5 3 4

All numbers $< 4$ in blue

All numbers $> 4$ in green

$$\Downarrow$$

1 2 6 5 3 4

Doesn't avoid 312 anymore!

# Avoidance of 312

## Lemma

*The permutations of* $\{1, 2, \ldots, k, k+1\}$ *ending in* $i$ *that avoid the pattern* 312 *are precisely those of the form,*

$$\pi_1 \pi_2 i$$

*the concatenation of* $\pi_1, \pi_2$*, and* $i$*, where* $\pi_1$ *is a permutation of* $\{1, 2, \ldots, i-1\}$ *that avoids the pattern* 312 *and* $\pi_2$ *is a permutation of* $\{i+1, \ldots, k+1\}$ *that avoids the pattern* 312*.*

# Avoidance of 312

## Definition

The Catalan numbers are the sequence of positive integers $C_i$ defined as follows,

$$C_0 = 1,\ C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \text{ for } n \geq 0$$

# Avoidance of 312

## Theorem

*The $n^{th}$ term of the sequence $a_n(312)$ is equal to $C_n$, the $n^{th}$ Catalan number, for $n > 0$.*

# Avoidance of 312

## Proof.

Assume that for all $i$ from 1 to $k$, the number of permutations of $\{1, 2, ..., i\}$ that avoid 312 is $C_i$.
It follows from the above lemma that the total number of permutations $\pi$ avoiding 312 and ending in $i$ is

$$C_{i-1} \cdot C_{k-i+1}$$

Summing over all possible values of $i$, the total number of permutations of $\{1, 2, ..., k+1\}$ that avoid 312 is equal to,

$$\sum_{i=1}^{k+1} C_{i-1} \cdot C_{k-i+1} = \sum_{i=0}^{k} C_i \cdot C_{k-i} = C_{k+1}$$

# The Reversing Lemma

## Definition (Reversing)

We define the *reverse* of a permutation $b_1 \cdots b_n$ to be the permutation $b_n \cdots b_1$. The reversing operator is denoted by $\mathcal{R}$.

# The Reversing Lemma

## Definition (Reversing)

We define the *reverse* of a permutation $b_1 \cdots b_n$ to be the permutation $b_n \cdots b_1$. The reversing operator is denoted by $\mathcal{R}$.

## Example

$\mathcal{R}(1324) = 4231.$

# The Reversing Lemma

## Definition (Reversing)

We define the *reverse* of a permutation $b_1 \cdots b_n$ to be the permutation $b_n \cdots b_1$. The reversing operator is denoted by $\mathcal{R}$.

## Example

$\mathcal{R}(1324) = 4231.$

## Example

$\mathcal{R}(1243) = 3421.$

# The Reversing Lemma

## Lemma (Reversing Lemma)

*The permutation $\sigma$ avoids the permutation $\pi$ iff $\mathcal{R}(\sigma)$ avoids $\mathcal{R}(\pi)$.*

## Corollary

*For a permutation $\pi$, $a_n(\pi) = a_n(\mathcal{R}(\pi))$.*

# The Flipping Lemma

## Definition (Flipping)

We define the *flip* of a sequence $b$ as the sequence $c$ with the same elements as $b$, but with the largest element swapped with the smallest element, the second largest element swapped with the second smallest element, etc. The flipping operator is denoted by $\mathcal{F}$.

# The Flipping Lemma

## Example

$$1 \quad 2 \quad 3 \quad 4 \quad \Longrightarrow \quad 4 \quad 3 \quad 2 \quad 1$$

$$\mathcal{F}(1234) = 4321$$

# The Flipping Lemma

## Example

$$1 \quad 2 \quad 3 \quad 4 \qquad \Longrightarrow \qquad 4 \quad 3 \quad 2 \quad 1$$

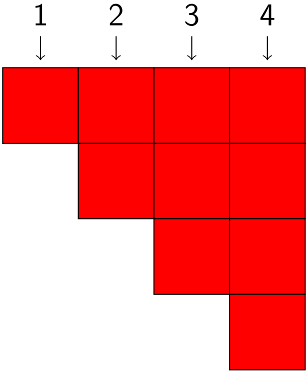
$$\mathcal{F}(1234) = 4321$$

## Example

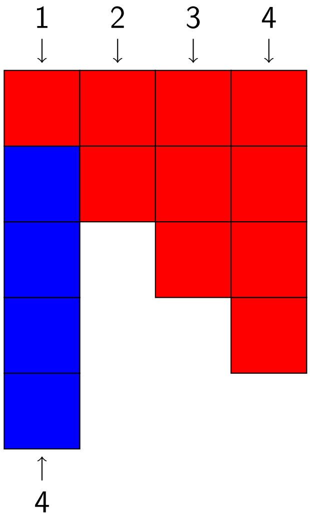
$$1 \quad 2 \quad 4 \quad 3 \qquad \Longrightarrow \qquad 4 \quad 3 \quad 1 \quad 2$$

$$\mathcal{F}(1243) = 4312$$

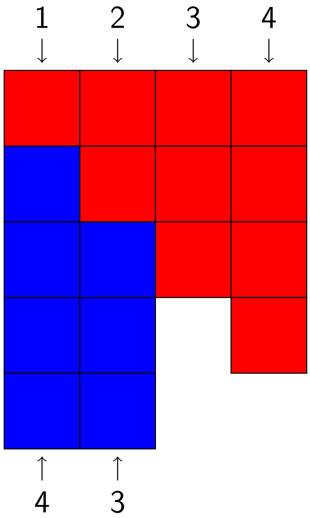
# The Flipping Lemma



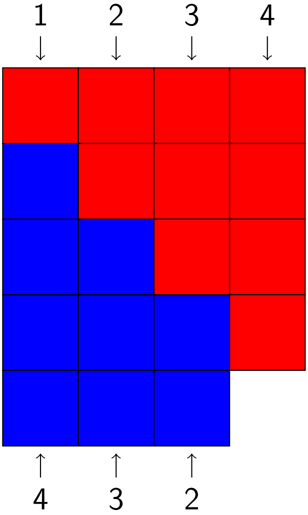
# The Flipping Lemma



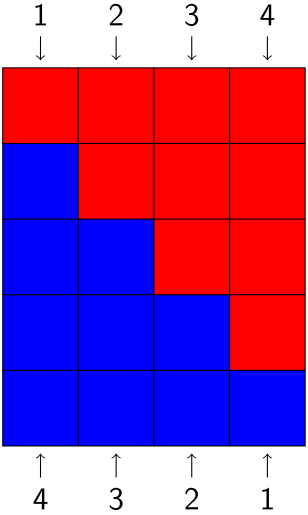
# The Flipping Lemma



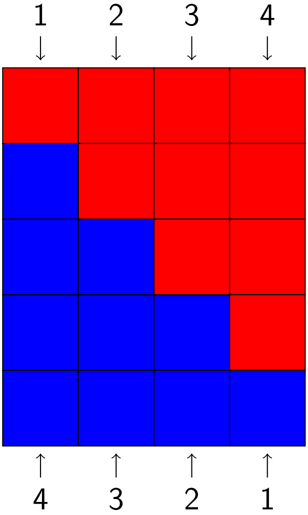
# The Flipping Lemma



# The Flipping Lemma

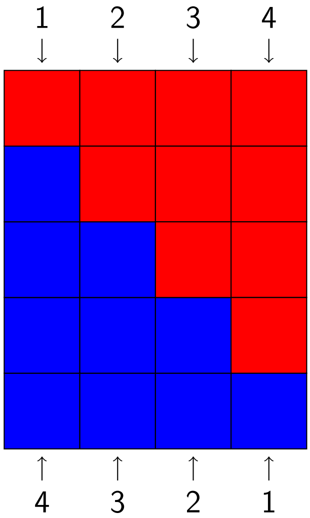


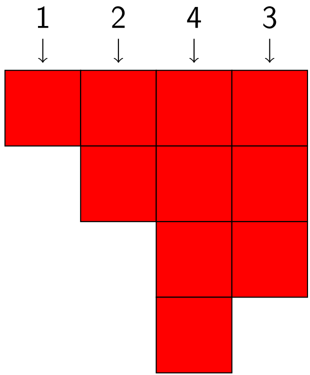
# The Flipping Lemma




$$\mathcal{F}(1234) = 4321$$

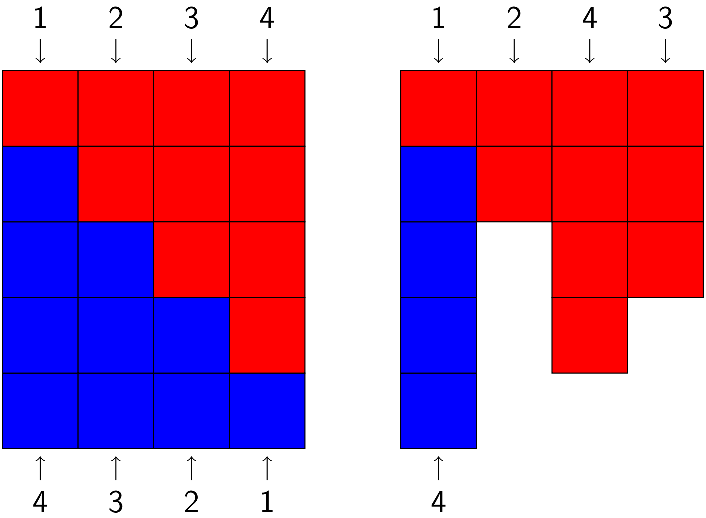
# The Flipping Lemma



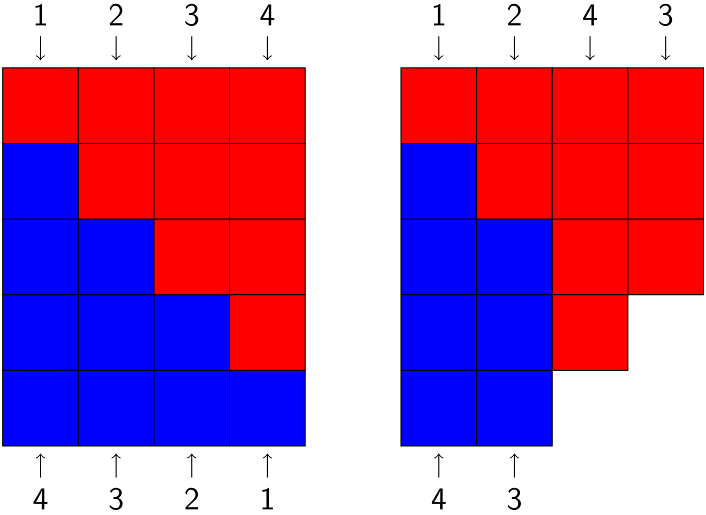

$$\mathcal{F}(1234) = 4321$$

# The Flipping Lemma



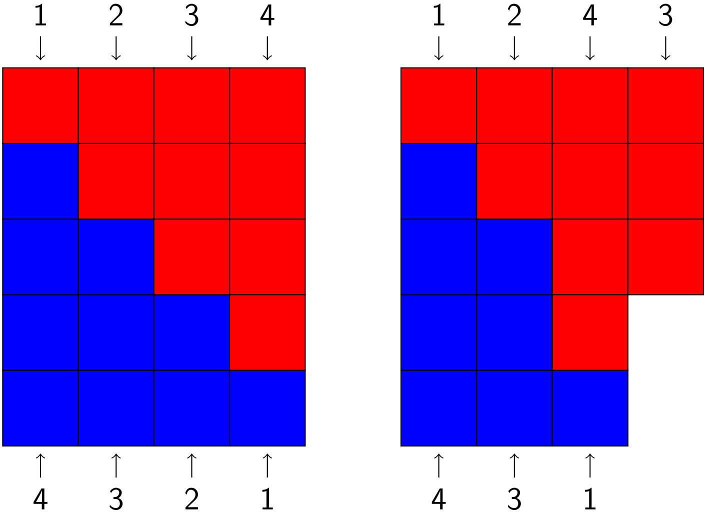

$$\mathcal{F}(1234) = 4321$$

# The Flipping Lemma



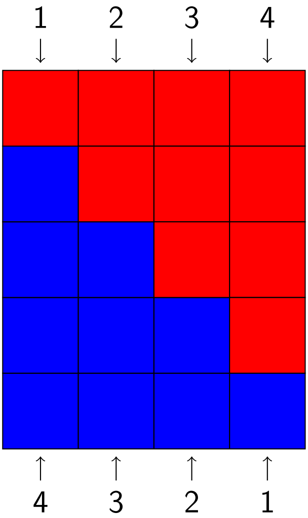

$$\mathcal{F}(1234) = 4321$$

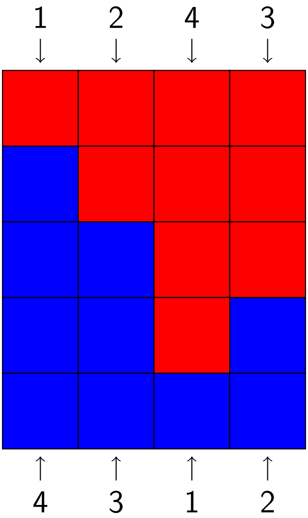
# The Flipping Lemma




$$\mathcal{F}(1234) = 4321$$

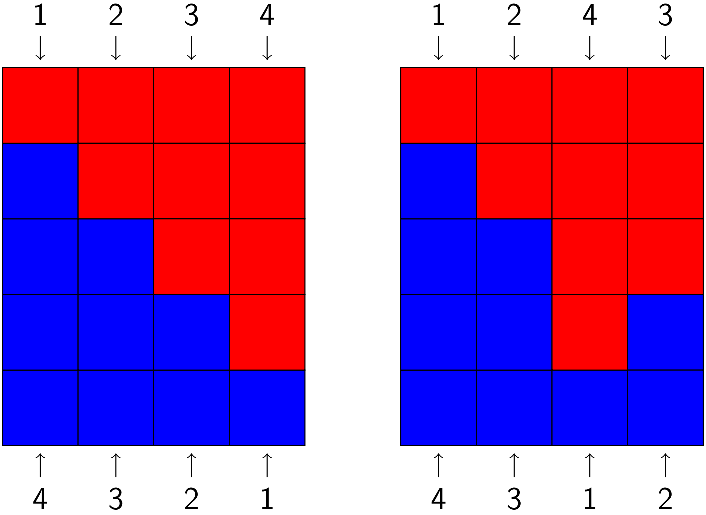
# The Flipping Lemma




$$\mathcal{F}(1234) = 4321$$

# The Flipping Lemma




$\mathcal{F}(1234) = 4321$ $\mathcal{F}(1243) = 4312$

# The Flipping Lemma

## Lemma (Flipping Lemma)

*The permutation $\sigma$ avoids the permutation $\pi$ iff $\mathcal{F}(\sigma)$ avoids $\mathcal{F}(\pi)$.*

## Corollary

*For a permutation $\pi$, $a_n(\pi) = a_n(\mathcal{F}(\pi))$.*

# Avoidance of other permutations in $S_3$

- From the Flipping Lemma and Reversing Lemmas, the sequences $(a_n(213))$, $(a_n(132))$ and $(a_n(231))$ are the sequence of Catalan numbers as well.
- However, it is much harder to prove that the sequences $(a_n(123))$ and $(a_n(321))$ are the sequence of Catalan numbers.

# Recall

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots \\ C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots \end{cases}$$

# Permutations in $S_4$

| | | | | | |
|---|---|---|---|---|---|
| 1234 | 1243 | 1324 | 1342 | 1423 | 1432 |
| 2134 | 2143 | 2314 | 2341 | 2413 | 2431 |
| 3124 | 3142 | 3214 | 3241 | 3412 | 3421 |
| 4123 | 4132 | 4213 | 4231 | 4312 | 4321 |

# Flipping and reversing buckets

$$\{1243, 4312, 2134, 3421\}, \{2413, 3142\},$$
$$\{1432, 4123, 2341, 3214\}, \{1234, 4321\},$$
$$\{4132, 1423, 2314, 3241\}, \{2143, 3412\},$$
$$\{4213, 1342, 3124, 2431\}, \{4231, 1324\}.$$

# Decomposition

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots \\ C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots \end{cases}$$

# Decomposition

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots \\ C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots \end{cases}$$

| B | A | C |
|---|---|---|
| {1243, 4312, 2134, 3421}, {1432, 4123, 2341, 3214}, {2143, 3412}, {1234, 4321} | {4132, 1423, 2314, 3241}, {4213, 1342, 3124, 2431}, {2413, 3142} | {4231, 1324} |

# Decomposition

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots \\ C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots \end{cases}$$

| B | A | C |
|---|---|---|
| {1243, 4312, 2134, 3421}, {1432, 4123, 2341, 3214}, {2143, 3412}, {1234, 4321} | {4132, 1423, 2314, 3241}, {4213, 1342, 3124, 2431}, {2413, 3142} | {4231, 1324} |

???

# Cycle notation

Take $\sigma \in S_5$ with

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

| One-line Notation | Cycle Notation |
|---|---|
| $\sigma = 45213$ | $\sigma = (14)(253)$ |

# Cycle Notation

Take $\sigma \in S_5$ with

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

| One-line Notation | Cycle Notation |
|---|---|
| $\sigma = 45213$ | $\sigma = (14)(253)$ |

# Cycle Notation

Take $\sigma \in S_5$ with

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

<u>One-line Notation</u>

$$\sigma = 45213$$

<u>Cycle Notation</u>

$$\sigma = (14)(253)$$

# Cycle Notation

Take $\sigma \in S_5$ with

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

| One-line Notation | Cycle Notation |
|---|---|
| $\sigma = 45213$ | $\sigma = (14)(253)$ |

# Buckets in cycle notation

$$\{(34),(12),(1423),(1324)\},\{(1243),(1342)\},$$
$$\{(24),(13),(1432),(1234)\},\{(1)(2)(3)(4),(14)(23)\},$$
$$\{(142),(243),(123),(134)\},\{(12)(34),(13)(24)\},$$
$$\{(143),(234),(132),(124)\},\{(14),(23)\}.$$

# Cycle decomposition

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots \\ C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots \end{cases}$$

# Cycle decomposition

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots \\ C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots \end{cases}$$

| B | A | C |
|---|---|---|
| {(34), (1423), (12), (1324)}, {(24), (1432), (13), (1234)}, {(12)(34), (13)(24)}, {(1)(2)(3)(4), (14)(23)} | {(142), (243), (123), (134)}, {(143), (234), (132), (124)}, {(1243), (1342)} | {(14), (23)} |

# Conjecture

## Conjecture.

- There are three possible sequences for $(a_n(\pi))$.
- Given two F&R buckets that look the same up to the cycle decompositions of their elements, they generate the same sequence.

# Other observations

$$(a_n(\pi)) = \begin{cases} A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots \\ B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots \\ C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots \end{cases}$$

- Many fewer buckets than expected
- Different growth rates?

# Avoidance in $T_n$

## Definition

Let $m$ be a positive integer. The set $T_{2m}$ is defined as all permutations in $S_{2m}$ such that:

- the odd numbers appear in increasing order,
- each even number $2i$ appears to the right of $2i - 1$.

## Example

The set $S_2$ is $\{12, 21\}$. The set $T_2$ is just $\{12\}$.

# Permutations in $T_4$

| | | | | | |
|---|---|---|---|---|---|
| 1234 | 1243 | 1324 | 1342 | 1423 | 1432 |
| 2134 | 2143 | 2314 | 2341 | 2413 | 2431 |
| 3124 | 3142 | 3214 | 3241 | 3412 | 3421 |
| 4123 | 4132 | 4213 | 4231 | 4312 | 4321 |

# Problem Statement

## Definition

Given a permutation $\pi \in S_k$, we define $b_m(\pi)$ as

$$b_m(\pi) = \#\{\sigma \in T_{2m} \mid \sigma \text{ avoids } \pi\}.$$

## Problem

*Let $\pi \in S_3$, and $m$ an arbitrary positive integer. Compute $b_m(\pi)$.*

# Values of $b_m(\pi)$

| $\pi$ | $m=1$ | $m=2$ | $m=3$ | $m=4$ | $m=5$ |
|---|---|---|---|---|---|
| 123 | 1 | 0 | 0 | 0 | 0 |
| 132 | 1 | 1 | 1 | 1 | 1 |
| 213 | 1 | 2 | 4 | 8 | 16 |
| 231 | 1 | 2 | 4 | 8 | 16 |
| 312 | 1 | 3 | 12 | 55 | 273 |
| 321 | 1 | 3 | 12 | 55 | 273 |

# Computing $b_m(123)$

| $\pi$ | $m = 1$ | $m = 2$ | $m = 3$ | $m = 4$ | $m = 5$ |
|---|---|---|---|---|---|
| 123 | 1 | 0 | 0 | 0 | 0 |
| 132 | 1 | 1 | 1 | 1 | 1 |
| 213 | 1 | 2 | 4 | 8 | 16 |
| 231 | 1 | 2 | 4 | 8 | 16 |
| 312 | 1 | 3 | 12 | 55 | 273 |
| 321 | 1 | 3 | 12 | 55 | 273 |

Pick any $\sigma \in T_{2m}$ with $m \geq 2$. Then:

- 3 comes after 1
- 4 comes after 3
- So 134 is a subsequence of $\sigma$.

# Computing $b_m(132)$

| $\pi$ | $m = 1$ | $m = 2$ | $m = 3$ | $m = 4$ | $m = 5$ |
|---|---|---|---|---|---|
| 123 | 1 | 0 | 0 | 0 | 0 |
| 132 | 1 | 1 | 1 | 1 | 1 |
| 213 | 1 | 2 | 4 | 8 | 16 |
| 231 | 1 | 2 | 4 | 8 | 16 |
| 312 | 1 | 3 | 12 | 55 | 273 |
| 321 | 1 | 3 | 12 | 55 | 273 |

Let $m \geq 2$, and pick any $\sigma \in T_{2m}$ avoiding 132. Then:

- 1 always comes first
- Each even integer $2i$ must come before $2i + 1$
- So $\sigma$ must be $1234\ldots(2m)$.

# On $b_m(213)$ and $b_m(231)$

| $\pi$ | $m=1$ | $m=2$ | $m=3$ | $m=4$ | $m=5$ |
|---|---|---|---|---|---|
| 123 | 1 | 0 | 0 | 0 | 0 |
| 132 | 1 | 1 | 1 | 1 | 1 |
| 213 | 1 | 2 | 4 | 8 | 16 |
| 231 | 1 | 2 | 4 | 8 | 16 |
| 312 | 1 | 3 | 12 | 55 | 273 |
| 321 | 1 | 3 | 12 | 55 | 273 |

## Theorem

$b_m(213) = 2^{m-1}$, *and* $b_m(231) = 2^{m-1}$.

# On $b_m(312)$ and $b_m(321)$

| $\pi$ | $m = 1$ | $m = 2$ | $m = 3$ | $m = 4$ | $m = 5$ |
|---|---|---|---|---|---|
| 123 | 1 | 0 | 0 | 0 | 0 |
| 132 | 1 | 1 | 1 | 1 | 1 |
| 213 | 1 | 2 | 4 | 8 | 16 |
| 231 | 1 | 2 | 4 | 8 | 16 |
| 312 | 1 | 3 | 12 | 55 | 273 |
| 321 | 1 | 3 | 12 | 55 | 273 |

## Theorem

$b_m(312) = b_m(321)$.

## Conjecture.

$b_m(312) = \binom{3m}{m} \cdot \frac{1}{2m+1}$.