Avoidance in permutations

Yajit Jain, Deepak Narayanan, Leon Zhang

November 6, 2014

1 Introduction

In this paper we investigate pattern avoidance in finite sequences of natural numbers. We begin with an example: consider the sequences 123 and 1324. Note the order relationships of the elements of 123: we have $1 < 2 < 3$. Notice in addition that there is an ordered subsequence of 1324 that follows the same general pattern, namely 124. In this situation we say that the sequence 1324 *does not avoid the pattern 123*. We can also examine the sequence 1432, and observe that none of its subsequences have the same order relationships as 123. We say that 1432 *avoids* 123.

With examples of a four element sequence that avoids 123 and a four element sequence that does not avoid 123, it is natural to ask the question: how many four element sequences avoid 123? And more generally, how many n-element sequences avoid 123? If we define $s_n(123)$ to be the number of n element sequences that avoid 123, then the question above is equivalent to identifying the sequence $(s_n(123))_n$. In this paper we consider specific cases of a very general avoidance problem, describing the sequence $(s_n(\sigma))_n$ where σ is any permutation of the elements of the set $\{1, 2, \cdots, k\} \subset \mathbb{N}$ for some k.

In Section 3 we consider sequences $s_n(\pi)$ when π is a permutation on the set $\{1, 2, 3\}$, and prove that the sequence $s_n(\pi)$ for all such π is in fact the sequence of Catalan numbers. In Section 5 we will consider sequences $s_n(\pi)$ when π is a permutation of $\{1, 2, 3, 4\}$. In Section 6 we consider a variation of the avoidance problem that considers the problem of avoiding sequences in a subset T_n of the permutations of n elements. Before proceeding, however, we formally introduce the notation that will be used through the rest of this paper.

1.1 Notation

Definition 1. *The set S_n is the collection of permutations of the set $\{1, 2, \cdots, n\}$.*

Example 1. $S_3 = \{123, 132, 213, 231, 312, 321\}$.

We never define w
a pattern is. Eithe
scrub the paper of
mentions or define
here

Remark 1. The notation used to describe the elements of S_n above is 'one-line' notation. Thus far we have ignored the fact that S_n has a group structure. When discussing this group structure it is convenient to use a different notation known as 'cycle' notation. This alternative notation will be used in Section 5 and an introduction to cycle notation can be found in Appendix [].

The notation used to describe the elements of S_n above is 'one-line' notation. Thus far we have ignored the fact that S_n has a group structure. When discussing this group structure it is convenient to use a different notation known as 'cycle' notation. This alternative notation will be used in Section 5 and an introduction to cycle notation can be found in Appendix [].

Definition 2. *Two finite sequences of distinct positive integers $a_1, \cdots a_k$ and $b_1, \cdots b_k$ have the same relative order if the ith largest entry of each unordered set $\{a_1, \cdots, a_k\}$ and $\{b_1, \cdots, b_k\}$ appears in the same position in $a_1 \cdots a_k$ and $b_1 \cdots b_k$ for each $1 \leq i \leq k$.*

Remark 2. Equivalently, two finite sequences of distinct positive integers have the same relative order if the elements in each position of the sequence satisfy the same set of inequalities.

Definition 3. *A finite sequence of distinct positive integers $a_1 a_2 \cdots a_n$ avoids another finite sequence of distinct positive integers $b_1 \cdots b_k$ with $n \geq k$ if no subsequence $a_{i_1} a_{i_2} \cdots a_{i_k}$ of $a_1 a_2 \cdots a_n$ has its terms in the same relative order as $b_1 \cdots b_k$.*

Definition 4. *A permutation $\sigma \in S_n$ avoids a permutation $\pi \in S_k$ if the sequence of numbers for the one-line notation for σ avoids the sequence of numbers for the one-line notation for π.*

Definition 5. *For a permutation $\pi \in S_k$, $s_n(\pi)$ represents the number of permutations in S_n that avoid π.*

2   Flipping and Reversing

In this section we will identify operations that preserve avoidance. Explicitly, we say that an operator $\mathcal{O}$ preserves avoidance if a permutation σ avoids π if and only if $\mathcal{O}(\sigma)$ avoids $\mathcal{O}(\pi)$. If $\mathcal{O}$ is an avoidance preserving operator then for any $\sigma \in S_k$, $s_n(\sigma) = s_n(\mathcal{O}(\sigma))$ for all n and k. Two such operators that we will identify are the 'flipping' and 'reversing' operators. Before discussing these operators, we will introduce generating trees, a tool which we use to study pattern avoidance and particularly to prove the avoidance preserving nature of these operators.

> [margin note] o should define formally

> [margin note] Jerison wanted us define this formall think

> [margin note] anged this definon... take a look see if you agree it

2.1 Generating Trees

The generating tree for the sequence $s_n(\pi)$ is the infinite tree whose vertices are elements of S_n that avoid π. The first node in the tree is always 1. To construct the tree, we define the children of an arbitrary node. Let $\omega \in S_k$ be a node of the generating tree of $s_n(\pi)$. Then the children of ω are elements of S_{k+1} that avoid π and that can be obtained by inserting the integer $k + 1$ between integers in ω. Figure 1 is an example of the generating tree of $s_n(312)$.
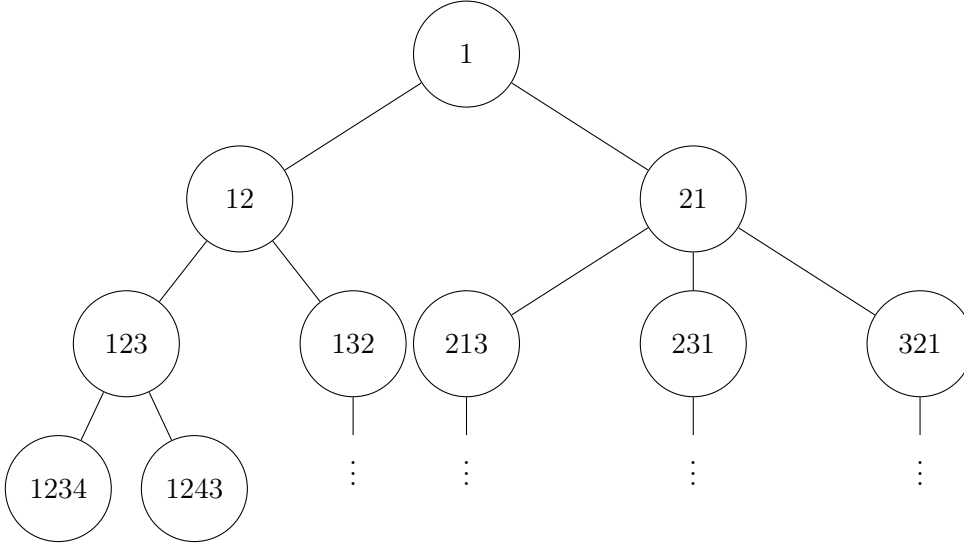
Figure 1: Generating tree of the sequence $s_n(312)$

2.2 Flipping

In this section we introduce the concept of flipping and use generating trees to prove the 'Flipping Lemma.'

Definition 6. *We define the* flip *of a sequence a as the sequence b with the same elements as a, but with the largest element swapped with the smallest element, the second largest element swapped with the second smallest element, etc. The flipping operator will be denoted by $\mathcal{F}$.*

Example 2. $\mathcal{F}(1324) = 4231$.

Lemma 7 (Flipping Lemma)**.** *The permutation σ avoids the permutation π if and only if $\mathcal{F}(\sigma)$ avoids $\mathcal{F}(\pi)$. I.e. $\mathcal{F}$ is an avoidance preserving operator.*

Proof. Notice that we only need to proof one direction of this statement. The opposite direction will follow since applying $\mathcal{F}$ twice returns the original permutation. We show

3

that if σ avoids π then $\mathcal{F}(\sigma)$ avoids $\mathcal{F}(\pi)$. Let $\pi = b_1 \cdots b_k$. We proceed by induction. Certainly the statement is true for elements of S_1. Suppose it is true for elements of S_{n-1}. Let $\sigma \in S_n$. If σ does avoids π then σ appears in the generating tree of π. Then $\sigma = a_1 \cdots a_n$ where $a_i \in \{1, \cdots, n\}$. Without loss of generality, let $a_\ell = n$. Then the parent node of σ is $\sigma_p = a_1 \cdots a_{\ell-1} a_{\ell+1} \cdots a_n$. The induction hypothesis allows us to conclude that $\mathcal{F}(\sigma_p)$ avoids $\mathcal{F}(\pi)$. Now suppose that $\mathcal{F}(\sigma)$ is not a child of $\mathcal{F}(\sigma_p)$. Let $\mathcal{F}(\sigma) = a'_1 \cdots a'_{\gamma-1} a_\gamma a'_{\gamma+1} \cdots a'_n$ and $\mathcal{F}(\sigma_p) = a'_1 \cdots a'_{\gamma-1} a'_{\gamma+1} \cdots a'_n$ with $a'_\gamma = n$. Then $\mathcal{F}(\sigma_p)$ avoids π while $\mathcal{F}(\sigma)$ does not. That means that there is a subsequence ω of $\mathcal{F}(\sigma)$ that has the same relative order as $\mathcal{F}(\pi)$. Let $\mathcal{F}(\pi) = b'_1 \cdots b'_k$ with $b'_i = 1$ and $b'_j = k$. So ω contains a'_γ, and a'_γ occupies the same position as b'_j. Now if we remove 1 and n from $\mathcal{F}(\sigma)$, then the result does not avoid $\mathcal{F}(\pi)$ with $b'_i = 1$ and $b'_j = k$ omitted. Then if we apply a flip, by the induction hypothesis we see that the result of flipping $\mathcal{F}(\sigma)$ with 1 and n omitted does not avoid π with 1 and k omitted. If we add in 1 and n in the appropriate places to reconstruct $\mathcal{F}(\mathcal{F}(\sigma)) = \sigma$ and if we reconstruct σ similarly, then σ does not avoid π, a contradiction. $\square$

Corollary 8. *For a permutation π, $s_n(\pi) = s_n(\mathcal{F}(\pi))$.*

Proof. This result follows because the flipping lemma gives a bijection between permutations that avoid π and permutations that avoid $\mathcal{F}(\pi)$. $\square$

2.3   Reversing

Lemma 9 (Reversing Lemma). *The permutation σ avoids the permutation π if and only if $\mathcal{R}(\sigma)$ avoids $\mathcal{R}(\pi)$. I.e. $\mathcal{R}$ is an avoidance preserving operator.*

Proof. Again we use the generating tree. Certainly the statement is true for elements of S_1. Suppose it is true for elements of S_{n-1}. Suppose that $\sigma \in S_n$ avoids π. Then if σ_p is the parent of σ, $\mathcal{R}(\sigma_p)$ avoids $\mathcal{R}(\pi)$. Now if we add n to the permutation $\mathcal{R}(\sigma_p)$ to construct $\mathcal{F}(\sigma)$, we see that $\mathcal{F}(\sigma)$ must avoid $\mathcal{F}(\pi)$, for if it did not, we could reverse one more time and obtain the result that $\sigma \in S_n$ does not avoid π, a contradiction. The other direction follows because $\mathcal{R}^2 = 1$. $\square$

Corollary 10. *For a permutation π, $s_n(\pi) = s_n(\mathcal{R}(\pi))$.*

Proof. This result follows because the reversing lemma gives a bijection between permutations that avoid π and permutations that avoid $\mathcal{F}(\pi)$. $\square$

3   Terms in the sequence $s_n(312)$

We claim that the sequence of numbers $s_n(312)$ is in fact the sequence of Catalan numbers. Before formally stating and proving the result, however, we prove the following useful lemma.

Lemma 11. *The permutations of $\{1, 2, \ldots, k, k+1\}$ ending in i that avoid the pattern 312 are precisely those of the form,*

$$\pi_1 \pi_2 i$$

the concatenation of $\pi_1, \pi_2,$ and i, where π_1 is a permutation of $\{1, 2, \ldots, i-1\}$ that avoids the pattern 312 and π_2 is a permutation of $\{i+1, \ldots, k+1\}$ that avoids the pattern 312.

Proof. It is easy to see that such a permutation is sufficient for avoiding 312 and ending in i. We will prove that the condition is necessary as well.

First, we define the sets $A = \{1, 2, \ldots, i-1\}$ and $B = \{i+1, i+2, \ldots, k+1\}$. For the sake of contradiction, let us assume that there exists some permutation π of $\{1, 2, ..., k, k+1\}$ that ends with value i such that some integer $x < i$ (that is, $x \in A$) is to the right of some integer $y > i$ ($y \in B$). Clearly, this permutation is not of the form described above, and the subsequence (y, x, i) is of the same relative order as 312; as a consequence, π does not avoid 312.

It follows that for any permutation $\pi \in S_{k+1}$ avoiding 312 and ending in i, all $x < i$ must be to the left of all $y > i$. Hence π must be the concatenation of three subsequences $\pi_1 \pi_2 i$, where π_1 is a permutation of $\{1, 2, \ldots, i-1\}$ and π_2 is a permutation of $\{i+1, i+2, \ldots, k+1\}$. Furthermore, it is clear that any subsequences of the permutation $\pi = \pi_1 \pi_2 i$ must avoid 312 if the entire permutation π is to avoid 312 as well; this implies that the permutations π_1 and π_2 must avoid 312 as well. $\square$

Before we move on to the proof of our main theorem for this section, we look at the sequence of Catalan numbers.

Definition 12. *The Catalan numbers are the sequence of positive integers C_i defined as follows,*

$$C_0 = 1, \, C_{n+1} = \sum_{i=1}^{n} C_i C_{n-i} \text{ for } n \geq 0$$

Theorem 13. *$s_n(312)$, the total number of permutations of $\{1, 2, 3, ..., n\}$ that avoid the pattern 312, is equal to C_n, the nth Catalan number.*

Proof. The inductive hypothesis holds for our base case of $\{1\}$, since the only permutation of $\{1\}$ trivially avoids 312.

We need to prove the inductive case. Let us first assume that for all i from 1 to k, the number of permutations of $\{1, 2, ..., i\}$ that avoid the order 312 as a subsequence is C_i.

Now, we want to prove the inductive hypothesis for $\{1, 2, ..., k, k+1\}$ as well. That is, we want to prove that the number of permutations of $\{1, 2, ..., k, k+1\}$ that avoid the order 312 as a subsequence is C_{k+1}.

Using our lemma, we count the number of permutations of $\{1, 2, ..., k, k+1\}$ that avoid 312 by enumerating through all possible values of the last term of a valid permutation. Consider π a permutation of $\{1, 2, \ldots, k, k+1\}$ ending in i. Let us define the

subsets A and B of the set $\{1, 2, ..., k+1\} \setminus \{i\}$ as the set of integers less than i and the set of integers greater than i respectively. It is clear from the definition of A and B that A and B are disjoint from each other.

From the above lemma, we know that all permutations of $\{1, 2, ..., k, k+1\}$ ending in i that avoid 312 are precisely those of the form

$$\pi = \pi_1 \pi_2 i$$

where π_1 is a permutation of A that avoids 312 and π_2 is a permutation of B that avoids 312.

It follows that the total number of permutations π avoiding 312 and ending in i is

$$C_{i-1} \cdot C_{k-i+1},$$

since by our induction hypothesis the total number of valid choices for π_1 is simply C_{i-1}, and similarly the total number of valid choices for π_2 is C_{k-1+1}.

Now, summing over all possible values of i, we see that the total number of permutations of $\{1, 2, ..., k+1\}$ that avoid 312 is equal to,

$$\sum_{i=1}^{k+1} C_{i-1} \cdot C_{k-i+1} = \sum_{i=0}^{k} C_i \cdot C_{k-i}$$

which is well-known to equal C_{k+1}, the $n+1$st Catalan number. The proof therefore follows by induction.

$\square$

It is easy to see that the above proof can be modified slightly to prove that the sequences $s_n(132)$, $s_n(213)$ and $s_n(231)$ are in fact the sequence of catalan numbers as well. In fact, we claim that if σ and π are both permutations in S_k for some arbitrary k, then if $\sigma = flip(\pi)$, $s_n(\pi) = s_n(\sigma)$. This is easy to see since if a is a permutation that avoids σ, then $flip(a)$ is a permutation that avoids π – since this can be done both ways, it is bijective and we can conclude that the cardinalities of the two sets are equal, or $s_n(\sigma) = s_n(\pi)$.

should move this
and make it clear
in the beginning
that this stuff gener-
es

4 Terms in the sequence $s_n(321)$

Unfortunately, there is no clear way to reformulate Lemma 11 for the case of avoiding the permutation 321 or 123. As a consequence, the proof in the previous section simply does not hold. Fortunately, however, we still have a closed form expression for $s_n(321)$ (recall that $s_n(321) = s_n(123)$ by our flipping principle). Surprisingly, in fact, the sequence remains the same: $s_n(321) = C_n$, the nth Catalan number. In order to prove this result, we need the machinery of Young tableaux. What follows is a brief exposition of the key results, adapted from Stanley's *Algebraic Combinatorics*.

Definition 14. *A Young diagram of a partition* $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *of the integer* $\sum_{i=1}^{n} \lambda_i$ *is a left-justified array of squares, with* λ_i *squares in the ith row.*

*Example* 3. The Young diagram of $(4, 3, 1, 1)$ looks like:
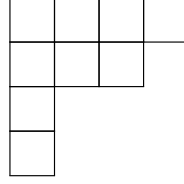
Definition 15. *A standard Young tableau (or SYT) consists of the Young diagram D of some partition λ of an integer n, together with the numbers $1, 2, \ldots, n$ inserted into the squares of D, so that each number appears exactly once, and every row and column is increasing. We call λ the* shape *of the SYT.*

Example 4. There are five SYT of the shape $(2, 2, 1)$. They are given by

1	2
3	4
5	

1	2
3	5
4	

1	3
2	4
5	

1	3
2	5
4	

1	4
2	5
3	
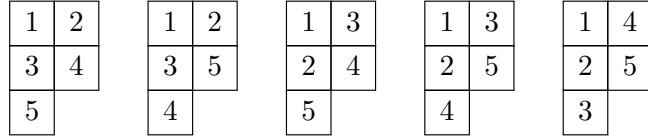
Definition 16. *Let u be a square of the Young diagram of the partition λ. Then the hook $H(u)$ is the set of all squares directly to the right of u or directly below u, including u itself. The size of $H(u)$ is called the* hook length *of u, and is denoted $h(u)$.*

Example 5. In the Young diagram of the partition $(4, 2, 2)$ below, each square u contains its hook length $h(u)$.

6	5	2	1
3	2		
2	1		
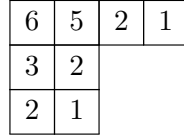
Theorem 17. *Let λ be a partition of n. Then f^λ, the number of SYT of shape λ, is given by*

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)},$$

where the notation $u \in \lambda$ means that u ranges over all squares of the Young diagram of λ.

The theorem above is called the *hook-length formula.*

Example 6. The diagram above for the hook lengths of $\lambda = (4, 2, 2)$ tells us that the number of SYT of shape λ is given by

$$\frac{8!}{6 \cdot 5 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 2 \cdot 1} = 56.$$

It is an easy application of the hook-length formula to prove the following theorem:

Theorem 18. *The number of SYT of shape (n, n) is given by C_n, the nth Catalan number.*

We need only one more result before we are ready to prove our central result.

Theorem 19. *There exists an algorithm (called the RSK algorithm) which defines a bijection between the set of permutations in $S(n)$ avoiding a decreasing subsequence of length 3 and the set of pairs of size-n SYT of the same shape and at most two rows.*

Using all this machinery and all these results, we are finally able to prove the closed-form expression for $s_n(321)$.

Theorem 20. *The total number of permutations of $\{1, 2, 3, \ldots, n\}$ that avoid the order 321 as a subsequence is C_n, where C_n is the nth Catalan number.*

Proof. By the first part of Theorem 12, there are $s_n(321)$ elements in the set A of pairs of size-n SYT of the same shape and at most two rows. By Theorem 11, the set B_n of SYT of shape (n, n) has its size given by C_n. We will construct a bijection between the sets A_n and B_n: because $|A_n| = s_n(321)$ and $|B_n| = C_n$, it will therefore follow that $s_n(321) = C_n$, as desired.

Given any pair of size n Young tableaux of the same shape and at most two parts, take the second tableau and invert its numbers: that is, send i to $2n + 1 - i$ for all i. Now rotate the Young tableau and 'fit' it into the first tableau. We demonstrate this process for a pair of Young tableaux with $n = 6$: we begin with

1	2	3	4
5	6		

1	3	4	5
2	6		

Inverting the numbers of the second tableau gives us

1	2	3	4
5	6		

12	10	9	8
11	7		

Finally, rotating the second Young tableau and fitting it into the first gives us

1	2	3	4	7	11
5	6	8	9	10	12

a valid Young tableau of shape $(6, 6)$.

Beginning with a pair of size n Young tableaux of shape λ and at most two parts, it is easy to see that this process always yields a valid Young tableau of shape (n, n). Indeed, we can define its inverse: given any Young tableau for the partition (n, n), we can find the shape formed by all numbers greater than n: split it off from the original tableau, rotate it, and invert the numbers to get a pair of size n Young tableau of the same shape and at most two parts. These processes can be easily checked to be inverses: hence the two sets are of the same size, and the proof follows. $\square$

5 Conjectures on $s_n(\pi)$ for $\pi \in S_4$

In this section we present our findings and conjectures about sequences $s_n(\pi)$ for $\pi \in S_4$. We computed values up to s_9. It appears that there are three sequences that appear, and we have sorted the permutations according to which sequence they produce and paired them with their flips.

We define the following sequences based on the first nine terms:

$$A := 1, 2, 6, 23, 103, 512, 2740, 15485, 91245 \ldots$$

$$B := 1, 2, 6, 23, 103, 513, 2761, 15767, 94359 \ldots$$

$$C := 1, 2, 6, 23, 103, 513, 2762, 15793, 94776 \ldots$$

The table below lists the elements of S_4 generate sequences A, B, C for the first 8 terms.

B	A	C
1234, 4321	4132, 1423	4231, 1324
1243, 4312	4213, 1342	
1432, 4123	2431, 3124	
2134, 3421	2413, 3142	
2143, 3412	2314, 3241	
2341, 3214		

There is no guarantee that these sequences will not diverge and produce more than three distinct sequences, however for now we hypothesize that A, B, and C are the only sequences that arise for $s_n(\pi)$ with $\pi \in S_4$. Assuming this hypothesis, if we write each length four permutation in cycle notation, a pattern begins to emerge.

B	A	C
(1),(14)(23)	(243),(142)	(23),(14)
(34),(1423)	(234),(143)	
(24),(1432)	(124),(132)	
(12),(1324)	(123),(134)	
(12)(34),(13)(24)	(1243),(1342)	
(1234),(13)		

So we conjecture that sequence A is characterized by elements of S_4 that are three cycles or four cycles when written in cycle notation. Sequence B seems to be characterized by either a pair of disjoint two cycles paired with a flip that is a 2-cycle, or a single 4-cycle paired with a flip that is a 2-cycle. Sequence C appears to be characterized by a 2-cycle paired with a 2-cycle as a flip.

6 Avoidance of permutations of T_n

We now impose further restrictions on the set of permutations S_n. This new set of permutations, T_n is defined follows.

Definition 21. *For even n, T_n is defined as the set of all permutations $\sigma \in S_n$ for which $1, 3, 5, \ldots, 2n - 1$ appear in increasing order, and $2i$ always appears to the right of $2i - 1$.*

Since T_n is only defined for even n, henceforth we shall refer to T_n as T_{2m} where m is any integer greater than or equal to 0.

Given the above definition of T_{2m}, we prove the following result about the cardinality of T_{2m}.

Theorem 22. $|T_{2m}| = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2m - 1)$

Proof. Observe that since $1, 3, 5, \ldots, 2m - 1$ must appear in increasing order in T_{2m}, our problem is now reduced to determining the relative order of $2, 4, 6, \ldots, 2m$ with respect to each other, as well as with respect to $1, 3, 5, \ldots, 2m - 1$.

Let us first try to insert $2m$ into the sequence $1, 3, 5, \ldots, 2m - 1$. Clearly, the way T_{2m} is defined, $2m$ can be inserted into only 1 slot, the one following $2m - 1$. (Here we define a slot as a gap between two existing elements of the sequence, or the gap that follows the last element of the sequence or the gap that precedes the first element of the sequence)

Now, let us try to insert $(2m - 2)$ into the sequence. $(2m - 2)$ can be inserted into 3 slots in the sequence – the one between $2m - 3$ and $2m - 1$, the one between $2m - 1$ and $2m$ or the one following $2m$.

Now, if we try to insert $(2m - 4)$ into this incompletely formed sequence, we will see that there are 5 possible locations into which this number can be inserted. Continuing this for all even numbers upto 2, we observe that the total number of ways such a sequence can be created is equal to $1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2m - 1)$, as desired. $\qquad \square$

Studying avoidance of permutations in T_n seems an interesting exercise. In the following section, we consider the sequence of numbers $t_n(\pi)$, defined as the number of permutations in T_n that avoid the permutation π. In particular, we consider permutations $\pi \in S_3$.

Before stating our conjectures, we present the computed sequences. Note that since the sequence T_n is defined only for even n, we only present the even terms of the sequence t_n.

$$t_n(123) = 1, 0, 0, 0, 0, \ldots$$

$$t_n(132) = 1, 1, 1, 1, 1, \ldots$$

$$t_n(213) = 1, 2, 4, 8, 16, \ldots$$

$$t_n(231) = 1, 2, 4, 8, 16, \ldots$$

$$t_n(312) = 1, 3, 12, 55, 273, \ldots$$

$$t_n(321) = 1, 3, 12, 55, 273, \ldots$$

It is easy to see why $t_n(123)$ and $t_n(132)$ behave the way they do – since all odd numbers in T_n are increasing, it is impossible to have a permutation in T_n that avoids 123 and the permutation consisting of all increasing integers is the only permutation of T_n that avoids 132.

More interesting is the equivalences between 213 and 231, and 312 and 321. We will prove the equivalence between 312 and 321 below, but before doing so, we first state and prove the following lemmas which will be useful in our proof.

Lemma 23. *If a and b are two elements that appear in a permutation in T_n such that a is to the left of b, then if $a > b$, b must be even.*

Proof. For the sake of contradiction, let us assume that b is odd. Then there exist two possibilities that need to be considered – one, that a is odd, and two, that a is even.

- **Case 1:** a is odd.

  Note that all permutations in T_n must have odd elements monotonically increasing. So if a appears to the left of b, and both a and b are odd, then a must be less than b – a contradiction. Hence, we conclude that a can't be odd.

- **Case 2:** a is even.

  For all permutations σ in T_n, every even element j must be to the right of all odd elements less than j. From this, we conclude that since a is even, it must be to the right of all odd numbers less than it. But b is an odd number less than a – we conclude that a can't be even either.

Since a can't be odd or even if b is odd, we conclude that b must be even, as desired. $\square$

Lemma 24. *If x, y, z is a three-element subsequence of a permutation π in T_n such that $x > y > z$ or $x > z > y$, then swapping y and z in the permutation π yields a permutation π' that also belongs to T_n.*

Proof. From Lemma 23 we see that both y and z must be even. Then there exist two cases that need to be covered.

- **Case 1:** x is odd.

  If x is odd, then it is clear that the permutation π' obtained from swapping y and z is still valid, since both y and z are to the right of the odd numbers $y - 1$ and $z - 1$ (since $y - 1$ and $z - 1$ are themselves to the left of x since they are strictly less than x)

- **Case 2:** x is even.

  If x is even, then $x - 1$ which is an odd number greater than both $y - 1$ and $z - 1$, must be to the left of x. Since $y - 1$ and $z - 1$ are already to the left of $x - 1$, we conclude that swapping y and z still ensures that y and z are both to the right of $y - 1$ and $z - 1$.

Hence, we see that regardless of whether x is even or odd, swapping y and z in a permutation $\pi \in T_n$ if $x > y > z$ or $x > z > y$ yields a permutation π' that also belongs to T_n. $\qquad\square$

Given these lemmas, we now present a proof for $t_n(312) = t_n(321)$.

Theorem 25. $t_n(312) = t_n(321)$

Proof. First observe from Lemma 23 that if x, y, z is a subsequence of three elements of a permutation in T_n in the relative order 312 or 321, then both y and z must be even.

Before proving that there exists a bijective mapping between the set of permutations in T_n that avoid 312 (call this set A) and the set of permutations in T_n that avoid 321 (call this set B), we establish a chain of transformation moves that can be used to transform an element $a \in A$ to an element $b \in B$.

Consider a permutation π in T_n that avoids 312. For each permutation $\pi \in T_n$ that avoids 312, we want to show that there exists a sequence of transformation moves that transforms π to a unique permutation π' that avoids 321.

If the original permutation π avoids 321 already, then clearly $\pi' = \pi$ avoids 321.

Now if π does not avoid 321, there must exist some sub-sequence x, y, z in π in the relative ordering 321. From Lemma 24, we see that swapping y and z in the permutation π yields another valid permutation π_1 in T_n – note that the permutation produced after this swap has x, y, z in the relative ordering 312 instead of 321. Note that π_1 may still have sub-sequences in the relative ordering 321, so we may have to perform the above swapping operation again to get some permutation $\pi_2 \in T_n$ such that a sub-sequence x_1, y_1, z_1 in π_1 that has a relative ordering 321 is in the relative ordering 312 in π_2. Continuing this process produces the following chain of permutations,

$$\pi \to \pi_1 \to \pi_2 \to \ldots \to \pi_k$$

Where the last permutation in the chain π_k does not have a single three-element sub-sequence in the relative ordering 321, that is $\pi' = \pi_k$ is a permutation in T_n that avoids 321.

This process is completely invertible, since we can take a permutation π' in T_n that avoids 321, and traverse through the above chain in reverse order to obtain a permutation that avoids 312.

We now prove that there exists a bijection between the sets A and B.

Let a be an arbitrary permutation in A. Clearly, by the definition of A, a must avoid 312. Now, we claim that for every element i in the permutation a, all even numbers in a to the right of i and less than i must be in decreasing order; if this were not

the case, then we would have a subsequence in the relative ordering 312, which would be a contradiction. It is easy to see that we can make a corresponding claim about permutation b, that is for every element i in the permutation a, all even numbers in a to the right of i and less than i must be in increasing order.

The above two claims shows there's actually an easy way to transform an arbitrary permutation $a \in A$ to a $b \in B$, and similarly, to transform an arbitrary permutation $b \in B$ to a $a \in A$ – to obtain a permutation in B from a permutation $a \in A$, for every element i in the permutation a, all even elements in a to the right of i less than i should be reversed (that is reordered to in increasing order); similarly to obtain a permutation in A from a permutation $b \in B$, for every element i in the permutation b, all even elements in b to the right of i, less than i should be reversed (that is reordered to in decreasing order). Since the mapping from A to B is clearly reversible, and hence invertible, we conclude that the mapping we just established between A and B is bijective.

Since there exists a bijective mapping between permutations in T_n that avoid 312 and permutations in T_n that avoid 321, we can conclude that $t_n(312) = t_n(321)$.

$\square$

Unfortunately, it turns out that proving the equivalence between 231 and 213 is not as straightforward as above, however we do prove that both the sequences $(t_n(231))$ and $(t_n(213)$ are equal to the sequence $1, 2, 4, 8, \ldots$, and hence equivalent, below.

Theorem 26. $t_n(231) = 1, 2, 4, 8, \ldots$

Proof. Proving the above theorem is equivalent to proving that the mth term of $t_n(231)$ is equal to 2^{m-1}, or that the number of permutations in T_{2m} that avoid 231 is equal to 2^{m-1}.

Consider a permutation σ in T_{2m} that avoids 231. Consider an arbitrary even element $2i$ in σ. For this element $2i$, define A to be the set of numbers greater than $2i$ and to its left in the permutation σ. First off, note that the set A cannot contain any even numbers, because if it did, then the permutation σ would contain a three-element sub-sequence in the relative ordering 231 – if a is an even number in the set A, then $(a - 1, a, 2i)$ would be a three-element sub-sequence of σ in the relative ordering 231 (Note that $a - 1$ must be to the left of a in the permutation σ because of the way the set T_{2m} was constructed)

Furthermore, the set A cannot contain any odd elements greater than $2i+1$, because if it did, then again the permutation σ would contain a three-element subsequence in the relative ordering 231 – if $2j + 1 > 2i + 1$ is an odd number in A, then $(2i + 1, 2j + 1, 2i)$ is a three-element sub-sequence of σ in the relative ordering 231 Therefore, we conclude that the set A either only contains the element $2i + 1$, or is empty.

From this we can conclude that the element $2i$ can be inserted into exactly two valid positions in a permutation of length $2m$ – at index $2i$ and at index $2i + 1$. It follows from the above result that when element $2i$ is at index $2i$, element $2i + 1$ is at index $2i + 1$ in the permutation, and when element $2i$ is at index $2i + 1$, element $2i + 1$ is at

index $2i$. Note that the element 1 must always be the first element in the permutation, and the element $2m$ must always be the last element.

Combining all these facts together, we see that the number of permutations σ in T_{2m} that avoid 231 must be equal to $2 \cdot 2 \cdot \ldots \cdot 2 = 2^{m-1}$, since every even element can be inserted in exactly two positions in σ.

$\square$

Theorem 27. $t_n(213) = 1, 2, 4, 8, \ldots$

Proof. As before, proving the above theorem is equivalent to proving that the mth term of $t_n(213)$ is equal to 2^{m-1}, or that the number of permutations in T_{2m} that avoid 213 is equal to 2^{m-1}.

First, let us define A_m as the set of all permutations in T_{2m} that avoid 213. It is easy to see that the mth term of $t_n(213)$ is $|A_m|$.

We now claim that for every permutation σ in A_m, the numbers $2m-1$ and $2m$ must be consecutive in the permutation σ. We argue that this must be true by contradiction. If this were not the case, then it is easy to see that the permutation σ would have a sub-sequence in the relative ordering 213 – let j be an element that's between $2m-1$ and $2m$ in σ. Clearly, $j < 2m - 1 < 2m$ (since every element in σ other than $2m$ is less than $2m - 1$), which means $(2m-1, j, 2m)$ are in the relative ordering 213. We conclude that $2m-1$ and $2m$ must be consecutive in σ.

Furthermore, we see that we can construct A_m from A_{m-1} by inserting the consecutive elements $2m-1, 2m$ into a permutation σ' in A_{m-1}. We claim that we get a valid permutation σ in A_m from σ' only if we insert the consecutive elements $2m-1, 2m$ right after either $2m-3$ or $2m-2$ in σ' (which themselves are consecutive in σ' from the above claim).

First, note that $2m-1, 2m$ must be inserted to the right of both $2m-3$ and $2m-2$, because of the way T_{2m} is defined. Furthermore, if we insert $2m - 1, 2m$ right after some a in σ not equal to $2m-3$ or $2m-2$, then we see that the produced permutation σ has a three-element sub-sequence in the relative ordering 213 – $(2m-2, a, 2m-1)$. Hence we conclude that $2m-1, 2m$ must be inserted to the right of either $2m-3$ or $2m-2$; since we can do this for every permutation σ' in A_{m-1}, we conclude that $|A_m| = 2|A_{m-1}|$.

Given that $|A_1|$ is trivially equal to 1, we conclude that $|A_m|$, or the mth term of $t_n(213)$ is indeed equal to 2^{m-1}, as desired. $\square$

Note that even though we proved that the sequences $t_n(231)$ and $t_n(213)$ are both equivalent, the methods used to prove that each sequence is equal to the sequence $1, 2, 4, 8, \ldots$ are vastly different – this is an interesting topic of research that we would like to pursue as future work.

7 Appendix A

Let S_n be the set of permutations of $\{1, \cdots, n\}$. There are two conventions commonly used to write elements of S_n. The first is *one-line* notation where permutations in S_n are written explicitly, e.g. 43215 is a permutation of $\{1, 2, 3, 4, 5\}$ in S_5.

The second is *cycle notation*, where a cycle $(a_1 \cdots a_n) \in S_n$ with $1 \leq a_i \leq n$ acts on a sequence by sending $a_1 \mapsto a_2$, $a_2 \mapsto a_3$, $\cdots$, $a_n \mapsto a_1$. For example, the cycle $(14)(23)$ sends 12345 to 43215.