Comprehensive Exam Notes

(1) Cholesky Decomposition:

- decomposition of a Hermitian, positive definite matrix into the product of a lower triangular matrix and its conjugate transpose
- $A = LL^T$ where L is lower triangular matrix with real and +ve diagonal enteries

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

- Formula: $$l_{ki} = \frac{a_{ki} - \sum_{j=1}^{i-1} l_{ij} \cdot l_{kj}}{l_{ii}}$$

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}$$

$$L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

eg. $$A = \begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix}$$

First check if matrix is +ve definite. That is, check if determinants of all upper-left sub-matrices are positive.

$$|6| = 6 \qquad \begin{vmatrix} 6 & 15 \\ 15 & 55 \end{vmatrix} = 105 \qquad \begin{vmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{vmatrix} = 19320 - 34650 + 19250 = 3920$$

Now,

$$l_{11} = \sqrt{a_{11}} = \sqrt{6} = 2.4495$$

$$l_{21} = \frac{a_{21} - \cancel{l_{10}} \cdot \cancel{l_{20}}}{l_{11}} = \frac{a_{21}}{l_{11}} = \frac{15}{2.4495} = 6.1237$$

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{55 - (6.12)^2} = \sqrt{55 - 37.5} = 4.1833$$

$$l_{31} = \frac{a_{31} - \cancel{l_{10}}\, \cancel{l_{30}}}{l_{11}} = \frac{a_{31}}{l_{11}} = \frac{55}{2.4495} = 22.4537$$

$$l_{32} = \frac{a_{32} - l_{21}\, l_{31}}{l_{22}} = \frac{225 - (22.4537)(6.1237)}{4.1833} = \frac{225 - 137.5}{4.1833} = 20.9165$$

$$l_{33} = \sqrt{a_{33} - l_{32}^2 - l_{31}^2} = \sqrt{979 - (20.92)^2 - (22.45)^2} = \sqrt{979 - 941.67} = 6.1101$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 2.45 & 0 & 0 \\ 6.12 & 4.18 & 0 \\ 22.45 & 20.92 & 6.11 \end{bmatrix}$$

(2) LU Decomposition :

- factorization of a given square matrix into two triangular matrices, one upper triangular and one lower triangular matrix.

- $A = LU \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Eg: $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & & \\ -0.75 & 1 & \\ 0.25 & & 1 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ 0 & 2.5 & -2.5 \\ 0 & -2.5 & 2.5 \end{bmatrix}$$

$R_2 - \left(-\frac{6}{8}\right) R_1$

$= R_2 - (-0.75) R_1$

$R_3 - \left(\frac{2}{8}\right) R_1$

$= R_3 - (0.25) R_1$

$$= \begin{bmatrix} 1 & & \\ -0.75 & 1 & \\ 0.25 & -1 & 1 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ 0 & 2.5 & -2.5 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 - (-1) R_1$

$= LU$

(3) QR Decomposition:

- decomposition of a matrix A into a product $A = QR$ of an orthogonal matrix Q and an upper triangular matrix R.

Applications

- Solve linear equations, least square problems
- gives simple formulas for p-inv. of matrix with linearly independent columns

- $A = [a_1 | a_2 | \cdots | a_n]$

Then $u_1 = a_1$, $e_1 = \frac{u_1}{\|u_1\|}$

$u_2 = a_2 - (a_2 \cdot e_1) e_1$, $e_2 = \frac{u_2}{\|u_2\|}$

$u_3 = a_3 - (a_3 \cdot e_1) e_1 - (a_3 (e_2) e_2$, $e_3 = \frac{u_3}{\|u_3\|}$

$u_{k+1} = u_{k+1} - (a_{k+1} \cdot e_1) e_1 - \cdots - (a_{k+1} \cdot e_k) e_k$ $\quad e_{k+1} = \frac{u_{k+1}}{\|u_{k+1}\|}$

- $A = [e_1 | e_2 | \cdots | e_k] \begin{bmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & \cdots & a_n \cdot e_1 \\ 0 & a_2 \cdot e_2 & \cdots & a_n \cdot e_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \cdot e_n \end{bmatrix} = QR$

eg: $A = \begin{bmatrix} 9 & -1 & 2 \\ -1 & 8 & 5 \\ 2 & 5 & 7 \end{bmatrix}$

Solving $Ax = b$

$Q(Rx) = b \Rightarrow Qy = b$

$y = Rx = Q^T b = c$

$\Rightarrow x = R^{-1} y$

$a_1 = \begin{bmatrix} 9 \\ -1 \\ 2 \end{bmatrix}$ $\quad \|a_1\| = \sqrt{9^2 + (-1)^2 + 2^2} = \sqrt{86} = 9.2736$

$e_1 = \dfrac{a_1}{\|a_1\|} = \begin{bmatrix} 0.9705 \\ -0.1078 \\ 0.2157 \end{bmatrix}$

$r_{11} = a_1 \cdot e_1 = \begin{bmatrix} 9 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.9705 \\ -0.1078 \\ 0.2157 \end{bmatrix} = 9.2736$

$r_{12} = a_2 \cdot e_1 = \begin{bmatrix} -1 & 8 & 5 \end{bmatrix} \begin{bmatrix} 0.9705 \\ -0.1078 \\ 0.2157 \end{bmatrix} = -0.7548$

$r_{13} = a_3 \cdot e_1 = \begin{bmatrix} 2 & 5 & 7 \end{bmatrix} \begin{bmatrix} e_1 \end{bmatrix} = 2.9115$

$a_2 = \begin{bmatrix} -1 \\ 8 \\ 5 \end{bmatrix}$ $\quad u_2 = a_2 - (a_2 \cdot e_1) e_1 = \begin{bmatrix} -1 \\ 8 \\ 5 \end{bmatrix} + 0.7548 \begin{bmatrix} +0.9705 \\ -0.1078 \\ 0.2157 \end{bmatrix} = \begin{bmatrix} -0.2674 \\ 7.9186 \\ 5.1628 \end{bmatrix}$

$e_2 = \dfrac{u_2}{\|u_2\|} = \dfrac{1}{9.4568} \begin{bmatrix} -0.2674 \\ 7.9186 \\ 5.1628 \end{bmatrix} = \begin{bmatrix} -0.0283 \\ 0.8373 \\ 0.5459 \end{bmatrix}$

$r_{22} = a_2 \cdot e_2 = 9.4568$

$r_{23} = a_3 \cdot e_2 = \begin{bmatrix} 2 & 5 & 7 \end{bmatrix} \begin{bmatrix} -0.0283 \\ 0.8373 \\ 0.5459 \end{bmatrix} = 7.9517$

$a_3 = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$, $\quad u_3 = a_3 - (a_3 \cdot e_1) e_1 - (a_3 \cdot e_2) e_2$

$= \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} - 2.9115 \begin{bmatrix} 0.9705 \\ -0.1078 \\ 0.2157 \end{bmatrix} - 7.9517 \begin{bmatrix} -0.0283 \\ 0.8373 \\ 0.5459 \end{bmatrix} = \begin{bmatrix} -0.6007 \\ -1.3444 \\ 2.0309 \end{bmatrix}$

$r_{33} = a_3 \cdot e_3$ $\qquad e_3 = \dfrac{u_3}{\|u_3\|} = \begin{bmatrix} -0.2395 \\ -0.5359 \\ 0.8096 \end{bmatrix}$

$= 2.5086$

$Q = [e_1 | e_2 | e_3] = \left[\begin{array}{c|c|c} 0.9705 & -0.0283 & -0.2395 \\ -0.1078 & 0.8373 & -0.5359 \\ 0.2157 & 0.5459 & -0.8096 \end{array}\right]$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = \begin{bmatrix} 9.2736 & -0.7548 & 2.9115 \\ 0 & 9.4568 & 7.9517 \\ 0 & 0 & 2.5086 \end{bmatrix}$$

(4) Newton - Raphson Method :

- powerful technique for solving equations numerically
- It is a root-finding algorithm which produces successively better approximations to the roots (or zeros) of a real-valued function.
- The most basic version starts with a single-variable function f defined for a real variable x, the function's derivative f' and an initial guess x_0 for root of f. If the function satisfies sufficient assumptions, then initial guess is close, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

is better approximation of the root than x_0

Approximating roots of polynomial equations of any order

- Geometrically $(x_1, 0)$ is the intersection of x-axis and tangent of graph of f at $(x_0, f(x_0))$, that is, improved guess is the unique root of the linear approximation at initial point.

The process is repeated as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until a sufficiently precise value is reached.

System of equations

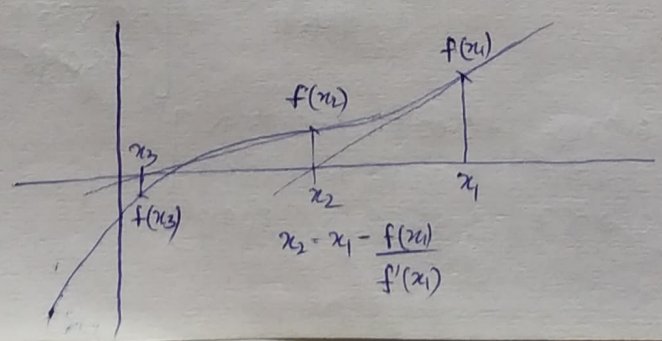
$f_1(x_1 \cdots x_n) = 0$
$f_2(x_1 \cdots x_n) = 0$
$\vdots$
$f_n(x_1 \cdots x_n) = 0$

$f(x^{(0)} + \Delta x^{(0)}) = f(x^{(0)}) + J^{(0)} \Delta x^{(0)}$

where J is $n \times n$ Jacobian

$$J = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{bmatrix}$$

- The <u>overall idea</u> is to start with an initial guess which is reasonably close to the true root, then to approximate the function by its tangent line using calculus and finally to compute the x-intercept of this tangent line by elementary algebra.



$$\Delta x^{(0)} = -(J^{(0)})^{-1} f(x^{(0)})$$

$$x^{(1)} = x^{(0)} + \Delta x^{(0)}$$

$$x^{(v+1)} = x^{(v)} - (J^{v})^{-1} f(x^{v})$$

or

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{J^{(n)}}}$$

eg. Find a root of an equation $f(x) = 2x^3 - 2x - 5$ with initial solution $x_0 = 2$ using a Newton Raphson method.

$f(x) = 2x^3 - 2x - 5$

$f'(x) = 6x^2 - 2$

$x_0 = 2$

First iteration

$f(x_0) = f(2) = 2 \cdot 2^3 - 2 \cdot 2 - 5 = 7$

$f'(x_0) = f'(2) = 6 \cdot 2^2 - 2 = 22$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{7}{22} = 1.6818$$

Second iteration:

$f(x_1) = f(1.6818) = 2 \cdot (1.6818)^3 - 2(1.6818) - 5 = 1.1505$

$f'(x_1) = f'(1.6818) = 6(1.6818)^2 - 2 = 14.9711$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.6818 - \frac{1.1505}{14.9711} = 1.605$$

Third iteration:

$f(x_2) = f(1.605) = 2 \cdot (1.605)^3 - 2(1.605) - 5 = 0.0587$

$f'(x_2) = f(1.605) = 6 \cdot (1.605)^2 - 2 = 13.4556$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.605 - \frac{0.0587}{13.4556} = 1.6006$$

Fourth iteration:

$f(x_3) = f(1.606) = 2 \cdot (1.606)^3 - 2 \cdot (1.6006) - 5 = 0.0002$

$f'(x_3) = f'(1.6006) = 6 \cdot (1.6006)^2 - 2 = 13.3718$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.6006 - \frac{0.0002}{13.3718} = 1.6006$$

n	x_0	$f(x_0)$	$f'(x_0)$	x_1	Update
1	2	7	22	1.6818	$x_0 = x_1$
2	1.6818	1.1505	14.9711	1.605	$x_0 = x_1$
3	1.605	0.0587	13.4556	1.6006	$x_0 = x_1$
4	1.6006	0.0002	13.3718	1.6006	$x_0 = x_1$

(5) Adams - Bashforth technique :

- Methods like Euler, Runge-Kutta have been presented so far are called <u>single-step methods</u> because they use only the information from one previous point to compute the successive point, that is, only the initial point (x_0, y_0) is used to compute (x_1, y_1) and in general, y_i is needed to compute y_{i+1}.
- After several points have been found, it is feasible to use several prior points in the calculation. This is the basis of <u>multi-step method</u>
- Adams-Bashforth is a four-step method, in which y_{i-3}, y_{i-2}, y_{i-1} and y_i is required in calculation of y_{i+1}
- <u>Adams-Bashforth formula:</u> (Predictor)

$$y_{i+1,p} = y_0 + \frac{h}{4}\left[55y'_i - 59y'_{i-1} + 37y'_{i-2} - 9y'_{i-3}\right]$$

<u>Adams-Moulton corrector formula:</u>

$$y_{i+1,c} = y_i + \frac{h}{24}\left[9f^p_{i+1} + 19f_i - 5f_{n-1} + f_{i-2}\right]$$

$c \rightarrow$ corrected value

$p \rightarrow$ predicted value

<u>Example:</u> Using Adams-Bashforth formula, find $y(t=0.4)$ for the following diff. equation:

$$y' = 1 - t + 4y \quad \text{with} \quad y(0) = 1 \text{ and step size } h = 0.1$$

<u>Sol:</u> First we need to find four starting values.

Using Euler Method,

$$y_{i+1} = y_i + h f(x_i, y_i) \qquad \text{Here } x = t$$

$x_0 = 0$, $y_0 = y(0) = 1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$, $y_1 = y_0 + h f(x_0, y_0)$

$= 1 + 0.1(1 - 0 + 4(1)) = 1.5$

$x_2 = 0.1 + 0.1 = 0.2$, $y_2 = y_1 + h f(x_1, y_1)$

$y_2 = 1.5 + 0.1(1 - 0.1 + 4(1.5))$

$= 2.19$

$x_3 = 0.3$, $y_3 = y_2 + h f(x_2, y_2)$

$= 2.19 + 0.1(1 - 0.2 + 4(2.19)) = 3.146$

four values are

x	y
0	1
0.1	1.5
0.2	2.19
0.3	3.146

Adam's Bashforth formula is

$$y_{i+1,p} = y_i + \frac{h}{24}(55 y'_i - 59 y'_{i-1} + 37 y'_{i-2} - 9 y'_{i-3})$$

Putting $i=3$

$$y_{4,p} = y_3 + \frac{h}{24}(55 y'_3 - 59 y'_2 + 37 y'_1 - 9 y'_0) \quad \text{——— (1)}$$

Now,

$y'_0 = 1 - x + 4y \big|_{x=0, y=1} = 1 - 0 + 4 = 5$

$y'_1 = 1 - x + 4y \big|_{x=0.1, y=1.5} = 1 - 0.1 + 4(1.5) = 6.9$

$y'_2 = 1 - x + 4y \big|_{x=0.2, y=2.19} = 9.56$

$y'_3 = 1 - x + 4y \big|_{x=0.3, y=3.146} = 13.284$

Putting in eq. (1)

$$y_{4,p} = 3.146 + \frac{0.1}{24}(55(13.284) - 59(9.56) + 37(6.9) - 9(5))$$

$= 4.7163$ (Predicted Value)

Now, using Adam's Corrector formula,

$$y_{i+1,c} = y_i + \frac{h}{24}(9 y'_{i+1} + 19 y'_i - 5 y'_{i-1} + y'_{i-2})$$

Putting $i = 3$

$$y_{4,c} = y_3 + \frac{h}{24}(9y'_4 + 19y'_3 - 5y'_2 + y'_1)$$

$$y_{4,c} = 3.146 + \frac{0.1}{24}(9y'_4 + 19(13.284) - 5(9.56) + 6.9) \quad \text{—— (2)}$$

$$y'_4 = 1 - x + 4y \big|_{x=0.4,\, y=4.7163} = 19.4653$$

Putting y'_4 in eq. (2)

$$y_{4,c} = 4.7572$$

$$\boxed{\therefore\ y(0.4) = 4.7572}$$

Singular Value Decomposition:

Let A be $m \times n$ matrix, then

$$A = U \Sigma V^T$$

Specifically,

- The ordering of vector comes from the ordering of singular values (largest to smallest)
- The columns of U are eigenvectors of AA^T
- The columns of V are eigenvectors of A^TA
- The diagonal elements of Σ are singular values, $\sigma_i = \sqrt{\lambda_i}$
- Pseudoinverse of A: $A^+ = VS^+U^T$

  S^+ is n by m matrix with diagonals as $\frac{1}{\sigma_1}$ ---- $\frac{1}{\sigma_r}$

Eg: $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$A \cdot A^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Eigen values of $A \cdot A^T$

$$|A \cdot A^T - \lambda I| = 0 \rightarrow \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 1, 3$$

Eigen vector for $\lambda = 3$

$$A \cdot A^T - \lambda I = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_1 - R_2} \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$$

$$v_1 = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} \quad \text{Let } x_2 = 1 \Rightarrow v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Eigen vector for $\lambda = 1$

$$A \cdot A^T - \lambda I = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

$$v_2 = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for v_1, $\|v_1\| = \sqrt{(-1)^2 + (1)^2} = 1.414$

$$u_1 = \left(\frac{-1}{1.414}, \frac{1}{1.414} \right) = (-0.707, 0.707)$$

for $v_2 = \|v_2\| = \sqrt{1^2 + 1^2} = 1.414$

$$u_2 = \left(\frac{1}{1.414}, \frac{1}{1.414} \right) = (0.707, 0.707)$$

$$\therefore U = [u_1 \ u_2] = \begin{bmatrix} -0.707 & 0.707 \\ 0.707 & 0.707 \end{bmatrix}$$

And

$$\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{1} & 0 \end{bmatrix} = \begin{bmatrix} 1.732 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

V is found using formula $v_i = \frac{1}{\sigma_i} A^T u_i$

$$v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -0.707 \\ 0.707 \end{bmatrix} = \begin{bmatrix} 0.4082 \\ -0.8164 \\ 0.4082 \end{bmatrix}$$

$$v_2 = \frac{1}{\sqrt{1}} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}$$

To find v_3

$$A^TA = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$|A^TA - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(1-\lambda)-1\right] + 1\left(-1\times(1-\lambda)\right) = 0$$

$$\Rightarrow (1-\lambda)(1-3\lambda+\lambda^2) + (-1+\lambda) = 0$$

$$\Rightarrow 1-4\lambda+4\lambda^2-\lambda^3+(-1+\lambda) = 0$$

$$\Rightarrow (-\lambda^3+4\lambda^2-3\lambda) = 0$$

$$\Rightarrow -\lambda(\lambda-1)(\lambda-3) = 0 \Rightarrow \lambda = 0, 1, 3$$

for $\lambda = 0$

$$A^TA - \lambda I = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 + R_2]{R_1 \leftarrow R_1 + R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \left.\begin{matrix} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{matrix}\right\} \begin{matrix} x_1 = x_3 \\ x_2 = x_3 \end{matrix}$$

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\|v_3\| = \sqrt{1^2+1^2+1^2} = \sqrt{3} = 1.732$$

$$v_3 = \left(\frac{1}{1.732}, \frac{1}{1.732}, \frac{1}{1.732}\right) = (0.577, 0.577, 0.577)$$

Then

$$V = [v_1 \ v_2 \ v_3]$$

$$V = \begin{bmatrix} 0.4082 & -0.7071 & 0.5773 \\ -0.8165 & 0 & 0.5773 \\ 0.4082 & 0.707 & 0.5773 \end{bmatrix}$$

Pseudo inverse of A is

$$A^+ = V S^+ U^T$$

$$= \begin{bmatrix} 0.4082 & -0.7071 & 0.5773 \\ -0.8165 & 0 & 0.5773 \\ 0.4082 & 0.7071 & 0.5773 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{1}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.707 & 0.707 \\ 0.707 & 0.707 \end{bmatrix}$$

$$= \begin{bmatrix} -0.667 & -0.333 \\ 0.333 & -0.333 \\ 0.333 & 0.667 \end{bmatrix}$$

Runge-Kutta Method

Let an initial value problem be specified as

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

Pick a step size $h > 0$ and define

$$y_{n+1} = y_n + \frac{1}{6} h (k_1 + 2k_2 + 2k_3 + k_4)$$

$$x_{n+1} = x_n + h$$

here,

$$k_1 = f(x_n, y_n)$$

$$k_2 = f\left(x_n + \frac{h}{2}, y_n + h\frac{k_1}{2}\right)$$

$$k_3 = f\left(x_n + \frac{h}{2}, y_n + h\frac{k_2}{2}\right)$$

$$k_4 = f(x_n + h, y_n + h k_3)$$

Euler's Method

$$y(x+h) = y(x) + h f(x,y)$$

$$y_1 = y_0 + h f(x_0, y_0)$$

$$\boxed{\begin{aligned} y_{n+1} &= y_n + h f(x_n, y_n) \\ x_{n+1} &= x_n + h \end{aligned}}$$

eg. By R-K method, solve $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$ for $x = 0.2$ using the initial condition $y = 1$ & $y' = 0$ when $x = 0$

Sol: Given eqn. $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$

Put $\frac{dy}{dx} = z$ and $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Then, $\frac{dz}{dx} = xz^2 - y^2$ with $y_0 = 1$ & $y_0' = 0$ when $x = 0$

We have

$$\frac{dy}{dx} = z = f(x,y,z)$$

$$\frac{dz}{dx} = xz^2 - y^2 = g(x,y,z)$$

$$y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \text{—— (1)}$$

$x_0 + h = 0.2 \Rightarrow h = 0.2 \quad \because x_0 = 0$

$k_1 = hf(x_0, y_0, z_0) = 0.2 f(0,1,0) = 0.2 \times 0 = 0$

$l_1 = hg(x_0, y_0, z_0) = 0.2\, g(0,1,0) = (0.2)(0.0^2 - 1^2) = -0.2$

$k_2 = hf(x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) = 0.2\, f(0.1, 1, -0.1) = (0.2)(-0.1) = -0.02$

$h_2 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}) = 0.2\, g(0.1, 1, -0.1) = (0.2)[(0.1)(-0.1)^2 - 1^2]$

$= -0.1998$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}) = 0.2 f(0.1, 0.99, -0.0999) = 0.2(-0.0999)$

$= -0.01998$

$l_3 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}) = 0.2\, g(0.1, 0.99, -0.0999)$

$= (0.2)[(0.1)(-0.0999)^2 - (0.99)^2] = -0.1958$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2 f(0.2, 0.98002, -0.1958) = 0.2(-0.1958)$

$= -0.03916$

$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2\, g(0.2, 0.98002, -0.1958)$

$= (0.2)[(0.2)(-0.1958)^2 - (0.98002)^2]$

$= -0.19055$

Substitute in eq. (1)

$$y(0.2) = 1 + \frac{1}{6}(0 + 2(-0.002) + 2(-0.01998) + (-0.03916))$$

$\therefore y(0.2) = 0.9801$

eg: Find the solution for 2nd order differential equation using R-K method for value of $x = 2$

$y'' - 3y' + 2y = 0$ with initial cond^n $y(0) = 4$, $y'(0) = 5$

Sol: $z = y'$ and $z' = y'$

Then, $z' - 3z + 2y = 0$

Now, $y' = z = f(y, z) = f(x, y, z)$

$z' = 3z - 2y = g(x, y, z)$

Let $h = 1$

$$y(0+1) = y(0) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \text{——(1)}$$

$k_1 = hf(x_0, y_0, z_0) = 1f(0, 4, 5) = 1 \times 5 = 5$

$l_1 = hg(x_0, y_0, z_0) = 1\,g(0, 4, 5) = 1(3 \times 5 - 2 \times 4) = 7$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = 1.f(0.5, 6.5, 8.5) = 8.5$

$l_2 = g(0.5, 6.5, 8.5) = 3 \times 8.5 - 2 \times 6.5 = 12.5$

$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = f(0.5, 8.25, 11.25) = 11.25$

$l_3 = g(0.5, 8.25, 11.25) = 3 \times 11.25 - 2 \times 8.25 = 17.25$

$k_4 = f(x_0 + h, y_0 + k_3, z_0 + l_3) = f(0.5, 15.25, 22.25) = 22.25$

$l_4 = g(0.5, 15.25, 22.25) = 3 \times 22.25 - 2 \times 15.25 = 36.25$

Put in (1)

$$y(1) = 4 + \frac{1}{6}(5 + (2 \times 8.5) + (2 \times 11.25) + 22.25) = 15.125$$

Now,

$$y(2) = y(1) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \text{——(2)}$$

$k_1 = f(x_1, y_1, z_1) = f(1, 15.125, z_1)$

$z(1) = z(0) + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 5 + \frac{1}{6}(7 + 2 \times 12.5 + 2 \times 17.25 + 36.25)$

$= 22.125$

$k_1 = f(1, 15.125, 22.125) = 22.125$

$l_1 = g(1, 15.125, 22.125) = 3 \times 22.125 - 2 \times 15.125 = 36.125$

$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right) = f(1.5, 15.0625, 23.0625) = 23.0625$

$l_2 = g(1.5, 15.0625, 23.0625) = 3 \times 23.0625 - 2 \times 15.0625 = 39.0625$

$k_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right) = f(1.5, 15.53125, 24.53125) = 24.53125$

$l_3 = g(1.5, 15.53125, 24.53125) = 3 \times 24.53125 - 2 \times 15.53125 = 42.53125$

$k_4 = f(x_1 + h, y_1 + k_3, z_1 + l_3) = f(2, 28.53125, 4$

$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right) = f(1.5, 26.1875, 40.1875) = 40.1875$

$l_2 = 3 \times 40.1875 - 2 \times 26.1875 = 68.1875$

$k_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right) = f(1.5, 35.219, 56.219) = 56.219$

$l_3 = 3 \times 56.219 - 2 \times 35.219 = 98.218$

$k_4 = f(x_1 + h,\ y_1 + k_3,\ z_1 + l_3) = f(2, 71.344, 120.343) = 120.343$

$l_4 = 3 \times 120.343 - 2 \times 71.344 = 218.341$

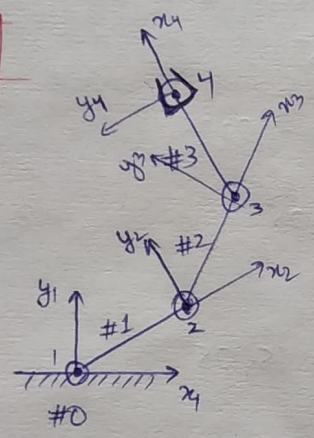
Put in eq. (2)

$$y(2) = 15.125 + \frac{1}{6}(22.125 + 2 \times 40.1875 + 2 \times 56.219 + 120.343)$$

$$= 71.0052 \quad \text{Ans.}$$

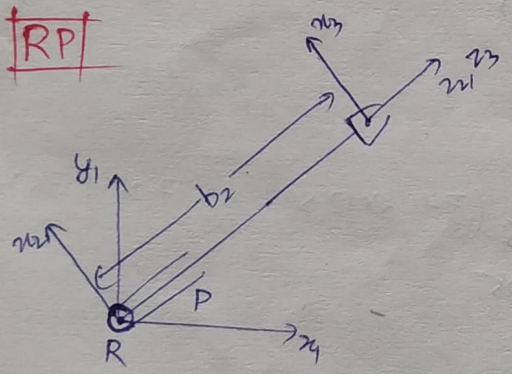
DH Parameters:

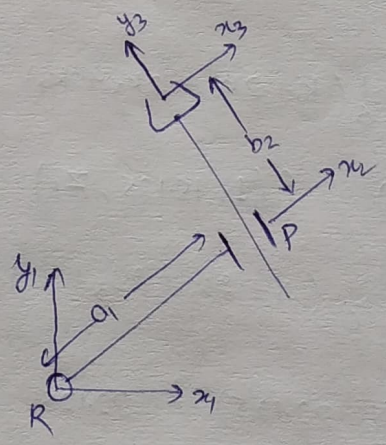
RRR



Link #	b_i	θ_i	a_i	α_i
1	0	θ_1	a_1	0
2	0	θ_2	a_2	0
3	0	θ_3	a_3	0

RP

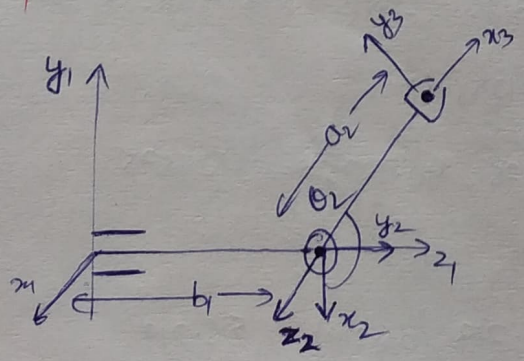




Link #	b_i	θ_i	a_i	α_i
1 (R)	0	θ_1	0	$\pi/2$
2 (P)	b_2	0	0	0

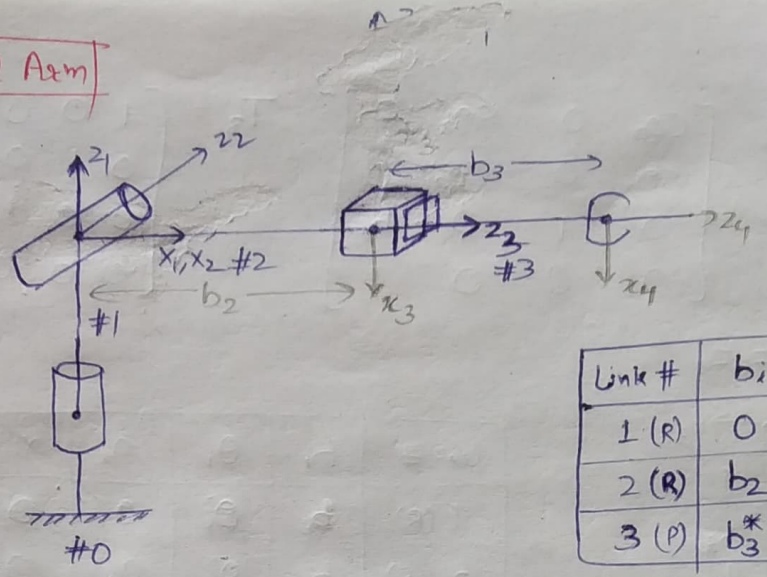
Link #	b_i	θ_i	a_i	α_i
1 (R)	0	θ_1	a_1	$-\pi/2$
2 (P)	b_2	0	0	$\pi/2$

PR



Link #	b_i	θ_i	a_i	α_i
1 (P)	b_1	$-\pi/2$	0	$\pi/2$
2 (R)	0	θ_2	a_2	0

Spherical Arm



Link #	b_i	θ_i	a_i	α_i
1 (R)	0	θ_1^*	0	$\pi/2$
2 (R)	b_2	θ_2^*	0	$\pi/2$
3 (P)	b_3^*	0	0	0

Homogeneous Transformation matrix:

$$T_i = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & b_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for RP(I)

$$T_1 = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 & 0 \\ s\theta_1 & 0 & -c\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 & b_2 s\theta_1 \\ s\theta_1 & 0 & -c\theta_1 & -b_2 c\theta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for RP(II)

$$T_1 = \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & a_1 c\theta_1 \\ s\theta_1 & 0 & c\theta_1 & a_1 s\theta_1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

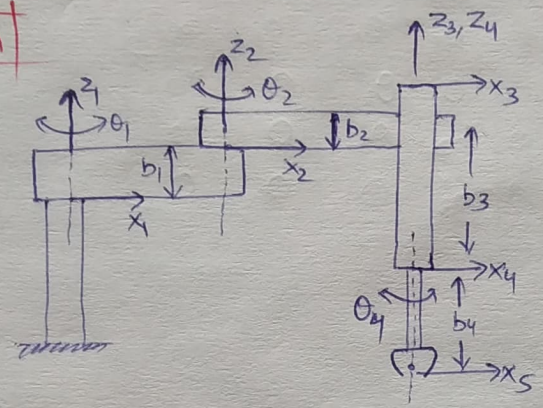
for PR

$$T_1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for Spherical arm,

$$T_1 = \begin{bmatrix} C\theta_1 & 0 & S\theta_1 & 0 \\ S\theta_1 & 0 & -C\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2 = \begin{bmatrix} C\theta_2 & 0 & S\theta_2 & 0 \\ S\theta_2 & 0 & -C\theta_2 & 0 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

SCARA

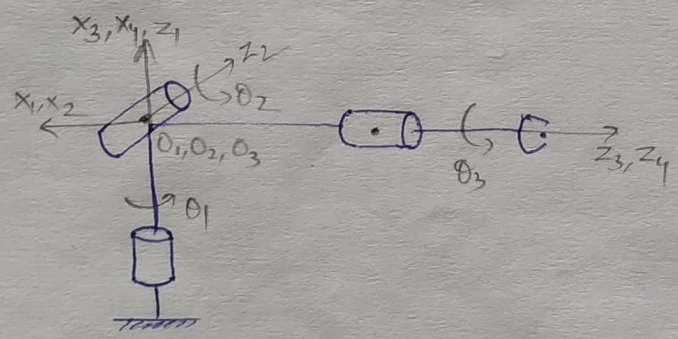


Link #	b_i	θ_i	a_i	α_i
1 (R)	b_1	θ_1^*	a_1	0
2 (R)	b_2	θ_2^*	a_2	0
3 (P)	b_3^*	0	0	0
4 (R)	$-b_4$	θ_4^*	0	0

$$T_1 = \begin{bmatrix} C_1 & -S_1 & 0 & a_1C_1 \\ S_1 & C_1 & 0 & a_1S_1 \\ 0 & 0 & 1 & b_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & b_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4 = \begin{bmatrix} C_4 & -S_4 & 0 & 0 \\ S_4 & C_4 & 0 & 0 \\ 0 & 0 & 1 & -b_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow T = \begin{bmatrix} C_{124} & -S_{124} & 0 & a_1C_1 + a_2C_{12} \\ S_{124} & C_{124} & 0 & a_1S_1 + a_2S_{12} \\ 0 & 0 & 1 & b_1 + b_2 + b_3 - b_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Spherical Wrist

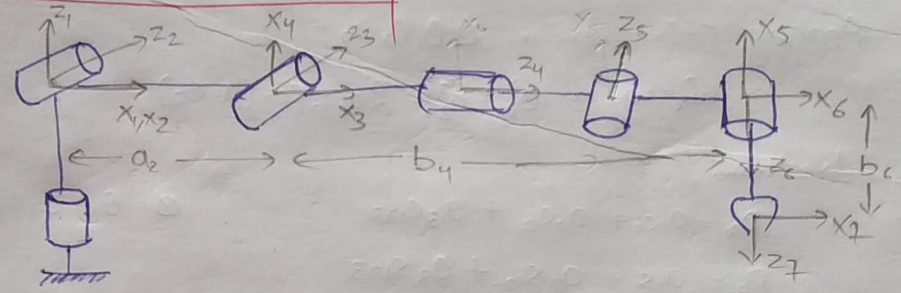


Link #	b_i	θ_i	a_i	α_i
1 (R)	0	θ_1^*	0	$\pi/2$
2 (R)	0	θ_2^*	0	$-\pi/2$
3 (R)	0	θ_3^*	0	0

$$T_1 = \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2 = \begin{bmatrix} C_2 & 0 & -S_2 & 0 \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_3 = \begin{bmatrix} C_3 & -S_3 & 0 & 0 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

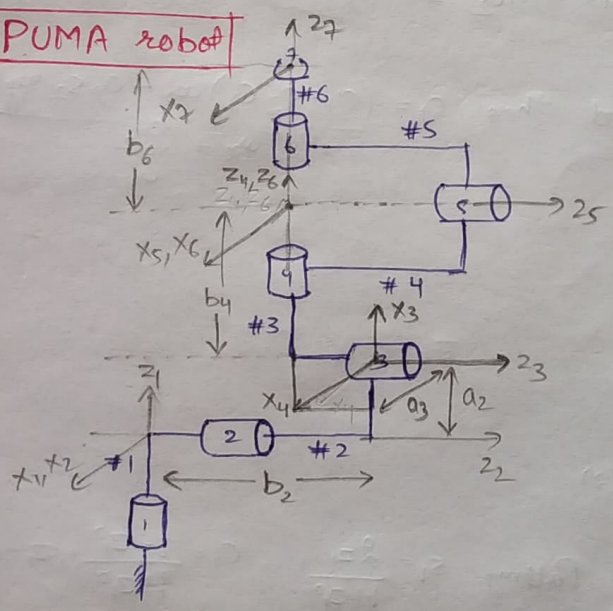
$$T = \begin{bmatrix} C_1C_2C_3 - S_1S_3 & -C_1C_2S_3 - S_1C_3 & -C_1S_2 & 0 \\ S_1C_2C_3 + C_1S_3 & -S_1C_2S_3 + C_1C_3 & -S_1S_2 & 0 \\ S_2C_3 & -S_2S_3 & C_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Articulated 6-dof robot



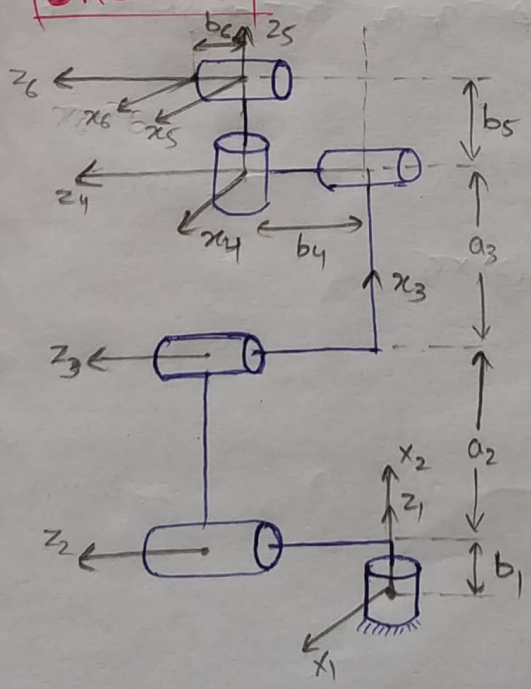
Link i	b_i	θ_i	a_i	α_i
1	0	θ_1^*	0	$-\pi/2$
2	0	θ_2^*	a_2	0
3	0	θ_3^*	0	$-\pi/2$
4	b_4	θ_4^*	0	$\pi/2$
5	0	θ_5^*	0	$-\pi/2$
6	b_6	θ_6^*	0	0

PUMA robot



Link #	b_i	θ_i	a_i	α_i
1	0	θ_1^* (0)	0	$-\pi/2$
2	b_2	θ_2^* ($-\pi/2$)	a_2	0
3	0	θ_3^* ($\pi/2$)	a_3	$\pi/2$
4	b_4	θ_4^* (0)	0	$-\pi/2$
5	0	θ_5^* (0)	0	$\pi/2$
6	b_6	θ_6^* (0)	0	0

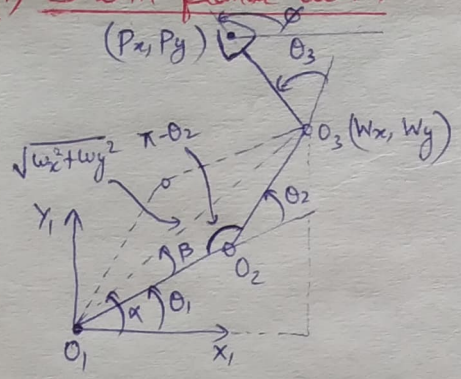
UR5 robot



Link #	b_i	θ_i	a_i	α_i
1	b_1	θ_1^*	0	$\pi/2$
2	0	θ_2^*	a_2	0
3	0	θ_3^*	a_3	0
4	b_4	θ_4^*	0	$-\pi/2$
5	b_5	θ_5^*	0	$\pi/2$
6	b_6	θ_6^*	0	0

Inverse Position Kinematics

(1) 3-link planar arm :



(P_x, P_y) & ϕ given

find $\theta_1, \theta_2, \theta_3$

Applying cosine law in $\Delta O_1 O_2 O_3$

$$W_x^2 + W_y^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\pi-\theta_2)$$

We know,

$$P_x = a_1c_1 + a_2c_{12} + a_3c_{123} \qquad \phi = \theta_1+\theta_2+\theta_3$$

$$P_y = a_1s_1 + a_2s_{12} + a_3s_{123}$$

$$\left.\begin{aligned} W_x &= P_x - a_3c_\phi \\ W_y &= P_y - a_3s_\phi \\ \Rightarrow W_x &= a_1c_1 + a_2c_{12} \\ W_y &= a_1s_1 + a_2s_{12} \end{aligned}\right\} \Rightarrow W_x^2+W_y^2 = a_1^2+a_2^2+2a_1a_2c_2$$

$$c_2 = \frac{W_x^2+W_y^2-a_1^2-a_2^2}{2a_1a_2}$$

$$s_2 = \pm\sqrt{1-c_2^2}$$

$$\boxed{\theta_2 = \operatorname{atan2}(s_2, c_2)}$$

To find θ_1,

$$W_x = a_1c_1 + a_2c_1c_2 - a_2s_1s_2 = (a_1+a_2c_2)c_1 - a_2s_1s_2 \quad \text{—(1)}$$

$$W_y = a_1s_1 + a_2s_1c_2 + a_2c_1s_2 = (a_1+a_2c_2)s_1 + a_2c_1c_2 \quad \text{—(2)}$$

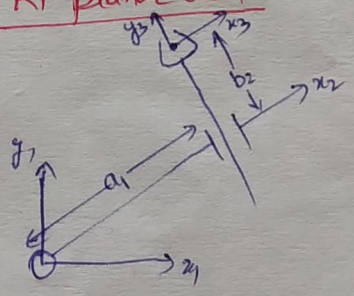
$(1)\times a_2s_2 - (2)\times(a_1+a_2c_2)$

$$s_1 = \frac{(a_1+a_2c_2)W_y - a_2s_2W_x}{W_x^2+W_y^2} \quad \text{and then} \quad c_1 = \frac{(a_1+a_2c_2)W_x + a_2s_2W_y}{W_x^2+W_y^2}$$

$$\theta_1 = \operatorname{atan2}(s_1, c_1)$$

$$\theta_3 = \phi - \theta_1 - \theta_2$$

(2) RP planar arm :



$$P_x = a_1c_1 - b_2s_1 \quad \text{—(1)}$$

$$P_y = a_1s_1 + b_2c_1 \quad \text{—(2)}$$

$(1)\times c_1 + (2)\times s_1$ Putting $s_1 = \frac{2z_1}{1+z_1^2}$, $c_1 = \frac{1-z_1^2}{1+z_1^2}$, $z_1 = \tan\frac{\theta_1}{2}$

$$P_xc_1 + P_ys_1 = a_1$$

$$\Rightarrow Az_1^2 + Bz_1 + C = 0 \quad \text{where } A = a_1+P_x,\ B = -2P_y,\ C = a_1 - P_x$$

$$z_{1,2} = \frac{-B \pm \sqrt{B^2-4AC}}{2A}$$

then $\theta_1^{(i)} = 2\tan^{-1}(z_1)$

Put θ_1 in (1) & (2), then $(1)\times s_1 - (2)\times c_1$

$$b_2^{(i)} = P_yc_1^{(i)} - P_xs_1^{(i)}$$

Jacobian:

$$\mathcal{t}_e = J\dot{\theta}$$

for every joint i:

$$T_i = \left[\begin{array}{ccc|c} x_i & y_i & z_i & O_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Then

for Revolute:

$$J[:, i] = \left[\begin{array}{c} z_i \times (O_i - O_{i-1}) \\ \hline z_i \end{array}\right] \qquad \text{if } i=1 \quad \left[\begin{array}{c} z_i \times (O_i - \vec{0}) \\ \hline z_i \end{array}\right]$$

for Prismatic:

$$J[:, i] = \left[\begin{array}{c} z_i \\ \hline 0 \end{array}\right] \qquad \text{if } i=1 \quad \left[\begin{array}{c} z_i \\ \hline 0 \end{array}\right]$$

Inverse velocity analysis:

$$\dot{\theta} = J^{-1}\mathcal{t}_e$$

The explicit inversion of J is never recommended as it is numerically expensive requiring order (n^3) computations.

Instead LU decomposition of J is used, followed by forward and backward substitutions.

$$J = LU$$

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

$$Ly = \mathcal{t}_e \quad \text{and} \quad U\dot{\theta} = y$$

y is first solved and then $\dot{\theta}$ by applying only forward and backward substitutions. Note that, LU decomposition requires only $O(n^3/3)$ arithematic operations followed by $O(n^2/2)$ operations for each forward & backward substitutions.

Dynamics

1) Euler-Lagrange formulation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = u_i$$

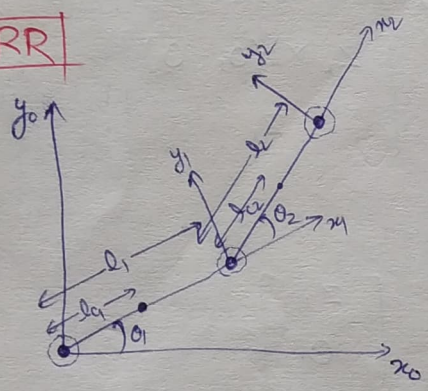
non-conservative (active / dissipative) generalized forces performing work on q_i coordinate

Lagrangian: $L = T - U$

K.E: $T = \frac{1}{2}\dot{q}^T M \dot{q} := \frac{1}{2} m v^T v + \frac{1}{2}\omega^T I \omega$

P.E: $U = g^T r_{c_i} m_i$

RR



$$r_{c_1} = \begin{bmatrix} l_{c_1} c_1 \\ l_{c_1} s_1 \\ 0 \end{bmatrix} \qquad r_{c_2} = \begin{bmatrix} l_1 c_1 + l_{c_2} c_{12} \\ l_1 s_1 + l_{c_2} s_{12} \\ 0 \end{bmatrix}$$

$$v_{c_1} = \begin{bmatrix} -l_{c_1} s_1 \dot{\theta}_1 \\ l_{c_1} c_1 \dot{\theta}_1 \\ 0 \end{bmatrix} \qquad v_{c_2} = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_{c_2} s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

$$v_{c_1}^T v_{c_1} = l_{c_1}^2 \dot{\theta}_1^2$$

$$v_{c_2}^T v_{c_2} = l_1^2 \dot{\theta}_1^2 + l_{c_2}^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 l_1 l_{c_2} s_1 s_{12} \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) + \underbrace{2 l_1 l_{c_2} c_1 c_{12} \dot{\theta}_1 (\dot{\theta}_1 + \theta_2)}$$

$$= 2 l_1 l_{c_2} c_2 \dot{\theta}_1^2 + 2 l_1 l_{c_2} c_2 \dot{\theta}_1 \dot{\theta}_2 + l_1^2 \dot{\theta}_1^2 + l_{c_2}^2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \qquad \omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{12} \end{bmatrix}$$

$$T_1 = \frac{1}{2} m_1 v_{c_1}^T v_{c_1} + \frac{1}{2} \omega^T I_{122} \omega$$

$$= \frac{1}{2} m_1 l_{c_1}^2 \dot{\theta}_1^2 + \frac{1}{2} I_{1,22} \dot{\theta}_1^2$$

$$T_2 = \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_{c_2}^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 l_1 l_{c_2} c_2 \dot{\theta}_1^2 + 2 l_1 l_{c_2} c_2 \dot{\theta}_1 \dot{\theta}_2 \right) + \frac{1}{2} I_{2,22} \dot{\theta}_{12}^2$$

$$U_1 = [0 \; -g \; 0] \begin{bmatrix} l_{c_1} c_1 \\ l_{c_1} s_1 \\ 0 \end{bmatrix} m_1 = -m_1 l_{c_1} g s_1$$

$$U_2 = -m_2 g (l_1 s_1 + l_{c_2} s_{12})$$

$$L_1 = T_1 - U_1$$

$$\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \tau_1$$

$$\frac{d}{dt}\left(m_1 l_{c_1}^2 \dot{\theta}_1 + \cancel{2} I_{1,22} \dot{\theta}_1 \right) + m_1 g l_{c_1} c_1 + m_2 g (l_1 c_1 + l_{c_2} c_{12}) = \tau_1$$

$$\left(+ m_2 l_1^2 \dot{\theta}_1 + m_2 l_{c_2}^2 (\dot{\theta}_1 + \dot{\theta}_2) + 2 l_1 l_{c_2} c_2 \dot{\theta}_1 + 2 l_1 l_{c_2} c_2 \dot{\theta}_2 + I_{2,22} (\dot{\theta}_1 + \dot{\theta}_2) \right)$$

$$m_1 l_{c_1}^2 \ddot{\theta}_1 + I_{1,22}\ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_{c_2}^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + (2 l_1 l_{c_2} c_2 \ddot{\theta}_1 + l_1 l_{c_2} c_2 \ddot{\theta}_2) m_2$$
$$+ I_{2,22}\ddot{\theta}_1 + I_{2,22}\ddot{\theta}_2 + (-2 l_1 l_{c_2} s_2 \dot{\theta}_1 \dot{\theta}_2 - l_1 l_{c_2} s_2 \dot{\theta}_2^2) m_2 = \tau_1 \quad \text{——(1)}$$

And,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = \tau_2$$

$$\frac{d}{dt}\left(m_2 l_{c_2}^2 (\dot{\theta}_1 + \dot{\theta}_2) + m_2 l_1 l_{c_2} c_2 \dot{\theta}_1\right) + m_2 g (l_{c_2} c_{12}) + m_2 l_1 l_{c_2} s_2 \dot{\theta}_1^2 + m_2 l_1 l_{c_2} s_2 \dot{\theta}_1 \dot{\theta}_2 = \tau_2$$

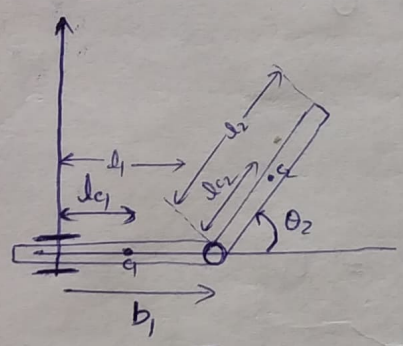
$$m_2 l_{c_2}^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_{c_2} c_2 \ddot{\theta}_1 + I_{2,22}\ddot{\theta}_2 + m_2 l_1 l_{c_2} s_2 \dot{\theta}_1^2 = \tau_2 \quad \text{——(2)}$$
$$\cancel{- m_2 l_1 l_{c_2} s_2 \dot{\theta}_1 \dot{\theta}_2} + \cancel{m_2 l_1 l_{c_2} s_2 \dot{\theta}_1 \dot{\theta}_2}$$

$$\begin{bmatrix} m_1 l_{c_1}^2 + I_{1,22} + m_2 (l_1^2 + l_{c_2}^2 + 2 l_1 l_{c_2} c_2) + I_{2,22} & m_2 l_{c_2}^2 + m_2 l_1 l_{c_2} c_2 + I_{2,22} \\ m_2 l_{c_2}^2 + m_2 l_1 l_{c_2} c_2 + I_{2,22} & m_2 l_{c_2}^2 + I_{2,22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -2 l_1 l_{c_2}^{m_2} s_2 \dot{\theta}_1 \dot{\theta}_2 - l_1 l_{c_2}^{m_2} s_2 \dot{\theta}_2^2 \\ m_2 l_1 l_{c_2} s_2 \dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} m_1 g l_{c_1} c_1 + m_2 g (l_1 c_1 + l_2 c_{12}) \\ m_2 g (l_{c_2} c_{12}) \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$I\ddot{\theta} + C + G = \tau$$

PR



$$r_{c_1} = \begin{bmatrix} b_1 - l_{c_1} \\ 0 \\ 0 \end{bmatrix} \qquad r_{c_2} = \begin{bmatrix} b_1 + l_{c_2} c_2 \\ l_{c_2} s_2 \\ 0 \end{bmatrix}$$

$$v_{c_1} = \begin{bmatrix} \dot{b}_1 \\ 0 \\ 0 \end{bmatrix} \qquad v_{c_2} = \begin{bmatrix} \dot{b}_1 - l_{c_2} s_2 \dot{\theta}_2 \\ l_{c_2} c_2 \dot{\theta}_2 \\ 0 \end{bmatrix} \qquad \omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix}$$

$$v_{c_1}^T v_{c_1} = \dot{b}_1^2 \qquad v_{c_2}^T v_{c_2} = \dot{b}_1^2 + l_{c_2}^2 \dot{\theta}_2^2 - 2 l_{c_2} s_2 \dot{b}_1 \dot{\theta}_2$$

$$T_1 = \frac{1}{2} m_1 v_{c_1}^T v_{c_1} = \frac{1}{2} m_1 \dot{b}_1^2$$

$$T_2 = \frac{1}{2} m_2 v_{c_2}^T v_{c_2} + \frac{1}{2} I_{2,22} \omega_2^T \omega_2 = \frac{1}{2} m_2 (\dot{b}_1^2 + l_{c_2}^2 \dot{\theta}_2^2 - 2 l_{c_2} s_2 \dot{b}_1 \dot{\theta}_2) + \frac{1}{2} I_{2,22} \dot{\theta}_2^2$$

$$U_1 = [0 \ -g \ 0] \begin{bmatrix} b_1 - l_{c_1} \\ 0 \\ 0 \end{bmatrix} m_1 = 0 \qquad T = T_1 + T_2 \qquad L = T - U$$
$$U = U_1 - U_2$$

$$U_2 = [0 \ -g \ 0] \, r_{c_2} m_2 = -m_2 g l_{c_2} s_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{b}_1}\right) - \frac{\partial L}{\partial b_1} = \tau_1$$

$$\frac{d}{dt}(m_1 \dot{b}_1 + m_2 \dot{b}_1 - 2 m_2 l_{c_2} s_2 \dot{\theta}_2) = \tau_1$$

$$\Rightarrow m_1 \ddot{b}_1 + m_2 \ddot{b}_1 - m_2 l_{c_2} s_2 \ddot{\theta}_2 - m_2 l_{c_2} c_2 \dot{\theta}_2^2 = \tau_1 \quad \text{——(1)}$$

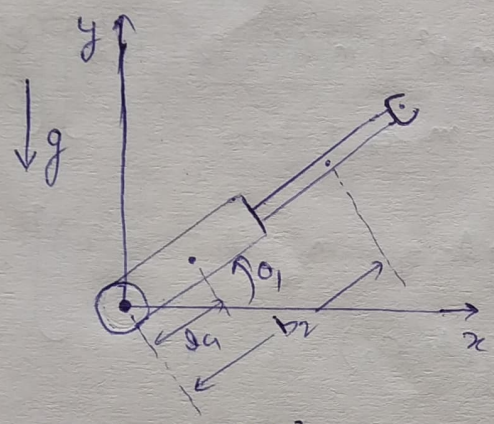
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = \tau_2$$

$$\Rightarrow \frac{d}{dt}\left(m_2 l_{c_2}^2 \dot{\theta}_2 - m_2 l_{c_2} s_2 \dot{b}_1 + I_{2,22}\dot{\theta}_2\right) + m_2 g l_{c_2} c_2 + m_2 l_{c_2} c_2 \dot{b}_1 \dot{\theta}_2 = \tau_2$$

$$\Rightarrow -m_2 l_{c_2} s_2 \ddot{b}_1 + I_{2,22}\ddot{\theta}_2 + m_2 l_{c_2}^2 \ddot{\theta}_2 - \cancel{m_2 l_{c_2} c_2 \dot{b}_1 \dot{\theta}_1} + m_2 g l_{c_2} c_2 + \cancel{m_2 l_{c_2} c_2 \dot{b}_1 \dot{\theta}_1} = \tau_2$$

$$\begin{bmatrix} m_1 + m_2 & -m_2 l_{c_2} s_2 \\ -m_2 l_{c_2} s_2 & I_{2,22} + m_2 l_{c_2}^2 \end{bmatrix} \begin{bmatrix} \ddot{b}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -m_2 l_{c_2} c_2 \dot{\theta}_2^2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ m_2 g l_{c_2} c_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$\boxed{\text{RP}}$



$$r_{c_1} = \begin{bmatrix} l_{c_1} c_1 \\ l_{c_1} s_1 \\ 0 \end{bmatrix} \qquad r_{c_2} = \begin{bmatrix} b_2 c_1 \\ b_2 s_1 \\ 0 \end{bmatrix}$$

$$v_{c_1} = \begin{bmatrix} -l_{c_1} s_1 \dot{\theta}_1 \\ l_{c_1} c_1 \dot{\theta}_1 \\ 0 \end{bmatrix} \qquad v_{c_2} = \begin{bmatrix} -b_2 s_1 \dot{\theta}_1 + \dot{b}_2 c_1 \\ b_2 c_1 \dot{\theta}_1 + \dot{b}_2 s_1 \\ 0 \end{bmatrix}$$

$$v_{c_1}^T v_{c_1} = l_{c_1}^2 \dot{\theta}_1^2$$

$$v_{c_2}^T v_{c_2} = b_2^2 s_1^2 \dot{\theta}_1^2 + \dot{b}_2^2 c_1^2 - \cancel{2 b_2 s_1 c_1 \dot{b}_2 \dot{\theta}_1} + b_2^2 c_1^2 \dot{\theta}_1^2 + \dot{b}_2^2 s_1^2 + \cancel{2 b_2 c_1 s_1 \dot{b}_2 \dot{\theta}_1}$$

$$= b_2^2 \dot{\theta}_1^2 + \dot{b}_2^2$$

$$T_1 = \frac{1}{2} m_1 l_{c_1}^2 \dot{\theta}_1^2 + \frac{1}{2} I_{1,22} \dot{\theta}_1^2$$

$$T_2 = \frac{1}{2} m_2 \left(b_2^2 \dot{\theta}_1^2 + \dot{b}_2^2\right) + \frac{1}{2} I_{2,22} \dot{\theta}_1^2$$

$$U_1 = -m_1 l_{c_1} g s_1 \qquad T = T_1 + T_2 \qquad L = T - U$$

$$U_2 = -m_2 b_2 g s_1 \qquad U = U_1 + U_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = \tau_1$$

$$\frac{d}{dt}\left(m_1 l_{c_1}^2 \dot{\theta}_1 + I_{1,22}\dot{\theta}_1 + I_{2,22}\dot{\theta}_1 + m_2 b_2^2 \dot{\theta}_1\right) + m_1 l_{c_1} g c_1 + m_2 b_2 g c_1 = \tau_1$$

$$\Rightarrow m_1 l_{c_1}^2 \ddot{\theta}_1 + \left(I_{1,22} + I_{2,22} + m_2 b_2^2\right)\ddot{\theta}_1 + m_1 g l_{c_1} c_1 + m_2 b_2 g c_1 = \tau_1 \quad \text{——— (1)}$$

$$+ 2 m_2 b_2 \dot{b}_2 \dot{\theta}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{b}_2}\right) - \frac{\partial L}{\partial b_2} = \tau_2$$

$$\frac{d}{dt}\left(m_2 \dot{b}_2\right) - m_2 b_2 \dot{\theta}_1^2 + m_2 g s_1 = \tau_2$$

$$m_2 \ddot{b}_2 - m_2 b_2 \dot{\theta}_1^2 + m_2 g s_1 = \tau_2 \quad \text{——— (2)}$$

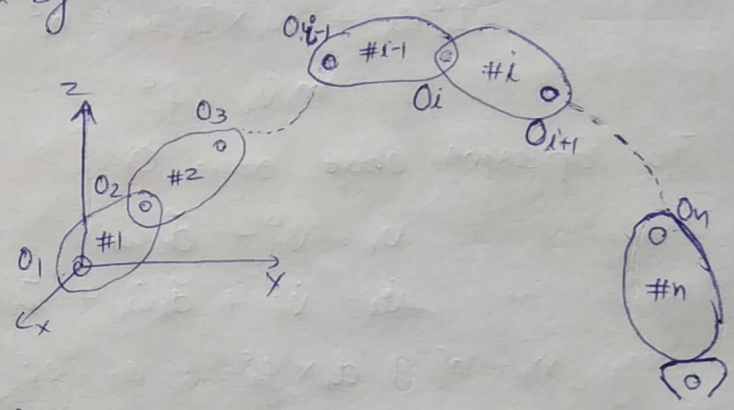
$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \left[\begin{array}{c|c} m_1 l_{c_1}^2 + I_{221} + I_{222} + m_2 b_2^2 & 0 \\ \hline 0 & m_2 \end{array}\right] \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{b}_1 \end{bmatrix} + \begin{bmatrix} 2 m_2 b_2 \dot{b}_2 \dot{\theta}_1 \\ -m_2 b_2 \dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} m_1 l_{c_1} g c_1 + m_2 b_2 g c_1 \\ m_2 g s_1 \end{bmatrix}$$

DeNOC-based Dynamic formulation:

Let us consider n rigid bodies coupled by n-single d-o-f kinematic joints.

The ith joint couples the link-i to link-$(i-1)$.

$$\dot{q} = [\dot{q}_1, \dots \dot{q}_i \dots \dot{q}_n]^T$$



Kinematic Constraints:

$$\omega_i = \omega_{i-1} + e_i \dot{q}_i \quad \text{——(1)}$$

$$v_i = a_{i,i-1} \times \omega_{i-1} + v_{i-1} \quad \text{——(2)}$$

where e_i is 3-D unit vector parallel to the axis of rotation or translation of ith joint

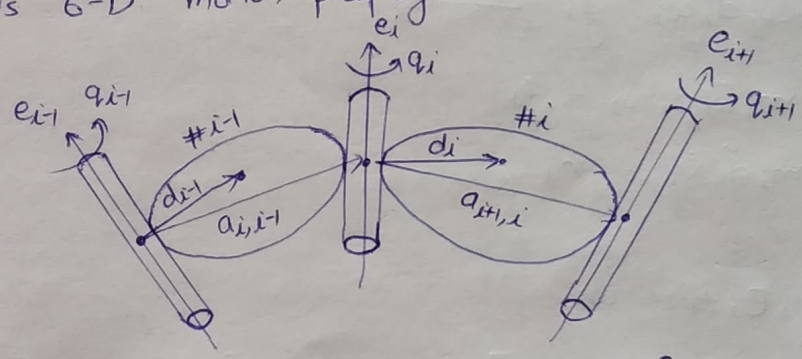
$a_{i,i-1}$ is 3D position vector of origin of $(i-1)^{st}$ from the origin of ith link

Combining (1) & (2)

$$t_i = A_{i,i-1}\, t_{i-1} + P_i \dot{q}_i$$

$A_{i,i-1}$ is 6×6 twist propagation matrix

P_i is 6-D motion propagation matrix



$$t_i = \begin{bmatrix} \omega_i \\ v_i \end{bmatrix}, \quad A_{i,i-1} = \begin{bmatrix} 1_{3\times3} & 0_{3\times3} \\ \tilde{a}_{i,i-1} & 1_{3\times3} \end{bmatrix}_{6\times6}, \quad P_i = \begin{bmatrix} e_i \\ 0_{3\times1} \end{bmatrix}_{6\times1} \text{ (for revolute)}$$

$$= \begin{bmatrix} 0_{3\times1} \\ e_i \end{bmatrix}_{6\times1} \text{ (for prismatic)}$$

where 1 and 0 are 3×3 identity & zero matrices

Now, $t = N\dot{q}$

where N is $6N \times n$ natural orthogonal complement (NOC)

$$N = N_l N_d$$

$$N_l = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ A_{2,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & \cdots & 1 \end{bmatrix}_{6n\times6n} \qquad N_d = \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_n \end{bmatrix}_{6n\times n}$$

The unconstrained or uncoupled NE EoM for i^{th} rigid-link can be

$$I_i^c \dot{\omega}_i + \omega_i \times I_i^c \omega_i = n_i^c \Rightarrow I_i^c \dot{\omega}_i + \tilde{\omega}_i I_i^c \omega_i = n_i^c \quad \text{—(1)}$$

$$m_i \dot{v}_i = f_i^c \quad \text{—(2)}$$

To represent about origin O_i

$$v_i = v_i^c + \tilde{d}_i \omega_i \quad \text{—(3)}$$

$$\dot{v}_i = \dot{v}_i^c + \tilde{d}_i \dot{\omega}_i + \tilde{\omega}_i \tilde{d}_i \omega_i \quad \text{—(4)}$$

llly $n_i = n_i^c + d_i \times f_i^c$; $f_i = f_i^c$ —(5)

$$I_i = I_i^c - m_i \tilde{d}_i^2 \quad \text{—(6)}$$

Substitute 3,4,5,6 in (1)& (2)

$$(1) \Rightarrow (I_i + m_i \tilde{d}_i^2)\dot{\omega}_i + \tilde{\omega}_i (I_i + m_i \tilde{d}_i^2)\omega_i = n_i - d_i \times f_i$$

$$I_i \dot{\omega}_i + m_i \tilde{d}_i^2 \dot{\omega}_i + \tilde{\omega}_i I_i \omega_i + \tilde{\omega}_i m_i \tilde{d}_i^2 \omega_i = n_i - d_i \times f_i$$

$$I_i \dot{\omega}_i + \tilde{\omega}_i I_i \omega_i + m_i (\tilde{d}_i^2 \dot{\omega}_i + \tilde{\omega}_i \tilde{d}_i^2 \omega_i) + d_i \times f_i = n_i$$

$$I_i \dot{\omega}_i + \tilde{\omega}_i I_i \omega_i + m_i \tilde{d}_i (\dot{v}_i - \dot{v}_i^c) + d_i \times f_i = n_i$$

$$I_i \omega_i + \tilde{\omega}_i I_i \omega_i + m_i \tilde{d}_i \dot{v}_i - \cancel{m_i \tilde{d}_i \dot{v}_i^c} + \cancel{\tilde{d}_i m_i \dot{v}_i^c} = n_i \quad \text{—(7)}$$

$$(2) \Rightarrow m_i (\dot{v}_i - \tilde{d}_i \dot{\omega}_i - \tilde{\omega}_i \tilde{d}_i \omega_i) = f_i$$

$$m_i \dot{v}_i - m_i \tilde{d}_i \dot{\omega}_i - m_i \tilde{\omega}_i \tilde{d}_i \omega_i = f_i \quad \text{—(8)}$$

<u>final eqns.</u>

$$m_i \dot{v}_i - m_i \tilde{d}_i \dot{\omega}_i - m_i \tilde{\omega}_i d_i \omega_i = f_i$$

$$I_i \dot{\omega}_i + \tilde{\omega}_i I_i \omega_i + m_i \tilde{d}_i \dot{v}_i = n_i$$

In compact form,

$$M_i \dot{\vartheta}_i + W_i M_i E_i \vartheta_i = \omega_i$$

$$M_i = \begin{bmatrix} I_i & m_i \tilde{d}_i \\ -m_i \tilde{d}_i & m_i 1 \end{bmatrix}_{6\times6} ; \quad W_i = \begin{bmatrix} \tilde{\omega}_i & 0 \\ 0 & \tilde{\omega}_i \end{bmatrix}_{6\times6} ; \quad \omega_i = \begin{bmatrix} n_i \\ f_i \end{bmatrix}_{6\times1} ; \quad E_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{6\times}$$

for n-bodies

$$\boxed{M\dot{\vartheta} + WME\vartheta = \omega} \quad \text{—(9)}$$

$$M = \text{diag}(M_1 \text{----} M_n) ; \quad W = \text{diag}[W_1 \text{-----} W_n] , \quad E = \text{diag}[E_1 \text{----} E_n]$$

$$\vartheta = [\vartheta_1^T \text{------} \vartheta_n^T] ; \quad \omega = [\omega_1^T \text{----} \omega_n^T]^T$$

$$\omega = \omega^E + \omega^C + \omega^D$$

$M \rightarrow$ Mass matrix

$W \rightarrow$ Angular velocity

$E \rightarrow$ Coupling

$W \rightarrow$ 6n-dimensional generalized vector of wrenches

$W^E, W^C, W^D \rightarrow$ wrenches due to external, constraint and driving moments & forces

The pre-multiplication of N to eq. (9) yields the minimal set of equations of motion eliminating the constraint wrenches, as constraint wrenches produces no work i.e $N_e^T N_d^T W^c = 0$.

$$N_e^T N_d^T (M\dot{\vartheta} + WME\vartheta) = N_e^T N_d^T (W^D + W^E)$$

Subs. $\vartheta = N_e N_d \dot{q}$ and its time derivative gives

$$I\ddot{q} + C\dot{q} = \tau + \tau^E \quad \text{or} \quad I\ddot{q} + h = \tau^g + \tau^a$$

(Easy)

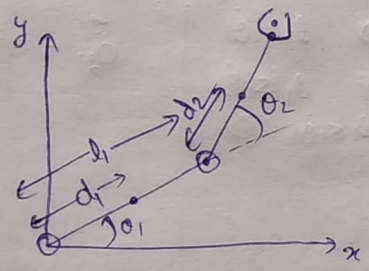
$$\vartheta = N_e N_d \dot{q}$$
$$\dot{\vartheta} = N_e N_d \ddot{q} + N_e \dot{N}_d \dot{q} + \dot{N}_e N_d \dot{q}$$
$$I = N_d^T N_e^T M N_e N_d$$
$$h = N_d^T N_e^T (M N_e \dot{N}_d + M \dot{N}_e N_d + WME N_e N_d)\dot{q}$$

Where $I = N^T M N$: Generalized inertia matrix (GIM)

$C = N^T (MN + WMEN)$; Matrix of convective inertia (MCI)

τ and τ^E ; vectors of generalizing driving forces and external forces

RR



for Joint 1, $M_1 \dot{\vartheta}_1 + W_1 M_1 E_1 \vartheta_1 = \omega_1$

$$\vartheta_1 = p_1 \dot{q}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} I_{3\times3} & O_{3\times3} \\ O_{3\times3} & O_{3\times3} \end{bmatrix}_{6\times6}$$

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \quad \tilde{\omega}_1 = \begin{bmatrix} 0 & -\dot{\theta}_1 & 0 \\ \dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} \tilde{\omega}_1 & O_{3\times3} \\ O_{3\times3} & \tilde{\omega}_1 \end{bmatrix}_{6\times6}$$

$$M_1 = \begin{bmatrix} I_1 & m_1 \tilde{d}_1 \\ -m_1 \tilde{d}_1 & m_1 1 \end{bmatrix} \quad d_1 = \begin{bmatrix} l_{c_1} c_1 \\ l_{c_1} s_1 \\ 0 \end{bmatrix} \quad \tilde{d}_1 = \begin{bmatrix} 0 & 0 & l_{c_1} s_1 \\ 0 & 0 & -l_{c_1} c_1 \\ -l_{c_1} s_1 & l_{c_1} c_1 & 0 \end{bmatrix}$$

$$I_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix} - m_1 \tilde{d}_1^2$$

for Joint 2 $M_2 \dot{\vartheta}_2 + W_2 M_2 E_2 \vartheta_2 = \omega_2$

$$\vartheta_2 = A_{21} \vartheta_1 + p_2 \dot{q}_2$$

$$A_{21} = \begin{bmatrix} I_{3\times3} & O_{3\times3} \\ \tilde{a}_{21} & I_{3\times3} \end{bmatrix}_{6\times6} \quad a_{12} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix} \quad \tilde{a}_{12} = \begin{bmatrix} 0 & 0 & l_1 s_1 \\ 0 & 0 & -l_1 c_1 \\ -l_1 s_1 & l_1 c_1 & 0 \end{bmatrix} \quad \tilde{a}_{21} = -\tilde{a}_{12}$$

$$p_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \dot{q}_2 = \dot{\theta}_2 \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{6\times6}$$

$$\omega_2 = \vartheta_2[1:3] \quad \tilde{\omega}_2 = \begin{bmatrix} 0 & -\vartheta_2(3) & \vartheta_2(2) \\ \vartheta_2(3) & 0 & -\vartheta_2(1) \\ -\vartheta_2(2) & \vartheta_2(1) & 0 \end{bmatrix} \quad W_2 = \begin{bmatrix} \tilde{\omega}_2 & O_{3\times3} \\ O_{3\times3} & \tilde{\omega}_2 \end{bmatrix}_{6\times6}$$

$$M_2 = \begin{bmatrix} I_2 & m_2 \tilde{d}_2 \\ -m_2 \tilde{d}_2 & m_2 1 \end{bmatrix} \quad d_2 = \begin{bmatrix} l_{c_2} c_{12} \\ l_{c_2} s_{12} \\ 0 \end{bmatrix} \quad \tilde{d}_2 = \begin{bmatrix} 0 & 0 & l_{c_2} s_{12} \\ 0 & 0 & -l_{c_2} c_{12} \\ -l_{c_2} s_{12} & l_{c_2} c_{12} & 0 \end{bmatrix}$$

$$I_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{222} \end{bmatrix} - m_2 \tilde{d}_2^2$$

for RR robot,

$$M = \begin{bmatrix} M_1 & 0_{6\times6} \\ 0_{6\times6} & M_2 \end{bmatrix}_{12\times12} \qquad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}_{12\times12} \qquad E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}_{12\times12}$$

DeNOC matrices

$$N = N_\ell N_d$$

$$N_d = \begin{bmatrix} p_1 & 0_{6\times1} \\ 0_{6\times1} & p_2 \end{bmatrix}_{12\times2} \qquad N_\ell = \begin{bmatrix} 1_{6\times6} & 0_{6\times6} \\ A_{21_{6\times6}} & 1_{6\times6} \end{bmatrix}_{12\times12} \Rightarrow [N]_{12\times2}$$

$$\dot{N}_d = [0]_{12\times2} \qquad \dot{N}_\ell = \begin{bmatrix} 0_{6\times6} & 0_{6\times6} \\ dA_{21_{6\times6}} & 0_{6\times6} \end{bmatrix}$$

$$dA_{21} = \begin{bmatrix} 0 & 0 \\ \tilde{\dot{a}}_{21} & 0 \end{bmatrix} \qquad \dot{a}_{12} = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 \\ l_1 c_1 \dot{\theta}_1 \\ 0 \end{bmatrix}, \quad \tilde{\dot{a}}_{12} = \begin{bmatrix} 0 & 0 & l_1 c_1 \dot{\theta}_1 \\ 0 & 0 & l_1 s_1 \dot{\theta}_1 \\ -l_1 c_1 \dot{\theta}_1 & -l_1 s_1 \dot{\theta}_1 & 0 \end{bmatrix}, \quad \tilde{\dot{a}}_{21} = -\tilde{\dot{a}}_{12}$$

Now,

$$N^T(M\dot{\vartheta} + WME\vartheta) = N^T(w^D + w^E)$$

$$\Rightarrow I\ddot{\theta} + C\dot{\theta} = \tau + \tau^E \Rightarrow I\ddot{\theta} + h = \tau$$

where $I = N^T M N = N_d' N_\ell' M N_\ell N_d$

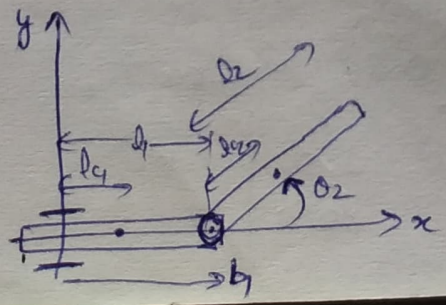
$$= \left[\begin{array}{c|c} m_1 l_{c1}^2 + I_{221} + m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} c_2) + I_{222} & m_2 l_{c2}^2 + m_1 l_1 l_{c2} c_2 + I_{222} \\ \hline m_2 l_c^2 + m_2 l_1 l_{c2} c_2 + I_{222} & m_2 l_{c2}^2 + I_{222} \end{array}\right]$$

$$h = N_d^T N_\ell^T M W E \vartheta + N_d^T N_\ell^T M N_\ell \underbrace{\dot{N}_d \dot{\theta}}_{0} + N_d^T N_\ell^T M \dot{N}_\ell N_d \dot{\theta}$$

$$h = N_d^T N_\ell^T W M E \vartheta + N_d^T N_\ell^T M \dot{N}_\ell N_d \dot{\theta}$$

$$h = \begin{bmatrix} -2 l_1 l_{c2} s_2 \dot{\theta}_1 \dot{\theta}_2 - l_1 l_{c2} s_2 \dot{\theta}_2^2 \\ m_2 l_1 l_{c2} s_2 \dot{\theta}_1^2 \end{bmatrix}$$

PR



Joint 1: $M_1 \dot{\vartheta}_1 + W_1 M_1 E_1 \vartheta_1 = w_1$

$$\vartheta_1 = p_1 \dot{q}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \dot{b}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dot{b}_1 \\ 0 \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}_{6\times6}$$

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \tilde{\omega}_1 = [0]_{3\times3} \quad W_1 = \begin{bmatrix} \tilde{\omega}_1 & 0 \\ 0 & \tilde{\omega}_1 \end{bmatrix}_{6\times6}$$

$$M_1 = \begin{bmatrix} I_1 & m_1\tilde{d}_1 \\ -m_1\tilde{d}_1 & m_1 1 \end{bmatrix}_{6\times6}, \quad I_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - m_1\tilde{d}_1^2, \quad d_1 = \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{d}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -b_1 \\ 0 & b_1 & 0 \end{bmatrix}$$

<u>for Joint 2</u>: $M_2\dot{\vartheta}_2 + W_2 M_2 E_2 \vartheta_2 = W_2$

$$\vartheta_2 = A_{21}\vartheta_1 + p_2\dot{q}_2$$

$$p_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \dot{q}_2 = \dot{\theta}_2, \quad A_{21} = \begin{bmatrix} 1 & 0 \\ \tilde{a}_{21} & 1 \end{bmatrix}, \quad a_{12} = \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{a}_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -b_1 \\ 0 & b_1 & 0 \end{bmatrix}, \quad \tilde{a}_{21} = -\tilde{a}_{12}$$

$$\omega_2 = \vartheta_2[1:3], \quad W_2 = \begin{bmatrix} \tilde{\omega}_2 & 0 \\ 0 & \tilde{\omega}_2 \end{bmatrix}_{6\times6}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} I_2 & m_2\tilde{d}_2 \\ -m_2\tilde{d}_2 & m_2 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{222} \end{bmatrix} - m_2\tilde{d}_2^2, \quad \tilde{d}_2 = \begin{bmatrix} l_{c_2}c_2 \\ l_{c_2}s_2 \\ 0 \end{bmatrix}, \quad \tilde{d}_2 = \begin{bmatrix} 0 & 0 & l_{c_2}s_2 \\ 0 & 0 & -l_{c_2}c_2 \\ -l_{c_2}s_2 & l_{c_2}c_2 & 0 \end{bmatrix}$$

<u>for PRRobot,</u>

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \quad W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \quad E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}$$

<u>DeNOC matrices</u>

$$N = N_l N_d$$

$$N_d = \begin{bmatrix} p_1 & 0_{6\times1} \\ 0_{6\times1} & p_2 \end{bmatrix}_{12\times2} \qquad N_l = \begin{bmatrix} 1_{6\times6} & 0_{6\times6} \\ A_{21\,6\times6} & 1_{6\times6} \end{bmatrix}_{12\times2}$$

$$\dot{N}_d = [0]_{12\times2} \qquad \dot{N}_l = \begin{bmatrix} 0_{6\times6} & 0_{6\times6} \\ \dot{A}_{21\,6\times6} & 0_{6\times6} \end{bmatrix}$$

$$\dot{A}_{21} = \begin{bmatrix} 0 & 0 \\ \dot{\tilde{a}}_{21} & 0 \end{bmatrix}, \quad \dot{\tilde{a}}_{21} = -\dot{\tilde{a}}_{12}, \quad \dot{a}_{12} = \begin{bmatrix} \dot{b}_1 \\ 0 \\ 0 \end{bmatrix}, \quad \dot{\tilde{a}}_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{b}_1 \\ 0 & \dot{b}_1 & 0 \end{bmatrix}$$

<u>Equation of Motion:</u>

$$M\dot{\vartheta} + WME\vartheta = \tau^g + \tau^a + \tau^c \quad \text{——} (1)$$

Multiply both sides by $N_d^T N_l^T$

$$N_d^T N_l^T (M\dot{\vartheta} + WME\vartheta) = \tau^a + \tau^g \qquad (N_d^T N_l^T \tau^c = 0)$$

$$\vartheta = N_l N_d \dot{q}$$

$$\dot{\vartheta} = N_l N_d \ddot{q} + N_d \dot{N}_d \dot{q} + \dot{N}_l N_d \dot{q}$$

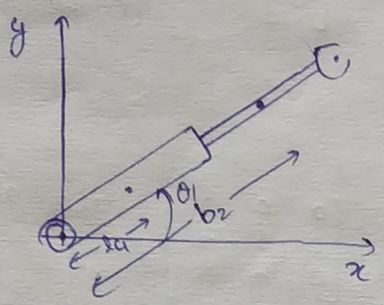
Substuting in (1) $\Rightarrow I\ddot{\theta} + h = \tau^a + \tau^g$

where, $I = N_d^T N_l^T M N_l N_d$, $h = N_d^T N_l^T (M N_l \dot{N}_d + M\dot{N}_l N_d + WME N_l N_d)\dot{q}$

$\Rightarrow$ On Solving

$$I = \begin{bmatrix} m_1 + m_2 & -m_2 l_{c_2} s_2 \\ -m_2 l_{c_2} s_2 & m_2 l_{c_2}^2 + I_{2,22} \end{bmatrix}, \quad h = \begin{bmatrix} -m_2 l_{c_2} c_2 \dot{\theta}_2^2 \\ 0 \end{bmatrix}, \quad \tau^g = \begin{bmatrix} 0 \\ m_2 g l_{c_2} c_2 \end{bmatrix}$$

RP (I)



Joint 1: $M_1\dot{\vartheta}_1 + W_1 M_1 E_1 \vartheta_1 = W_1$

$$\vartheta_1 = p_1\dot{q}_1 = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}\dot{\theta}_1 \quad , \quad E_1 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}_{6\times6}$$

$$\omega_1 = \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix} , \quad \tilde{\omega}_1 = \begin{bmatrix}0 & -\dot{\theta}_1 & 0\\\dot{\theta}_1 & 0 & 0\\0 & 0 & 0\end{bmatrix}_{3\times3} , \quad W_1 = \begin{bmatrix}\tilde{\omega}_1 & 0_{3\times3}\\0_{3\times3} & \tilde{\omega}_1\end{bmatrix}_{6\times6}$$

$$M_1 = \begin{bmatrix}I_1 & m_1\tilde{d}_1\\-m_1\tilde{d}_1 & m_1 1\end{bmatrix}_{6\times6} , \quad d_1 = \begin{bmatrix}l_{c_1}c_1\\l_{c_1}s_1\\0\end{bmatrix} , \quad \tilde{d}_1 = \begin{bmatrix}0 & 0 & l_{c_1}s_1\\0 & 0 & -l_{c_1}c_1\\-l_{c_1}s_1 & l_{c_1}s_1 & 0\end{bmatrix}$$

$$I_1 = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & I_{221}\end{bmatrix}_{3\times3} - m_1\tilde{d}_1^2$$

Joint 2: $M_2\dot{\vartheta}_2 + W_2 M_2 E_2 \vartheta_2 = W_2$

$$\vartheta_2 = A_{21}\vartheta_1 + p_2\dot{q}_2 \quad , \quad p_2 = \begin{bmatrix}0\\0\\0\\c_1\\s_1\\0\end{bmatrix} , \quad q_2 = b_2 , \quad A_{21} = \begin{bmatrix}1 & 0\\\tilde{a}_{21} & 1\end{bmatrix}_{6\times6} , \quad \tilde{a}_{21} = -\tilde{a}_{12}$$

$$a_{12} = 0_{12} = \begin{bmatrix}0\\0\\0\end{bmatrix} , \quad \tilde{a}_{12} = \begin{bmatrix}0 & -a_{12z} & a_{12y}\\a_{12z} & 0 & -a_{12x}\\-a_{12y} & a_{12x} & 0\end{bmatrix}$$

$$\omega_2 = \vartheta_2[1:3] , \quad \tilde{\omega}_2 = \begin{bmatrix}0 & -\vartheta_2(3) & \vartheta_2(2)\\\vartheta_2(3) & 0 & -\vartheta_2(1)\\-\vartheta_2(2) & \vartheta_2(1) & 0\end{bmatrix} , \quad W = \begin{bmatrix}\tilde{\omega}_2 & 0\\0 & \tilde{\omega}_2\end{bmatrix}_{6\times6} , \quad E_2 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$M_2 = \begin{bmatrix}I_2 & m_2\tilde{d}_2\\-m_2\tilde{d}_2 & m_2 1\end{bmatrix} , \quad I_2 = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & I_{222}\end{bmatrix} - m_2\tilde{d}_2^2 , \quad d_2 = \begin{bmatrix}b_2c_1\\b_2s_1\\0\end{bmatrix} , \quad \tilde{d}_2 = \begin{bmatrix}0 & 0 & b_2s_1\\0 & 0 & -b_2c_1\\-b_2s_1 & b_2c_1 & 0\end{bmatrix}$$

DeNOC Matrices

$$N = N_l N_d$$

$$N_d = \begin{bmatrix}p_1 & 0_{6\times1}\\0_{6\times1} & p_2\end{bmatrix}_{12\times2} \qquad N_l = \begin{bmatrix}1_{6\times6} & 0_{6\times6}\\A_{21_{6\times6}} & 0_{6\times6}\end{bmatrix}_{12\times12}$$

$$\dot{N}_d = \begin{bmatrix}\dot{p}_1 & 0\\0 & \dot{p}_2\end{bmatrix} , \qquad \dot{N}_l \begin{bmatrix}0 & 0\\\dot{A}_{21} & 0\end{bmatrix} , \quad \dot{A}_{21} = \begin{bmatrix}0 & 0\\\dot{\tilde{a}}_{21} & 0\end{bmatrix}$$

$$\dot{p}_1 = [0]_{6\times1} \quad \dot{p}_2 = [0\ 0\ 0\ -s_1\dot{\theta}_1\ \ c_1\dot{\theta}_1\ \ 0]^T \qquad \dot{\tilde{a}}_{21} = -\dot{\tilde{a}}_{12} = [0]_{3\times3} , \quad \dot{N}_l = [0]_{12\times12}$$

$$\dot{A}_{21} = [0]_{6\times6}$$

Equation of motion:

$$M\dot{\vartheta} + WME\vartheta = \tau^g + \tau^a + \tau^c$$

$$M = \begin{bmatrix}M_1 & 0\\0 & M_2\end{bmatrix} , \quad W = \begin{bmatrix}W_1 & 0\\0 & W_2\end{bmatrix} , \quad E = \begin{bmatrix}E_1 & 0\\0 & E_2\end{bmatrix}$$

$$N_d^T N_l^T (M\dot{\vartheta} + WME\vartheta) = N_d^T N_l^T (\tau^g + \tau^a)$$

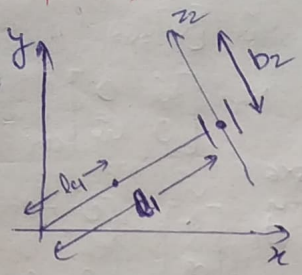
$$\Rightarrow I\ddot{\theta} + h = \tau^g + \tau^a$$

$$I = N_d^T N_l^T M N_l N_d = \left[\begin{array}{c|c}m_1 l_{c_1}^2 + I_{221} + I_{222} + m_2 b_2^2 & 0\\\hline 0 & m_2\end{array}\right]$$

$$h = N_d^T N_l^T (M N_l \dot{N}_d + M \dot{N}_l N_d + WME N_l N_d)\dot{q}$$

$$h = \begin{bmatrix}2m_2 b_2 \dot{b}_2 \dot{\theta}_1\\-m_2 b_2 \dot{\theta}_1^2\end{bmatrix}$$

RP (II)



Joint 1 $\quad M_1\dot{\vartheta}_1 + W_1 M_1 E_1 \vartheta_1 = W_1$

$$\vartheta_1 = p_1\dot{q}_1 = \begin{bmatrix}0\\0\\\dot{\theta}_1\\0\\0\\0\end{bmatrix}, \quad \omega_1 = \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}, \quad \tilde{\omega}_1 = \begin{bmatrix}0 & -\dot{\theta}_1 & 0\\ \dot{\theta}_1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad W = \begin{bmatrix}\tilde{\omega}_1 & 0\\ 0 & \tilde{\omega}_1\end{bmatrix}_{6\times6}$$

$$E_1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}_{6\times6} \qquad d_1 = \begin{bmatrix}l_{c1}c_1\\ l_{c1}s_1\\ 0\end{bmatrix}, \quad \tilde{d}_1$$

$$M_1 = \begin{bmatrix}I_1 & m_1\tilde{d}_1\\ -m_1\tilde{d}_1 & m_1 1\end{bmatrix}, \quad I_1 = \begin{bmatrix}0&0&0\\0&0&0\\0&0&I_{ZZ1}\end{bmatrix} - m_1\tilde{d}_1^2,$$

Joint 2: $\quad M_2\dot{\vartheta}_2 + W_2 M_2 E_2 \vartheta_2 = W_2$

$$\vartheta_2 = A_{21}\vartheta_1 + p_2\dot{q}_2, \quad p_2 = \begin{bmatrix}0\\0\\0\\-s_1\\c_1\\0\end{bmatrix}, \quad q_2 = \dot{b}_2, \quad A_{21} = \begin{bmatrix}1 & 0\\ \tilde{a}_{21} & 0\end{bmatrix}, \quad \tilde{a}_{21} = -\tilde{a}_{12}, \quad a_{12} = \begin{bmatrix}l_1c_1\\ l_1s_1\\ 0\end{bmatrix}$$

$$\omega_2 = \vartheta_2[1:3], \quad W_2 = \begin{bmatrix}\tilde{\omega}_2 & 0\\ 0 & \tilde{\omega}_2\end{bmatrix}, \quad M_2 = \begin{bmatrix}I_2 + m_2\tilde{d}_2^2 & \\ -m_2\tilde{d}_2 & m_2 1\end{bmatrix}, \quad I_2 = [0]_{3\times3} - m_2\tilde{d}_2^2$$

$$d_2 = \begin{bmatrix}-b_2s_1\\ b_2c_1\\ 0\end{bmatrix}, \quad \tilde{d}_2 = \begin{bmatrix}0 & 0 & b_2c_1\\ 0 & 0 & b_2s_1\\ -b_2c_2 & -b_2s_2 & 0\end{bmatrix}$$

DeNOC Matrices

$$N = N_l N_d$$

$$N_d = \begin{bmatrix}p_1 & 0\\ 0 & p_2\end{bmatrix}_{12\times2}, \quad N_l = \begin{bmatrix}1 & 0\\ A_{21} & 0\end{bmatrix}_{12\times12}$$

$$\dot{N}_d = \begin{bmatrix}\dot{p}_1 & 0\\ 0 & \dot{p}_2\end{bmatrix}, \quad \dot{N}_l = \begin{bmatrix}0 & 0\\ \dot{A}_{21} & 0\end{bmatrix}, \quad \dot{A}_{21} = \begin{bmatrix}0 & 0\\ \dot{\tilde{a}}_{21} & 0\end{bmatrix}, \quad \dot{\tilde{a}}_{21} = -\dot{\tilde{a}}_{12}$$

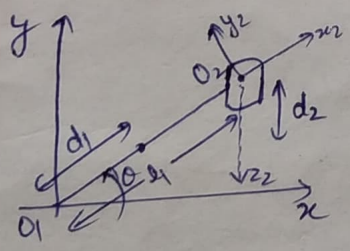
$$\dot{p}_1 = [0]_{6\times1} \quad \dot{p}_2 = \begin{bmatrix}0\\0\\0\\-c_1\dot{\theta}_1\\-s_1\dot{\theta}_1\\0\end{bmatrix} \qquad \dot{a}_{12} = \begin{bmatrix}-l_1s_1\dot{\theta}_1\\ l_1c_1\dot{\theta}_1\\ 0\end{bmatrix}, \quad \dot{\tilde{a}}_{12} = \begin{bmatrix}0 & 0 & l_1c_1\dot{\theta}_1\\ 0 & 0 & l_1s_1\dot{\theta}_1\\ -l_1c_1\dot{\theta}_1 & -l_1s_1\dot{\theta}_1 & 0\end{bmatrix}$$

Equation of motion:

$$I = N_d^T N_l^T M N_l N_d = \begin{bmatrix}m_1l_{c1}^2 + m_2l_1^2 + m_2b_2^2 + I_{ZZ1}, & m_2l_1\\ m_2l_1 & m_2\end{bmatrix}$$

$$h = \begin{bmatrix}m_2b_2\dot{b}_2\dot{\theta}_1\\ -m_2b_2\dot{\theta}_1^2\end{bmatrix}$$

RP (Spatial)



$$p_1 = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix} \qquad p_2 = \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix} \qquad N_d = \begin{bmatrix}p_1 & 0\\ 0 & p_2\end{bmatrix}$$

$$a_{12} = \begin{bmatrix}l_1c_\theta\\ l_1s_\theta\\ 0\end{bmatrix} \qquad \tilde{a}_{21} = -\tilde{a}_{12} = \begin{bmatrix}0 & 0 & -l_1s_\theta\\ 0 & 0 & l_1c_\theta\\ l_1s_\theta & -l_1c_\theta & 0\end{bmatrix}, \quad A_{21} = \begin{bmatrix}1 & 0\\ \tilde{a}_{21} & 1\end{bmatrix}$$

$$N_l = \begin{bmatrix}1 & 0\\ A_{21} & 1\end{bmatrix}$$

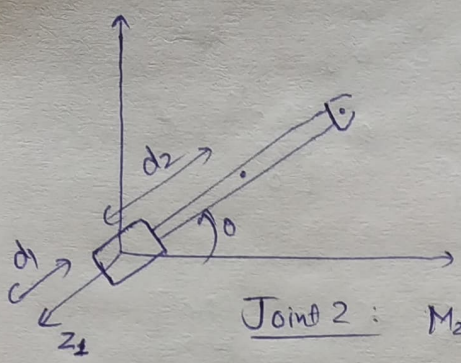
$$\vartheta_1 = \begin{bmatrix}0\\0\\\dot{\theta}_1\\0\\0\\0\end{bmatrix}, \quad \vartheta_2 = A_{21}\vartheta_1 + p_2\dot{d}_2 \qquad d_1 = \begin{bmatrix}d_1c_\theta\\ d_1s_\theta\\ 0\end{bmatrix}, \quad \tilde{d}_1 = \begin{bmatrix}0 & 0 & d_1s_\theta\\ 0 & 0 & -d_1c_\theta\\ -d_1s_\theta & d_1c_\theta & 0\end{bmatrix} \qquad h = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\vec{d}_2 = \begin{bmatrix}d_1c_\theta\\ d_1s_\theta\\ d_2\end{bmatrix} \quad \tilde{d}_2 = \begin{bmatrix}0 & -d_2 & d_1s_\theta\\ d_2 & 0 & -d_1c_\theta\\ -d_1s_\theta & d_1c_\theta & 0\end{bmatrix}$$

On Solving,

$$M = \begin{bmatrix}I_{ZZ1} + m_1d_1^2 + 4m_2l_1^2 & 0\\ 0 & m_2\end{bmatrix}$$

PRI (Spatial)



Joint 1: $M_1\dot{\vartheta}_1 + W_1 M_1 E_1 \vartheta_1 = W_1$

$$\vartheta_1 = p_1\dot{q}_1 = \begin{bmatrix}0\\0\\0\\0\\0\\1\end{bmatrix}\dot{d}_1 = \begin{bmatrix}0\\0\\0\\0\\0\\\dot{d}_1\end{bmatrix}, \quad \omega_1 = \begin{bmatrix}0\\0\\0\end{bmatrix}, \quad \tilde{\omega}_1 = [0]_{3\times3}$$

$$E = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}_{6\times6} \qquad W_1 = \begin{bmatrix}\tilde{\omega}_1 & 0\\0 & \tilde{\omega}_1\end{bmatrix}_{6\times6}$$

$$M_1 = \begin{bmatrix}I_1 & m_1\tilde{d}_1\\-m_1\tilde{d}_1 & m_1 1\end{bmatrix}, \quad I_1 = [0]_{3\times3} - m_1\tilde{d}_1^2, \quad \tilde{d}_1 = \begin{bmatrix}\tilde{0}\\0\\d_1\end{bmatrix} = \begin{bmatrix}0 & -d_1 & 0\\d_1 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Joint 2: $M_2\dot{\vartheta}_2 + W_2 M_2 E_2 \vartheta_2 = W_2$

$$\vartheta_2 = A_{21}\vartheta_1 + p_2\dot{q}_2, \quad p_2 = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}, \quad \dot{q}_2 = \dot{\theta}, \quad A_{21} = \begin{bmatrix}1 & 0\\\tilde{a}_{21} & 1\end{bmatrix}, \quad \tilde{a}_{21} = -\tilde{a}_{12}, \quad \tilde{a}_{12} = \begin{bmatrix}\tilde{0}\\0\\0\end{bmatrix} = [0]_{3\times3}$$

$$\vartheta_2[1:3] = \omega_2, \quad W_2 = \begin{bmatrix}\tilde{\omega}_2 & 0\\0 & \tilde{\omega}_2\end{bmatrix}, \quad E_2 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \quad M_2 = \begin{bmatrix}I_2 & m_2\tilde{d}_2\\-m_2\tilde{d}_2 & m_2 1\end{bmatrix}$$

$$I_2 = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & I_{222}\end{bmatrix} - m_2\tilde{d}_2^2, \quad \tilde{d}_2 = \begin{bmatrix}d_2C_\theta\\d_2S_\theta\\0\end{bmatrix} = \begin{bmatrix}0 & 0 & d_2S_\theta\\0 & 0 & -d_2C_\theta\\-d_2S_\theta & d_2C_\theta & 0\end{bmatrix}$$

DeNoC matrices: $N = N_\ell N_d$

$$N_d = \begin{bmatrix}p_1 & 0\\0 & p_2\end{bmatrix}_{12\times2} \qquad N_\ell = \begin{bmatrix}1 & 0\\A_{21} & 1\end{bmatrix}_{12\times12}$$

$$\dot{N}_d = [0]_{12\times2} \qquad \dot{N}_\ell = \begin{bmatrix}0 & 0\\\dot{A}_{21} & 0\end{bmatrix}, \quad \dot{A}_{21} = \begin{bmatrix}0 & 0\\\dot{\tilde{a}}_{21} & 0\end{bmatrix}, \quad \dot{\tilde{a}}_{21} = -\dot{\tilde{a}}_{12} = [0]_{3\times3}$$

$$\Rightarrow \dot{N}_\ell = [0]_{12\times2}$$

Equation of motion

$$M\dot{\vartheta} + WME\vartheta = \tau^a + \tau^g + \tau^c \quad \text{——} (1)$$

$$(1) \times N_d^T N_\ell^T \Rightarrow I\ddot{\theta} + h = \tau^a + \tau^g$$

$$I = N_d^T N_\ell^T M N_\ell N_d = \left[\begin{array}{c|c}m_1 + m_2 & 0\\\hline 0 & m_2 d_2^2 + I_{222}\end{array}\right]$$

$$h = N_d^T N_\ell^T (M N_\ell \dot{N}_d + M \dot{N}_\ell N_d + WME N_\ell N_d)\dot{q} = \begin{bmatrix}0\\0\end{bmatrix}$$

Force control: The basic aim of any force/torque control algorithm is to efficiently synthesize the information of force/torque senses at the joints or at the proximal end i.e end-effector or even at the base in some cases, to actively control the forces exerted by robot to the environment with which it interacts

Types:

(1) **Stiffness control**: In this end-effector produces a wrench W_E that is proportional to the displacement from the desired pose Δp. The const. of proportionality is known as mechanical stiffness of manipulator

Considering the robot's behaviour be governed by closed-loop dynamics given by

$$I\ddot{\theta} + h = \tau + J^T W_E$$

$$\Delta q = \begin{bmatrix} \Delta p \\ \Delta \varphi \end{bmatrix}, \quad K_p = \begin{bmatrix} K_{pp} & 0 \\ 0 & K_{p\varphi} \end{bmatrix}$$

In case of stiffness control,

$$\tau = \tau_p - \tau_v - \tau_E - \tau_g$$

$$J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}, \quad t_e = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

where $\tau_p = J^T K_p \Delta q$

$\tau_v = K_v J^{-1} t_e = K_v \dot{\theta}$

$\tau_E = J^T W_E$

$$W_E = \begin{cases} K_E (p - p_E) & \forall\ p > p_E \\ 0 & \forall\ p \le p_E \end{cases}$$

Here, K_E denotes the environmental stiffness.

(2) **Impedence control**: In this, system is so designed that the end-effector applies a wrench W_e which is proportional to the end-effector's twist t_e i.e $W_e = K_v t_e$. The constant of proportionality is known as mechanical impedence of manipulator. A robot with impedence controller is desirable for an application where robot is demanded not to just track the trajectory alone but also to regulate the mechanical impedence of manipulator.

In contrast to stiffness control, the driving torque is given by

$$\tau = \tau_{qv} - \tau_v - \tau_E - \tau_g$$

where $\tau_{qv} = J^T (K_p \Delta q + K_v \Delta t_e)$

(3) **Admittance control**: Admittance A is defined as the inverse of impedance

$$A = \text{diag}\,[A_\varphi, A_p]_{6\times 6}$$

such that $t_e = A W_E$

W_E = sensed wrench vector

where $A = K_v^{-1}$

This control allows a desired wrench input W_D to be commanded explicitly to a pose controlled robot. This is quite analogous to a position controlled robot being controlled for force using an external force control loop. The pose q_a is fed to the robot controller corresponding to the desired wrench input W_D. This is given as

$$q_a = \int A (\omega_D - W_E)\, dt$$

for a robot accepting torque commands at its joints, the contribution of torque due to the deviation of pose from the value as calculated earlier and the desired pose input q_D is given as

$$\tau_a = K_{pa} J^{-1} \Delta X$$

where $\Delta X = q_D + q_a - q$ and K_{pa} is $n \times n$ diagonal matrix of proportional control gain.

$$\tau = \tau_a - \tau_v - \tau_E - \tau_g$$

Admittance control focuses more on desired wrench tracking as compared to impedance control where pose and twist are tracked. Such systems are more suitable where the system demands maintaining a const. reaction wrench on making a contact with object surface e.g. grinding, polishing, buffing, etc.

(4) Hybrid Control: Hybrid position/force control combines two independent control systems into a single controller based on two complimentary orthogonal spaces on displacement and force. The 6×6 diagonal compliant selection matrix S determines the subspace for which the wrench or the pose is to be controlled. Both can't be simultaneously controlled. The diagonal element S_{ii} of the matrix S is set to 1, if the ith d-o-f of end-effector is to be wrench controlled.

The control torque is given by

$\tau_p \rightarrow$ Torque due to pose error

$\tau_f \rightarrow$ " " " force error

$$\tau = \tau_p + \tau_f - \tau_v - \tau_E - \tau_g$$

Other Notes:

- State Space model to Transfer function:

$$\dot{x} = Ax + Bu \qquad x \in R^n,\ u \in R^m,\ A \in R^{n \times n},\ B^{n \times m},\ y \in R^p$$

$$y = Cx + Du \qquad C \in R^{p \times n}$$

Then $G(s) = C(SI - A)^{-1} B + D$

MATLAB ssf(A,B,C,D) return $\frac{b}{a} = \frac{b_1 s^3 + b_2 s^2 + b_3 s + b_4}{a_1 s^3 + a_2 s^2 + a_3 s + a_3}$

- System poles & Stability:

find poles of $SI - A$ using $|SI - A|$

- If all poles of sys. lie in left half plane, the sys. is stable.
- If any poles of sys. lie in right half plane, the sys. is unstable.
- If poles lies on imaginary axis, sys. is marginally stable

- State Transition matrix:

$$x(t) = \phi(t)\, x(0)$$

Method 1: $\phi(t) = L^{-1}\left[(SI-A)^{-1}\right]$

Method 2: $\phi(t) = e^{At} = P e^{\Lambda t} P^{-1}$ where P is a matrix of eigen vectors

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

If λ_i ~~&~~ distinct

$\lambda_1 = \lambda_2 = \lambda_3$

$P = [v_1 \; v_2 \; v_3] \Rightarrow P^{-1}AP$ (Jordan form)

$(\lambda_1 I - A) v_1 = 0$

$(\lambda_1 I - A) v_2 = -v_1$

$(\lambda_1 I - A) v_3 = -v_2$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & t e^{\lambda_1 t} & \frac{t^2}{2} e^{\lambda_1 t} \\ & e^{\lambda_1 t} & t e^{\lambda_1 t} \\ & & e^{\lambda_1 t} \end{bmatrix}$$

eg: Equation of motion in state space

$$\dot{x} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} x$$ find State transition matrix?

Sol: $$|\lambda I - A| = \begin{vmatrix} \lambda & 0 & +2 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda-3 \end{vmatrix} = \lambda(\lambda-1)(\lambda-3) + 2(+(\lambda-1))$$

$$= \lambda(\lambda^2 - 4\lambda + 3) + 2\lambda - 2 = \lambda^3 - 4\lambda^2 + 5\lambda + 2$$

$$= \lambda(\lambda^2 - 4\lambda + 5) + 2$$

$$= \cancel{\lambda(\lambda^2 - 5\lambda + \lambda - 5)}$$

$$= \cancel{\lambda(\lambda+1)(\lambda+5)} + 2$$

$\lambda = 2, 1, 1$

for $\lambda_1 = 2$

$$\begin{bmatrix} 2 & 0 & +2 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\downarrow R_3 \to 2R_3 + R_1$

$$\begin{bmatrix} 2 & 0 & +2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 2x_1 + 2x_3 = 0 \Rightarrow 2x_1 = -2x_3 \\ x_2 = 0 \end{matrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

for $\lambda_2 = 1$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ -1 & 0 & -2 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + 2x_3 = 0 \\ x_1 = -2x_3 \end{matrix} \Rightarrow \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

for $\lambda_3 = 1$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} +2 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + 2x_3 = +2 \Rightarrow x_1 = 2 - 2x_3 \\ x_3 = 1, x_1 = 0 \end{matrix}$$ (Not possible)

Thus, Choosing third arbitrarily as $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ to make P independent

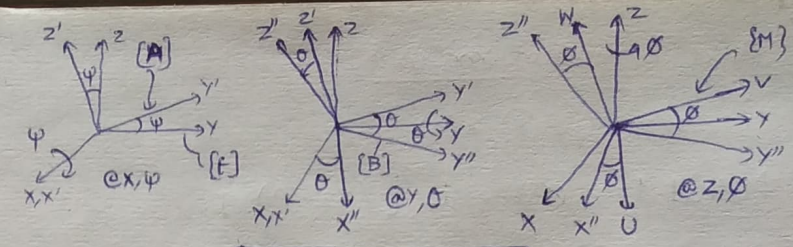
$$P = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \qquad e^{\Lambda t} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix}$$

$$\phi(t) = P^{-1} e^{\Lambda t} P = \begin{bmatrix} -1 & 0 & -2 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{t}-e^{2t} & 0 & 2e^{t}-2e^{2t} \\ 0 & e^{t} & 0 \\ e^{2t}-e^{t} & 0 & 2e^{2t}-e^{t} \end{bmatrix}$$

Given $x_0 = 0$

$$x(t) = \phi(t) \times x_0 = \begin{bmatrix} 0 \\ e^{t} \\ 0 \end{bmatrix}$$

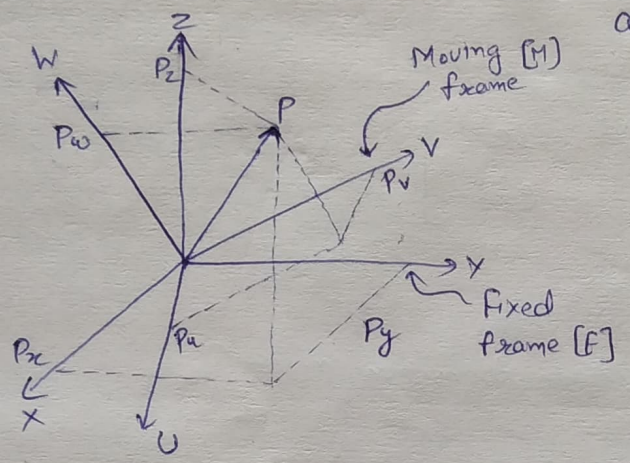


Fixed axis rotations

$Q = Q_z Q_y Q_x$ (Pre-multiplications)

- **Orientation Representations:**

1) **Direction Cosine representation:** Consider fixed and moving frame with point P as shown in fig.



Here,
$$\left.\begin{aligned} u &= u_x x + u_y y + u_z z \\ v &= v_x x + v_y y + v_z z \\ w &= w_x x + w_y y + w_z z \end{aligned}\right\} \text{— (1)}$$

Point P in moving frame

$$P = P_u u + P_v v + P_w w \quad \text{— (2)}$$

Sub. (1) in (2)

$$P = (P_u u_x + P_v v_x + P_w w_x)x + (P_u u_y + P_v v_y + P_w w_y)y + (P_u u_z + P_v v_z + P_w w_z)z$$

$$[P]_F = Q[P]_M$$

where $[P]_F = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$, $[P]_M = \begin{bmatrix} P_u \\ P_v \\ P_w \end{bmatrix}$, $Q = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} = \begin{bmatrix} u^T x & v^T x & w^T x \\ u^T y & v^T y & w^T y \\ u^T z & v^T z & w^T z \end{bmatrix}$

$Q^T Q = Q Q^T = I$

$|Q| = 1$, $Q^{-1} = Q^T$

Drawback:

- Require 9 parameters
- 6 constraint equations; reduces to 3 parameters
- difficult to choose set of 3 independent parameters.

2) **Fixed axis Rotations** This represents rotations about three fixed axes of an inertial frame (F) to represent the orientation of frame (M). They constitute minimal set representation of orientation. Rotation about fixed X, Y and Z-axes are also referred to as roll, pitch, yaw angles, respectively. $Q = Q_z^{\phi} Q_y^{\theta} Q_x^{\psi}$

3) **Euler angles representations:** Euler angles also constitute a minimal representation of orientation obtained by composing three elementary rotations with respect to axes of current frame. Unlike fixed-axes rotations, here rotations are performed w.r.t current frames. $Q = Q_x^{\phi} Q_{y'}^{\theta} Q_{z''}^{\psi}$

Drawbacks:

- Fails to find solution of inverse problem i.e given Q, find ϕ, θ, ψ

eg. $Q = Q_{z \times z} = \begin{bmatrix} \cdot & \cdot & S_\theta C_\phi \\ \cdot & \cdot & S_\theta S_\phi \\ \cdot & \cdot & C_\theta \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$

$\phi = \operatorname{atan2}\left(\frac{q_{23}}{S\theta}, \frac{q_{13}}{S\theta}\right)$

It only exists when $S\theta \neq 0$ i.e when $\theta \neq 0$ or $n\pi$

This non existence of solution is known as numerical singularity

If $S_\theta = 0$, then $Q = \begin{bmatrix} q_{11} & q_{12} & 0 \\ q_{21} & q_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} = \begin{bmatrix} C_{\phi+\psi} & -S_{\phi+\psi} & 0 \\ S_{\phi+\psi} & C_{\phi+\psi} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$

Only sum of $\phi+\psi$ can be determined in this case and there are infinitely many solutions.

- If $S_\theta \neq 0$, then $C_\theta = q_{33}$, $S_\theta = \pm\sqrt{1-q_{33}^2}$, so

$$\theta = a\tan 2\left(q_{33}, \pm\sqrt{1-q_{33}^2}\right)$$

Thus there are 2 solutions depending on the sign of θ choosen.

4) <u>Euler Parameters</u>: This is a four parameter representation of rotation matrix. It is also known as unit quaternion, defined by

$$r = e\left(\sin\frac{\alpha}{2}\right), \quad r_0 = \cos\frac{\alpha}{2}$$

$r = [r_x, r_y, r_z] \rightarrow$ scaled vector ||lel to axis of rotation e

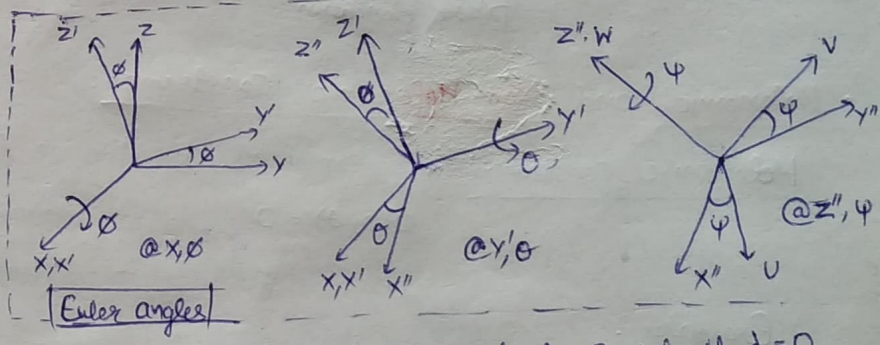
$$r_x^2 + r_y^2 + r_z^2 + r_0^2 = 1$$

$$Q = (r_0^2 - r^T r)1 + 2rr^T + 2r_0 r \times 1$$

$$r_0 = \frac{1}{2}\sqrt{1+q_{11}+q_{22}+q_{33}}, \quad r_x = \frac{1}{4r_0}(q_{32}-q_{23}), \quad r_y = \frac{1}{4r_0}(q_{13}-q_{31}), \quad r_z = \frac{1}{4r_0}(q_{21}-q_{12})$$

<u>Drawback</u>:

When $\alpha = \pi$, r_0 vanishes.



Euler angles

- <u>Singular Matrix</u>:

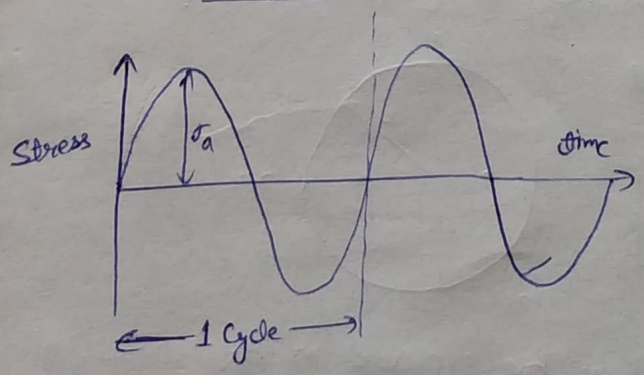
If A is a singular matrix:

a) $\det(A) = 0$

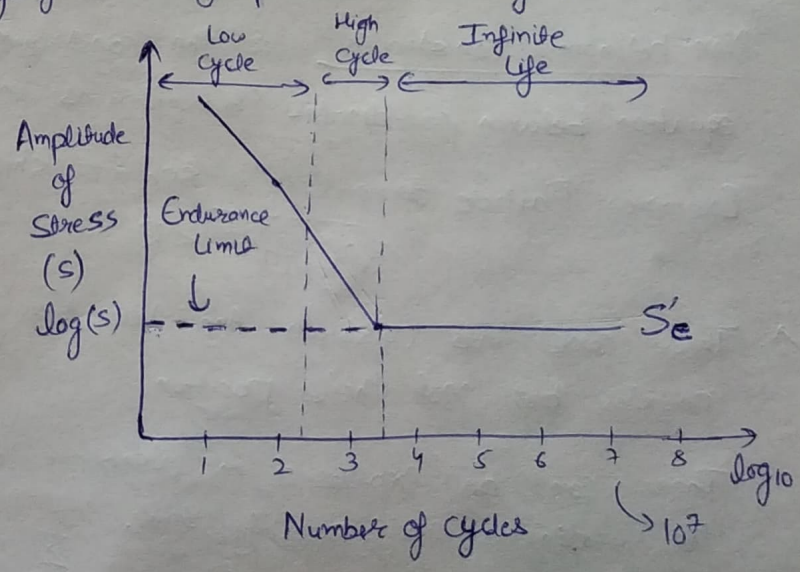
b) A^{-1} doesn't exist

c) 0 eigen values and 0 eigen vector $\because$ $\det(A-\lambda I) = 0$, $\det A = 0$ only if $\lambda = 0$

- <u>SN Curve (Fatigue Curve)</u>: It is a graph between stress and no. of cycles plotted for fatigue loading. Here fatigue failure is considered. It means a material is subjected to alternate tensile and compressive stresses at the same time. The results of fatigue testing plotted on a graph is called as '<u>SN Curve</u>'.



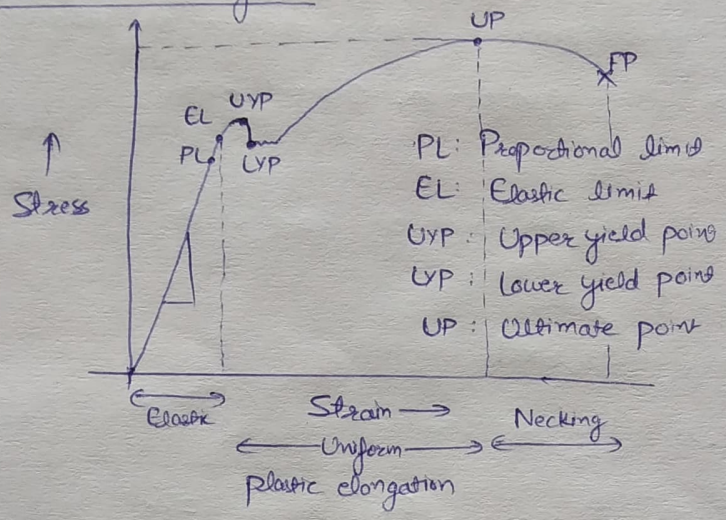
Dynamic loading



The fatigue or endurance limit of a material is defined as the maximum amplitude of completely reversed stress that the standard specimen can sustain for an unlimited number of cycles without fatigue failure.

In such a loading, material can fail in the elastic region or beyond by increasing the number of cycles. This is due to Baushinger effect which changes the yield point of material as the load direction changes.

Stress - Strain diagram



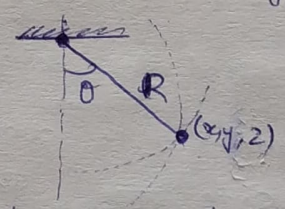
- **Holonomic and Non-Holonomic Constraints:**

Holonomic constraints are expressed directly in terms of position (and time) as

$\phi(x, t) = 0$ — Stationary or scleronomic
— Moving or rheonomic

Examples:

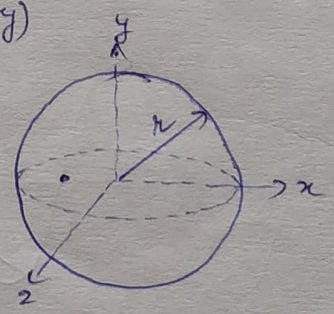
- particles in a plane connected by rigid rod (scleronomic) $x^2+y^2=l^2$
- particles connected by a rod with specified length variation (rheonomic) $x^2+y^2 \leq l^2$
- Spherical pendulum (scleronomic) $x^2+y^2+z^2=R^2$
- particles on ~~rotating~~ hoop (rheonomic)



Non-holonomic constraints are the constraints which can't be expressed as equation between coordinates (but often as inequality)

Example: Particle trapped in a spherical shell

$$\sqrt{x^2+y^2+z^2} < R$$

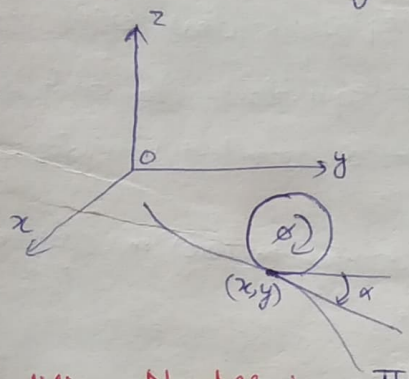


(✓) These constraints constrain the velocities of particles but not their position

These can't be expressed in algebraic form but expressed in differentials of the coordinates and possibly time.

$$\sum_{i=1}^{n} a_{ji}\, dq_i + a_{jt}\, dt = 0 \quad ; j = 1, 2, \ldots m$$

Example: Disk rolling on a plane



$$\left.\begin{aligned} dx &= ds \sin\alpha \\ dy &= ds \cos\alpha \\ ds &= r\,d\phi \end{aligned}\right\} \quad \begin{aligned} dx - r\,d\phi \sin\alpha &= 0 \\ dy - r\,d\phi \cos\alpha &= 0 \end{aligned}$$

(Two Non-holonomic Constraints)

$\{x, y, \phi, \alpha\}$ are generalized constraints

- **Condition Number:** The condition number of the matrix measures the ratio of the maximum relative stretching to the maximum relative shrinking that matrix does to any non-zero vectors.

In other words, condition no. of a matrix measures the amount of distortion of unit sphere under the transformation by the matrix. Larger the C.N, more distorted (relatively long and thin) the unit sphere becomes when transformed by matrix.

The C.N is a measure of how close a matrix is to being singular; a matrix with large C.N is nearly singular, whereas a matrix with C.N close to 1 is far from being singular.

Properties:

1) $\text{cond}(A) \geq 1$
2) $\text{cond}(I) = 1$
3) $\text{cond}(A) = \text{cond}(A^{-1})$
4) for diagonal matrix D, $\text{cond}(D) = (\max|d_i|)/(\min|d_i|)$
5) $\text{cond}_1(A) = \|A\|_1 \|A^{-1}\|_1$ where $\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

$$\text{cond}(A) = \frac{\lambda_{max}}{\lambda_{min}} \qquad \lambda \rightarrow \text{eigen values}$$

- **Euler Beam Assumptions:**

1) The beam is initially straight and has a constant cross-section
2) The beam is made of homogeneous material and the beam has longitudinal plane of symmetry
3) Resultant of the applied loads lie in the plane of symmetry
4) The geometry of overall member is such that bending is the primary cause of failure and not buckling.
5) The stress level attained during tension and compression do not exceed the elastic limit and it is well within the linear graphical region where young's modulus is obeyed.
6) The plane cross-section remain plane before and after bending

- **DAE Vs ODE:**

Differential Algebraic Equations (DAE) is a system of equations that either contains differential equations and algebraic equations or is equivalent to such a system

$$\dot{x}(t) = f(x(t), y(t), t) \qquad (x,y) \text{ vector of dependent variables}$$
$$0 = g(x(t), y(t), t)$$

Example: The behaviour of a pendulum of length L with center in (0,0) in cartesian coordinates (x,y) is described by Euler-Lagrange equations.

$$\dot{x} = u, \ \dot{y} = v \qquad \text{where } \lambda \text{ is Lagrange multiplier}$$
$$\dot{u} = \lambda x, \ \dot{v} = \lambda y - g$$
$$x^2 + y^2 = L^2$$

Ordinary Differential Equation (ODE) is a set of differential equations involving functions of only one independent variable and one or more of their derivatives with respect to that variable.

Example: 1) If $x = \sin\theta$ and $y = \cos\theta$ in above equation, then

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

2) The relationship between displacement x and time t of an object under force F is given by,

$$\frac{m d^2 x(t)}{dt^2} = F(x(t))$$

General defination:

Given F, a function of x, y and derivatives of y. Then an equation of form

$$F(x, y, y', \ldots y^{(n-1)}) = y^{(n)}$$ is called an explicit ODE of order n

The term ordinary is used in contrast with the term partial diff. eqn. which may be with respect to more than one independent variable.

- **Factor of Safety (FoS):** FoS is expressed as a ratio that compares absolute strength to actual applied load. It is a constant value that a structure must meet or exceed according to law, specification, contract or standard

for eg: FoS of 2 means the structure will fail at twice the design load

$$FoS = \frac{\text{yield stress}}{\text{working stress}}$$

for dynamic loading (such as bridges) must use large FoS than statically loaded structures like building or tower.

• Dexterity and Manipulability:

Dexterity $(w_D) = \det(J)$

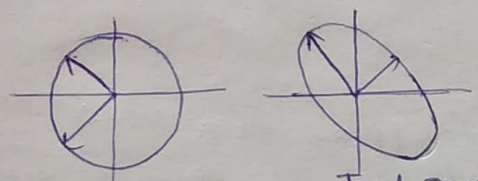
Manipulability $(w_M) = \sqrt{\det(JJ^T)} = \sqrt{\lambda_1 \lambda_2 \cdots \lambda_m} = \sigma_1 \sigma_2 \cdots \sigma_m$

From the kinetostatic point of view, a dextrous robot with good manipulability would mean that the manipulator can reach all points within the workspace with all possible orientations of its end-effector and can transmit motion & force comfortably.

Condition number is defined as the ratio of max. to min. singular values of Jacobian matrix and can measure a singularity and w_D, w_M.

Velocity and force ellipsoids:

$\dot{\theta}_e^T \dot{\theta}_e = 1 \Rightarrow \dot{\theta}^T J^T J \dot{\theta} = 1 \Rightarrow$ represents ellipsoid in n-dimensional joint space.



End-effector space Joint space

The closer the velocity ellipsoid to a sphere, better the transformation characteristics are. If the lengths of principal axis are equal, transformation is called isometric (CN = 1)

$\tau^T \tau = 1 \Rightarrow w_e^T J J^T w_e = 1$

• Poles and zeros of dynamical system:

The transfer function provides a basis for determining poles and zeros of dynamical system.

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{N(s)}{D(s)} = K \frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

$K = \frac{b_m}{a_n}$

$N(s) = 0$ where z_i's are roots and called as zeros

$D(s) = 0$ where p_i's are roots and called as poles

eg: $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 2\frac{du}{dt} + 1 \Rightarrow H(s) = \frac{2s+1}{s^2+5s+6} \Rightarrow H(s) = \frac{1}{2}\frac{s+\frac{1}{2}}{(s+3)(s+2)} = \frac{1}{2}\frac{s-(-\frac{1}{2})}{(s-(-3))(s-(-2))}$

zero at $s = -\frac{1}{2}$

real poles at $s = -3, -2$

Applications:

1) System stability: The exact magnitude of excitation/damping of oscillation depends on the relative distance b/w poles and zeros. Pole increases the oscillations and zero dampens the oscillation

2) Response of system: Smaller magnitude of zero makes the system response faster and increase the overshoot/undershoot. Similarly, a smaller magnitude of pole makes the system response slower.

Practice Questions

Q. Using ZXY Euler angles find out the orientation matrix. Write the angles for which its inverse calculation are not possible.

Sol:- $Q_{z\phi}\, Q_{x'\theta}\, Q_{y''\varphi} = Q$

$$\begin{bmatrix} C\phi & -S\phi & 0 \\ S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} \begin{bmatrix} C\varphi & 0 & S\varphi \\ 0 & 1 & 0 \\ -S\varphi & 0 & C\varphi \end{bmatrix}}_{\begin{bmatrix} C\varphi & 0 & S\varphi \\ S\theta S\varphi & C\theta & -S\theta C\varphi \\ -C\theta S\varphi & S\theta & C\theta C\varphi \end{bmatrix}}$$

$$\begin{bmatrix} C\phi C\varphi - S\phi S\theta S\varphi & -S\phi C\theta & C\phi S\varphi + S\phi S\theta C\varphi \\ S\phi C\varphi + C\phi S\theta S\varphi & C\phi C\theta & S\phi S\varphi - C\phi S\theta C\varphi \\ -C\theta S\varphi & S\theta & C\theta C\varphi \end{bmatrix}$$

$\phi = \operatorname{atan2}\left(\frac{q_{12}}{C\theta}, \frac{q_{22}}{C\theta}\right)$

Only exist when $C\theta \neq 0$

i.e when $\theta \neq \frac{\pi}{2}$ or $\frac{n\pi}{2}$

If $\theta = \frac{\pi}{2}$,

$$\begin{bmatrix} C\phi C\varphi - S\phi S\varphi & 0 & C\phi S\varphi + S\phi C\varphi \\ S\phi C\varphi + C\phi S\varphi & 0 & S\phi S\varphi - C\phi C\varphi \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} C\phi+\varphi & 0 & S\phi+\varphi \\ S\phi+\varphi & 0 & -C\phi+\varphi \\ 0 & 1 & 0 \end{bmatrix}$$

Only sum of $\phi+\varphi$ can be determined and there are infinitely many solutions

Q. Define degree of freedom of mechanical system.

Sol: The Degree of Freedom (DOF) of a mechanical system is the number of independent parameters that define its configuration or state. The mechanical system or body represents a set of rigid bodies that are constrained by joints connecting these bodies.

Consider a serial robot manipulator has n links, then each link has 6 dof. and a robot has $6n$ dof measured relative to fixed base. In order to count the dof of this robot, we can include the fixed link in count of links, so that dof is independent of choice of link that is treated as fixed link. Then,

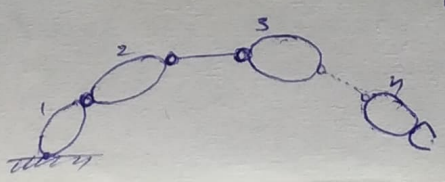
$$M = 6n = 6(N-1) \quad \text{because fixed link has 0 dof.}$$

In General,

$$M = 6n - \sum_{i=1}^{j} \underbrace{(6-f_i)}_{c_i} \quad \text{where } f_i \text{ represents joints dof}$$

$c_i \leftarrow$ Constraints imposed by joint i

eg.



$$M = 6n - 5n = n \text{ dof}$$

Constraints by each revolute or prismatic joint

Q. What are different errors in numerical methods?

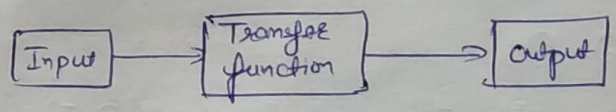
Sol: Truncation error: The word 'Truncate' means to 'shorten'. It refers to an error in a method which occurs because some number/series of steps (finite or infinite) is truncated (shortened) to a fewer number. Such errors are essentially algorithmic errors and we can predict the extent of the error that will occur in the method. for example, if we approximate the sine function by the first two non-zero term of its Taylor series, as in $\sin(x) \approx x - \frac{1}{6}x^3$ for small x, the resulting error is truncation error. It is present even with infinite-precision arithematic, because it is caused by truncation of infinite Taylor series to form the algorithm.

Roundoff error: It occurs because of the inability of computing device to deal with certain numbers. Such numbers need to be rounded off to some near approximation which is dependent on word size used to represent number of device. for eg: value of pi is rounded off.

Q. Define Transfer function in control theory.

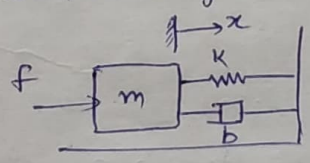
Sol: The transfer function of control system is defined as the ratio of laplace transform of the output variable to laplace transform of input variable assuming all initial condition to be zero.

$$G(s) = \frac{C(s)}{R(s)}$$



TF changes differential equations to algebraic ones in order to simplify the solution strategies

eg



$m\ddot{x} + b\dot{x} + Kx = f$ or $\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = f'$

$f' = \frac{1}{m}f$, $\omega_n = \sqrt{\frac{K}{m}}$, $\xi = \frac{b}{2\sqrt{Km}}$

$$G(s) = \frac{x(s)}{f(s)} = \frac{1}{ms^2+bs+K} = \frac{1/m\,\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Poles of system

$$s_{1,2} = \frac{-b \pm \sqrt{b^2-4mK}}{2m} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

Q Define Pseudo inverse of Matrix. Explain its two applications.

Sol: Pseudo inverse is a generalization of matrix inverse when the matrix may not be invertible.

$$\underbrace{A^{-1} = \frac{1}{|A|}\,\text{adj}A}_{\text{for square and non singular matrix}}$$

$A^+ = (A^TA)^{-1}A^T$ (Left P-inv)

$A^+ = A^T(AA^T)^{-1}$ (Right P-inv)

Applications:

1) Linear Least-squares: A common use is to compute a best fit (least square) solution to sys. of linear equations that lacks a solution

$Ax = b \qquad A \in \mathbb{R}^{m \times n}$

$\forall x \in \mathbb{R}^n$, we have $\|Ax - b\|_2 \geq \|Az - b\|_2$ where $z = A^+ b$

2) Obtaining all solutions of linear system:

$x = A^+ b + (I - A^+ A)\omega$ for any arbitrary vector ω

If solutions exist but A doesn't have full column rank, then we have an indeterminate system, all of whose solutions are given by above equation

3) Minimum norm solution to linear system: Another application is to find the minimum norm solution to system of linear equations with multiple solutions. (under-determined systems)

If $Ax = b$, then $z = A^+ b$ is a solution and satisfies $\|z\|_2 \leq \|x\|_2$ for all solutions

4) Condition Number: $\text{Cond}(A) = \|A\| \, \|A^+\|$

Q Calculate Cholesky decomposition of $\begin{bmatrix} 4 & 10 & 14 \\ 10 & 41 & 59 \\ 41 & 59 & 94 \end{bmatrix}$

Sol: $A = LL^T$ where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and $l_{ki} = \dfrac{a_{ki} - \sum_{j=1}^{i-1} l_{ij} l_{kj}}{l_{ii}}$

$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}$

$l_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$

$l_{21} = \dfrac{a_{21}}{l_{11}} = \dfrac{10}{2} = 5$

$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{41 - (5)^2} = \sqrt{41 - 25} = 4$

$l_{31} = \dfrac{a_{31}}{l_{11}} = \dfrac{14}{2} = 7$

$l_{32} = \dfrac{a_{32} - l_{21} l_{31}}{l_{22}} = \dfrac{59 - 7 \times 5}{4} = 6$

$l_{33} = \sqrt{a_{33}^2 - l_{31}^2 - l_{32}^2} = \sqrt{94 - 7^2 - 6^2} = \sqrt{94 - 85} = 3$

$L = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 4 & 0 \\ 7 & 6 & 3 \end{bmatrix}$ Ans

Q Write in state-space form and find its stability?

$\dot{x}_1 = -x_1 - 2x_2$

$\dot{x}_2 = x_1 - 4x_2$

Sol: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -1-\lambda & -2 \\ 1 & -4-\lambda \end{vmatrix} = 0 \Rightarrow (-1-\lambda)(-4-\lambda) + 2 = 0$

$\Rightarrow 4 + 5\lambda + \lambda^2 + 2 = 0 \Rightarrow \lambda^2 + 5\lambda + 6 = 0$

$\Rightarrow (\lambda + 3)(\lambda + 2) = 0$

$\Rightarrow \lambda = -3, -2$

All λ's are -ve, so system is stable

Q Find linear and quadratic approximation of $f(x) = x^3 - 4x^2 + 3x + 8$ at $x_0 = 2$

Sol: Linear approximation: $L(x) \simeq f(x_0) + f'(x_0) \cdot (x - x_0)$

$f(x_0) = (2)^3 - 4(2)^2 + 3(2) + 8 = 8 - 16 + 6 + 8 = 6$

$f'(x) = 3x^2 - 8x + 3 \Rightarrow f'(x_0) = 3(2)^2 - 8(2) + 3 = 12 - 16 + 3 = -1$

$\Rightarrow L(x) = 6 + (-1)(x - 2) = -x + 8$ Ans

Quadratic approximation: $Q(x) \simeq f(x_0) + (x-x_0)f'(x_0) + \frac{1}{2}(x-x_0)^2 f''(x_0)$

$f''(x) = 6x-8 \Rightarrow f''(x_0) = 6\times 2 - 8 = 4$

$Q(x) = 6 + (x-2)(-1) + \frac{1}{2}(x-2)^2 4 = 6 - x + 2 + 2(x^2+4-4x) = 6-x+2+2x^2+8-8x$

$Q(x) = 2x^2 - 9x + 16$ Ans

Q) Find minimum of following function using Steepest descent method.

$f(x_1, x_2) = x_1^2 + 2x_2^2 - 4x_1 - 2x_1x_2$ Initial $(x_1, x_2) = (1,1)$

Sol:-

$\nabla f = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 4 - 2x_2 \\ 4x_2 - 2x_1 \end{bmatrix} \Rightarrow \nabla f_{(1,1)} = \begin{bmatrix} 2-4-2 \\ 4-2 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

$x_1 = x_0 - \alpha \nabla f(x_0)$ choosing $\alpha = 0.5$

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$f(x_0) = 1+2-4-2 = -3$

$f(x_1) = (3)^2 - (4\times 3) = 9-12 = -3$

$\nabla f(x_1) = \begin{bmatrix} 2\times 3 - 4 \\ -2\times 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

$x_2 = x_1 - \frac{1}{2}\nabla f(x_1) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$f(x_2) = (2)^2 + 2(-3)^2 - 4(2) - 2(2)(-3) = 4+18-8+12 = 26$

Step size is wrong here, so function not minimizing

To choose step size,

$\phi(\alpha) = f(x_0 - \alpha \nabla f(x_0)) = f((1,1) - \alpha(3,0)) = f(1-3\alpha, 1)$

$\phi'(\alpha) = -\nabla f(1-3\alpha, 1) \cdot (3,0)$

$= -[(2(1-3\alpha) - 4 - 2), 4 - 2(1-3\alpha)] \cdot (3,0)$

WRONG $= -3(2-6\alpha-6) = -3(-6\alpha-4) = +18\alpha + 12 = 0 \Rightarrow \alpha = \frac{-12}{18} = \frac{-2}{3}$

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2.67 \\ 1.33 \end{bmatrix} = \begin{bmatrix} -1.67 \\ -0.33 \end{bmatrix}$

$f(x_1) = (-1.67)^2 + 2(-0.33)^2 - 4(-0.167) - 2(-1.67)(-0.33)$

$= 2.79 + 0.218 + 0.668 - 1.1022$

$= 2.5738$

$\phi(\alpha) = f(x_0 - \alpha \nabla f(x_0)) = f((1,1) - \alpha(-4,2)) = f(1+4\alpha, 1-2\alpha)$

$= (1+4\alpha)^2 + 2(1-2\alpha)^2 - 4(1+4\alpha) - 2(1+4\alpha)(1-2\alpha)$

$= 1 + 16\alpha^2 + 8\alpha + 2 + 8\alpha^2 - 8\alpha - 4 - 16\alpha - 2(1 + 2\alpha - 8\alpha^2)$

$= 1 + 16\alpha^2 + 8\alpha + 2 + 8\alpha^2 - 8\alpha - 4 - 16\alpha - 2 - 4\alpha + 16\alpha^2$

$= 40\alpha^2 - 20\alpha - 3$

$\phi'\alpha = 80\alpha - 20 = 0 \Rightarrow \alpha = 1/4 = 0.25$

$x_1 = x_0 - \alpha \nabla f(x_0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$

$\nabla f(x_1) = \begin{bmatrix} 4-4-1 \\ 2-4 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$$x_2 = x_1 - \alpha \nabla f(x_1) = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix} - \alpha \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2+\alpha \\ 0.5+2\alpha \end{bmatrix}$$

$$\phi(x_2) = f(x_2) = (2+\alpha)^2 + 2(0.5+2\alpha)^2 - 4(2+\alpha) - 2(2+\alpha)(0.5+2\alpha)$$

$$= 4+\alpha^2+4\alpha + 0.5+8\alpha^2+4\alpha - 8-4\alpha - 2(1+4.5\alpha+2\alpha^2)$$

$$= 4+\alpha^2+4\alpha + 0.5 + 8\alpha^2+4\alpha - 8 - 4\alpha - 2 - 9\alpha - 4\alpha^2$$

$$= 5\alpha^2 - 5\alpha - 5.5$$

$$\phi'(x_2) = 10\alpha - 5 = 0 \Rightarrow \alpha = \frac{1}{2}$$

$$x_2 = \begin{bmatrix} 2.5 \\ 1.5 \end{bmatrix}$$

note

Steepest Descent algorithm

1) Given x^0, set $k=0$
2) $d^k = -\nabla f(x^k)$. If $d^k = 0$, stop
3) Solve $\min_\alpha f(x^k + \alpha d^k)$ for the stepsize α^k
4) Set $x^{k+1} \leftarrow x^k + \alpha^k d^k$, $k \leftarrow k+1$.

Q Using Runge-Kutta 4th order method, find $y(t=0.4)$ for $y' = 1 - t + 4y$ with $y(0)=1$ and $h = 0.1$

Sol:- $y' = 1 - t + 4y = f(t,y)$, $y(t=0) = 1$, $h = 0.1$

$$y_{n+1} = y_n + \frac{1}{6} h (k_1 + 2k_2 + 2k_3 + k_4)$$

$$x_{n+1} = x_n + h$$

here, $k_1 = f(t_n, y_n)$

$k_2 = f(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2})$

$k_3 = f(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2})$

$k_4 = f(t_n + h, y_n + h k_3)$

$k_1 = hf(t_0, y_0) = 0.1 f(0,1) = 0.1(5) = 0.5$

$k_2 = hf(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1 f(0.05, 1.25) = 0.1(5.95) = 0.595$

$k_3 = hf(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1 f(0.05, 1.2975) = 0.1(6.14) = 0.614$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 f(0.1, 1.614) = 0.1(7.356) = 0.7356$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{1}{6}(0.5 + 2\times 0.595 + 2\times 0.614 + 0.7356) = 1.6089 = y(0.1)$

$k_1 = hf(t_1, y_1) = 0.1 f(0.1, 1.6089) = (0.1)\cdot(7.3357) = 0.7336$

$k_2 = hf(t_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1 f(0.15, 1.9757) = 0.1(8.7529) = 0.8753$

$k_3 = hf(t_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.1 f(0.15, 2.0466) = 0.1(9.0363) = 0.9036$

$k_4 = hf(t_1 + h, y_1 + k_3) = 0.1 f(0.2, 2.5126) = 0.1(10.8503) = 1.085$

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.6089 + \frac{1}{6}(0.7336 + 2\times 0.8753 + 2\times 0.9036 + 1.085) = 2.505 = y(0.2)$

$k_1 = h f(t_2, y_2) = 0.1 f(0.2, 2.505) = 0.1 (10.82) = 1.082$

$k_2 = h f(t_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}) = 0.1 f(0.25, 3.046) = 0.1 (12.934) = 1.2934$

$k_3 = h f(t_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}) = 0.1 f(0.25, 3.1517) = 0.1 (13.3568) = 1.3357$

$k_4 = h f(t_2 + h, y_2 + k_3) = 0.1 f(0.3, 3.8407) = 0.1 (16.0628) = 1.6063$

$y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2.505 + \frac{1}{6}(1.082 + 2(1.2934) + 2(1.3357) + 1.6063) = 3.8294$
$= y(0.3)$

$k_1 = h f(t_3, y_3) = 0.1 f(0.3, 3.8294) = 0.1 (16.0177) = 1.6018$

$k_2 = h f(t_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}) = 0.1 f(0.35, 4.6303) = 0.1 (19.1712) = 1.9171$

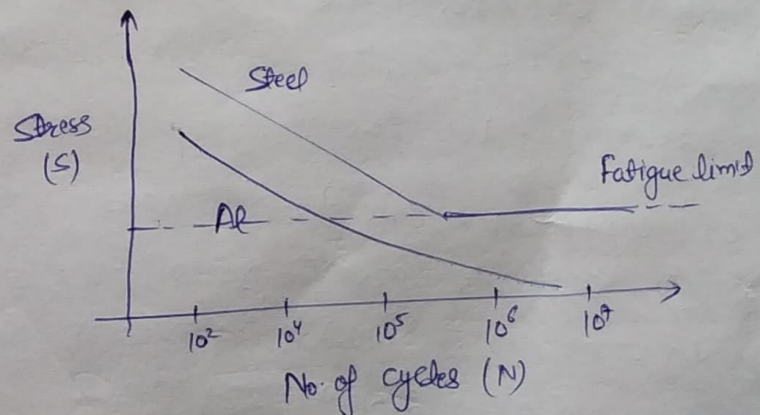
$k_3 = h f(t_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}) = 0.1 f(0.35, 4.788) = 0.1 (19.8019) = 1.9802$

$k_4 = h f(t_3 + h, y_3 + k_3) = 0.1 f(0.4, 5.8096) = 0.1 (23.8384) = 2.3838$

$y_4 = y_3 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 3.8294 + \frac{1}{6}(1.6018 + 2 \times 1.9171 + 2 \times 1.9802 + 2.3838)$
$\Rightarrow y(0.4) = 5.7928$ ★

Q. In case of fatigue failure, what is basis for design?

Sol: Fatigue is the weaking of a material subjected to repeated loads and is progressive and localised structural damage. Fatigue life is a term describing how long a product will last before the complete failure. There are a no. of different factors that can affect the fatigue-life of a product including material type; its structure, working temperature and environmental conditions. The stress value that can cause the fatigue damage may be much less than the strength of material typically quoted as the yield stress or ultimate stress. The material performance is normally characterised by S-N graph illustrating the magnitude of a cycle stress (S) against the logarithmic scales of the number of cycles to failure. S-N curves are derived from physical experiments on samples of material. Shown below are example S-N curve for aluminium and steel.



The curve shows how many cycles the material is expected to survive at

a given stress. Obviously, as the working stress is increased, the no. of cycles that the component can last will decrease. The product is safe to use if and only if working stress is below the curve. With some materials, such as steel, there is what is known as endurance limit or a fatigue limit. If the working stress is below the fatigue limit, the component will never fail. If the stress is above the limit, its life will be finite. Thus material prone to fatigue failure need to be designed within the safe zone based upon the S-N curve.

Q (a) Find the rank of matrix. (b) Find LU decomposition.

$$\begin{bmatrix} 9 & 2 & 3 & 12 \\ 9 & 0 & 5 & 12 \\ 0 & 1 & -4 & 0 \end{bmatrix}$$

Sol: (a) $R_2 \leftarrow R_2 - R_1$

$$\begin{bmatrix} 9 & 2 & 3 & 12 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix} \quad R_3 \leftarrow 2R_3 + R_2 \quad \begin{bmatrix} 9 & 2 & 3 & 12 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix}$$

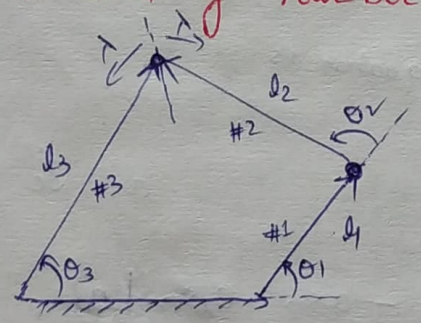
$r(A) = 3$ (No. of all non-zero rows)

(b) $R_2 \leftarrow R_2 - (1) R_1$ $\qquad$ $R_3 \leftarrow R_3 - (-0.5) R_2$

$$\begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} 9 & 2 & 3 & 12 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} 9 & 2 & 3 & 12 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$

Q Find the EoM of Four-bar Mechanism using (a) EL (b) DeNOC approach

Sol.



for link #1 and #2, treat it as RR-robot and derive the equation.

In (a) EL:

$$T_3 = \frac{1}{2} m_3 V_{c_3}^T V_{c_3} + \frac{1}{2} I_3 \omega_3^T \omega_3$$

$$V_{c_3}^T V_{c_3} = l_{c_3}^2 \dot{\theta}_3^2$$

$$\Rightarrow T_3 = \frac{1}{2} m_3 l_{c_3}^2 \dot{\theta}_3^2 + \frac{1}{2} I_{3,22} \dot{\theta}_3^2$$

and $U_3 = [0 \ -g \ 0] \begin{bmatrix} l_{c_3} c_1 \\ l_{c_3} s_1 \\ 0 \end{bmatrix} m_3 = -m_3 l_{c_3} g s_1$

In (b) DeNOC:

Again link #1 and #2, treat it as RR and derive the same equation for link #3, do the same as for joint 1 (link 1)

DeNOC matrices

$$N = N_e N_d$$

$$N_d = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}_{18\times 3}, \quad N_e = \begin{bmatrix} 1 & 0 & 0 \\ A_{21} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{18\times 18}$$

$$\dot{N}_d = \begin{bmatrix} \dot{p}_1 & 0 & 0 \\ 0 & \dot{p}_2 & 0 \\ 0 & 0 & \dot{p}_3 \end{bmatrix}, \quad \dot{N}_e = \begin{bmatrix} 0 & 0 & 0 \\ \dot{A}_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \dot{p}_3 = [0]_{6\times 1}$$

EoM

$$M = \begin{bmatrix} M_1 & & \\ & M_2 & \\ & & M_3 \end{bmatrix}, \quad W = \mathrm{diag}(W_1, W_2, W_3), \quad E = \mathrm{diag}(E_1, E_2, E_3)$$

$$M\dot{t} + WMEt = \omega^D + \omega^E$$

$$N^T(M\dot{t} + WMEt) = N^T(\omega^D + \omega^E)$$

$$\Rightarrow I\ddot{\theta} + h = \tau$$

$$I = \begin{bmatrix} I_{RR} & 0 \\ 0 & m_3 l_{c3}^2 + I_{322} \end{bmatrix}_{3\times 3} \quad h = \begin{bmatrix} h_{RR} \\ 0 \end{bmatrix}_{3\times 1} \quad g = \begin{bmatrix} g_{RR} \\ m_3 g l_{c3} c_3 \end{bmatrix}_{3\times 1}$$

For forward dynamics simulation, use cut joint method.

$$I\ddot{\theta} + h = \tau + J_c^T \lambda$$

where J_c is the constrained jacobian matrix and is $J_c\dot{\theta} = 0$. λ is the vector of lagrange multipliers representing the constraint forces at joint.

Using $J_c\ddot{\theta} = -\dot{J}_c\dot{\theta}$

$$\begin{bmatrix} I & J_c^T \\ J_c & 0 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ -\lambda \end{bmatrix} = \begin{bmatrix} \phi \\ -\dot{J}_c\dot{\theta} \end{bmatrix} \quad \text{where } \phi = \tau + (-h)$$

$$\lambda = -S_I^{-1}(J_c I^{-1}\phi + \dot{J}_c\dot{\theta}) \quad \text{where } S_I = J_c I^{-1} J_c^T$$

In case of 4-bar mechanism, loop-closure equation is

$$a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) - a_3\cos\theta_3 = 0$$
$$a_1\sin\theta_1 + a_2\sin(\theta_1+\theta_2) - a_3\sin\theta_3 = 0$$

and $\dot{J}_c$ given in code

Diff. wrt time

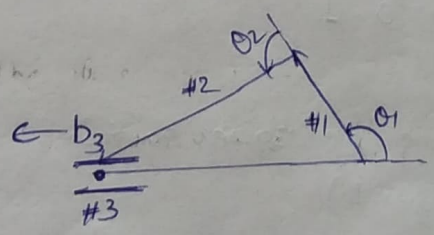
$$-a_1 s_1\dot{\theta}_1 - a_2 s_{12}\dot{\theta}_{12} + a_3 s_3\dot{\theta}_3 = 0$$
$$a_1 c_1\dot{\theta}_1 + a_2 c_{12}\dot{\theta}_{12} - a_3 c_3\dot{\theta}_3 = 0$$

$$\Rightarrow \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} & a_3 s_3 \\ a_1 c_1 + a_2 c_{12} & +a_2 c_{12} & -a_3 c_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = 0$$

$$J_c\dot{\theta} = 0$$

Q Find the EoM of Slider Crank mechanism using EL and DeNOC

Sol:



EL $\quad T_3 = \frac{1}{2} m_3 \dot{b}_3^2$

DeNOC: for 3, do same as prismatic joint

$$I = \begin{bmatrix} I_{RR} & 0 \\ 0 & m_3 \end{bmatrix} \quad h = \begin{bmatrix} h_{RR} \\ 0 \end{bmatrix} \quad g = \begin{bmatrix} g_{RR} \\ 0 \end{bmatrix}$$

Loop closure equation:

$$a_1c_1 + a_2c_{12} + b_1 = 0 \qquad\qquad -a_1s_1\dot{\theta}_1 - a_2s_{12}\dot{\theta}_{12} + \dot{b}_1 = 0$$
$$a_1s_1 + a_2s_{12} + 0 = 0 \quad \Rightarrow \quad a_1c_1\dot{\theta}_1 + a_2c_{12}\dot{\theta}_{12} = 0$$

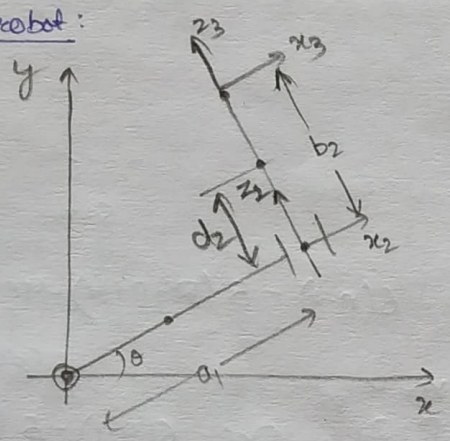
$$J_c\dot{\theta} = 0$$

$$\Rightarrow \left[\begin{array}{c|c|c} -a_1s_1 - a_2s_{12} & -a_2s_{12} & 1 \\ \hline a_1c_1 + a_2c_{12} & a_2c_{12} & 0 \end{array}\right] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{b}_2 \end{bmatrix} =$$

- Newton-Euler formulation:

- Refer Saha sir's book for detailed study
- RR-robot is solved there as an example
- Here I will solve for RP robot.

RP robot:



$\theta_2 = \theta_1 + 90$

$$Q_1 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Q_2 = \begin{bmatrix} -s_1 & -c_1 & 0 \\ c_1 & -s_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[d_1]_1 = \left[\tfrac{1}{2}a_1c_1 \;\; \tfrac{1}{2}a_1s_1 \;\; 0\right]^T \qquad [r_1]_2 = \left[\tfrac{a_1}{2} \;\; 0 \;\; 0\right]^T$$

$$[d_2]_2 = \left[-b_2s_1 \;\; b_2c_1 \;\; 0\right]^T \qquad [r_2]_3 = \left[0 \;\; 0 \;\; \tfrac{b_2}{2}\right]^T$$

$$[I_1]_2 = \frac{m_1a_1^2}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [I_2]_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

forward Computation

Link 1 $[\omega]_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$, $[\dot{\omega}_1]_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}$, $[\dot{c}_1]_1 = \dot{\theta}_1 \frac{a_1}{2}\begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}$, $[\ddot{c}_1]_1 = \ddot{\theta}_1 \frac{a_1}{2}\begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} - \dot{\theta}_1^2 \frac{a_1}{2}\begin{bmatrix} c_1 \\ s_1 \\ 0 \end{bmatrix}$

$[g]_1 = [0 \;\; -g \;\; 0]^T$

Link 2 Prismatic

$$[\omega_i]_i = Q_{i-1}^T[\omega_{i-1}]_{i-1}$$
$$[\dot{\omega}_i]_i = Q_{i-1}^T[\dot{\omega}_{i-1}]_{i-1}$$

$$[\omega_2]_2 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$[\dot{\omega}_2]_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}$$

$$[\ddot{c}_i]_i = Q_{i-1}^T\left[\ddot{c}_{i-1} + \underbrace{\dot{\omega}_{i-1}\times r_{i-1}} + \omega_{i-1}\times(\omega_{i-1}\times r_{i-1})\right]_{i-1}$$
$$+\left[\dot{\omega}_i\times d_i + \omega_i\times(\omega_i\times d_i) + \ddot{b}_i e_i + 2\dot{b}_i\omega_i\times e_i\right]_i$$

$$\dot{\omega}_{i-1}\times(r_{i-1})_{i-1} = \begin{bmatrix} i & j & k \\ 0 & 0 & \ddot{\theta}_1 \\ \frac{1}{2}a_1c_1 & \frac{1}{2}a_1s_1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}a_1s_1\ddot{\theta}_1 \\ \frac{1}{2}a_1c_1\ddot{\theta}_1 \\ 0 \end{bmatrix}$$

Similarly calculate all terms

formulation used (Revolute)

$$\omega_i = \omega_{i-1} + \dot{\theta}_i e_i$$
$$[\omega_i]_i = Q_{i-1}^T[\omega_{i-1}]_{i-1} + \dot{\theta}_i[e_i]_i$$

$$Q_{i-1}^T = \begin{bmatrix} C\theta_{i-1} & S\theta_{i-1} & 0 \\ -S\theta_{i-1}C\alpha_{i-1} & C\theta_{i-1}C\alpha_{i-1} & S\alpha_{i-1} \\ S\theta_{i-1}S\alpha_{i-1} & -C\theta_{i-1}S\alpha_{i-1} & C\alpha_{i-1} \end{bmatrix}$$

$$\dot{\omega}_i = \dot{\omega}_{i-1} + \ddot{\theta}_i e_i + \dot{\theta}_i\omega_i\times e_i$$
$$[\dot{\omega}_i]_i = Q_{i-1}^T[\dot{\omega}_{i-1}]_{i-1} + \ddot{\theta}_i[e_i]_i + \dot{\theta}_i[\omega_i]_i\times[e_i]_i$$

$$[\dot{c}_i]_i = Q_{i-1}^T[\dot{c}_{i-1}]_{i-1} + [\dot{\omega}_{i-1}]_{i-1}\times[r_{i-1}]_{i-1} + [\omega_i]_i\times[d_i]_i$$

$$[\ddot{c}_i] = Q_{i-1}^T\left[\ddot{c}_{i-1} + \dot{\omega}_{i-1}\times r_{i-1} + \omega_{i-1}\times(\omega_{i-1}\times r_{i-1})\right]_{i-1} + \dot{\omega}_i\times d_i + \omega_i\times(\omega_i\times d_i)]$$

$$[g]_i = Q_{i-1}^T[g]_{i-1}$$

Backward computation:

$$[f_i]_i = m_i[\ddot{c}_i]_i$$

$$[n_i]_i = [I_i]_i[\dot{\omega}_i]_i + [\omega_i]_i \times [I_i]_i[\omega_i]_i$$

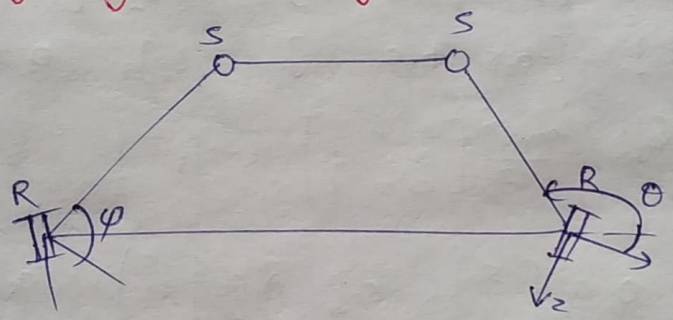
$$[f_{i-1,i}]_i = [f_i]_i + [f_{i,i+1}]_i - m_i[g]_i$$

$$[n_{i-1,i}]_i = [n_i]_i + [n_{i,i+1}]_i + [d_i]_i \times [f_{i-1,i}]_i + [r_i]_i \times [f_{i,i+1}]_i$$

Joint Torques and forces

$$\tau_i = \begin{cases} [e_i]_i^T [n_{i-1,i}]_i & \text{Revolute} \\ [e_i]_i^T [f_{i-1,i}]_i & \text{Prismatic} \end{cases}$$

• Degree of Freedom of RSSR Mechanism



$$DOF = 6(n-1) - 5(R+P+H) - 4C - 3(S+E)$$

Screw, planar pairs, cylindrical

$n = 4$
$R = 2$
$S = 2$

$$DOF = 6(4-1) - 5(2) - 3(2) = 18 - 10 - 6 = 2$$

• Difference between N-E and Lagrange EoM

Euler Lagrange (EL)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q_i$$

- EoM are derived from Newton's 2nd law which relates force and momentum as well as torque and angular momentum
- The constraint forces involved in the system are automatically eliminated
- The closed form dynamics equations can be derived in any coordinate system
- Lagrangian is easily extensible to hydrodynamics, electric circuits etc. due to generalized coordinates

Newton-Euler (NE)

$$I_i^c \dot{\omega}_i + \omega_i \times I_i^c \omega_i = n_i^c$$

$$m_i v_i^c = f_i^c$$

- Lagrangian EoM are derived from principle of least equation or from Newtonian mechanics. It describes the dynamic systems in terms of work and energy stored in system rather than forces and moments of individual members
- The resulting expression involve constraint forces
- The constraint forces need to be eliminated to form closed form equation
- It is quite complex to model other system with NE because of consideration of constraints and geometrical nature

Behaviour of dynamic system:

$$m\ddot{x} + b\dot{x} + kx = f$$

$$ms^2 + bs + k = 0$$

$$s = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

Case I: Real and unequal roots (s_1, s_2)

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + c_p$$

Case II: for complex roots $(s_{1,2} = \alpha \pm i\beta)$

$$x(t) = e^{\lambda t}(\alpha_1 \cos(\beta t) + \alpha_2 \sin(\beta t)) + c_p$$

Case III: for real and equal roots $(s_1 = s_2 = s)$

$$x(t) = (c_1 + c_2 t) e^{\frac{-b}{2m} t} + c_p$$

Q. Consider a dynamic system

$$m\ddot{x} + b\dot{x} + kx = f \qquad m = 1, b = 4, k = 3$$

(a) Solve for x when $f = \sin 2t$

$$s^2 + 4s + 3 = 0$$

$$s^2 + 3s + s + 3 = 0$$

$$(s+3)(s+1) = 0 \Rightarrow s_1 = -3, \; s_2 = -1$$

$$x(t) = c_1 e^{-2t} + c_2 e^{-3t} + c_p$$

Initial guess for c_p:

$$x_p(t) = A \cos 2t + B \sin 2t$$

$$x_p'(t) = -2A \sin 2t + 2B \cos 2t$$

$$x_p''(t) = -4A \cos 2t + (-4B \sin 2t)$$

$$3A \cos 2t + 3B \sin 2t - 8A \sin 2t + 8B \cos 2t - 4A \cos 2t - 4B \sin 2t = \sin 2t$$

$$-A \cos 2t + 8B \cos 2t - 8A \sin 2t - B \sin 2t = \sin 2t$$

$$\Rightarrow \begin{aligned} -A + 8B &= 0 \\ -8A - B &= 1 \end{aligned}$$

$$-65A = 8 \Rightarrow A = \frac{-8}{65}, \; B = \frac{-1}{65}$$

$$x(t) = c_1 e^{-2t} + c_2 e^{-3t} - \frac{8}{65} \cos 2t + \left(\frac{-1}{65} \sin 2t\right)$$

(b) Propose a PD control law to make system critically damped

$$\ddot{x} + 4\dot{x} + 3x = -k_p x - k_v \dot{x}$$

$$\ddot{x} + (4 + k_v)\dot{x} + (3 + k_p)x = 0$$

$$\ddot{x} + b'\dot{x} + k'x = 0$$

for critical damping,

$$b' = 2\sqrt{mk'} \Rightarrow b' = 2\sqrt{k'} \Rightarrow \boxed{4 + k_v = 2\sqrt{3 + k_p}}$$

- **Holonomic and Non-Holonomic System:**

Holonomic systems are systems for which all constraints are integrable into positional constraints eg. a robot can move in any direction in configuration space.

Non-holonomic systems are systems which have constraints that are non-integrable into positional constraints eg. systems where velocities (magnitude or direction) and other derivatives of position are constraint.

- **Generalized Coordinates and forces:**

The coordinates that specify the configuration (i.e. position and orientation) of all the bodies or links of a mechanical system completely are called generalized coordinates.

- Number of independent generalized coordinates is equal no. of DoF of robot.
- Coordinate is referred to as complete when its capable of describing all geometric configuration at all times.

If the work required to increase the coordinate q_j by δq_j is $Q_j \delta q_j$ then Q_j is the generalized force associated with coordinate q_j.

- A generalize force need not always be dimensionaly equivalent to force. for eg. if generalized force is an angle, the corresponding generalized force will be a torque.

- **Grashof's Criteria:** It states that for a 4-bar linkage system, if sum of length of shortest and longest of a planar quadrilateral linkage is less than or equal to sum of remaining two links, then the shortest link can rotate freely with respect to neighbouring link

$$S+L \leq P+Q$$

If $S+L < P+Q$; S is fixed ⇒ double crank

opposite to S is fixed ⇒ double rocker

link adjacent to S is fixed ⇒ crank-rocker

- **Gruebler and Kutzbach criteria:**

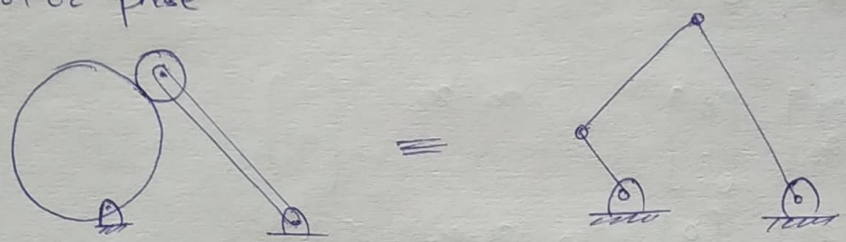
$$m = 3(n-1) - 2J_1 - J_2$$

J_1 ← lower pairs (1 DoF)

J_2 ← higher pairs (2 DoF)

Equivalent Linkage of mechanism: An equivalent linkage may be defined as one that produces identical motion as the part being analyzed for a given position or phase

eg:



Rank of a matrix: defined as the max. no. of linearly independent column vectors in the matrix or row vectors in the matrix. for $r \times c$ matrix if $r < c$, then max. rank is r

Nullity of matrix: The dimension of the nullspace $N(A)$ is called the nullity of matrix. The Nullspace of a matrix $N(A)$ is the set of all n-dimensional column vectors x such that $Ax = 0$

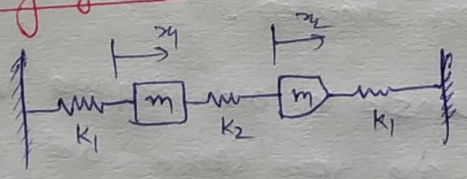
Methods of calculating SVD

(1) Direct method is first transforming the original matrix to bidiagonal matrix and then getting its SVD via the eigen pairs of tridiagonal matrix

(2) The Jacob method is to get the SVD of biadiagonal matrix through Jacobi rotation

(3) Combined built in method is to get the SVD of bidiagonal matrix via the built in svd function and combined squaring method is employing the squaring/indered method to get the SVD of bidiagonal matrix and finally get the SVD of original matrix

Mass-Spring Systems



$$\ddot{X} = KX \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \ddot{X} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}$$

$$K = \begin{bmatrix} \frac{-K_1+K_2}{m_1} & \frac{-K_2}{m_1} \\ \frac{K_2}{m_2} & \frac{-K_2+K_3}{m_2} \end{bmatrix}$$

Given $K_1 = K_2 = K_3 = m = 1$

$$K = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

We will seek the solution in the form of harmonic oscillation, that is

as $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e^{i\omega t}$

$$\det(k+\omega^2 I) = 0 \Rightarrow \begin{vmatrix} -2+\omega^2 & 1 \\ 1 & -2+\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow (-2+\omega^2)(-2+\omega^2) - 1 = 0$$

$$\Rightarrow 4 - 4\omega^2 + \omega^4 - 1 = 0 \Rightarrow \omega^4 - 4\omega^2 + 3 = 0 \Rightarrow (\omega^2-1)(\omega^2-3) = 0 \Rightarrow \omega^2 = 1, 3$$

Find eigen vector

$\omega = 1$, $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = x_2 \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\omega = 3$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = -x_2 \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$X(t) = c_1 e^{i\omega_1 t} v_1 + c_2 e^{-\omega_1 t} v_1 + c_3 e^{i\omega_2 t} v_2 + c_4 e^{-i\omega_2 t} v_2$$

$$c_1 = \bar{c}_2 \Rightarrow \alpha_1 + i\beta_1 = \alpha_2 - i\beta_2 \quad \Big\} \begin{matrix} \alpha_1 = \alpha_2 \\ \beta_1 = -\beta_2 \end{matrix}$$

$$c_3 = \bar{c}_4 \Rightarrow \alpha_3 + i\beta_3 = \alpha_4 - i\beta_4 \quad \Big\} \begin{matrix} \alpha_3 = \alpha_4 \\ \beta_3 = -\beta_4 \end{matrix}$$

$$X(t) = 2\left[\alpha_1 \cos(\omega_1 t) - \beta_1 \sin \omega_1 t)\right] v_1 + 2\left[\alpha_3 \cos(\omega_2 t) - \beta_3 \sin(\omega_2 t)\right] v_2$$

Introducing phase angles

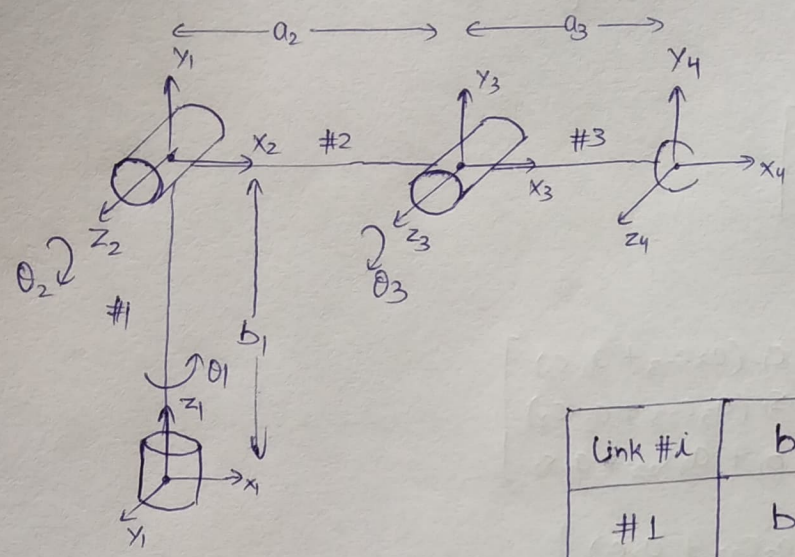
$$X(t) = A_1 \cos(\omega_1 t + \phi_1) v_1 + A_2 \cos(\omega_2 t + \phi_2) v_2 \quad \star$$

- **Implicit and Explicit numerical methods for solving non-linear differential equations:**

- Explicit scheme is one where there is a simple updating procedure that does not depend on other values at current level while an implicit one contains information at current level which requires the solving of simultaneous equations.
- Explicit schemes are easy to program with one simple calculations performed at each timestep while implicit schemes are more difficult and can involve many iterations per timestep.
- Explicit have a tendency to be unstable and require very small timesteps while implicit schemes are very stable and can have much larger timesteps.
- Explicit eg: Forward Euler ($y_{n+1} = y_n + f(x_n, y_n)\Delta x$), Implicit Runge-Kutta method, second order Adams-Bashforth (AB2) method

  Implicit eg: Backward Euler ($y_{n+1} = y_n + f(x_n, y_{n+1})\Delta x$), Explicit Runge-Kutta method, Second order Adams-Moulton method (AM2)

Articulated RRR arm



DH Parameters

Link #i	b_i	θ_i	a_i	α_i
#1	b_1	θ_1^*	0	$\pi/2$
#2	0	θ_2^*	a_2	0
#3	0	θ_3^*	a_3	0

Homogeneous Transformation matrix:

$$T_i = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & b_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = T_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = T_1 \times T_2 = T_1 \times \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 c_2 & -c_1 s_2 & s_1 & a_2 c_1 c_2 \\ s_1 c_2 & -s_1 s_2 & -c_1 & a_2 c_2 s_1 \\ s_2 & c_2 & 0 & b_1 + s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = A_2 \times T_3 = A_2 \times \begin{bmatrix} c_3 & -s_3 & 0 & a_3 c_3 \\ s_3 & c_3 & 0 & a_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 c_{23} & -s_{23} c_1 & s_1 & c_1(a_3 c_{23} + a_2 c_2) \\ c_{23} s_1 & -s_{23} s_1 & -c_1 & s_1(a_3 c_{23} + a_2 c_2) \\ s_{23} & c_{23} & 0 & b_1 + a_3 s_{23} + a_2 s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Jacobian :

$$J[:,i] = \begin{bmatrix} z_{i-1} \times (O_e - O_{i-1}) \\ \hline z_{i-1} \end{bmatrix}$$

if $i=1$

$$= \begin{bmatrix} z_{i-1} \times (O_e - \vec{0}) \\ \hline z_{i-1} \end{bmatrix}$$

We have,

$$T_i = \left[\begin{array}{ccc|c} x_i & y_i & z_i & O_i \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

for $i=1$,

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad O_e = O_3 = \begin{bmatrix} c_1 (a_3 c_{23} + a_2 c_2) \\ s_1 (a_3 c_{23} + a_2 c_2) \\ b_1 + a_3 s_{23} + a_2 s_2 \end{bmatrix}$$

$$J_1 = \begin{bmatrix} z_0 \times O_e \\ \hline z_0 \end{bmatrix} = \begin{bmatrix} -s_1 (a_3 c_{23} + a_2 c_2) \\ c_1 (a_3 c_{23} + a_2 c_2) \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{bmatrix}$$

for $i=2$,

$$z_1 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}, \quad O_3 - O_1 = O_3 - \begin{bmatrix} 0 \\ 0 \\ b_1 \end{bmatrix} = \begin{bmatrix} c_1 (a_3 c_{23} + a_2 c_2) \\ s_1 (a_3 c_{23} + a_2 c_2) \\ a_3 s_{23} + a_2 s_2 \end{bmatrix}$$

$$J_2 = \begin{bmatrix} z_1 \times (O_3 - O_1) \\ \hline z_1 \end{bmatrix} = \begin{bmatrix} -c_1 (a_3 s_{23} + a_2 s_2) \\ -s_1 (a_3 s_{23} + a_2 s_2) \\ a_3 c_{23} + a_2 c_2 \\ \hline s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

for $i=3$,

$$z_2 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}, \quad O_3 - O_2 = O_3 - \begin{bmatrix} a_2 c_1 c_2 \\ a_2 c_2 s_1 \\ b_1 + a_2 s_2 \end{bmatrix} = \begin{bmatrix} c_1 a_3 c_{23} \\ s_1 a_3 c_{23} \\ a_3 s_{23} \end{bmatrix}$$

$$J_3 = \begin{bmatrix} z_2 \times (O_3 - O_2) \\ \hline z_2 \end{bmatrix} = \begin{bmatrix} -a_3 s_{23} c_1 \\ -a_3 s_{23} s_1 \\ a_3 c_{23} \\ \hline s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

$$J = [\, J_1 \mid J_2 \mid J_3 \,]$$

$$J = \left[\begin{array}{c|c|c} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \\ \hline 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{c} J_v \\ \hline J_\omega \end{array}\right]$$

To find the singularities,

$$\det(J_v) = |J_v| = -(a_2c_2 + a_3c_{23})\left[a_3s_1^2s_{23}(a_2c_2 + a_3c_{23}) + a_3c_1^2s_{23}(a_2c_2 + a_3c_{23})\right]$$
$$+ a_3c_{23}\left[s_1^2(a_2c_2 + a_3c_{23})(a_2s_2 + a_2s_{23}) + c_1^2(a_2c_2 + a_3c_{23})(a_2s_2 + a_3s_{23})\right]$$

$$= -(a_2c_2 + a_3c_{23})\left[a_3s_{23}(a_2c_2 + a_3c_{23})\right] + a_3c_{23}\left[(a_2c_2 + a_3c_{23})(a_2s_2 + a_3s_{23})\right]$$

$$= (a_2c_2 + a_3c_{23})\left[-a_3s_{23}(a_2c_2 + a_3c_{23}) + a_3c_{23}(a_2s_2 + a_3s_{23})\right]$$

$$= (a_2c_2 + a_3c_{23})\left[-a_2a_3c_2s_{23} - a_3^2\cancel{c_{23}}s_{23} + a_3a_2s_2c_{23} + a_3^2\cancel{s_{23}}c_{23}\right]$$

$$= (a_2c_2 + a_3c_{23})\left[a_2a_3\underbrace{(-c_2s_{23} + s_2c_{23})}\right]$$

$$-c_2(s_2c_3 + c_2s_3) + s_2(c_2c_3 - s_2s_3)$$
$$-\cancel{c_2s_2}c_3 - c_2^2s_3 + \cancel{s_2c_2}c_3 - s_2^2s_3$$
$$-s_3(c_2^2 + s_2^2)$$
$$-s_3$$

$$= -a_2a_3s_3(a_2c_2 + a_3c_{23})$$

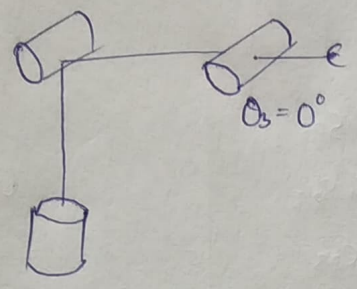
The determinant will be zero when

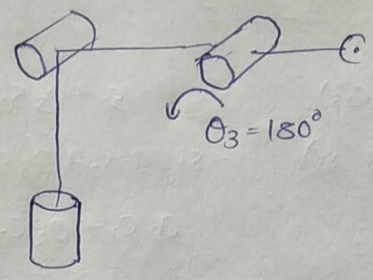
$$s_3 = 0 \Rightarrow \theta_3 = 0 \text{ or } \pi \qquad \text{—(1)}$$

and

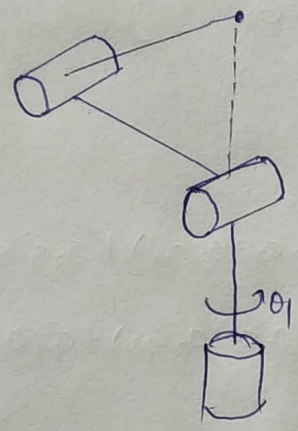
$$a_2c_2 + a_3c_{23} = 0 \qquad \text{—(2)}$$

Eq. (1) and (2) represent singular configurations of the arm





Eq. (1) : <u>Elbow singularities</u>



<u>Eq (2)</u>: Singularity occurs when end-effector center intersects the axis of base rotation.

Position vector for COM of links

$$C_1 = \begin{bmatrix} 0 \\ 0 \\ l_{c_1} \end{bmatrix}, \quad C_2 = \begin{bmatrix} l_{c_2} c_1 c_2 \\ l_{c_2} s_1 c_2 \\ l_1 + l_{c_2} s_2 \end{bmatrix}, \quad C_3 = \begin{bmatrix} l_2 c_1 c_2 + l_{c_3} c_{23} c_1 \\ l_2 s_1 c_2 + l_{c_3} c_{23} s_1 \\ l_1 + l_2 s_2 + l_{c_3} s_{23} \end{bmatrix}$$

where $l_{c_1} = l_1/2 = b/2$

$l_{c_2} = l_2/2 = a_2/2$

$l_{c_3} = l_3/3 = a_3/2$

$$V_1 = \dot{C}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad V_2 = \dot{C}_2 = \begin{bmatrix} -l_{c_2} c_2 s_1 \dot{\theta}_1 - l_{c_2} c_1 s_2 \dot{\theta}_2 \\ l_{c_2} c_1 c_2 \dot{\theta}_1 - l_{c_2} s_1 s_2 \dot{\theta}_2 \\ l_{c_2} c_2 \dot{\theta}_2 \end{bmatrix}$$

$$, \quad V_3 = \dot{C}_3 = \begin{bmatrix} -\dot{\theta}_1 (l_{c_3} c_{23} s_1 + l_2 c_2 s_1) - \dot{\theta}_2 (l_{c_3} s_{23} c_1 + l_2 c_1 s_2) - \dot{\theta}_3 l_{c_3} s_{23} c_1 \\ \dot{\theta}_1 (l_{c_3} c_{23} c_1 + l_2 c_1 c_2) - \dot{\theta}_2 (l_{c_3} s_{23} s_1 + l_2 s_1 s_2) - \dot{\theta}_3 l_{c_3} s_{23} s_1 \\ \dot{\theta}_2 (l_{c_3} c_{23} + l_2 c_2) + \dot{\theta}_3 l_{c_3} c_{23} \end{bmatrix}$$

Kinetic Energy

$$K_i = \frac{1}{2} m_i V_i^T V_i + \frac{1}{2} I_{zzi} \omega_i^T \omega_i$$

$$K_1 = \frac{1}{2} I_{zz1} \dot{\theta}_1^2 \qquad (\because \omega_1 = [0 \ 0 \ \dot{\theta}_1]^T) \qquad \text{——(1)}$$

$$K_2 = \frac{1}{2} m_2 V_2^T V_2 + \frac{1}{2} I_{zzz} \omega_2^T \omega_2$$

$$= \frac{1}{2} m_2 l_{c_2}^2 (\dot{\theta}_1^2 - 2\dot{\theta}_1^2 s_2^2 + \dot{\theta}_2^2) + \frac{1}{2} I_{zz2} (\dot{\theta}_{12}^2) \qquad \text{——(2)} \qquad \Big| \ \omega_2 = [0 \ 0 \ \dot{\theta}_{12}]^T$$

$$K_3 = \frac{1}{2} m_3 V_3^T V_3 + \frac{1}{2} I_{322} \omega_3^T \omega_3 \qquad \text{——(3)}$$

$$= m_3 (\dot{\theta}_1^2 l_2^2 + 2\dot{\theta}_2^2 l_2^2 + \dot{\theta}_1^2 l_{c_3}^2 + 2\dot{\theta}_2^2 l_{c_3}^2 + 2\dot{\theta}_3^2 l_{c_3}^2 + \dot{\theta}_1^2 l_2^2 c(2\theta_2) + \dot{\theta}_1^2 l_{c_3}^2 2c_{2\theta_2+2\theta_3}$$
$$+ 4\dot{\theta}_2 \dot{\theta}_3 l_{c_3}^2 + 2\dot{\theta}_1^2 l_2 l_{c_3} c_{(2\theta_2+\theta_3)} + 2\dot{\theta}_1^2 l_2 l_{c_3} c_3 + 4\dot{\theta}_2^2 l_2 l_{c_3} c_3$$
$$+ 4\dot{\theta}_2 \dot{\theta}_3 l_2 l_{c_3} c_3)/4 + \frac{1}{2} I_{322} \dot{\theta}_{123}^2$$

Potential energy

$$P_i = -m_i g_i^T C_i \qquad g_i^T = [0 \ -g \ 0]^T$$

$$P_1 = 0$$

$$P_2 = m_2 g l_{c_2} c_2 s_1 \qquad \text{——(4)}$$

$$P_3 = m_3 g s_1 (l_{c_3} c_{23} + l_2 c_2)$$

Lagrangian

$$L = T - P = (T_1 + T_2 + T_3) - (P_1 + P_2 + P_3)$$

Equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i \qquad q \in [\dot{\theta}_1, \theta_2, \theta_3]$$

for $i = 1$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = \tau_1$$

$$\frac{d}{dt}\Big(I_{221}\dot{\theta}_1 + I_{222}\dot{\theta}_1 + I_{222}\dot{\theta}_2 + I_{223}\dot{\theta}_{123} + \frac{1}{2}m_3 l_2^2 \dot{\theta}_1 + \frac{1}{2}m_2 l_{c_2}^2 \dot{\theta}_1 + \frac{1}{2}m_3 l_{c_3}^2 \dot{\theta}_1$$
$$+ \frac{1}{2}m_3 l_2^2 C_{2\theta_2}\dot{\theta}_1 + \frac{1}{2}m_2 l_{c_2}^2 C_{2\theta_2}\dot{\theta}_1 + \frac{1}{2}m_3 l_{c_3}^2 C_{(2\theta_2+2\theta_3)}\dot{\theta}_1 + m_3 l_{c_3} l_2 C_{(2\theta_2+\theta_3)}\dot{\theta}_1$$
$$+ m_3 l_{c_3} l_2 C_3 \dot{\theta}_1\Big) - \Big(\cancel{m_3 l_2 g S_1 S_2} + \cancel{\frac{1}{2} m_2 l_{c_2}^2 \dot{\theta}_1^2 S_{(2\theta_2)}} - \cancel{\frac{1}{2} m_3 l_2^2 \dot{\theta}_1^2 S_{(2\theta_2)}}$$
$$+ \cancel{m_2 l_{c_2} g S_1} - \cancel{m_3 l} - g C_1 (l_2 m_3 C_2 + m_2 l_{c_2} C_2 + m_3 l_{c_3} C_{23})\Big) = \tau_1$$

$$\Rightarrow \Big(I_{221} + I_{222} + I_{223} + \frac{1}{2}m_3 l_2^2 + \frac{1}{2}m_2 l_{c_2}^2 + \frac{1}{2}m_3 l_{c_3}^2 + \frac{1}{2}m_3 l_{c_3}^2 C_{(2\theta_2+2\theta_3)} + \frac{1}{2}m_3 l_2^2 C_{2\theta_2}$$
$$+ \frac{1}{2}m_2 l_{c_2}^2 C_{(2\theta_2)} + m_3 l_{c_3} l_2 C_3 + l_2 l_{c_3} m_3 C_{(2\theta_2+\theta_3)}\Big)\ddot{\theta}_1 + (I_{222} + I_{223})\ddot{\theta}_2$$
$$+ I_{223}\ddot{\theta}_3 - \dot{\theta}_1\Big(\dot{\theta}_2 l_2^2 m_3 S_{(2\theta_2)} + \dot{\theta}_2 l_{c_2}^2 m_2 S_{(2\theta_2)} + \dot{\theta}_2 l_{c_3}^2 m_3 S_{(2\theta_2+2\theta_3)}$$
$$+ 2\dot{\theta}_2 l_2 l_{c_3} m_3 S_{(2\theta_2+\theta_3)} + \dot{\theta}_3 l_2 l_{c_3} m_3 \sin(2\theta_2+\theta_3) + l_2 l_{c_3}\dot{\theta}_3 m_3 S_3\Big)$$
$$- \text{-----} = \tau_1 \qquad \text{——— (5)}$$

for $i = 2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = \tau_2$$

$$\frac{d}{dt}\Big(I_{222}\dot{\theta}_{12} + I_{223}\dot{\theta}_{123} + \dot{\theta}_2 l_2^2 m_3 + \dot{\theta}_2 l_{c_2}^2 m_2 + \dot{\theta}_2 l_{c_3}^2 m_3 + \dot{\theta}_3 l_{c_3}^2 m_3 + 2\dot{\theta}_2 l_2 l_{c_3} m_3 C_3$$
$$+ \dot{\theta}_3 l_2 l_{c_3} m_3 C_3\Big) - \frac{\partial L}{\partial \theta_2} = \tau_2$$

$$\Rightarrow (I_{222} + I_{223})\ddot{\theta}_1 + (m_3 l_2^2 + 2m_3 C_3 l_2 l_{c_3} + m_2 l_{c_2}^2 + m_3 l_{c_3}^2 + I_{222} + I_{223})\ddot{\theta}_2$$
$$+ (m_3 l_{c_3}^2 + m_3 l_2 C_3 l_{c_3} + I_{223})\ddot{\theta}_3 - \dot{\theta}_3 l_2 l_{c_3} m_3 S_3 (2\dot{\theta}_2 + \dot{\theta}_3) - g l_2 m_3 S_1 S_2$$
$$- \frac{1}{2}\dot{\theta}_1^2 l_{c_2}^2 m_2 S_{(2\theta_2)} - \frac{1}{2}\dot{\theta}_1^2 l_2^2 m_3 S_{(2\theta_2)} + g l_{c_2} m_2 S_1 S_2 - \frac{1}{2}\dot{\theta}_1^2 l_{c_3}^2 m_3 C_{2\theta_2} \cdot S_{2\theta_3}$$
$$- \frac{1}{2}\dot{\theta}_1^2 l_{c_3}^2 m_3 C_{(2\theta_3)} S_{(2\theta_2)} + g l_{c_3} m_3 C_2 S_1 S_3 + g l_{c_3} m_3 C_3 S_1 S_2 - \dot{\theta}_1^2 l_2 l_{c_3} m_3 C_{(2\theta_2)} \cdot S_{(3)}$$
$$- \dot{\theta}_1^2 l_2 l_{c_3} m_3 S_{(2\theta_2)} + C_3 = \tau_2 \qquad \text{——— (6)}$$

for $i = 3$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) - \frac{\partial L}{\partial \theta_3} = \tau_3$$

$$\frac{d}{dt}\left(I_{223}\dot{\theta}_{123} + \dot{\theta}_2 l_{c_3}^2 m_3 + \dot{\theta}_3 l_{c_3}^2 m_3 + \dot{\theta}_2 l_2 l_{c_3} m_3 c_3\right) - \frac{\partial L}{\partial \theta_3} = \tau_3$$

$$\Rightarrow I_{223}\ddot{\theta}_1 + (m_3 l_{c_3}^2 + m_3 l_2 c_3 l_{c_3} + I_{223})\ddot{\theta}_2 + (m_3 l_{c_3}^2 + I_{223})\ddot{\theta}_3 - \dot{\theta}_2\dot{\theta}_3 l_2 l_{c_3} m_3 s_3$$

$$- \left(-l_{c_3} m_3 (g c_{123} - g c_{(2-1+3)} + \dot{\theta}_1^2 l_{c_3} s_{(2\theta_2+2\theta_3)} + \dot{\theta}_1^2 l_2 s_3 + 2\dot{\theta}_2^2 l_2 s_3\right.$$

$$\left. + \dot{\theta}_1^2 l_2 s_{(2\theta_2+\theta_3)} + 2\dot{\theta}_2\dot{\theta}_3 l_2 s_3\right)/2 = \tau_3 \qquad \text{——— (7)}$$

Now,

$$I\ddot{\theta} + h = \tau$$

where $I = \begin{bmatrix} i_{11} & i_{12} & i_{13} \\ i_{21} & i_{22} & i_{23} \\ i_{31} & i_{32} & i_{33} \end{bmatrix}$, $\ddot{\theta} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix}$, $h = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$, $\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$

Here

$$i_{11} = I_{221} + I_{222} + I_{23} + (l_2^2 m_3)/2 + \frac{1}{2} m_2 l_{c_2}^2 + \frac{1}{2} m_3 l_{c_3}^2 + \frac{1}{2} m_3 l_{c_3}^2 c_{(2\theta_2+2\theta_3)}$$

$$+ \frac{1}{2} m_3 l_2^2 c_{(2\theta_2)} + \frac{1}{2} m_2 l_{c_2}^2 c_{(2\theta_2)} + m_3 l_2 l_{c_3} c_3 + l_2 l_{c_3} m_3 c_{(2\theta_2+\theta_3)}$$

$$i_{12} = i_{21} = I_{222} + I_{223}$$

$$i_{13} = i_{31} = I_{223}$$

$$i_{22} = m_3 l_2^2 + 2 m_3 c_3 l_2 l_{c_3} + m_2 l_{c_2}^2 + m_3 l_{c_3}^2 + I_{222} + I_{223}$$

$$i_{23} = m_3 l_{c_3}^2 + l_2 m_3 c_3 l_{c_3} + I_{223} = i_{32}$$

$$i_{33} = m_3 l_c^2 + I_{223}$$