Assignment 1: Kalman Decomposition Matlab Code

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1 Introduction

The Kalman decomposition provides a mathematical means to convert the representation of any Linear Time-Invariant (LTI) system to a form in whih the system can be decomposed into

- Controllable and unobservable states ($C\tilde{O}$)
- Controllable but observable states (CO)
- Uncontrollable and unobservable states. ($\tilde{C}\tilde{O}$)
- Uncontrollable but observable states ($\tilde{C}O$)

Let the LTI sytem is given as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D} \end{aligned} \tag{1}$$

The Kalman decomposition brings the system to the form

$$\begin{aligned} \dot{\bar{\boldsymbol{x}}} &= \bar{\boldsymbol{A}}\bar{\boldsymbol{x}} + \bar{\boldsymbol{B}}\boldsymbol{u} \\ \boldsymbol{y} &= \bar{\boldsymbol{C}}\bar{\boldsymbol{x}} + \bar{\boldsymbol{D}} \end{aligned} \tag{2}$$

This form can be achieved by transformation $\boldsymbol{T}$ to a tuple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ to $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ as given below:

$$\begin{aligned} \bar{\boldsymbol{A}} &= \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} \\ \bar{\boldsymbol{B}} &= \boldsymbol{T}\boldsymbol{B} \\ \bar{\boldsymbol{C}} &= \boldsymbol{C}\boldsymbol{T}^{-}1 \end{aligned} \tag{3}$$

The transformation matrix is $n \times n$ ivertible matrix and is given as

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{T}_{C\tilde{O}} & \boldsymbol{T}_{CO} & \boldsymbol{T}_{\tilde{C}\tilde{O}} & \boldsymbol{T}_{\tilde{C}O} \end{bmatrix} \tag{4}$$

2 Methodology

There methodology used for calculating the kalman decomposition of LTI system is explained in the subsequent subsections.

2.1 Subspaces calcuation

The first step in finding the Kalman decomposition is to determine the controllable($\boldsymbol{S}_C$) /uncontrollable($\boldsymbol{S}_{\tilde{C}}$) and observable($\boldsymbol{S}_O$) /unobservable($\boldsymbol{S}_{\tilde{O}}$) subspaces. These subspaces can be obtained from the range ($\mathcal{R}$) and null spaces ($\mathcal{N}$) of controllability matrix ($\boldsymbol{\mathcal{C}}$) and observabilty matrices ($\boldsymbol{\mathcal{O}}$) as given below:

$$\boldsymbol{S}_C = \mathcal{R}(\boldsymbol{\mathcal{C}}), \boldsymbol{S}_{\tilde{C}} = \mathcal{N}(\boldsymbol{\mathcal{C}}^T)$$
$$\boldsymbol{S}_O = \mathcal{R}(\boldsymbol{\mathcal{O}}), \boldsymbol{S}_{\tilde{O}} = \mathcal{N}(\boldsymbol{\mathcal{O}}^T) \tag{5}$$

There are various methods to obtain range and null space of the matrix such as Singular value decomposition (SVD), QR Factorization etc. I have used QR factorization method as this method gives results similar to MATLAB in-built functions for controllabilty and observability decomposition.

QR factorization: For matrix $\boldsymbol{A} \in \mathcal{R}^{m\times n}$, the QR factorization can be written as

$$\boldsymbol{A} = \boldsymbol{Q} * \boldsymbol{R} \tag{6}$$

where $\boldsymbol{Q} = \begin{bmatrix}\boldsymbol{q}_1 & ... & \boldsymbol{q}_r & \boldsymbol{q}_{r+1} & ... & \boldsymbol{q}_n\end{bmatrix}$. The columns $\boldsymbol{q}_1, ..., \boldsymbol{q}_r$ is an orthonormal basis for $range(\boldsymbol{A})$ where $r = rank(\boldsymbol{A})$. The columns $\boldsymbol{q}_{r+1}, ..., \boldsymbol{q}_n$ is an orthonormal basis for $null(\boldsymbol{A}^T)$. Using QR factorization, the controllable/uncontrollable and observable/unobservable subspaces are obtained as follows

$$qr(\boldsymbol{\mathcal{C}}) = \begin{bmatrix}\boldsymbol{S}_C & \boldsymbol{S}_{\tilde{C}}\end{bmatrix} * \boldsymbol{R}$$
$$qr(\boldsymbol{\mathcal{O}}^T) = \begin{bmatrix}\boldsymbol{S}_O & \boldsymbol{S}_{\tilde{O}}\end{bmatrix} * \boldsymbol{R} \tag{7}$$

2.2 Transformation matrix

In this section, the method to obtain the transformation matrix is explained. Since the column $\boldsymbol{T}_{C\tilde{O}}$ is given as

$$\boldsymbol{T}_{C\tilde{O}} = \mathcal{R}(\boldsymbol{\mathcal{C}}) \cap \mathcal{N}(\boldsymbol{\mathcal{O}}^T)$$

Using (5), it can be written as

$$\boldsymbol{T}_{C\tilde{O}} = \boldsymbol{S}_C \cap \boldsymbol{S}_{\tilde{O}}$$

Similarly, other columns of $\boldsymbol{T}$ can be written as

$$\boldsymbol{T}_{CO} = \boldsymbol{S}_C \cap \boldsymbol{S}_O$$
$$\boldsymbol{T}_{\tilde{C}\tilde{O}} = \boldsymbol{S}_{\tilde{C}} \cap \boldsymbol{S}_{\tilde{O}}$$
$$\boldsymbol{T}_{\tilde{C}O} = \boldsymbol{S}_{\tilde{C}} \cap \boldsymbol{S}_O \tag{8}$$

3 MATLAB Code

3.1 System requirements

- MATLAB R2019b with System Control Toolbox

3.2 How to run the code

- `[Abar,Bbar,Cbar,T] = getKalmanDec(A,B,C)`

3.3 Algorithim

1. The input to the function is tuplet A,B,C .
2. First the state of system is checked by finding the rank of A matix, controllabilty and observabilty matrix as n, r_c and r_o, respectively.
3. If $r_c < n$ and $r_o = n$, then controllable decomposition is done.
   If $r_c = n$ and $r_o < n$, then observable decomposition is done.
   If $r_c < n$ and $r_o < n$, then kalman decomposition is done.
4. Returns a transformation matrix T and transformed matrices Abar, Bbar, Cbar.

3.4 Code snippets

- To find QR decomposition of matrix

```
[Q,R] = qr(A)
```

- To find intersaction of two subspaces

```
function [inters,dim] = getIntersect(V,W)
dim_v = min(size(V));
dim_w = min(size(W));
inters = V*[eye(dim_v)  zeros(dim_v,dim_w)]*null([V W]);
dim= size(inters,2);
end
```

3.5 Example

```
clc; clear all;
A=[1 0 0 0;0 -2 1 0;1 0 -1 0;0 0 0 3];
B=[1;1;0;0];
C=[0 0 1 1];
[Abar,Bbar,Cbar,T] = getKalmanDec(A,B,C)
```

```
Result:
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Kalman decomposition of LTI System
Contributor: Deepak Raina (2019MEZ8497) PhD@IITD
---------------------------------------------------
The system is both uncontrollable and unobservable
Obtaining Kalman decomposition
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Abar =

   -2.0000   -0.9317   -0.3632         0
    0.0000   -1.0746    0.1914         0
   -0.0000   -0.8086    1.0746         0
         0         0         0    3.0000

Bbar =

    1.4142
    0.5136
   -1.3177
         0

Cbar =

    0.0000   -0.6588   -0.2568    0.7071

T =

    0.0000    0.2568   -0.6588         0
    0.7071    0.0000    0.0000         0
    0.0000   -0.6588   -0.2568         0
         0         0         0    0.7071
```

4 References

[1] http://ee263.stanford.edu/lectures/qr.pdf

[2] NPTEL - Linear Systems Theory Course [https://nptel.ac.in/courses/108106150/]