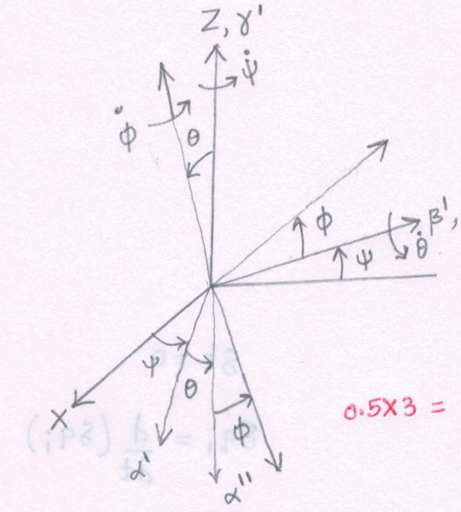
MCL731 2019-2020 Minor Test-II solution

1.

$$\begin{bmatrix} \omega_{\alpha'} \\ \omega_{\beta'} \\ \omega_{\gamma'} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix}$$



$$\begin{bmatrix} \omega_{\alpha''} \\ \omega_{\beta''} \\ \omega_{\gamma''} \end{bmatrix} = \underbrace{\begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}}_{Q_\theta^T} \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix} + \begin{bmatrix} 0 \\ \dot\theta \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\dot\psi s\theta \\ \dot\theta \\ \dot\psi c\theta \end{bmatrix}$$ 1 Mark

0.5X3 = 1.5 Mark

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\dot\psi s\theta \\ \dot\theta \\ \dot\psi c\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\phi \end{bmatrix}$$

$$= \begin{bmatrix} -\dot\psi s\theta c\phi + \dot\theta s\phi \\ \dot\psi s\theta s\phi + \dot\theta c\phi \\ \dot\psi c\theta + \dot\phi \end{bmatrix} = \begin{bmatrix} -s\theta c\phi & s\phi & 0 \\ s\theta s\phi & c\phi & 0 \\ c\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot\psi \\ \dot\theta \\ \dot\phi \end{bmatrix}$$

1 Mark 1 Mark

2. The actual path in configuration space followed by a Holonomic dynamical system during the fixed time interval t_1 to t_2 is such that the integral

$$I = \int_{t_1}^{t_2} L(q, \dot q, t)\, dt$$

is stationary with respect to path variations which vanish at the end points. 1.5 Mark

Mathematical Form $\delta I = 0$, i.e., $\delta \int_{t_1}^{t_2} L\, dt = 0$ 0.5 Mark

3. From Hamilton's principle, we have $\delta I = \delta\int_{t_1}^{t_2} L\,dt = 0$

$$\Rightarrow \delta \int_{t_1}^{t_2} L(q_1, q_2, \dots, q_n ; \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t)\, dt = 0$$

For holonomic system we have

$$\int_{t_1}^{t_2} \delta L(q_1, q_2, \dots, q_n ; \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t)\, dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left\{ \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L}{\partial t} \cancelto{0}{\delta t} \right\} dt = 0$$

1 Mark

$\delta t = 0$

$\delta \dot{q}_i = \frac{d}{dt}(\delta q_i)$

$$\Rightarrow \int_{t_1}^{t_2} \left\{ \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt}(\delta q_i) \right\} dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta q_i \, dt + \sum_{i=1}^{n} \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt}(\delta q_i)\, dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta q_i \, dt + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Bigg|_{t=t_1}^{t=t_2} - \sum_{i=1}^{n} \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i \, dt = 0$$

1 Mark

$$\Rightarrow \int_{t_1}^{t_2} \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta q_i \, dt - \int_{t_1}^{t_2} \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i \, dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \sum_{i=1}^{n} \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \right\} \delta q_i \, dt = 0$$

1 Mark

From Fundamental lemma, the integrand should vanish. Further, since δq_i are all independent, we get 1 Mark (Reasoning)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0.$$

4. Constraint : $r\,d\theta - ds = 0$ —① 0.5 Mark

Kinetic Energy $T = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}mr^2\dot{\theta}^2$

Potential Energy $V = mg(L-s)\sin\phi + r\cos\phi$ } 0.5 Mark

where L is the length of the inclined plane

Lagrangian $L = T - V$

$$= \frac{1}{2}m\dot{s}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mg(L-s)\sin\phi$$ ~~0.5M~~

The two Lagrange's equations of sekond-kind are

$$m\ddot{s} - mg\sin\phi = -\mu \quad —②$$

$$mr^2\ddot{\theta} = \mu r \quad —③$$

2 Mark

Equations ①,②,③ constitote three equations in unknowns θ, s, λ

From ① $\quad r\dot{\theta} - \dot{s} = 0$

Differentiating, we get $\quad r\ddot{\theta} - \ddot{s} = 0$ —— ④

From ③ and ④ weget $\quad m\ddot{s} = \mu$ — ⑤

From ⑤ and ②, ③ $\quad \ddot{s} = \frac{g\sin\phi}{2}$

$$\mu = \frac{mg\sin\phi}{2}$$ 1.0 Mark

$$\ddot{\theta} = \frac{g\sin\phi}{2r}$$

Friction Force and Moment enforce the constraint $\dot{s} - r\dot{\theta} = 0$

Constraint Forces

$$C_\theta = \frac{Mg\sin\phi}{2}r$$ 0.5 Mark

$$C_s = -\frac{Mg\sin\phi}{2}$$ 1 Mark

$C_\theta = \mu r$

$C_s = -\mu$ 0.5 Mark

C_θ is frictional torque about C.M.

C_s is frictional force at contact

5. $$H = \frac{p^2}{2\alpha} - bqpe^{-\alpha t} + \frac{ba}{2}q^2e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{kq^2}{2}$$

$\dot{q} = \frac{\partial H}{\partial p} \Rightarrow \dot{q} = \frac{p}{\alpha} - bqe^{-\alpha t}$ —① 0.5 Mark

From ① $\quad p = \alpha\{\dot{q} + bqe^{-\alpha t}\}$ —② 0.5 Mark

$H = p\dot{q} - L \Rightarrow L = p\dot{q} - H$ 0.5 Mark

$$\Rightarrow L = p\dot{q} - \left\{\frac{p^2}{2\alpha} - bqpe^{-\alpha t} + \frac{ba}{2}q^2e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{kq^2}{2}\right\}$$ 0.5 Mark

Now substitute $p = \alpha\{\dot{q} + bqe^{-\alpha t}\}$

$$\Rightarrow L = \alpha\dot{q}\{\dot{q} + bqe^{-\alpha t}\} - \left\{\frac{1}{2\alpha}\alpha^2(\dot{q} + bqe^{-\alpha t})^2 - bqe^{-\alpha t}\alpha(\dot{q} + bqe^{-\alpha t}) + \frac{ba}{2}q^2e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{kq^2}{2}\right\}$$ 0.5 Mark