Assignment 2: Controller-Observer Design

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1 Introduction

When the full-state observer is used in conjunction with the feedback control gain matrix, the result is an n^{th} order dynamic controller. The system's input and output signals are measured by the observer, and an estimate of the complete state is generated. This estimate is used by the control gain matrix as if it were the true state, and the control law is $u(t) = F\hat{x}$. With this controller, the dimension of the complete system is $2n$.

Let the LTI system is given as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D} \tag{1}$$

where $\boldsymbol{x} \in \mathcal{R}^n, \boldsymbol{u} \in \mathcal{R}^m, \boldsymbol{y} \in \mathcal{R}^p, \boldsymbol{A} \in \mathcal{R}^{n\times n}, \boldsymbol{B} \in \mathcal{R}^{n\times m}, \boldsymbol{C} \in \mathcal{R}^{p\times n}$

1.1 State-feedback controller

An important element of the control design is the performance specification. The simplest performance specification is that of stability i.e. in the absence of any disturbances, we would like the equilibrium point of the system to be asymptotically stable. This can be achieved by designing a state-feedback controller such that the closed-loop eigen values are stable. Generally, a linear state feedback with gain matrix $\boldsymbol{F}$ is used. The input $\boldsymbol{u}$ is then given as

$$\boldsymbol{u} = \boldsymbol{F}\boldsymbol{x}$$

Substituting it in eq. (1)

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F})\boldsymbol{x}$$

Thus the state feedback matrix is designed in such a way that matrix $(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F})$ is stable.

1.2 Linear functional observer

Linear functional observers take advantage of recurring theme in state feedback control. That is, frequently in feedback control applications, only a linear combinations of the state variables is required i.e. $\boldsymbol{Fx}(t)$ is required, rather than a complete knowledge of state vector $\boldsymbol{x}(t)$. Linear functional observers are constructed to estimate a linear combination of some of the unmeasurable states. These are further reduced order, simpler in structure and stable.

Let $g \in \mathcal{R}^r$ be a vector that is required to be estimated such that $g = Fx$ and $F \in \mathcal{R}^{r\times n}$ is a known matrix. To reconstruct the state function g, the following observer structure of order q ($\geqslant m(n-p)/p$) is proposed

$$\begin{aligned}\dot{\hat{\boldsymbol{\eta}}} &= \boldsymbol{N}\hat{\boldsymbol{\eta}} + \boldsymbol{J}\boldsymbol{y} + \boldsymbol{H}\boldsymbol{u}\\ \hat{\boldsymbol{g}} &= \boldsymbol{K}\boldsymbol{y} + \boldsymbol{D}\hat{\boldsymbol{\eta}}\end{aligned} \tag{2}$$

where $\boldsymbol{\eta} = \boldsymbol{Tx} \in \mathcal{R}^q$ and $\hat{\boldsymbol{g}} \in \mathcal{R}^r$ is the estimate of g. Observer matrices $\boldsymbol{D} \in \mathcal{R}^{r\times q}, \boldsymbol{K} \in \mathcal{R}^{r\times p}, \boldsymbol{N} \in \mathcal{R}^{q\times q}, \boldsymbol{J} \in \mathcal{R}^{q\times p}, \boldsymbol{H} \in \mathcal{R}^{q\times m}$ are to be determined such that $\hat{\boldsymbol{g}}$ converges asymptotically to $\boldsymbol{g}$. If $\boldsymbol{g}$ is a control signal i.e. $\boldsymbol{g} = \boldsymbol{Fx}$ to be estimated for feedback control, then role of functional observer is to reconstruct the control law to be directly fed back in to the system. Then the following conditions [3] hold

$$\boldsymbol{NT} + \boldsymbol{JC} - \boldsymbol{TA} = 0 \tag{3}$$

$$\boldsymbol{F} - \boldsymbol{DT} - \boldsymbol{KC} = 0 \tag{4}$$

$$\boldsymbol{H} - \boldsymbol{LB} = 0 \tag{5}$$

2 Methodology

The methodology used for calculating the design parameters ($\boldsymbol{F}, \boldsymbol{N}, \boldsymbol{J}, \boldsymbol{H}, \boldsymbol{K}, \boldsymbol{D}$) of observer-based controller for a LTI system is explained in the subsequent subsections.

2.1 Controller design

In the controller design, the state-feedback control gains need to be calculated. First, it is verified that system is controllable. If it is not controllable, the system must be stabilizable. The PBH test [1] has been used for checking stabalizability of the system. Then the state feedback gain matrix ($\boldsymbol{F}$) needs to be calculated. This can be done by two methods (1) Linear Quadratic Regulation (LQR) method which will optimize the control effort ad error, and (2) Ackermann's method which will place the closed-loop poles at desired locations.

2.2 Observer design

To design a functional observer for estimating the linear combination of states, first it needs to be checked whether a state feedback gain spans the observable space of the system. This can be ensured if the below given condition holds:

$$rank\left(\begin{bmatrix}\boldsymbol{F}\\ \boldsymbol{\mathcal{O}}\end{bmatrix}\right) = rank(\boldsymbol{\mathcal{O}}) \tag{6}$$

where $\boldsymbol{\mathcal{O}}$ is the observability matrix. For obtaining the observer parameters, the eq. (3) and (4) are simplified by post multiplying them with a invertible matrix $P \in \mathcal{R}^{n\times n}$ which is given as

$$\boldsymbol{P} = \begin{bmatrix}\boldsymbol{C}^{\dagger} & \boldsymbol{C}^{\perp}\end{bmatrix}$$

Here, $\boldsymbol{C}^{\dagger} \in \mathcal{R}^{n\times p}$ is the Moore-Penrose inverse of $\boldsymbol{C}$ and $\boldsymbol{C}^{\perp} \in \mathcal{R}^{n\times(n-p)}$ denotes an orthogonal basis for null-space of $\boldsymbol{C}$. The modified eq. (3) after post-multiplication can be written as

$$\boldsymbol{NT} + \boldsymbol{JC} - \boldsymbol{TP}^{-1}\boldsymbol{AP} = 0$$

$$\boldsymbol{N}\begin{bmatrix}\boldsymbol{T1} & \boldsymbol{T2}\end{bmatrix} + \boldsymbol{J}\begin{bmatrix}\boldsymbol{I}_p & \boldsymbol{O}\end{bmatrix} - \boldsymbol{T}\begin{bmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}\end{bmatrix} = 0 \tag{7}$$

Similarly, the modified form eq. (4) can be written as,

$$\boldsymbol{FP} - \boldsymbol{DTP} - \boldsymbol{KCP} = 0$$

$$\begin{bmatrix}\boldsymbol{F1} & \boldsymbol{F2}\end{bmatrix} - \boldsymbol{D}\begin{bmatrix}\boldsymbol{T1} & \boldsymbol{T2}\end{bmatrix} - \boldsymbol{K}\begin{bmatrix}\boldsymbol{I}_p & \boldsymbol{O}\end{bmatrix} = 0 \tag{8}$$

where $\boldsymbol{F}_1 \in \mathcal{R}^{r\times p}, \boldsymbol{F}_2 \in \mathcal{R}^{r\times(n-p)}, \boldsymbol{T}_1 \in \mathcal{R}^{q\times p}, \boldsymbol{T}_2 \in \mathcal{R}^{q\times(n-p)}$, $\boldsymbol{A}_{11} \in \mathcal{R}^{p\times p}$, $\boldsymbol{A}_{12} \in \mathcal{R}^{p\times(n-p)}$, $\boldsymbol{A}_{21} \in \mathcal{R}^{(n-p)\times p}$, $\boldsymbol{A}_{22} \in \mathcal{R}^{(n-p)\times(n-p)}$

To solve for these matrices, we have to arbitrarily choose matrix $\boldsymbol{N}$ and $\boldsymbol{D}$. The matrix $\boldsymbol{N}$ is $q \times q$ convergent matrix with all its eigen values $(\lambda_1, \lambda_2, ..., \lambda_q)$ in the stable region $(Re(\lambda) < 0)$. That is,

$$\boldsymbol{N} = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_q\end{bmatrix}$$

Here stable eigen values for matrix $\boldsymbol{N}$ are choosen as the minimum of the real part of eigen values of matrix $[\boldsymbol{A} - \boldsymbol{BF}]$. That is,

$$\lambda_1 = \lambda_2 = ... = \lambda_q = min(Re(eig(\boldsymbol{A} - \boldsymbol{BF})))$$

The matrix $\boldsymbol{D}$ is choosen as first $r \times q$ elements of the $\boldsymbol{F}$. The eqs. (5), (7) and (8) are then solved using the procedure given in [2] and [3], and then observer design parameters $(\boldsymbol{N}, \boldsymbol{J}, \boldsymbol{H}, \boldsymbol{K}, \boldsymbol{D})$ have been obtained.

3 MATLAB Code

3.1 System requirements

- MATLAB R2019b with System Control Toolbox

3.2 How to run the code

- Specify the matrix $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ in *getCtrlObsDesign.m* file. The sytem can also be specified in *LTI_SYS.m*.

- Run the file *getCtrlObsDesign.m*
- The other parameters like user-defined poles, controller tunning can be done through *inputs.m* file

  Note: This code has been tested on 15 LTI systems given in *LTI_SYS.m* file

3.3 Algorithm

1. The input to the function is tuplet A,B,C .
2. First the state of system is checked by finding the rank of A matrix, controllability and observability matrix as n, r_c and r_o, respectively.
3. If $r_c = n$ and $r_o = n$, then state-feedback (SFB) gain is obtained. If SFB gain is observable, observer parameters are calculated as described in section 2.1 and 2.2 respectively.
4. If $r_c < n$ and $r_o = n$, then stabilizability of system is checked. If stabilizable, state-feedback gain is computed. If SFB gain is observable, observer parameters are obtained.
5. If $r_c = n$ and $r_o < n$, state-feedback gain is computed. The detectability of system is checked. If SFB gain is observable, observer parameters are estimated.
6. If $r_c < n$ and $r_o < n$, both stabilizability and detectability is checked. If system is stabilizable, state-feedback gain is obtained. If detectable and SFB gain is observable, observer design parameters are returned.

3.4 Code snippets

- Function to find state-feedback gain

```
function K = getFeedbackGain(A,B,C,mthd,des_p,P)
if mthd==1
    K=place(A,B,des_p);
elseif mthd==2
    Q = P*C'*C; R = eye(size(B,2));
    [K,~,~] = lqr(A,B,Q,R);
end
end
```

- Function to check if system is stabilizable or not

```
function Stablizable = checkStabilizable(A,B)
Stablizable = true;
n=size(A,1)
eval=eig(A);
```

```
pos_eval=eval(real(eval)>=0);
for i=1:size(pos_eval)
    pbh_rank = rank([pos_eval(i)*eye(n)-A B]);
    if pbh_rank<n
        Stablizable = false;
        break;
    end
end
end
```

- Function to check if system is detectable or not

```
function Detectable = checkDetectable(A,C)
Detectable = true;
n=size(A,1);
eval=eig(A);
pos_eval=eval(real(eval)>=0);
for i=1:size(pos_eval)
    pbh_rank = rank([pos_eval(i)*eye(n)-A' C']);
    if pbh_rank<n
        Detectable = false;
        break;
    end
end
end
```

- Function to check if SFB is observable

```
function SFBObservable = checkSFBObservable(F,A,C)
OM = obsv(A,C);
rc=rank(OM);
rfc=rank([F;OM]);
if rfc==rc
    SFBObservable=true;
else
    SFBObservable=false;
end
end
```

3.5 Example

(I) **Controllable/Observable System (taken from [4])**

```
A =
```

```
   -0.5239    1.4189   -0.3685    0.7399    0.9059   -0.0207
   -1.1815   -0.6443   -0.8353    0.2801   -0.2886    0.5353
    1.1412    0.0331   -1.4217    0.3034   -0.2047   -0.5148
   -0.7425    0.3843   -0.2520   -0.9369    0.0413    0.3482
   -0.3108    0.2047   -0.8974    0.2738   -1.0119    0.2390
   -0.4142   -0.5814    0.0021   -0.3042   -0.3582   -0.9751


B =

         0
         0
   -0.9865
   -0.0716
         0
   -0.6943


C =

    0.0371    0.0631   -0.0054    0.0333    0.0369   -0.9961

---------------------------------------------------
Controller-Observer design of LTI system
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---------------------------------------------------
---------------------------------------------------
The given system is controllable and observable
---------------------------------------------------
Designing a Full State Feedback (SFB) Controller
---------------------------------------------------
SFB Gain Matrix

F =

   -0.0310    0.1176   -0.0540    0.0611    0.0461   -0.2802

---------------------------------------------------
The SFB gain is observable.
---------------------------------------------------
Designing a Linear Functional Observer
---------------------------------------------------
Observer design parameters (N,D,J,H,K)
```

```
N =

   -2.1917         0         0         0         0
         0   -2.1917         0         0         0
         0         0   -2.1917         0         0
         0         0         0   -2.1917         0
         0         0         0         0   -2.1917


D =

    0.2894    0.1018   -0.0527    0.0528    0.0369


J =

   -0.0254
   -0.0090
    0.0046
   -0.0046
   -0.0032


K =

    0.2964


H =

   -0.0178
   -0.0062
    0.0032
   -0.0032
   -0.0023
```

(II) **Uncontrollable-but-stabilizable/Unobservable-but-detectable System**

```
A =

         0         0         0         0
         0   -2.0000         0         0
    2.5000    2.5000   -1.0000         0
    2.5000    2.5000    3.0000         0
```

```
B =

     1
     1
     0
     0


C =

     0     0     1     1

---------------------------------------------------
The system is both uncontrollable and unobservable
---------------------------------------------------
The given LTI system is stabilizable
---------------------------------------------------
Designing a Full State Feedback (SFB) Controller
---------------------------------------------------
SFB Gain Matrix

F =

    2.5869    1.8852    1.7541    1.0000

---------------------------------------------------
The given LTI system is detectable
---------------------------------------------------
The SFB gain is observable.
---------------------------------------------------
Designing a Linear Functional Observer
---------------------------------------------------
Controller-Observer design parameters (N,D,J,K,H)

N =

   -6.6034         0         0
         0   -6.6034         0
         0         0   -6.6034


D =

    1.3770    1.8852   -1.4522
```

```
J =

   -3.3718
   -4.6161
    3.5557


K =

    3.6880


H =

    0.0063
    0.0087
   -0.0067
```

(III) **Uncontrollable-but-stabilizable/Observable System**

```
A =

     1     1     1
     1    -2     1
     0     0    -1


B =

     1
     0
     0


C =

     1     0     0

---------------------------------------------------
The system is uncontrollable but observable
---------------------------------------------------
```

```
---------------------------------------------------
The given LTI system is stablizable
Designing a Full State Feedback (SFB) Controller
---------------------------------------------------
SFB Gain Matrix

F =

    2.9001    0.8053         0

---------------------------------------------------
The SFB gain is observable.
---------------------------------------------------
Designing a Linear Functional Observer
---------------------------------------------------
Controller-Observer design parameters (N,D,J,K,H)

N =

   -2.3942         0
         0   -2.3942


D =

    2.9001    0.8053


J =

   -0.3206
   -0.0890


K =

    3.4282


H =

   -0.1690
   -0.0469
```

4 References

[1] Hespanha, Joao P. Linear systems theory. Princeton university press, 2018.

[2] Trinh, Hieu, and Tyrone Fernando. Functional observers for dynamical systems. Vol. 420. Springer Science and Business Media, 2011.

[3] `https://shodhganga.inflibnet.ac.in/bitstream/10603/16049/15/15_chapter%209.pdf`

[4] Janardhanan, S., and Inamdar, S. (2012). Computationally Efficient Functional Observer for LTI System based on A Multirate Output sampling Algorithm. In Advanced Materials Research (Vol. 403, pp. 3875-3883). Trans Tech Publications.