2019-2020 MCL 731: Minor Test-I Solution

1\. (a) $Q_\theta = \sum_{i=1}^{N} \bar{F}_i \cdot \frac{\partial \bar{r}_i}{\partial \theta}$ 0.5 Mark

$\bar{r}_1 = x\hat{i} + 0\hat{j} = l\cos\theta\,\hat{i}$

$\bar{r}_2 = 0\hat{i} + y\hat{j} = l\sin\theta\,\hat{j}$

$\bar{F}_1 = \bar{0}$

$\bar{F}_2 = -m_2 g\,\hat{j}$

$$Q_\theta = \bar{0}\cdot\frac{\partial \bar{r}_1}{\partial \theta} + -m_2 g\,\hat{j}\cdot\left(\frac{\partial}{\partial \theta} l\sin\theta\,\hat{j}\right)$$

$$\Rightarrow \boxed{Q_\theta = -m_2 g l\cos\theta}$$ 1.5 Mark

(b) Lagrangian function

$$L = T - V$$

$x = l\cos\theta,\ y = l\sin\theta$

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2$$

$$\Rightarrow T = \frac{1}{2}m_1 l^2\sin^2\theta\,\dot{\theta}^2 + \frac{1}{2}m_2 l^2\cos^2\theta\,\dot{\theta}^2$$ 0.5 Mark

$$V = m_2 g y = m_2 g l\sin\theta$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial T}{\partial \dot{\theta}} = m_1 l^2\sin^2\theta\,\dot{\theta} + m_2 l^2\cos^2\theta\,\dot{\theta}$$

0.5 Mark

$$\Rightarrow P_\theta = m_1 l^2\sin^2\theta\,\dot{\theta} + m_2 l^2\cos^2\theta\,\dot{\theta}$$ 1 Mark

2. Constraint $\phi \equiv x^2+y^2-r^2=0$ —— (1)

Lagrange's equation of motion of first-kind are

$$m_i \ddot{\bar{r}}_i = \bar{F}_i^{(a)} + \sum_{j=1}^{m} \lambda_j \frac{\partial \phi_j}{\partial \bar{r}_i} + \sum_{k=1}^{l} \mu_k \bar{a}_{ki} \quad \text{—— (2)}$$

0.5 Mark

So we get

$$m\ddot{x} = 0 + \lambda \frac{\partial \phi}{\partial x} \Rightarrow m\ddot{x} = 2\lambda x \quad \text{—— (3)}$$

1 Mark

$$m\ddot{y} = \cancel{0} + \lambda \frac{\partial \phi}{\partial y} - mg \qquad m\ddot{y} = -mg + 2\lambda y \quad \text{—— (4)}$$

From (3) and (4), eliminating λ we get

$$\frac{m\ddot{x}}{2x} = \frac{m\ddot{y}+mg}{2y}$$

$$\Rightarrow \boxed{y\ddot{x} - x\ddot{y} - gx = 0.} \quad \text{—— (5)}$$

1.5 Mark

(b) Equations (1), (3), (4) should be solved for $x(t)$, $y(t)$ and $\lambda(t)$

Differentiating (1) gives

$$2x\dot{x} + 2y\dot{y} = 0 \quad \text{—— (6)}$$

Further differentiation gives

$$2\dot{x}^2 + 2x\ddot{x} + 2\dot{y}^2 + 2y\ddot{y} = 0 \quad \text{—— (7)}$$

You cannot use $m\ddot{x} = 2\lambda x$ and say constraint force is $m\ddot{x}$ bcz above eqn says $2\lambda x$ force equals mass times acceleration. You need to find λ otherway.

0.5 Mark

Substituting expressions for $\ddot{x}$ and $\ddot{y}$ from (3) and (4) in (7) gives

0.5 Mark

$$2\dot{x}^2 + 2x\left(\frac{2\lambda x}{m}\right) + 2\dot{y}^2 + 2y\left(\frac{-mg+2\lambda y}{m}\right) = 0$$

$$\Rightarrow 2\dot{x}^2 + \frac{4\lambda x^2}{m} + 2\dot{y}^2 - 2gy + \frac{4\lambda y^2}{m} = 0$$

$$\Rightarrow \dot{x}^2 + \dot{y}^2 - gy + \frac{2\lambda}{m}(x^2+y^2) = 0$$

$$\Rightarrow \dot{x}^2 + \dot{y}^2 - gy + \frac{2\lambda}{m} r^2 = 0$$

$$\Rightarrow \boxed{\lambda = -\frac{m}{2r^2}\left(\dot{x}^2 + \dot{y}^2 - gy\right)} \quad \text{—— (8)}$$

1 Mark

Constraint Force

$R_x = 2\lambda x$ —— (9)

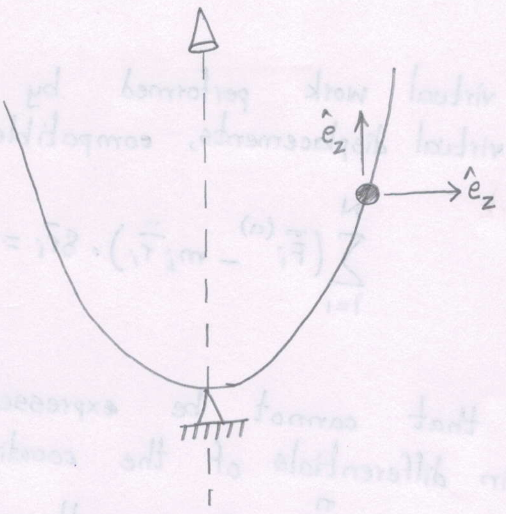
1 Mark

$R_y = 2\lambda y$ —— (10)

where λ is given by (8)

Once you know λ, using (3) and (4) we obtain two coupled differential equations in x and y which can solved in closed form or numerically for $x(t)$ and $y(t)$.

3. $\bar{\rho} = D\hat{e}_r + H\hat{e}_z$



$(\dot{\bar{\rho}})_r = \dot{D}\hat{e}_r + \dot{H}\hat{e}_z$

$(\ddot{\bar{\rho}})_r = \ddot{D}\hat{e}_r + \ddot{H}\hat{e}_z$

$\bar{\omega} = \omega\hat{e}_z$

$\bar{\alpha} = \dot{\bar{\omega}} = \bar{0}$

$\ddot{\bar{R}} = \bar{0}$

$$\bar{a} = \ddot{\bar{R}} + \dot{\bar{\omega}}\times\bar{\rho} + \bar{\omega}\times(\bar{\omega}\times\bar{\rho}) + (\ddot{\bar{\rho}})_r + 2\bar{\omega}\times(\dot{\bar{\rho}})_r$$

$$\bar{a} = \bar{0} + \bar{0} + \omega\hat{e}_z\times\{\omega\hat{e}_z\times(D\hat{e}_r + H\hat{e}_z)\} + (\ddot{D}\hat{e}_r + \ddot{H}\hat{e}_z) + 2\omega\hat{e}_z\times(\dot{D}\hat{e}_r + \dot{H}\hat{e}_z)$$

$$\Rightarrow \bar{a} = \omega\hat{e}_z\times(\omega D\hat{e}_\theta) + \ddot{D}\hat{e}_r + \ddot{H}\hat{e}_z + 2\omega\dot{D}\hat{e}_\theta$$

$$\Rightarrow \bar{a} = -\omega^2 D\hat{e}_r + \ddot{D}\hat{e}_r + \ddot{H}\hat{e}_z + 2\omega\dot{D}\hat{e}_\theta \quad \text{———} (1) \quad \text{3 Mark}$$

Since the particle slides down a parabola, D and H, $\dot{D}$ and $\dot{H}$, $\ddot{D}$ and $\ddot{H}$ are related. We have

$D^2 = 4aH$ ——— (2)

Differentiating gives

$2D\dot{D} = 4a\dot{H}$

$\Rightarrow D\dot{D} = 2a\dot{H}$ ——— (3)

0.5 Mark

We have

$\dot{D}^2 + \dot{H}^2 = v^2$ ——— (4)

From (3) and (4) we get

$$\dot{D}^2 = \frac{4a^2v^2}{4a^2 + D^2} \quad \text{———} (5)$$

$$\Rightarrow \dot{D} = -\sqrt{\frac{4a^2v^2}{4a^2 + D^2}} \quad \text{———} (6) \quad \text{0.5 Mark}$$

–ve sign because D is decreasing as particle slides "down".

Differentiating (3) we get

$$\dot{D}^2 + D\ddot{D} = 2a\ddot{H} \quad \Rightarrow \quad \ddot{H} = \frac{\dot{D}^2 + D\ddot{D}}{2a} \quad \text{———} (7) \quad \text{0.5 Mark}$$

Differentiating (5)

$$2\dot{D}\ddot{D} = \frac{-8a^2v^2D}{(4a^2 + D^2)^2}$$

$$\Rightarrow \ddot{D} = \frac{-4a^2v^2D}{\dot{D}(4a^2 + D^2)^2} \quad \text{———} (8) \quad \text{0.5 Mark}$$

Using (6), (7) and (8) all the expressions in (1) can be written in terms of a, D, ω.

4. The total virtual work performed by the applied and inertia forces through infinitesimal virtual displacements, compatible with the system constraints, is zero 0.5 Mark

$$\sum_{i=1}^{N} \left(\bar{F}_i^{(a)} - m_i \ddot{\bar{r}}_i \right) \cdot \delta \bar{r}_i = 0$$ 0.5 Mark

5. Constraints that cannot be expressed in algebraic form but expressed in differentials of the coordinates and possibly time

$$\sum_{i=1}^{n} a_{ji}\, dq_i + a_{jt}\, dt = 0 \qquad j = 1, 2, \cdots, m$$

provided the above Pfaffian form ceases to be written in an exact differential form 1 Mark

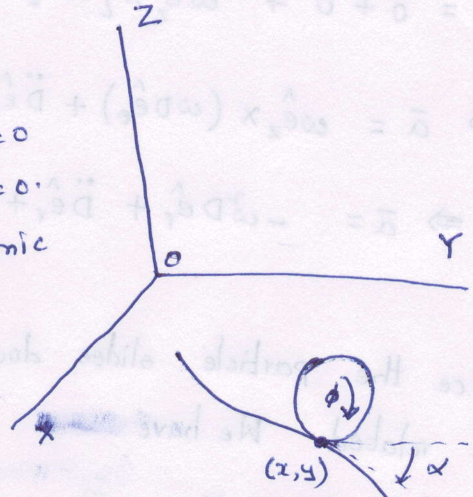
Example

$$\left.\begin{aligned} dx &= ds \sin\alpha \\ dy &= ds \cos\alpha \\ ds &= r\, d\phi \end{aligned}\right\} \Rightarrow \begin{aligned} dx - r\, d\phi \sin\alpha &= 0 \\ dy - r\, d\phi \cos\alpha &= 0. \end{aligned}$$

Two non-holonomic constraints

$\{x, y, \phi, \alpha\}$ are generalized coordinates

1 Mark



Rolling of a circular disc

6. $dx + dy + \cos t\, dt = 0$ Notice that $a_{jt} \neq 0$ 1 Mark

$\Rightarrow d\{x + y + \sin t\} = 0$

7. Actual displacement occurs in time interval dt, during which the forces and constraints may be changing. 0.5 Mark

Virtual displacement occurs if the system was frozen in its motion at time t, and the system was then moved without violating any of the forces and constraints operating on the system at that instant 0.5 Mark

$$\underset{\substack{\uparrow \\ \text{virtual}}}{\delta \bar{r}_i} = \underset{\substack{\uparrow \\ \text{Actual}}}{d\bar{r}_i}\Big|_{dt=0}$$