Analytical Dynamics
Project 2
Double Link Pendulum Dynamics Simulation

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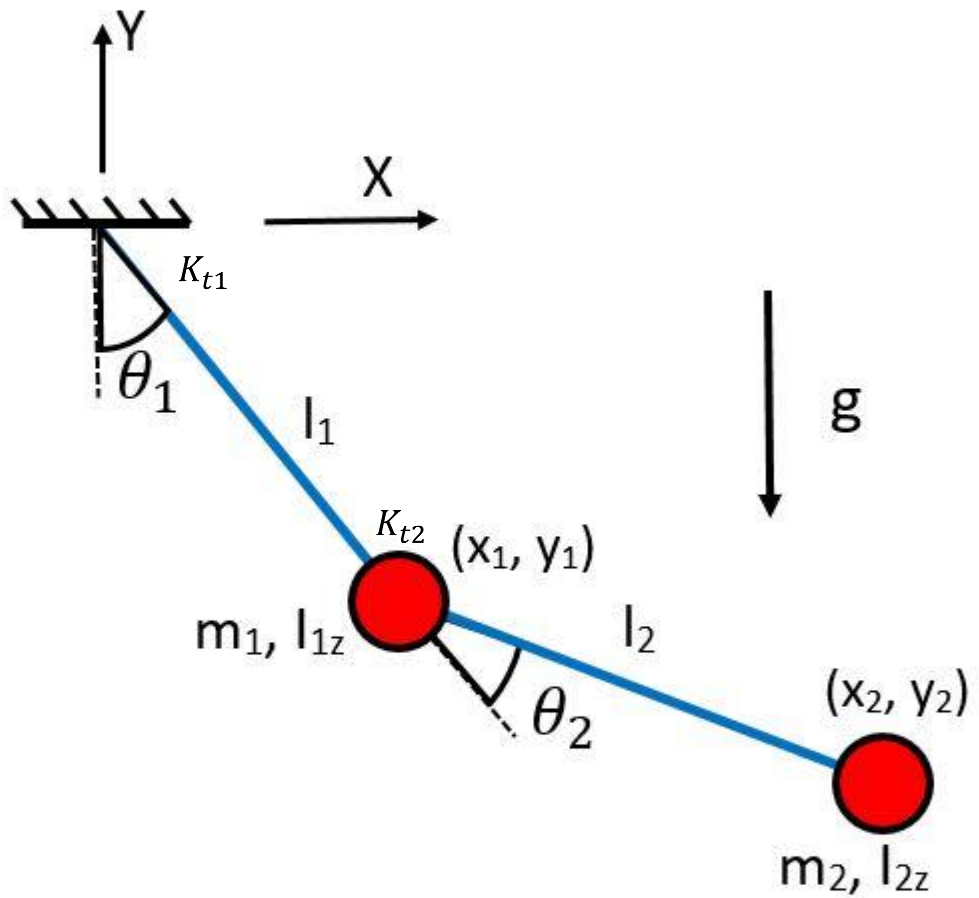
Outline

- Problem Statement
- Equation of Motion (EoM)
- Conversion to reduced order EoM
- Differential Equation Solver
- Simulation Results
- Analysis

Problem Statement

- To simulate dynamics of double link pendulum under the effect of gravity
  - Considering two point masses
  - Considering two rods
  - With or without torsional spring
  - Linearizing the equation of motion

Case I: With 2 point masses



Assumptions:

1. Massless rods
2. Planar motion

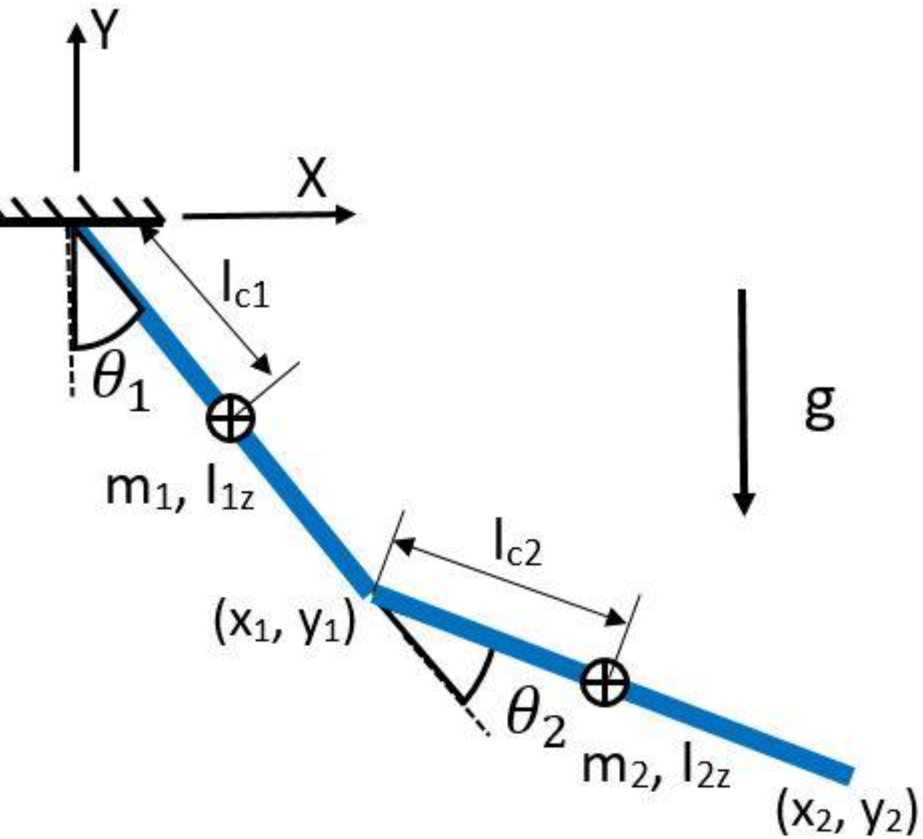
Inertia of masses

$$I_{1z} = m_1 l_1^2$$
$$I_{2z} = m_2 l_2^2$$

Position of masses

$$x_1 = l_1 \sin\theta_1$$
$$y_1 = -l_1 \cos\theta_1$$
$$x_2 = x_1 + l_2 \sin(\theta_1 + \theta_2)$$
$$y_2 = y_1 - l_2 \cos(\theta_1 + \theta_2)$$

Case II: With 2 rods



Assumptions:

1. COM lies at center of link

Inertia of masses

$$I_{1z} = m_1 l_1^2/3$$
$$I_{2z} = m_2 l_2^2/3$$

Position of masses

$$x_1 = l_{c1} \sin\theta_1$$
$$y_1 = -l_{c1} \cos\theta_1$$
$$x_2 = l_1(\sin\theta_1) + l_{c2}/2 (\sin\theta_2 + \theta_1)$$
$$y_2 = -l_1(\cos\theta_1) - l_{c2}/2 (\cos\theta_2 + \theta_1)$$

Equation of Motion (EoM)

- Lagrangian

$$\boxed{\mathrm{L} = \mathrm{T} - \mathrm{V}}$$

$$= \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 l_1 l_2\dot{\theta}_1\big(\dot{\theta}_1 +$$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau}$$

Equation of Motion (EoM)

$$\boxed{M(q)\ddot{q} + C(q,\dot{q}) + G(q) = \tau}$$

Where, q = $[\theta_1 \quad \theta_2]^T$

$$M = \begin{bmatrix} m_1 l_1^2 + m_2 l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 \cos\theta_2 + I_{1z} + I_{2z} & m_2 l_2^2 + m_2 l_1 l_2 \cos\theta_2 + I_{2z} \\ m_2 l_2^2 + m_2 l_1 l_2 \cos\theta_2 + I_{2z} & m_2 l_2^2 + I_{2z} \end{bmatrix}$$

$$C = \begin{bmatrix} -2m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2 - \dot{m}_2 l_1 l_2 \dot{\theta}_2^2 \sin\theta_2 \\ m_2 l_1 l_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix}$$

$$G = \begin{bmatrix} m_1 l_1 g \sin\theta_1 + m_2 l_1 g \sin\theta_1 + m_2 l_2 g \sin(\theta_1 + \theta_2) + k_{t1}\theta_1 \\ m_2 l_2 g \sin(\theta_1 + \theta_2) + k_{t2}\theta_2 \end{bmatrix}$$

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Reduced order EoM

$$M(q)\ddot{q} + C(q,\dot{q}) + G(q) = \tau$$

$$\boxed{\ddot{q} = M^{-1}[\tau - C(q,\dot{q}) - G(q)]}$$

Defining the state vector as $[\boldsymbol{r}\ \ \dot{\boldsymbol{r}}]^T$
Then, reduced order equation of motion are

$$\begin{bmatrix} r \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -M^{-1}[C(q,\dot{q}) - G(q)] \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \tau$$

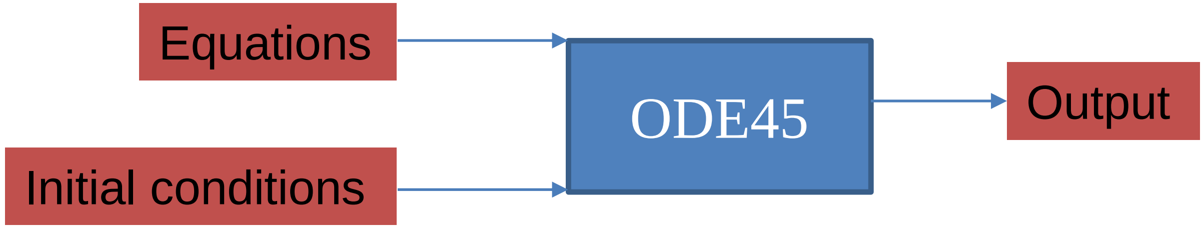
Where $\boldsymbol{q} = [\theta_1\ \ \theta_2]^T$

Differential Equation Solver

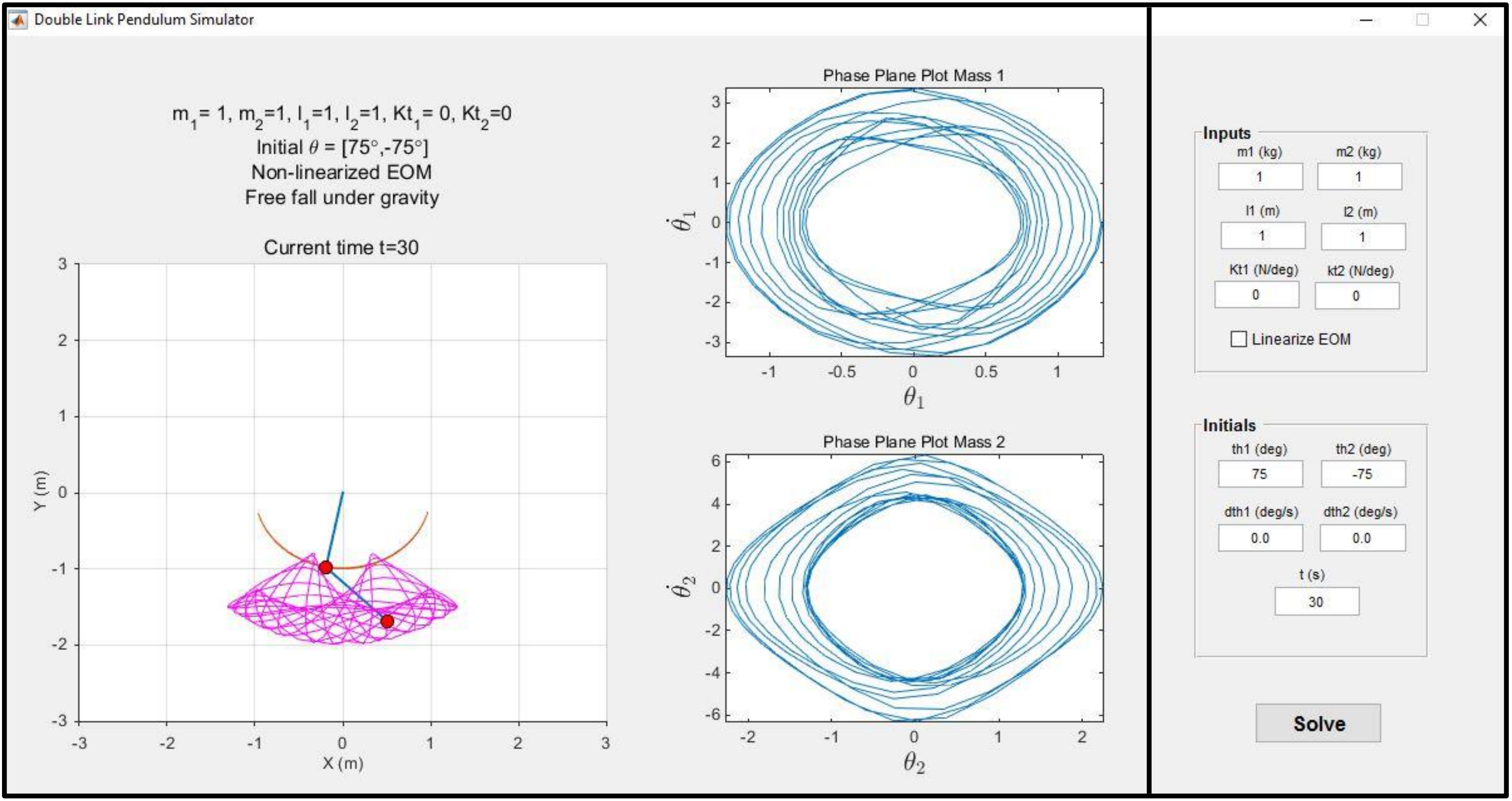
ODE45:

- This function implements a Runge-Kutta method
- Syntax

[t,x] = ode45(@f_name, t_span, x_init, options)



MATLAB GUI


Double Link Pendulum Simulator
m1= 1, m2=1, l1=1, l2=1, Kt1= 0, Kt2=0
Initial θ = [75°,-75°]
Non-linearized EOM
Free fall under gravity
Current time t=30
Y (m)
X (m)
Phase Plane Plot Mass 1
Phase Plane Plot Mass 2
Inputs
m1 (kg)
m2 (kg)
l1 (m)
l2 (m)
Kt1 (N/deg)
kt2 (N/deg)
Linearize EOM
Initials
th1 (deg)
th2 (deg)
dth1 (deg/s)
dth2 (deg/s)
t (s)
Solve

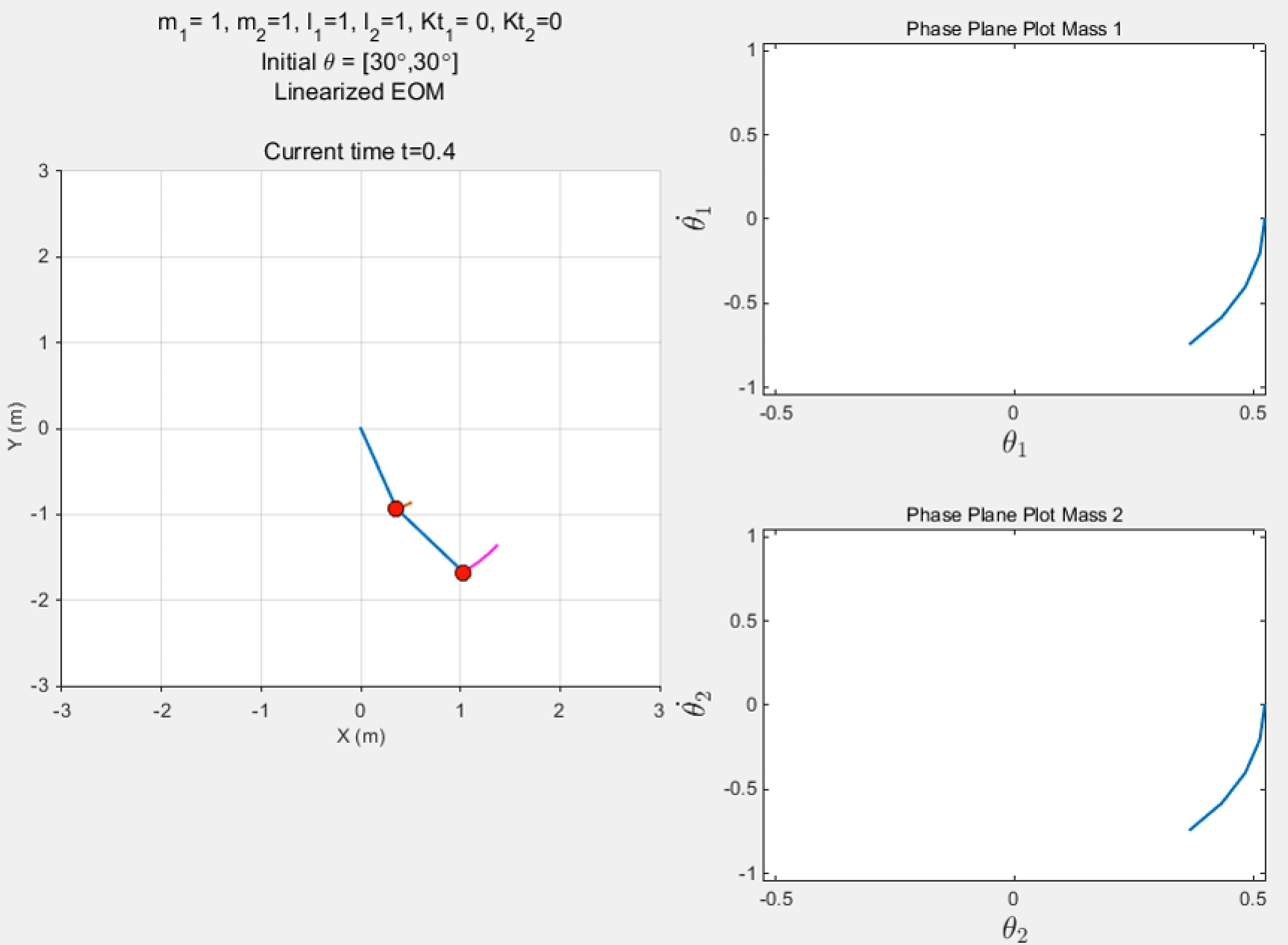
Simulation Results – for 2 point mass

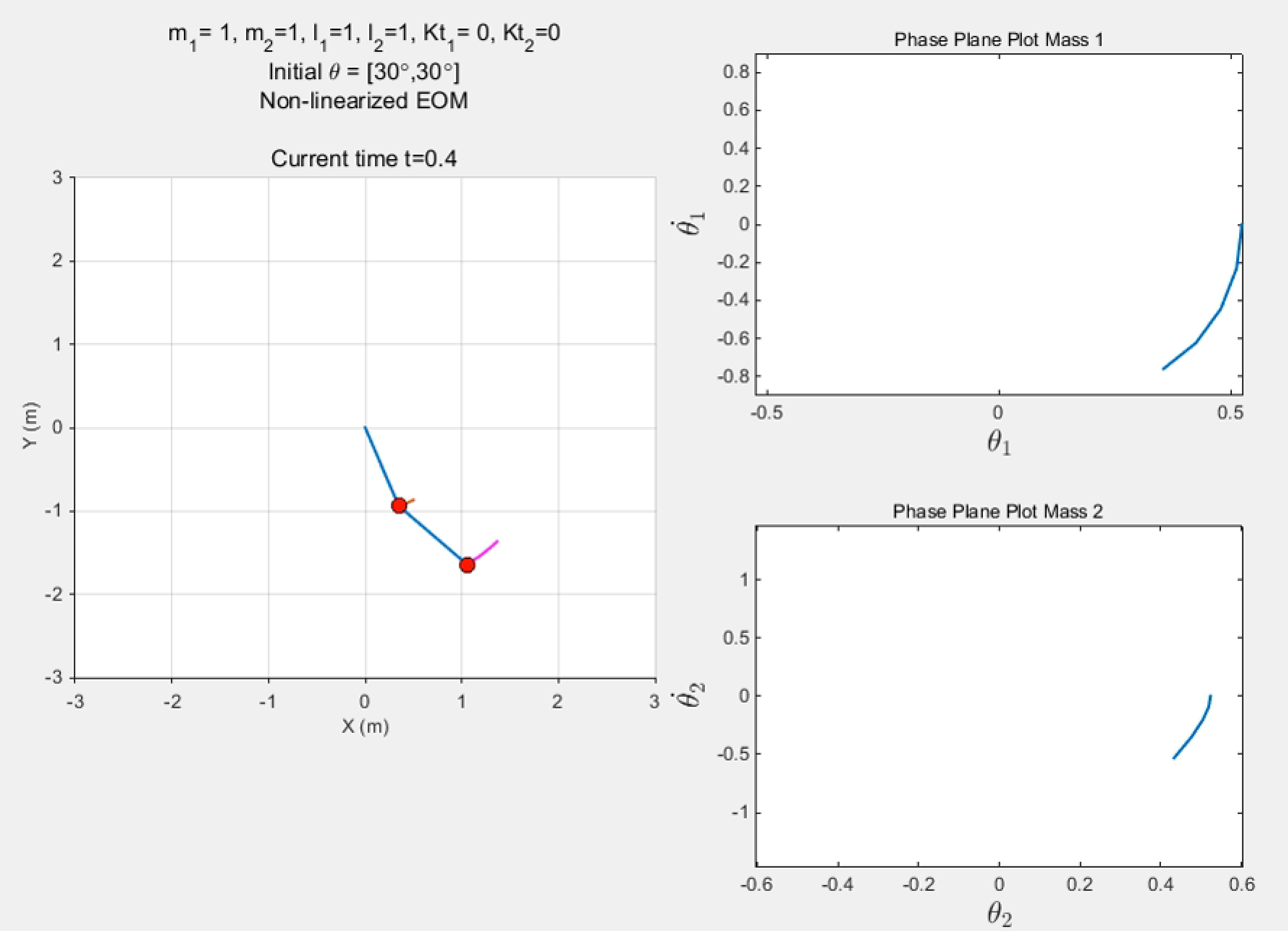
- Simulation under gravity ($\tau_1 = \tau_2 = 0$)
  - without torsional springs

$$K_{t1} = 0 \,, K_{t2} = 0$$

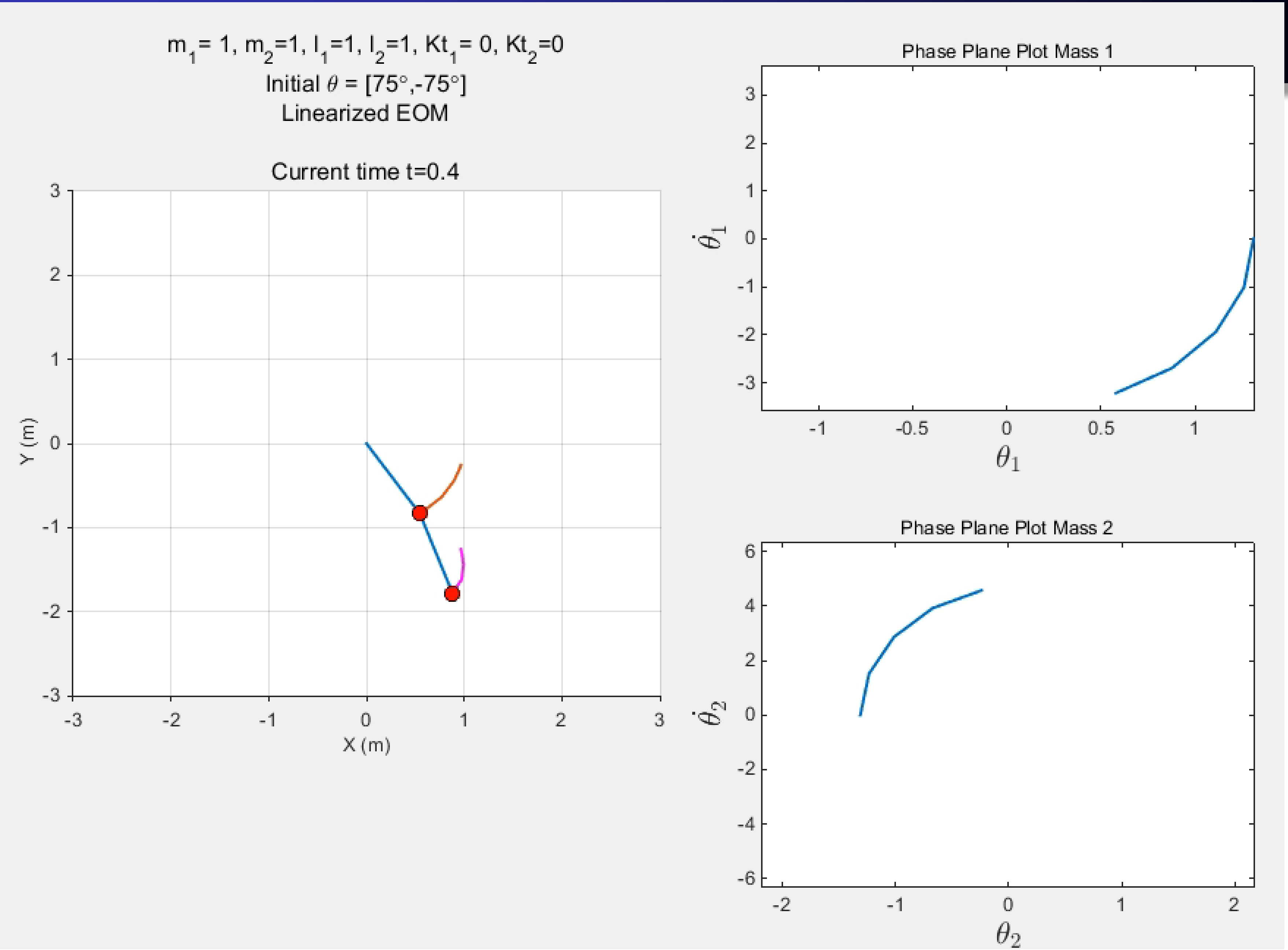
  - Linearizing the equation of motion as

$$\sin(\theta) \approx \theta \,, \cos(\theta) \approx 1$$

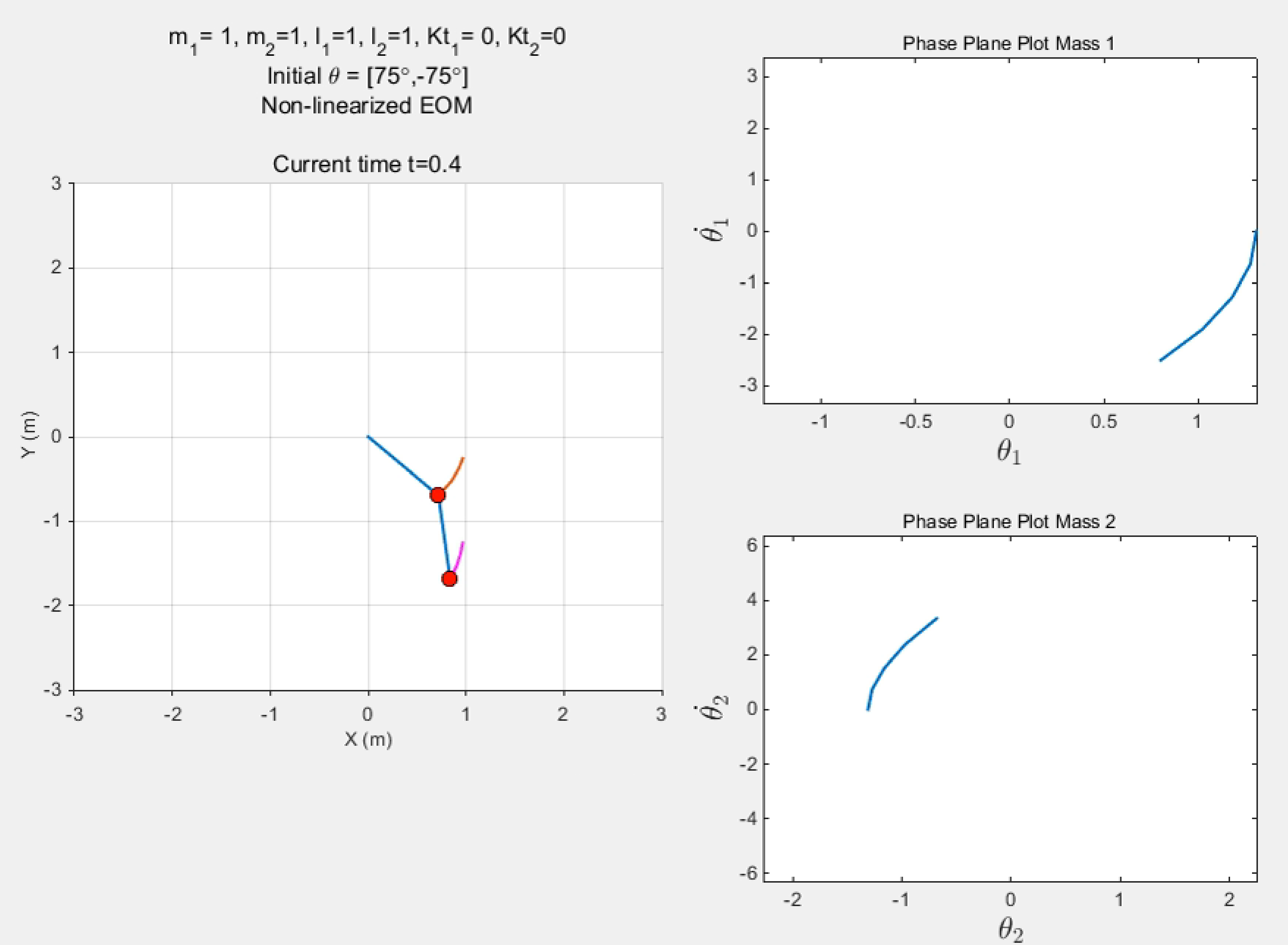

m1= 1, m2=1, l1=1, l2=1, Kt1= 0, Kt2=0
Initial θ = [30°,30°]
Linearized EOM
Current time t=0.4
Y (m)
X (m)
3
2
1
0
-1
-2
-3
Phase Plane Plot Mass 1
Phase Plane Plot Mass 2
-0.5
0
0.5
1
-1
θ̇1
θ1
θ̇2
θ2

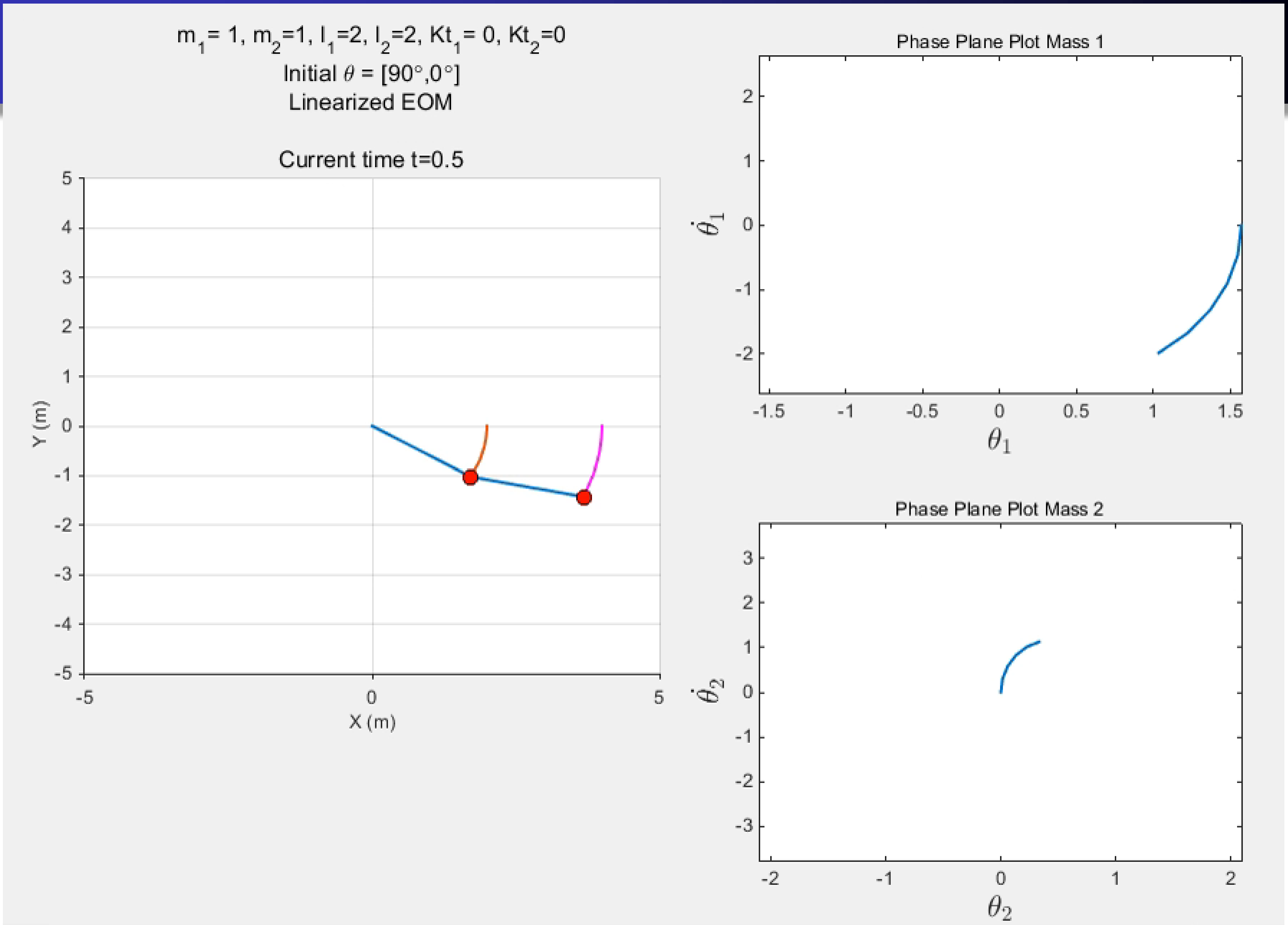


m_1= 1, m_2=1, l_1=1, l_2=1, Kt_1= 0, Kt_2=0
Initial θ = [30°,30°]
Non-linearized EOM
Current time t=0.4
Y (m)
X (m)
Phase Plane Plot Mass 1
$\dot{\theta}_1$
θ_1
Phase Plane Plot Mass 2
$\dot{\theta}_2$
θ_2

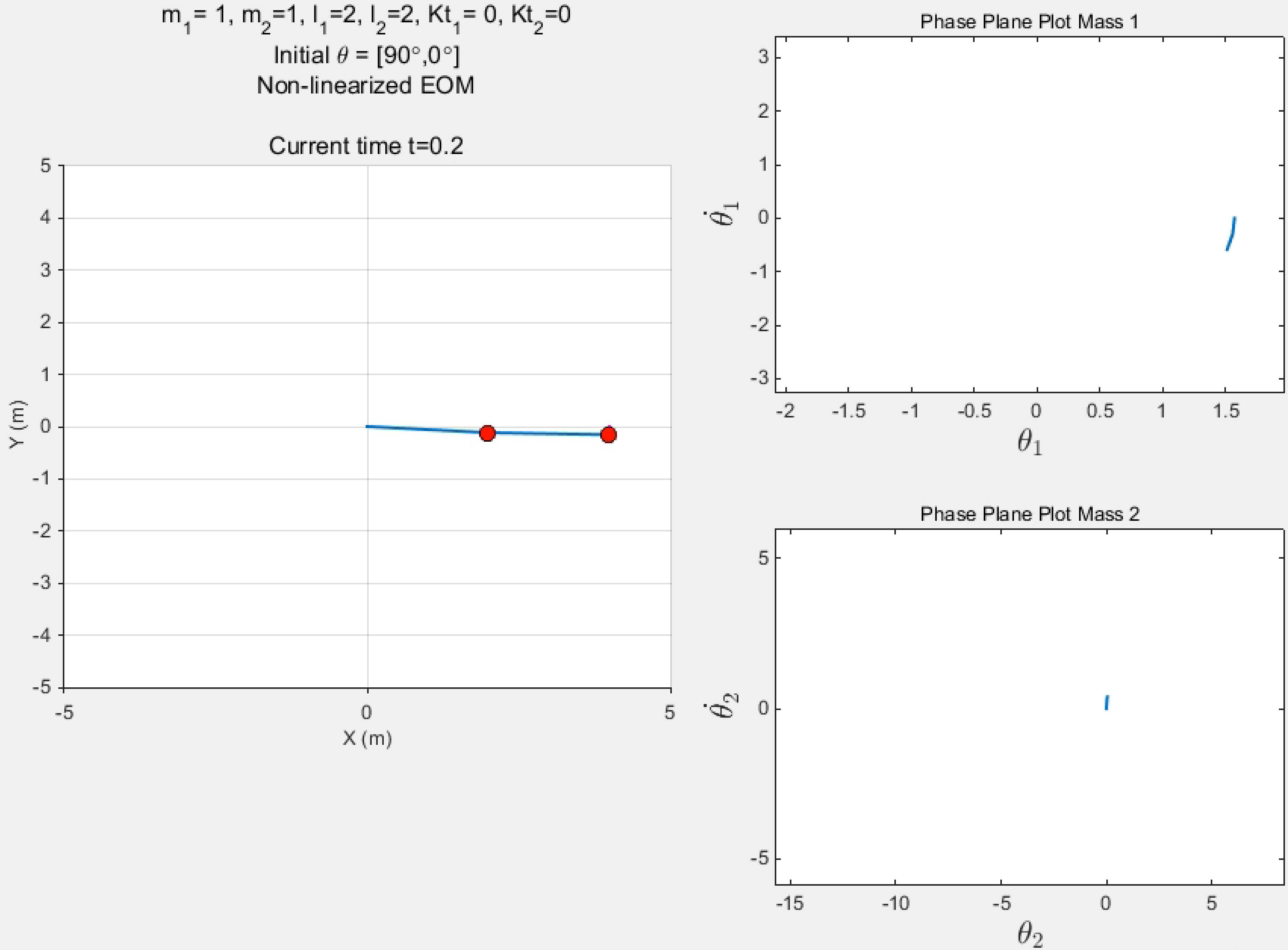


m1= 1, m2=1, l1=1, l2=1, Kt1= 0, Kt2=0
Initial θ = [75°,-75°]
Linearized EOM
Current time t=0.4
Y (m)
X (m)
Phase Plane Plot Mass 1
Phase Plane Plot Mass 2


m1= 1, m2=1, l1=1, l2=1, Kt1= 0, Kt2=0
Initial θ = [75°,-75°]
Non-linearized EOM
Current time t=0.4
Y (m)
X (m)
Phase Plane Plot Mass 1
$\dot{\theta}_1$
θ_1
Phase Plane Plot Mass 2
$\dot{\theta}_2$
θ_2



m1= 1, m2=1, l1=2, l2=2, Kt1= 0, Kt2=0
Initial θ = [90°,0°]
Linearized EOM
Current time t=0.5
Y (m)
X (m)
Phase Plane Plot Mass 1
Phase Plane Plot Mass 2


m1= 1, m2=1, l1=2, l2=2, Kt1= 0, Kt2=0
Initial θ = [90°,0°]
Non-linearized EOM
Current time t=0.2
Y (m)
X (m)
Phase Plane Plot Mass 1
Phase Plane Plot Mass 2

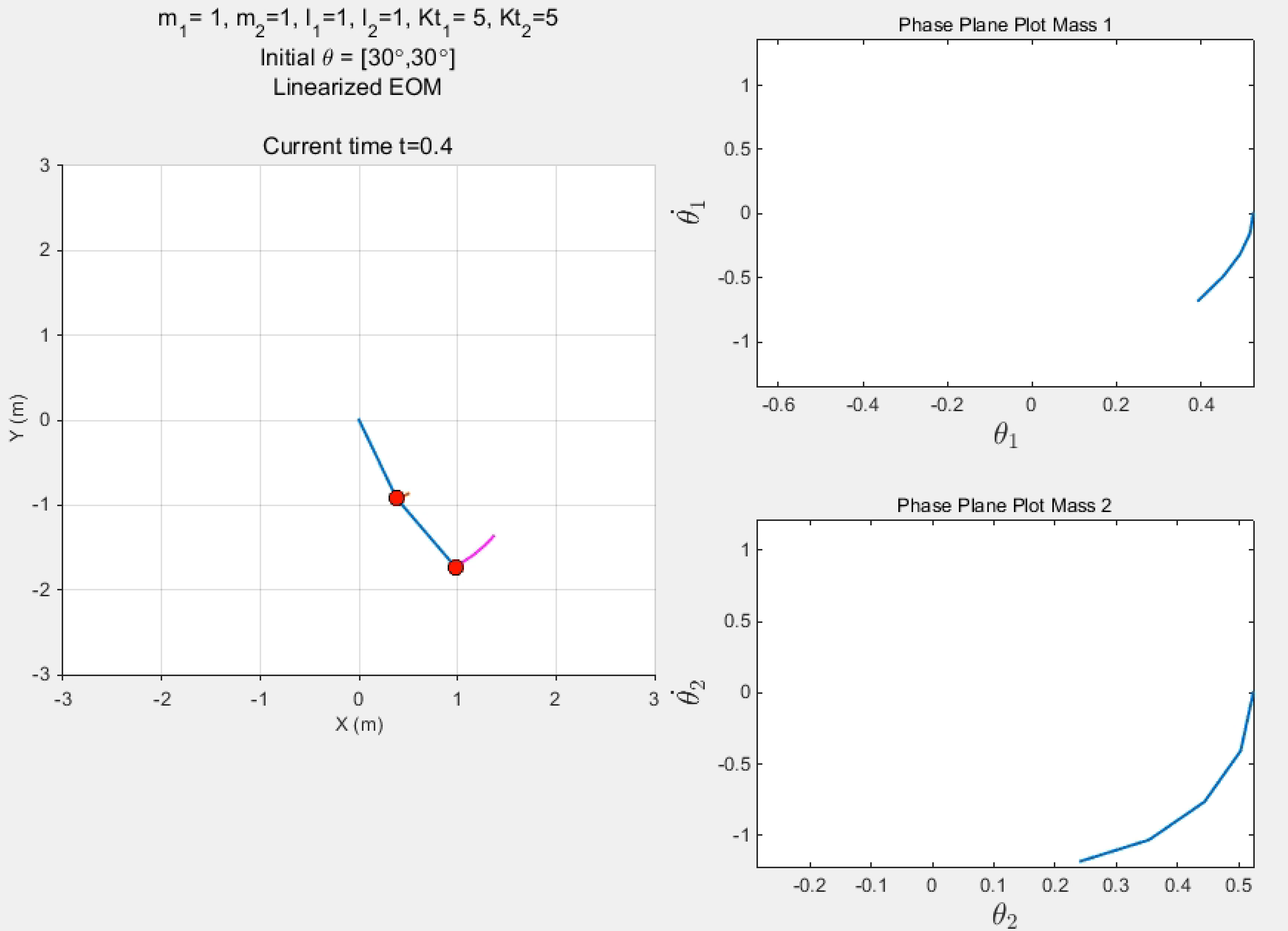
Simulation Results

- Simulation under gravity
  - with torsional springs

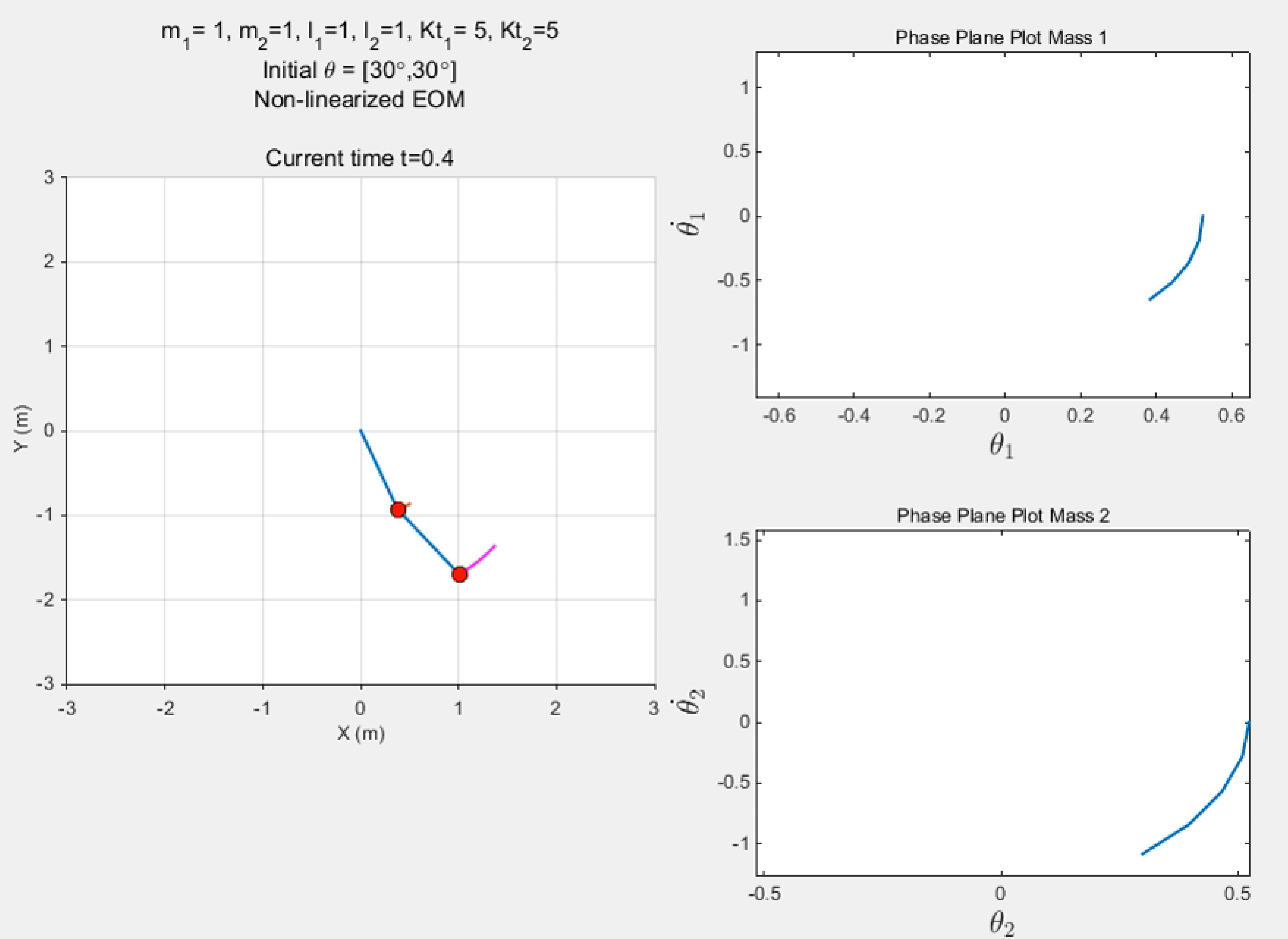
$$K_{t1} = 5\,, K_{t2} = 5\ (Nm/rad)$$

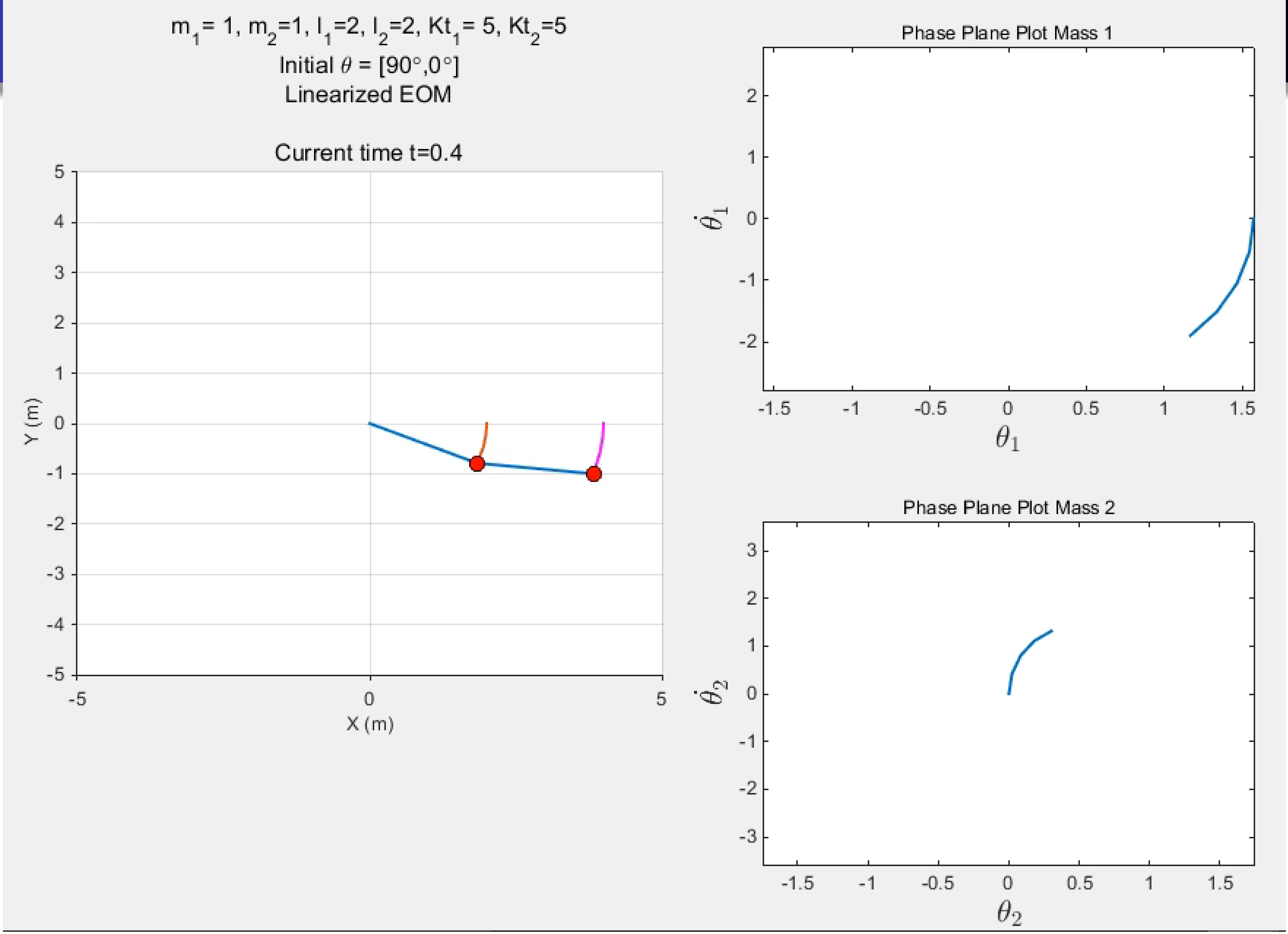
  - Linearizing the equation of motion as

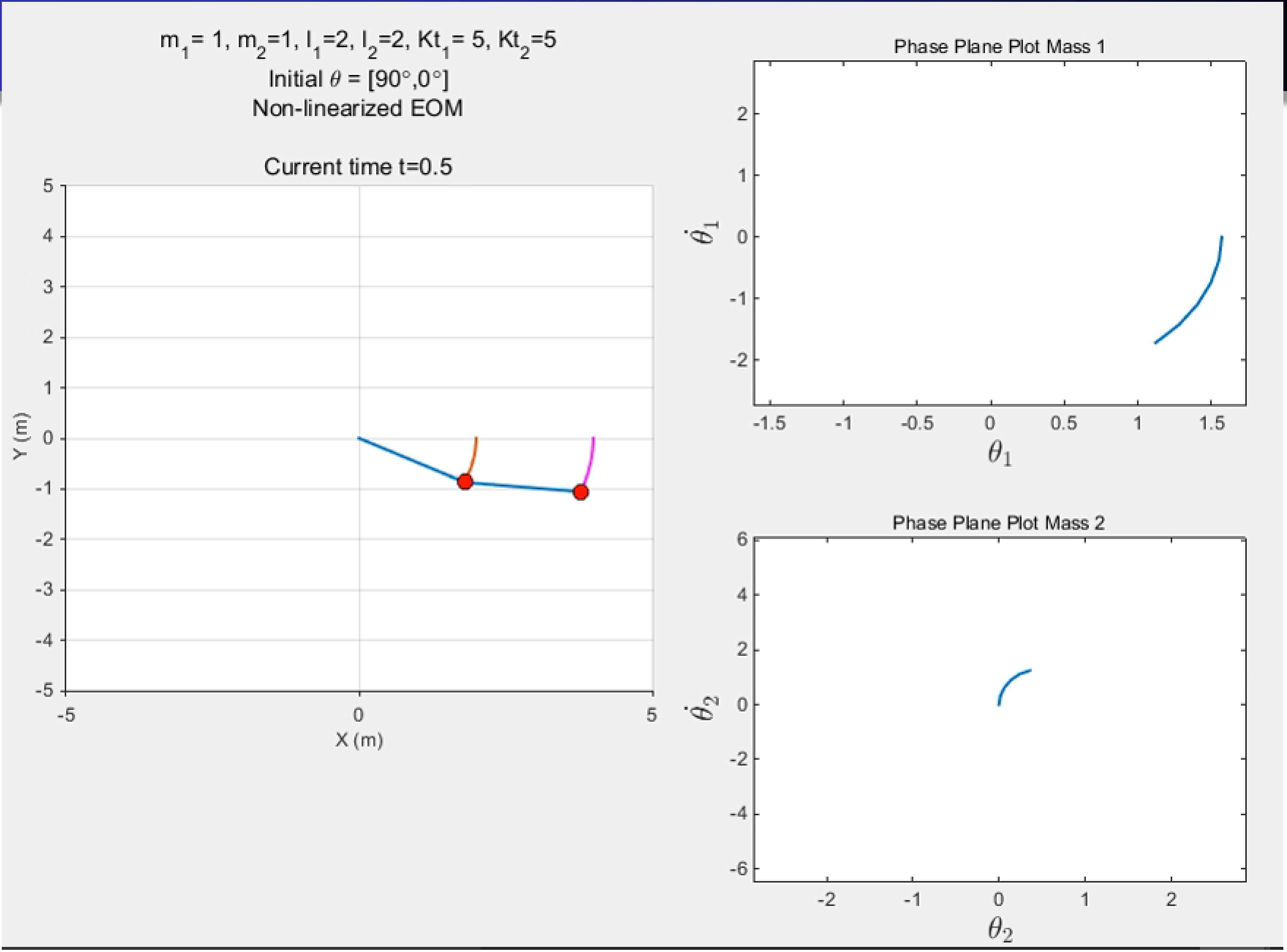
$$\sin(\theta) \approx \theta\,, \cos(\theta) \approx 1$$



m_1= 1, m_2=1, l_1=1, l_2=1, Kt_1= 5, Kt_2=5
Initial θ = [30°,30°]
Linearized EOM
Current time t=0.4
Y (m)
X (m)
3
2
1
0
-1
-2
-3
Phase Plane Plot Mass 1
$\dot{\theta}_1$
θ_1
-0.6
-0.4
-0.2
0.2
0.4
0.5
Phase Plane Plot Mass 2
$\dot{\theta}_2$
θ_2
-0.1
0.1
0.3


m1= 1, m2=1, l1=1, l2=1, Kt1= 5, Kt2=5
Initial θ = [30°,30°]
Non-linearized EOM
Current time t=0.4
Y (m)
X (m)
Phase Plane Plot Mass 1
Phase Plane Plot Mass 2


m_1= 1, m_2=1, l_1=2, l_2=2, Kt_1= 5, Kt_2=5
Initial θ = [90°,0°]
Linearized EOM
Current time t=0.4
Y (m)
X (m)
Phase Plane Plot Mass 1
$\dot{\theta}_1$
θ_1
Phase Plane Plot Mass 2
$\dot{\theta}_2$
θ_2


m_1= 1, m_2=1, l_1=2, l_2=2, Kt_1= 5, Kt_2=5
Initial θ = [90°,0°]
Non-linearized EOM
Current time t=0.5
Y (m)
X (m)
Phase Plane Plot Mass 1
Phase Plane Plot Mass 2

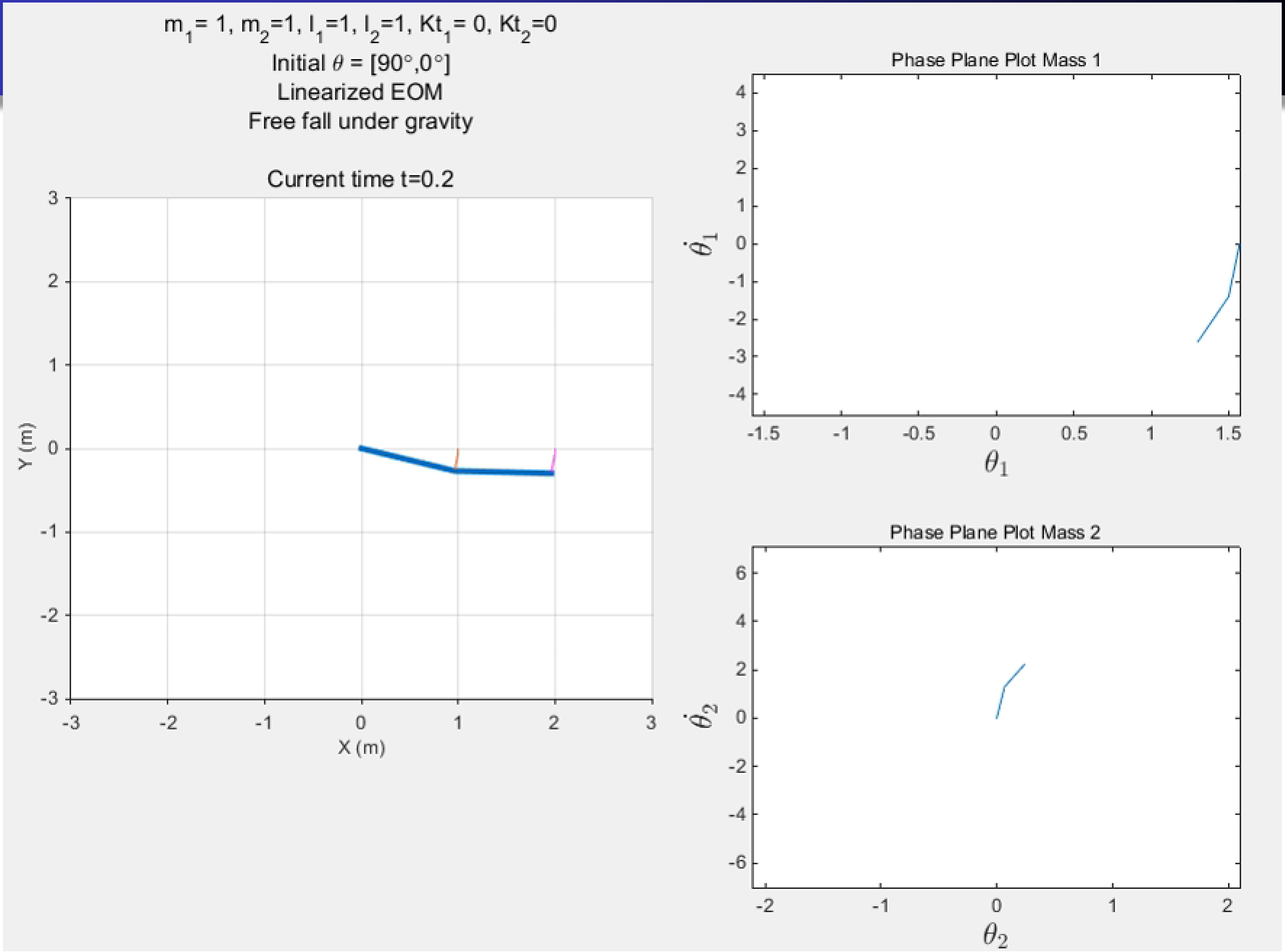
Simulation Results – for 2 rods

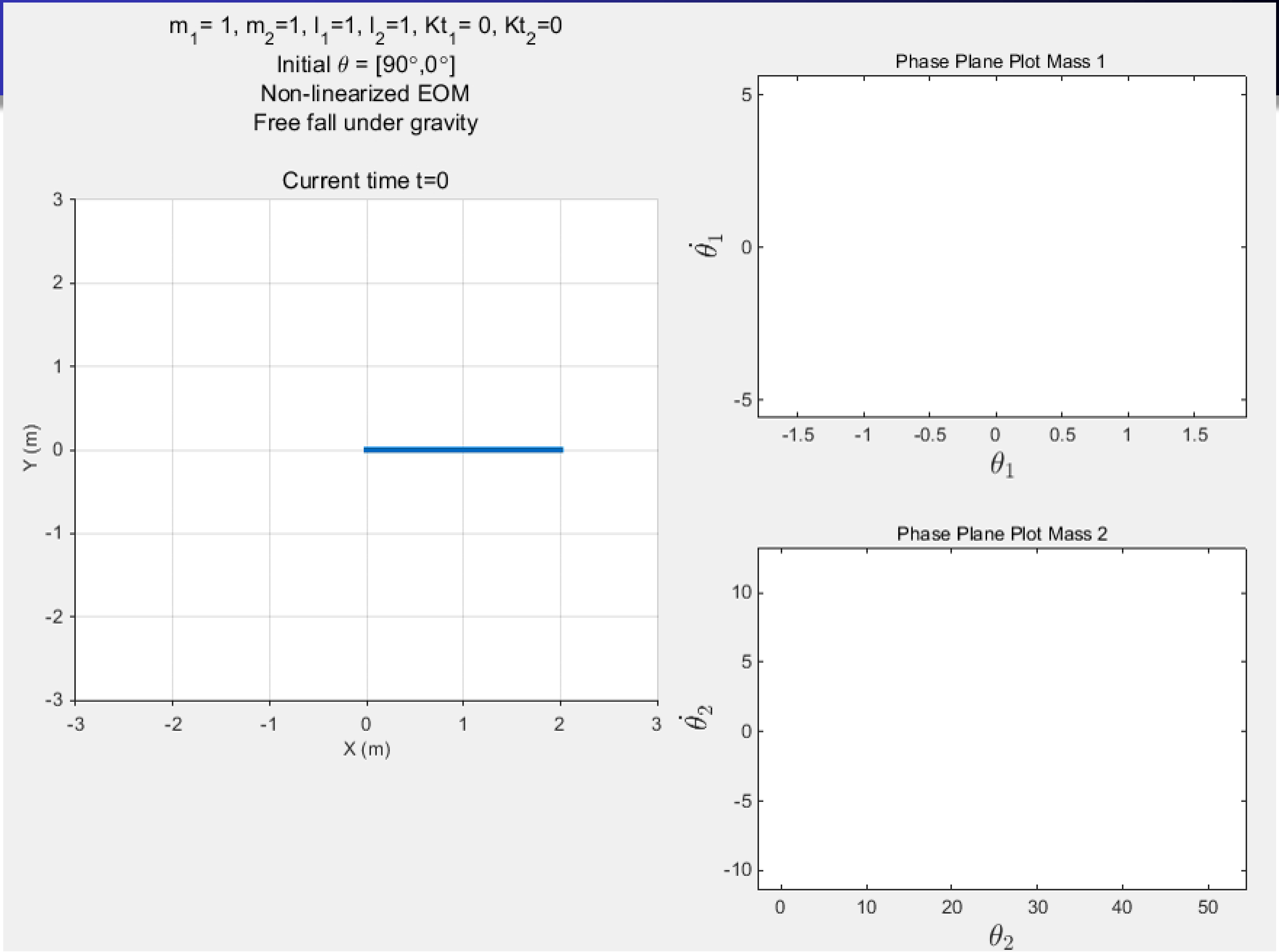
- Simulation under gravity
  - without torsional springs

$$K_{t1} = 0\,, K_{t2} = 0$$

  - Linearizing the equation of motion as

$$\sin(\theta) \approx \theta\,, \cos(\theta) \approx 1$$


m_1= 1, m_2=1, l_1=1, l_2=1, Kt_1= 0, Kt_2=0
Initial θ = [90°,0°]
Linearized EOM
Free fall under gravity
Current time t=0.2
X (m)
Y (m)
Phase Plane Plot Mass 1
Phase Plane Plot Mass 2


m_1= 1, m_2=1, l_1=1, l_2=1, Kt_1= 0, Kt_2=0
Initial θ = [90°,0°]
Non-linearized EOM
Free fall under gravity
Current time t=0
Y (m)
X (m)
Phase Plane Plot Mass 1
$\dot{\theta}_1$
θ_1
Phase Plane Plot Mass 2
$\dot{\theta}_2$
θ_2

Analysis of simulations

- For small angles, chaotic motion is absent, for large angle chaos motion is present.

- For non-linearized case, there is more random motion otherwise its smooth.

- Presence of torsional spring restrict the large value of angles, hence prevents chaotic motion.

- **Chaotic behavior**
  - Small changes in initial conditions can lead to large changes in the long-term behavior of the system
  - Motion of the system is difficult to predict
  - Double pendulum is a chaotic system

Prediction of Chaotic Behavior

- **Lyapunov exponent**
  - Used to characterize the chaos of dynamic system
  - Mathematically, it is given by λ as:

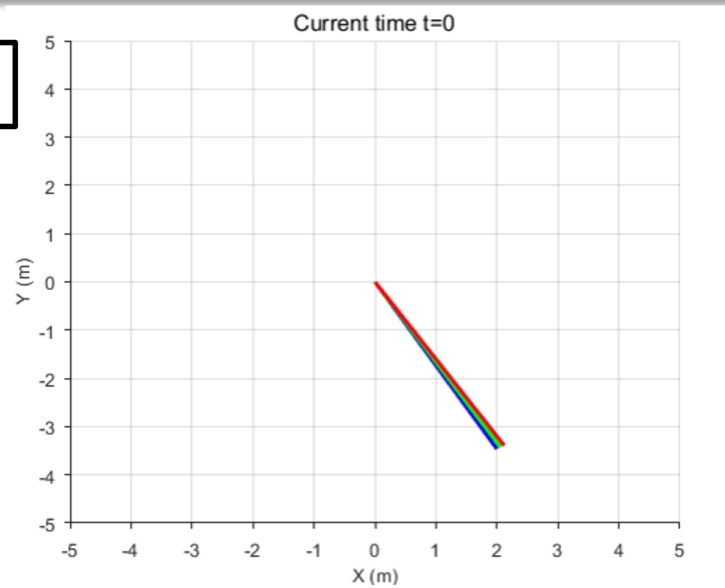
$$e^{\lambda t} \approx |\varphi(t) - (\varphi + \delta\varphi)(t)|$$

  - For n-dimensional phase-space, we have n λ's
  - If average value of λ is positive, then system will show chaos
  - If average value of λ is negative, then system is non-chaotic

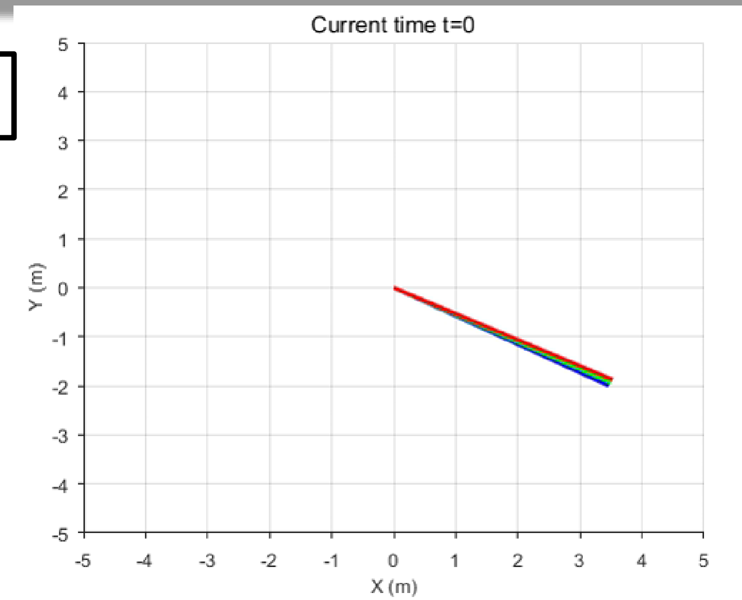
Reference: Levien, R. B., and S. M. Tan. "Double pendulum: An experiment in chaos." American Journal of Physics 61.11 (1993): 1038-1044.

Effect of initial condition

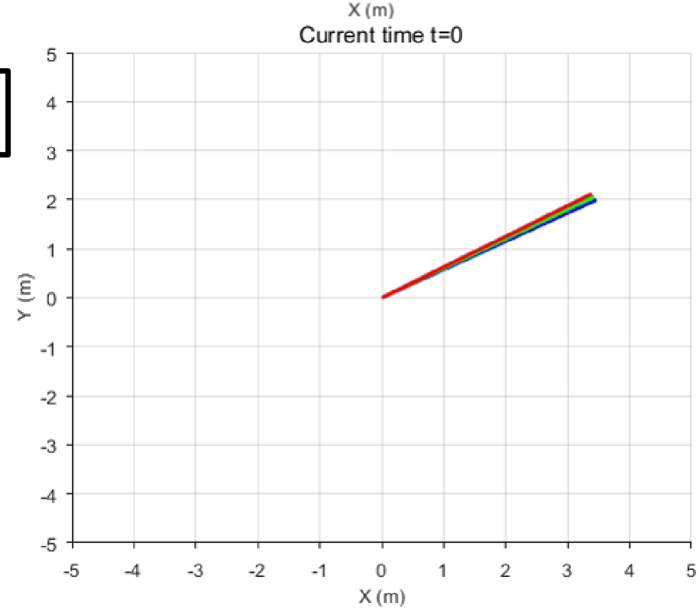
1



2



3



Initial Joint angles (relative)	**Cases**		
	1	**2**	**3**
θ_1	30, 31, 32	60, 61, 62	120, 121, 122
θ_2	0, 0, 0	0, 0, 0	0, 0, 0

Analysis of chaotic behavior

- Effect of initial condition:
  - Varying initial angle of first link

$$\boxed{m_1 = 1, m_2 = 1,\ l_1 = 1,\ l_2 = 1, \theta_2 = \ \dot{\theta}_1 = \dot{\theta}_2 = 0, t = 100s}$$

Initial $\boldsymbol{\theta_1}$ (in degrees)	$\lambda_{\boldsymbol{\theta_1}}$	$\lambda_{\boldsymbol{\theta_2}}$	$\lambda_{\dot{\boldsymbol{\theta}}_1}$	$\lambda_{\dot{\boldsymbol{\theta}}_2}$	**Behavior**
30	-3.8165	-3.1433	-2.7859	-2.009	Non-chaotic
45	-2.7030	-1.9831	-1.7236	-0.8644	Non-chaotic
55	-1.1814	-0.5964	-0.4184	0.1949	Chaotic
60	-1.7082	-0.8446	-0.5228	0.1923	Chaotic
90	-0.5015	2.54522	0.3260	0.6802	Chaotic
120	1.6308	2.1625	0.3544	0.7550	Chaotic

Analysis of chaotic behavior

- Effect of initial condition:
  - Varying mass of links

$$\boxed{\theta_1 = 45\,^\circ, \theta_1 = 0, \qquad \dot{\theta}_1 = \dot{\theta}_2 = 0, t = 100s}$$

Parameter	$\lambda_{\boldsymbol{\theta_1}}$	$\lambda_{\boldsymbol{\theta_2}}$	$\lambda_{\dot{\boldsymbol{\theta}}_1}$	$\lambda_{\dot{\boldsymbol{\theta}}_2}$	**Behavior**
$m_1 = 1, m_2 = 1,$ $l_1 = 1, l_2 = 1$	-2.7030	-1.9831	-1.7236	-0.8644	Non-chaotic
$m_1 = 10, m_2 = 1,$ $l_1 = 1, l_2 = 1$	-3.6473	-2.4418	-2.8975	-1.7634	Non-chaotic
$m_1 = 15, m_2 = 1,$ $l_1 = 1, l_2 = 1$	-2.5056	1.2026	-1.7559	-0.0898	Chaotic
$m_1 = 1, m_2 = 15,$ $l_1 = 1, l_2 = 1$	-3.4115	-3.2332	-2.0998	-1.7351	Non-chaotic

Extra Results

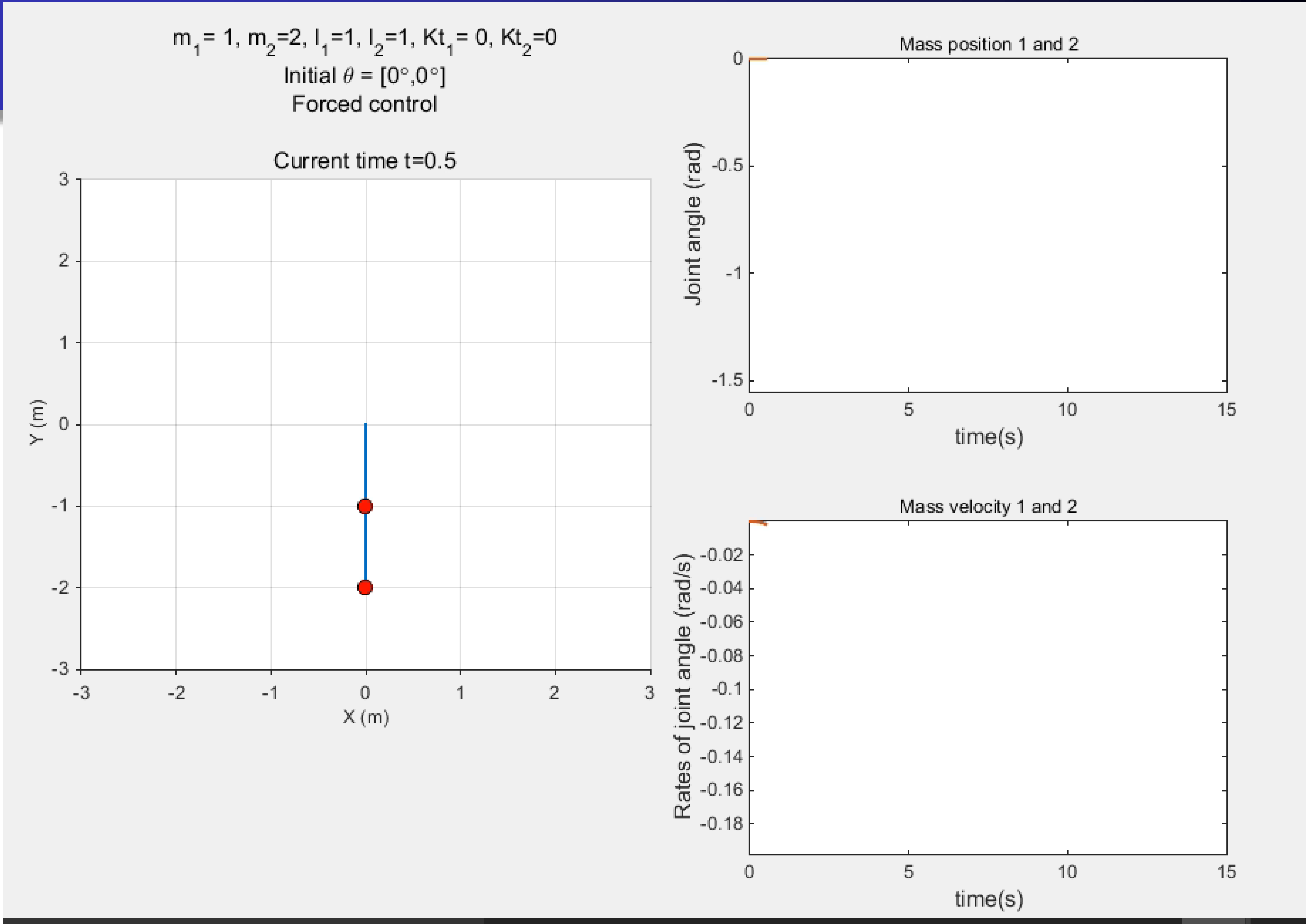
- Forced Control
  - Applied Torque

$$\boxed{\tau = k_p(\theta - \theta_d) + k_d(\dot{\theta} - \dot{\theta}_d)}$$

  - Desired Trajectory: Cycloidal

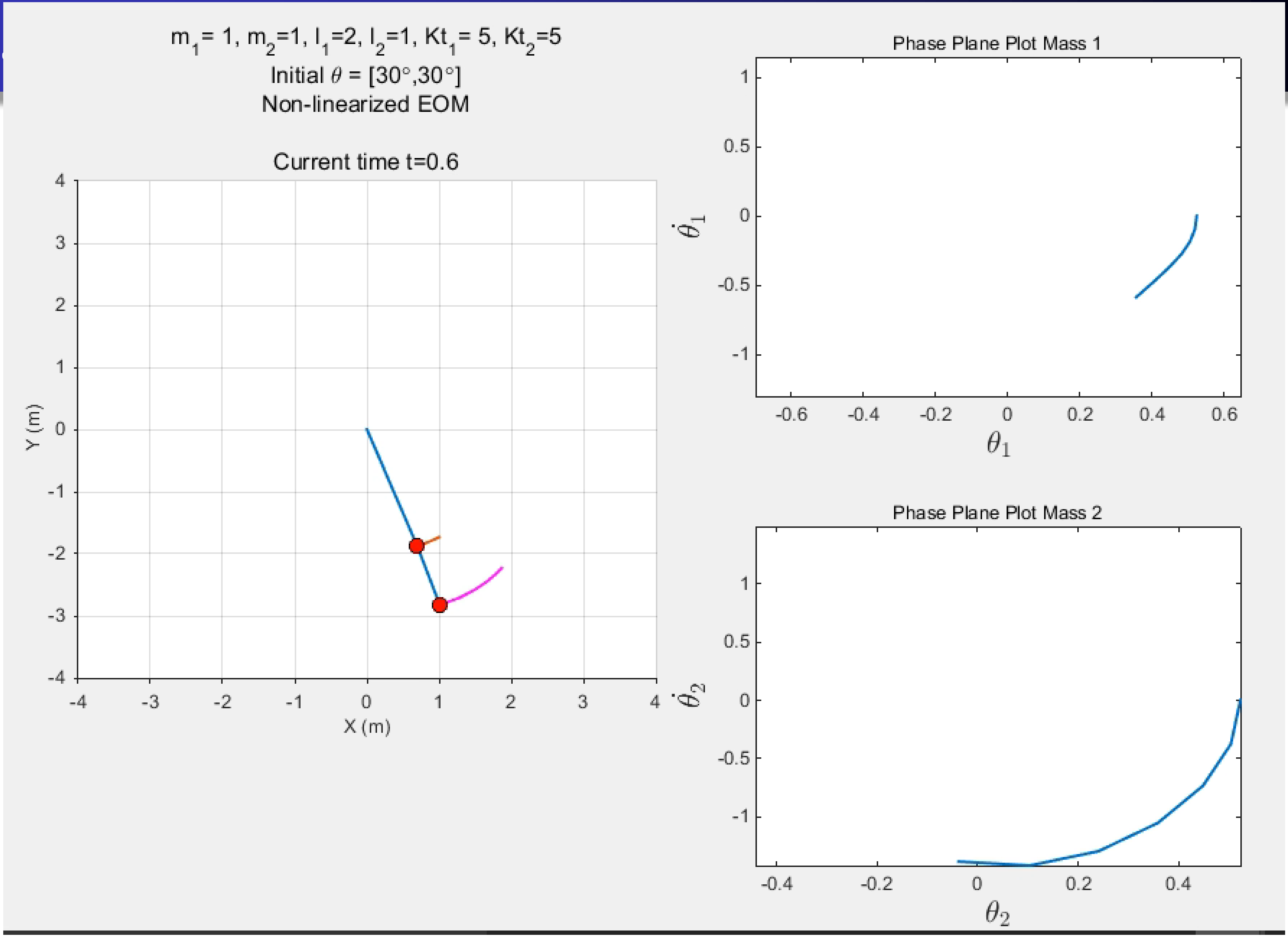
$$\theta_d = \theta_i + \frac{\theta_f - \theta_i}{T_p}\left(t - \frac{T_p}{2\pi}\sin\frac{2\pi}{T_p}t\right)$$

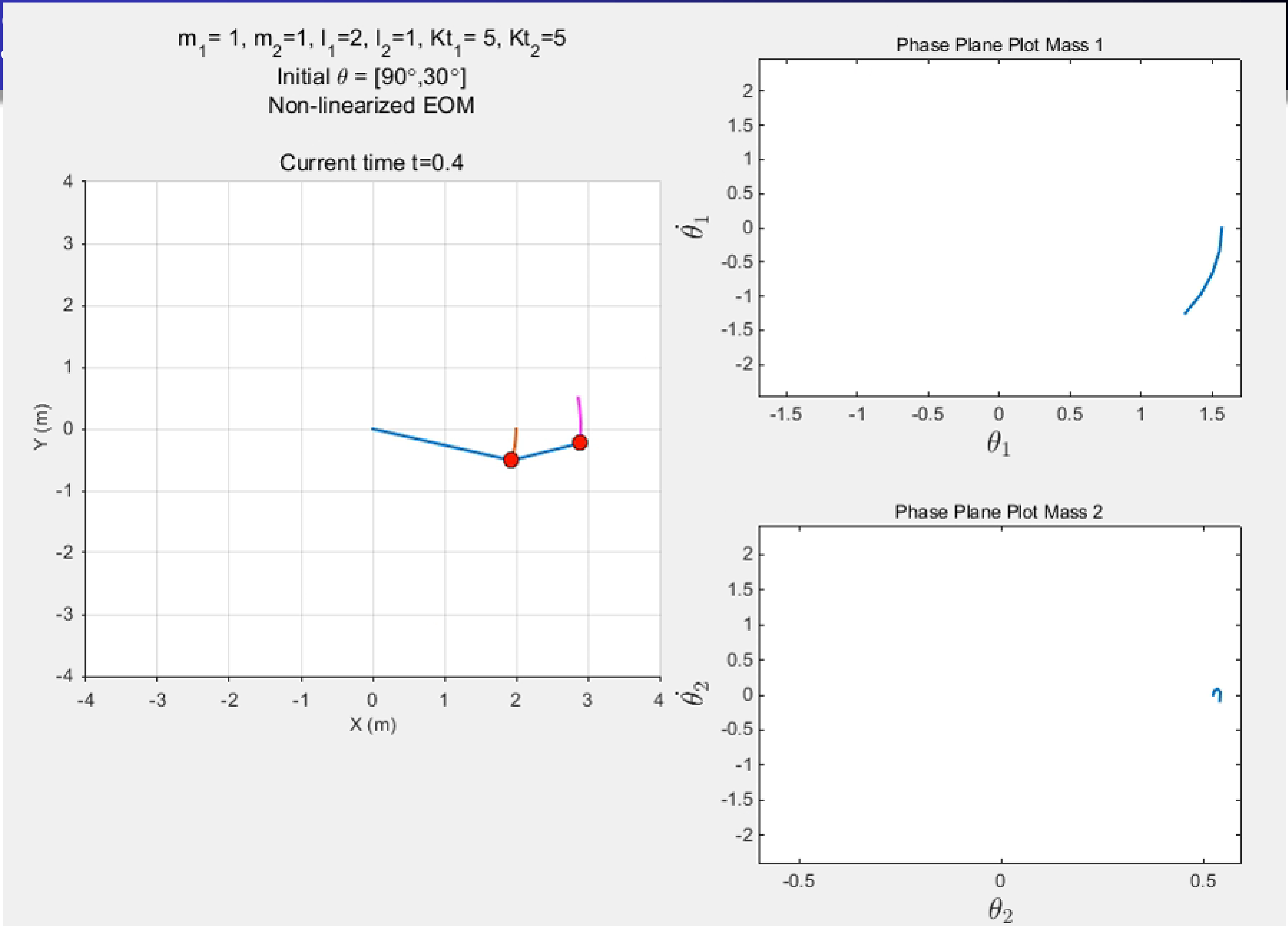
$$\dot{\theta}_d = \frac{\theta_f - \theta_i}{T_p}\left(1 - \cos\frac{2\pi}{T_p}t\right)$$


m_1= 1, m_2=2, l_1=1, l_2=1, Kt_1= 0, Kt_2=0
Initial θ = [0°,0°]
Forced control
Current time t=0.5
Y (m)
X (m)
Mass position 1 and 2
Joint angle (rad)
time(s)
Mass velocity 1 and 2
Rates of joint angle (rad/s)
time(s)

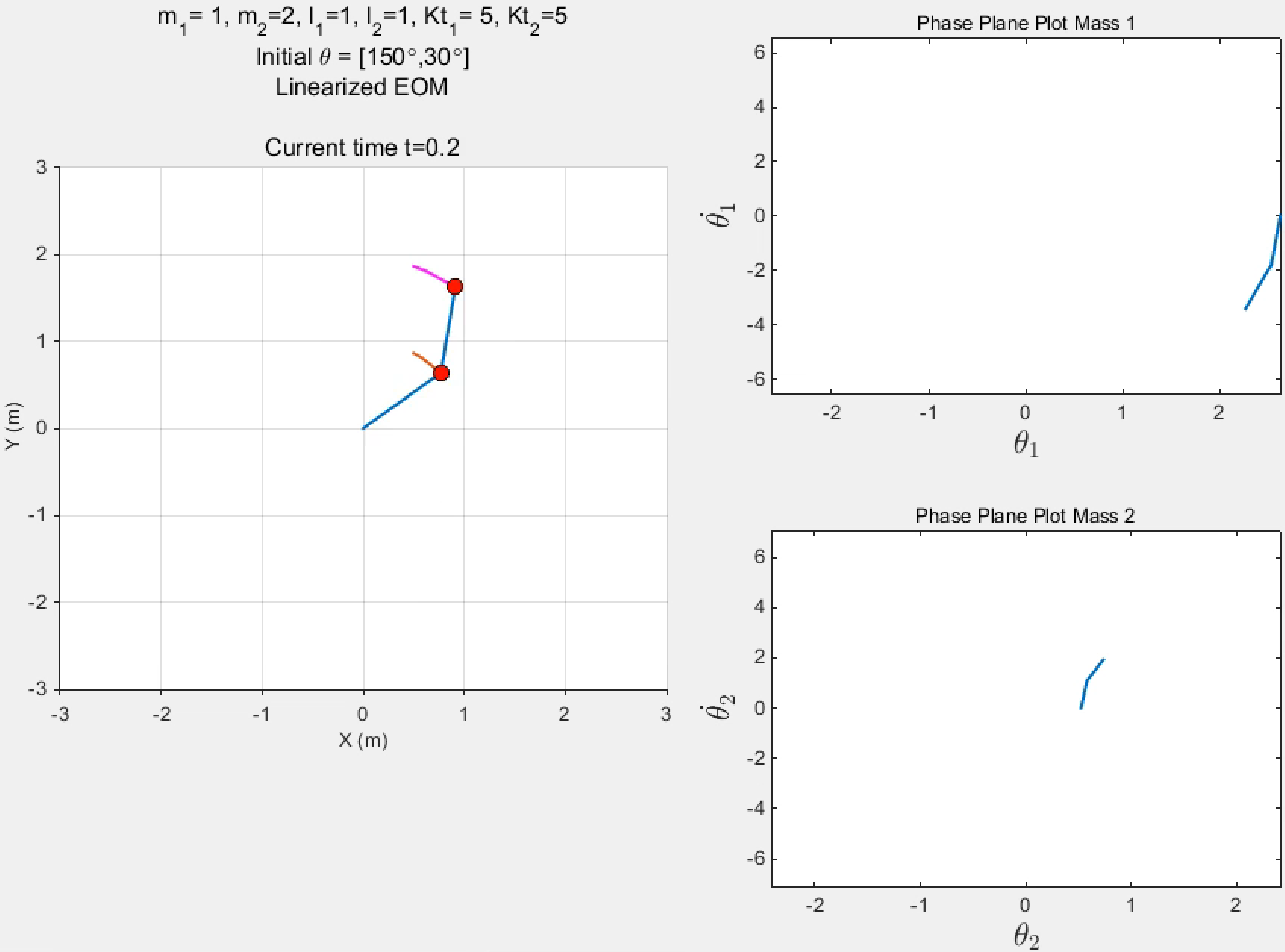
Thank You

Some more results


m_1= 1, m_2=1, l_1=2, l_2=1, Kt_1= 5, Kt_2=5
Initial θ = [30°,30°]
Non-linearized EOM
Current time t=0.6
Y (m)
X (m)
Phase Plane Plot Mass 1
$\dot{\theta}_1$
θ_1
Phase Plane Plot Mass 2
$\dot{\theta}_2$
θ_2



m1= 1, m2=1, l1=2, l2=1, Kt1= 5, Kt2=5
Initial θ = [90°,30°]
Non-linearized EOM
Current time t=0.4
Y (m)
X (m)
Phase Plane Plot Mass 1
$\dot{\theta}_1$
θ_1
Phase Plane Plot Mass 2
$\dot{\theta}_2$
θ_2



m_1= 1, m_2=2, l_1=1, l_2=1, Kt_1= 5, Kt_2=5
Initial θ = [150°,30°]
Linearized EOM
Current time t=0.2
X (m)
Y (m)
Phase Plane Plot Mass 1
θ_1
$\dot{\theta}_1$
Phase Plane Plot Mass 2
θ_2
$\dot{\theta}_2$