COL864: Homework 1

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1 Probabilistic graphical model

1.1 State space:

The state space consists of unknown location of the robot in a discrete grid world at time t denoted by state variable $\boldsymbol{X}_t$. The domain of state variable $\boldsymbol{X}_t$ is set of all possible locations in the world, that is

$$\{x_0 \quad x_1 \quad \dots \quad x_{n-1}\} \tag{1}$$

In the given problem, the world is a lake and assumed as a $2D$ grid of varying sizes. For instance, lake decriticized as 5×5 grid is shown below:

x_0	x_5	x_{10}	x_{15}	x_{20}
x_1	x_6	x_{11}	x_{16}	x_{21}
x_2	x_7	x_{12}	x_{17}	x_{22}
x_3	x_8	x_{13}	x_{18}	x_{23}
x_4	x_9	x_{14}	x_{19}	x_{24}

1.2 Observation space:

The observation space consists of two evidence variables, one is rotor sound and second is bump sound, as given below:

$$\{RotorSound_t \quad BumpSound_t\} \tag{2}$$

1.3 Transition model:

The transition model is the probability of a transitions from state x_i to state x_j. This can be mathematically written as:

$$\boldsymbol{T}_{i,j} = \boldsymbol{P}(\boldsymbol{X}_t = x_j | \boldsymbol{X}_{t-1} = x_i) \tag{3}$$

The transition matrix contains all the possible transitions from state x_i to state x_j and will have $n \times n = n^2$ entries as shown below:

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{T}_{x_0,x_0} & \boldsymbol{T}_{x_1,x_0} & \dots & \boldsymbol{T}_{x_n,x_0} \\ \boldsymbol{T}_{x_0,x_1} & \boldsymbol{T}_{x_1,x_1} & \dots & \boldsymbol{T}_{x_n,x_1} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{T}_{x_0,x_n} & \boldsymbol{T}_{x_1,x_n} & \dots & \boldsymbol{T}_{x_n,x_n} \end{bmatrix} \tag{4}$$

Here, the robot can take 4 discrete actions of moving forward, backwards, left or right to an adjacent grid locations and these movements are equally likely within the grid. Let $N(x_i)$ be the number of feasible states adjacent to x_i that can be reached by feasible action $\boldsymbol{Move} \in \{MoveUp, MoveDown, MoveLeft, MoveRight\}$. Thus, the transition model for $\boldsymbol{Move}$ action says that the robot is equally likely to end up at feasible locations. Mathematically,

$$\boldsymbol{T}_{x_i,x_j} = \begin{cases} 1/N(x_i), & \text{if } x_j \in feasibleStates(x_i) \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

Since we do not know the initial position of the robot, we will assume a uniform probability distribution over all locations of the grid. That is,

$$P(\boldsymbol{X}_0) = \begin{bmatrix} 1/n & 1/n & \dots & 1/n \end{bmatrix}^T$$

1.4 Observation model:

Observation model consists of evidence, which allows to make inference about the position of robots in the world. It denotes the probability that the sensor returns reading e_j given that we are in state x_i. That is,

$$\boldsymbol{O}_{e_j,x_i} = P(\boldsymbol{E} = e_j | \boldsymbol{X} = x_i) \tag{6}$$

For two independent observations, $\boldsymbol{E}_{rotor}$ and $\boldsymbol{E}_{bump}$, a new visible observation variable is formed with 4 states using the below expression:

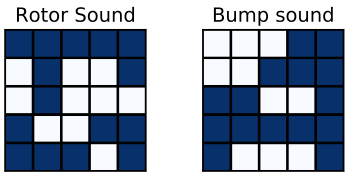
$$\boldsymbol{O}_{e_{rotor,bump},x_i} = P(\boldsymbol{E}_{rotor}, \boldsymbol{E}_{bump} | \boldsymbol{X} = x_i) = P(\boldsymbol{E}_{rotor} | \boldsymbol{X} = x_i) * P(\boldsymbol{E}_{bump} | \boldsymbol{X} = x_i) \tag{7}$$

Here, $e_{rotor,bump} = e_{rotor} \otimes e_{bump} \in (0,1) \otimes (0,1)$ is the observation with $0 \equiv$ *rotor/bump* and $1 \equiv$ *no rotor/no bump*. As such there would be 4 states of observation for the given spatial likelihood of absence or presence of sound as shown in Figure 1.

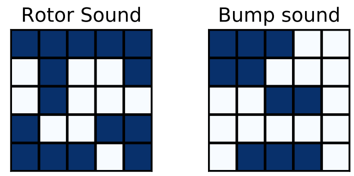
Each case of observation is represented as a diagonal matrix of $\boldsymbol{O}$ of same shape as transition matrix. For case 1, the observation matrix looks as follows:

$$\boldsymbol{O}_{e_{00}} = \begin{bmatrix} O_{e_{00},x_0} & 0 & \dots & 0 \\ 0 & O_{e_{00},x_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & O_{e_{00},x_n} \end{bmatrix} \tag{8}$$

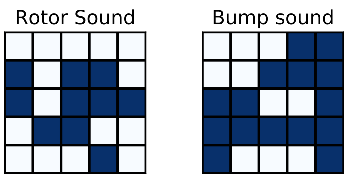
Each diagonal term can be calculated using the probability values for given spatial likelihood. For example, $O_{e_{00},x_0} = 0.1 * 0.9$



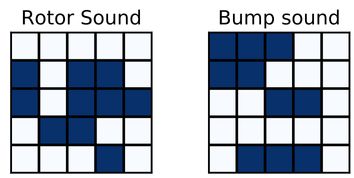
(a) $\boldsymbol{O}_{e_{00}}$=[Rotor, Bump]



(b) $\boldsymbol{O}_{e_{01}}$=[Rotor, NoBump]



(c) $\boldsymbol{O}_{e_{10}}$=[NoRotor, Bump]



(d) $\boldsymbol{O}_{e_{11}}$=[NoRotor, NoBump]

Figure 1: The 4 states of observation for the given spatial likelihood of hearing the presence or absence of a rotor sound (left) or a bump sound (right) over the 5×5 grid. The light squares represent probability 0.9 and dark square represents 0.1

1.5 Graphical model:

The graphical model of the given robot localization problem is shown in Fig. 2

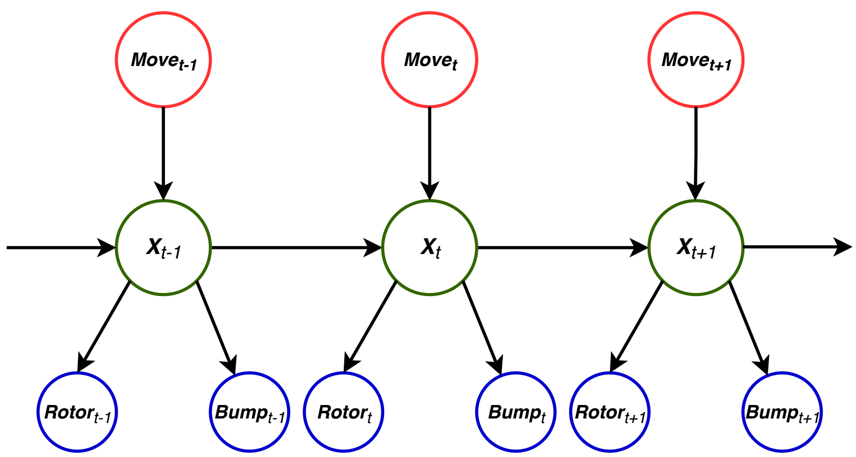


Figure 2: Graphical model where $\boldsymbol{X}_t$ denotes state, $Move_t$ denotes feasible action and $Rotor_t$, $Bump_t$ denotes rotor and bump sound observation at time t

1.6 Conditional independence assumptions:

The following conditional independence assumptions have been considered:

1. The current state depends only on the previous state and not on earlier states. In other words, state provides enough information to make the future conditionally independent of the past. Thus transition model is $\boldsymbol{P}(\boldsymbol{X}_t|\boldsymbol{X}_{0:t-1}) = \boldsymbol{P}(\boldsymbol{X}_t|\boldsymbol{X}_{t-1})$

2. The evidence variables depend only on the state variables $\boldsymbol{X}_t$ from the same time t. Thus observation model is $\boldsymbol{P}(\boldsymbol{E}_t|\boldsymbol{X}_{0:t}, \boldsymbol{E}_{0:t-1}) = \boldsymbol{P}(\boldsymbol{E}_t|\boldsymbol{X}_t)$

1.7 Joint distribution:

For any time t, the complete joint distribution over all the variables is given as:

$$\boldsymbol{P}(\boldsymbol{X}_{0:t}, \boldsymbol{E}_{1:t}) = \boldsymbol{P}(\boldsymbol{X}_0) \prod_{i=1}^{t} \boldsymbol{P}(\boldsymbol{X}_i|\boldsymbol{X}_{i-1}) \boldsymbol{P}(\boldsymbol{E}_i|\boldsymbol{X}_i) \tag{9}$$

1.8 Inference task:

The following inference task have been addressed in this problem:

1. **Filtering:** This is the task of estimating the current state $\boldsymbol{X}_t$ at time t given the sequence of observation $\boldsymbol{E}_{1:t}$. That is , we compute $\boldsymbol{P}(\boldsymbol{X}_t|\boldsymbol{E}_{1:t})$. It can be computed recursively as follows:
$$\boldsymbol{f}_{1:t} = \boldsymbol{\alpha} * \boldsymbol{O}_{e_t} * \boldsymbol{T} * \boldsymbol{f}_{1:t-1} \tag{10}$$
where $\boldsymbol{\alpha}$ is a normalizing constant. In addition to filtering, the **log-likelihood** $\boldsymbol{l}_t$ of the state can be computed by summing the filtered state over x_t and taking log of it.
$$\boldsymbol{l}_{1:t} = \log \sum_{x_t} \boldsymbol{f}_{1:t} \tag{11}$$

2. **Prediction:** This is the task of computing the posterior distribution over the future state, given all evidence to date. That is, we compute $\boldsymbol{P}(\boldsymbol{X}_{t+k}|\boldsymbol{E}_{1:t})$ for some $k > 0$. It can be computed recursively as follows::
$$\boldsymbol{p}_{1:t} = \boldsymbol{\alpha} * \boldsymbol{T} * \boldsymbol{p}_{1:t-1} \tag{12}$$

3. **Smoothing:** This is the task of computing the posterior distribution over a past state, given all evidence up to the present. Thus, we compute $\boldsymbol{P}(\boldsymbol{X}_{t+k}|\boldsymbol{E}_{1:t})$ for some k such that $0 \leq k < t$. The smoothed state $\boldsymbol{s}_t$ can be computed recursively using the forward-backward algorithm as follows:
$$\boldsymbol{s}_t = \alpha * \boldsymbol{f}_t * \boldsymbol{b}_t \tag{13}$$
where $\boldsymbol{b}_t$ is the backward message and can be computed recursively as follows:
$$\boldsymbol{b}_{1:t-1} = \boldsymbol{T} * \boldsymbol{O}_{e_t} * \boldsymbol{b}_{1:t} \tag{14}$$
Here, the backward message at final time step T is initiated with vector of 1s ($\boldsymbol{b}_T = 1$)

4. **Most likely sequence:** This task is used for finding the sequence of states that is most likely to have generated the given sequence of observations. That is, we compute $\arg\max_{x_{1:t}} \boldsymbol{P}(\boldsymbol{X}_{1:t}|\boldsymbol{E}_{1:t})$. The *Viterbi Algorithim* [2, 3] has been used for finding

the most likely sequence. The recursive formulation for this algorithm starts with computing the following:

$$\boldsymbol{m}_{1:t-1} = \max_{x_{1:t}} \boldsymbol{O}_{e_t} * \boldsymbol{T} * \boldsymbol{m}_{1:t} \tag{15}$$

Here $\boldsymbol{m}_T = 1/n$ Once we get all $\boldsymbol{m}$, we can backtrack to find the most likely path $x^*_{1:t}$ using the following relation:

$$x^*_{1:t} = \arg\max_{x_{1:t}} \boldsymbol{O}_{e_t} * \boldsymbol{T} * \boldsymbol{m}_{1:t} \tag{16}$$

1.9 Simulation:

The robot has been simulated for 10 time steps and observations have been noted down, as shown in Fig. 3 below. The initial state of robot is chosen randomly and the subsequent states have been chosen by random number generation using transition probabilities of previous state as probability distribution. A particular case of observation has been picked in a similar manner.

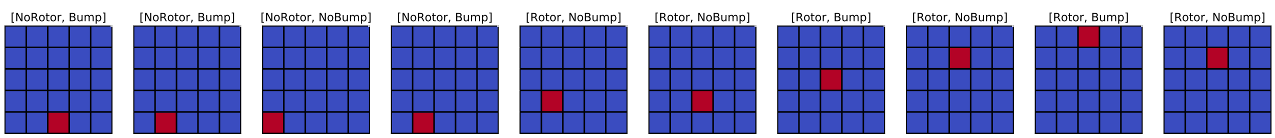


Figure 3: **Ground truth** of robot (red) moving in the 5×5 grid lake (blue) and generating the *Bump* or *NoBump* and *Rotor* or *NoRotor* sound observations for 10 time steps

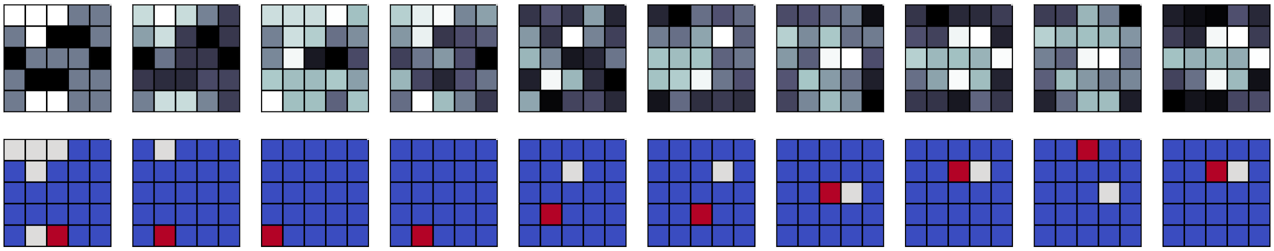
2 Results for inference tasks in 5×5 grid

The location of robot has been estimated using various inference tasks described in section 1.8 and visualized in a grid as shown in Fig. 5 and 6. The most likely locations has also been plotted against ground truth for each of the inference tasks.

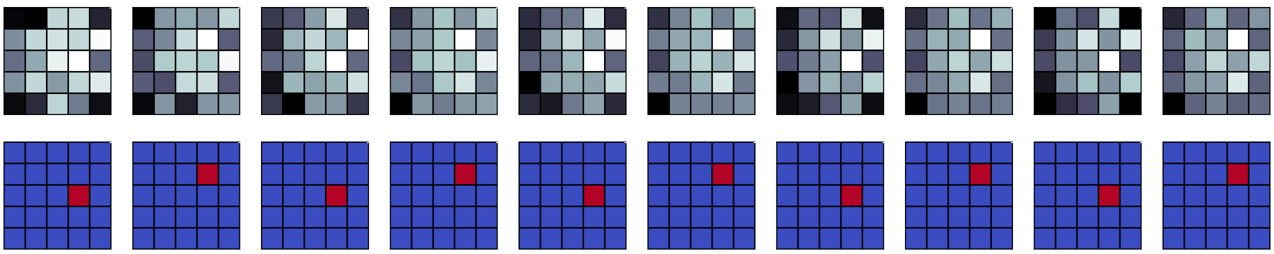
$$\text{True states} = \left\{x_{14} \quad x_9 \quad x_4 \quad x_9 \quad x_8 \quad x_{13} \quad x_{12} \quad x_{11} \quad x_{10} \quad x_{11}\right\}$$
$$\text{Most likely path} = \left\{x_{14} \quad x_9 \quad x_4 \quad x_9 \quad x_8 \quad x_{13} \quad x_{12} \quad x_{11} \quad x_{12} \quad x_{11}\right\}$$
$$\text{Error} = \textstyle\sum_i |(True\ path)_i - (Most\ likely\ path)_i| = 2 \text{ units}$$

Color-map of grid: In Fig. 5, 6, 8 and 9, *bone* color-map [1] has been used. The intensities are in the range $[0, 1]$ and the color scheme looks like Fig. 4 below.

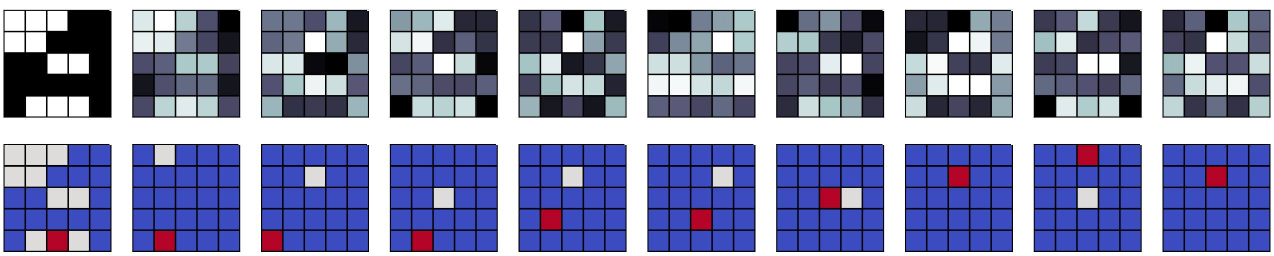
Figure 4: Bone colormap array. Black color represents 0 and white represents 1



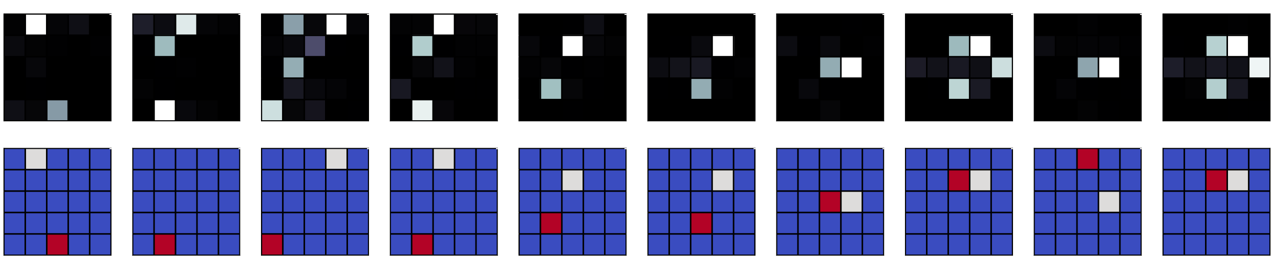
(a) **Filtering:** Estimated log-likelihood over the grid locations at each time step (above) and plot of most likely location against ground truth (below)



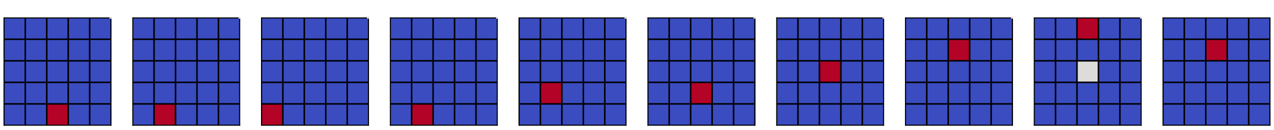
(b) **Prediction:** Estimated log-likelihood over the agents future location ($t > 10$) given all the evidence till now (above) and predicted location for next 10 time steps (below)



(c) **Filtering with only bump observation:** Estimated log-likelihood over the grid locations at each time step (above) and plot of most likely location against ground truth (below)

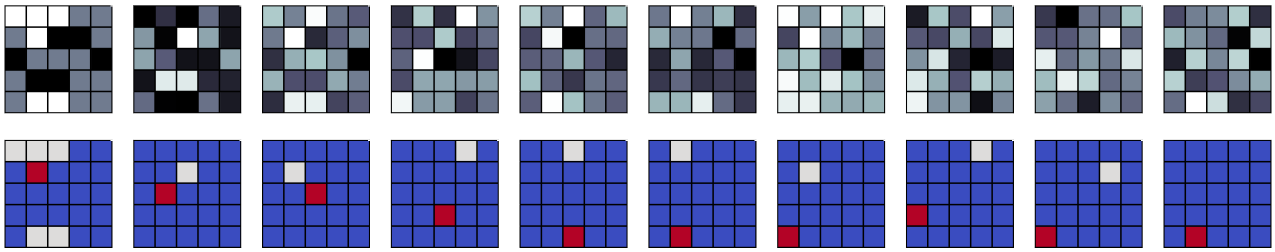


(d) **Smoothing:** Estimating the robots past locations given the evidence up to present time step and plot of most likely location against ground truth (below)

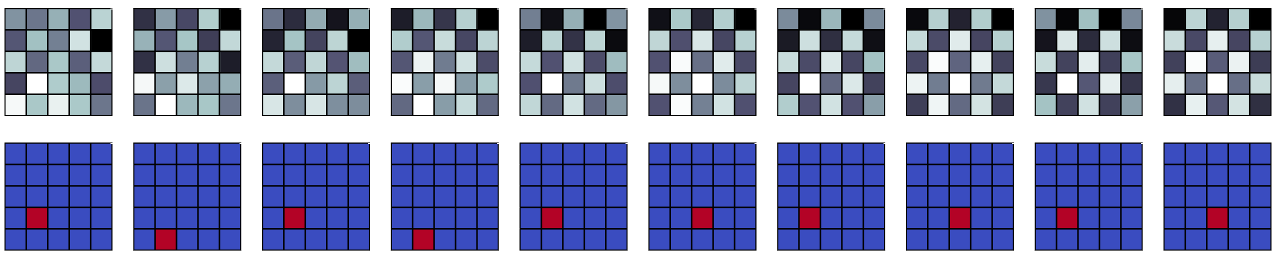


(e) **Most likely path:** $\{x_{14} \quad x_9 \quad x_4 \quad x_9 \quad x_8 \quad x_{13} \quad x_{12} \quad x_{11} \quad x_{12} \quad x_{11}\}$. **Error** $= 2$ units

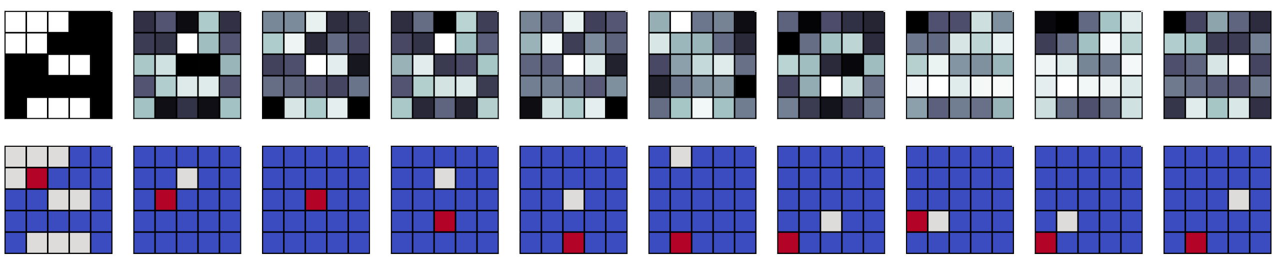
Figure 5: Results of inference tasks for estimating the location of robot in a lake discretized as grid of size 5×5



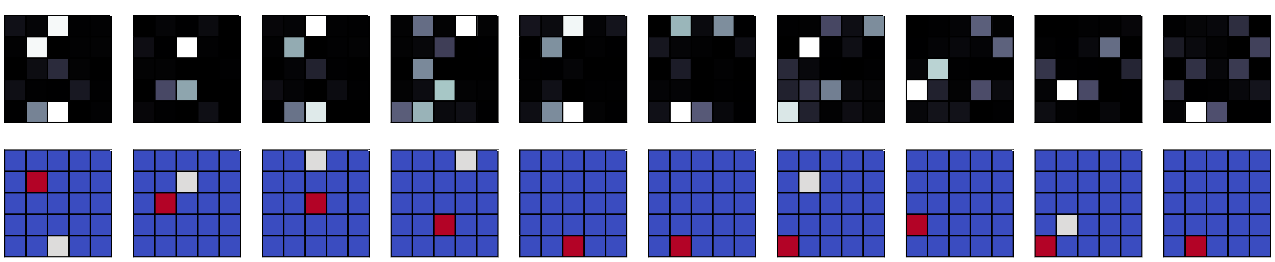
(a) **Filtering:** Estimated log-likelihood over the grid locations at each time step (above) and plot of most likely location against ground truth (below)



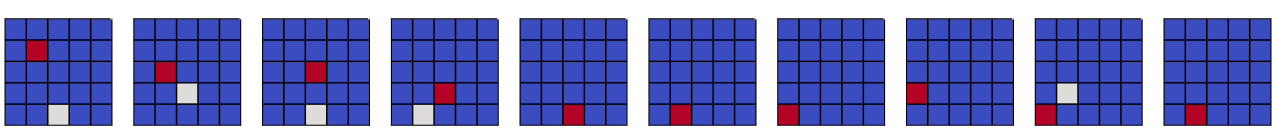
(b) **Prediction:** Estimated log-likelihood over the agents future location ($t > 10$) given all the evidence till now (above) and predicted location for next 10 time steps (below)



(c) **Filtering with only bump observation:** Estimated log-likelihood over the grid locations at each time step (above) and plot of most likely location against ground truth (below)



(d) **Smoothing:** Estimating the robots past locations given the evidence up to present time step and plot of most likely location against ground truth (below)



(e) **Most likely path:** $\{x_{14} \quad x_{13} \quad x_{14} \quad x_9 \quad x_{14} \quad x_9 \quad x_4 \quad x_3 \quad x_8 \quad x_9\}$. **Error** $= 24$ units

Figure 6: Results of inference tasks for estimating the location of robot in a lake discretized as grid of size 5×5

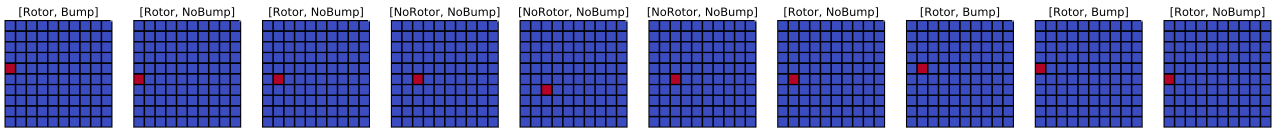
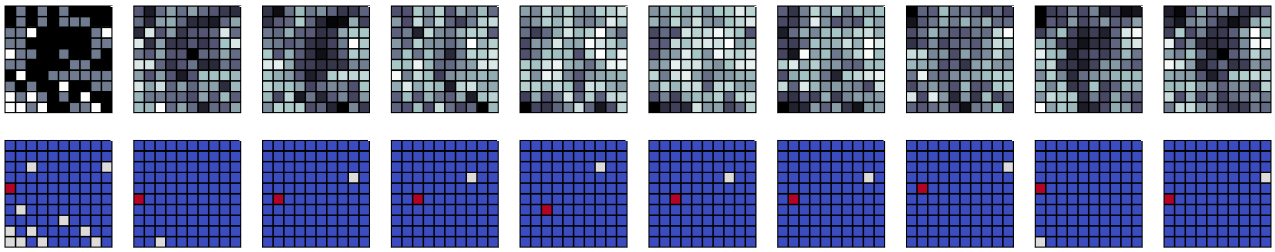
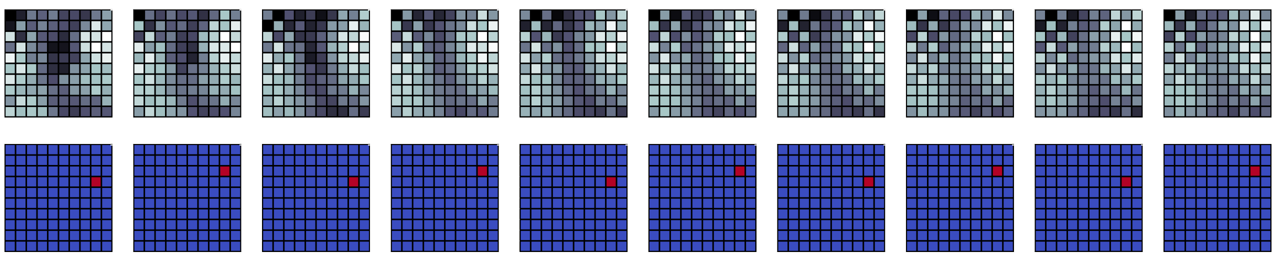


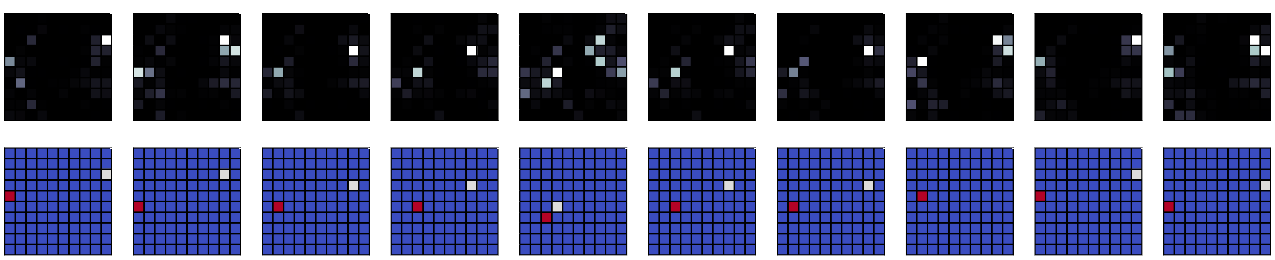
Figure 7: Ground truth of robot in the 10×10 grid



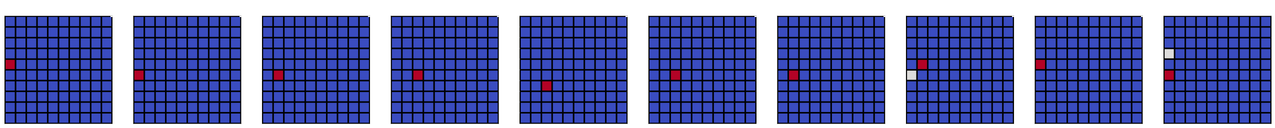
(a) **Filtering:** Estimated log-likelihood over the grid locations at each time step (above) and plot of most likely location against ground truth (below)



(b) **Prediction:** Estimated log-likelihood over the agents future location ($t > 10$) given all the evidence till now (above) and predicted location for next 10 time steps (below)



(c) **Smoothing:** Estimating the robots past locations given the evidence up to present time step and plot of most likely location against ground truth (below)



(d) **Most likely path:** $\left\{x_4 \quad x_5 \quad x_{15} \quad x_{25} \quad x_{26} \quad x_{25} \quad x_{15} \quad x_5 \quad x_4 \quad x_3\right\}$. **Error** $= 11$ units

Figure 8: Ground truth and results of inference tasks for estimating the location of robot in a lake discretized as grid of size 10×10

3 Results for inference tasks in 10×10 grid

The ground truth is shown in Fig. 7. The location of robot has been estimated using various inference tasks and visualized as shown in Fig. 8.

4 Results for inference tasks in 25×25 grid

The location of robot has been estimated using filtering task described in section 1.8 and visualized in a 25×25 grid as shown in Fig. 9. The location has been plotted against ground truth and compared with the most likely path. The error has also been estimated using Manhattan distance metric as given below:

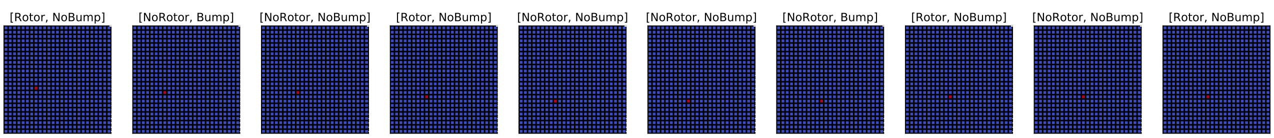
$$\text{True path} = \{x_{189} \quad x_{190} \quad x_{215} \quad x_{216} \quad x_{217} \quad x_{242} \quad x_{267} \quad x_{266} \quad x_{291} \quad x_{266}\}$$

$$\text{Most likely path} = \{x_{148} \quad x_{149} \quad x_{174} \quad x_{199} \quad x_{224} \quad x_{249} \quad x_{248} \quad x_{233} \quad x_{224} \quad x_{199}\}$$

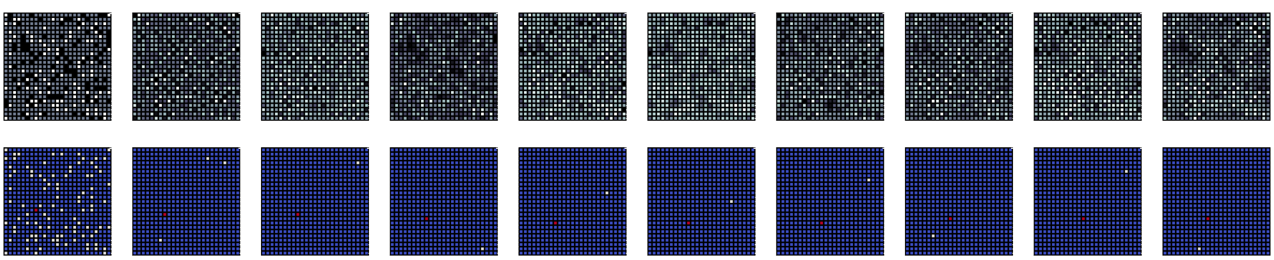
$$\text{Error} = \textstyle\sum_i |(True\ path)_i - (Most\ likely\ path)_i| = 350 \text{ units}$$

Space and time complexity: The space and time complexity of filtering task is given as:

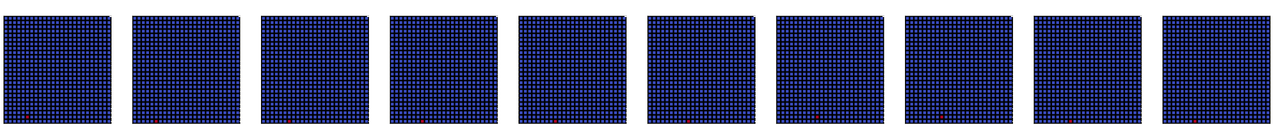
$$Time : \mathcal{O}(n^2 T)$$
$$Space : \mathcal{O}(nT)$$



(a) **Ground truth states:** $\{x_{189} \quad x_{190} \quad x_{215} \quad x_{216} \quad x_{217} \quad x_{242} \quad x_{267} \quad x_{266} \quad x_{291} \quad x_{266}\}$



(b) **Filtering:** Estimated log-likelihood over the grid locations at each time step (above) and most likely location (below)



(c) **Most likely path:** $\{x_{148} \quad x_{149} \quad x_{174} \quad x_{199} \quad x_{224} \quad x_{249} \quad x_{248} \quad x_{233} \quad x_{224} \quad x_{199}\}$

Figure 9: Ground truth and results of filtering tasks for estimating the location of robot in a lake discretized as grid of size 25×25

5 Discussion

The following observations has been noted down from the inferencing tasks for discrete worlds of different sizes.

a) The error (manhattan distance metric) between the ground truth and most likely path increases with increase in size of state space (n). Also, error is observed to have less value when robot remains within a small portion of large grid world.

b) The magnitude of error is dependent upon random spatial distribution of observation likelihood

c) The space complexity increases linearly while time complexity increases quadraticly with size of state space (n).

d) From Fig. 6c, it is clear that the uncertainty in robot's location increases with decrease in number of observations. Thus, more observation variables increases the certainty of localizing the robot.

6 How to run the code

The syntax for running the code is:

python run_prob_inf.py int grid_size bool DoFiltering bool DoPrediction bool DoFilteringSingleObs bool DoSmoothing bool DoMostLikelyPath

For example, to run all the inference tasks on 5×5 grid, use the following syntax:

python run_prob_inf.py 5 1 1 1 1 1

References

[1] Choosing Colormaps in Matplotlib,. `https://matplotlib.org/3.1.0/tutorials/colors/colormaps.html`. Accessed: 23-Feb-2020.

[2] David Barber. *Bayesian reasoning and machine learning*. Cambridge University Press, 2012.

[3] Stuart J Russell and Peter Norvig. *Artificial intelligence: a modern approach*. Malaysia; Pearson Education Limited,, 2016.