5)

(a) $T(n) = 2T(n/2) + n\log(\log n)$

let $n=2^k$

let $S(k)=T(2^k)$

$S(0) = T(1) = d$

$S(k) = 2S(k-1) + 2^k\log(\log 2^k)$

$= 2S(k-1) + 2^k\log(k)$

$= 2\{2S(k-2) + 2^{k-1}\log(k-1)\} + 2^k\log k$

$= 4S(k-2) + 2^k\log(k-1) + 2^k\log k$

$=8S(k-3) + 2^k\log(k-2) + 2^k\log(k-1) + 2^k\log k$

let $T(1) = c$, Therefore $S(0) = d$; //some constant

$S(k) = 2^kS(0) + 2^k\{\log (k*(k-1)*(k-2)*(k-3)*...*1)\}$

$S(k) = d2^k + 2^k\log(k!)$

$\log(k!) = \log 1 +\log 2 +\log 3+...+\log k <= \log k + \log k+\log k +...$ k times

therefore, $\log(k!)<=k\log k$

$S(k) = dn + n\log n\log(\log n)$ € $\mathbf{O(n\log n(\log(\log n)))}$

(b) $T(n) = 5T(n/4) + n\log n$

Work done at i^{th} level = $5^i (n/4^i) (\log n-2i)$

Let $T(1)=d$;

Let $k = \log_4 n$

Therefore work at kth level = $5^k d = 5^{\log_4 n} = n^{\log_4 5}$

Total work done = $\sum_{i=0}^{k-1} (5/4)^i(n\log_2 n- 2ni) + n^{\log_4 5}$

$= ((5/4)^k-1)/((5/4)-1)\ n\log_2 n -2n \sum_{i=0}^{k-1} (5/4)^i\, i + n^{\log_4 5}$ // substituting for $k = \log_4 n$

$<=(n^{\log_4 5}/n)n\log_2 n + n^{\log_4 5}$ //Removing all terms that were negative to get the upper bound

$<=n^{\log_4 5}*\log(n) + n^{\log_4 5}$ € $\mathbf{O((n^{\log_4 5})\log n)}$

(c) $T(n) = 2T(n/2)+n\log^2 n$

let $n=2^k$

let $T(1)=c$ //constant

let $S(k)=T(2^k)$

$S(0) = T(1) = c$

$S(k) = 2S(k-1) + 2^k\log^2(2^k)$

$= 2S(k-1) + 2^k(k)^2$

$= 2[2S(k-2)+2^{k-1}\log^2(2^{k-1})]$

$= 4S(k-2) + 2^k(k^2 + (k-1)^2)$

$= 8S(k-3) + 2^k(k^2 + (k-1)^2 + (k-2)^2)$

$= 2^kS(0) + 2^k(\sum_{i=1}^{k} i^2)$

$= nc + n(k)(k+1)(2k+1)/6$
$= nc + n(2k^3 + k^2 + 2k +1)/6$
$= nc + n(2\log^3 n + \log^2 n +2\log n +1)/6 \in \Theta(n\log^3 n)$

(d) $T(n) = 2T(n/2)+n^2/\log^2 n$

let $n=2^k$
let $T(1)=c$ //constant
let $S(k)=T(2^k)$
$S(0) = T(1) = c$
$S(k) = 2S(k-1) + 2^{2k}/k^2$
$S(k) = 2(2S(k-2)+2^{2k}/4(k-1)^2) + 2^{2k}/k^2$
$= 2^2S(k-2) + 2^{2k}/4(k-1)^2 + 2^{2k}/k^2$
$= 2^3S(k-3) + 2^{2k}/2^3(k-2)^2 + 2^{2k}/2^2(k-1)^2 + 2^{2k}/k^2$
$= 2^kS(0) + 2^{2k} \sum_{i=1}^{k}(1/(2^\wedge(2(k-i))(i^\wedge 2)))$
But $\sum_{i=1}^{k}(1/(2^\wedge(2(k-i))(i^\wedge 2)))<= \sum_{i=1}^{\infty}(1/i^\wedge 2) = \pi^2/6$ -- a constant
$S(k) <= 2^kc + 2^{2k}(\pi^2/6) <= nc+ n^2(\pi^2/6) \in$ **$O(n^2)$**

(e) $T(n) = 4T(n/2)+n^2/\log^2 n$

let $n=2^k$
let $T(1)=c$ //constant
let $S(k)=T(2^k)$
$S(0) = T(1) = c$
$S(k) = 4S(k-1) + 2^{2k}/k^2$
$= 4(4S(k-2)+2^{2k}/4(k-1)^2) + 2^{2k}/k^2$
$= 4^2S(k-2) + 2^{2k}/(k-1)^2 + 2^{2k}/k^2$
$= 4^kS(0) + 2^{2k}(\sum_{i=1}^{k}(1/i^\wedge 2))$
But $\sum_{i=1}^{k}(1/i^\wedge 2) <= \sum_{i=1}^{\infty}(1/i^\wedge 2) = \pi^2/6$
Therefore $S(k) = 4^kc + 4^k(\pi^2/6)$
$S(k) = n^2c + n^2(\pi^2/6) \in$ **$\Theta(n^2)$**

f. $T(n) = 27\ T(n/3) + n^3$

This is of the form $T(n) = a\ T(n/b) + f(n)$. In the above case, $a = 27$, $b = 3$, $f(n) = n^3$.
So, $n^{\log_b a}= n^{\log_3 27}= n^4 = f(n)$.

Since, $f(n) \in \Theta(n^{\log_b a})$, using Master Theorem, we can say $T(n) \in \Theta(n^{\log_b a} \log n)$. Hence, in our case, $T(n) \in \Theta(n^4 \log n)$.

g. $T(n) = 6\,T(n/2) + n^3$

This is of the form $T(n) = a\,T(n/b) + f(n)$. In the above case, $a = 6$, $b = 2$, $f(n) = n^3$.

So, $n^{\log ba} = n^{\log 26} = n^{2.58}$.

Since, $f(n) = n^3$ and $n^{\log_b a} = n^{2.58}$ we can conclude that $f(n) \in \Omega(n^{\log_b a})$. And by using Master Theorem, we can say $T(n) \in \Theta(f(n))$. Hence, in our case,

$T(n) \in \Theta(n^3)$.