SML-A1

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1.) $P(x|\omega_1) \sim N(\mu_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma_0 = I)$

$P(x|\omega_2) \sim N(\mu_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \Sigma = I)$

$P(x|\omega_3) \sim 0.5 N(\mu_3 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \Sigma = I) + 0.5 N(\mu_4 = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}, \Sigma = I)$

given equiprobable categories.

$\therefore P(\omega_1) = \frac{1}{3}, P(\omega_2) = \frac{1}{3}, P(\omega_3) = \frac{1}{3}$

For minimum probability of error classification.

$$P(\text{error}|x) = \min \begin{cases} P(\omega_2|x) + P(\omega_3|x) & \text{if } x \in \omega_1 \\ P(\omega_1|x) + P(\omega_3|x) & \text{if } x \in \omega_2 \\ P(\omega_1|x) + P(\omega_2|x) & \text{if } x \in \omega_3 \end{cases}$$

Now, $P(\omega_i|x) \equiv P(x|\omega_i) \cdot P(\omega_i) = P(x|\omega_i) \cdot \frac{1}{3}$.

$x = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}$

$$P(x|\omega_i) = \frac{1}{(2\pi)} \cdot \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu_i)^T \cdot \Sigma^{-1} \cdot (x-\mu_i)\right\}$$

$$\Rightarrow P(x|\omega_1) = \frac{1}{2\pi} \cdot \exp\left\{-\frac{1}{2} \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}\right\}$$

$$= \frac{1}{2\pi} \exp\{-0.09\} = 0.145456.$$

$$P(x|\omega_2) = \frac{1}{(2\pi)} \cdot \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2} \cdot \begin{bmatrix} -0.7 & 0 \\ 0 & -0.7 \end{bmatrix} \begin{bmatrix} -0.7 \\ -0.7 \end{bmatrix}\right\}$$

$$= \frac{1}{(2\pi)} \cdot \frac{1}{1} \exp\{-0.49\} = 0.0975025.$$

$$P(x|\omega_3) = \frac{0.5}{2\pi} \cdot \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2} \cdot \begin{bmatrix} -0.2 & 0 \\ 0 & -0.2 \end{bmatrix} \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix}\right\}$$

$$+ \frac{0.5}{2\pi} \cdot \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2} \begin{bmatrix} 0.8 & 0 \\ 0 & -0.2 \end{bmatrix} \begin{bmatrix} 0.8 \\ -0.2 \end{bmatrix}\right\}$$

$\Rightarrow P(x|\omega_3) = \frac{1}{4\pi} \exp\{-0.04\} + \frac{1}{4\pi} \exp\{-0.34\}$.

$= 0.133098$

Now putting $P(x|\omega_1)$, $P(x|\omega_2)$ and $P(x|\omega_3)$ in $P(\text{error}|x)$ we get.

$$P(\text{error}|x) = \min \begin{cases} \frac{1}{3}(0.2306005) & \text{if } x \in \omega_1 \\ \frac{1}{3}(0.278554) & \text{if } x \in \omega_2 \\ \frac{1}{3}(0.2429585) & \text{if } x \in \omega_3 \end{cases}$$

$$\Rightarrow P(\text{error}|x) = \min \begin{cases} 0.076866833 & \text{if } x \in \omega_1 \\ 0.09285133 & \text{if } x \in \omega_2 \\ 0.08098616 & \text{if } x \in \omega_3 \end{cases}$$

$\therefore$ x belongs to class ω_1.

2.) Given $P(x|\omega_i) = \frac{1}{2b} e^{-\frac{|x-a_i|}{b}}$ for $i = 1,2$

$a_1 = 0, a_2 = 1$

$b = 1$

and equally probable classes.

$\therefore P(\omega_1) = \frac{1}{2}$, $P(\omega_2) = \frac{1}{2}$.

So $P(x|\omega_1) = \frac{1}{2} e^{-|x|}$, $P(x|\omega_2) = \frac{1}{2} e^{-|x-1|}$

(a) Likelihood ratio : $\frac{P(x|\omega_1)}{P(x|\omega_2)}$.

$$\Rightarrow \frac{P(x|\omega_1)}{P(x|\omega_2)} = \frac{\frac{1}{2} e^{-|x|}}{\frac{1}{2} e^{-|x-1|}} = e^{-|x| + |x-1|} = e^{-|x| + |x-1|}.$$

Now plot of $e^{-|x| + |x-1|}$ against x.

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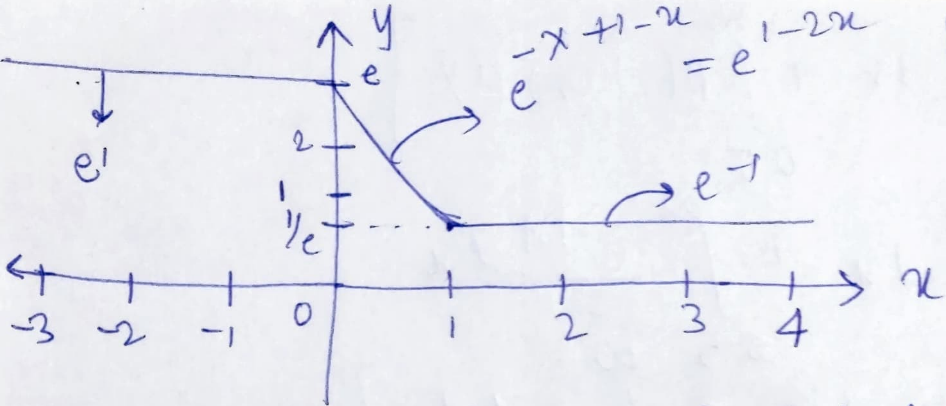


b) Optimal minimum error rate decision boundary is $P(\omega_1|x) = P(\omega_2|x)$

$\Rightarrow P(x|\omega_1) \cdot P(\omega_1) = P(x|\omega_2) \cdot P(\omega_2)$

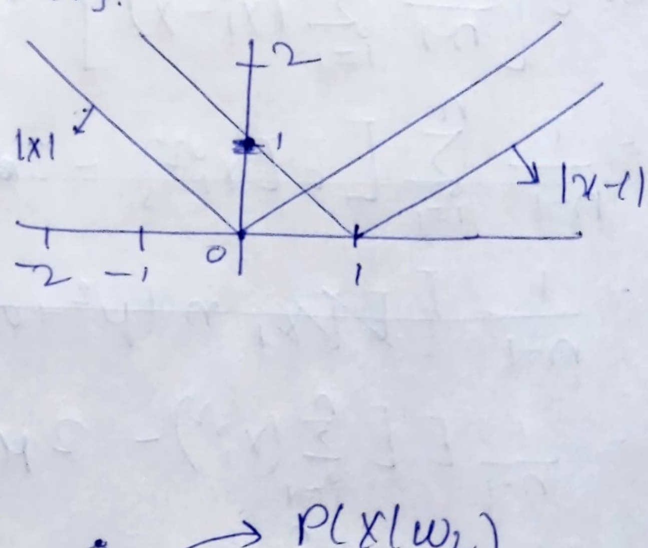
$\Rightarrow P(x|\omega_1) = P(x|\omega_2)$. [$\because$ equally probable ω_1 & ω_2]

$\Rightarrow \frac{1}{2b} \exp\{-|x|\} = \frac{1}{2b} \exp\{-|x-1|\}$

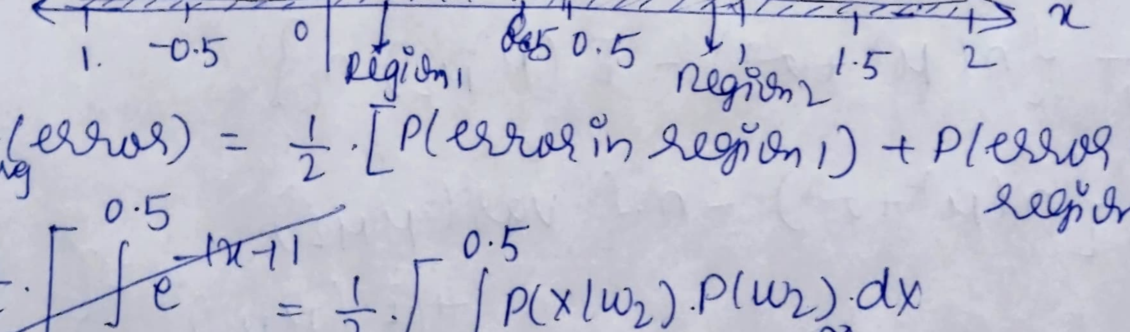
$\Rightarrow \exp\{-|x|\} = \exp\{-|x-1|\}$. $\Rightarrow |x| = |x-1|$.

$\Rightarrow x = 1-x \Rightarrow x = \frac{1}{2}$

$\therefore x = \frac{1}{2}$ is decision boundary.



c) Plotting $P(x|\omega_1)$, $P(x|\omega_2)$.



$\therefore P_{avg}(\text{error}) = \frac{1}{2} \cdot [P(\text{error in region 1}) + P(\text{error in region 2})]$

$= \frac{1}{2} \cdot \cancel{\left[\int_{-\infty}^{0.5} e^{-|x-1|} \right.}$ $= \frac{1}{2} \cdot \left[\int_{-\infty}^{0.5} P(x|\omega_2) \cdot P(\omega_2)\, dx + \int_{0.5}^{\infty} P(x|\omega_1) \cdot P(\omega_1)\, dx \right.$

$$= \frac{1}{4}\left[\int_{-\infty}^{0.5} p(x|\omega_2).dx + \int_{0.5}^{\infty} p(x|\omega_1).dx \right]$$

$$= \frac{1}{4}\left[\int_{-\infty}^{0.5} \frac{e^{-|x-1|}}{2}.dx + \int_{0.5}^{\infty} \frac{e^{-|x|}}{2}.dx \right]$$

$$= \frac{1}{4}\cdot\frac{1}{2}\left[\int_{-\infty}^{0.5} e^{x-1}.dx + \int_{0.5}^{\infty} e^{-x}.dx \right].$$

$$= \frac{1}{8}\left[\left[e^{x-1}\right]_{-\infty}^{0.5} + (-1)\left[e^{-x}\right]_{0.5}^{\infty} \right] = \frac{1}{4} \times 0.60653065$$

$$= 0.151632664$$

3.) Given estimate of variance $\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$

$\bar{x}$ is μ_{MLE}.

So $E\left[\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2\right] = \frac{1}{n-1} E\left[\sum_{i=1}^{n} x_i^2 - 2\bar{x}.x_i + \bar{x}^2\right]$

$$\cancel{= \frac{1}{n-1}\sum_{i=1}^{n}\left[E[x_i^2] - 2\bar{x}.E[x_i] + E[\bar{x}^2]\right]}$$

$$\cancel{= \frac{1}{n-1}\{E[x_i\ n(\mu^2+\sigma^2) - 2n\ x}$$

$$= \frac{1}{n-1} E\left[\sum_{i=1}^{n}(x_i^2) - 2\mu_{ML}^2.n + n\mu_{ML}^2\right]$$

$$= \frac{1}{n-1}.E\left[\sum_{i=1}^{n}(x_i^2) - \mu_{ML}^2.n\right].$$

$$= \frac{1}{n-1}.\{n(\mu^2+\sigma^2) - E.(\mu^2_{ML}).n\}$$

$$= \frac{1}{n-1}\{n(\mu^2+\sigma^2) - n.\{var(\mu_{ML}) + [E.(\mu_{ML})]^2\}\}$$

$$= \frac{1}{n-1}.\{n\mu^2 + n\sigma^2 - n.(var(\sum_{i=1}^{n}\frac{1}{n}.x_i)) - n.\mu^2\}.$$

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$$= \frac{1}{n-1}\left\{n\sigma^2 - \frac{n}{n^2}\cdot\sum_{i=1}^{n} var(x_i)\right\}$$

$$= \frac{1}{n-1}\left\{n\sigma^2 - \frac{n\cdot\sigma^2}{n^2}\cdot n\right\} = \frac{1}{n-1}\cdot\left\{n\sigma^2 - \sigma^2\right\} = \sigma^2$$

$\therefore E\left[\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2\right] = \sigma^2$ $\therefore$ The given estimate of variance is unbiased.

4.) Given $E[X] = \mu$, $Var[X] = \sigma^2$

a) $E[aX+b] = E[aX] + E[b] = a\,E[X] + b = a\mu + b$

b) $Var(aX+b) = a^2 Var(X) = a^2\sigma^2$.

5.) Given $P(c|x) = 1 - P(error|x)$.

Now, $P(c) = \int_x P(c,x)\cdot dx = \int_x P(c|x)\cdot P(x)\cdot dx$

$\Rightarrow P(c) = \int_x (1 - P(error|x))\cdot P(x)\cdot dx = \int_x P(x)\cdot dx - \int_x P(error|x)\cdot P(x)\cdot dx$

$\Rightarrow P(c) = 1 - \int_x P(error|x)\cdot P(x)\cdot dx$ ......(1)

Now in Cauchy distribution example of class.

$$P(x) = \frac{1}{\pi b}\cdot\frac{1}{1+\left(\frac{x-a}{b}\right)^2}$$

Likelihood: $P(x|w_1) = \frac{1}{\pi b\left(1+\left(\frac{x-a_1}{b}\right)^2\right)}$, $P(x|w_2) = \frac{1}{\pi b\cdot\left(1+\left(\frac{x-a_2}{b}\right)^2\right)}$

Priors: $P(w_1) = P(w_2) = \frac{1}{2}$.

Now for zero-one-loss case we've decision boundary $P(w_1|x) = P(w_2|x)$

$\Rightarrow P(x|\omega_1) \cdot P(\omega_1) = P(x|\omega_2) \cdot P(\omega_2)$

$$\Rightarrow \frac{1}{\pi b} \cdot \frac{1}{1+\left(\frac{x-a_1}{b}\right)^2} = \frac{1}{\pi b} \cdot \frac{1}{1+\left(\frac{x-a_2}{b}\right)^2} \Rightarrow (x-a_1)^2 = (x-a_2)^2$$

$$\Rightarrow (x-a_1)^2 - (x-a_2)^2 \Rightarrow (x-a_1+x-a_2)(x-a_1-x+a_2) = 0$$

$$\Rightarrow x = \frac{a_1+a_2}{2}.$$

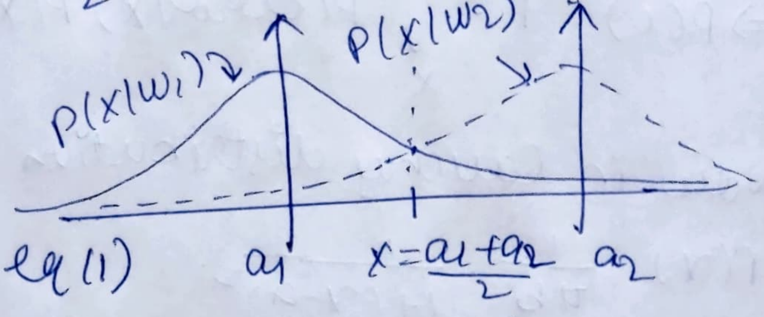
$$\therefore \int_X P(\text{error}|x) \cdot P(x) \cdot dx = \int_X \min[P(\omega_1|x), P(\omega_2|x)] \cdot P(x) \cdot dx$$

$$= \int_{-\infty}^{\frac{a_1+a_2}{2}} P(\omega_2|x) \cdot P(x)\, dx + \int_{\frac{a_1+a_2}{2}}^{\infty} P(\omega_1|x) \cdot P(x) \cdot dx$$

$$= \int_{-\infty}^{\frac{a_1+a_2}{2}} P(x|\omega_2) \cdot P(\omega_2) \cdot dx + \int_{\frac{a_1+a_2}{2}}^{\infty} P(x|\omega_1) \cdot P(\omega_1) \cdot dx$$

$$= \frac{1}{2}\left\{ \int_{-\infty}^{\frac{a_1+a_2}{2}} \frac{1}{\pi b \cdot \left[1+\left(\frac{x-a_2}{b}\right)^2\right]} \cdot dx + \int_{\frac{a_1+a_2}{2}}^{\infty} \frac{1}{\pi b\left[1+\left(\frac{x-a_1}{b}\right)^2\right]} \cdot dx \right.$$

$$= \frac{1}{2} - \frac{1}{\pi} \tan^{-1}\left(\frac{a_2-a_1}{2b}\right). \quad \ldots (2)$$



Now plugging (2) into eq (1)

we get $P(c) = 1 - \left\{ \frac{1}{2} - \frac{1}{\pi} \tan^{-1}\left(\frac{a_2-a_1}{2b}\right) \right\}$

$$\Rightarrow P(c) = \frac{1}{2} + \frac{1}{\pi} \cdot \tan^{-1}\left(\frac{a_2-a_1}{2b}\right).$$

I have directly used results discussed in class in this question.