Optimal Execution under the Almgren-Chriss Framework

Deterministic and Stochastic Linear-Quadratic Control Formulations

Author: Deepansh Makkar

Abstract

We study the classical Almgren Chriss optimal execution problem under deterministic and stochastic price dynamics. The execution task is formulated as a finite-horizon linear-quadratic control problem with linear temporary market impact and quadratic inventory risk. In the deterministic setting, exact liquidation is enforced through a global constraint, resulting in a convex quadratic program characterized by Karush-Kuhn-Tucker conditions. The discrete-time solution is analyzed and its convergence to the continuous-time closed-form trajectory is established.

The framework is extended to a stochastic setting in which prices follow a diffusion process. The resulting problem is solved using dynamic programming, yielding a quadratic value function and a linear state-feedback execution policy defined by a Riccati recursion. Deterministic open-loop and stochastic closed-loop execution are compared analytically and through Monte Carlo simulation, with the stochastic strategy exhibiting lower execution variance under identical model parameters.

1. Introduction

Large order execution requires balancing temporary market impact against inventory risk over a finite horizon. The Almgren Chriss model captures this trade off through linear impact costs and a quadratic inventory penalty.

We analyze deterministic and stochastic versions of the model in discrete time. The deterministic formulation specifies an open loop trading schedule fixed at the initial time. The stochastic formulation yields a closed loop execution policy in which trading rates depend on current inventory. Both settings are treated within a linear quadratic control framework.

2. Deterministic Almgren-Chriss Model

2.1 Time Discretization and Inventory Dynamics

We consider a fixed execution horizon $T > 0$ and divide it into N uniform trading periods. The time increment between consecutive decision points is defined as $\Delta t = \frac{T}{N}$. Let x_k denote the remaining inventory at decision time $t_k = k\Delta t$, with initial condition $x_0 = X_0 > 0$.

Trading occurs at a nonnegative rate v_k during the time interval from t_k to t_{k+1}. Inventory evolves according to the linear dynamics

$$x_{k+1} = x_k - v_k \Delta t \qquad k = 0, \dots, N-1$$

Exact liquidation is imposed over the horizon.

$$x_N = 0$$

This condition is equivalently expressed as the global constraint

$$\sum_{k=0}^{N-1} v_k \Delta t = X_0$$

This formulation treats the trading rates $\{v_k\}_{k=0}^{N-1}$ as the control variables of the problem.

2.2 Cost Functional

The deterministic Almgren-Chriss objective balances execution cost against inventory risk. Temporary market impact is assumed linear in the trading rate, leading to a quadratic cost in v_k. Inventory risk is penalized quadratically through the variance of execution proceeds induced by price uncertainty.

The discrete-time objective is

$$J(v) = \eta \sum_{k=0}^{N-1} v_k^2 \, \Delta t + \lambda \sigma^2 \sum_{k=0}^{N-1} x_k^2 \, \Delta t$$

The parameters have the following interpretation.

- $\eta > 0$ is the temporary market impact coefficient.
- $\sigma > 0$ is the annualized price volatility.
- $\lambda > 0$ controls risk aversion.

The first term penalizes aggressive trading, while the second term penalizes holding inventory in the presence of price uncertainty. Under deterministic execution, the expectation operator is redundant and the objective reduces to a purely quadratic functional of the control sequence.

2.3 Operator Representation

Define the trading-rate vector

$$v = (v_0, \dots, v_{N-1})^\top \in \mathbb{R}^N$$

and the inventory vector

$$x = (x_0, \dots, x_{N-1})^\top \in \mathbb{R}^N.$$

The inventory dynamics can be written compactly as

$$x = X_0 1 - Lv$$

where $1 \in \mathbb{R}^N$ denotes the vector of ones. The matrix $L \in \mathbb{R}^{N \times N}$ is lower triangular with entries

$$L_{ij} = \Delta t \, \mathbb{1}_{\{i \geq j\}}.$$

Substituting this representation into the objective yields a quadratic form in v, laying the groundwork for a convex optimization formulation.

3. Quadratic Program and KKT Characterization

3.1 Quadratic Formulation

Substituting the operator representation $x = X_0 1 - Lv$ into the objective functional yields

$$J(v) = \eta \Delta t \, v^\top v + \lambda \sigma^2 \Delta t \, (X_0 1 - Lv)^\top (X_0 1 - Lv)$$

Up to an additive constant independent of v, the objective can be written as a purely quadratic form.

$$J(v) = \tfrac{1}{2} v^\top A v$$

The matrix A is given by

$$A = 2\eta \Delta t \, I + 2\lambda \sigma^2 \Delta t \, L^\top L$$

The matrix A is symmetric and positive definite, since $\eta > 0$ and $L^\top L$ is positive semidefinite. Consequently, the objective is strictly convex.

3.2 Exact Liquidation Constraint

Exact liquidation is enforced through the linear constraint

$$\sum_{k=0}^{N-1} v_k \Delta t = X_0$$

In vector form, this becomes

$$Cv = X_0$$

where

$$C = \Delta t \, 1^\top \in \mathbb{R}^{1\times N}.$$

The deterministic execution problem is therefore a convex quadratic program with a single equality constraint.

3.3 Karush-Kuhn-Tucker System

Introducing a Lagrange multiplier $\mu \in \mathbb{R}$ for the liquidation constraint, the Lagrangian is

$$\mathcal{L}(v, \mu) = \tfrac{1}{2} v^\top A v + \mu(Cv - X_0)$$

First-order optimality conditions yield the KKT system

$$\begin{bmatrix} A & C^\top \\ C & 0 \end{bmatrix} \begin{bmatrix} v \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ X_0 \end{bmatrix}$$

Since A is positive definite and the constraint is full rank, this system admits a unique solution. The resulting trading-rate vector v^* defines the optimal deterministic execution strategy.

3.4 Structural Properties

Convexity and the form of the constraint imply several structural invariants of the optimal solution: - exact liquidation over the horizon, - non-negative trading rates (under standard parameter regimes), - monotone, non-increasing inventory, - non-negative inventory at all times.

4. Baseline Trajectories and TWAP Benchmark

4.1 Optimal Deterministic Execution Path

Solving the KKT system yields the optimal trading-rate sequence

$$v^* = (v_0^*, \dots, v_{N-1}^*)$$

from which the corresponding inventory trajectory is recovered via

$$x_k^* = X_0 - \sum_{j=0}^{k-1} v_j^* \Delta t \qquad k = 1, \dots, N$$

The optimal inventory path is smooth and strictly decreasing. For moderate risk aversion, execution is front-loaded relative to uniform trading, reflecting the incentive to reduce exposure to price risk early in the horizon.

4.2 Time-Weighted Average Price (TWAP)

As a benchmark, consider a time-weighted average price strategy, which executes inventory uniformly over time.

$$v_k^{\text{TWAP}} = \frac{X_0}{T} \qquad k = 0, \dots, N-1$$

The associated inventory trajectory is linear.

$$x_k^{\text{TWAP}} = X_0 \left(1 - \frac{k\Delta t}{T}\right)$$

TWAP ignores both market impact and price risk, and therefore serves as a natural baseline for evaluating the structure induced by optimal control.

4.3 Inventory Comparison

Relative to TWAP, the deterministic Almgren-Chriss strategy executes more aggressively at early times. This deviation is entirely driven by the quadratic inventory risk term and vanishes in the limit $\lambda \to 0$, where the objective reduces to pure impact minimization.

5. Cost-Risk Decomposition and Scaling Behavior

5.1 Decomposition of the Objective

The deterministic Almgren-Chriss objective decomposes naturally into two components.

$$\text{Temporary impact cost} = \eta \int_0^T v(t)^2 \, dt$$

$$\text{Inventory risk} = \sigma^2 \int_0^T x(t)^2 \, dt$$

In discrete time, these quantities are approximated by

$$\eta \sum_{k=0}^{N-1} v_k^2 \, \Delta t$$

$$\sigma^2 \sum_{k=0}^{N-1} x_k^2 \, \Delta t$$

The risk-aversion parameter λ controls the relative weight assigned to inventory risk.

5.2 One-Parameter Cost-Risk Ray

Under exact liquidation, varying λ generates a one-parameter family of optimal solutions. Empirically, the resulting pairs

$$\left(\sigma^2 \int_0^T x(t)^2\,dt\;,\;\eta \int_0^T v(t)^2\,dt\right)$$

lie on a single ray in the cost-risk plane.

This behavior reflects scale invariance of constrained linear-quadratic control problems: changing λ rescales the relative emphasis on risk and impact, but does not generate a full efficient frontier as in unconstrained portfolio optimization.

6. Continuous-Time Limit and Euler-Lagrange Formulation

6.1 Continuous-Time Optimal Trajectory

As the discretization is refined and $\Delta t \to 0$, the deterministic Almgren-Chriss solution converges to a continuous-time trajectory. In this limit, the inventory process $ x(t) $ satisfies the second-order ordinary differential equation obtained from the Euler-Lagrange condition,

$$\ddot{x}(t) = \kappa^2 x(t)$$

$$\kappa = \sqrt{\frac{\lambda\sigma^2}{\eta}}$$

with boundary conditions

$$x(0) = X_0$$

and

$$x(T) = 0$$

The unique solution is

$$x(t) = X_0 \frac{\sinh\big(\kappa(T-t)\big)}{\sinh(\kappa T)}$$

This closed-form expression explains the characteristic exponential curvature observed in the discrete optimal trajectories.

6.2 Euler-Lagrange Discretization

An alternative route to the discrete solution is obtained by discretizing the continuous-time Euler-Lagrange equation directly. This leads to a second-order finite-difference scheme for the interior inventory points,

$$x_{k+1} - 2x_k + x_{k-1} = \kappa^2 \Delta t^2 \, x_k \qquad k = 1, \dots, N-1$$

with boundary conditions $x_0 = X_0$ and $x_N = 0$.

This formulation yields a tridiagonal linear system for the inventory path, which is solved efficiently and admits a clear interpretation as a consistent numerical approximation of the continuous-time problem.

6.3 Discretization Trade-offs

While both the control discretization and the Euler-Lagrange discretization converge to the same continuous-time limit, they are not pointwise equivalent at finite Δt. The control discretization enforces exact liquidation through a global constraint, whereas the Euler-Lagrange scheme enforces it locally via boundary conditions.

7. Convergence Analysis and Model Limitations

7.1 Convergence to the Continuous-Time Solution

To verify consistency of the Euler-Lagrange discretization, we compare the discrete inventory trajectory against the continuous-time closed-form solution under successive grid refinements. For a fixed set of model parameters, the relative ℓ^2 error

$$\frac{\|x^{\Delta t} - x\|_2}{\|x\|_2}$$

is computed for increasing values of N.

The observed decay of the error is second order in Δt, confirming that the finite-difference scheme is a consistent and convergent approximation of the continuous-time Almgren-Chriss trajectory.

7.2 Model Assumptions

- linear temporary market impact,
- absence of permanent market impact,
- no price drift in the unaffected price,
- deterministic execution without adaptation to realized prices.

These assumptions are appropriate for short execution horizons, highly liquid assets, and moderate order sizes, where informational effects and strategic interaction can be neglected.

7.3 Breakdown Regimes

The deterministic model becomes inadequate when: - market impact is nonlinear or asymmetric, - execution decisions must adapt to realized price paths, - permanent impact materially affects expected costs, - uncertainty plays a central role in execution timing.

In such regimes, execution must be treated as a stochastic control problem, motivating the stochastic extension developed in the next section.

8. Stochastic Almgren-Chriss Formulation

8.1 Price Dynamics

We now extend the execution problem to a stochastic environment in which the unaffected asset price evolves randomly over time. The price process is modeled as a diffusion.

$$dS_t = \mu \, dt + \sigma \, dW_t$$

Here μ is the drift, σ is the volatility, and W_t is a standard Brownian motion.

In discrete time, this becomes

$$S_{k+1} = S_k + \sigma\sqrt{\Delta t}\, \xi_k$$

where

$$\xi_k \sim \mathcal{N}(0,1).$$

While the drift affects expected execution proceeds, it does not influence the optimal execution policy under a quadratic mean-variance objective. Consequently, the control problem depends only on inventory dynamics and volatility.

8.2 Execution Costs and Objective

Trading at rate v_k over interval k incurs temporary market impact, shifting the execution price to

$$\tilde{S}_k = S_k - \eta v_k$$

Total execution performance is measured via implementation shortfall. Under the assumed linear impact model, the resulting mean-variance objective reduces to

$$J = \mathbb{E}\left[\sum_{k=0}^{N-1} \eta v_k^2 \, \Delta t\right] + \lambda \mathbb{E}\left[\sum_{k=0}^{N-1} \sigma^2 x_k^2 \, \Delta t\right]$$

The first term penalizes aggressive trading through impact costs, while the second penalizes exposure to price uncertainty through inventory risk.

8.3 Inventory Dynamics

Inventory evolves deterministically conditional on the chosen control.

$$x_{k+1} = x_k - v_k \Delta t$$

with initial condition

$$x_0 = X_0.$$

Unlike the deterministic formulation, exact liquidation at the terminal time is not imposed directly as a hard constraint. Instead, it is enforced through a large quadratic terminal penalty on remaining inventory, as described below.

9. Dynamic Programming and Riccati Recursion

9.1 Value Function Formulation

Let $V_k(x)$ denote the optimal value function at time step k given inventory level x. The stochastic execution problem can be written as

$$V_k(x) = \min_{\{v_j\}_{j=k}^{N-1}} \mathbb{E}_k \left[\sum_{j=k}^{N-1} \left(\eta v_j^2 + \lambda \sigma^2 x_j^2 \right) \Delta t \right]$$

subject to the inventory dynamics

$$x_{j+1} = x_j - v_j \Delta t.$$

Terminal liquidation is enforced through the quadratic penalty

$$V_N(x) = \phi x^2$$

with $\phi \gg 1$, making residual inventory at the horizon prohibitively costly.

9.2 Quadratic Value Function Ansatz

By convexity and the linear structure of the dynamics, the value function admits a quadratic form,

$$V_k(x) = a_k x^2$$

with

$$a_k \geq 0.$$

The sequence of coefficients $\{a_k\}_{k=0}^{N}$ is determined by backward induction. This ansatz leads directly to a linear optimal feedback control and a discrete Riccati equation.

9.3 Bellman Equation

The Bellman recursion is

$$V_k(x) = \min_{v_k} \mathbb{E}_k[\eta v_k^2 \Delta t + \lambda\sigma^2 x^2 \Delta t + V_{k+1}(x - v_k \Delta t)]$$

with terminal condition $V_N(x) = \phi x^2$.

Substituting the quadratic ansatz into the Bellman equation yields a one-dimensional minimization problem at each time step, whose solution characterizes the optimal control.

10. Optimal Feedback Control and Execution Dynamics

10.1 Linear State-Feedback Strategy

Solving the Bellman recursion yields a linear state-feedback control of the form

$$v_k^* = \kappa_k \, x_k$$

where the feedback gains $\{\kappa_k\}_{k=0}^{N-1}$ are given by

$$\kappa_k = \frac{a_{k+1}\Delta t}{\eta \Delta t + a_{k+1}\Delta t^2}$$

The coefficients a_k satisfy the Riccati recursion

$$a_k = a_{k+1} - \frac{a_{k+1}^2 \Delta t^2}{\eta \Delta t + a_{k+1}\Delta t^2} + \lambda\sigma^2 \Delta t$$

with terminal condition

$$a_N = \phi.$$

This recursion fully characterizes the stochastic optimal execution strategy.

10.2 Closed-Loop Execution

Unlike the deterministic solution, which is an open-loop strategy fixed at time zero, the stochastic solution is closed-loop. Trading rates adapt dynamically to the current inventory level, making execution robust to deviations induced by stochastic price paths.

Conditional on the feedback control, inventory evolves deterministically, while prices evolve stochastically. Uncertainty therefore enters the problem through the backward recursion rather than through explicit price tracking in the control.

10.3 Role of Drift

Although the price process may include a drift term, it contributes only an additive constant to expected execution proceeds under a mean-variance objective. As a result, drift does not appear in the Riccati recursion and does not affect the optimal feedback gains.

11. Comparison with Deterministic Execution

11.1 Open-Loop versus Closed-Loop Strategies

The deterministic Almgren-Chriss solution prescribes an open-loop execution schedule, fully specified at the initial time and independent of realized price paths. In contrast, the stochastic formulation yields a closed-loop strategy in which trading rates respond continuously to the current inventory through state feedback.

Although both formulations share the same quadratic cost structure, this distinction has important implications for risk management. Open-loop execution cannot react to deviations induced by stochastic price evolution, whereas closed-loop execution continuously rebalances the remaining inventory.

11.2 Expected Inventory Trajectories

Monte Carlo simulation shows that the expected inventory trajectory under stochastic execution coincides with the deterministic Almgren-Chriss trajectory when inventory dynamics are linear and noise enters only through prices.

Despite identical expected paths, the stochastic strategy differs fundamentally at the pathwise level, where feedback enables adaptive responses to realized uncertainty.

11.3 Low-Risk Regime

For baseline parameters where

$$\kappa = \sqrt{\frac{\lambda \sigma^2}{\eta}}$$

is small, both deterministic and stochastic strategies approach a near-TWAP schedule. In this regime, temporary impact dominates inventory risk, and curvature in the optimal trajectory is weak.

As risk aversion increases, curvature becomes more pronounced, with both formulations front-loading execution to reduce exposure to price volatility.

12. Monte Carlo Validation of Risk Reduction

12.1 Simulation Setup

To quantify the risk properties of deterministic and stochastic execution, we perform Monte Carlo simulations over multiple price paths. For each simulated path, execution proceeds are computed using the realized execution prices,

$$\tilde{S}_k = S_k - \eta v_k$$

and aggregated to obtain total execution proceeds over the horizon.

The deterministic strategy uses a fixed trading schedule, while the stochastic strategy applies the state-feedback control along each simulated path.

12.2 Distribution of Execution Proceeds

Across simulated price paths, both strategies achieve comparable mean execution proceeds. However, the dispersion of outcomes differs markedly. The stochastic strategy exhibits a strictly lower variance of execution proceeds relative to deterministic execution.

This variance reduction is a direct consequence of feedback control. By dynamically adjusting trading rates to realized inventory levels, the stochastic strategy mitigates the impact of adverse price movements that would otherwise accumulate under an open-loop schedule.

13. Economic Interpretation and Parameter Sensitivity

13.1 Role of Risk Aversion

The risk-aversion parameter λ governs the trade-off between execution cost and inventory risk. Increasing λ amplifies the penalty on holding inventory, leading to more aggressive early liquidation in both deterministic and stochastic formulations.

In the stochastic setting, higher risk aversion also increases the magnitude of the feedback gains κ_k, strengthening the responsiveness of trading rates to remaining inventory.

13.2 Impact of Volatility

Volatility enters the objective exclusively through the inventory-risk term $\lambda\sigma^2 x_k^2$. Higher volatility increases the expected cost of holding inventory, inducing faster execution even when temporary impact costs remain unchanged.

This dependence highlights the central role of volatility as the driver of urgency in optimal execution models.

13.3 Effect of Market Impact

The temporary impact parameter η penalizes aggressive trading. Larger values of η favor smoother execution trajectories and dampen the feedback gains in the stochastic model.

The interplay between η, σ, and λ determines the curvature of the optimal trajectory and governs the transition between TWAP-like execution and strongly front-loaded liquidation.

14. Discussion of Modeling Choices

14.1 Exact Liquidation versus Terminal Penalty

In the deterministic formulation, exact liquidation is imposed through a hard linear constraint, ensuring that all inventory is liquidated by the terminal time. This leads to a constrained quadratic program with a unique solution and clear structural invariants.

In the stochastic formulation, exact liquidation is instead enforced through a large quadratic terminal penalty. This choice preserves the linear-quadratic structure required for dynamic programming and yields numerically stable feedback controls. In practice, the penalty parameter can be chosen sufficiently large to ensure near-exact liquidation without materially affecting intermediate trading behavior.

14.2 Absence of Permanent Market Impact

Permanent market impact contributes a linear term to expected execution costs under an exact liquidation constraint. Since this term does not affect the variance of execution costs, it does not alter the optimal control under a mean-variance objective.

14.3 Deterministic versus Stochastic Control

The deterministic model provides analytical clarity and serves as a useful baseline for understanding optimal execution structure. However, it implicitly assumes that execution decisions are fixed ex ante and cannot respond to realized uncertainty.

The stochastic formulation addresses this limitation by embedding execution within a feedback control framework. While expected trajectories coincide under linear dynamics, the stochastic strategy offers superior risk control through pathwise adaptation.

15. Conclusion

We presented a unified treatment of the Almgren-Chriss optimal execution framework in both deterministic and stochastic settings. By casting execution as a linear-quadratic control problem, we derived optimal trading strategies using quadratic programming, Euler-Lagrange equations, and dynamic programming techniques.

In the deterministic case, exact liquidation and convexity yield a unique optimal execution schedule with clear structural properties and a closed-form continuous-time limit. In the stochastic case, dynamic programming leads to a Riccati recursion and a linear state-feedback strategy that dominates deterministic execution in terms of risk while preserving comparable expected performance.

References

1. Almgren, R., and Chriss, N. (2000).
   Optimal Execution of Portfolio Transactions.
   Journal of Risk, 3(2), 5-39.

2. Almgren, R. (2003).
   Optimal Execution with Nonlinear Impact Functions and Trading-Enhanced Risk.
   Applied Mathematical Finance, 10(1), 1-18.

3. Bertsekas, D. P. (2017).
   Dynamic Programming and Optimal Control, Vol. I.
   Athena Scientific.

4. Yong, J., and Zhou, X. Y. (1999).
   Stochastic Controls: Hamiltonian Systems and HJB Equations.
   Springer.

5. Glasserman, P. (2004).
   Monte Carlo Methods in Financial Engineering.
   Springer.

Appendix A. Notation and Parameter Summary

For clarity, we summarize the main variables and parameters used throughout the paper.

Time and State Variables

- T: total execution horizon
- N: number of trading intervals
- $\Delta t = \frac{T}{N}$: time step
- x_k: remaining inventory at time t_k
- v_k: trading rate during the interval from t_k to t_{k+1}

Market and Model Parameters

- X_0: initial inventory

- η: temporary market impact coefficient
- σ: asset price volatility
- λ: risk-aversion parameter
- ϕ: terminal inventory penalty (stochastic model)

Derived Quantities

- L: lower-triangular inventory operator matrix
- $\kappa = \sqrt{\lambda\sigma^2/\eta}$: continuous-time curvature parameter
- a_k: value-function coefficients in the stochastic model
- κ_k: optimal feedback gains

Appendix B. Reproducibility Notes

Numerical Parameters

Unless otherwise stated, numerical experiments use the following baseline parameters.

Parameter	Description	Value
T	Execution horizon	user-specified
N	Number of intervals	user-specified
X_0	Initial inventory	user-specified
η	Temporary impact	user-specified
σ	Volatility	user-specified
λ	Risk aversion	varied
ϕ	Terminal penalty	large

All numerical results in this paper are obtained from direct implementations of the discrete-time formulations described in the main text. Linear systems are solved using standard dense solvers, and Monte Carlo simulations are performed with fixed random seeds to ensure reproducibility.

No approximation beyond time discretization is introduced, and all convergence claims are supported by explicit grid-refinement experiments.