Theoretical planetary orbits around binary stars

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Fourth order Runge-Kutta method is used to solve a restricted three body problem of a binary star system and a planet, to find theoretical stable orbits for the planet. Same star mass and different star mass (3M and M) systems are explored. It is found that at large external orbits, the stable orbits behave as a single star system, where both prograde and retrograde motion are viable. When the external orbit is at small enough distance, the gravitational field fluctuations can cause apsidal precession of the eccentric orbit around the system centre. If the distance is reduced further the periodic fluctuation of gravitational field can cause external quasi-circular orbits. Small external rounded square orbit was found. Such simple orbits without trajectory intersections are possible if the period of the planet is in multiples of the period of star rotation. "Squeezed balloon" orbit was found for the same mass system, with period one (period of star rotation). If the period is slightly less, this will cause the rotation of the original shape and over time create a complex orbit with symmetry at the barycentre. The internal orbit of a different mass system is dominated by the more massive star, where a complex orbit was found closer to the more massive star at the barycentre. For internal orbitals and small external orbits, only retrograde orbits were stable. All complicated orbits looked circular simple in a rotating frame axis.

I. INTRODUCTION

Newtons gravitational theory on a two body problem is well understood and can be analytically solved. This paper will explore a three body problem with a planet in an orbit of a system of binary stars. A three body problem in general can't be solved analytically except for special cases[1]. In this case the system is simplified, where the orbit of binary stars is solved and the motion of the planet is calculated using the Fourth Order Runge-Kutta method[2]. Various binary star systems are explored to find stable orbits for the planet. These orbits are classified and are explained relative to the motion of the binary stars.

II. ASSUMPTIONS

To simplify the calculation the following assumptions shall be used:

- **•:** The planet and the the binary stars are point particles
- **•:** The mass of the planet is negligible compared to the mass of the stars
- **•:** The binary stars orbit a common barycentre
- **•:** The planet orbit is on the same plane as the binary stars
- **•:** There are no tidal effects

Due to our assumptions our system is simplified to a restricted three body problem[3] The planet of negligible mass exerts no force on the binary stars, which themselves move in two body motion. The motion of the binary star system can be classified as: two equal masses in common circular orbit around a barycentre (Fig.1), two equal masses in orbit around a common barycentre in separate orbits (Fig.2), two bodies with different masses in orbit around a common barycentre but a different radius circular orbits (Fig.3)

III. POSSIBLE ORBITS FOR BINARY STARS

Due to negligible mass of the planet the orbit calculation becomes a restricted three body problem, which is highly dependent on the orbit of binary stars. Solving for the orbit for binary stars we may treat it as a two body problem, where: x_1 and x_2 are positions of two stars and m_1 and m_2 are the corresponding masses. From Newton's second law

$$\begin{aligned} F_{12}(x_1, x_2) &= m_1 \ddot{x_1} \\ F_{21}(x_1, x_2) &= m_2 \ddot{x_2} \end{aligned} \tag{1}$$

Adding the two equations, applying newtons third law F_{12} = -F_{21} and letting R be the centre of mass (barycentre) of the system we have:

$$m_1\ddot{x_1} + m_1\ddot{x_1} = (m_1 + m_2)\ddot{R} = F_{12} + F_{21} = 0 \tag{2}$$

Thus

$$\ddot{R} = \frac{m_1\ddot{x_1} + m_2\ddot{x_2}}{m_1 + m_2} = 0 \tag{3}$$

Integrating this shows that the centre of mass velocity is constant, hence total momentum of the system $m_1\dot{x_1}$ + $m_2\dot{x_2}$=constant and is conserved. Solving for the position of stars using the inverse square law without external forces shows that for stable orbit the two stars must orbit a barycentre conserving momentum[8]. Three main possibles orbits of stars of similar masses are shown and will be investigated for three body stable planetary orbit

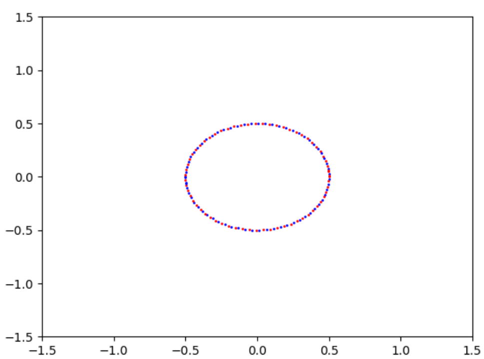

FIG. 1. Possible orbit of equal mass binary stars around a barycentre of common circular orbit

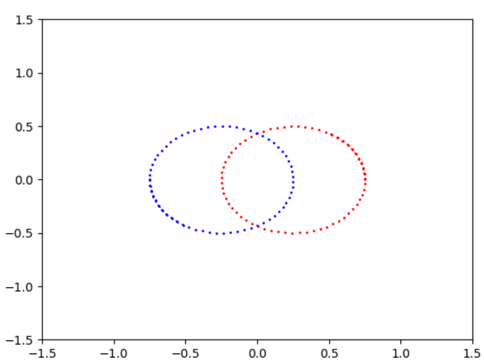

FIG. 2. Possible orbit of equal mass binary stars around a barycentre of different circular orbit

IV. CALCULATION

Consider initially the first restricted three body problem of binary stars of equal masses moving around a com-

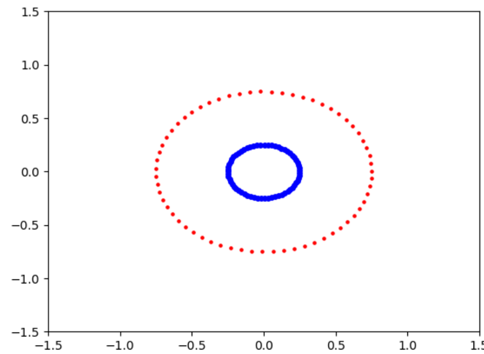

FIG. 3. Possible orbit of binary stars of masses M and 3M around a common barycentre

mon barycentre with radius R. Working in Cartesian coordinates, the position of the stars (X,Y) will be:

$$\begin{aligned} X_1 &= Rcos(\Omega t) \quad Y_1 = Rsin(\Omega t) \\ X_2 &= -Rcos(\Omega t) \quad Y_2 = -Rsin(\Omega t) \end{aligned} \tag{4}$$

Where Ω is the frequency of orbit. We want to use dimensionless coordinates and hence use Kepler's third law relationship GM=R^3 Ω^2[7]

If the planet is at position *(x,y)* it will be a distance r from the stars where

$$\begin{aligned} r_1^2 &= (x - X_1)^2 + (y - Y_1)^2 \\ r_2^2 &= (x - X_2)^2 + (y - Y_2)^2 \end{aligned} \tag{5}$$

Applying it to Newtons Universal Law of gravitation we have equations of motion for the planet as:

$$\begin{aligned} \ddot{x} &= -GM\left(\frac{1}{r_1^2}\frac{x - X_1}{r_1} + \frac{1}{r_2^2}\frac{x - X_2}{r_2}\right) \\ \ddot{y} &= -GM\left(\frac{1}{r_1^2}\frac{y - Y_1}{r_1} + \frac{1}{r_2^2}\frac{y - Y_2}{r_2}\right) \end{aligned} \tag{6}$$

Dimensionless variables will be used when calculating position. The units of distance will be in terms of R and units of time in terms of period of binary star system. In this case R=0.5 and Ω =1. All units will be relative to this, hence:

One time unit= orbit period of binary stars

One unit of distance= Distance between two binary stars.

For any binary system the stable orbit of a planet can be classified into two cases[5].

Internal orbit: the planet is close to either one or both of the stars and follows an orbit to either around one or both of the stars.

External orbit: the planet is far away from both of the stars and orbits both of them as a single massive star

Furthermore for the classification of orbits it is important to note the direction of motion of the planet relative to the binary system where:

Retrograde Orbit: Orbit of the planet is in opposite direction of the stars. It is found in the simulation that most stable planetary orbits follow this

Prograde Orbit: Orbit of the the planet is in same direction of the stars. It is found this is generally stable for external orbits

The calculation for orbits is heavily depended on the Runge-Kutta method, hence it is important to discuss to discuss the error in he numerical method.

A. Runge-Kutta Error

The Fourth Order Runge-Kutta method will generate a numerical solution for the position of the planet step by step, with a local truncation error of $O(h^5)$ and a total accumulated error of $O(h^4)$[4]

Therefore, it is important to reduce the interval size in the iteration to reduce accumulation of error. In the external orbits this could be in order of 0.1, but has to be in order of 0.01 maximum for internal orbits due to fast period of stable orbits. This has to be relative to the computing power at hand as small intervals can take long calculation times.

B. Rotating Frame Axis

Many internal orbits viewed from a fixed axis will look complicated and difficult to explain. It is therefore useful to view them from a rotating frame relative to the rotating stars. The subsequent orbits would look simple. Both axis will be used to analyse stable orbits. The rotating frame axis is calculated as follows:

$$\begin{aligned} \xi &= xcos(\Omega t) + ysin(\Omega t) \\ \eta &= -xsin(\Omega t) + ycos(\Omega t) \end{aligned} \tag{7}$$

V. LARGE EXTERNAL ORBITS

For all the binary star systems mentioned before, if the planet is placed far away from the stars it will see the two stars as one massive star with a sum of the two masses. This is because the fluctuation of the field strength due to the motion of the stars is negligible and will be similar to the field of a single star as in a two body problem.

The motion of a planet around a binary system at large distances, as a motion around one massive star can be observed when comparing the field strength of a single star to a fluctuating field strength of common circular orbit binary system in Table 1

The planetary orbits in the table were initialised to be in circular orbit by setting the initial position at distance x and an initial velocity where the gravitational attraction is equal to centripetal force.

$$\begin{aligned} \frac{GMm}{r^2} &= \frac{mv^2}{r} \\ v &= \sqrt{\frac{GM}{r}} \end{aligned} \tag{8}$$

TABLE I. Table to show variation of field strength of a binary star system compared to an equivalent single star system. Distance in reference to distance between stars (1 unit). Gravitational Field Strength in reference to a single star strength at Distance=20. Period of orbit of planet around binary system in units of a single period of a binary system i.e. Period=20 equivalent to 20 internal orbits of stars for one planetary orbit.

Distance	Single	Binary Mean	Minimum	Maximum	Period
20	1.000	1.001	0.999	1.002	178.25
18	1.234	1.236	1.235	1.238	152.31
16	1.563	1.565	1.562	1.568	127.48
14	2.041	2.044	2.041	2.049	89.76
12	2.778	2.785	2.778	2.792	82.76
10	4.000	4.015	4.000	4.031	63.03
8	6.250	6.284	6.250	6.327	44,7
6	11.110	11.251	11.111	11.374	28.9
4	25.0	25.711	25.001	26.383	15.56
2	100.0	112.684	100.020	131.875	4.934

From the table it can be seen that at large distances the mean field strength of a binary system is equivalent to a single star system. The gravitational field strength for a single star system is constant. For a binary star system fluctuates with a period of 0.5, which is the half the period of the binary system. The internal motion of the stars causes the fluctuation in gravitational field strength. However, the difference between maximum field strength and minimum field strength is negligible and has little to no effect for large distances.

At the large distances both the prograde and retrograde orbits are stable, but at distance of less than 4, only the retrograde orbit is possible. This is because for prograde orbits, the planet is aligned in motion with one of stars, thus increasing the period of fluctuation. This destabilises the orbit because the motion of the planet is away from binary star system.

The period of the planet greatly decreases with distance. At R=20 the stars completed 178.25 orbits for one planetary orbit. While at R=4 the stars only complete 4.934 orbits for one planetary orbit

It is also important to note that at small distances such as R=2 the maximum and minimum field strength differences are large and this causes the deviation of orbit from the circular path.

The table initialised the orbits to follow a circular path with eccentricity=0 but for large external orbits elliptical orbits are also possible, same as a single star system. Two possible large external orbits are shown: an elliptical (Fig.5) and circular orbits (Fig.4). Overall we can see how the three body problem simplifies to a simple two body approximation at large distances.

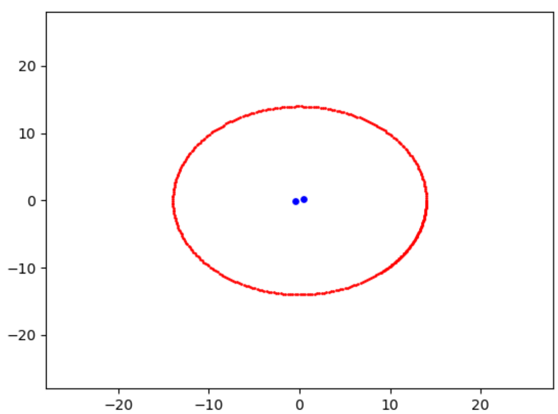

FIG. 4. Circular orbit at R=14

Note that the eccentricity for both orbits starting from the same initial position R=14 is dependent on the initial velocity when initiating Runge-Kutta method.

VI. APSIDAL PRECESSION OF ECCENTRIC EXTERNAL ORBITS

Apsidal precession is the gradual rotation of the line connecting line of apsides[6]. This can be observed for external eccentric orbits, when the radius of planets is close enough to feel the fluctuations of gravitational field. The apsidal precession can be negative or positive depending if the orbits axis rotation is in the same or opposite direction of orbit motion. The apsidal precession is caused by the fact that the gravitational field strength does not follow the inverse square law at small distances. As shown

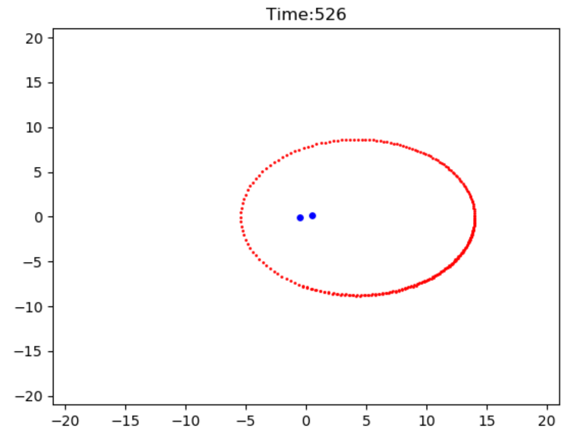

FIG. 5. Elliptical orbit at R=14

in Table 1. An example can be shown below with R=4 and period=15.1 (Fig.6)

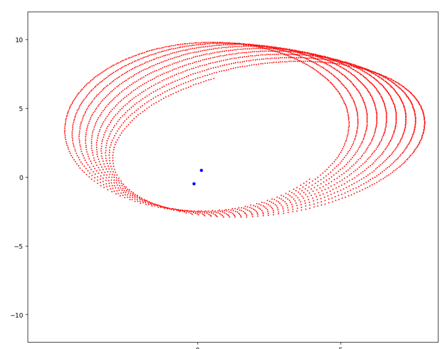

FIG. 6. Positive Apsidal Precession of Eccentric Orbit Initial Radius=4 Period=15.1

Given enough time the Apsidal precession of the orbit would make a full circle around the binary stars and return to the initial trajectory

VII. FLUCTUATING EXTERNAL ORBITS

For external orbits with a small initial radius, the gravitational field fluctuations have a strong effects which alter traditional elliptical trajectories.

Let us define a stable orbit as an orbit with a regular trajectory over a long time without leaving the binary star system and having a closed loop orbit. At distances R=4 or below it is difficult initialise a stable orbit and many trajectories of the planet are chaotic and only retrograde orbits are stable. One such chaotic orbit can be seen in (Fig.7)

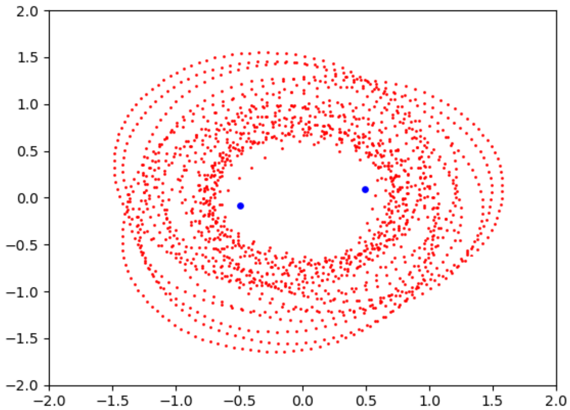

FIG. 7. Chaotic orbit initialised at distance R=1

The chaotic trajectories have very small period and if left for a long time will become unstable and eventually leave the binary star system.

The stable orbit of a planet close to a binary system at initial distances of R <=2 no longer have circular geometries. They either have quasi-circular (Almost, but not quite circular) orbit with regular fluctuations around a virtual circular orbit due to fluctuations of the gravitational field, or they have complex orbits with points of conjunction.

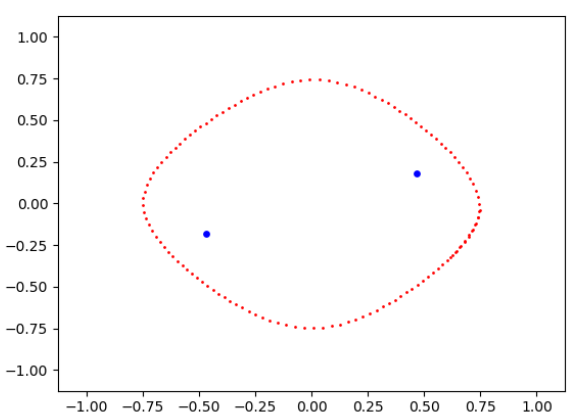

FIG. 8. Rounded Square Orbit Period=1 R=0.75

The second type can be seen in (Fig 8). The orbit has four rounded edges due to exactly four points of conjunction (The star and the planet are aligned). At these points, the planet is at maximum distance from the system centre. This orbit is perfectly stable and has no deviations. It is important to note that stable orbits with a simple path and a trajectory which does not intersect itself, such as this one, are only possible if the period of the planet is equal to the period of the binary star system. This is because only for equal periods, will the planet experience a periodic force. If the planet has a minor difference in the period it will experience an offset in force every orbit, hence slight offset in is trajectory every rotation. Furthermore, it can be observed that for a Rotating Frame axis view, the orbit appears simple and circular of radius R=0.75 (Fig.9)

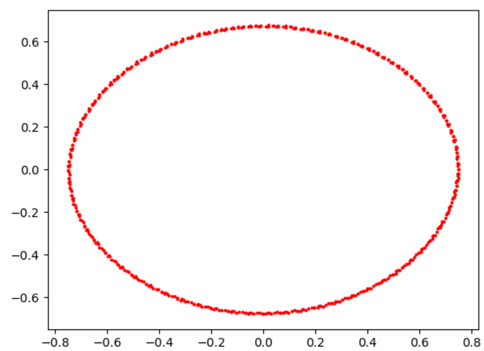

FIG. 9. Rounded Fixed Axis View Square Orbit Period=1 R=0.75

A different type quasi-circular orbit can also be observed (Fig.10), which consist of regular fluctuations around a virtual circular path due fluctuations in the gravitational field. The conjunction of the planet has the lowest gravitational field strength and the planet is at the furthest distance from the system centre. When the planet is 45° from the system centre to the stars i.e. equidistant from both it will experience the greatest force and be at the closest point to the system centre.

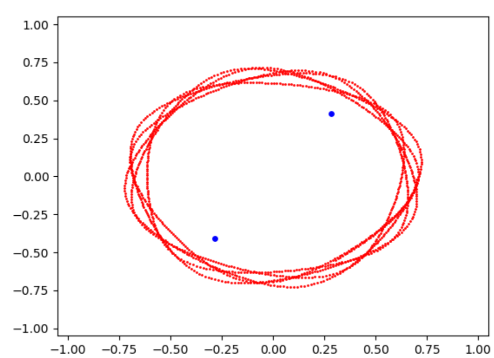

FIG. 10. Fluctuating Quasi-circular Orbit Period=0.875 R=0.7

From the fixed frame axis this orbit again looks simple (Fig.11). One can observe that the period of the planet 0.875 is less than the period of the binary stars.

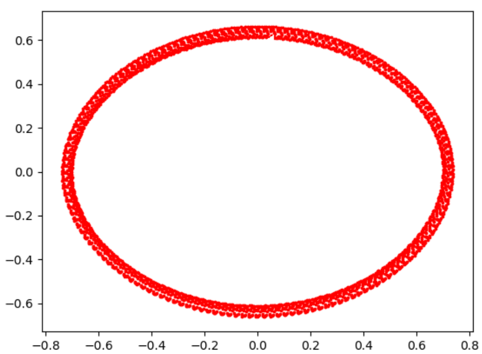

FIG. 11. Fixed Axis Fluctuating Quasi-circular Orbit Period=0.875 R=0.7

The difference in binary stars orbit causes the differences in periodicity and hence the regular fluctuations due to conjunctions of the planet and the stars. The fluctuations of the radial distance from the centre are in the range from 0.77 maximum and 0.64 minimum. Further different quasi-circular orbits can be observed at different periods where a pattern was observed with a greater range in distances from the system centre for smaller periods. For the small external orbits one can see a pattern occurring that stable orbits with no deviations and exact conjunctions are possible if the period of the planet is in multiples of period of the binary stars. In addition, stable quasi-circular orbits are possible if period of the planet is less than the period of binary stars. Two instances were shown where the planet was initialised at R=0.75 (Fig. 8 , Fig.10) but is suspected that different orbits are possible, such as exactly three conjunction points (instead of four shown) for different variety of radii.

VIII. INTERNAL ORBITS

A. Internal Orbit Around Two Stars

Internal orbits consist of a planet in motion around one or sometimes both of the stars, whereas the external orbit is a deformation of a circular orbit. The internal orbit can have very complex shapes as the planet is constantly under the force of both of the stars. Due to the symmetry of rotation of the stars one can predict that if the planet is under influence of both of the stars, the orbit itself must be symmetric. Internal orbits which are stable are hard to find and initialise, because if the path is slightly asymmetrical the planet will tend towards one of the stars and eventually shoot off out of the system.

Working from previous findings mentioned before, for orbits with simple paths without orbit intersections the period of the planet has to equal the period of the stars and projection has to be in retrograde motion. Furthermore, one can predict that the orbit must tend through the system centre as it is one of the few viable paths for symmetry, without turning into an external orbit. After some trial and error one can find such orbit (Fig.12).

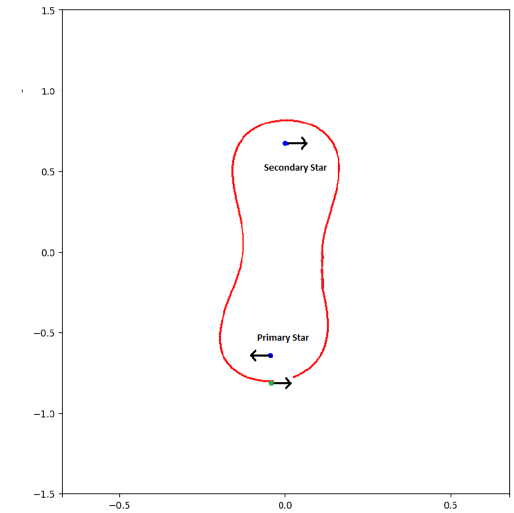

FIG. 12. Simple Internal Orbit with period=1 R=1.5 around common circle binary system. The planet is initialised with velocity in retrograde motion to primary star

The stable orbit has period=1 and symmetry as predicted. The planet is initialised with retrograde motion to the closest star (primary star) towards the system centre. However, the primary star itself follows its own orbit hence, by the time the planet reaches the closest point to the system centre, the two stars would have rotated by 0.25 of the period. Hence the planet is at the closest point towards the other star and is attracted towards it, forcing the motion downwards and in mirror to the path of arrival as seen in (Fig.13). By the time the stars rotate another 0.25 of the period, the planet is at the furthest distance from the initial position. It returns in a symmetrical path to the initial position and completes this periodic orbit. This internal orbit is only possible because the periods of stars and the planet are equal and the subsequent motion is symmetrical. Overall this follows the shape of a "squeezed balloon" at the centre.

More complex internal orbits are possible if the period of the planetary orbit is less than the period of binary stars. In this case stable orbits are possible which are the precession of the original "squeezed balloon orbit". This change is caused by the slight offset in periodicity, and the original shape is tessellated by a small angle. Many such orbits are possible for different initial radius and initial velocity, the one shown below is the tessellation of the original orbit at R=1.5 with a period =0.971 (Fig.15). Note a circular radius in the centre of the system which

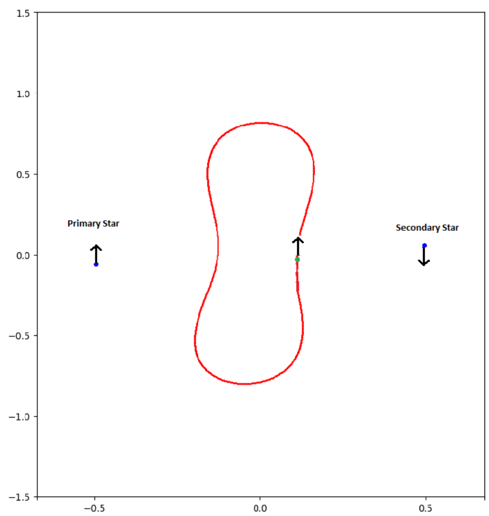

FIG. 13. Simple Internal Orbit with period=1 R=1.5 around common circle binary system. At 0.25 of the period the planet is closest point to the system centre, while the stars have rotated 90° The planet is not closest to the secondary star.

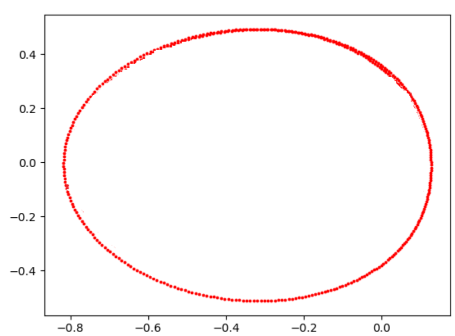

FIG. 14. Simple Internal Orbit with period=1 R=1.5 around common circle binary system.Rotating Frame Axis View

the planet never crosses. It is dependent on the initial velocity of the planet, where the lower the velocity the closer it can get to the centre. Stable orbits such as these precess the original shape in a circle until returned to the initial position. This complex orbit looks simple in a rotating frame axis view (Fig.16).

B. Internal Orbit Around a Single Star

In previous examples of internal orbit, the planet orbits both of the stars. However, if the planet is very close to one of the stars, sufficiently so that the gravitational field strength of the closest star makes other star field strength negligible, completely stable internal orbits around a single star are possible. Thus

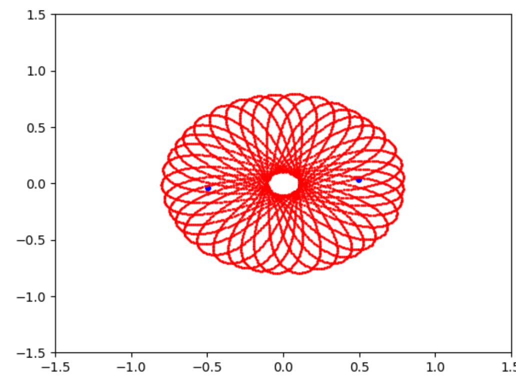

FIG. 15. Complex Internal Orbit with period=0.971 R=1.5 The offset in the period causes the precession of the original shape

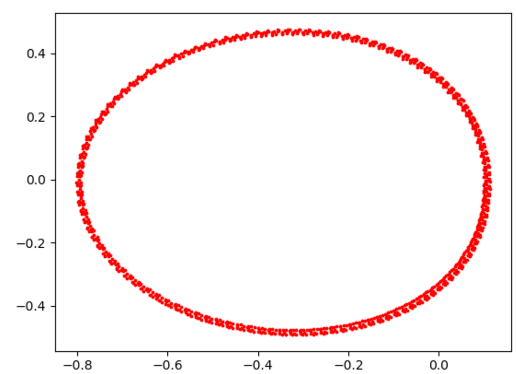

FIG. 16. Complex Internal Orbit with period=0.971 R=1.5 Rotating Frame Axis View

the orbit will behave as a single star system, around a primary star, while in orbit around the system centre. Hence, the planet will have two periods. First period around its primary star, and the period around the system centre, which is the same as the period of binary stars. For this to occur the planet must be very close to the primary star. In the example below the planet is R=0.55 from the centre (0.05 from primary star) with a period=0.18 around its primary star (Fig.17).

In the Rotating Frame this looks simple (Fig.18), but note that the circular trajectory is offset and hence has noise. This is because the planet isn't initialised perfectly and has a slight offset in its orbit.

In this case internal orbit for a single star is done for a binary system with equal masses around the circular orbit, but this is possible for any binary star system provided, that the planet is sufficiently close a single star. In addition, retrograde and prograde orbits are both possible.

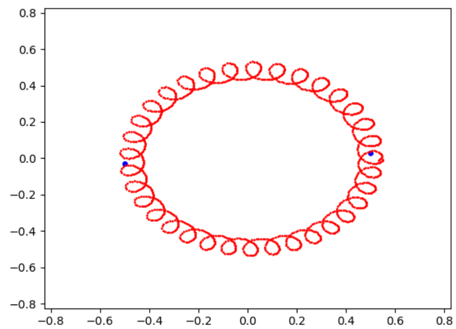

FIG. 17. Internal Orbit Around a Single Star R=0.55 Period=0.18

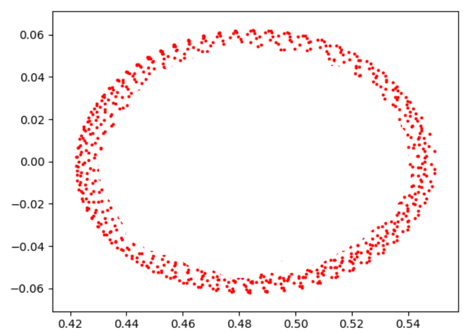

FIG. 18. Internal Orbit Around a Single Star R=0.55 Period=0.18 Rotating Frame Axis

IX. STABLE ORBITS FOR DIFFERENT MASS BINARY STAR SYSTEMS

If we consider a binary star system with masses M and 3M we no longer have a common orbit for both of the stars. Due to conservation of momentum as shown before, the stars will have separate orbits around a common barycentre. From conservation of momentum (Equation 3) it follows that the barycentre position follows the equation:

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{9}$$

Where m_1 and m_2 are the masses of two stars and x_1 and x_2 the positions. With masses of M and 3M it follows that the star of mass M must be 0.75 units away from the barycentre and star of mass 3M must be 0.25 units away from the barycentre. Where unit of 1 is the distance between binary stars. The subsequent motion of the stars will be of two circular orbits of different radius around a common barycentre as in (Fig.3).

For this motion large external orbits and small internal orbits around a single star are possible and will be the same of any binary star. It is more interesting to discuss complex internal stable orbits.

Due to large discrepancies in mass, one can find that the motion of the planet is dominated by the more massive star in most cases. The majority of non-chaotic orbits found can be described as internal orbits around a single massive star which is elongated by the attraction of a smaller star. Such orbit can be seen in (Fig.19)

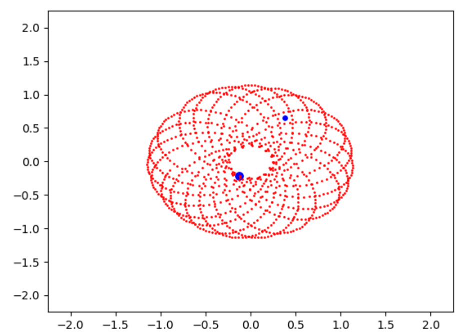

FIG. 19. Internal Orbit Around a 3M and M mass binary system

Such an orbit arises due to a planet initially having an internal elliptic orbit around the massive star which is elongated creating a "petal" shape (Fig. 20) . The planet is attracted to a smaller star but never reaches it due to the internal rotation of the massive star bringing it closer to the planet, thus overpowering the smaller star. This motion causes rotation of the "petal" shape in a circle.

The maximum and minimum distance from the centre are 1.14 and 0.83 respectively. The period is 0.25, which is the time taken to complete one orbit but enough time for the massive star to move by about 45° hence allowing the massive star to be close enough to rotate the initial petal shape. This orbit is very similar to a complex internal orbits of binary systems with stars in common orbit. However, notice that the centre of the planet orbit is no longer in between the stars, but at the barycentre of the binary star system. Again, the rotating frame axis is simple and circular.

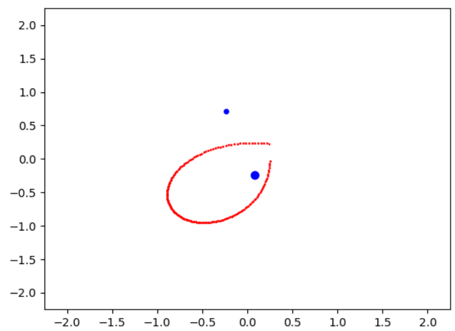

FIG. 20. Initial "Petal" of Internal Orbit Around a 3M and M mass binary system. Notice how the petal is rotated around in a circle due to motion of the heavier star, giving rise to its full orbit in Fig.19

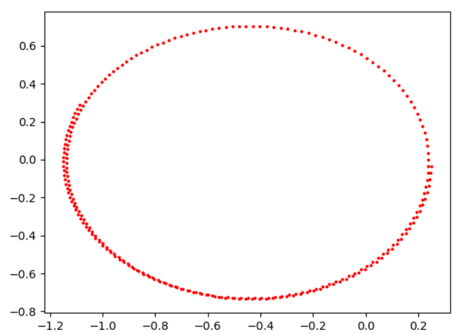

FIG. 21. Rotating frame axis for internal orbit of 3M and M mass binary system

X. CONCLUSION

Two binary star systems were explored to find stable planetary orbits: An equal mass star system around a common circular orbit and a different star mass system (M and 3M) around a common barycentre with different radius orbits. It was found that for both systems, large external orbits were possible, where the motion of the planet is as of around a single star system when the planet is at large distances from the system centre. This is because the gravitational field fluctuations are negligible and the orbits follow the inverse square law.

Apsidal Precession of eccentric external orbits was observed for both binary star systems.This occurs when the distance is small enough for the planet to feel the gravitational fluctuations, but also to be in an external orbit. Apsidal precession causes the rotation of the orbit in a circle around the system centre. The precession period takes a very long time to return to the original orbit and is caused because the gravitational field strength doesn't follow the inverse square law.

Internal orbits around a single star at small distances were also observed for both binary star systems, where the planet orbits a single star while simultaneously being in orbit around the system centre parallel to the orbit of the star. For external and internal single star orbits, both the retrograde and prograde motions are viable.

The main variation in difference of orbits for binary stars arises in small internal orbits. In internal orbits retrograde motion is required for stability. Stable internal orbits are difficult to achieve, are complicated and have an inherent symmetry. For a same mass system, the simplest orbit without trajectory intersection is found when the period of the planet is the same as the period of star rotation. Such orbit has a "squeezed balloon" shape and is only possible if periods of stars and planet are equal. If the period is less than the period of the stars, after one star orbit the planet will not be in the same initial position and the shape can rotate in a circle. This rotation is followed by every star rotation and can generate a complex tessellated orbit of the original shape. For a different mass system, the motion of the planet is overpowered by the greater mass star and causes "petal" shaped orbit due to elongation of the elliptical orbit by the smaller star. This orbit is rotated by further star rotations and generates a stable complex internal orbit. The centre of such orbit is found to be at the barycentre of the stars.

For the same mass system, small external orbits showed fluctuations in the field. This generated two types of orbits: A planet orbit with a rounded square shape due to exactly four conjunctions. Also a heavily fluctuated quasi-circular orbit around a virtual circle. For all cases the complex orbits looked simple and circular in the rotating frame axis. This paper found the general stable orbits in a three body problem, but there may be other which were not mentioned.

The Runge-Kutta method was used effectively with relevant accuracy, but it must be noted that it was difficult to find stable internal orbits, but this is expected with a three body problem. Hopefully these theoretical stable orbits found are useful and will be observed in nature.

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