Principle of Mathematical Induction[1]

- **Formal Setting.**

  Mathematical Induction is used to prove theorems of the form $\forall n \in \mathbb{N} : P(n)$ where P is some predicate with the natural numbers as the domain of discourse. Formally, it is stated as follows

$$\Big(P(1) \wedge (\forall k \in \mathbb{N} : P(k) \Rightarrow P(k+1))\Big) \Rightarrow (\forall n \in \mathbb{N} : P(n)) \tag{PMI}$$

  In plain English, it asserts that to prove the statement "$P(n)$ is true for all $n \in \mathbb{N}$.", it suffices to prove

  – **The Base Case:**(*often easy*) Prove that $P(1)$ is true; and

  – **The Inductive Case:**(*the meat!*) For any natural number k, if $P(k)$ is true, then prove that $P(k+1)$ is true.

  In the inductive case, the *assumption* that "$P(k)$ is true" is called the Induction Hypothesis.

- **Arithmetic Series**

Theorem 1. For all positive integers n, $\sum_{i=1}^{n} i = n(n+1)/2$

The predicate $P(n)$ takes the value true if $\sum_{i=1}^{n} i = n(n+1)/2$ and false otherwise. Theorem 1 asserts that $P(n)$ is true for all natural numbers.

Proof. To prove $\forall n \in \mathbb{N} : P(n)$, the principle of mathematical induction (or simple induction, henceforth) asks us to check/prove the following.

Base Case: Let us verify that $P(1)$ is true. Indeed, $\sum_{i=1}^{1} i = 1$ and $\frac{1(1+1)}{2} = 1$, and thus $P(1)$ is true.

Inductive Case: Fix any natural number k. The induction hypothesis is that $P(k)$ is true. We need to prove $P(k+1)$ is true.

$P(k)$ is true implies

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \tag{Induction Hypothesis}$$

To prove $P(k+1)$ is true, that is, we need to show

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2} \tag{Need to Show}$$

[1] *Lecture notes by Deeparnab Chakrabarty. Last modified : 28th Aug, 2021*
These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

We establish this by noting that the LHS of (Need to Show) is

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) = (k+1) \cdot \left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2}$$

where in the second inequality we have used the (Induction Hypothesis). Thus, we have established (Need to Show), and thus $\forall n \in \mathbb{N} : P(n)$ follows from induction. $\square$

Exercise: *Using induction, prove* $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$ *for any integer* $a > 1$ *and non-negative integer* n.

- **A Divisibility Fact.** We now prove the following fact by induction.

Theorem 2. For all $n \in \mathbb{N}$, 3 divides $n^3 - n$.

Proof. Let $P(n)$ be the predicate representing the truth value of the statement given in the theorem for a fixed natural number n. We proceed to prove $\forall n \in \mathbb{N} : P(n)$ by induction.

Base Case: Let us verify $P(1)$. We need to verify that 3 divides $1^3 - 1 = 0$. Indeed, 3 times 0 is 0.

Inductive Case: Let us now assume for a *fixed* $k \in \mathbb{N}$ that $P(k)$ is true. That is, 3 divides $k^3 - k$. We need to show $P(k+1)$ is true, that is, 3 divides $(k+1)^3 - (k+1)$. To do so, we expand $(k+1)^3$, to get

$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1) = (k^3 - k) + 3(k^2 + k)$$

$3(k^2 + k)$ is divisible by 3, and by the *induction hypothesis* (that is, $P(k)$ is true), $k^3 - k$ is divisible by 3. Therefore, $(k+1)^3 - (k+1)$ is divisible by 3. That is, $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

Exercise: *Does* 4 *divide* $n^4 - n$ *for all non-negative integers* n*? Mimic the above proof.*

- **Another Divisibility Fact.** We now prove the following fact by induction.

Theorem 3. For all $n \in \mathbb{N}$, 7 divides $3^{2n} - 2^n$.

Proof. Let $P(n)$ be the predicate representing the truth value of the statement given in the theorem for a fixed natural number n. We proceed to prove $\forall n \in \mathbb{N} : P(n)$ by induction.

Base Case: Let us verify $P(1)$. We need to verify that 7 divides $3^2 - 2^1 = 7$. Indeed it does. Therefore $P(1)$ is true.

Inductive Case: Let us now assume for a *fixed* $k \in \mathbb{N}$ that $P(k)$ is true. That is, 7 divides $3^{2k} - 2^k$. We need to show $P(k+1)$ is true, that is, 7 divides $3^{2(k+1)} - 2^{(k+1)}$. Indeed observe,

$$\begin{aligned}
3^{2(k+1)} - 2^{(k+1)} &= 3^2 \cdot 3^{2k} - 2 \cdot 2^k \\
&= 9 \cdot 3^{2k} - 2 \cdot 2^k \\
&= 9 \cdot 3^{2k} - 9 \cdot 2^k + 9 \cdot 2^k - 2 \cdot 2^k && (1) \\
&= 9 \cdot \left(3^{2k} - 2^k\right) + 7 \cdot 2^k && (2)
\end{aligned}$$

7 divides $3^{2k} - 2^k$, by the induction hypothesis. 7 clearly divides $7 \cdot 2^k$. Therefore, 7 divides $3^{2(k+1)} - 2^{k+1}$. That is, $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

This proof was (slightly) tricky. Line 3 is where the trick was where we subtracted and added $9 \cdot 2^k$. Why did we do that? Well, we knew something about $3^{2k} - 2^k$, but when we expanded out we got $9 \cdot 3^{2k} - 2 \cdot 2^k$. If the "coefficients" of 3^{2k} and 2^k were same we would be done (but it isn't), so we just added and subtracted so that the coefficients of one became the same. The other we had an "happy accident" (of $9 - 2 = 7$). Indeed, the person who devised this theorem (in this case, me) probably worked backwards to come up with the statement. $\square$

Exercise: *Can you come up with statements like above? Can you guess which number will always divide $4^{3n} - 3^{2n}$ for all natural numbers n? After guessing, can you prove that guess using induction.*

Remark: *Sometimes, the induction principle may look as follows: (a) The base case may involve proving $P(1), P(2), \ldots, P(c)$ for some finite c, and (b) The inductive case may be possible only for numbers $k \geq c$. Note this is also perfectly OK to establish $\forall n : P(n)$. We will see such an example in class and problem sets.*

Answers to Exercises.

- **Exercise:** *Using induction, prove $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$ for any integer $a > 1$ and non-negative integer n.*

Proof. Fix any real $a > 1$. Let $P(n)$ be the predicate which takes the value true if $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$. We need to prove $\forall n \in \mathbb{N} \cup \{0\} : P(n)$. We proceed by inductions.

Base Case. We need to prove $P(0)$ is true. That is, $\sum_{i=0}^{0} a^i = \frac{a-1}{a-1}$. Indeed, both LHS and RHS are 1.

Inductive Case. Fix $k \geq 0$ and suppose $P(k)$ is true. That is, $\sum_{i=0}^{k} a^i = \frac{a^{k+1}-1}{a-1}$. We need to prove $P(k+1)$ is true.

Now note,

$$\sum_{i=0}^{k+1} a^i = a^{k+1} + \sum_{i=0}^{k} a^i \underbrace{=}_{P(k)} a^{k+1} + \frac{a^{k+1}-1}{a-1} \underbrace{=}_{\text{algebra}} \frac{\left(a^{k+2} - a^{k+1}\right) + \left(a^{k+1} - 1\right)}{a-1}$$

And now we see that the RHS is $\frac{a^{k+2}-1}{a-1}$, thereby establishing $P(k+1)$. And thus, we have proved the statement by induction. $\square$

- **Exercise:** *Does 4 divide $n^4 - n$ for all non-negative integers n? Mimic the above proof.*

Actually, 4 **does not** divide all $n^4 - n$. Rather than giving you a counterexample, let me actually take you down a "proof", which will **fail** and thus give us a counter example.

"Proof" Let $P(n)$ be the predicate representing the truth value of the statement given in the theorem for a fixed natural number n. We proceed to prove $\forall n \in \mathbb{N} : P(n)$ by induction.

Base Case: Let us verify $P(1)$. We need to verify that 4 divides $1^4 - 1 = 0$. Indeed, 4 times 0 is 0.

Inductive Case: Let us now assume for a *fixed* $k \in \mathbb{N}$ that $P(k)$ is true. That is, 4 divides $k^4 - k$. We need to show $P(k+1)$ is true, that is, 4 divides $(k+1)^4 - (k+1)$. To do so, we expand $(k+1)^4$, to get

$$(k+1)^4 - (k+1) = (k^4 + 4k^3 + 6k^2 + 4k + 1) - (k+1) = (k^4 - k) + 4(k^3 + k) + 6k^2$$

And now we see our problem. To assert 4 divides $(k+1)^4 - (k+1)$, we see that 4 must divide $6k^2$. This is because 4 does divide $k^4 - k$ (by induction hypothesis) and 4 divides $4(k^3 + k)$. But does 4 divide $6k^2$ always? No! Not when $k = 1$. And so, it suggests for $k = 1$, $P(k+1)$ **may not** be true. That is $P(2)$ may not be true.

Indeed, 4 **does not** divide $2^4 - 2 = 14$. Ta da!

- **Exercise:** *Can you come up with statements like above? Can you guess which number will always divide $4^{3n} - 3^{2n}$ for all natural numbers n? After guessing, can you prove that guess using induction.*

Did you guess? It's $55 = 4^3 - 3^2$.

Proof. Let $P(n)$ be the predicate representing the truth value of the statement given in the theorem for a fixed natural number n. We proceed to prove $\forall n \in \mathbb{N} : P(n)$ by induction.

Base Case: Let us verify $P(1)$. We need to verify that 55 divides $4^3 - 3^2 = 55$. Indeed it does. Therefore $P(1)$ is true.

Inductive Case: Let us now assume for a *fixed* $k \in \mathbb{N}$ that $P(k)$ is true. That is, 55 divides $4^{3k} - 3^{2k}$. We need to show $P(k+1)$ is true, that is, 55 divides $4^{3(k+1)} - 3^{2(k+1)}$. Indeed observe,

$$\begin{aligned}
4^{3(k+1)} - 3^{2(k+1)} &= 4^3 \cdot 4^{3k} - 3^2 \cdot 3^{2k} \\
&= 64 \cdot 4^{3k} - 9 \cdot 3^{2k} \\
&= 64 \cdot 4^{3k} - 64 \cdot 3^{2k} + 64 \cdot 3^{2k} - 9 \cdot 3^{2k} \qquad (3) \\
&= 64 \cdot \left(4^{3k} - 3^{2k}\right) + 55 \cdot 3^{2k} \qquad (4)
\end{aligned}$$

55 divides $4^{3k} - 3^{2k}$, by the induction hypothesis. 55 clearly divides $55 \cdot 3^{2k}$. Therefore, 55 divides $4^{3(k+1)} - 3^{2(k+1)}$. That is, $P(k+1)$ is true. By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$