Graphs : Weak Duality between Flows and Cuts[1]

1 Flows and Cuts: A Duality

One of the most fascinating results in algorithms is that the above max s, t flow problem and the min s, t cut problem are actually one and the same. Or more correctly, they are two sides of the same coin. They are duals of each other. We are going to build this over the next two lectures. In this lecture, we are going to look at the "easy" direction. We show that given any network (G, s, t, u), the value of *any* feasible s, t flow must be at most the capacity of *any* s, t cut. And in particular, the *max-flow is at most the min-cut*.

Lemma 1. Let f be any feasible s, t flow and $\partial^+ S$ be any s, t cut. Then

$$\mathsf{val}(f) := \ \mathsf{excess}_f(t) \leq u(\partial^+ S) \ \ =: \mathsf{cap}(S)$$

Proof. Since we know that $\mathsf{excess}_f(t) = -\mathsf{excess}_f(s)$, the lemma is equivalently asking us to show that

$$u(\partial^+ S) + \mathsf{excess}_f(s) \geq 0$$

To get this, let's add the excesses for every $v \in S$. Why? This is because we wish to argue about the relation between $\mathsf{excess}_f(s)$ and $u(\partial^+ S)$, and the edges participating in the latter may be "far away" from the vertex s. Rather, they involve vertices on the "boundary" of the set S and somehow we need to "propagate" their information to the vertex s which may be deep inside the set. We

$$\sum_{v \in S} \mathsf{excess}_f(v) = \sum_{v \in S} \left(\sum_{(u,v) \in E} f(u,v) - \sum_{(v,w) \in E} f(v,w) \right) = \sum_{(x,y) \in E} \left(f(x,y) \cdot \mathbf{1}_{y \in S} - f(x,y) \cdot \mathbf{1}_{x \in S} \right)$$

where $\mathbf{1}_{x \in S}$ is the indicator variable for $x \in S$ and takes value 1 if $x \in S$ and 0 if $x \notin S$. Now note that since f is feasible, the LHS is precisely $\mathsf{excess}_f(s)$. On the other hand, the RHS is precisely $f(\partial^- S) - f(\partial^+ S)$, that is, the total flow on the $\partial^- S$ edges minus the total flow on the $\partial^+ S$ edges. Together, we get,

$$\mathsf{excess}_f(s) = f(\partial^- S) - f(\partial^+ S)$$

See Figure 1, and the caption below the picture, for an illustration of this.
To finish the proof of the lemma, we use the following observations.

Observation 1. (a) Since $f(e) \geq 0$, we get $f(\partial^- S) \geq 0$, (b) since $f(e) \leq u(e)$, we get $f(\partial^+ S) \leq u(\partial^+ S)$.

Taking this together, we get

$$\mathsf{excess}_f(s) \geq -u(\partial^+ S) \ \ \Rightarrow \ \ u(\partial^+ S) + \mathsf{excess}_f(s) \geq 0$$

which completes the proof of the lemma. □

The obvious corollary to the above fact is that the maximum feasible s, t flow in any graph is at most the minimum s, t cut.

[1]*Lecture notes by Deeparnab Chakrabarty. Last modified : 19th Mar, 2022*
These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

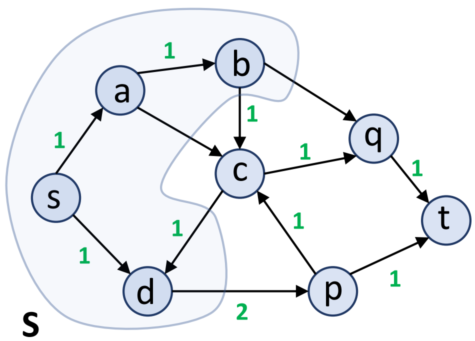

Figure 1: The set $S = \{s, a, b, d\}$. The flow is shown in green numbers. If an edge has no number, the flow is 0. The $\mathsf{excess}_f(s)$ is precisely -2, the negative of the flow leaving s. $f(\partial^+ S) = 3$, on the edges (d, p) and (b, c). The incoming flow into S is $f(\partial^- S) = 1$ on the edge (c, d). The difference is the excess at s.

Theorem 1 (Weak Duality)**.** In any graph network (G, s, t, u), the maximum s, t flow is at most the minimum s, t cut value.

There is an important consequence of the (proof of) the above lemma which will be used to prove the magical fact: that the maximum s, t flow ***equals*** the minimum s, t cut.

Theorem 2 (Corollary to Lemma 1)**.** Suppose f is a feasible s, t flow f, and S is an s, t cut S such that

a. $f(e) = u(e)$ for all $e \in \partial^+ S$
b. $f(e) = 0$ for all $e \in \partial^- S$

Then f is a ***maximum*** s, t flow, S is a ***minimum*** s, t cut, and their values are the same.

Proof. The assumptions above imply that the inequalities in Observation 1 are indeed equalities. That would imply $\mathsf{excess}_f(s) + u(\partial^+ S) = 0$ implying $\mathsf{val}(f) = \mathsf{val}(S)$. $\square$