Homework 1

Due: May 21st, 2009

Solutions

1. To detect if the graph contains directed cycles we can use DFS. The algorithm runs in $O(m)$, where m is the number of arcs in the digraph. An alternative is using topological ordering.

2. The first and fourth measures are regular, the others are not. We first do the proof for the regular ones. Let S, S' be two schedules with completition times $C_1^S, \dots, C_n^S$ and $C_1^{S'}, \dots, C_n^{S'}$ respectively. Suppose $C_j^S \le C_j^{S'}$ for all $j = 1, \dots, n$. To show that a performance measure γ is regular we need to show that $\gamma(S) \le \gamma(S')$.

   Let γ_1 and γ_2 be regular performance measures. Define $\gamma = w_1\gamma_1 + w_2\gamma_2$, for $w_1, w_2 \ge 0$. Then $\gamma(S) = w_1\gamma_1(S) + w_2\gamma_2(S) \le w_1\gamma_1(S') + w_2\gamma_2(S') = \gamma(S')$, where the inequality follows from the fact that γ_1, γ_2 are regular and $w_1, w_2 \ge 0$. Therefore the first performance measure is regular.

   Now define $\gamma = e^{c\gamma_1}$, for some $c > 0$. As e^x is an increasing function and γ_1 is regular we have $\gamma(S) = e^{c\gamma_1(S)} \le e^{c\gamma_1(S')} = \gamma(S')$. Therefore the fourth performance measure is regular.

   To show that the other performance measures are not regular we only need to provide a counterexample. We will use the same counterexample for all of them. Let S be a schedule with $C_1^S = 3, C_2^S = 1$ and S' a schedule with $C_1^{S'} = 3, C_2^{S'} = 2$. So $C_j^S \le C_j^{S'}$ for $j = 1, 2$. Define the performance measures $\gamma_i = C_i$ for $i = 1, 2$. It is easy to check that γ_1, γ_2 are regular. With these schedule, $\gamma_1(S) - \gamma_2(S) = 2 > 1 = \gamma_1(S') - \gamma_2(S')$, hence the second performance measure is not regular. Similarly we can check that the other performance measures are not regular.

3. Consider a schedule with completition times $C_1, \dots, C_n$. By eventually relabelling the jobs, we may assume $C_1 \le C_2 \le \ldots \le C_n$. Define $C_0 := 0$. Note that at time $t \in [C_{i-1}, C_i)$ we have $N_u(t) = n - i + 1$. Then we have

$$\overline{N_u} := \frac{1}{C_{max}} \sum_{i=1}^{n} \int_{C_{i-1}}^{C_i} N_u(t)dt = \frac{1}{C_{max}} \sum_{i=1}^{n} \int_{C_{i-1}}^{C_i} (n-i+1)dt$$

$$= \frac{1}{C_{max}} \sum_{i=1}^{n} (n-i+1)(C_i - C_{i-1}) = \frac{1}{C_{max}} \left(\sum_{i=1}^{n} (n-i+1)C_i - \sum_{i=1}^{n} (n-i+1)C_{i-1} \right)$$

$$= \frac{1}{C_{max}} \left(\sum_{i=1}^{n} (n-i+1)C_i - \sum_{i=0}^{n-1} (n-i)C_i \right) = \frac{1}{C_{max}} \sum_{i=1}^{n} C_i$$

   .

   This performance measure is not regular (take for example the schedules S with $C_1^S = 1, C_2^S = 2$ and S' with $C_1^{S'} = 1, C_2^{S'} = 3$).

4. Suppose by contradiction that this is not the case, that is, every optimal solution to $(J \mid \mid \gamma)$ is not active. Let S be an optimal solution such that $\sum_{i=1}^n C_i^S$ is minimized. As S is not active, there exists another schedule S' such that $C_i^{S'} \le C_i^S$ for all $i = 1, \dots, n$ and $C_j^{S'} < C_j^S$ for at least one $j \in \{1, \dots, n\}$. Therefore $\sum_{i=1}^n C_i^{S'} < \sum_{i=1}^n C_i^S$. Moreover, as γ is regular, $\gamma(S') \le \gamma(S)$. Hence S' is an optimal schedule that contradicts the choice of S.

5. The Traveling Salesman Problem can be formulated as follows. For city 0 (the one we start from) make two jobs, 0 and $n+1$. For every other city j make job j. Define all the processing times to be equal to zero.

Now define:

(1) $s_{j,0} = \infty$ for all $j \in \{1, \dots, n+1\}$

(2) $s_{j,k} = d_{j,k}$ for all $j \neq k, \quad j \in \{0, \dots, n\}$ and $k \in \{1, \dots, n\}$

(3) $s_{j,n+1} = d_{j,0}$ for all $j \in \{0, \dots, n\}$

(4) $s_{n+1,k} = \infty$ for all $k \in \{0, \dots, n\}$.

Now solve $(1 \mid s_{jk} \mid C_{max})$.

A schedule for this problem corresponds to assigning an ordering to the jobs on the machine, which corresponds to an order in which the travelling salesman will visit the cities. Definition (1) ensures that job 0 is the first one and (4) ensures that job $n+1$ is the last one, so we make sure to start and end in the right city. The definitions in (2) and (3) ensure that C_{max} is the length of the tour.

6. First consider the problem $(O2 \mid \mid C_{max})$. Consider the schedule in which the order of the jobs is the following. On machine one, $3, 2, 4, 1$. On machine two, $1, 3, 2, 4$. The makespan of this schedule is 30. For each machine, the makespan of any schedule is at least the sum of the processing times of all jobs on that machine. Hence the optimal is at least $\max\{\sum_{j=1}^{4} p_{1j}, \sum_{j=1}^{4} p_{2j}\} = 30$. Therefore the given schedule is optimal.

   Now consider $(F2 \mid \mid C_{max})$, where each job needs to be processed first on machine 1 and then on machine 2. Let P_2 be the sum of the processing times of the jobs on machine 2. As we cannot start processing any job on machine 2 before it has been processed by machine 1, we have that any schedule has makespan at least $\min\{p_{j2}\} + P_1 = 4 + 30 = 34$. Now consider the following schedule. On machine one the order is $3, 2, 4, 1$, on machine two the order is $3, 2, 4, 1$ (so there are 4 units of idle time on machine 2 at the beginning, while job 3 is processed on machine 1). This schedule has a makespan of 34, hence it is optimal.