E0234 Randomized Algorithms

Midterm

29th Feb, 2016. 2pm to 5pm.

Good luck!

0. Write your name.

1. (a) Prove or disprove: for independent random variables X, Y, $\mathbf{Var}[XY] = \mathbf{Var}[X] \cdot \mathbf{Var}[Y]$.

2. Prove that for any two random variables X and Y (not necessarily independent), $\mathbf{Exp}[XY] \leq \sqrt{\mathbf{Exp}[X^2]}\sqrt{\mathbf{Exp}[Y^2]}$. Assume X and Y are discrete.

3. X is a random variable with expectation $\mu > 0$ and standard deviation σ. Prove $\mathbf{Pr}[X > 0] \geq \frac{\mu^2}{\sigma^2+\mu^2}$. You are allowed to use the statement from the previous question without proof.

4. Suppose you can draw independent samples of a real random variable X that has expectation 0 and standard deviation σ. Explain how to use only $O(\log n)$ samples from this source to generate a random variable Y with expectation μ such that $\mathbf{Pr}[|Y - \mu| > 2\sigma] < 1/n$.

5. Consider the following algorithm for the independent set problem on an n-node graph. Sample a random permutation σ of $\{1, 2, \ldots, n\}$. Initialize I to $\varnothing$. For $i = 1$ to n, place $\sigma(i)$ in I if it doesn't have an edge to any vertex in I. What is the best lower bound you can prove on the expected size of the independent set picked?

6. In d-dimensions, there can be at most $(d + 1)$ unit vectors which are orthogonal to each other, that is, $u^\top v = 0$ for any two. Call a pair of unit vectors ε-orthogonal if $|u^\top v| \leq \varepsilon$. How large (in cardinality) a set of pairwise ε-orthogonal unit vectors can you construct in d-dimensions?

7. A hypergraph is k-uniform if each hyperedge has exactly k vertices, and is called k-regular if every vertex is in exactly k different hyperedges. A hypergraph is 2-colorable if every vertex can be coloured either red or blue such that there are no monochromatic hyperedges. Prove that for $k > 10$, every k-uniform, k-regular hypergraph is 2-colourable.

8. (A Markov chain question.)