More Big-Oh : Fun with Factorials and Logarithms[1]

Recall the factorial function defined as $n! = 1 \cdot 2 \cdot 3 \cdots n$. What is its relation with n^n?

Theorem 1. $n! = O(n^n)$

Proof. Let's look at $n!/n^n$. This equals $(1/n) \cdot (2/n) \cdots (n/n)$. As $n \to \infty$, each of the products goes to 0. Thus, the claim follows. $\square$

Theorem 2. $n^n \neq O(n!)$

Proof. Again the flipped rational function $n^n/n! = (n/1) \cdot (n/2) \cdots (n/n)$ which is an increasing function of n. Thus the limit is ∞ implying $n^n \neq O(n!)$. $\square$

Now let us take logarithms.

Theorem 3. $\log_2(n!) = \Theta(n \log_2 n)$.

Thus, we see that although $n! \neq \Theta(n^n)$, taking logs satisfies the $\Theta()$ relation. Again, this should not be surprising – we have seen $f(n) = \Theta(g(n))$ but $2^{f(n)} \neq O(2^{g(n)})$.

Proof. Let's now prove the claim. Once again, let's look at the rational function $r(n) := \frac{\log_2(n!)}{n \log_2 n}$.

First we note that the numerator is precisely (using log of product is sum of logs)

$$\log_2(n!) = \log_2(1 \times 2 \times \cdots \times n) = \log_2 1 + \log_2 2 + \cdots + \log_2 n = \sum_{i=1}^{n} \log_2 i$$

Now, since $\log_2 i \leq \log_2 n$ for all $i \leq n$ (since $\log_2$ is an increasing function), we get that the numerator of $r(n)$ is $\leq n \log_2 n$. That is, the denominator of $r(n)$. This implies $r(n) \leq 1$ for all n. And in particular, $\lim_{n\to\infty} r(n) \leq 1$.

We now need to show that $\lim_{n\to\infty} r(n) >$ some positive number. This will show $\log_2(n!) \in \Theta(n \log_2 n)$. To see this, we note that for all $i \geq n/2$, $\log_2 i > \log_2(n/2)$. Therefore, we get that the numerator of $r(n)$ is

$$\sum_{i=1}^{n} \log_2 i = \log_2 1 + \log_2 2 + \cdots + \log_2 n \geq \log_2(n/2) + \log_2(n/2+1) + \cdots + \log_2 n \geq (n/2) \cdot \log_2(n/2) = \frac{n}{2} \cdot (\log_2 n - 1)$$

Thus $r(n) \geq \frac{1}{2} \cdot \frac{\log_2 n - 1}{\log_2 n}$. As $n \to \infty$, the second fraction $\to 1$. Thus, $\lim_{n\to\infty} r(n) \geq 1/2$. $\square$

Recall that given any natural number N, the number of bits required to describe it is $\lceil \log_2(N+1) \rceil$. In your problem set, you will show that $\lceil x \rceil = \Theta(x)$. Therefore, one requires $\Theta(\log_2 N)$ bits to describe a natural number N. The above theorem, therefore, gives the following corollary.

[1]*Lecture notes by Deeparnab Chakrabarty. Last modified : 19th Mar, 2022*
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Theorem 4. For any number n, the number of bits required to write $n!$ is $\Theta(n \log_2 n)$.