Approximation Algorithms for the Firefighter Problem: Cuts over Time and Submodularity

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Abstract. We provide approximation algorithms for several variants of the Firefighter problem on general graphs. The Firefighter problem models the case where an infection or another diffusive process (such as an idea, a computer virus, or a fire) is spreading through a network, and our goal is to stop this infection by using targeted vaccinations. Specifically, we are allowed to vaccinate at most B nodes per time-step (for some budget B), with the goal of minimizing the effect of the infection. The difficulty of this problem comes from its temporal component, since we must choose nodes to vaccinate at every time-step while the infection is spreading through the network, leading to notions of "cuts over time".

We consider two versions of the Firefighter problem: a "non-spreading" model, where vaccinating a node means only that this node cannot be infected; and a "spreading" model where the vaccination itself is an infectious process, such as in the case where the infection is a harmful idea, and the vaccine to it is another infectious idea. We give complexity and approximation results for problems on both models.

1 Introduction

Faced with an epidemic that is spreading through a population, and a limited supply of vaccine (or simply a lack of time to administer it), it is necessary to decide whom to vaccinate. Questions about the spread of disease and epidemics in a social network have often been modeled using graph theory (e.g. [3, 11]), and correspond to fundamental graph-theoretic concepts [22]. Moreover, these graph theoretic principles can be applied to many diffusive network processes, including epidemics in computer networks, the spread of innovations and ideas, and viral marketing [23]. In this paper, we focus specifically on inhibiting the spread of an epidemic or an idea by using vaccination.

Model and the Firefighter problem We model our network of agents as a directed[3] graph $G = (V, E)$ and a source node s. All nodes in the graph are in one of three

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[3] We use a directed graph since it is more general – an undirected graph is just a directed graph with two arcs per edge.

states: they can be *infected*, *vaccinated*, or *vulnerable*, that is neither vaccinated nor infected. At time $\tau = 0$, all nodes are vulnerable, except node s, which is infected. At each $\tau > 0$, any vulnerable vertex v which is connected to an infected node u, such that $(u, v) \in E$, gets infected at time $\tau + 1$, unless it is vaccinated at time step τ. Infected and vaccinated nodes stay infected and vaccinated, respectively. We call a node *saved* if it is either vaccinated or if all paths from any infected node to it contains at least one vaccinated node.

Definition 1. *A vaccination strategy is a set $\Psi \subseteq V \times J$ where V is the set of vertices of graph G and $J = \{1, 2, \ldots |V|\}$. The vertex v is vaccinated at time $\tau \in J$ by the vaccination strategy Ψ if $(v, \tau) \in \Psi$. A vaccination strategy Ψ is valid with respect to budget B, if the following two conditions are satisfied:*

i. if $(v, \tau) \in \Psi$ then v is not infected at time τ,
ii. let $\Psi_\tau = \{(v, \tau) \in \Psi\}$; then $|\Psi_\tau| \leq B \quad for \quad \tau = 1 \ldots |V|$.

The first condition implies we can only vaccinate vulnerable nodes, and the second requires that no more than B nodes are vaccinated at any time-step.

We consider two objectives in this paper. The first objective, which we call MaxSave, is to maximize the number of non-infected nodes at the end, given a fixed budget B. The second objective, which we call MinBudget, is to minimize the budget B needed per time instant in order to save a given set of nodes, $T \subseteq V$. They can be formally described as follows.

MaxSave(G, B, s, T)
Instance: A rooted graph $(G(V, E), s)$, integer $B \geq 1$ and $T \subseteq V$
Objective: Find a valid vaccination strategy Ψ such that if s is the only infected node at time 0, then at the end of the above process the number of non-infected nodes that belong to T is maximized.

This problem is also referred to as the Firefighter problem in the literature when $T = V$ [16, 20].

MinBudget(G, s, T)
Instance: A rooted graph $(G(V, E), s)$, and $T \subseteq V$
Objective: Find a valid vaccination strategy Ψ with minimum possible budget B, such that if s is the only infected node at time 0, then at the end of the above process all nodes in T are saved.

We also consider a variant of the above model, where the vaccination is also a process that spreads through the network. In the case of ideas propagating through a social network, this represents the fact that an antidote to a harmful idea is often another idea, which can be just as infectious. In disease propagation, this represents the fact that vaccines can be infectious as well, since they are often an attenuated version of the actual disease. In this *Spreading Vaccination Model*, if at time step $\tau > 0$ a node u is vaccinated and there is a vulnerable node v such that $(u, v) \in E$, then at time $\tau + 1$, the node v also gets vaccinated. Note that a vulnerable node may be adjacent to both an infected node and a vaccinated node, in which case we assume that the vaccine prevails over the infection and the vulnerable node is vaccinated in the subsequent time step (assuming the opposite does not change the quality of our results). In the spreading model, we will say that a node is vaccinated *directly* when it is vaccinated by the vaccination strategy, and it is vaccinated *indirectly* when it is vaccinated by the spread of the vaccine through the network.

Example 1. To gain some intuition about this problem, consider the example shown in Figure 1 using the non-spreading model of vaccination.

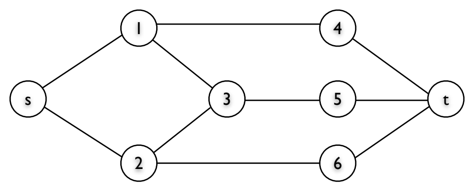

Fig. 1. This example shows that sometimes vaccinating nodes far away from the infection is the only way to save all the required nodes.

Consider the MinBudget objective for this example. The infection begins at node s, and the goal is to find the smallest number B of nodes that need to be vaccinated at every time step so that we can save the node t, which we assume cannot itself be vaccinated. If we were only allowed to cut nodes during the first time-step, this would be equivalent to the minimum s-t node-cut problem. However, unlike previous works, such as [22], which examine the *static* problem of vaccinating a 'cut' before the infection has started spreading, we need to find the "best" *cut over time* (where best depends on the considered objective). This temporal nature of the problem, complicates matters: intuitively, the tradeoff is between vaccinating a small set of nodes close to the infection source early, or spreading out (over time) the vaccination of a larger set of nodes which are farther away from the source.

For instance, in the above example, a minimum s-t node-cut is $\{1, 2\}$, which requires $B = 2$. However, there *is* a solution to the above problem with $B = 1$, but the final set of vaccinated nodes does not form a minimum s-t node-cut. One such solution is to vaccinate vertices 4, 6, and 5 at time steps 1, 2, and 3 respectively, leading to the final set of vaccinated nodes being $\{4, 5, 6\}$ which is not a minimum cut. In fact, it is not hard to come up with examples where the optimal value of B is much smaller than the size of a minimum node s-t cut *and* the final set of vaccinated nodes is much larger than the size of a minimum node s-t cut (e.g., take a graph where s has k neighbors, each of which is connected to t via k long internally node-disjoint paths). Thus, this "cuts over time" problem is quite different from the classical min-cut problem, and in fact is known to be NP-hard (even when the graph is a tree!) [15].

Our Results In Section 2, we consider the model of spreading vaccinations. In general, our results show that this model is more tractable than the model with non-spreading vaccinations. For MaxSave we show that this problem reduces to maximizing a submodular function with a matroid constraint. Therefore a simple greedy algorithm provides a 2-approximation, and a recent result of [6] lets us prove a $(1 - 1/e)$-factor approximation. For MinBudget we give a $O(\log n)$ approximation algorithm, and show that this approximation ratio is tight, by showing a set-cover hardness.

The non-spreading model, on the other hand, does not yield itself to good approximation algorithms. In fact, we show in Section 3 that it is NP-hard to approximate MaxSave in general graphs by a factor of n^{α}, for any $\alpha < 1$. For MinBudget, we give a $O(\sqrt{n})$ factor approximation algorithm for general graphs based on a natural LP relaxation for the problem. For *directed* layered graphs with ℓ layers, we give a $O(H_\ell) = O(\log \ell)$ approximation algorithm. The latter algorithm is combinatorial and requires just one max-flow computation. We show that this result is tight by proving that the integrality gap of the LP is $\Omega(\log \ell)$ for ℓ-layered directed graphs.

Section 3.1 is devoted to vaccination strategies when the underlying graph is a tree. This special case has received a lot of attention [21, 26], is computationally difficult [15], and is in fact a generalization of a complex scheduling problem (details in the full version [1]). For this special case we show that both the spreading and the non-spreading models are equivalent, so the stronger results from Section 2 hold for the non-spreading model as well. In addition, our algorithm for layered graphs also implies a $O(\log h)$ approximation algorithm for MinBudget on trees with height h. Note that this is stronger than the $O(\log n)$ algorithm we have for general graphs in the spreading model.

Related Work Questions about epidemic propagation have been studied in several fields, (e.g., [4,29]), although most of this research models the epidemic as a dynamic system and ignores the effect of the network structure. Recently, a few groups have considered the spread of viruses or ideas on Internet-like topologies, such as small-world networks [32] and preferential attachment models [5, 25]. The papers [10, 13] study targeted vaccinations in this context, and show that they can be used to significantly reduce the effect of an epidemic. These studies assume certain properties of the networks (based on where these networks arise from).

Various recent papers consider modeling vaccination by using graph cuts. For example, the work of Hayrapetyan et al. [22] and others [3, 12] fully utilizes the social-network structure to "cut off" and contain various diffusive processes in a social network. As mentioned earlier, all this work is only concerned with vaccinating a set of nodes before the infection begins, however, and does not have the temporal component of the Firefighter problem. A lot more work has been done on maximizing the spread of an infection (instead of trying to stop it using vaccinations), by selecting the best nodes to infect initially [11, 23].

The *Firefighter problem* was first introduced by B. Hartnell [20], and there has been much work on this problem; see, e.g., [16] for a survey. However, much of the work has focused on special graph structures, such as grids [9,18,31], and that too usually with the MaxSave objective. The Firefighter problem is NP-complete even when the underlying graph is a tree [15], although [21] and [26] give approximation algorithms for this case, and [28] shows how to solve the problem in in polynomial time for special cases of trees.

We have recently learnt that, independent of our work, Chalermsook and Chuzhoy [7] obtain some similar results for the non-spreading version of the MinBudget problem. In particular, they also give an $O(\log \ell)$-approximation for ℓ-layered directed graphs, and obtain an improved approximation for trees.

2 Spreading Vaccination Model

We first show a few simple hardness results about this model, and then give approximation algorithms for both our objectives. Due to lack of space, detailed proofs appear in the full version [1] of the paper.

2.1 General Properties

We make certain useful observations about this model. Let $N(v,i)$ be the set of all the nodes that are a distance of at most i from v.

Lemma 1. *At time τ, all nodes in the neighborhood $N(s,\tau)$ will either be vaccinated or infected.*

Now, since all the nodes in the neighborhood $N(s,\tau)$ will be either infected or vaccinated by time τ, any optimal vaccination strategy would not vaccinate any node in this neighborhood at time $> \tau$. Since any valid strategy can vaccinate only B nodes at any time-step, it means that an optimal strategy would vaccinate at most $B \cdot \tau$ nodes directly in the neighborhood $N(s,\tau)$.

We define a set $\Gamma(v)$ for every node $v \in V$ by

$$\Gamma(v) = \{(u,\tau) | u \in V \text{ and } 0 < \tau \leq (d(s,v) - d(u,v))\}$$

The tuple (v,τ) essentially represents the event of vaccinating node v at time τ.

Theorem 1. *A node $v \in V$ is vaccinated by the vaccination strategy Ψ iff $\Psi \cap \Gamma(v) \neq \emptyset$.*

This theorem tells us that vaccinating an element of $\Gamma(v)$ is exactly what is needed to save a node v, and this provides insight into the structure of the problem.

2.2 Approximation for MaxSave

As we explain in the full version of the paper, the MaxSave problem can be modeled as a problem of maximizing a submodular set function on a collection of sets that form a partition matroid. On the basis of this knowledge, techniques like the greedy algorithm [17] can be used to obtain a $\frac{1}{2}$ approximation for MaxSave, while the randomized algorithm of [6] can be used to obtain a $(1-1/e)$ approximation.

Theorem 2. *There is a randomized algorithm which gives with high probability a $(1-1/e)$-approximation for the* MaxSave *problem. Additionally, a simple greedy algorithm gives a $\frac{1}{2}$-approximation.*

The gist of the proof is as follows. A partition matroid consists of disjoint sets $E_1, \ldots, E_k$, and a set S is called independent if $|S \cap E_i| \leq \ell_i$, for some given numbers $\ell_1, \ldots, \ell_k$. We argue that one can actually consider vaccination strategies that satisfy only property (ii) in Definition 1. This set of strategies forms a partition matroid, since we can only choose at most B nodes at every time-step to vaccinate (so $\ell_i = B$ for all i). We next show that the function $f(\Psi)$ defined (suitably) as the number of nodes saved by using the (possibly invalid) vaccination strategy Ψ is submodular, by using Theorem 1, and if Ψ satisfies the budget-constraint then there is a valid vaccination strategy Ψ' such that $f(\Psi) = f(\Psi')$. Thus, we can use the results of [6,17] on maximizing a submodular function subject to a matroid constraint to obtain the desired approximations.

We note that a $(1-1/e)$-approximation can also be obtained by applying a randomized rounding technique similar to [26] to a modified version of the MaxSave problem. We believe, however, that modeling the problem using partition matroids and submodular functions is fruitful since it provides further insight into the problem, and also yields a rather simple and efficient, combinatorial, deterministic, $\frac{1}{2}$-approximation algorithm.

2.3 Approximation for MinBudget

Consider an instance of MinBudget. First, suppose that we know the optimal budget B that is needed in order to save all nodes of T. Below we give an algorithm that saves all nodes in T using a budget of at most $B \log n$.

By slight adjustments to the proof of Theorem 2, we know that by running the greedy algorithm with budget B, we save at least half of the nodes in T. The greedy algorithm in this case chooses the nodes to vaccinate in each time-step one at a time, always picking the node that saves the most nodes of T. For this purpose the greedy algorithm needs to know exactly which nodes will be saved if a node u is vaccinated at time τ, which we can compute in poly-time. Once finished with the first time-step, the algorithm goes on to the second, and so on.

The complete algorithm is as follows. Repeat the following steps $\log n$ times:

- vaccinate nodes in graph G using the greedy algorithm with budget B.
- Construct graph G_1 from G by removing all the vertices that were vaccinated directly and indirectly in the previous step. Let T_1 be the nodes of T that are in G_1.
- Set $G = G_1$ and $T = T_1$.

It is clear that the new graph G_1 will always contain the original source node s as it is never vaccinated by the greedy algorithm. The $\log n$ applications of the greedy algorithm yield an algorithm that vaccinates $B \log n$ nodes at each step, since at each step, we can simply combine the (at most) B nodes that the greedy algorithm vaccinates at that time step in the $\log n$ runs. We call the resulting algorithm, the *RepGreedy* algorithm.

Theorem 3. *The* RepGreedy *algorithm saves all nodes of T by vaccinating at most $B \log n$ nodes per time-step.*

Finally, to obtain an $O(\log n)$-approximation algorithm without knowing B, we simply do a binary search on B, and run *RepGreedy* for every choice of B.

We complement the above result with the following inapproximability result.

Theorem 4. *The* MinBudget *problem is as hard as set cover, and hence, cannot be approximated in poly-time to a factor better than* $\log n$ *unless P=NP.*

3 Non-Spreading Vaccination Model

The non-spreading model is considerably more difficult than the spreading model. One of the main reasons is that Lemma 1 (or any simple modification of it) is no longer true. The MaxSave problem is NP-complete for bipartite graphs [28] and for cubic graphs (3-regular) [24]. The MaxSave problem is NP-complete even when restricted to trees with maximum degree three [15]. We prove the following about the inapproximability of MaxSave

Theorem 5. *The* MaxSave*(G(V,E),s,B,T) problem cannot be approximated in poly-time to the factor of* n^{α} *where* $n = |V|$ *and* $\alpha < 1$*, unless P=NP.*

We introduce an auxiliary problem, Save-t, which asks whether a specified node t can be saved by vaccinating one node (other than t) at a time. The NP-completeness of this problem follows from known NP-completeness proofs. We then give a gap introducing reduction from the Save-t problem to the MaxSave problem such that if there exists any n^{α} approximation for the MaxSave problem then we can solve the Save-t problem in polynomial time. The proof appears in full version.

In the remainder of the section, we focus on the MinBudget problem. Note that we need to save *all* the nodes in a set T with the minimum number of vaccinations required per time instant. To simplify notation, we consider the following equivalent problem: we add a new node t with edges from all nodes in T to t, and consider the problem of saving t with minimum budget *under the additional constraint that t itself cannot be vaccinated.* We call s the source and t the sink. Let $\mathcal{P}$ denote the collection of all s-t paths.

$$\begin{aligned} \text{Minimize} \quad & B && \text{(Primal)} \\ \text{s.t.} \quad & \sum_{v \in V} x_v^{\tau} \le B && \forall \tau = 1, \dots, n \\ & \sum_{i=1}^{k} \sum_{\tau=1}^{i} x_{v_i}^{\tau} \ge 1 && \forall (s, v_1, \cdots, v_k, t) \in \mathcal{P} \\ & x \ge 0. \end{aligned}$$

$$\begin{aligned} \text{Maximize} \quad & \sum_{P \in \mathcal{P}} f_P && \text{(Dual)} \\ \text{s.t.} \quad & \sum_{\tau=1}^{n} z_{\tau} \le 1 \\ & \sum_{P \in \mathcal{P} : v \in P^{(\tau)}} f_P \le z_{\tau} && \forall v \in V, \tau = 1, \dots, n \\ & z, f \ge 0. \end{aligned}$$

(Primal) gives the LP relaxation of the problem, and (Dual) is the dual of this LP. The primal LP has a variable x_v^τ that indicates whether vertex v is vaccinated at time τ or not. We consider τ going up to n since it is clear that there is no need to vaccinate any vertex after time n. The first constraint bounds the number of vaccinations at every time step. The second constraint says that for every path $(s, v_1, \cdots, v_k, t)$ to the sink t, one of the nodes, say v_i, must be vaccinated *by* time i. This is a necessary and sufficient condition for this path not to transmit the infection to t. In the dual, we have a flow-variable f_P for every s-t path P. The second constraint in the dual is a bit subtle: it says that for every τ, the total flow through a vertex v via paths such that v lies at a distance τ or more from s on the path, is at most z_τ. We use $P^{(\tau)}$ to denote the portion of the path from the τth vertex to t; that is, if $P = (s, v_1, \ldots, v_\tau, \ldots, v_k, t)$, then $P^{(\tau)} = (v_\tau, \ldots, v_k, t)$.

Although the primal LP above has exponentially many constraints, it can be solved in polynomial time since one can give an efficient separation oracle for the LP. Strictly speaking the LP (Primal) may have an integrality gap of $n = |V|$. However note that if OPT denotes the optimal value of (Primal), then in fact $\lceil OPT \rceil$ is a lower bound on the minimum budget, and by comparing the budget of our solution against this lower bound, we prove the following theorems. Similarly, when we say "integrality gap" below, we mean the worst-case ratio of the (integer) optimum budget and $\lceil OPT \rceil$.

Theorem 6. *In the non-spreading model, there is a $2\sqrt{n}$-approximation for the* MinBudget *problem in general graphs.*

(Proof Sketch) At a high level, the algorithm recognizes the set of vertices to be vaccinated by time i by looking at the fraction vaccinated by time i. If this fraction is larger than $1/\sqrt{n}$, then the node is vaccinated by day i. We can then show that in the remaining graph, infection can reach t only using paths of length longer than $\sqrt{n}$, and thus there is a cut of size $\sqrt{n}$ which separates s and t. Thus vaccinating this cut as well completes the algorithm. The analysis is slightly subtle and is deferred to the full version.

We now present an improved approximation algorithm for layered graphs. An *s-t directed layered graph* with ℓ layers is one where (i) s has only outgoing edges, t has only incoming edges; (ii) all nodes except t can be partitioned into sets $L_0 := \{s\}, L_1, L_2, \ldots, L_\ell$ such that for every node $v \in L_{i+1}$ (so $v \neq t$) and every incoming edge (u, v) of v, we have $u \in L_i$. Note that now we only need to consider $\tau = 1, \ldots, \ell$ in (Primal) and (Dual). Let $H_r = 1 + 1/2 + \cdots + 1/r$.

Theorem 7. *There is an H_ℓ-approximation for the* MinBudget *problem in s-t directed ℓ-layered graphs. Furthermore, there are ℓ-layered instances showing that the integrality gap of* (Primal) *is at least $H_\ell = \Omega(\log \ell)$.*

(Proof Sketch) The algorithm sets capacity $1/iH_\ell$ on each vertex of layer i, for all i, and simply computes a minimum s-t vertex cut. It then divides the cut into ℓ pieces, corresponding to the vertices vaccinated on day i. Using the dual LP, we can show that our solution is within H_ℓ of the LP optimum. The integrality gap example is a similar layered graph. We defer the details to the full version.

3.1 Vaccination on Trees

When G is a tree rooted at s, the following observation establishes the equivalence between the spreading model and non-spreading model. For the spreading model on general graphs we defined a function $\Gamma(v)$ as a set of all tuples (u, τ) such that if u is vaccinated directly at time τ then the node v will be saved. For a tree, it is easy to observe that a node v will be saved if any of its ancestors is vaccinated directly before the infection reaches v. Therefore, the optimal strategy will be the same on a given tree irrespective of the vaccination model being spreading or non-spreading. This implies that all the positive results from Section 2 also hold for trees. Since the MinBudget problem on trees with height h yields an instance of MinBudget on an s-t directed graph with h layers, we immediately obtain the following Corollary of Theorem 7.

Corollary 1. *There is an $O(\log h)$-approximation for* MinBudget *on trees, where the set T is the set of leaves and h is the height of the tree.*

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1. Full version can be found at `www.cs.rpi.edu/~eanshel/pubs.html`.
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