1 Writing Arguments

Writing a mathematical argument (aka proof) up which convinces others is a non-trivial task. It is best to think of such an argument as providing a map to a reader. For this, you need to

- Clearly state what you want to prove (your destination).
- Clearly state what you already know (your source).
- Describe the logical path from your source to destination, explaining every "turn" on the way.

Let us take an example. Consider the problem.

Problem 1. For any $x \geq 1$, using first principles prove that $\lceil x \rceil = \Theta(x)$.

Please *write* a solution yourself before reading on. Let us first consider the following solution[1].

Solution 1:

For any $x \geq 1$, to prove that $\lceil x \rceil = \Theta(x)$, we should prove:

a. $\lceil x \rceil = \Omega(x) \quad \rightarrow \lceil x \rceil \geq a_1 x \quad$ for $x \geq 1$.
   Since we know that $\lceil x \rceil \geq x$, then $a_1 = 1$.
   So, for $a_1 = 1, b = 1 \quad \lceil x \rceil = \Omega(x)$.

b. $\lceil x \rceil = O(x) \quad \rightarrow \lceil x \rceil \leq a_2 x \quad$ for $x \geq 1$.
   $2x \geq \lceil x \rceil$
   $2x \geq x + 1 \quad$ "since $\lceil x \rceil \leq x + 1$"
   $x \geq 1 \quad$ "subtract x from both sides."
   which is given.
   Then for $a_2 = 2$ and $x \geq 1 \quad \lceil x \rceil = O(x)$.
   Hence $\lceil x \rceil = \Theta(x)$.

Next, consider the following re-writing.

Solution 2:

To prove that $\lceil x \rceil = \Theta(x)$, we need to prove two things: $\lceil x \rceil = \Omega(x)$ and $\lceil x \rceil = O(x)$.

a. $\lceil x \rceil = \Omega(x)$. To prove $\lceil x \rceil = \Omega(x)$, we need to show two non-negative constants a, b with $a > 0$, such that for every $x \geq b$, we have $\lceil x \rceil \geq a \cdot x$. We assert that picking $a = 1$ and $b = 1$ suffices. Indeed, $\lceil x \rceil \geq x$ for every $x \geq 1$, since $\lceil x \rceil$ is the ceiling. Thus, $\lceil x \rceil = \Omega(x)$.

b. $\lceil x \rceil = O(x)$. To prove $\lceil x \rceil = O(x)$, we need to show two non-negative constants a, b with $a > 0$, such that for every $x \geq b$, we have $\lceil x \rceil \leq a \cdot x$. We claim that $b = 1$ and $a = 2$ suffices. Indeed,

$$\lceil x \rceil \leq x + 1 \leq x + x = 2x$$

where the first inequality follows from the property of the ceiling (the ceiling never exceeds by one more), and the second follows since $x \geq 1$. Thus, $\lceil x \rceil = O(x)$.

Hence $\lceil x \rceil = \Theta(x)$.

[1] This is not any single person's solution. Rather a mish-mash of three actual solutions written by your predecessors.

At the "idea" level, both solutions are same: they both have "solved" the problem. But I would hope you would agree the second writing is clearer. Let me spend some time dissecting why.

Solution 1 starts well: it clearly states in the first line what we need to do, that is, we need to prove the two things. Solution 2 mimics that. However, while trying to prove $\lceil x \rceil = \Omega(x)$, Solution 1 doesn't follow the three principles mentioned above. Or rather, it tries to do so, but fails. In particular, consider the line in Solution 1

$$\lceil x \rceil = \Omega(x) \qquad \rightarrow \lceil x \rceil \geq a_1 x \qquad \text{for } x \geq 1.$$

This line, if you read it, just doesn't make sense. What is a_1 here? (You all are computer scientists, right? Well, in the CS lingo: this code doesn't compile and would give an error of "undefined variable"). The same is repeated in part 2. This is a ***common fallacy*** found in many submissions. *Please make sure you are using things which you have already defined.*

There is another common fallacy found in part 2 of the Solution 1. It is in the *order of arguments*. The solution is trying to assert "for $x \geq 1$, one has $\lceil x \rceil \leq 2x$." But the proof is given "backwards". It starts from where it wants to be and walks backwards to what it knows ($x \geq 1$). *This way of reasoning is just not sound.* In particular, consider the two lines: "$2x \geq \lceil x \rceil$" which is followed by "$2x \geq x + 1 \quad$ since $\lceil x \rceil \leq x + 1$". Does the first line imply the second line? No! The inequalities go the wrong way. More precisely, $2x \geq \lceil x \rceil$, in general, doesn't imply $2x \geq x + 1$. In fact, the solution wants the second line to imply the first – but that's not how one gives directions, right? *Please make sure you give the right order of logical arguments.*

Here is a tip: after you write an argument, don't immediately submit it. Read your solution. Read it the next day. See if it reads well. Maybe get a TA (in private) to read it and critique it.

Writing well takes a LOT of time — so don't be disheartened. In fact, although Solution 2 is "better" than Solution 1, one can do even better. Solution 2 is clear but *not* concise. There is a lot of "verbiage" and rather needless repetition of the definitions. Here is a crisper version of the same.

Solution 3:

> By definition of the ceiling function, $\lceil x \rceil$ is $\geq x$ for any $x \geq 0$, and therefore $\lceil x \rceil = \Omega(x)$. Also by definition of ceiling, we know that $\lceil x \rceil \leq x + 1$. Therefore, if $x \geq 1$, one gets $\lceil x \rceil \leq 2x$. Therefore, $\lceil x \rceil = O(x)$ as well. Thus, $\lceil x \rceil = \Theta(x)$.

Of course, Solution 3 might seem confusing to the reader who has just learned about the Big-Oh notation; Solution 3 is written for someone who knows what the Big-Oh notation is and doesn't need reminding of the a's and b's in the definition.