Inference for Single Proportions using the Normal Distribution

STAT 120

Sections 6.1

Day 16

Background

- **Resampling** inference methods like the bootstrap (CI) and randomization tests require the use of computers!
- We can achieve the same using **statistical theory**
  - Why are most resampling distributions bell-shaped?
  - **CLT**: when n is big enough, means and proportions behave like a normal distribution.
  - Today we will compute SE using formulas derived from probability theory
- The inference methods in ch. 6+ are "**classical**" methods that *could* be done just with pen and paper.

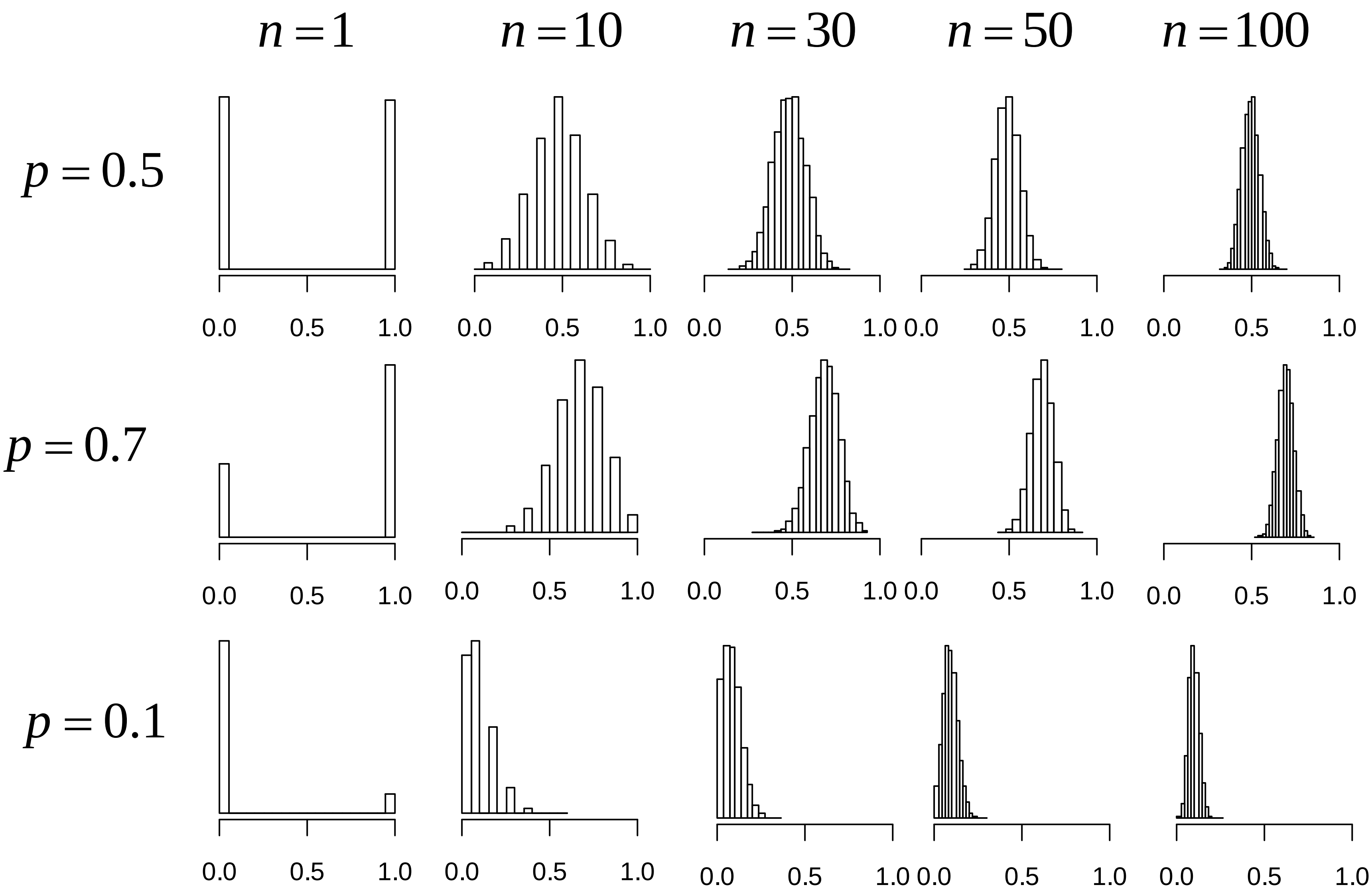
The big question:
Resampling vs. Classical methods

- Once we complete ch. 6 you will usually have two choices of methods for inference
  - Results are often very similar (no practical difference)
- Resampling methods are intuitive and don't require lots of statistical theory/background.
- But in your research fields you will likely only see classical methods used
  - In the "olden days", classical methods were the only thing taught in stats methods classes.
  - Plus more advanced methods usually do rely on classical theory due to their complexity.

The Central Limit Theorem applies to the distribution of the

1. statistic
2. parameter
3. null value
4. data
5. standard error

Distribution of sample proportions



The SE for a Sample Proportion

The standard error for $\hat{p}$ is

$$SE_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$$

- The larger the sample size, the smaller the SE

Central Limit Theorem

For a *sufficiently large sample size*, the distribution of sample statistics for a mean or a proportion is normal

One sample proportion: The sampling distribution for a sample proportion is approximately normally distributed:

$$\hat{p} \approx N\left(p, \sqrt{\frac{p(1-p)}{n}} \right)$$

- Need *n* large enough so $np \geq 10$ and $n(1 - p) \geq 10$

Election polling

- President Biden won 52.4% of the popular vote in Minnesota in the 2020 election.
- If we had sampled 100 likely voters just prior to the election, what would be the SE for the sample proportion of voters for Biden?

$$SE = \sqrt{\frac{0.524 \times 0.476}{100}} \approx 0.05$$

Margin of Error

For a single proportion, what is the **margin of error**?

$$\hat{p} \pm z^* \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

1. $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
2. $z^* \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
3. $2 \times z^* \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Margin of Error and Sample Size

$$\boxed{ME = z^{*} \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$$

You can choose your sample size in advance, depending on your desired margin of error!

Given this formula for margin of error, solve for *n*.

$$n = \left(\frac{z^{*}}{ME}\right)^{2} \hat{p}(1-\hat{p})$$

Margin of Error and Sample Size

$$n = \left(\frac{z^*}{ME} \right)^2 \hat{p}(1 - \hat{p})$$

Neither p nor $\hat{p}$ is known in advance.

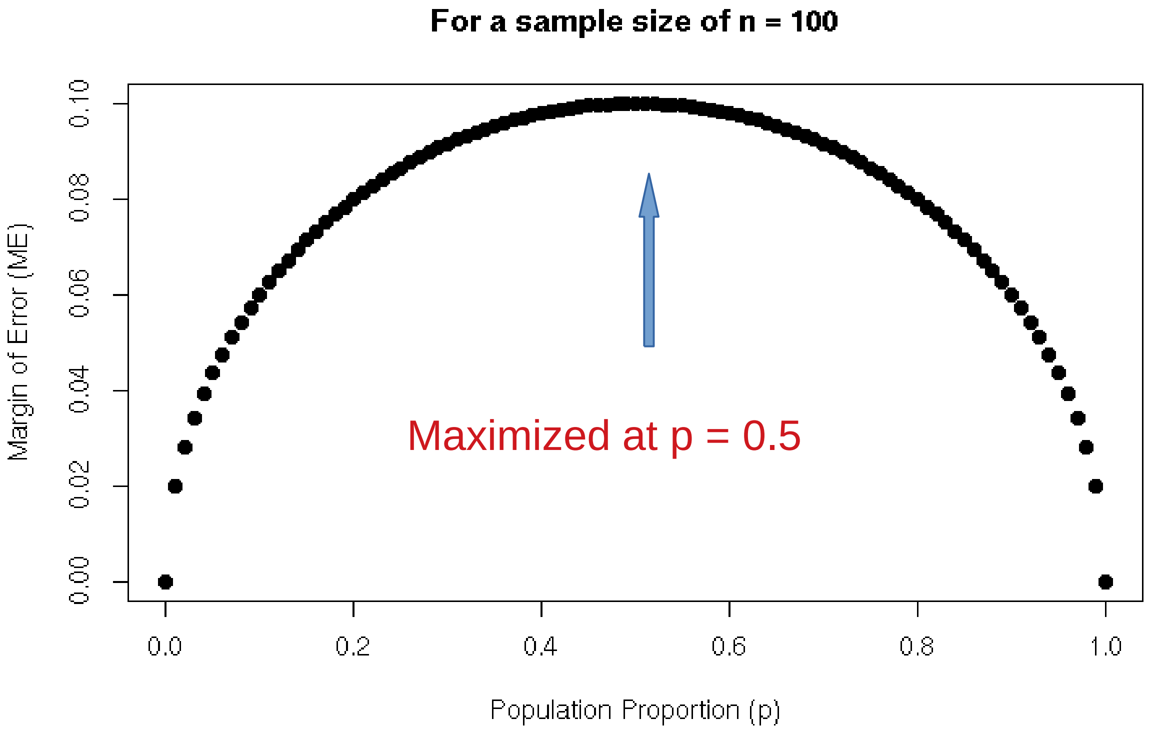
To be conservative, use $p = 0.5$.

For a 95% confidence interval, $z^* \approx 2$

$$n \approx \frac{1}{ME^2}$$

Margin of Error and p

$$n = \left(\frac{z^*}{ME}\right)^2 \hat{p}(1-\hat{p})$$



$$n \approx \frac{1}{ME^2}$$

Margin of Error and n

Suppose we want to estimate a proportion with a margin of error of 0.03 with 95% confidence.

How large a sample size do we need?

1. About 100
2. About 500
3. About 1000
4. About 5000

$$n \approx \frac{1}{ME^2}$$

Election polling continued..

- What should n be to get a margin of error of 3%?

$$0.03 = 2 \text{ ´ } SE$$

$$0.015 = SE = \sqrt{\frac{0.482 \times 0.518}{n}}$$

$$n = \frac{0.524 \times 0.476}{0.015^2} \approx 1109$$

Test for a Single Proportion: Standardized Test Stat and P-value

$$\boxed{\mathrm{H}_0 : p = p_0}$$

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}}$$

- If $np_0 \geq 10$ and $n(1 - p_0) \geq 10$, then the p-value can be computed as the area in the tail(s) of a standard normal beyond z.

Global Warming

Do a majority of Americans believe in global warming?

$H_0 : p = 0.50$

$H_A : p > 0.50$

$p =$ proportion of all Americans who believe in global warming

A survey on 2,251 randomly selected individuals conducted in October 2010 found that 1328 answered "Yes" to the question

"Is there solid evidence of global warming?"

Source: "Wide Partisan Divide Over Global Warming", Pew Research Center, 10/27/10. s

Global Warming

A survey on 2,251 randomly selected individuals conducted in October 2010 found that 1328 answered "Yes" to the question

"Is there solid evidence of global warming?"

Sample proportion: $\hat{p} = \frac{1328}{2251} = 0.590$

Standardized test stat: $z = \frac{0.590 - 0.50}{\sqrt{\frac{0.50(0.50)}{2251}}} = \frac{0.09}{0.0105} = 8.54$

P-value: proportion above z=8.54 on a N(0,1) curve.

```
> 1-pnorm(8.54,0,1)
[1] 0
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Global Warming

Do a majority of Americans believe in global warming?

Yes, there is strong evidence that the percentage of Americans that believe in global warming is greater than 50% (z=8.51, p<0.0001).

How much greater? Want a CI…

But what proportion do we use to compute the SE?

$$SE = \sqrt{\frac{p\ ´\ (1 - p)}{n}}$$

Estimate the SE with the sample proportion

$$\hat{p} = \frac{1328}{2251} = 0.590$$

Confidence Interval for *p*

$$\boxed{statistic \pm z^{*} \cdot SE}$$

For large enough n:

$$\hat{p} \pm z^{*} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$$

Global Warming

How much greater? Want a CI…

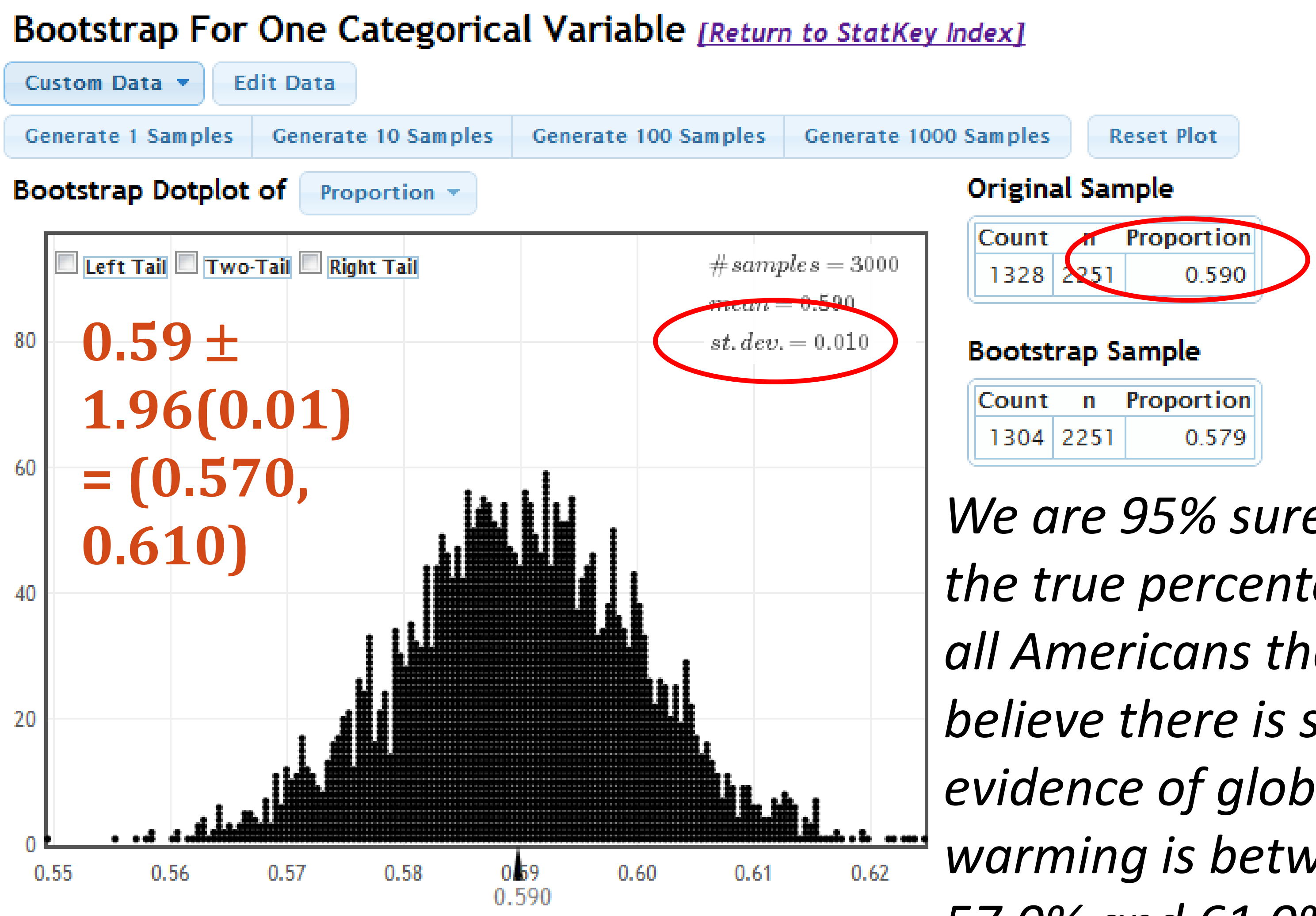
$$0.59 \pm 1.96\sqrt{\frac{0.59 \text{ ´ } (1 - 0.59)}{2251}} = 0.59 \pm 1.96 \text{ ´ } 0.0104$$

$$= (0.570, 0.610)$$

We are 95% confident that between 57% and 61% of Americans believe in global warming.

Does this agree with the bootstrap CI?
Yes!

Global Warming: ch. 3 example



Bootstrap For One Categorical Variable [Return to StatKey Index]

Bootstrap Dotplot of Proportion ▾

□ Left Tail □ Two-Tail □ Right Tail

$\# samples = 3000$

$mean = 0.590$

$st.dev. = 0.010$

Original Sample

Count	n	Proportion
1328	2251	0.590

Bootstrap Sample

Count	n	Proportion
1304	2251	0.579

0.59 ± 1.96(0.01) = (0.570, 0.610)

We are 95% sure that the true percentage of all Americans that believe there is solid evidence of global warming is between 57.0% and 61.0%

Summary

- **Standard error** for a sample proportion:
- **Central Limit Theorem for a proportion:** If counts for each category are at least 10 (meaning *np* ≥ 10 and n(1 – p) ≥ 10), then .
  - **For a CI**, use p-hat in place of p:
  - **For a Hypothesis Test**, use p_0 in place of p when calculating the standardized statistic: