RANDOMIZATION DISTRIBUTIONS & P-VALUES

Sections 4.2 & 4.3

Day 13

STATISTICAL HYPOTHESES

Null Hypothesis (H_0): Claim that there is no effect or difference.

Alternative Hypothesis (H_a): Claim for which we seek evidence.

- Always claims about population parameters.

STATISTICAL SIGNIFICANCE

When results as extreme as the observed sample statistic are *unlikely* to occur by random chance alone (assuming the null hypothesis is true), we say the sample results are ***statistically significant***

- If our sample is **statistically significant**, we have convincing evidence against H_0, in **favor of $\boldsymbol{H_a}$**
- If our sample is **not statistically significant**, our test is **inconclusive. The null hypothesis may be true (or maybe not).**

KEY QUESTION

How unusual is it to see a sample statistic as extreme as that observed, if H_0 is true?

EXTRASENSORY PERCEPTION (EXAMPLE 1)

p = Proportion of correct guesses

H_0: $\boldsymbol{p = 1/5}$

H_a: $\boldsymbol{p > 1/5}$

- Suppose we try this n=10 times and get 3 correct guesses.
- What kinds of statistics (sample proportions) would we observe just by chance, if the null were true and ESP does not exist?
- How can we generate this distribution?

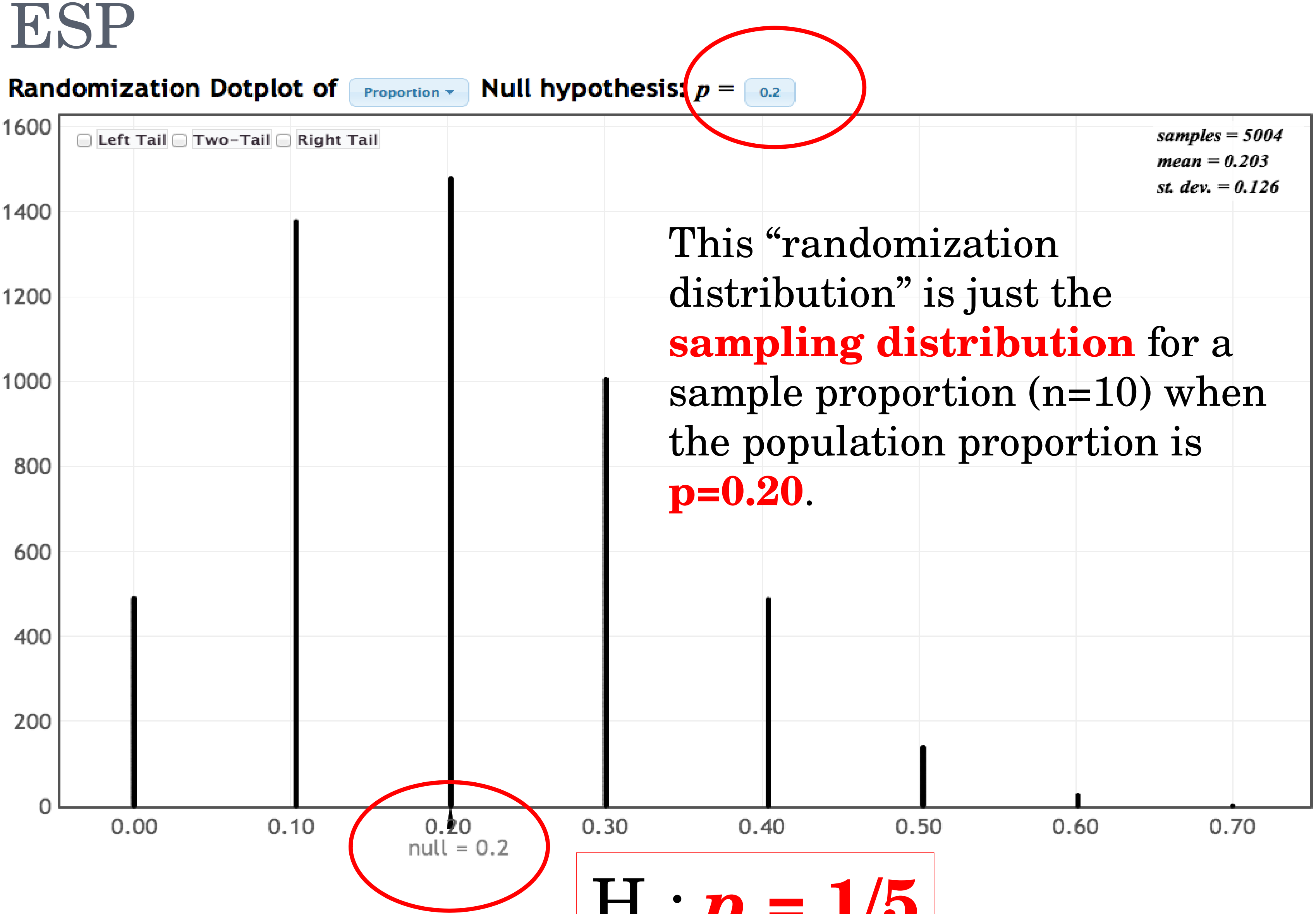
Simulate many samples of size n=10 with p=0.2 and look at the distribution of sample proportions.

RANDOMIZATION DISTRIBUTION

A ***randomization distribution*** is a collection of statistics from samples simulated assuming the **null hypothesis is true**

- Also known as a **permutation distribution**.
- A randomization distribution is **centered** at the value of the **parameter given in the null hypothesis**.

RANDOMIZATION DISTRIBUTION FOR ESP



Randomization Dotplot of Proportion ▾ Null hypothesis: p = 0.2

samples = 5004
mean = 0.203
st. dev. = 0.126

This "randomization distribution" is just the **sampling distribution** for a sample proportion (n=10) when the population proportion is **p=0.20**.

H_0: $\boldsymbol{p = 1/5}$

KEY QUESTION

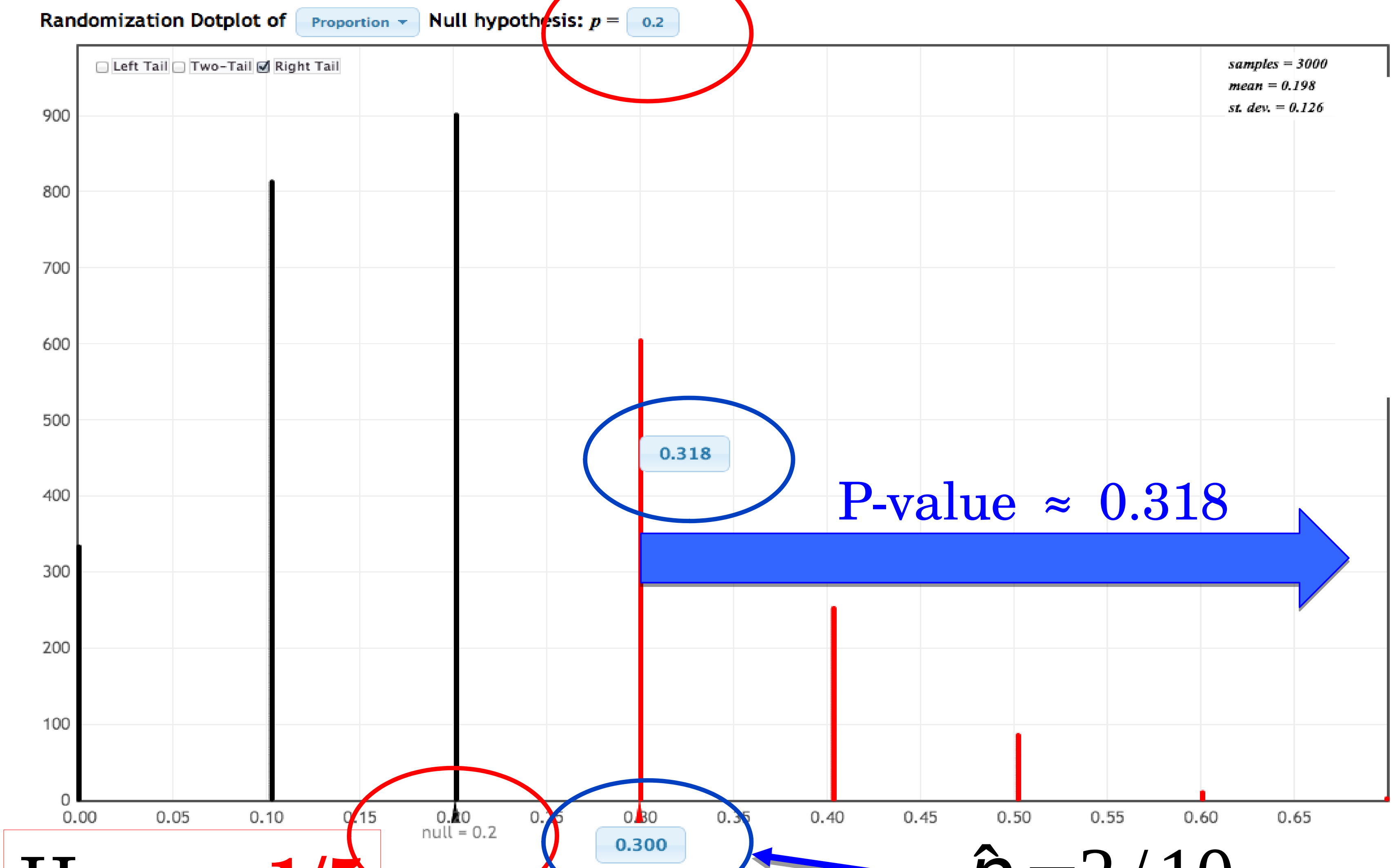
How unusual is it to see a sample statistic as extreme as that observed, if H_0 is true?

P-VALUE

The ***p-value*** is the chance of obtaining a sample statistic as extreme (or more extreme) than the observed sample statistic, if the null hypothesis is true

- The p-value can be calculated as the proportion of statistics in a randomization distribution that are **as extreme (or more extreme) than the observed sample statistic**
- "**extreme**" is determined by the alternative hypothesis

RANDOMIZATION DISTRIBUTION FOR ESP



Randomization Dotplot of Proportion ▾ Null hypothesis: p = 0.2

□ Left Tail □ Two-Tail ☑ Right Tail

samples = 3000
mean = 0.198
st. dev. = 0.126

0.318

0.300

null = 0.2

P-value ≈ 0.318

H_0: ***p* = 1/5**

$\hat{p} = 3/10$

P-VALUE FOR ESP (EXAMPLE 1)

- The ***p-value*** is the chance of getting **at least 3 out of 10 guesses correct**, if $p = 0.2$.
  - P-value is about 0.318.
  - About 31% of the time we would get at least 3 out 10 guesses correct just by chance (no ESP). (interpretation)
  - Which conclusion does this p-value support?

[✓ A.] Inconclusive, little evidence that supports ESP (H_a)

B. Borderline, weak evidence for ESP (H_a)

C. Strong statistically significant evidence for ESP (H_a)

P-VALUE AND H_0

- If the **p-value is small**, then a statistic as extreme as that observed would be unlikely if the null hypothesis were true, providing evidence against H_0 and in **favor of the alternative**
- **Small p-value**
  - Results are statistically significant
  - Reject the null in favor of the alternative
- **Large p-value**
  - Results are not statistically significant
  - Do not reject the null in favor of the alternative

P-VALUE (EXAMPLE 2)

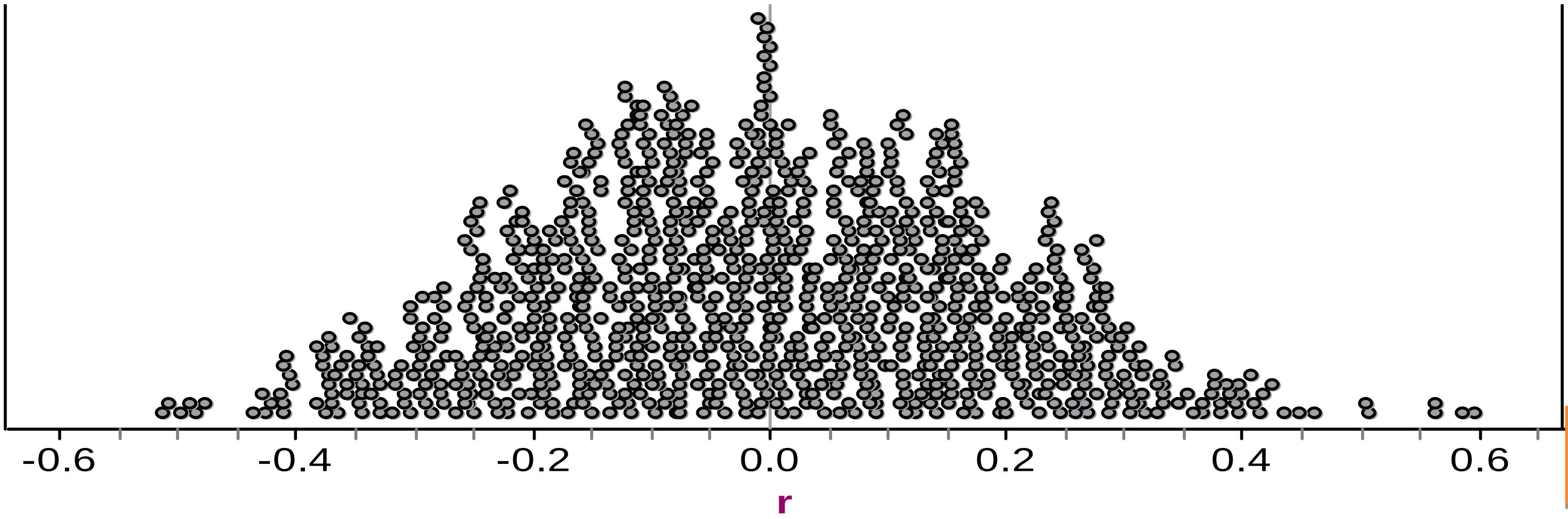
Using the randomization distribution below to test

$H_0 : \quad = 0$ vs $H_a : \quad > 0$

Match the sample correlation and p-values:

Sample Correlation: r = 0.1, r = 0.3, or r = 0.5

P-values: 0.005 , 0.15, or 0.35



SLEEP VERSUS CAFFEINE (EXAMP

- Recall the sleep versus caffeine experiment
- μ_s and μ_c are the true mean number of words recalled after sleeping and after caffeine.
  - $H_0: \mu_s = \mu_c$
  - $H_a: \mu_s \neq \mu_c$

→

$$\boxed{\begin{array}{l} H_0: \mu_s - \mu_c = 0 \\ H_a: \mu_s - \mu_c \neq 0 \end{array}}$$

- How can we create a randomization distribution consistent with the null?
  - What statistic do we compute?
  - Sample difference: $\bar{x}_S - \bar{x}_C$
  - Where is the distribution centered?
  - Distribution centered at a difference of 0 (null)

Sleep versus Caffeine Data

Words	Group
9	sleep
11	sleep
13	sleep
14	sleep
14	sleep
15	sleep
16	sleep
17	sleep
17	sleep
18	sleep
18	sleep
21	sleep

$\bar{x}_s = 15.25$

Words	Group
6	caffeine
7	caffeine
10	caffeine
10	caffeine
12	caffeine
12	caffeine
13	caffeine
14	caffeine
14	caffeine
15	caffeine
16	caffeine
18	caffeine

$\bar{x}_c = 12.25$

$\bar{x}_s - \bar{x}_c = 3$

What kinds of results would you see, just by random chance, ***if sleep or caffeine were equivalent for memory****?*

Rerandomize sleep/caffeine*, but do not change the number of words recalled.*

Sleep versus Caffeine – one **rerandomized** data set (under H_0)

Words	Group
9	sleep
11	caffeine
13	caffeine
14	sleep
14	sleep
15	caffeine
16	sleep
17	caffeine
17	sleep
18	sleep
18	caffeine
21	sleep

Words	Group
6	caffeine
7	sleep
10	sleep
10	caffeine
12	caffeine
12	caffeine
13	caffeine
14	caffeine
14	sleep
15	sleep
16	sleep
18	caffeine

$\bar{x}_S = 14.25$ $\bar{x}_C = 13.25$

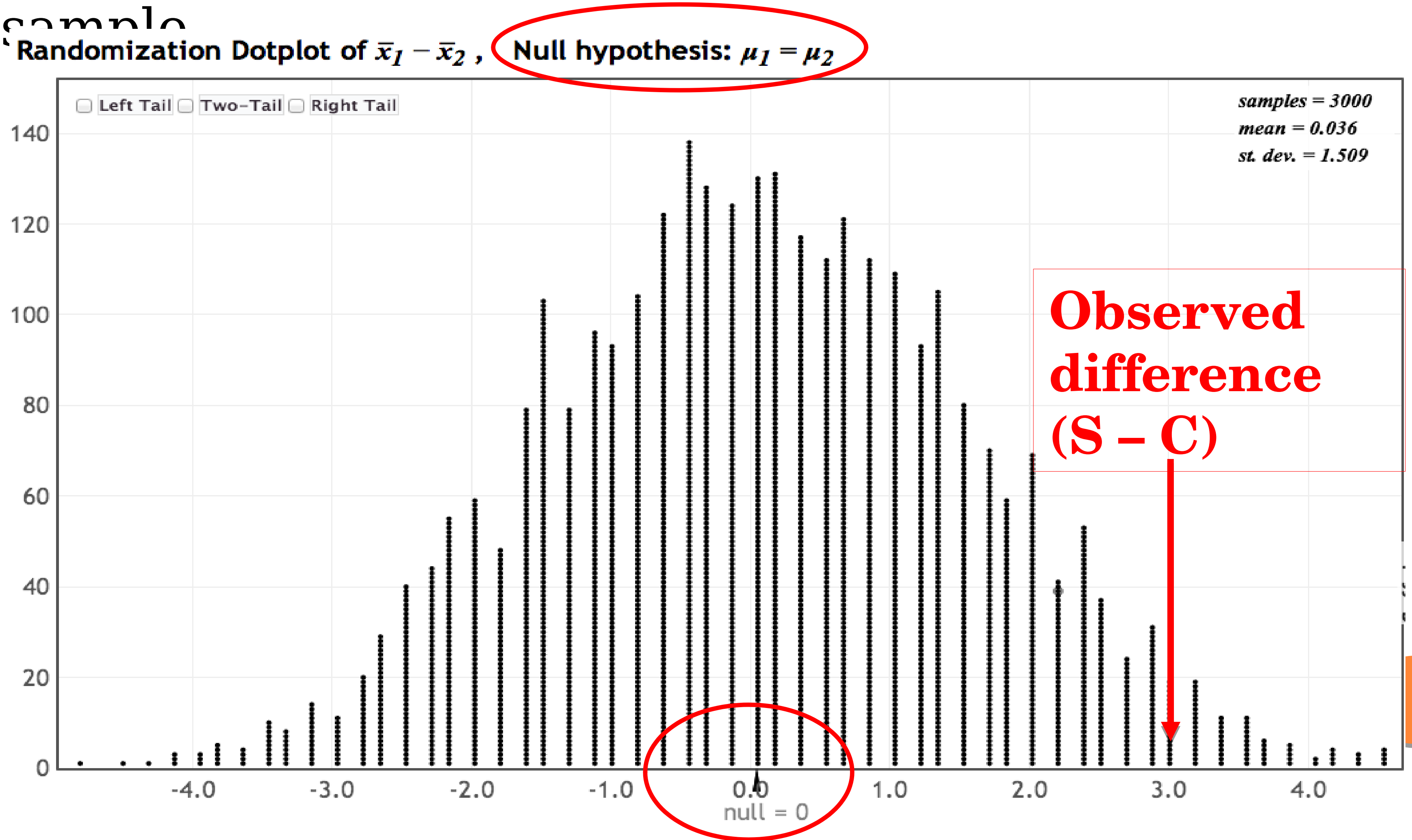
$\bar{x}_S - \bar{x}_C = 1$

What kinds of results would you see, just by random chance, ***if sleep or caffeine were equivalent for memory****?*

Rerandomize sleep/caffeine*, but do not change the number of words recalled.*

Sleep vs. Caffeine: Randomization Distribution

- Rerandomize many, many times.
- Compute difference in means for each rerandomized sample



Randomization Dotplot of $\bar{x}_1 - \bar{x}_2$, Null hypothesis: $\mu_1 = \mu_2$

□ Left Tail □ Two-Tail □ Right Tail

samples = 3000
mean = 0.036
st. dev. = 1.509

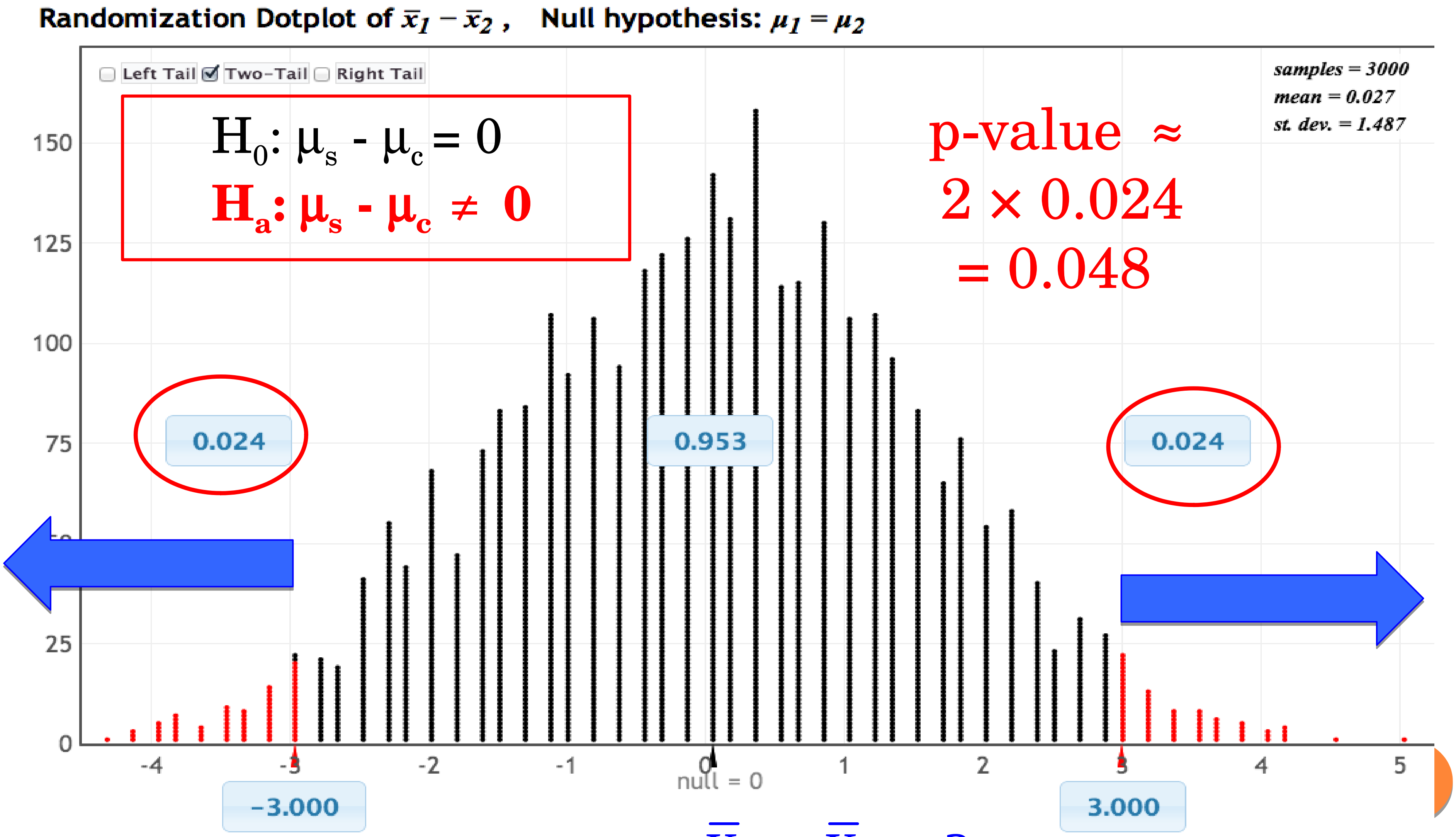
Observed difference (S – C)

SLEEP VERSUS CAFFEINE

$$H_0: \mu_s - \mu_c = 0$$
$$H_a: \mu_s - \mu_c \neq 0$$

- The observed difference is 3 words.
- The p-value is the proportion of samples that yield a **difference in means of 3 or more words** (under randomization model).
  - Two-sided alternative: no direction specified!

Sleep versus Caffeine



Randomization Dotplot of $\bar{x}_1 - \bar{x}_2$, Null hypothesis: $\mu_1 = \mu_2$

□ Left Tail ☑ Two-Tail □ Right Tail

samples = 3000
mean = 0.027
st. dev. = 1.487

H_0: $\mu_s - \mu_c = 0$

H_a: $\mu_s - \mu_c \neq 0$

p-value ≈ 2 × 0.024 = 0.048

$$\bar{X}_S - \bar{X}_C = 3$$

SLEEP VERSUS CAFFEINE (EXAMPLE 3)

$$H_0: \mu_s - \mu_c = 0$$
$$H_a: \mu_s - \mu_c \neq 0$$

- P-value is about 0.048
- About 4.8% of samples will yield a difference in means of 3 or more words if sleep and caffeine have the same influence on memory.
- Which hypothesis does this p-value support?

A. Inconclusive, little evidence that suggests treatments differ

[✓ B.] Borderline, weak evidence that suggests treatments differ

C. Strong statistically significant evidence that suggests treatments differ

Alternative Hypothesis

- The p-value is the proportion in the tail in the direction specified by H_a

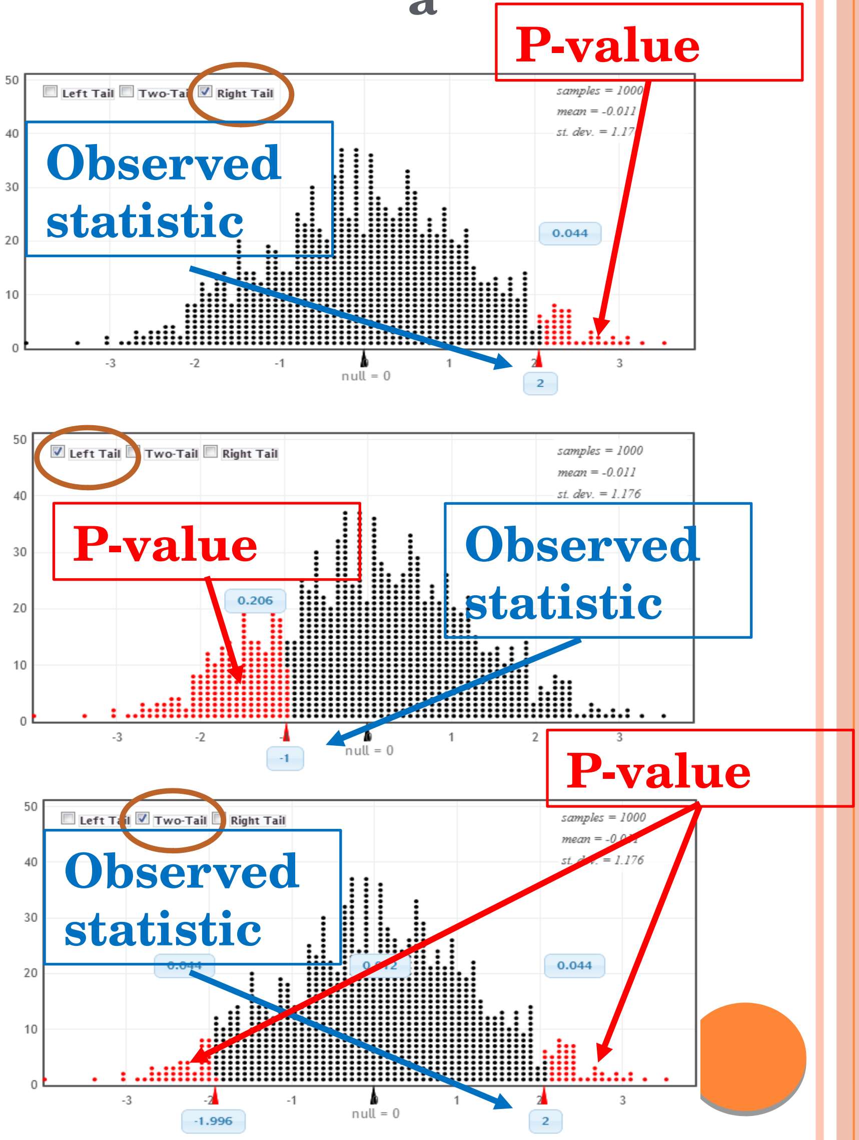
- For a two-sided alternative, the p-value is twice the proportion in the smallest tail

Summary: p-value and H_a

Upper-tail
(Right Tail)
H_a: parameter > null value

Lower-tail
(Left Tail)
H_a: parameter < null value

Two-tailed
H_a: parameter ≠ null value



SUMMARY: RANDOMIZATION DISTRIBUTION FOR ONE PROPORTION

- Null: H_0: p = p_0 where p_0 is the null value of the population parameter p
- Creating a randomization distribution consistent with H_0:
  - Generate a sample of size n from a population with proportion p_0
  - Compute the sample proportion
  - Repeat lots of times

SUMMARY: RANDOMIZATION DISTRIBUTION FOR COMPARING TWO GROUPS 1 AND 2

- Null: H_0: $\mu_1 - \mu_2 = 0$ OR H_0: $p_1 - p_2 = 0$
- Creating a randomization distribution consistent with H_0: **group membership arbitrary (no affect on response)**
  - Randomly permute (re-randomize) the group assignment for all cases
  - Compute the sample mean/proportion for each group and find the difference $\bar{x}_1 - \bar{x}_2$ OR $\hat{p}_1 - \hat{p}_2$ and repeat lots of times

Original data

Case	response	Group
1	x_1	1
2	x_2	1
3	x_3	1
4	x_4	2
5	x_5	2
6	x_6	2
7	x_7	2

Permute groups →

Case	response	**Group**
1	x_1	**2**
2	x_2	**2**
3	x_3	**1**
4	x_4	**2**
5	x_5	**2**
6	x_6	**1**
7	x_7	**1**

SUMMARY: RANDOMIZATION DISTRIBUTION FOR COMPARING TWO GROUPS 1 AND 2

- Comment:
  - Equivalently, we can permute (re-randomized) the response for all cases but leave the group assignments fixed.
  - Will get the same randomization distribution for the difference in means or proportions either way.

Original data

Case	response	Group
1	x_1	1
2	x_2	1
3	x_3	1
4	x_4	2
5	x_5	2
6	x_6	2
7	x_7	2

Permute responses →

Case	**response**	Group
1	x_6	1
2	x_7	1
3	x_3	1
4	x_5	2
5	x_4	2
6	x_1	2
7	x_2	2

SUMMARY: RANDOMIZATION DISTRIBUTION FOR CORRELATION OR SLOPE

- Null: H_0: $\rho = 0$ OR H_0: $\beta = 0$
- Creating a randomization distribution consistent with H_0: **no association between x and y**
  - Randomly permute (re-randomize) one of the variables (either or x or y)
  - Compute the sample correlation/slope r or b.
  - Repeat lots of times

Original data

Case	x variable	y variable
1	x_1	y_1
2	x_2	y_2
3	x_3	y_3
4	x_4	y_4
5	x_5	y_5
6	x_6	y_6
7	x	y

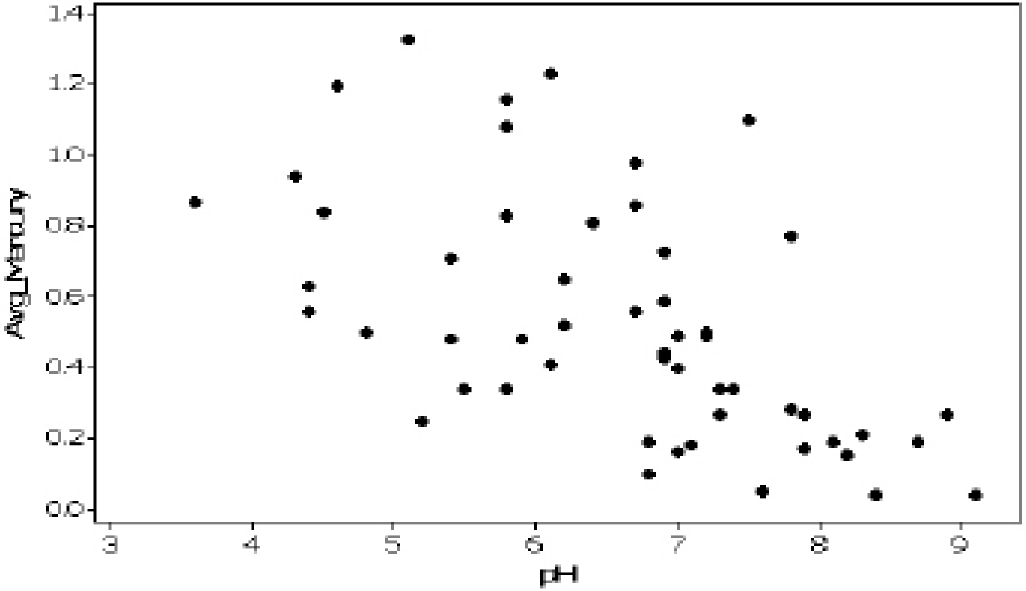
Permute y variable

Case	x variable	**y variable**
1	x_1	y_5
2	x_2	y_2
3	x_3	y_4
4	x_4	y_1
5	x_5	y_6
6	x_6	y_5
7	x_7	y_3

Mercury and pH in Lakes

- For Florida lakes, are lower pH levels (more acidity) associated with higher mercury levels?

$H_0: \beta = 0$ vs. $H_a: \beta < 0$

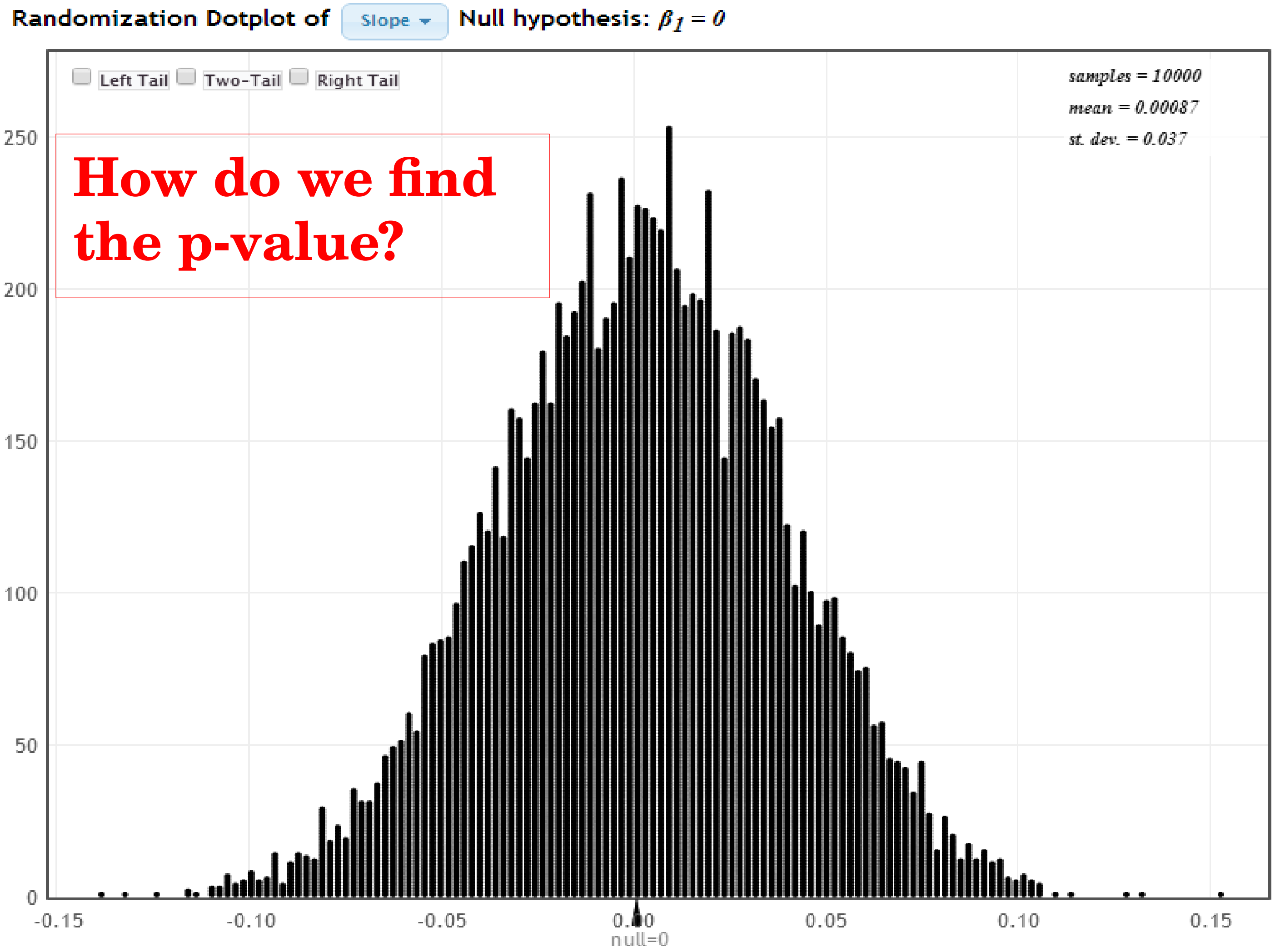


The regression line slope is b = -0.152.

Lange, Royals, and Connor, Transactions of the American Fisheries Society (1993)

Mercury and pH in Lakes

$H_0: \beta = 0$ vs. $H_a: \beta < 0$



Randomization Dotplot of Slope ▾ Null hypothesis: $\beta_1 = 0$

□ Left Tail □ Two-Tail □ Right Tail

samples = 10000

mean = 0.00087

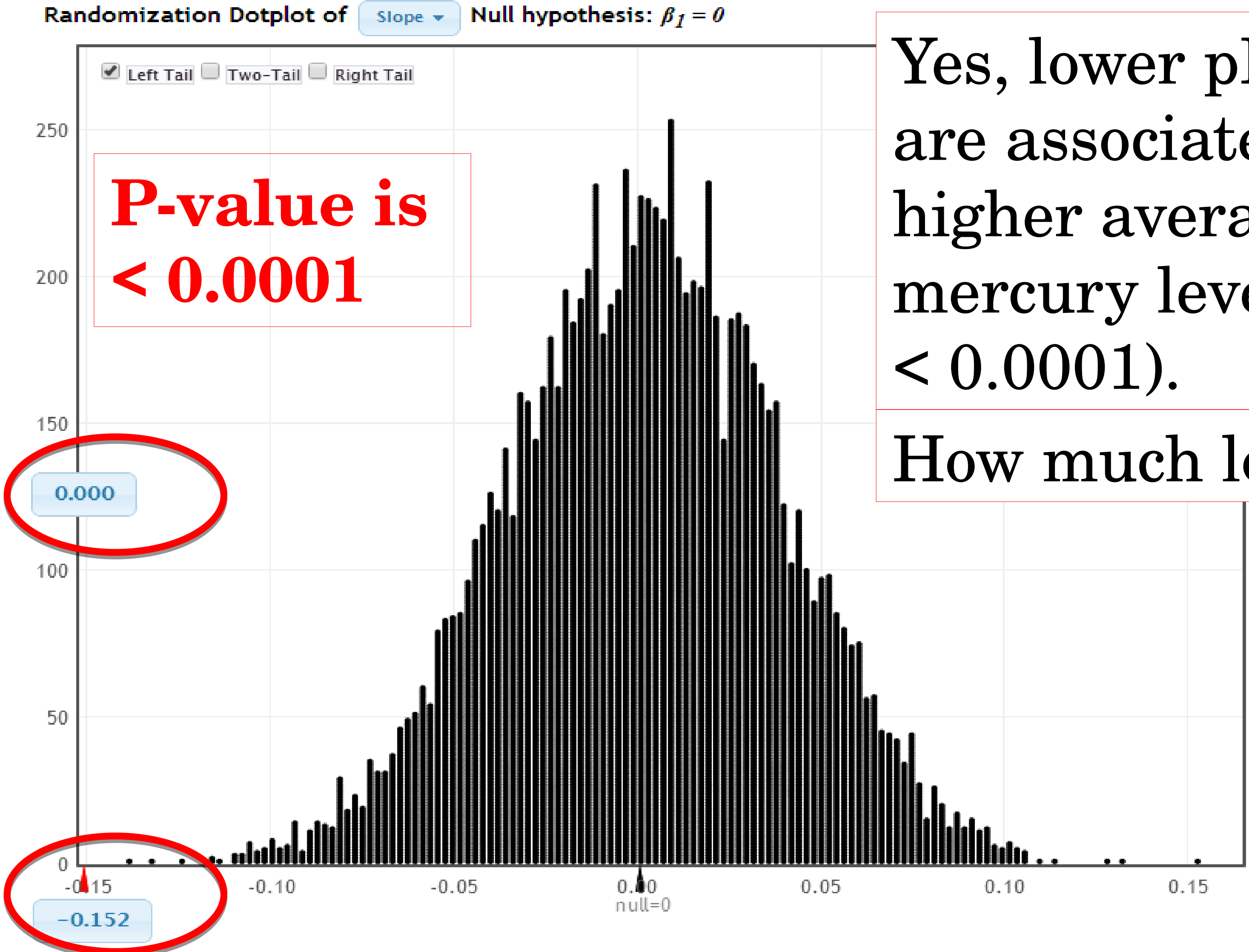
st. dev. = 0.037

How do we find the p-value?

Chance of getting a slope as small, or smaller than, the observed slope of b = -0.152.

Mercury and pH in Lakes

$H_0: \beta = 0$ vs. $H_a: \beta < 0$



Randomization Dotplot of Slope ▾ Null hypothesis: $\beta_1 = 0$

☑ Left Tail □ Two-Tail □ Right Tail

0.000

−0.152

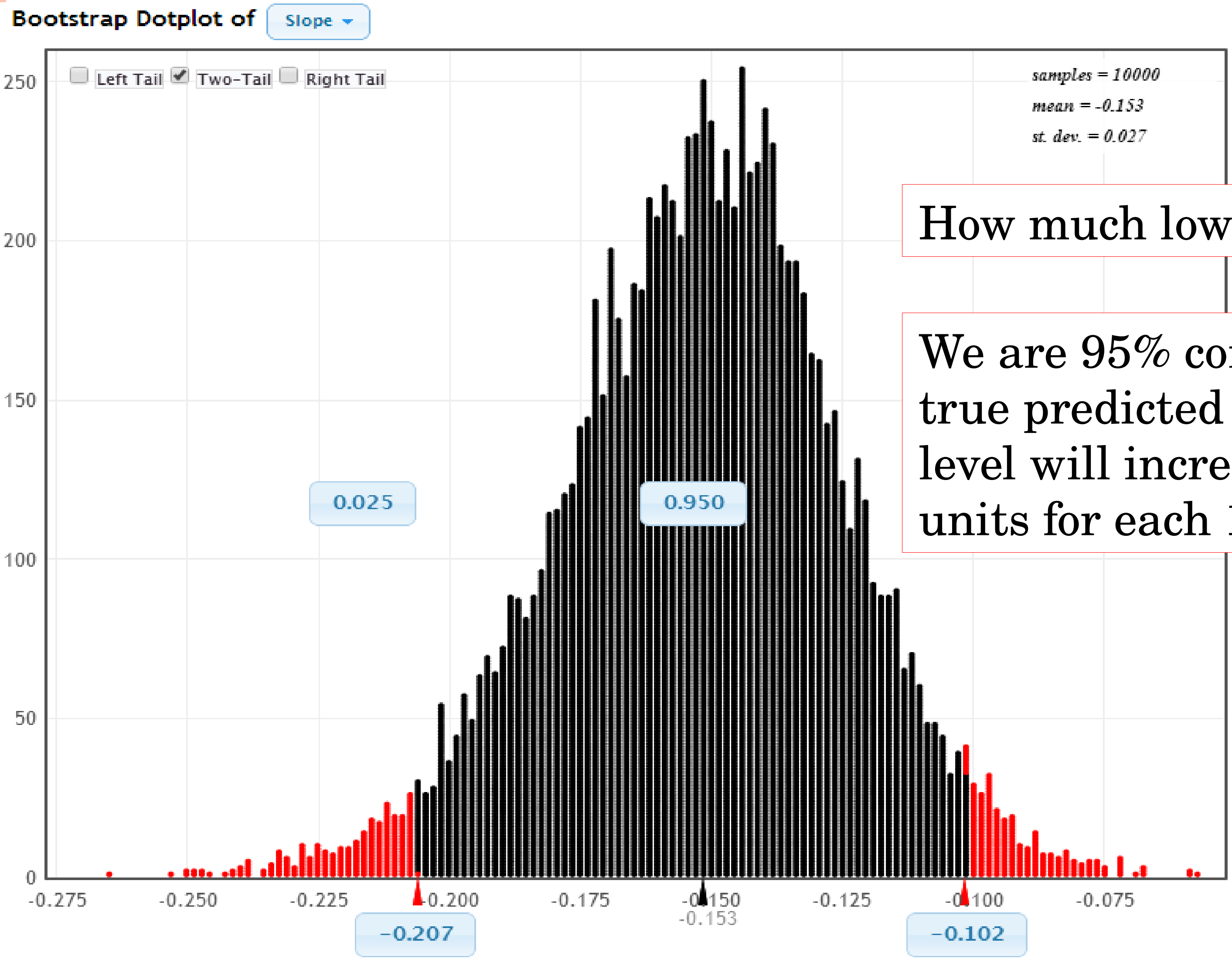
P-value is < 0.0001

Yes, lower pH levels are associated with higher average mercury levels (p-value < 0.0001).

How much lower?

Mercury and pH in Lakes

Bootstrap distribution for the slope



samples = 10000

mean = -0.153

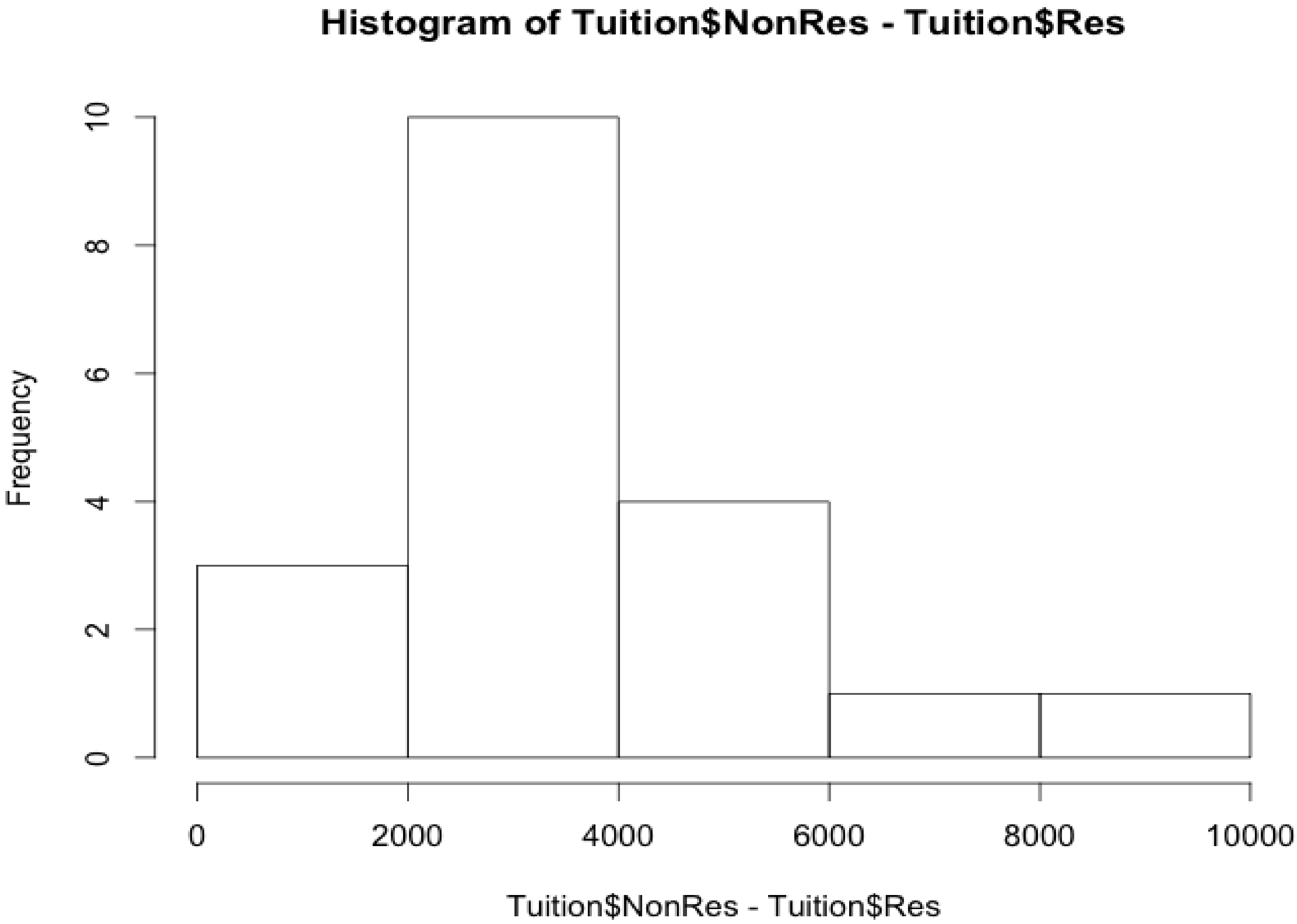
st. dev. = 0.027

How much lower?

We are 95% confident that the true predicted average mercury level will increase by 0.1 to 0.2 units for each 1 unit drop in pH.

TUITION: RESIDENT VS. NON-RESIDENT

- Tuition2006 data from the lab manual section 4.5
- We want to know if the average tuition charged to non-residents is higher than residents for all state colleges and universities
- Population: all state colleges and universities
- Parameters: μ = mean tuition (resident or non-resident) for all colleges and universities
- H_0: $\mu_{non\text{-}resident} - \mu_{resident} = 0$
- H_a: $\mu_{non\text{-}resident} - \mu_{resident} > 0$
- Data: **paired** tuition amounts (resident, non-resident) from a random sample of n=19 schools


Histogram of Tuition$NonRes - Tuition$Res
Frequency
0
2
4
6
8
10
0
2000
4000
6000
8000
10000
Tuition$NonRes - Tuition$Res

TUITION: RESIDENT VS. NON-RESIDENT

- $H_0: \mu_{non\text{-}resident} - \mu_{resident} = 0$
- $H_a: \mu_{non\text{-}resident} - \mu_{resident} > 0$
- How can we create a randomization distribution for paired data?

Original
Data (first 7 cases)

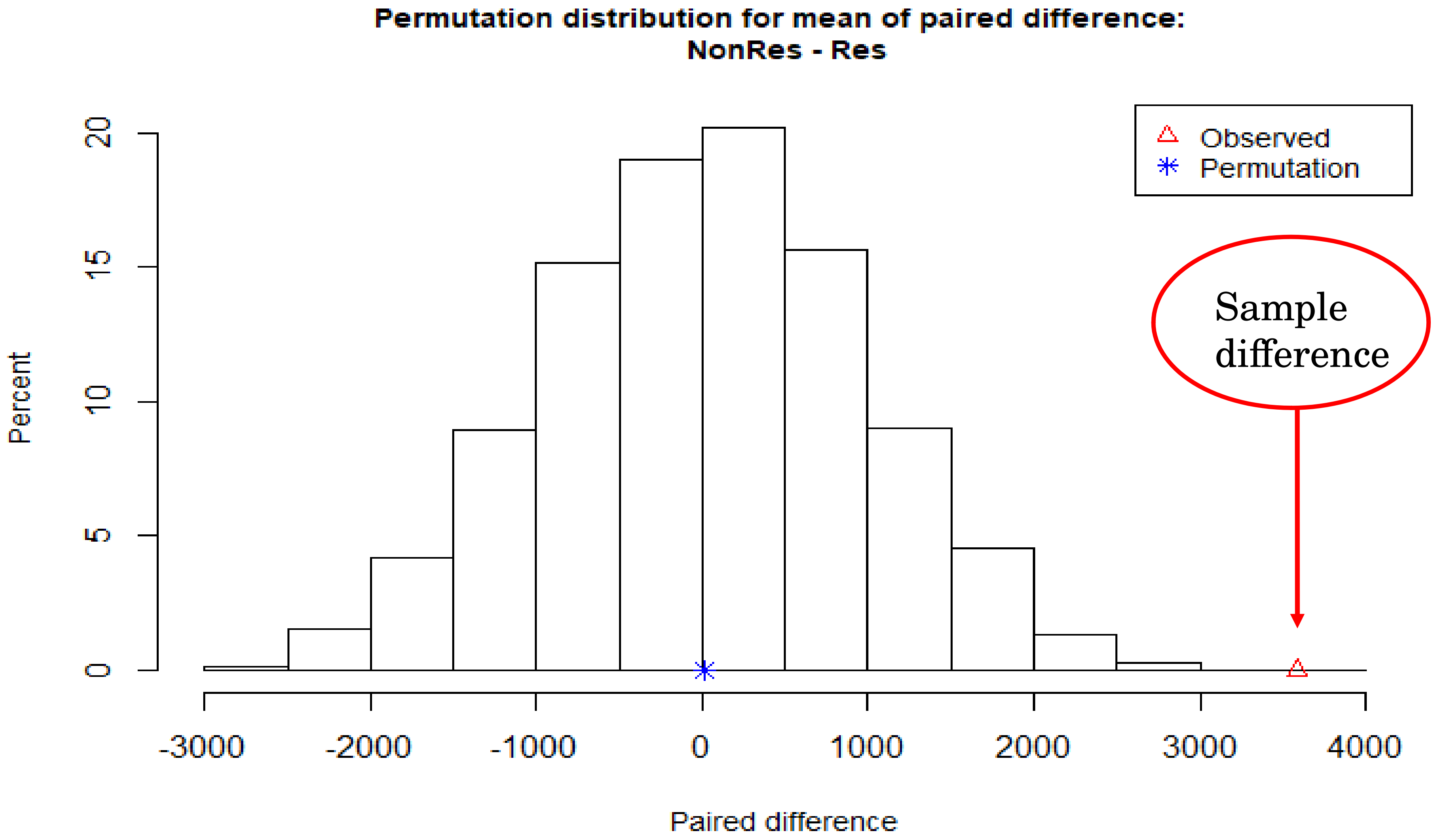
Case	Non-resident tuition	Resident tuition
1	8800	4200
2	3600	1900
3	8600	3400
4	7000	3200
5	12700	3400
6	5700	2600
7	5900	3300

Randomly assign tuition amounts to resident or non-resident for each case

Case	Non-resident tuition	Resident tuition
1	**4200**	**8800**
2	**1900**	**3600**
3	**8600**	**3400**
4	**7000**	**3200**
5	**12700**	**3400**
6	**2600**	**5700**
7	**5900**	**3300**

TUITION: RESIDENT VS. NON-RESIDENT

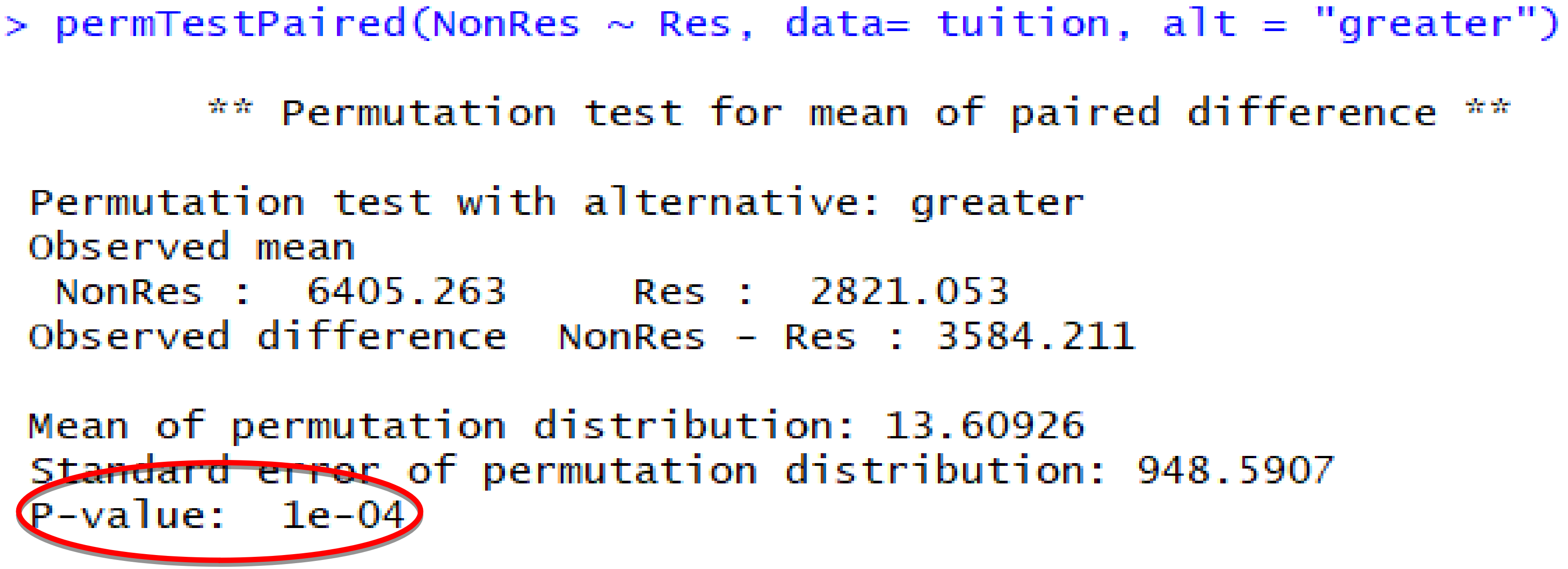
- H_0: $\mu_{non\text{-}resident} - \mu_{resident} = 0$
- H_a: $\mu_{non\text{-}resident} - \mu_{resident} > 0$
- How can we create a randomization distribution for paired data?
  - For each case: Randomly re-assign tuition amounts to resident or non-resident
  - Compute the difference in tuition for non-residents and residents
  - Calculate the mean difference $\bar{x}_{difference}$
  - Repeat lots of times
- Use R to get this randomization distribution



Sample difference

TUITION: RESIDENT VS. NON-RESIDENT

- H_0: $\mu_{non\text{-}resident} - \mu_{resident} = 0$
- H_a: $\mu_{non\text{-}resident} - \mu_{resident} > 0$



```
> permTestPaired(NonRes ~ Res, data= tuition, alt = "greater")

        ** Permutation test for mean of paired difference **

 Permutation test with alternative: greater
 Observed mean
  NonRes :  6405.263     Res :  2821.053
 Observed difference  NonRes - Res : 3584.211

 Mean of permutation distribution: 13.60926
 Standard error of permutation distribution: 948.5907
 P-value:  1e-04
```

- If there was no difference in mean tuition, we would see a mean difference (NR-R) of at least $3584 less than 0.01% of the time. We have very strong evidence that mean tuition for non-residents is higher than for residents.

Formal Decisions

A formal hypothesis test has only two possible conclusions:

1. The p-value is small: reject the null hypothesis in favor of the alternative

2. The p-value is not small: do not reject the null hypothesis

How small?

Significance Level

- The ***significance level***, , is the threshold below which the p-value is deemed small enough to reject the null hypothesis

p-value < Reject H_0

p-value $\geq$ Do not Reject H_0

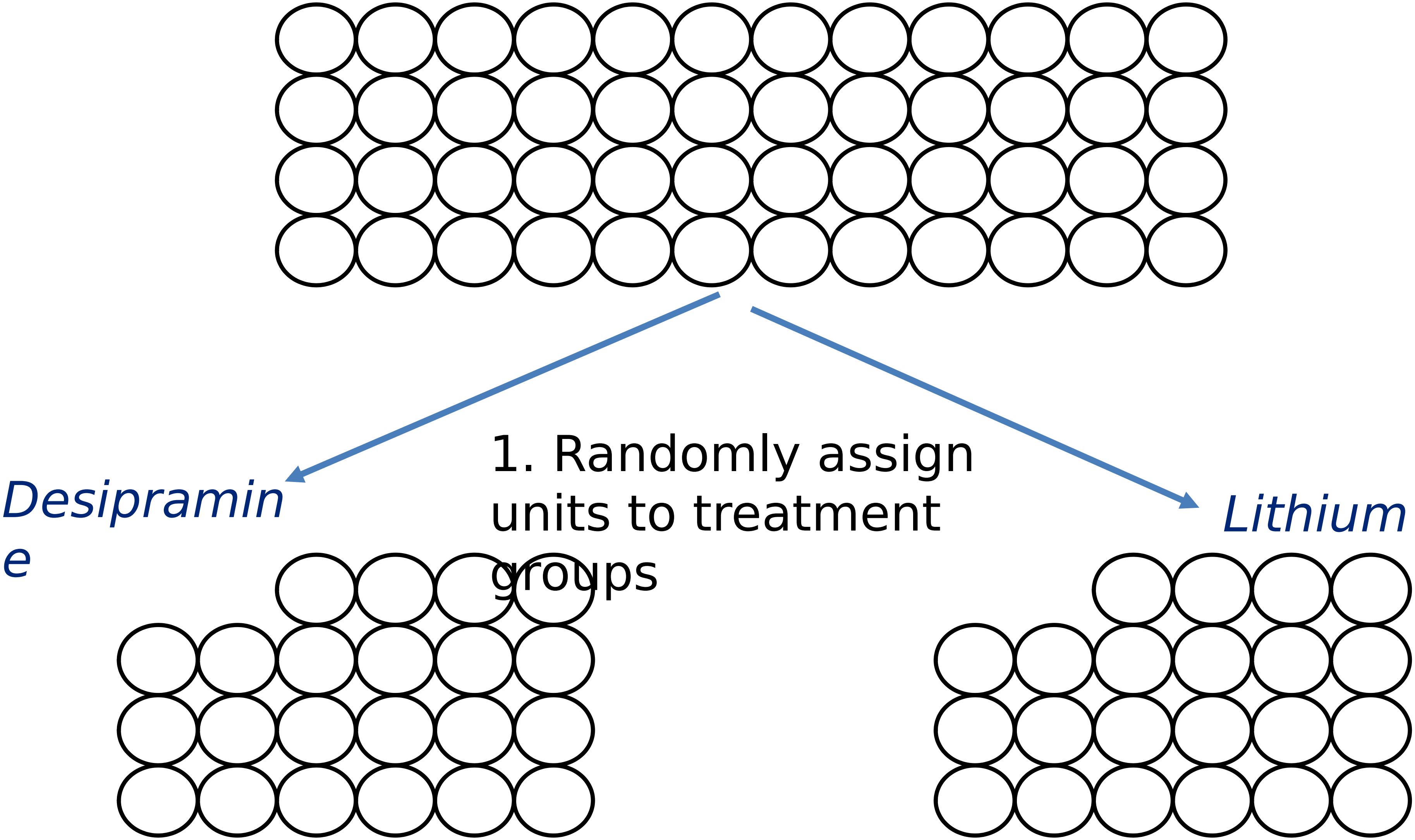
Significance Level

- If the p-value is ***less than*** , the results are ***statistically significant***, and we reject the null hypothesis in favor of the alternative
- If the p-value is ***not*** less than , the results are ***not*** statistically significant, and our test is inconclusive
- Often = 0.05 by default, unless otherwise specified

Cocaine Addiction

- In a randomized experiment on treating cocaine addiction, 48 people were randomly assigned to take either Desipramine (a new drug), or Lithium (an existing drug), and then followed to see who relapsed
- Question of interest:

- We are testing to see if desipramine is better than lithium at treating cocaine addiction.



1. Randomly assign units to treatment groups

Desipramine

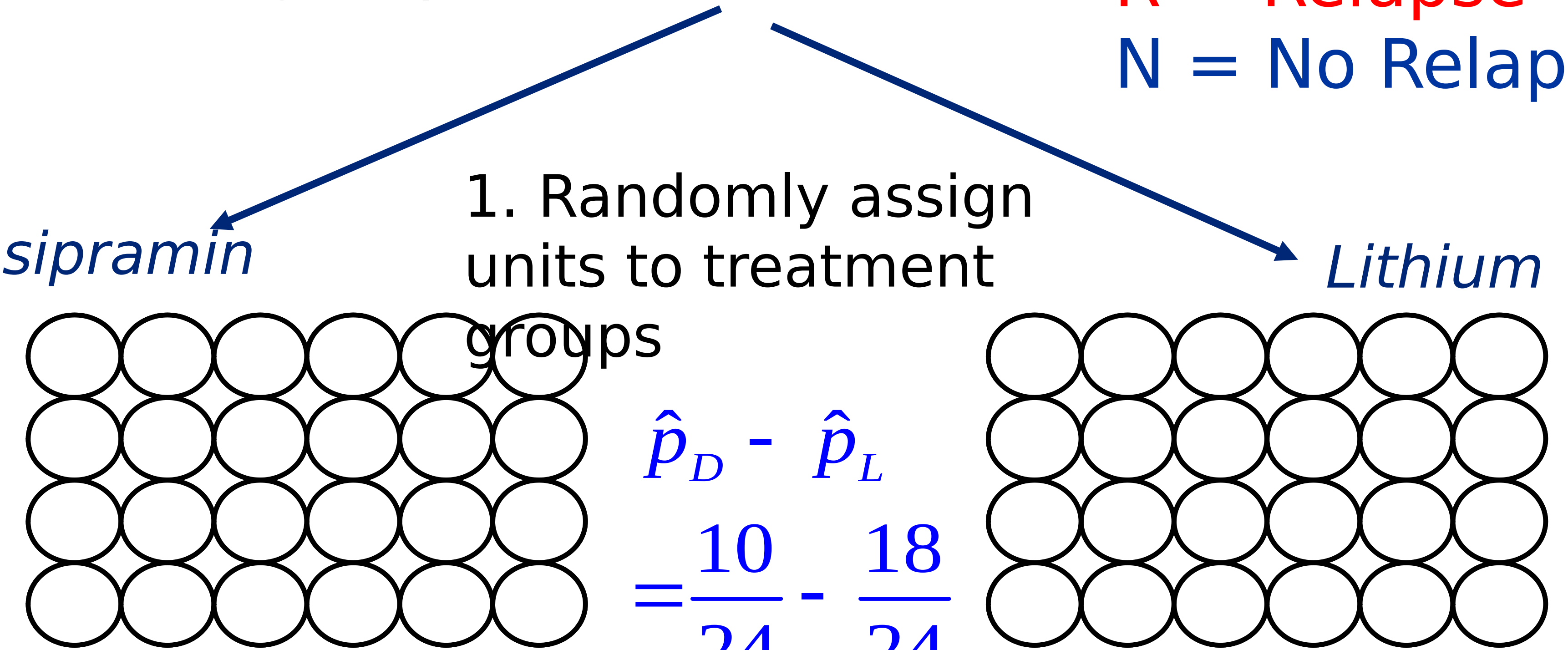
Lithium

2. Conduct experiment

3. Observe relapse counts in each group

R = Relapse

N = No Relapse



1. Randomly assign units to treatment groups

Desipramine

Lithium

$$\hat{p}_D - \hat{p}_L$$

$$= \frac{10}{24} - \frac{18}{24}$$

$$= -.333$$

10 relapse, 14 no relapse

18 relapse, 6 no relapse