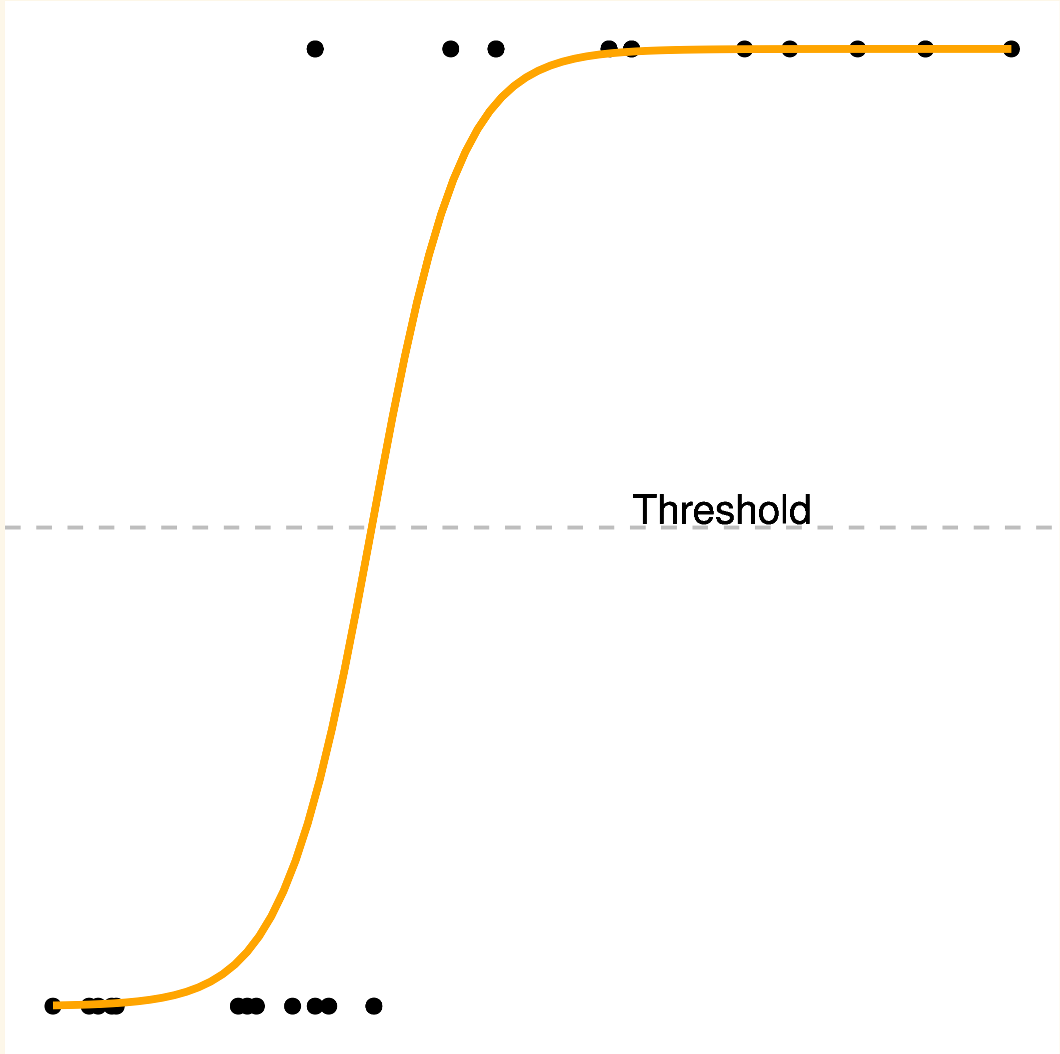
Logistic regression for binomial response

Stat 230

May 18 2022

Overview



Today:

Binomial responses

Logistic model for binomial responses

EDA and inference

Binomial responses

- **Binomial** counts are defined as
  - $Y_i =$ number of successes in m_i independent Bernoulli (success/failure) trials for case (unit) i
- Each case has predictor values:
  - $X_i = (x_{1,i}, \dots, x_{p,i})$ be the predictors for this case
- We want to use a logistic model for:
  - $\pi\left(X_i\right)$ is the probability of success for each of the m_i trials

Case study 21.1: Krunnit Island

In 1949, scientists recorded the number of "at risk" species on each island. Ten years later, they recorded how many of these species were extinct. We want to model the extinction rate for each island as a function of the island size.

- `Island` gives us an identifier of each island (case)
- `Area` is our predictor x_i for each island
- `AtRisk` is m_i, the number of animals available for extinction
- `Extinct` is y_i, the number (out of m_i) that went extinct

```
island <- case2101
glimpse(island)
Rows: 18
Columns: 4
$ Island  <fct> Ulkokrunni, Maakrunni, Ris
$ Area    <dbl> 185.80, 105.80, 30.70, 8.5
$ AtRisk  <int> 75, 67, 66, 51, 28, 20, 43
$ Extinct <int> 5, 3, 10, 6, 3, 4, 8, 3, 5
```

Binomial responses

Our Binomial counts are modeled by a **Binomial** distribution:

$$Y_i \mid X_i \overset{\text{indep.}}{\sim} \text{Binom}(m_i, \pi(X_i))$$

The mean response for each case is

$$E(Y_i \mid X_i) = \mu_{y|x} = m_i \pi(X_i)$$

and the standard deviation is

$$SD(Y_i \mid X_i) = \sigma_{y|x} = \sqrt{m_i \pi(X_i)(1 - \pi(X_i))}$$

Logistic model for Binomial responses

- Logistic function for probabilities:

$$\pi\left(X_i\right) = \frac{e^{\eta_i}}{1+e^{\eta_i}} = \frac{e^{\beta_0+\beta_1 x_{1,i}+\cdots+\beta_p x_{p,i}}}{1+e^{\beta_0+\beta_1 x_{1,i}+\cdots+\beta_p x_{p,i}}}$$

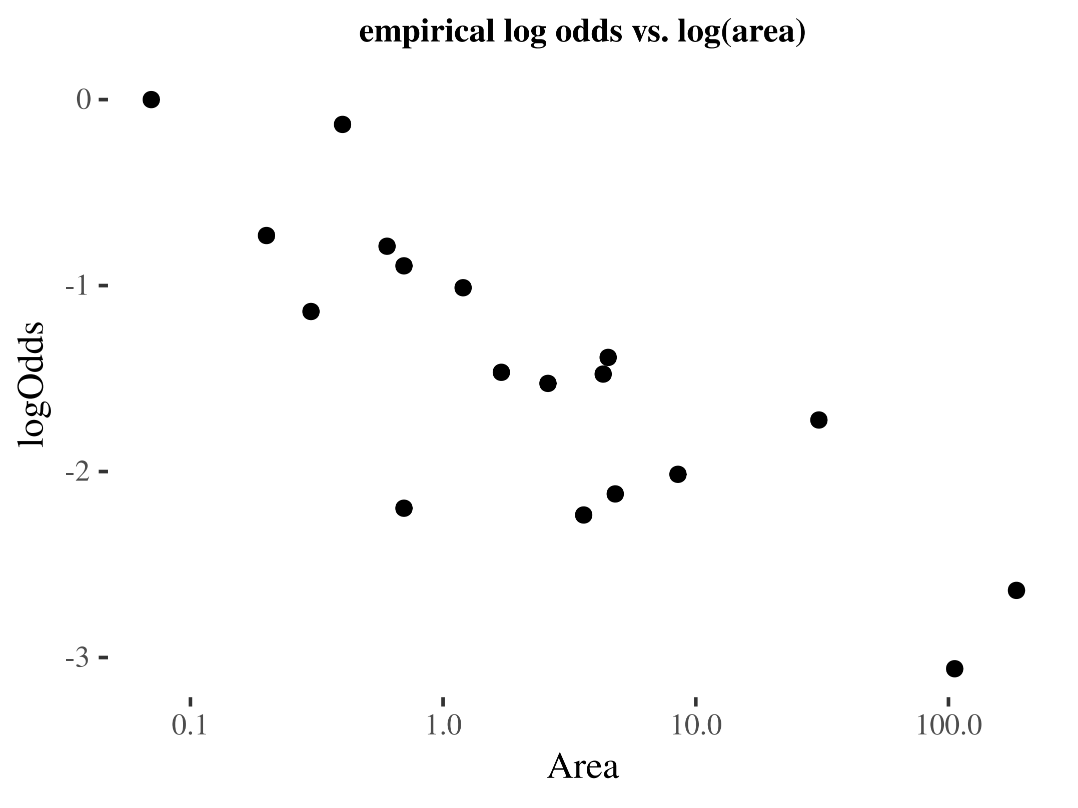
- logit (log odds) function for predictors:

$$\eta_i = \beta_0 + \beta_1 x_{1,i} + \cdots + \beta_p x_{p,i} = \ln\left(\frac{\pi\left(X_i\right)}{1-\pi\left(X_i\right)}\right)$$

- Interpretation of a logistic model for a binomial response is **exactly the same** as a binary response model.

Case study 21.1: Krunnit Island

```
ggplot(island, aes(x = Area, y = logOdds)) +
  geom_point() +
  scale_x_log10() +
  labs(title="empirical log odds vs. log(area)") +
  theme(plot.title = element_text(hjust=0.5, size=9,
                                  face='bold'))
```



Case study 21.1: Krunnit Island

- **Logistic model:** given an island size area_i, Y_i, the number of extinctions on island i, is a binomial variable

$$Y_i \mid \text{ area }_i \sim \text{Binom}(m_i, \pi(\text{ area }_i))$$

where $\pi(\text{ area }_i)$ is the probability of extinction for each of the m_i at risk species on island i.

- Based on the EDA, we will fit the logit model:

$$\log\left(\frac{\pi(\text{ area }_i)}{1-\pi(\text{ area }_i)}\right) = \beta_0 + \beta_1 \ln(\text{ area }_i)$$

Inference for Binomial response models

Similarities between binomial and binary response logistic models:

- Interpretation of β in the two models is the same.
- Inference for models is the same
- "Wald" z-tests and confidence intervals for β parameters are the same.
- Drop-in-deviance model comparison tests are the same.
- R functions of fitted, predict, augment are the same.

Inference for Binomial response models

Differences between binomial and binary response logistic models:

- The formula for deviance G^2 is different because our probability model is (slightly) different.
- In binary models, Wald inference relies on "large n ". In binomial models, we either need "large n " and/or large m_i values. E.g. In the binomial model, $n = 5$ is fine if all $m_i = 1000$.
- The R function `glm` wants a response equal to the empirical proportion of successes, Y_i/m_i, along with the number of binomial trials m_i in our `glm` specification:

$$\text{glm}(y/m \sim x1 + x2, \text{family } = \text{ binomial, weights } = m, \text{data } =)$$

Case study 21.1: Krunnit Island

- **response:** the ratio of y (Extinct) to m (AtRisk)
- **weights:** m (AtRisk):

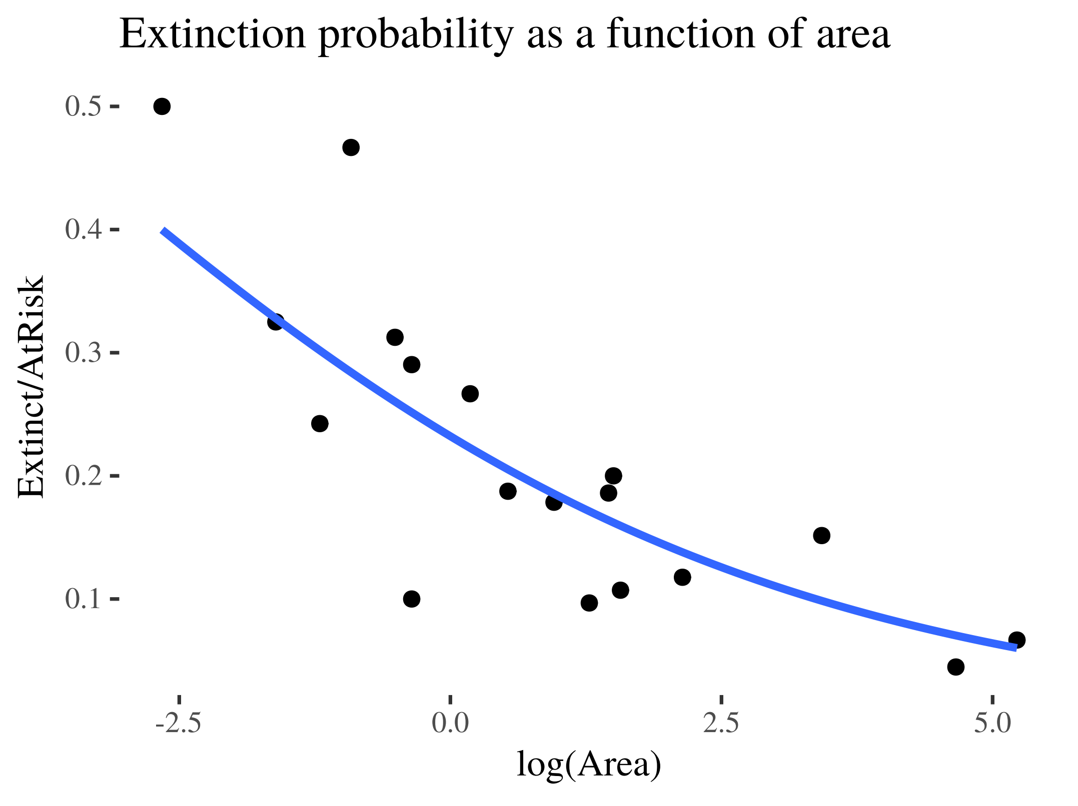
```
krunnit.glm <- glm(Extinct/AtRisk ~ log(Area), family="binomial", weights=AtRisk, data=island)
tidy(krunnit.glm, conf.int=TRUE)
# A tibble: 2 × 7
  term        estimate std.error statistic  p.value conf.low conf.high
  <chr>          <dbl>     <dbl>     <dbl>    <dbl>    <dbl>     <dbl>
1 (Intercept)   -1.20     0.118     -10.1  5.58e-24   -1.43     -0.968
2 log(Area)     -0.297    0.0549     -5.42 6.08e- 8   -0.408    -0.192
```

- The estimated estimated odds of extinction is

$$\frac{\hat{\pi}}{1-\hat{\pi}} = e^{-1.1962-0.29710\ln(\text{ Area })} = e^{-1.1962}\text{Area}^{-0.29710}$$

Case study 21.1: Krunnit Island

```
ggplot(island, aes(x=log(Area), y = Extinct/AtRisk, weight = AtRisk)) +
 geom_point() +
 geom_smooth(method="glm", se=FALSE,
 method.args = list(family="binomial")) +
labs(title="Extinction probability as a function of area")
```



Case study 21.1: Krunnit Island

```
tidy(krunnit.glm, conf.int=TRUE)
# A tibble: 2 × 7
  term        estimate std.error statistic  p.value conf.low conf.high
  <chr>          <dbl>     <dbl>     <dbl>    <dbl>    <dbl>     <dbl>
1 (Intercept)   -1.20     0.118      -10.1  5.58e-24  -1.43     -0.968
2 log(Area)     -0.297    0.0549      -5.42 6.08e- 8  -0.408    -0.192
```

- The estimated odds of extinction for a $5\ \mathrm{km}^2$ island is

$$\log\left(\frac{\hat{\pi}(x=5)}{1-\hat{\pi}(x=5)}\right) = -1.1962 - 0.29710\log(5) = -1.674364$$

```
predict(krunnit.glm, newdata=list(Area=5))
        1
-1.674365
```

- The estimated probability of extinction for at risk species on a $5\ \mathrm{km}^2$ island is

$$\hat{\pi}(x=5) = \frac{e^{-1.1962-0.29710\log(5)}}{1+e^{-1.1962-0.29710\log(5)}} = \frac{e^{-1.674364}}{1+e^{-1.674364}} = 0.1578432$$

```
predict(krunnit.glm, newdata=list(Area=5), type="resp
       1
0.157843
```

Case study 21.1: Krunnit Island

```
tidy(krunnit.glm, conf.int=TRUE)
# A tibble: 2 × 7
  term        estimate std.error statistic  p.value conf.low conf.high
  <chr>          <dbl>     <dbl>     <dbl>    <dbl>    <dbl>     <dbl>
1 (Intercept)   -1.20     0.118     -10.1  5.58e-24   -1.43     -0.968
2 log(Area)     -0.297    0.0549     -5.42 6.08e- 8   -0.408    -0.192
```

- The effect of area on extinction rates is statistically signicant (Wald $\mathrm{z} = -5.416, \mathrm{p} < 0.0001$).
- Effect of doubling island size?

$$OR = \frac{\hat{odds}(2 \times \text{ Area })}{\hat{odds}(\text{ Area })} = 2^{-0.29710} = 0.814, \quad 100\%(0.814 - 1) = -18.6$$

- Doubling an island area is associated with an 18.6% reduction in the odds of extinction of at risk species.
- We are 95% confident that doubling the area of an island is associated with anywhere from a 12.3% to 24.5% decrease in extinction rates.

Case study 21.1: Krunnit Island

```
tidy(krunnit.glm, conf.int=TRUE)
# A tibble: 2 × 7
  term        estimate std.error statistic  p.value conf.low conf.high
  <chr>          <dbl>     <dbl>     <dbl>    <dbl>    <dbl>     <dbl>
1 (Intercept)   -1.20     0.118     -10.1  5.58e-24   -1.43     -0.968
2 log(Area)     -0.297    0.0549     -5.42 6.08e- 8   -0.408    -0.192
```

- Effect of a 10% increase in island size?

$$OR = 1.1^{-0.29710} = 0.972, \quad 1.1^{-0.408} = 0.962, \quad 1.1^{-0.192} = 0.982$$

- We are 95% confident that a 10% increase in the area of an island is associated with a 1.8% to 3.8% decrease in the odds of extinction for at risk species.

Example: NES revisited

- Suppose that the NES data was collected via a **stratified** random sample
  - in 1980: separate random samples of people were taken within each region
  - in 2000: separate random samples of people were taken within each region
- In any given year/region combo $i = 1, \ldots, 8$
  - $m_i =$ number of people surveyed in that year/region (fixed sample size)
  - $Y_i =$ number of Democrat leaning people surveyed in that year/region

Example: NES revisited

```
nes_binom_data <- nes %>%
  group_by(region, year) %>%  # sample size by year/region
  summarize(m = n(), Dem_count = sum(dem))  # number Dem
nes_binom_data
```

```
# A tibble: 8 × 4
# Groups:   region [4]
  region year         m Dem_count
  <chr>  <chr>    <int>     <int>
1 NC     year1980   262       128
2 NC     year2000   301       162
3 NE     year1980   226       112
4 NE     year2000   201       122
5 S      year1980   380       228
6 S      year2000   426       210
7 W      year1980   186        96
8 W      year2000   250       136
```

- m: the number of people per year/region ($n()$ counts the number of rows for each group)
- `Dem_count:` the number of Democrats per year/region
- Binomial count data with region and year predictors

Example: NES revisited

$$Y_i \mid \text{ year, region } \sim \text{Binom}(m_i, \pi_i)$$

where π_i is the probability of Dem in that year/region combo i.

- We can model π as a function of year and region
- we get the same results for both the binomial and binary versions of this data!

Example: NES revisited

Binary version: using person-level responses and our `dem` indicator of Democrat

```
nes_glm_binary <- glm(dem ~ region*year,
                      data = nes,
                      family = binomial)
```

Binomial version: using aggregated year/region level counts and our `Dem_count/m` proportion Democrats for each year/region combo

```
nes_glm_binom <- glm(Dem_count/m ~ region*year,
                     weights = m,
                     data = nes_binom_data,
                     family = binomial)
```

Example: NES revisited

```
summary(nes_glm_binary)

Call:
glm(formula = dem ~ region * year, family = binomial, data = nes)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.3666  -1.2049   0.9993   1.1131   1.1969

Coefficients:
                      Estimate Std. Error z value Pr(>|z|)
(Intercept)           -0.04581    0.12359  -0.371  0.71090
regionNE               0.02811    0.18159   0.155  0.87698
regionS                0.45127    0.16199   2.786  0.00534 **
regionW                0.11035    0.19184   0.575  0.56515
yearyear2000           0.19893    0.16924   1.175  0.23982
regionNE:yearyear2000  0.25334    0.25923   0.977  0.32842
regionS:yearyear2000  -0.63257    0.22136  -2.858  0.00427 **
regionW:yearyear2000  -0.08701    0.25748  -0.338  0.73540
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 3083.3  on 2231  degrees of freedom
Residual deviance: 3065.5  on 2224  degrees of freedom
AIC: 3081.5

Number of Fisher Scoring iterations: 4
```

Example: NES revisited

```
summary(nes_glm_binom)

Call:
glm(formula = Dem_count/m ~ region * year, family = binomial,
    data = nes_binom_data, weights = m)

Deviance Residuals:
[1]  0  0  0  0  0  0  0  0

Coefficients:
                      Estimate Std. Error z value Pr(>|z|)
(Intercept)           -0.04581    0.12359  -0.371  0.71090
regionNE               0.02811    0.18159   0.155  0.87698
regionS                0.45127    0.16199   2.786  0.00534 **
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regionS:yearyear2000  -0.63257    0.22136  -2.858  0.00427 **
regionW:yearyear2000  -0.08701    0.25748  -0.338  0.73540
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 1.7792e+01  on 7  degrees of freedom
Residual deviance: 1.7764e-14  on 0  degrees of freedom
AIC: 64.266

Number of Fisher Scoring iterations: 2
```

Recall: Inference for Binomial response models

Differences between binomial and binary response logistic models:

- The formula for deviance G^2 is different because our probability model is (slightly) different.
- Binomial version assumes that the year/region sample size counts m_i are fixed
- Binomial version we have a sample size of $n = 8$ year/region observations
- In Binomial version we have no degrees of freedom left with 8 parameters in the interaction model!
- The predicted probability $\hat{\pi}_i$ is just the sample proportion of Dem for each year/region

Example: NES revisited

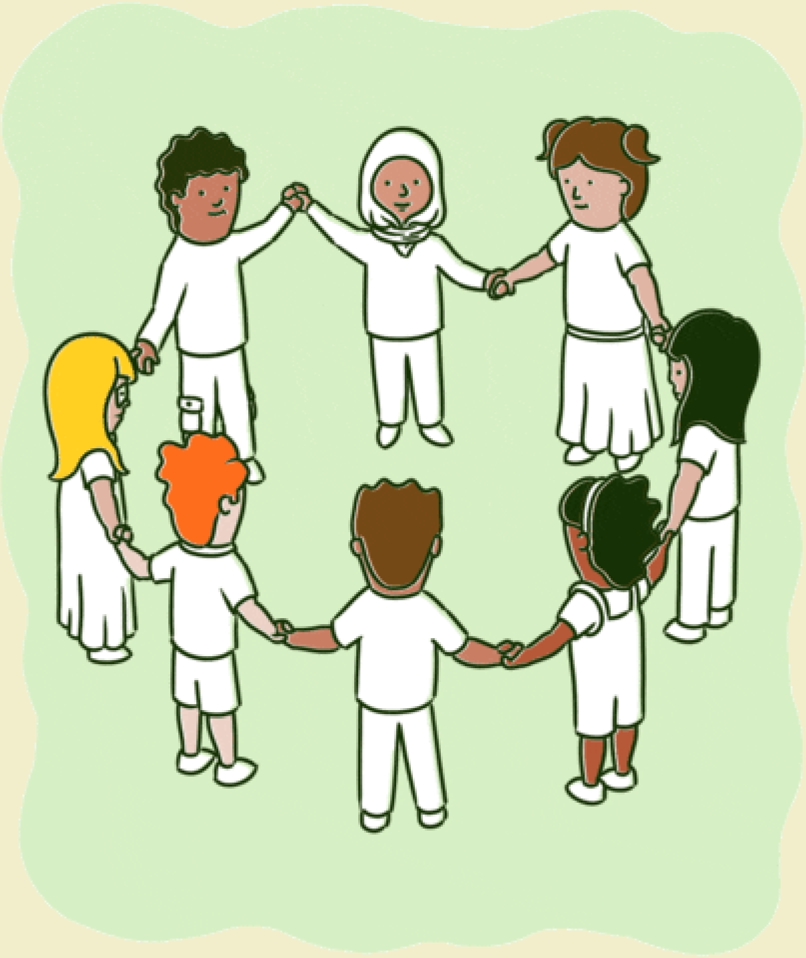
- Which model (binary vs binomial) to use?
- Assuming m_i are fixed sample sizes (e.g. a stratified design), doesn't much matter*
- unless you'd like to incorporate additional individual level predictors of a person's probability of voting Democratic $\rightarrow$ use binary

```
head(nes) # more individual level data!
```

```
      year age gender  race region     income union dem       educ
1 year1980  70   male black      S  lower 1/3    no   1 HS or less
2 year1980  67   male white     NC middle 1/3   yes   1 HS or less
3 year1980  47 female black      S  lower 1/3    no   1 HS or less
4 year1980  52 female white      W  upper 1/3   yes   0    College
5 year1980  30 female white     NC  upper 1/3    no   1 HS or less
6 year1980  37   male black     NC  upper 1/3    no   1    College
```

Your Turn 1

05:00



- Go over to the in class activity file
- Go over the class activity in your group