Inference for means using the t-distribution

Stat 120

May 09 2022

The SE for means

- The standard error for $\bar{x}$ is

$$SE_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

  where σ is the population SD of your response

- The standard error for $\bar{x}_1 - \bar{x}_2$ is

$$SE_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

- But we usually do not know σ !
  - Estimate σ with the sample SD s

Central Limit Theorem for means: One sample

The sampling distribution for a sample mean is approximately $N\,(\mu, SE_{\bar{x}})$

When is this approximately "good"?

- if $X \sim N(\mu, \sigma)$ then $\bar{X} \sim N(\mu, \sigma/\sqrt{n})$
- if $X \nsim N(\mu, \sigma)$ then $\bar{X} \sim N(\mu, \sigma/\sqrt{n})$ if $n \geqslant 30$

Problem!

- The estimated SE varies from sample to sample, along with $\bar{x}$!
- In z, only $\bar{x}$ varies from sample to sample

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

- In t, both $\bar{x}$ and s vary from sample to sample

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} \sim ???$$

Central Limit Theorem for means: Two independent samples:

The sampling distribution for a difference of two independent sample means is approximately $N\left(\mu_1 - \mu_2, SE_{\bar{x}_1 - \bar{x}_2}\right)$

When is this approximately "good"?

- need both n_1 and n_2 samples sizes big enough for the one-sample condition

Inference for means

Check: Check the one- or two-sample size conditions for the CLT

Tests: Use t-ratios of the form

$$t = \frac{\text{stat} - \text{null value}}{\text{SE}}$$

P-values computed from a t-distribution with appropriate df

- `pt(t, d f=)` gives the area to the left of t

Confidence intervals: CI of the form

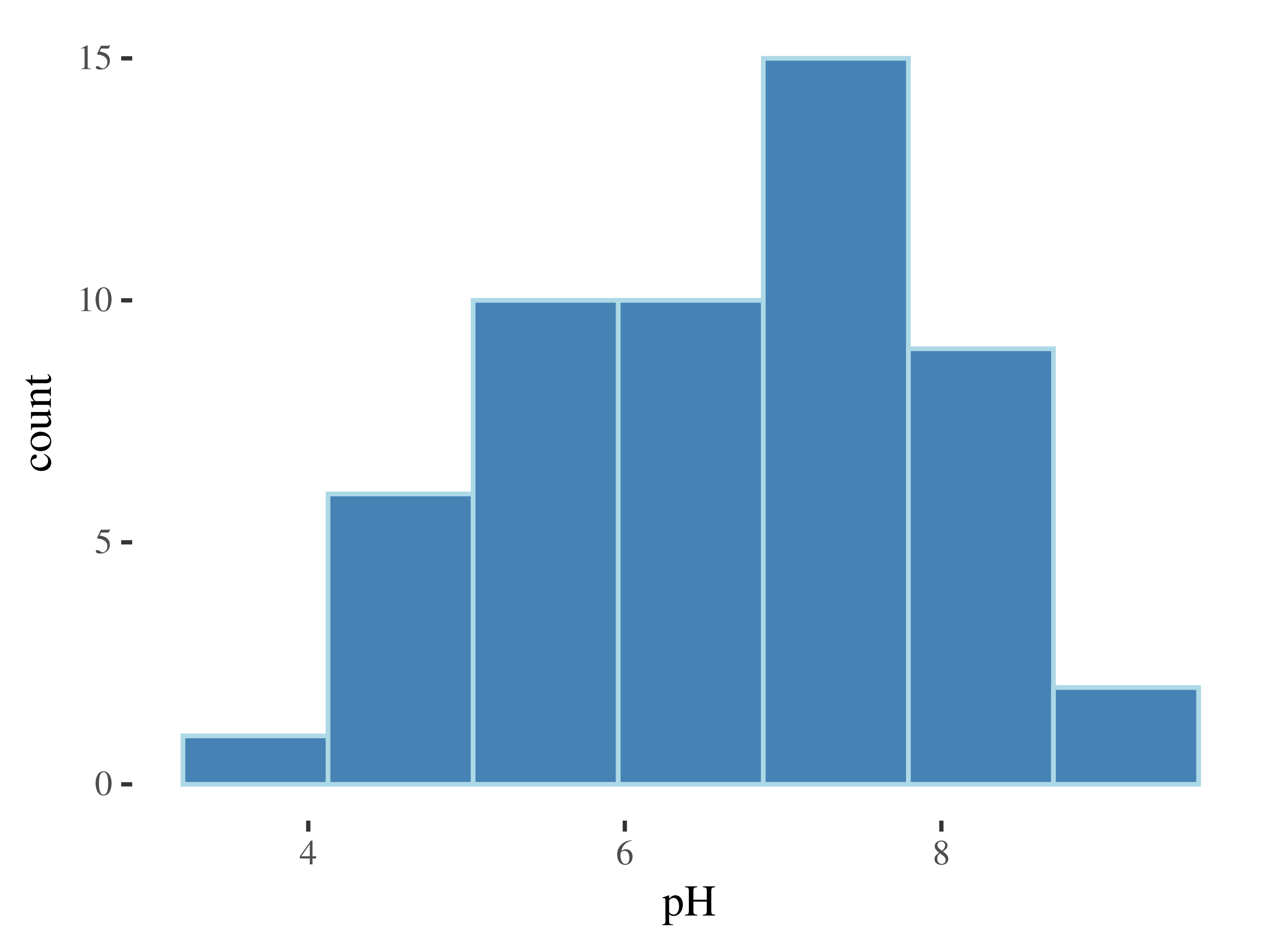
$$\text{stat} \pm t^{*}\text{SE}$$

The t^{*} multiplier comes from a t-distribution with appropriate df

- `qt(0.975, df=)` gives t^{*} for 95% confidence

Florida lakes

```r
library(ggplot2)
lakes <- read.csv("http://www.lock5stat.com/data
ggplot(lakes, aes(x = pH)) +
  geom_histogram(fill = "steelblue",
                 bins = 7,
                 col = "lightblue")
```



Florida lakes

$$H_0 : \mu = 7 \quad H_A : \mu \neq 7$$

- Data: The average pH was $\bar{x} = 6.591$ with a standard deviation of $s = 1.288$.

```
mean(lakes$pH)
[1] 6.590566
sd(lakes$pH)
[1] 1.288449
```

- The t-test stat is

$$t = \frac{6.591 - 7}{1.288/\sqrt{53}} \approx -2.31$$

- **Interpret t:** The observed mean of 6.591 is 2.31 SEs below 7.

Florida lakes

$$H_0 : \mu = 7 \quad H_A : \mu \neq 7$$

- **p-value** $2 \times P(t < -2.31)$, or double left tail area below -2.31
  - use t-distribution with $df = 53 - 1 = 52$

```
2*pt(-2.31, df=53-1)  # df = n-1
[1] 0.02489032
```

- **Interpret:** The p-value is 0.025. If the mean pH of all lakes is 7, then we would see a sample mean that is at least 2.31 SEs away from 7 about 2.5% of the time in samples of 53 lakes.
- **Conclusion:** There is a statistically significant difference between the observed mean pH of 6.591 and the hypothesized mean of 7 (t=-2.31, df=52, p=0.025).

Florida lakes: `t.test` in R

- We can also use `t.test` in R !

```
t.test(lakes$pH, mu  = 7)
```

```

    One Sample t-test

data:  lakes$pH
t = -2.3134, df = 52, p-value = 0.02469
alternative hypothesis: true mean is not equal to 7
95 percent confidence interval:
 6.235425 6.945707
sample estimates:
mean of x
 6.590566
```

Florida lakes

How different is the population mean from 7?

- 95% CI for μ:

$$6.591 \pm 2.0066 \frac{1.288}{\sqrt{53}} = 6.591 \pm 0.355 = 6.236, 6.946$$

where t^* corresponds to 95% confidence (97.5th percentile):

```
qt(.975, df=53-1)
[1] 2.006647
```

We are 95% confident that the mean pH of all lakes is between 6.236 and 6.946 (slightly acidic)

Academic Performance Index (API)

Academic Performance Index (API) is a number reflecting a school's performance on a statewide standardized test

- simple random sample of $n = 200$ schools
- variable `growth` measures the growth in API from 1999 to 2000 (API 2000 - API 1999).

```
# read data
api <- read.csv("https://raw.githubusercontent.com/deepbas/statdatasets/main/API.csv")
```

Academic Performance Index (API)

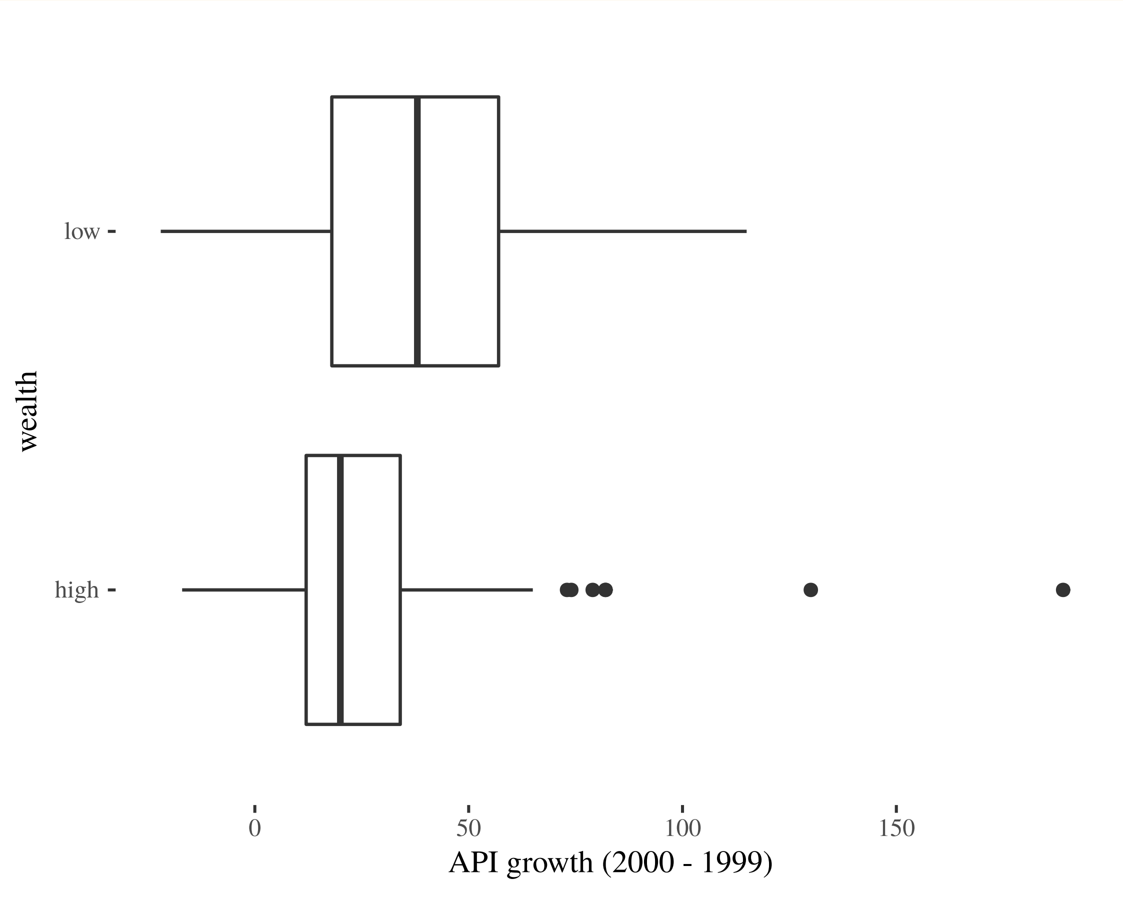
```
api$wealth <- ifelse(api$meals > 50, "low","high")
table(api$wealth)

high  low
 102   98
```

```
library(dplyr)
api %>%
  group_by(wealth) %>%
  summarize(mean(growth), sd(growth))
# A tibble: 2 × 3
  wealth `mean(growth)` `sd(growth)`
  <chr>           <dbl>        <dbl>
1 high             25.2         28.8
2 low              38.8         30.0
```

API

```
ggplot(api, aes(x = wealth, y = growth)) + geom_boxplot() +
  labs(y ="API growth (2000 - 1999)") + coord_flip()
```



Hypothesis Test

Can we use t-inference methods to compare mean growths?

- both samples sizes (98 and 102) can be deemed large
- No severe skewness (but two extreme outliers)

- Estimated Standard Error

$$SD_{\bar{x}_h - \bar{x}_l} = \sqrt{\frac{28.75380^2}{102} + \frac{29.95048^2}{98}} = 4.1544$$

- Test statistics

$$t = \frac{(25.24510 - 38.82653) - 0}{4.154404} = -3.2692$$

The observed mean difference is 3.3 SEs below the hypothesized mean difference of 0

Two-sample t-test

```
t.test(growth ~ wealth, data = api)
```

```

	Welch Two Sample t-test

data:  growth by wealth
t = -3.2692, df = 196.71, p-value = 0.001273
alternative hypothesis: true difference in means between group high and group low is not equ
95 percent confidence interval:
 -21.774321  -5.388544
sample estimates:
mean in group high  mean in group low 
          25.24510           38.82653 
```

The p-value is 0.001273. If there is no difference between mean growth in the two populations, then there is just a 0.13% chance of seeing a sample mean difference that is 3.27 standard errors or more away from 0.

Outliers

```
which(api$growth > 120 )
[1]  74 119
```

```
api %>% slice(74,119)
           cds stype            name                 sname snum
1 5.471911e+13     E Lincoln Element    Lincoln Elementary 5873
2 1.975342e+13     E Washington Elem Washington Elementary 2543
                   dname dnum       cname cnum flag pcttest api00 api99 target
1 Exeter Union Elementary  226      Tulare   53   NA      98   693   504     15
2   Redondo Beach Unified  585 Los Angeles   18   NA     100   745   615      9
  growth sch.wide comp.imp both awards meals ell yr.rnd mobility acs.k3 acs.46
1    189      Yes      Yes  Yes    Yes    50  18   <NA>        9     18     NA
2    130      Yes      Yes  Yes    Yes    41  20   <NA>       16     19     30
  acs.core pct.resp not.hsg hsg some.col col.grad grad.sch avg.ed full emer
1       NA       93      28  23       27       14        8   2.51   91    9
2       NA       81      11  26       32       16       16   2.99  100    3
  enroll api.stu    pw  fpc wealth
1    196     177 30.97 6194   high
2    391     313 30.97 6194   high
```

Remove Outliers

```
t.test(growth ~ wealth, data = api, subset = -c(74,119))
```

```
    Welch Two Sample t-test

data:  growth by wealth
t = -4.395, df = 174.97, p-value = 1.916e-05
alternative hypothesis: true difference in means between group high and group low is not equ
95 percent confidence interval:
 -23.571116  -8.961945
sample estimates:
mean in group high  mean in group low 
          22.56000           38.82653 
```

How does removing outliers influence `t-test` stat and p-value?

Confidence Interval

95% Confidence Interval from the output:

- Without Outliers: (-23.57, -8.96)
- With Outliers: (-21.77, -5.39)

Removing Outliers:

- the difference in means shifted further away from 0
- CI shifted further from a difference of 0
- decrease the SE of our sample difference

Interpretation: We are 95% confident that the mean API growth between 1999 and 2000 for all low wealth schools is anywhere from 8.96 points to 23.57 points higher than the mean API growth for all high wealth schools in California.

Paired Data

- Data are paired if the data being compared consists of paired data values
- Common paired data examples:
  - Two measurements on each case
  - natural pairs (twins, spouses, etc)

- Use paired data to reduce natural variation in the response when comparing the two groups/treatments
  - comparing group 1 and 2 responses among similar individuals
  - reduces the effects of confounding variables
  - reduces the SE for the mean difference!

Analyzing paired data

- Look at the difference between responses for each unit (pair)

$$d_i = x_{1,i} - x_{2,i}$$

- Analyze the **mean of these differences** rather than the average difference between two groups

$$\text{sample mean difference: } \bar{d}$$

$$\text{sample SD of difference: } s_d$$

$$\text{population mean difference: } \mu_d$$

- Use **one sample** inference methods for these differences

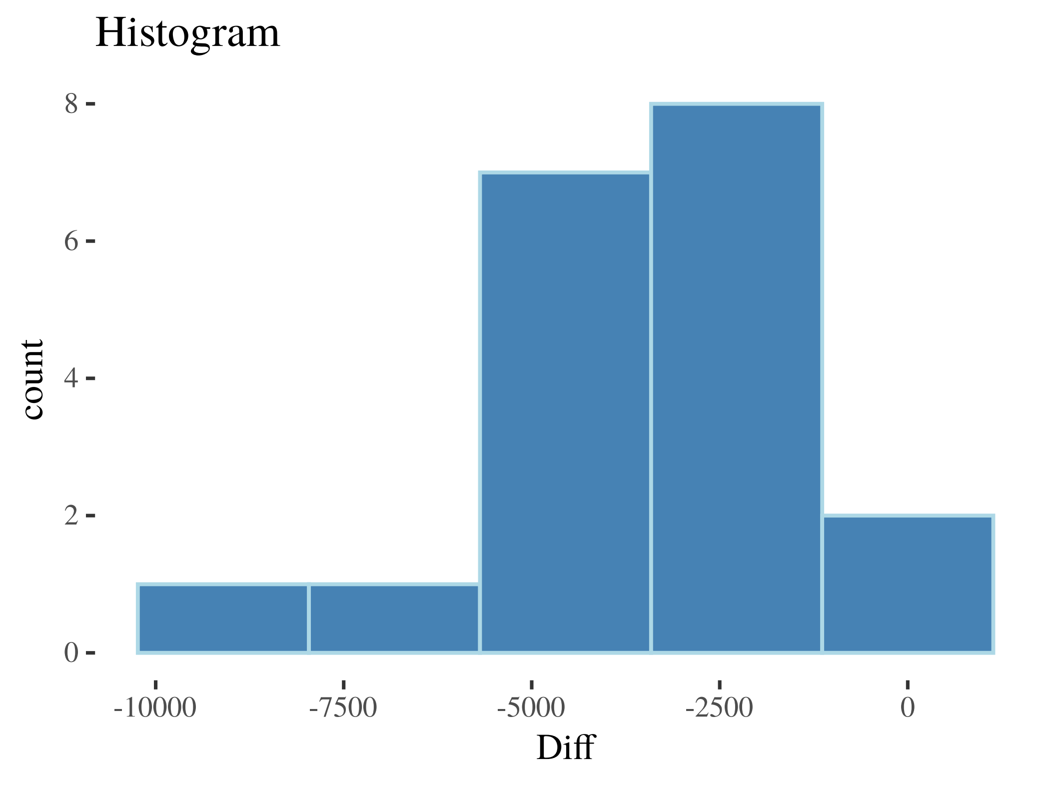
Tuition example

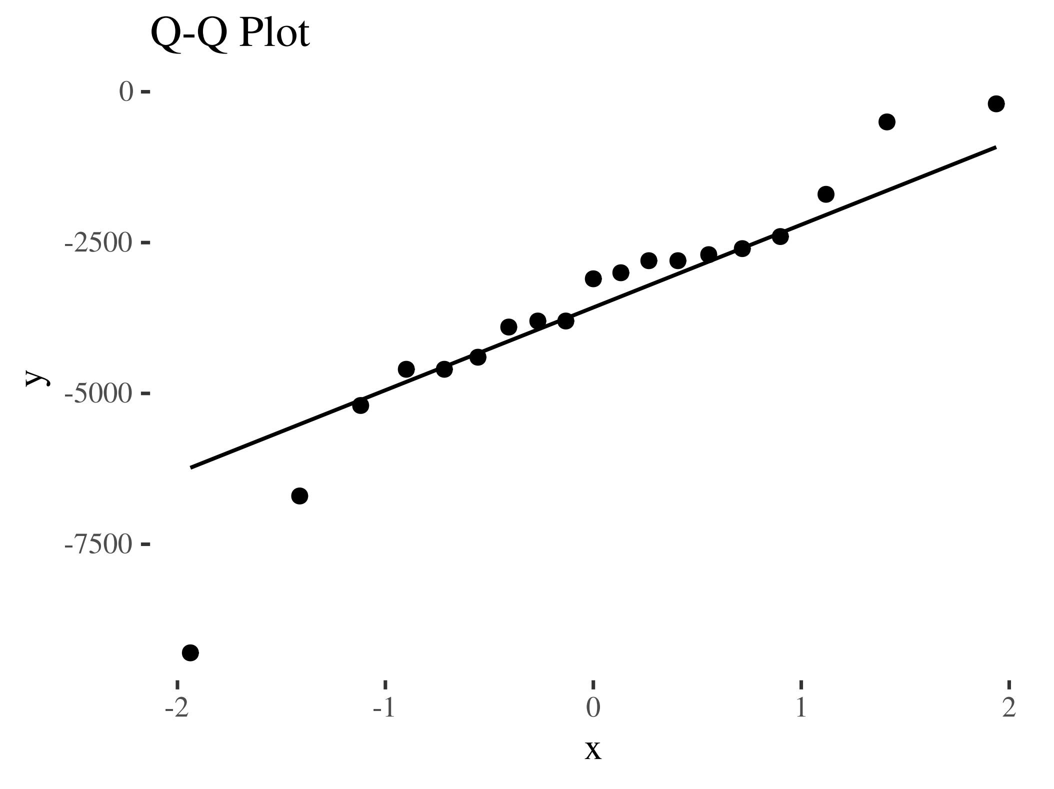
- How much higher is non-resident tuition, on average, compared to resident tuition?
- Use the `Tuition2006.csv` lab manual data
  - the variable `Diff` computues the difference `Res` - `NonRes`

```
tuition <- read.csv("http://math.carleton.edu/Stats215/RLabManual/Tuition2006.csv")
str(tuition)
'data.frame':	19 obs. of  5 variables:
 $ X          : int  1 2 3 4 5 6 7 8 9 10 ...
 $ Institution: chr  "Univ of Akron (OH)" "Athens State (AL)" "Ball State (IN)" "Bloomsburg
 $ Res        : int  4200 1900 3400 3200 3400 2600 3300 2900 2200 3400 ...
 $ NonRes     : int  8800 3600 8600 7000 12700 5700 5900 3400 4600 7300 ...
 $ Diff       : int  -4600 -1700 -5200 -3800 -9300 -3100 -2600 -500 -2400 -3900 ...
```

Tuition example

- Smaller sample size (n=19) and slightly left-skewed distribution or roughly symmetric with one low case!





Tuition example

We are 95% confident that the mean tuition for non-residents is $2,585 to $4584 higher than mean tuition for residents

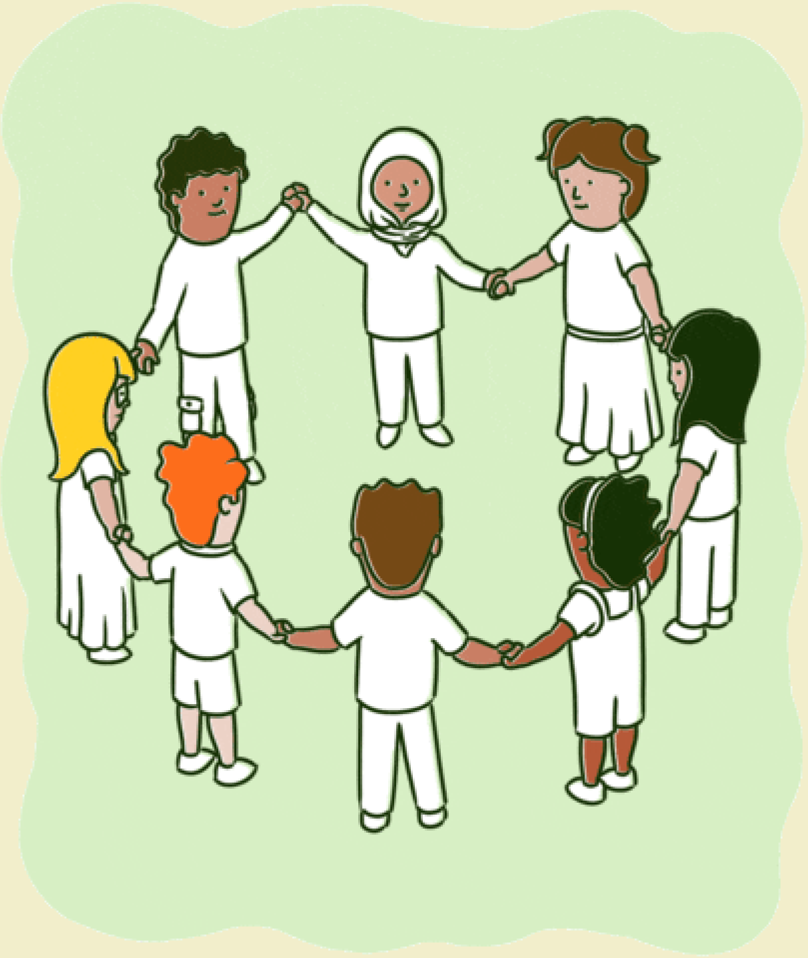
```
t.test(tuition$Diff)

    One Sample t-test

data:  tuition$Diff
t = -7.5349, df = 18, p-value = 5.69e-07
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 -4583.580 -2584.841
sample estimates:
mean of x
-3584.211
sd(tuition$Diff)/sqrt(19)  # SE for mean diff
[1] 475.6813
```

Your Turn 1

05:00



- Go over to the in class activity file
- Complete the activity in your group