Regression and ANOVA

Stat 120

May 22 2022

Simple Linear Model

The population/true simple linear model is:

$$y = \beta_0 + \beta_1 x + \varepsilon$$

- β_0 and β_1 are unknown parameters corresponding to the y-intercept and the slope, respectively
- ε is the random error
- Estimate with b_0 and b_1 from the least squares line $\hat{y} = b_0 + b_1 x$

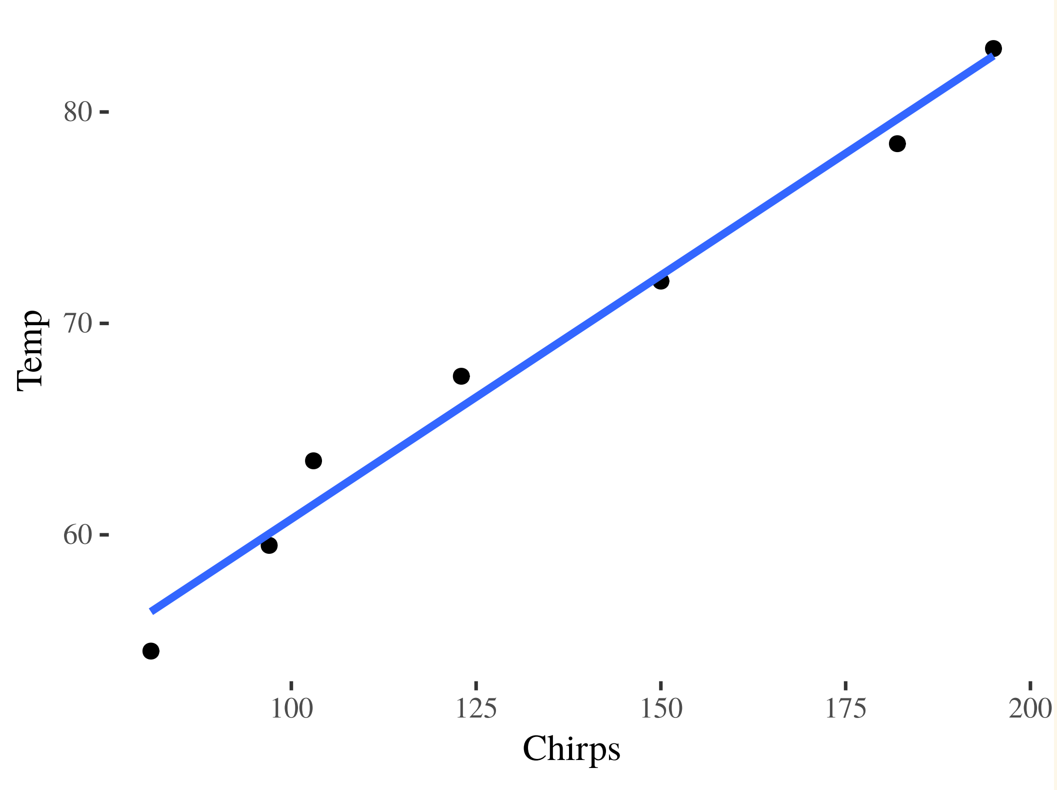
How accurate are the estimates?

Recall: Least Square Regression

- X = Cricket chirp rate
- Y = Temperature

Chirps	Temp
81	54.5
97	59.5
103	63.5
123	67.5
150	72.0
182	78.5
195	83.0

$$\widehat{\text{Temp}} = 37.7 + 0.23 \text{ Chirps}$$



What are the parameters being estimated?

Inference for the Slope

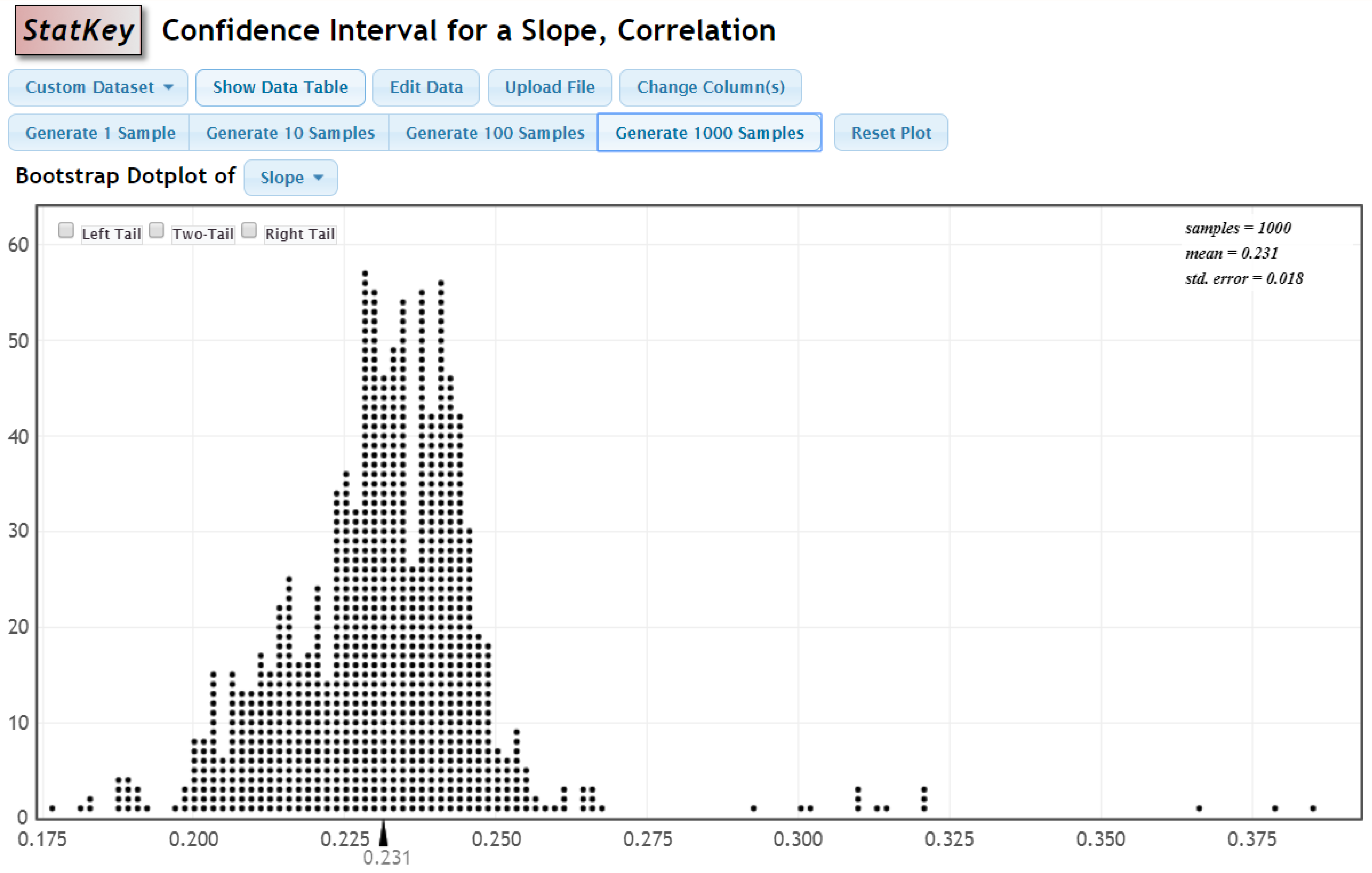
Confidence intervals and hypothesis tests for the slope can be done using the familiar **formulas**:

$$b_1 \pm t^* \cdot SE \qquad\qquad t = \frac{b_1 - \text{ null slope}}{SE}$$

But how do we estimate the **standard error**?

- Bootstrap/Randomization distributions
- Computer output

Standard Error Using a Bootstrap Distribution



Technology Examples

Slope estimate and Standard Error

```
chirps.lm <- lm(Temp ~ Chirps, data = data)
summary(chirps.lm)
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 37.67858    1.97817   19.05 7.35e-06 ***
Chirps       0.23067    0.01423   16.21 1.63e-05 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.528 on 5 degrees of freedom
Multiple R-squared:  0.9813
```

Confidence Interval for Slope

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 37.67858    1.97817   19.05 7.35e-06 ***
Chirps       0.23067    0.01423   16.21 1.63e-05 ***
```

We can use the values for b_1 and SE from the regression output to form a confidence interval in the usual way:

$$b_1 \pm t^* \cdot SE$$

- Here, t^* uses $n-2$ degrees of freedom, since we are estimating two parameters in the simple linear model.

Confidence Interval for Slope

Find a 95% confidence interval for the slope of the cricket temperature model.

```
Temperature  =  37.7 + 0.231 Chirps
Predictor         Coef     SE Coef       T     Pr(>|t|)
Constant      37.67858     1.97817   19.05   7.35e-06 ***
Chirps         0.23067     0.01423   16.21   1.63e-05 ***
```

$$b_1 \pm t^* \cdot SE$$

Hypothesis Test for Slope

- Population Simple Linear Model: $y = \beta_0 + \beta_1 x + \varepsilon$

$$\begin{aligned} H_0 : \beta_1 &= 0 \qquad \Longrightarrow \text{No linear relationship} \\ H_a : \beta_1 &\neq 0 \qquad \Longrightarrow \text{Some relationship} \end{aligned}$$

$$t = \frac{\text{statistic-null}}{SE} = \frac{b_1 - 0}{SE} = \frac{b_1}{SE}$$

- Again, $\boldsymbol{b}_1$ and SE come from R output.
- We find the p-value by using a t distribution with $n - 2\text{df}$.

Hypothesis Test for Slope

Confirm the **p-value** given by the regression output for testing the slope of the cricket chirp model.

```
Temperature   =  37.7 + 0.231 Chirps
Predictor          Coef     SE Coef       T     Pr(>|t|)
Constant     37.67858     1.97817   19.05  7.35e-06 ***
Chirps        0.23067     0.01423   16.21  1.63e-05 ***
```

$$H_0: \beta_1 = 0$$
$$H_a: \beta_1 \neq 0$$

$$t = \frac{b_1}{SE}$$

Standard Error of the Slope

Although we generally rely on technology to obtain the SE for the slope, we can also obtain it as follows:

$$SE = \frac{s_\varepsilon}{s_x\sqrt{n-1}}$$

where s_ε is the standard deviation of the error term and s_x is the standard deviation for the sample values of the predictor.

SE for Slope

```
The regression equation is
Temperature  =  37.7 + 0.231 Chirps

Predictor        Coef     SE Coef      T     Pr(>|t|)
Constant     37.67858     1.97817  19.05  7.35e-06 ***
Chirps        0.23067     0.01423  16.21  1.63e-05 ***

s = 1.52778    R-Sq = 98.1%    R-Sq(adj) = 97.8%
```

	Chirps
Mean	133.00000
Standard Dev.	43.84442
n	7.00000

$$SE = \frac{s_{\varepsilon}}{s_x\sqrt{n-1}} = \frac{\sqrt{MSE}}{s_x\sqrt{n-1}} =$$

Hypothesis Test for Correlation

How else can we measure the strength of association between two quantitative variables?

Recall: r = sample correlation, ρ = population correlation

$$H_0 : \rho = 0$$
$$H_a : \rho \neq 0$$

Find the p-value using a t-distribution with n - 2 df

$$t = \frac{\text{statistic -null}}{SE} = \frac{r - 0}{\sqrt{\frac{1-r^2}{n-2}}}$$

$$= \frac{r}{\frac{\sqrt{1-r^2}}{\sqrt{n-2}}} = \left(\frac{r\sqrt{n-2}}{\sqrt{1-r^2}}\right)$$

Hypothesis Test for Correlation

The correlation for the n = 7 cricket chirp data points is r = 0.99062. Compute the t-statistic for the test:

$$H_0 : \rho = 0$$
$$H_a : \rho \neq 0$$

$$t = \left(\frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \right) = \frac{0.99062\sqrt{7-2}}{\sqrt{1-0.99062^2}} = 16.21$$

Coefficient of Determination, R^2

Recall that for correlation: $-1 \leq r \leq 1$

If we square the correlation, we get the **coefficient of determination**, which is a number between 0 and 1 that can be interpreted as a proportion or percentage.

$R^2 =$ proportion of **variability** in the response variable, Y, that is "explained" by the explanatory variable, X.

- By convention we use a capital R^2, although the value is just r^2 for a single explanatory variable.

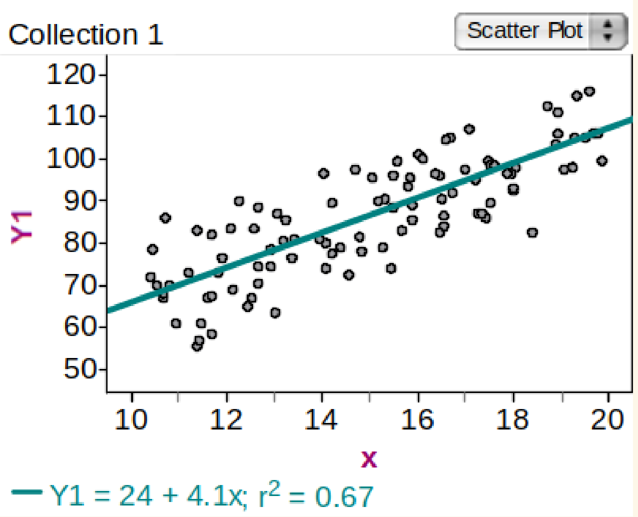
Checking Condition

$$y = \beta_0 + \beta_1 x + \varepsilon$$

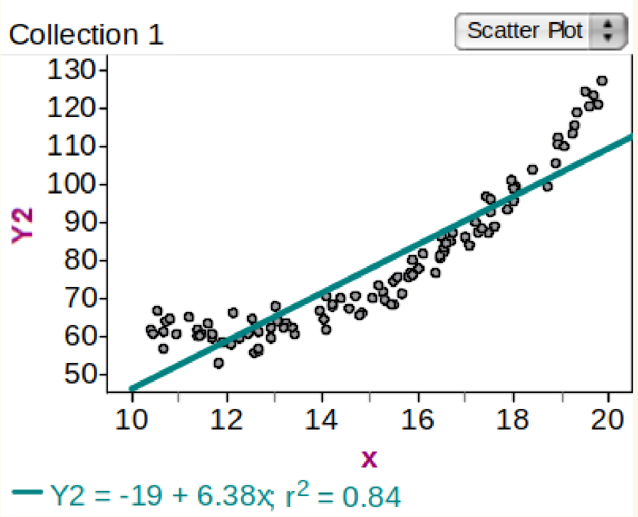
For a simple linear model, we assume the errors (ε) are randomly distributed above and below the line.

- **Quick check** : Look at a scatterplot with regression line on it.
- Watch out for:
  - Curved (nonlinear) patterns in the data
  - Consistently changing variability
  - Outliers and influential points

Scatterplot with Regression Line



Good

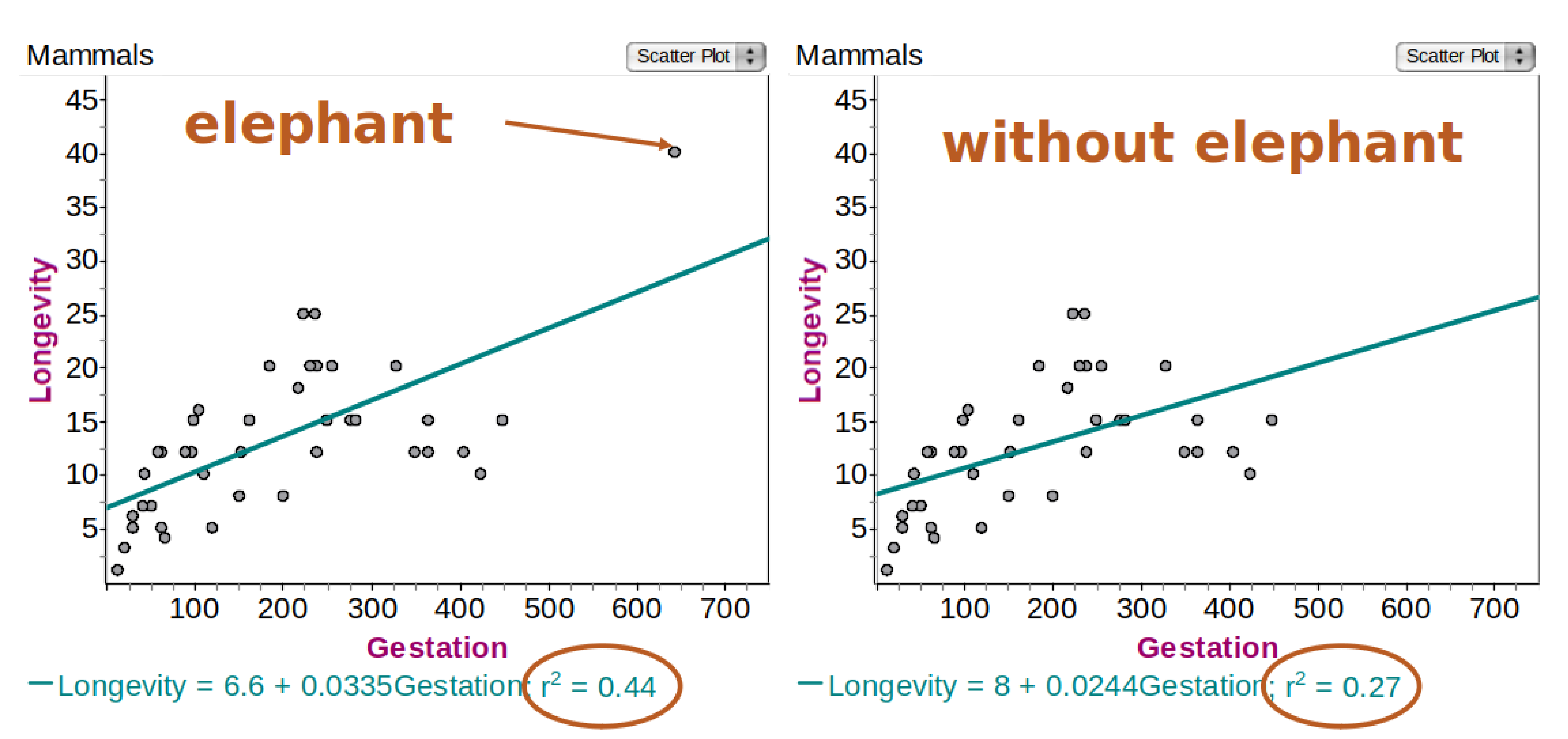


Problem

- Check linearity
- Check consistent variability

Scatterplot with Regression Line

- Check for outliers or influential points

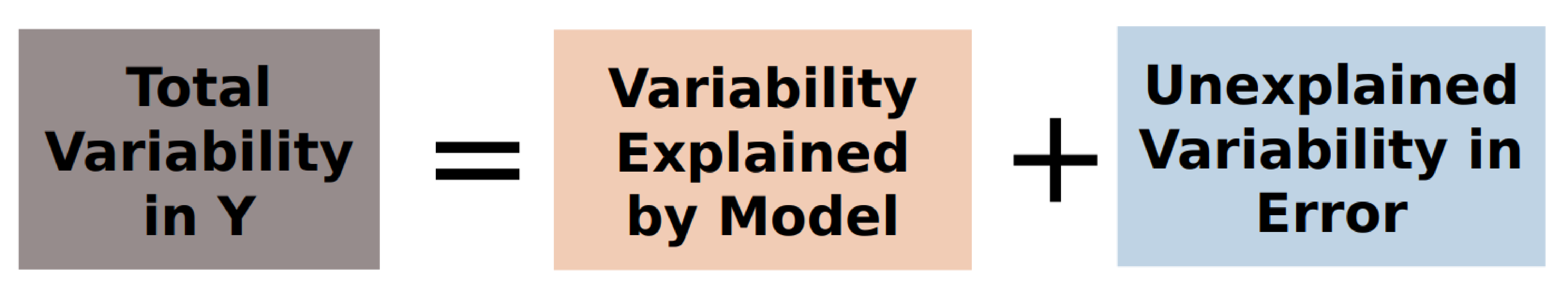


Partitioning Variability

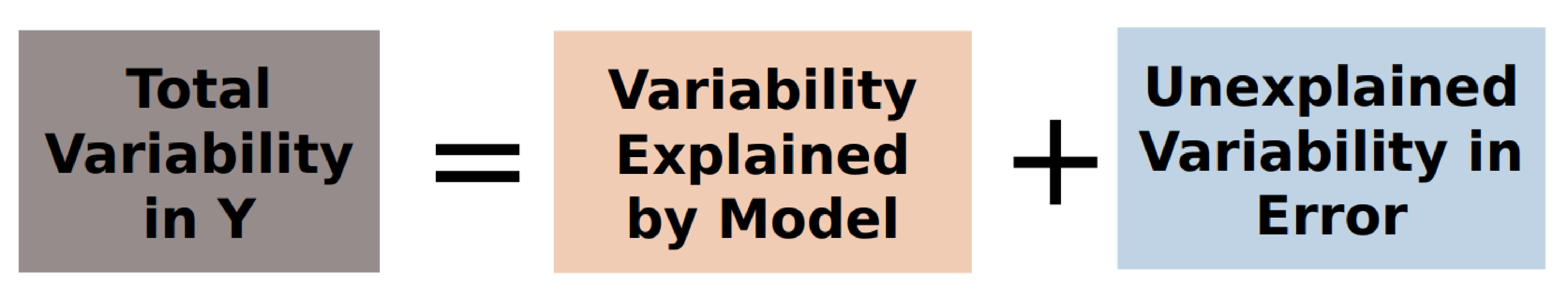
$$Y = \beta_0 + \beta_1 X + \varepsilon$$

$$\text{Data} \;=\; \text{Model} \;+\; \text{Error}$$

Split the total variability in Y into two pieces, variability explained by the model + unexplained (residual error) variability



Measuring Variability

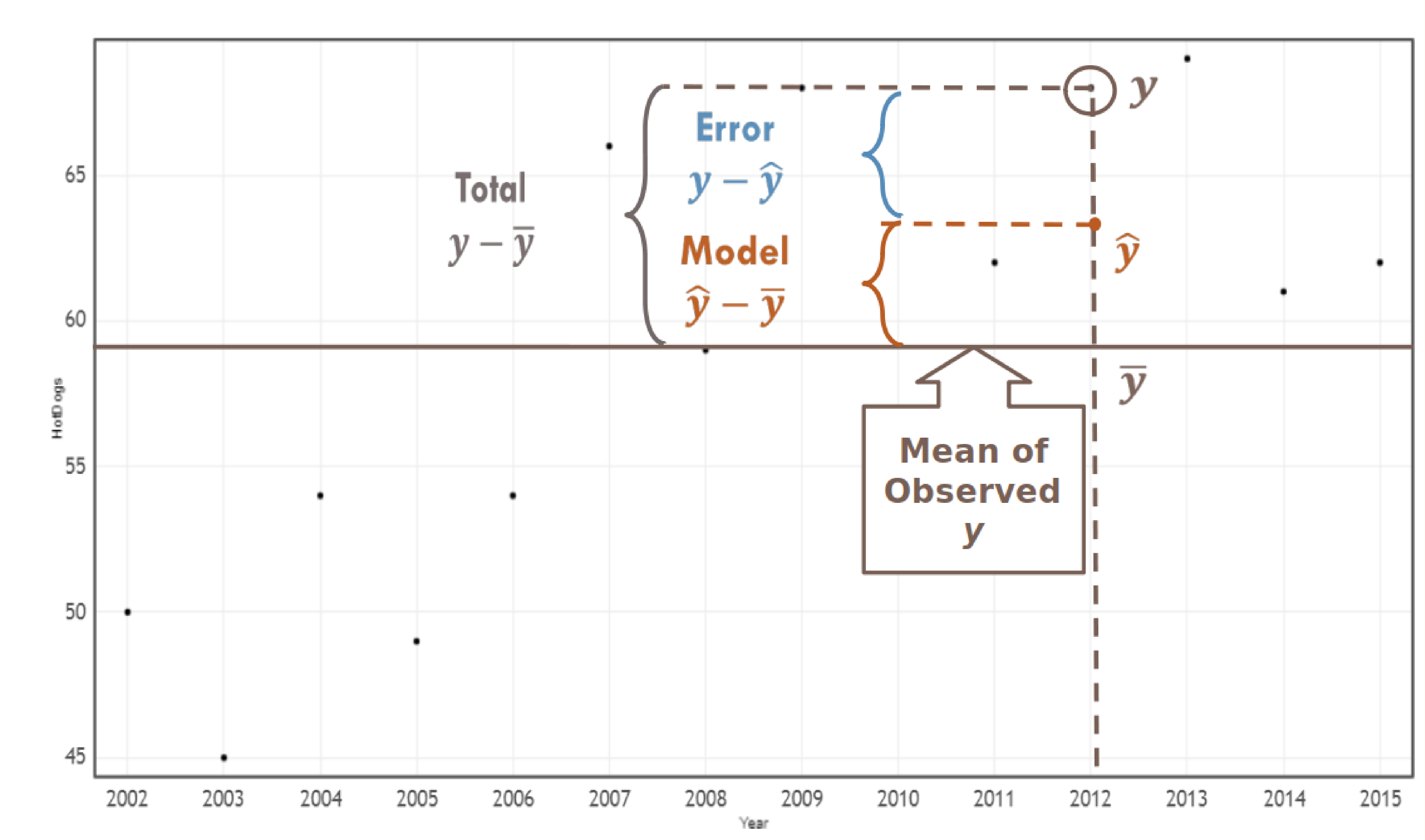


Total variability in Y : SSTotal $= \sum(y - \bar{y})^2$

Explained variability: SSModel $= \sum(\hat{y} - \bar{y})^2$

Unexplained variability: $SSE = \sum(y - \hat{y})^2$

$$Y = \beta_0 + \beta_1 X + \varepsilon \qquad \text{Data} = \text{Model} + \text{Error}$$



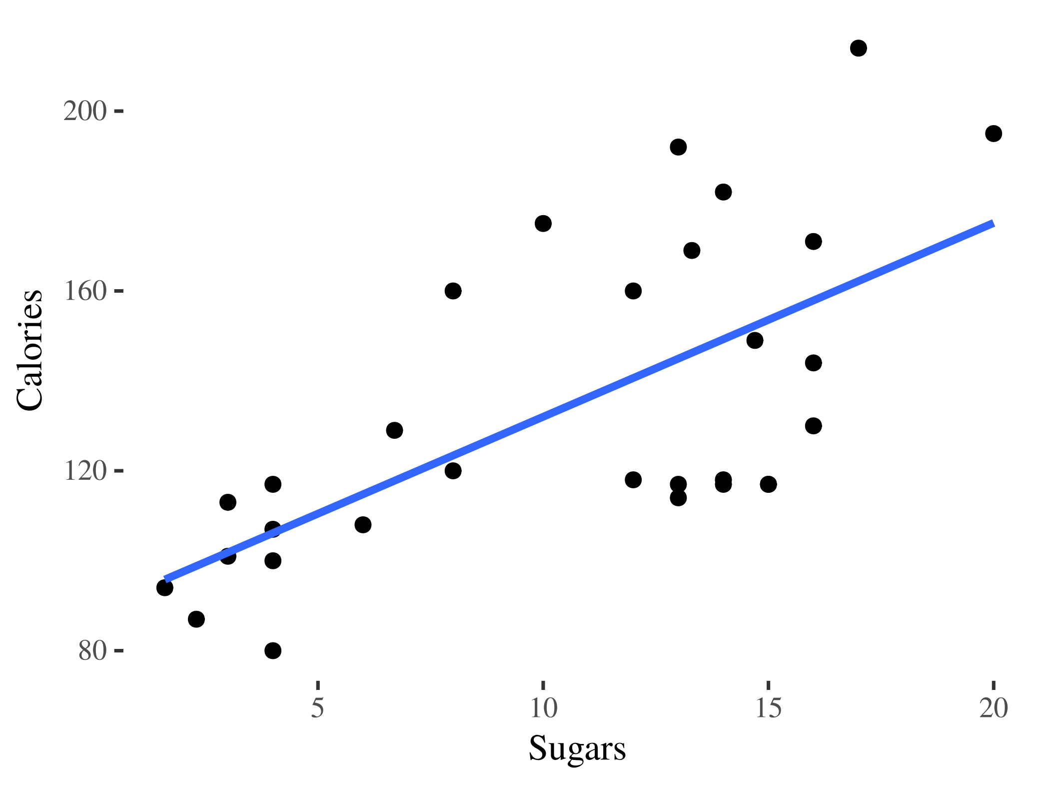
R: Calories Vs. Sugars

```r
library(Lock5Data)
mod <- lm(Calories~Sugars, data = Cereal)
anova(mod)
```

```
Analysis of Variance Table

Response: Calories
          Df Sum Sq Mean Sq F value    Pr(
Sugars     1  15316 15316.5  21.623 7.217e
Residuals 28  19834   708.3
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*
```

```r
ggplot(Cereal, aes(x = Sugars, y = Calorie
  geom_point() +
  geom_smooth(method="lm", se = FALSE)
```



R^2 in Regression

```r
library(Lock5Data)
mod <- lm(Calories~Sugars, data = Cereal)
anova(mod)
```

```
Analysis of Variance Table

Response: Calories
          Df Sum Sq Mean Sq F value     Pr(
Sugars     1  15316 15316.5  21.623 7.217e
Residuals 28  19834   708.3
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*
```

$$R^2 = \frac{\text{Variabilit y explained by Model}}{\text{Total variability in Y}}$$

$$R^2 = \frac{\text{SSModel}}{\text{SSTotal}}$$

$$R^2 =$$

R: Calories Vs. Sugars

```
mod <- lm(Calories~Sugars, data = Cereal)
summary(mod)
```

```
Call:
lm(formula = Calories ~ Sugars, data = Cereal)

Residuals:
    Min      1Q  Median      3Q     Max 
-36.574 -25.282  -2.549  17.796  51.805 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  88.9204    10.8120   8.224 5.96e-09 ***
Sugars        4.3103     0.9269   4.650 7.22e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 26.61 on 28 degrees of freedom
Multiple R-squared:  0.4357,    Adjusted R-squared:  0.4156 
F-statistic: 21.62 on 1 and 28 DF,  p-value: 7.217e-05
```

ANOVA for Regression

$$H_0 : \beta_1 = 0$$
$$H_a : \beta_1 \neq 0$$

$$H_0 : \text{ The model is ineffective}$$
$$H_a : \text{ The model is effective}$$

Source	**df**	**Sum of Squares**	**Mean Square**	**F-statistic**	**p-value**
Model	1	SSModel	$\frac{SSModel}{1}$	$F = \frac{MSModel}{MSE}$	$F_{1,n-2}$
Error	n - 2	SSE	$\frac{SSE}{n-2}$		
Total	n-1	SSTotal			

P-value for Regression ANOVA

$$F = \frac{\text{MSModel}}{MSE}$$

How do we know when the F-statistic is “significant”?

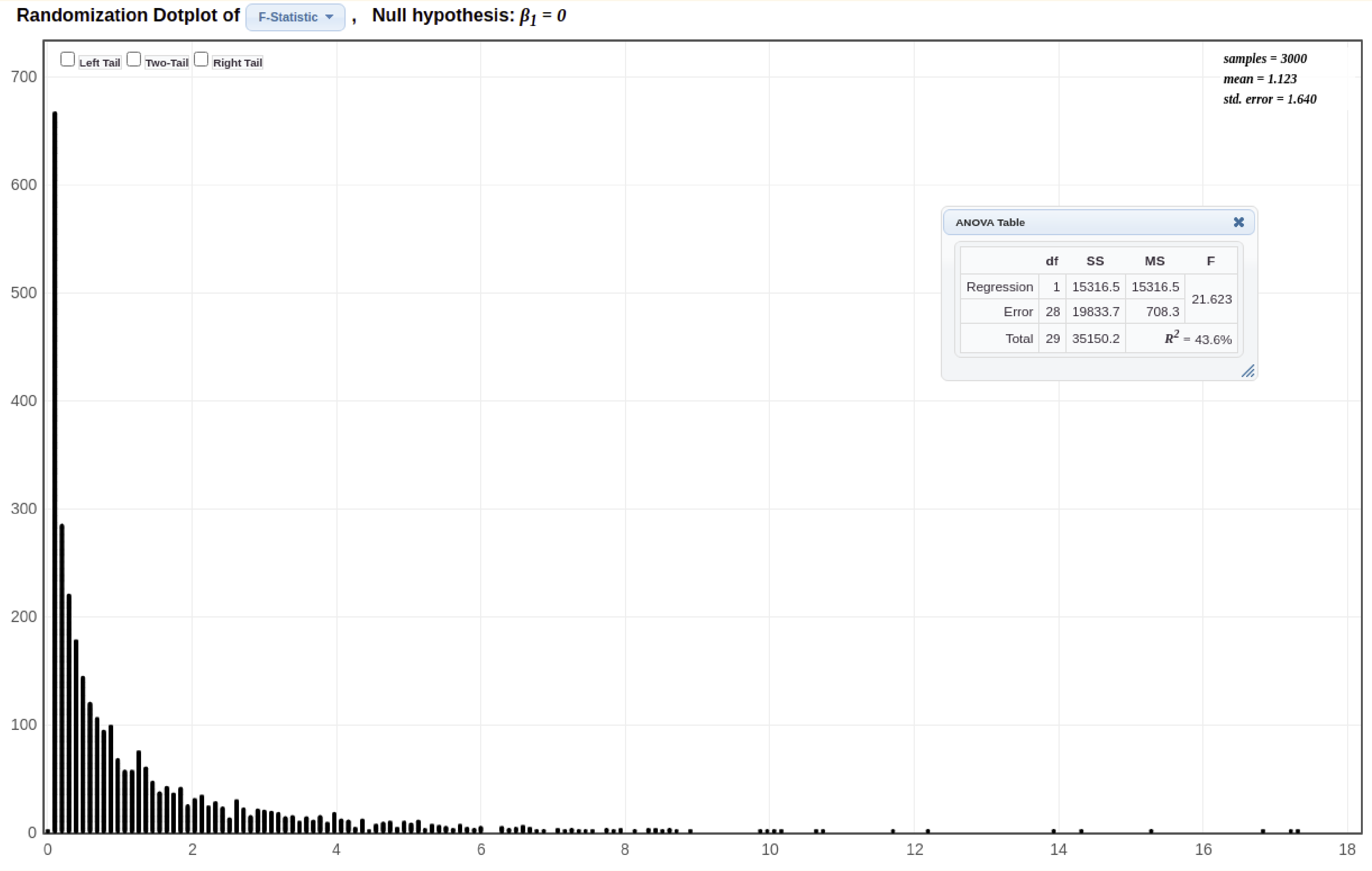
To find a p-value for the ANOVA F-statistic:

- Create a randomization distribution, OR
- Use a theoretical distribution

For a randomization distribution, we need to obtain samples where $\mathrm{H}_0 : \beta_1 = 0$ is true:

- Randomly scramble the response (Y) values
- Compute the ANOVA F-statistic for each sample

Randomization distribution



	df	SS	MS	F
Regression	1	15316.5	15316.5	21.623
Error	28	19833.7	708.3	
Total	29	35150.2	R^2 = 43.6%	

Theoretical Distribution: F-distribution

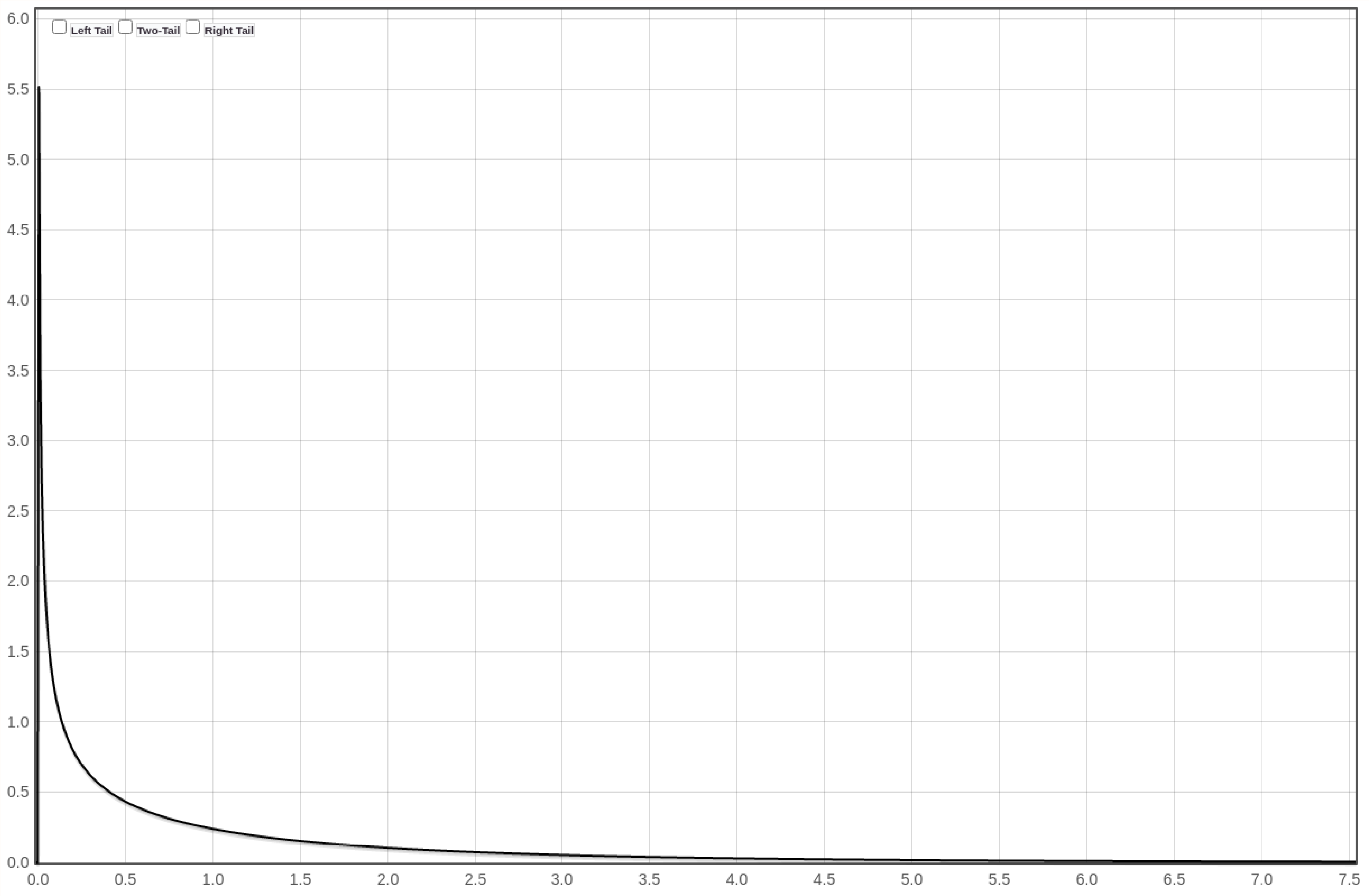
For testing an F-statistic (which is a ratio of variances)

- Use an F-distribution, specifying the degrees of freedom for both the numerator and the denominator.

When performing an ANOVA for Regression with a single predictor:

- 1 df (numerator)
- $n - 2$ df (denominator)
- Then use the upper tail beyond the F-statistic

F-distribution: Stat-key



Std. Dev. of Error in Regression

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

Condition: The random errors (ε) have a common standard deviation, σ_ε

Estimation:

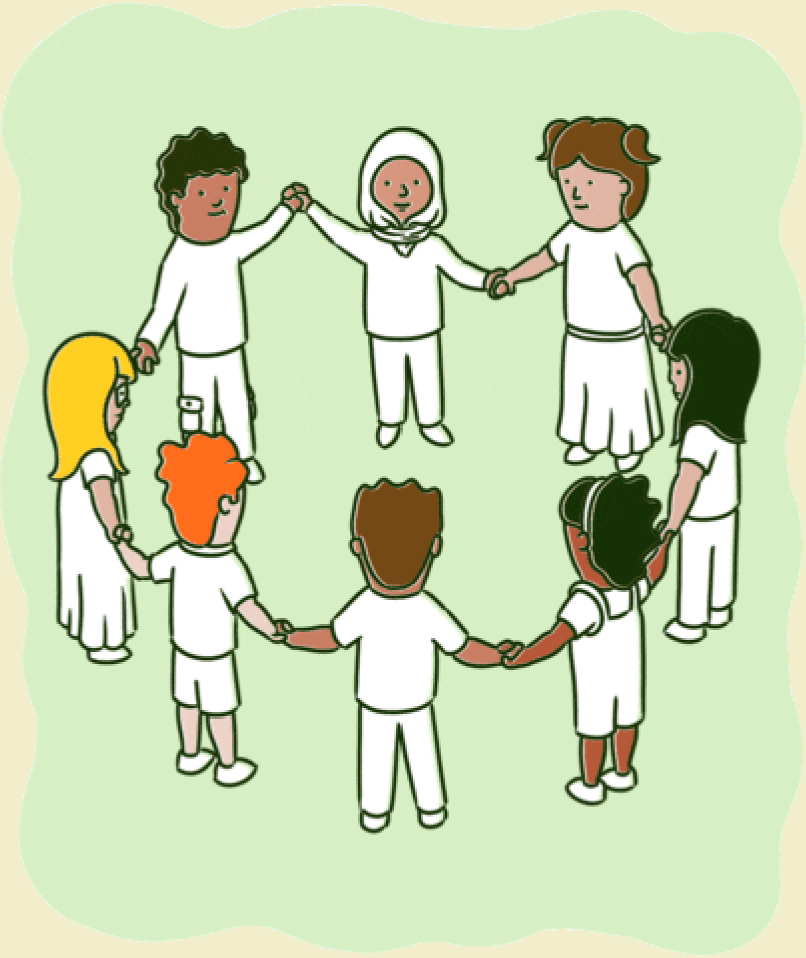
$$S_\varepsilon = \sqrt{\frac{\sum(y - \hat{y})^2}{n-2}} = \sqrt{\frac{SSE}{n-2}} = \sqrt{MSE}$$

For Calorie model:

$$S_\varepsilon =$$

Your Turn 1

05:00



- Go over to the in class activity file
- Complete the remaining activity