Frequency Estimators

Outline for Today

- ***Randomized Data Structures***
  - Our next approach to improving performance.
- ***Count-Min Sketches***
  - A simple and powerful data structure for estimating frequencies.
- ***Count Sketches***
  - Another approach for estimating frequencies.

Randomized Data Structures

Tradeoffs

- Data structure design is all about tradeoffs:
  - Trade preprocessing time for query time.
  - Trade asymptotic complexity for constant factors.
  - Trade worst-case per-operation guarantees for worst-case aggregate guarantees.

Randomization

- Randomization opens up new routes for tradeoffs in data structures:
  - Trade worst-case guarantees for average-case guarantees.
  - Trade exact answers for approximate answers.
- Over the next few lectures, we'll explore two families of data structures that make these tradeoffs:
  - Today: ***Frequency estimators.***
  - Next Week: ***Hash tables.***

Preliminaries: ***What is a Hash Function?***

Hashing in Practice

- In most programming languages, each object has “a” hash code.
  - C++: `std::hash`
  - Java: `Object.hashCode`
  - Python: `__hash__`
- To store objects in a hash table, you just go and implement the appropriate function or type.
- In other words, hash functions are ***intrinsic*** properties of objects.

Hashing in Theoryland

- In Theoryland, a hash function is a function from some domain called the ***universe*** (typically denoted $\mathscr{U}$) to some codomain.
- The codomain is usually a set of the form {0, 1, 2, 3, …, m – 1}, which we'll denote **[*m*]**.
- We often will grab lots of different hash functions from the same universe $\mathscr{U}$ to some codomain, and we'll assume we have access to as many of them as we need.
- In other words, hash functions are ***extrinsic*** to objects, and it's possible to have multiple different hash functions available at the same time.

Families of Hash Functions

- A ***family*** of hash functions is a set $\mathscr{H}$ of hash functions with the same domain and codomain.
- The data structures we'll explore will assume that we have access to certain families of hash functions with nice properties.
- We'll then sample uniformly-random choices $h \in \mathscr{H}$ to use as needed.

Sampling Random Functions

- Here's a family of hash functions $\mathscr{H}$ from $\mathbb{N}$ to $[137]$:

$$\mathscr{H} = \{\, f(n) = (an + b) \bmod 137 \mid a, b \in [137] \,\}$$

- In Theoryland, we'd model picking a uniformly-random hash function from $\mathscr{H}$ as just that – sampling some $h \in \mathscr{H}$ uniformly.
- In The Real World, we'd probably model picking such a function like this:

```
int a = rand() % 137;
int b = rand() % 137;

int hash(int value) {
    return (a * value + b) % 137;
}
```

Characterizing Hash Functions

- Different algorithms and data structures require different guarantees from their hash functions.
- In CS161, you explored ***universal hash functions*** in the context of chained hash tables.
- For what we'll be doing in CS166, we're going to need hash functions with slightly stronger probabilistic guarantees.

Pairwise Independence

- Let $\mathscr{H}$ be a family of hash functions from $\mathscr{U}$ to some set $\mathscr{C}$.
- We say that $\mathscr{H}$ is a ***2-independent family of hash functions*** if, for any distinct distinct $x, y \in \mathscr{U}$, if we choose a hash function $h \in \mathscr{H}$ uniformly at random, the following hold:

$h(x)$ and $h(y)$ are uniformly distributed over $\mathscr{C}$.

$h(x)$ and $h(y)$ are independent.

- 2-independent hash functions are great hash functions when we want a nice distribution over the output space even after fixing some specific element.

3-Independence

- Let $\mathscr{H}$ be a family of hash functions from $\mathscr{U}$ to some set $\mathscr{C}$.
- We say that $\mathscr{H}$ is a ***3-independent family of hash functions*** if, for any distinct distinct x, y, $z \in \mathscr{U}$, if we choose a hash function $h \in \mathscr{H}$ uniformly at random, the following hold:

$h(x)$, $h(y)$, and $h(z)$ are uniformly distributed over $\mathscr{C}$.

$h(x)$, $h(y)$, and $h(z)$ are independent.

- As you'll see, in many cases, making stronger assumptions about our hash functions makes it possible to simplify tricky probabilistic expressions.
- (As you can probably guess, this generalizes even further to k-independence, which we'll see on Tuesday.)

Frequency Estimation

Frequency Estimators

- A ***frequency estimator*** is a data structure supporting the following operations:
  - ***increment***(x), which increments the number of times that x has been seen, and
  - ***estimate***(x), which returns an estimate of the frequency of x.
- Using BSTs, we can solve this in space $\Theta(n)$ with worst-case $O(\log n)$ costs on the operations.
- Using hash tables, we can solve this in space $\Theta(n)$ with expected $O(1)$ costs on the operations.

Frequency Estimators

- Frequency estimation has many applications:
  - Search engines: Finding frequent search queries.
  - Network routing: Finding common source and destination addresses.
- In these applications, $\Theta(n)$ memory can be impractical.
- ***Goal:*** Get *approximate* answers to these queries in sublinear space.

Some Terminology

- Let's suppose that all elements x are drawn from some set $\mathscr{U} = \{ x_1, x_2, \ldots x_n \}$.
- We can interpret the frequency estimation problem as follows:

  Maintain an n-dimensional vector $\boldsymbol{a}$ such that $\boldsymbol{a}_i$ is the frequency of x_i.

- We'll represent $\boldsymbol{a}$ implicitly in a format that uses reduced space.

Vector Norms

- Let $\boldsymbol{a} \in \mathbb{R}^n$ be a vector.
- The ***L_1 norm of a***, denoted $\|\boldsymbol{a}\|_1$, is defined as

$$\|\boldsymbol{a}\|_1 = \sum_{i=1}^{n} |\boldsymbol{a}_i|$$

- The ***L_2 norm of a***, denoted $\|\boldsymbol{a}\|_2$, is defined as

$$\|\boldsymbol{a}\|_2 = \sqrt{\sum_{i=1}^{n} \boldsymbol{a}_i^2}$$

Properties of Norms

- The following property of norms holds for any vector $\boldsymbol{a} \in \mathbb{R}^n$. It's a good exercise to prove this on your own:

$$\|\boldsymbol{a}\|_2 \leq \|\boldsymbol{a}\|_1 \leq \Theta(n^{1/2}) \cdot \|\boldsymbol{a}\|_2$$

- The first bound is tight when exactly one component of $\boldsymbol{a}$ is nonzero.
- The second bound is tight when all components of $\boldsymbol{a}$ are equal.

Where We're Going

- Today, we'll see two data frequency estimation data structures.
- Each is parameterized over two quantities:
  - An ***accuracy*** parameter $\varepsilon \in (0, 1)$ determining how close to accurate we want our answers to be.
  - A ***confidence*** parameter $\delta \in (0, 1]$ determining how likely it is that our estimate is within the bounds given by ε.

Where We're Going

- The ***count-min sketch*** provides estimates with error at most $\varepsilon\|\boldsymbol{a}\|_1$ with probability at least $1 - \delta$.
- The ***count sketch*** provides estimates with an error at most $\varepsilon\|\boldsymbol{a}\|_2$ with probability at least $1 - \delta$.
  - (Notice that lowering ε and lower δ give better bounds.)
- Count-min sketches will use less space than count sketches for the same ε and δ, but provide slightly weaker guarantees.

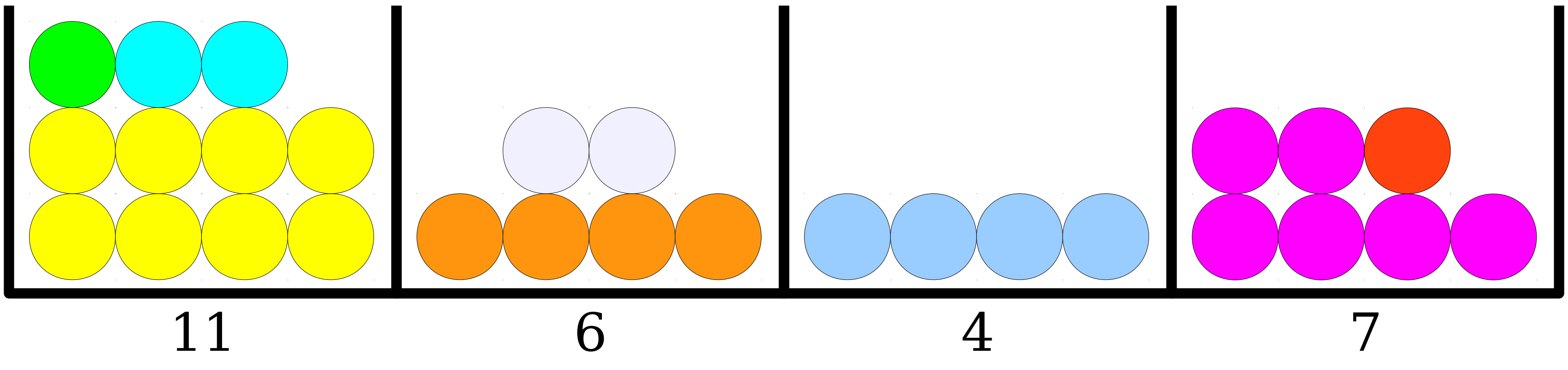
The Count-Min Sketch

The Count-Min Sketch

- Rather than diving into the full count-min sketch, we'll develop the data structure in phases.
- First, we'll build a simple data structure that *on expectation* provides good estimates, but which does not have a high probability of doing so.
- Next, we'll combine several of these data structures together to build a data structure that has a high probability of providing good estimates.

Revisiting the Exact Solution

- In the exact solution to the frequency estimation problem, we maintained a single counter for each distinct element. This is too space-inefficient.
- ***Idea:*** Store a fixed number of counters and assign a counter to each $x_i \in \mathscr{U}$. Multiple x_i's might be assigned to the same counter.
- To ***increment***(x), increment the counter for x.
- To ***estimate***(x), read the value of the counter for x.



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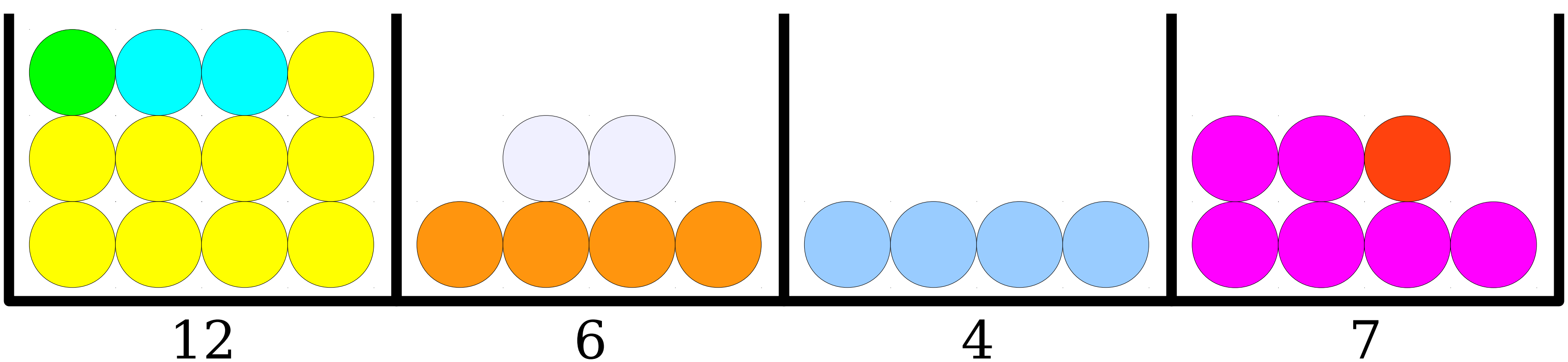

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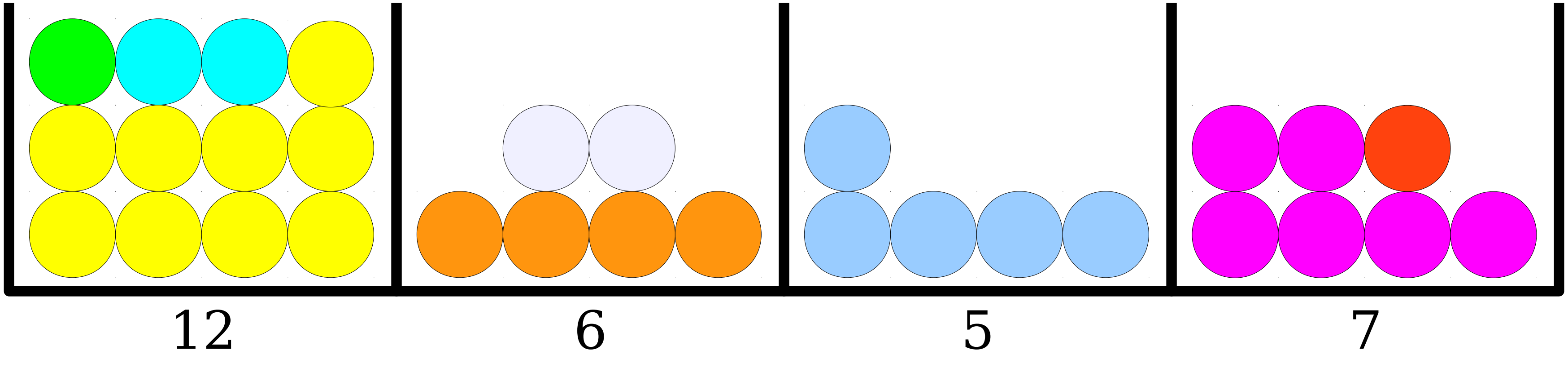
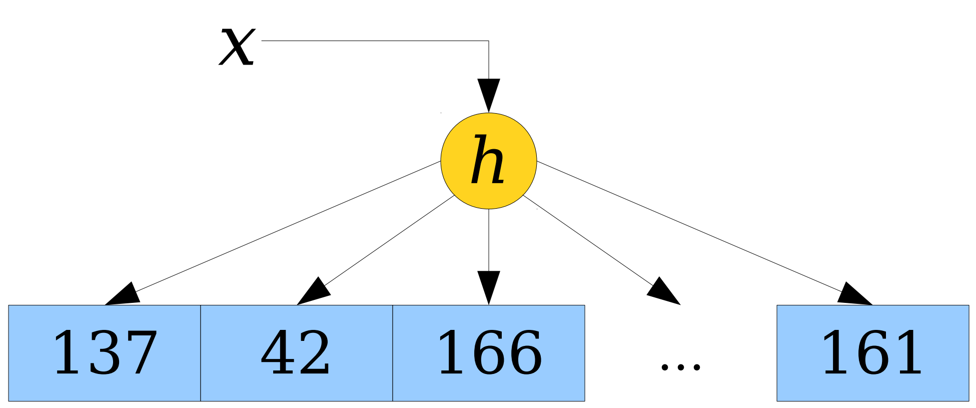


Our Initial Structure

- We can model "assigning each x_i to a counter" by using hash functions.
- Choose, from a family of 2-independent hash functions $\mathscr{H}$, a uniformly-random hash function $h : \mathscr{U} \to [w]$.
- Create an array **count** of w counters, each initially zero.
  - We'll choose w later on.
- To ***increment***(x), increment **count**$[h(x)]$.
- To ***estimate***(x), return **count**$[h(x)]$.

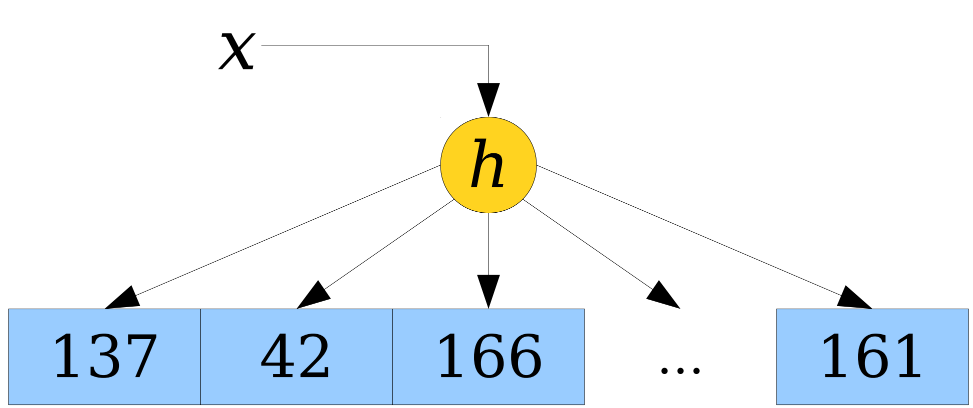


Analyzing this Structure

- ***Recall:*** $\boldsymbol{a}$ is the vector representing the true frequencies of the elements.
  - $\boldsymbol{a}_i$ is the frequency of element x_i.
- Denote by $\hat{\boldsymbol{a}}_i$ the value of ***estimate***(x_i). This is a random variable that depends on the true frequencies $\boldsymbol{a}$ (out of our control, but not random) and the hash function h (truly chosen at random.)
- ***Goal:*** Show that on expectation, $\hat{\boldsymbol{a}}_i$ is not far from $\boldsymbol{a}_i$.

Analyzing this Structure

- Intuitively, what do we expect $\hat{\boldsymbol{a}}_i$ to be?
- There are $\|\boldsymbol{a}\|_1$ total elements spread out across w buckets.
- Assuming they're well-distributed, we'd probably expect $\|\boldsymbol{a}\|_1 / w$ of them to be in each bucket.
- So a reasonable guess would be that $\hat{\boldsymbol{a}}_i$ should probably end up being something like $\boldsymbol{a}_i + \|\boldsymbol{a}\|_1 / w$.
- Let's see if we can formalize this.



Analyzing this Structure

- Let's look at $\hat{\boldsymbol{a}}_i$ = `count`[$h(x_i)$] for some choice of x_i.
- For each element x_j:
  - If $h(x_i) = h(x_j)$, then x_j contributes $\boldsymbol{a}_j$ to `count`[$h(x_i)$].
  - If $h(x_i) \neq h(x_j)$, then x_j contributes 0 to `count`[$h(x_i)$].

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- To pin this down precisely, let's define a set of random variables $X_1, X_2, \ldots,$ as follows:

$$X_j = \begin{cases} 1 & \text{if } h(x_i) = h(x_j) \\ 0 & \text{otherwise} \end{cases}$$

Each of these variables is called an ***indicator random variable***, since it "indicates" whether some event occurs.

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- The value of $\hat{\boldsymbol{a}}_i$ is then given by

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- The value of $\hat{\boldsymbol{a}}_i$ is then given by

$$\hat{\boldsymbol{a}}_i = \sum_j \boldsymbol{a}_j X_j = \boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j X_j$$

$$E[\hat{\boldsymbol{a}}_i] = E[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j X_j]$$

$$\begin{aligned} E[\hat{\boldsymbol{a}}_i] &= E[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j X_j] \\ &= E[\boldsymbol{a}_i] + E[\sum_{j \neq i} \boldsymbol{a}_j X_j] \end{aligned}$$

This follows from ***linearity of expectation***. We'll use this property extensively over the next few days.

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i] &= \mathrm{E}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j X_j] \\
&= \mathrm{E}[\boldsymbol{a}_i] + \mathrm{E}[\sum_{j \neq i} \boldsymbol{a}_j X_j] \\
&= \boldsymbol{a}_i + \sum_{j \neq i} \mathrm{E}[\boldsymbol{a}_j X_j]
\end{aligned}
$$

The actual value of $\boldsymbol{a}_i$ is not a random variable. The randomness here is in our choice of hash function, not the choice of the data.

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i] &= \mathrm{E}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j X_j] \\
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$$
\mathrm{E}[X_j] = 1 \cdot \Pr[h(x_i) = h(x_j)] + 0 \cdot \Pr[h(x_i) \neq h(x_j)]
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\end{aligned}
$$

If X is an indicator variable for some event $\mathcal{E}$, then **E[X] = Pr[$\mathcal{E}$]**. This is really useful when using linearity of expectation!

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i] &= \mathrm{E}[\boldsymbol{a}_i + \sum_{j\neq i} \boldsymbol{a}_j X_j] \\
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\mathrm{E}[X_j] &= 1\cdot\Pr[h(x_i)=h(x_j)] + 0\cdot\Pr[h(x_i)\neq h(x_j)] \\
&= 1\cdot\Pr[h(x_i)=h(x_j)]
\end{aligned}
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Any two hash codes from a randomly-chosen 2-independent hash function are independent, uniformly-random variables.

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&= \boldsymbol{a}_i + \sum_{j \neq i} \frac{\boldsymbol{a}_j}{w} \\
&\leq \boldsymbol{a}_i + \frac{\|\boldsymbol{a}\|_1}{w}
\end{aligned}
$$

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&= 1 \cdot \Pr[h(x_i) = h(x_j)] \\
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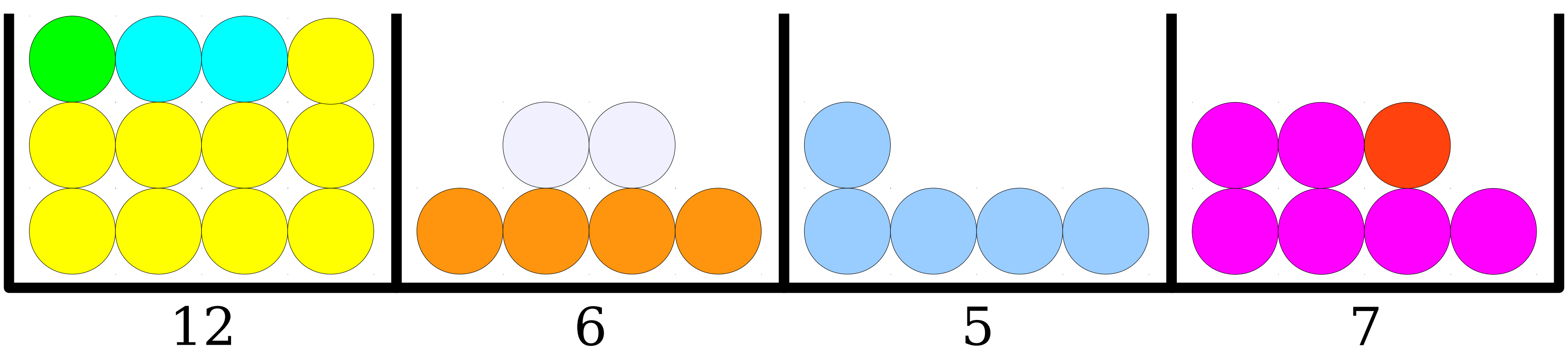
\end{aligned}
$$

Interpreting our Analysis

- On expectation, the value of ***estimate***(x_i) is at most $\|\boldsymbol{a}\|_1 / w$ greater than a_i.
  - That matches our intuition from before! Yay!
- From a practical perspective:
  - Increasing w increases memory usage, but improves accuracy.
  - Decreasing w decreases memory usage, but decreases accuracy.

One Problem

- We have shown that *on expectation*, the value of ***estimate***(x_i) can be made close to the true value.
- However, this data structure may give wildly inaccurate results for most elements.
  - Any low-frequency elements that collide with high-frequency elements will have overreported frequency.



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- However, this data structure may give wildly inaccurate results for most elements.
  - Any low-frequency elements that collide with high-frequency elements will have overreported frequency.
- ***Question:*** Can we bound the probability that we overestimate the frequency of an element?

A Useful Observation

- Notice that regardless of which hash function we use or the size of the table, we always have $\hat{a}_i \geq a_i$.
- This means that $\hat{a}_i - a_i \geq 0$.
- We have a ***one-sided error***; this data structure will never underreport the frequency of an element, but it may overreport it.

Bounding the Error Probability

- If X is a nonnegative random variable, then ***Markov's inequality*** states that for any $c > 0$, we have

$$\Pr[X > c \cdot \mathrm{E}[X]] \leq 1/c$$

- We know that

$$\mathrm{E}[\hat{\boldsymbol{a}}_i] \leq \boldsymbol{a}_i + \|\boldsymbol{a}\|_1 / w$$

- Therefore, we see that

$$\mathrm{E}[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i] \leq \|\boldsymbol{a}\|_1 / w$$

- By Markov's inequality, for any $c > 0$, we have

$$\Pr[\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i > \frac{c\|\boldsymbol{a}\|_1}{w}] \leq 1/c$$

- Equivalently:

$$\Pr[\hat{\boldsymbol{a}}_i > \boldsymbol{a}_i + \frac{c\|\boldsymbol{a}\|_1}{w}] \leq 1/c$$

Bounding the Error Probability

- For any $c > 0$, we know that
$$\Pr[\hat{\boldsymbol{a}}_i > \boldsymbol{a}_i + \frac{c\|\boldsymbol{a}\|_1}{w}] \leq 1/c$$
- In particular:
$$\Pr[\hat{\boldsymbol{a}}_i > \boldsymbol{a}_i + \frac{e\|\boldsymbol{a}\|_1}{w}] \leq 1/e$$
- Given an accuracy parameter $\varepsilon, \in (0, 1]$, let's set $w = \lceil e / \varepsilon \rceil$. Then we have
$$\Pr[\hat{\boldsymbol{a}}_i > \boldsymbol{a}_i + \varepsilon\|\boldsymbol{a}\|_1] \leq 1/e$$
- This data structure uses $O(\varepsilon^{-1})$ space and gives estimates with error at most $\varepsilon\|\boldsymbol{a}\|_1$ with probability at least $1 - 1 / e$.

Tuning the Probability

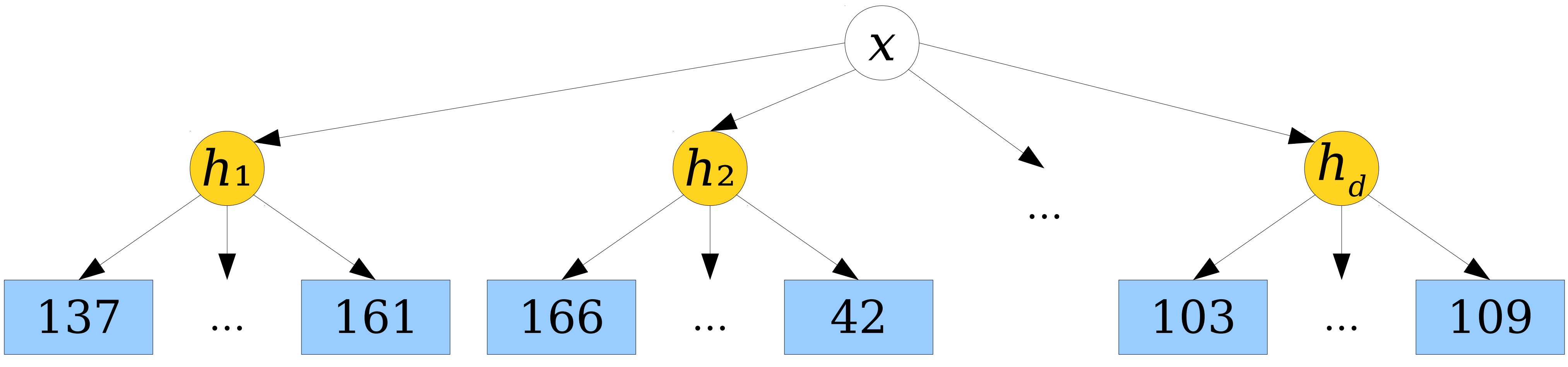
- Right now, we can tune the *accuracy* ε of the data structure, but we can't tune our *confidence* in that answer (it's always $1 - 1 / e$).
- ***Goal:*** Update the data structure so that for any confidence $0 < \delta < 1$, the probability that an estimate is correct is at least $1 - \delta$.

Tuning the Probability

- A single copy of our data structure has a decently good chance of providing an estimate that isn't too far off the true value.
- Intuitively, having *lots* of copies of this data structure would make it more likely that at least one of them gets a good estimate.
- ***Idea:*** Combine together multiple copies of this data structure to boost confidence in our estimates.

Running in Parallel

- Let's suppose that we run d independent copies of this data structure. Each has its own independently randomly chosen hash function.
- To ***increment***(x) in the overall structure, we call ***increment***(x) on each of the underlying data structures.
- The probability that at least one of them provides a good estimate is quite high.
- ***Question:*** How do you know which one?



Recognizing the Answer

- ***Recall:*** Each estimate $\hat{\boldsymbol{a}}_i$ is the sum of two independent terms:
  - The actual value $\boldsymbol{a}_i$.
  - Some "noise" terms from other elements colliding with x_i.
- Since the noise terms are always nonnegative, larger values of $\hat{\boldsymbol{a}}_i$ are less accurate than smaller values of $\hat{\boldsymbol{a}}_i$.
- ***Idea:*** Take, as our estimate, the minimum value of $\hat{\boldsymbol{a}}_i$ from all of the data structures.

The Final Analysis

- For each independent copy of this data structure, the probability that our estimate is within $\varepsilon||\boldsymbol{a}||_1$ of the true value is at least $1 - 1 / e$.
- Let $\mathcal{E}_i$ be the event that the ith copy of the data structure provides an estimate within $\varepsilon||\boldsymbol{a}||_1$ of the true answer.
- Let $\mathcal{E}$ be the event that the aggregate data structure provides an estimate within $\varepsilon||\boldsymbol{a}||_1$.
- ***Question:*** What is $\Pr[\mathcal{E}]$?

The Final Analysis

- Since we're taking the minimum of all the estimates, if *any* of the data structures provides a good estimate, our estimate will be accurate.
- Therefore,

$$\Pr[\mathcal{E}] = \Pr[\exists i.\ \mathcal{E}_i]$$

- Equivalently:

$$\Pr[\mathcal{E}] = 1 - \Pr[\forall i.\ \overline{\mathcal{E}}_i]$$

- Since all the estimates are independent:

$$\Pr[\mathcal{E}] = 1 - \Pr[\forall i.\ \overline{\mathcal{E}}_i] \geq 1 - 1/e^d.$$

The Final Analysis

- We now have that

$$\Pr[\mathcal{E}] \geq 1 - 1/e^{d}.$$

- If we want the confidence to be 1 – δ, we can choose δ such that

$$1 - \delta = 1 - 1/e^{d}$$

- Solving, we can choose $d = \ln \delta^{-1}$.
- If we make $\ln \delta^{-1}$ independent copies of our data structure, the probability that our estimate is off by at most $\varepsilon\|\boldsymbol{a}\|_1$ is at least 1 – δ.

The Count-Min Sketch

- This data structure is called the ***count-min sketch***.
- Given parameters ε and δ, choose

$$w = \lceil e \,/\, \varepsilon \rceil \qquad d = \lceil \ln \delta^{-1} \rceil$$

- Create an array **`count`** of size $w \times d$ and for each row i, choose a hash function $h_i : \mathscr{U} \to [w]$ uniformly and independently from a 2-independent family of hash functions $\mathscr{H}$.
- To ***increment***(x), increment **`count`**$[i][h_i(x)]$ for each row i.
- To ***estimate***(x), return the minimum value of **`count`**$[i][h_i(x)]$ across all rows i.

The Count-Min Sketch

- Update and query times are $\Theta(d)$, which is $\Theta(\log \delta^{-1})$.
- Space usage: $\Theta(\varepsilon^{-1} \cdot \log \delta^{-1})$ counters.
  - This can be *significantly* better than just storing a raw frequency count!
- Provides an estimate to within $\varepsilon \|\boldsymbol{a}\|_1$ with probability at least $1 - \delta$.

Some Generalizable Ideas

- Many of the techniques and ideas from this analysis will show up in other places.
- First, the idea of using ***indicator variables*** and ***linearity of expectation*** to simplify expected value calculations.
- Second, relying on the ***independence guarantees*** of our hash function to simplify some of the intermediate steps.
- Third, the fact that being good *on expectation* isn't the same as being good *with high probability* and using ***concentration inequalities*** to quantify spread.
- Finally, the fact that *confidence* and *accuracy* aren't the same, and running ***multiple parallel copies*** of a data structure to boost confidence.

Time-Out for Announcements!

Final Project Proposal

- Final project proposals were due today at 2:30PM.
- We’re going to run a matchmaking algorithm soon and get back to everyone with their assigned topics.
- We’re looking forward to seeing what everyone has come up with!

Problem Sets

- Problem Set Four is due next Thursday at 2:30PM.
- Have questions? As always, you can
  - stop by office hours, or
  - ask on Piazza!
- We hope you have fun with this one!

Back to CS166!

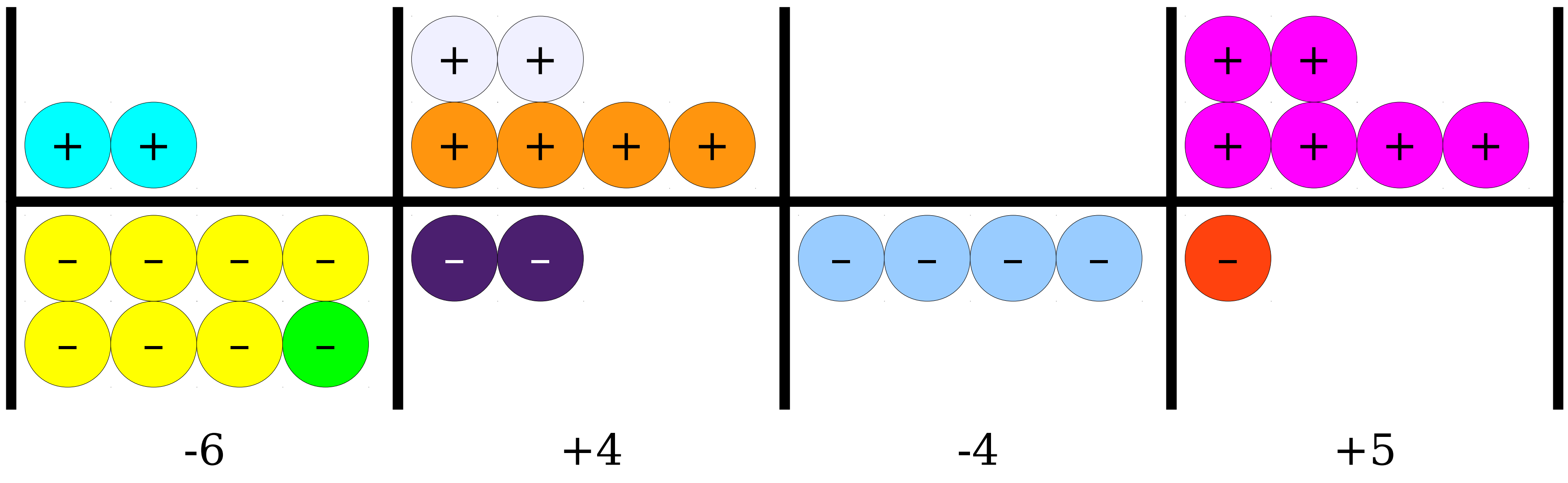
An Alternative: Count Sketches

The Motivation

- *(Note: This is historically backwards; count sketches came before count-min sketches.)*
- In a count-min sketch, errors arise when multiple elements collide.
- Errors are strictly additive; the more elements collide in a bucket, the worse the estimate for those elements.
- ***Question:*** Can we try to offset the “badness” that results from the collisions?

The Setup

- As before, for some parameter w, we'll create an array **count** of length w.
- As before, choose a hash function $h : \mathscr{U} \to [w]$ from a family $\mathscr{H}$.
- For each $x_i \in \mathscr{U}$, assign x_i either +1 or -1.
- To ***increment***(x), go to **count**$[h(x)]$ and add ±1 as appropriate.
- To ***estimate***(x), return **count**$[h(x)]$, multiplied by ±1 as appropriate.



The Setup

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- As before, choose a hash function $h : \mathscr{U} \to [w]$ from a family $\mathscr{H}$.
- For each $x_i \in \mathscr{U}$, assign x_i either +1 or -1.
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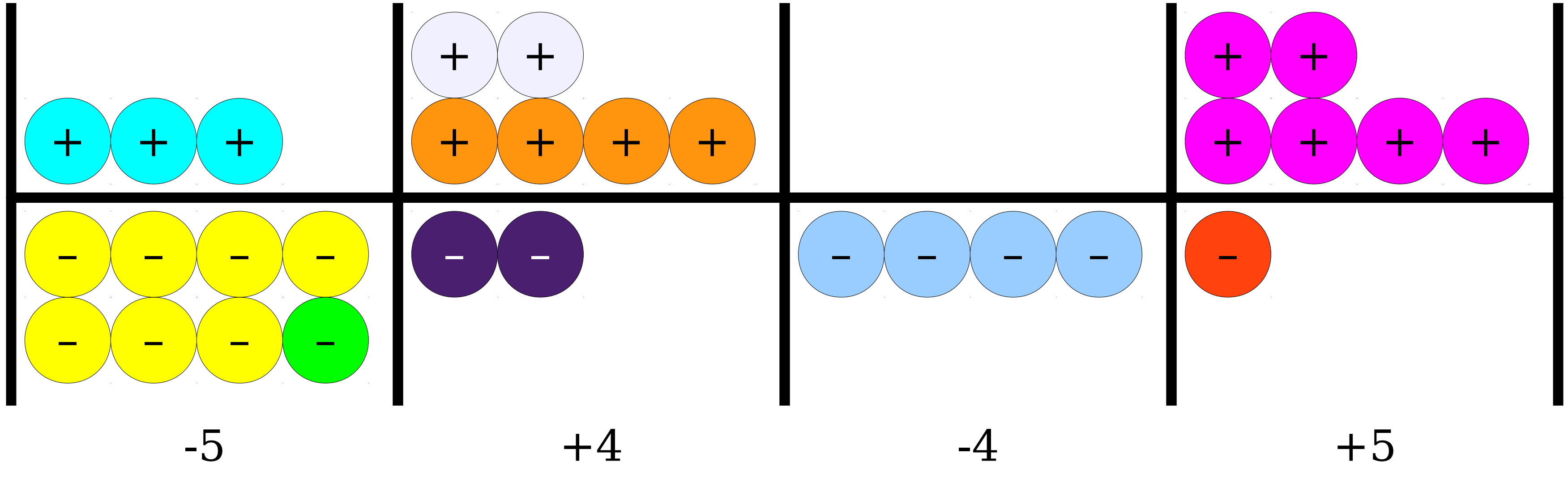

The Setup

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- As before, choose a hash function $h : \mathscr{U} \to [w]$ from a family $\mathscr{H}$.
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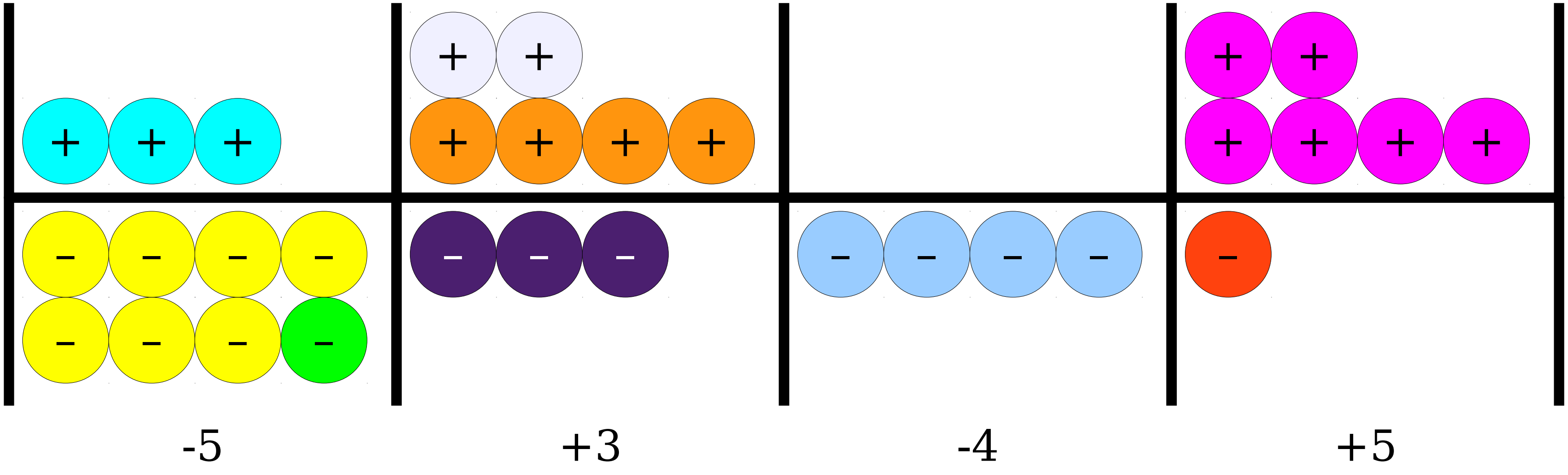
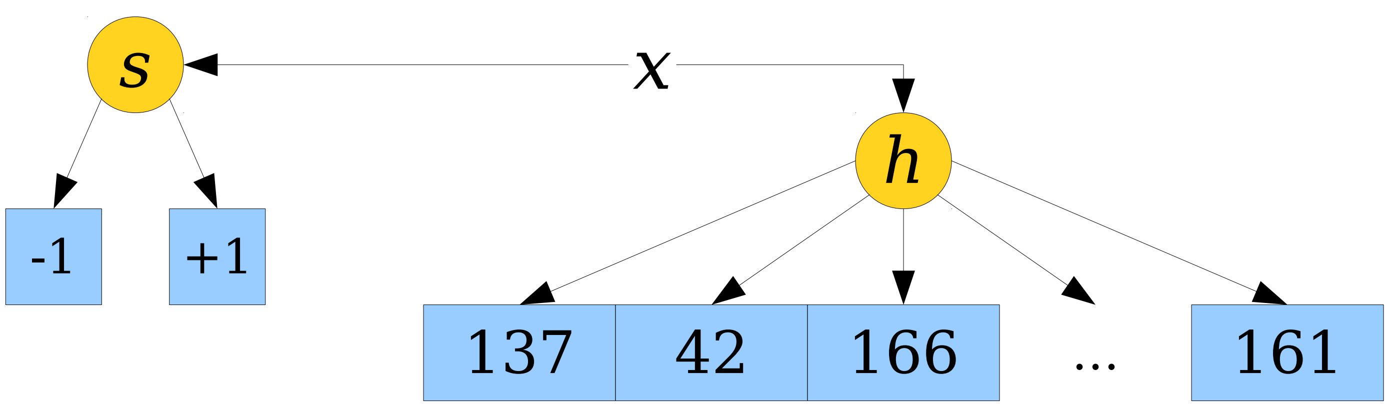


The Intuition

- Think about what introducing the ± 1 term does when collisions occur.
- If an element x collides with a frequent element y, we're not going to get a good estimate for x (but we wouldn't have gotten one anyway).
- If x collides with multiple infrequent elements, the collisions between those elements will partially offset one another and leave a better estimate for x.

More Formally

- Let's have $h \in \mathscr{H}$ chosen uniformly at random from a ***3-independent*** family of hash functions from $\mathscr{U}$. to w.
- Choose $s \in \mathscr{U}$ uniformly randomly and independently of h from a ***3-independent*** family from $\mathscr{U}$ to $\{-1, +1\}$.
  - (Note: The more traditional analysis uses 2-independence rather than 3-independence. I'm showing you a slightly simplified version.)
- To ***increment***(x), add $s(x)$ to **`count`**$[h(x)]$.
- To ***estimate***(x), return $s(x) \cdot$ **`count`**$[h(x)]$.



How accurate is our estimation?

Formalizing the Intuition

- As before, define $\hat{\boldsymbol{a}}_i$ to be our estimate of $\boldsymbol{a}_i$.
- As before, $\hat{\boldsymbol{a}}_i$ will depend on how the other elements are distributed. Unlike before, it now also depends on signs given to the elements by s.
- Specifically, for each other x_j that collides with x_i, the error contribution will be

$$s(x_i) \cdot s(x_j) \cdot \boldsymbol{a}_j$$

- Why?
  - The counter for x_i will have $s(x_j)\ \boldsymbol{a}_j$ added in.
  - We multiply the counter by $s(x_i)$ before returning it.

Formalizing the Intuition

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- Specifically, for each other x_j that collides with x_i, the error contribution will be

$$s(x_i) \cdot s(x_j) \cdot \boldsymbol{a}_j$$

- Or:
  - If $s(x_i)$ and $s(x_j)$ point in the same direction, the terms add to the total.
  - If $s(x_i)$ and $s(x_j)$ point in different directions, the terms subtract from the total.

Formalizing the Intuition

- In our quest to learn more about $\hat{\boldsymbol{a}}_i$, let's have X_j be a random variable indicating whether x_i and x_j collided with one another:

$$X_j = \begin{cases} 1 & \text{if } h(x_i) = h(x_j) \\ 0 & \text{if } h(x_i) \neq h(x_j) \end{cases}$$

Formalizing the Intuition

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- We can then express $\hat{\boldsymbol{a}}_i$ in terms of the signed contributions from the items it collides with:

$$\hat{\boldsymbol{a}}_i = \sum_j \boldsymbol{a}_j s(x_i) s(x_j) X_j$$

This is how much the collision impacts our estimate.

We only care about items we collided with.

Formalizing the Intuition

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- We can then express $\hat{\boldsymbol{a}}_i$ in terms of the signed contributions from the items it collides with:

$$\hat{\boldsymbol{a}}_i = \sum_j \boldsymbol{a}_j s(x_i) s(x_j) X_j = \boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j$$

This is how much the collision impacts our estimate.

We only care about items we collided with.

$$\mathrm{E}[\hat{\boldsymbol{a}}_i] \quad = \quad \mathrm{E}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j]$$

$$E[\hat{\boldsymbol{a}}_i] = E[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j]$$

$$= E[\boldsymbol{a}_i] + E[\sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j]$$

Hey, it's linearity of expectation!

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i] &= \mathrm{E}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&= \mathrm{E}[\boldsymbol{a}_i] + \mathrm{E}[\sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&= \boldsymbol{a}_i + \sum_{j \neq i} \mathrm{E}[\boldsymbol{a}_j s(x_i) s(x_j) X_j]
\end{aligned}
$$

Remember that $\boldsymbol{a}_i$ and the like aren't random variables.

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i] &= \mathrm{E}\Big[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j\Big] \\
&= \mathrm{E}[\boldsymbol{a}_i] + \mathrm{E}\Big[\sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j\Big] \\
&= \boldsymbol{a}_i + \sum_{j \neq i} \mathrm{E}[\boldsymbol{a}_j s(x_i) s(x_j) X_j]
\end{aligned}
$$

We chose the hash functions *h* and *s* independently of one another.

$$
X_j = \begin{cases} 1 & \text{if } h(x_i) = h(x_j) \\ 0 & \text{if } h(x_i) \neq h(x_j) \end{cases}
$$

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\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i] &= \mathrm{E}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&= \mathrm{E}[\boldsymbol{a}_i] + \mathrm{E}[\sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&= \boldsymbol{a}_i + \sum_{j \neq i} \mathrm{E}[\boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&= \boldsymbol{a}_i + \sum_{j \neq i} \mathrm{E}[s(x_i) s(x_j)] \mathrm{E}[\boldsymbol{a}_j X_j]
\end{aligned}
$$

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$$
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\end{aligned}
$$

Remember that s is drawn from a 3-independent family of hash functions, so $s(x_i)$ and $s(x_j)$ are independent random variables.

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i] &= \mathrm{E}[\boldsymbol{a}_i + \sum_{j\neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&= \mathrm{E}[\boldsymbol{a}_i] + \mathrm{E}[\sum_{j\neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&= \boldsymbol{a}_i + \sum_{j\neq i} \mathrm{E}[\boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
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\end{aligned}
$$

$\mathrm{E}[s(x_i)] =$

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i] &= \mathrm{E}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
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\end{aligned}
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$\mathrm{E}[s(x_i)] =$

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\end{aligned}
$$

$\mathrm{E}[s(x_i)] =$

s is drawn from a 3-independent family of hash functions.

$s(x_i)$ is uniform over {-1, +1}

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i] &= \mathrm{E}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
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\end{aligned}
$$

$\mathrm{E}[s(x_i)] =$

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$s(x_i)$ is uniform over $\{-1, +1\}$

$\Pr[s(x_i) = -1] = \frac{1}{2}$ $\Pr[s(x_i) = +1] = \frac{1}{2}$

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i] &= \mathrm{E}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&= \mathrm{E}[\boldsymbol{a}_i] + \mathrm{E}[\sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
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\end{aligned}
$$

$\mathrm{E}[s(x_i)] = ½ \cdot (-1) + ½ \cdot (+1)$

s is drawn from a 3-independent family of hash functions.

$s(x_i)$ is uniform over $\{-1, +1\}$

$\Pr[s(x_i) = -1] = ½$ $\Pr[s(x_i) = +1] = ½$

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i] &= \mathrm{E}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
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\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[s(x_i)] &= \tfrac{1}{2} \cdot (-1) + \tfrac{1}{2} \cdot (+1) \\
&= 0
\end{aligned}
$$

s is drawn from a 3-independent family of hash functions.

$s(x_i)$ is uniform over $\{-1, +1\}$

$\Pr[s(x_i) = -1] = \tfrac{1}{2}$ $\Pr[s(x_i) = +1] = \tfrac{1}{2}$

$$
\begin{aligned}
E[\hat{\boldsymbol{a}}_i] &= E[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&= E[\boldsymbol{a}_i] + E[\sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&= \boldsymbol{a}_i + \sum_{j \neq i} E[\boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&= \boldsymbol{a}_i + \sum_{j \neq i} E[s(x_i) s(x_j)] E[\boldsymbol{a}_j X_j] \\
&= \boldsymbol{a}_i + \sum_{j \neq i} E[s(x_i)] E[s(x_j)] E[\boldsymbol{a}_j X_j] \\
&= \boldsymbol{a}_i + \sum_{j \neq i} 0
\end{aligned}
$$

$$
\begin{aligned}
E[s(x_i)] &= \tfrac{1}{2} \cdot (-1) + \tfrac{1}{2} \cdot (+1) \\
&= 0
\end{aligned}
$$

s is drawn from a 3-independent family of hash functions.

$s(x_i)$ is uniform over $\{-1, +1\}$

$\Pr[s(x_i) = -1] = \tfrac{1}{2}$ $\Pr[s(x_i) = +1] = \tfrac{1}{2}$

$$
\begin{aligned}
\mathrm{E}[\hat{\boldsymbol{a}}_i] &= \mathrm{E}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
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&= \boldsymbol{a}_i + \sum_{j \neq i} 0 \\
&= \boldsymbol{a}_i
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[s(x_i)] &= \tfrac{1}{2} \cdot (-1) + \tfrac{1}{2} \cdot (+1) \\
&= 0
\end{aligned}
$$

s is drawn from a 3-independent family of hash functions.

$s(x_i)$ is uniform over $\{-1, +1\}$

$\Pr[s(x_i) = -1] = \tfrac{1}{2}$ $\Pr[s(x_i) = +1] = \tfrac{1}{2}$

Expecting the Unexpected

- We’ve just seen that $E[\hat{\boldsymbol{a}}_i] = \boldsymbol{a}_i$, so on expectation our estimate is perfectly correct!
- However, we have no idea how likely it is that we’re going to get an estimate like this.
- Let’s see if we can bound the likelihood that we stray far from $\boldsymbol{a}_i$.

A Hitch

- In the count-min sketch, we used Markov's inequality to bound the probability that we get a bad estimate.
- This worked because we had a ***one-sided error***: the distance $\hat{a}_i - a_i$ from the true answer was nonnegative.
- However, with the count sketch, we have a ***two-sided error***: $\hat{a}_i - a_i$ can be negative in the count sketch because collisions can *decrease* the estimate $\hat{a}_i$ below the true value a_i.
- We'll need to use a different technique to bound the error.

Chebyshev to the Rescue

- ***Chebyshev's inequality*** states that for any random variable X with finite variance, given any $c > 0$, the following holds:

$$\Pr\left[\ |X - \mathrm{E}[X]| \geq c\sqrt{\mathrm{Var}[X]}\ \right] \leq \frac{1}{c^2}$$

- Equivalently:

$$\Pr\left[\ |X - \mathrm{E}[X]| \geq c\ \right] \leq \frac{\mathrm{Var}[X]}{c^2}$$

- If we can get the variance of $\hat{\boldsymbol{a}}_i$, we can bound the probability that we get a bad estimate with our data structure.

Computing the Variance

- Let's try computing the variance of our estimate $\hat{\boldsymbol{a}}_i$:

$$\mathrm{Var}[\hat{\boldsymbol{a}}_i] = \mathrm{Var}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j]$$

$$\boxed{\mathrm{Var}[a + X] = \mathrm{Var}[X]}$$

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- Variance is not a linear operator, but it *is* linear if the underlying random variables are independent of one another.
- ***Claim:*** Each term of the sum is independent of the others.

Independence Day

- We want to show that these two terms are independent:

$$\boldsymbol{a}_j\ s(x_i)\ s(x_j)\ X_j \qquad\qquad \boldsymbol{a}_k\ s(x_i)\ s(x_k)\ X_k$$

- Imagine we know $\boldsymbol{a}_j\ s(x_i)\ s(x_j)\ X_j$.
- Whether $\boldsymbol{a}_k\ s(x_i)\ s(x_k)\ X_k = 0$ depends on whether $h(x_i) = h(x_k)$.
  - The values $h(x_i)$, $h(x_j)$, and $h(x_k)$ are uniformly-random and independent because h is 3-independent.
  - Knowing whether $h(x_i) = h(x_j)$ doesn't impact the probability that $h(x_i) = h(x_k)$, since all three values are uniform and independent.
- The sign of $\boldsymbol{a}_k\ s(x_i)\ s(x_k)\ X_k$ depends on $s(x_i) \cdot s(x_k)$.
  - $s(x_i)$, $s(x_j)$, and $s(x_k)$ are uniformly-random and independent because s is 3-independent.
  - There's an equal chance that $s(x_i) \cdot s(x_k) = 1$ and $s(x_i) \cdot s(x_k) = -1$, since even with $s(x_i) \cdot s(x_j)$ fixed, $s(x_k)$ is independently and uniformly distributed over $\{+1, -1\}$.

$$
\begin{aligned}
\operatorname{Var}[\hat{\boldsymbol{a}}_i] &= \operatorname{Var}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&= \operatorname{Var}[\sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j]
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$$

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&= \sum_{j \neq i} \mathrm{Var}[\boldsymbol{a}_j s(x_i) s(x_j) X_j]
\end{aligned}
$$

The “Sum-o’-Var”
Samovar!

☺

$$
\begin{aligned}
\mathrm{Var}[\hat{\boldsymbol{a}}_i] &= \mathrm{Var}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
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\end{aligned}
$$

$$
\boxed{\begin{aligned} \mathrm{Var}[Z] &= \mathrm{E}[Z^2] - \mathrm{E}[Z]^2 \\ &\leq \mathrm{E}[Z^2] \end{aligned}}
$$

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&= \sum_{j \neq i} \mathrm{Var}[\boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&\leq \sum_{j \neq i} \mathrm{E}[(\boldsymbol{a}_j s(x_i) s(x_j) X_j)^2]
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$s(x) = \pm 1$,

so

$s(x)^2 = 1$

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\boxed{X_j = \begin{cases} 1 & \text{if } h(x_i) = h(x_j) \\ 0 & \text{if } h(x_i) \neq h(x_j) \end{cases}}
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&= \sum_{j \neq i} \boldsymbol{a}_j^2 \mathrm{E}[X_j^2]
\end{aligned}
$$

Useful Fact:
If X is an indicator variable, then $X^2 = X$.

$$
X_j^{\mathbf{2}} = \begin{cases} 1 & \text{if } h(x_i) = h(x_j) \\ 0 & \text{if } h(x_i) \neq h(x_j) \end{cases}
$$

$$
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\mathrm{Var}[\hat{\boldsymbol{a}}_i] &= \mathrm{Var}[\boldsymbol{a}_i + \sum_{j \neq i} \boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
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&= \sum_{j \neq i} \mathrm{Var}[\boldsymbol{a}_j s(x_i) s(x_j) X_j] \\
&\leq \sum_{j \neq i} \mathrm{E}[(\boldsymbol{a}_j s(x_i) s(x_j) X_j)^2] \\
&= \sum_{j \neq i} \mathrm{E}[\boldsymbol{a}_j^2 s(x_i)^2 s(x_j)^2 X_j^2] \\
&= \sum_{j \neq i} \boldsymbol{a}_j^2 \mathrm{E}[X_j^2] \\
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&= \sum_{j \neq i} \boldsymbol{a}_j^2 \mathrm{E}[X_j^2] \\
&= \sum_{j \neq i} \boldsymbol{a}_j^2 \mathrm{E}[X_j] \\
&= \sum_{j \neq i} \boldsymbol{a}_j^2 / w
\end{aligned}
$$

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$$\boxed{\sqrt{\sum_j \boldsymbol{a}_j^2} = \|\boldsymbol{a}\|_2}$$

$$
\begin{aligned}
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&= \sum_{j \neq i} \boldsymbol{a}_j^2 \mathrm{E}[X_j] \\
&= \sum_{j \neq i} \boldsymbol{a}_j^2 / w \\
&\leq \|\boldsymbol{a}\|_2^2 / w
\end{aligned}
$$

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&= \sum_{j \neq i} \boldsymbol{a}_j^2 \mathrm{E}[X_j] \\
&= \sum_{j \neq i} \boldsymbol{a}_j^2 / w \\
&\leq \|\boldsymbol{a}\|_2^2 / w
\end{aligned}
$$

I know this might look really dense, but many of these substeps end up being really useful techniques. These ideas generalize, I promise.

Harnessing Chebyshev

- Chebyshev's Inequality says

$$\Pr\left[\ \left|X - \mathrm{E}[X]\right| \geq c\sqrt{\mathrm{Var}[X]}\ \right] \leq 1/c^2$$

- Applying this to $\hat{\boldsymbol{a}}_i$ yields

$$\Pr\left[\ \left|\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i\right| \geq \frac{c\left\|\boldsymbol{a}\right\|_2}{\sqrt{w}}\ \right] \leq 1/c^2$$

- Given error parameter ε, pick $w = \lceil e / \varepsilon^2 \rceil$, so

$$\Pr\left[\ \left|\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i\right| \geq \frac{c\,\varepsilon\left\|\boldsymbol{a}\right\|_2}{\sqrt{e}}\ \right] \leq 1/c^2$$

- Therefore, choosing $c = e^{1/2}$ gives

$$\Pr\left[\ \left|\hat{\boldsymbol{a}}_i - \boldsymbol{a}_i\right| \geq \varepsilon\left\|\boldsymbol{a}\right\|_2\ \right] \leq 1/e$$

The Story So Far

- We now know that, by setting $\varepsilon = (e / w)^{1/2}$, the estimate is within $\varepsilon\|\boldsymbol{a}\|_2$ with probability at least $1 - 1 / e$.
- Solving for w, this means that we will choose $w = \lceil e / \varepsilon^2 \rceil$.
- Space usage is now $O(\varepsilon^{-2})$, but the error bound is now $\varepsilon\|\boldsymbol{a}\|_2$ rather than $\varepsilon\|\boldsymbol{a}\|_1$.
- As before, the next step is to reduce the error probability.

Repetitions with a Catch

- As before, our goal is to make it possible to choose a bound $0 < \delta < 1$ so that the confidence is at least $1 - \delta$.
- As before, we'll do this by making d independent copies of the data structure and running each in parallel.
- Unlike the count-min sketch, errors in count sketches are two-sided; we can overshoot or undershoot.
- Therefore, it's not meaningful to take the minimum or maximum value.
- How do we know which value to report?

Working with the Median

- ***Claim:*** If we output the median estimate given by the data structures, we have high probability of giving the right answer.
- ***Intuition:*** The only way we report an answer more than $\varepsilon\|\boldsymbol{a}\|_2$ is if at least half of the data structures output an answer that is more than $\varepsilon\|\boldsymbol{a}\|_2$ from the true answer.
- Each individual data structure is wrong with probability at most $1 / e$, so this is highly unlikely.

The Setup

- Let X denote a random variable equal to the number of data structures that produce an answer *not* within $\varepsilon \|\boldsymbol{a}\|_2$ of the true answer.
- Since each independent data structure has failure probability at most $1 / e$, we can upper-bound X with a $\text{Binom}(d, 1 / e)$ variable.
- We want to know $\Pr[X > d / 2]$.
- How can we determine this?

Chernoff Bounds

- The ***Chernoff bound*** says that if $X \sim \text{Binom}(n, p)$ and $p < 1/2$, then

$$\Pr[X > n/2] < e^{\frac{-n(1/2-p)^2}{2p}}$$

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- In our case, $X \sim \text{Binom}(d, 1/e)$, so we know that

$$\Pr\left[X > \frac{d}{2}\right] \leq e^{\frac{-d(1/2-1/e)^2}{2(1/e)}}$$

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$$\begin{aligned}\Pr\left[X > \frac{d}{2}\right] &\leq e^{\frac{-d(1/2-1/e)^2}{2(1/e)}} \\ &= e^{-k \cdot d} \qquad \textit{(for some constant k)}\end{aligned}$$

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- Therefore, choosing $d = k^{-1} \cdot \log \delta^{-1}$ ensures that $\Pr[X > d / 2] \leq \delta$.

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- Therefore, the success probability is at least $1 - \delta$.

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- In our case, $X \sim \text{Binom}(d, 1/e)$, so we know that

The specific constant factor here matters, since it's an exponent! To implement this data structure, you'll need to work out the exact value.

$$e^{\frac{-d(1/2-1/e)^2}{2(1/e)}}$$

$$e^{-k \cdot d}$$ *(for some constant k)*

- Therefore, choosing $d = k^{-1} \cdot \log \delta^{-1}$ ensures that $\Pr[X > d / 2] \leq \delta$.
- Therefore, the success probability is at least $1 - \delta$.

The Overall Construction

- The ***count sketch*** is the data structure given as follows.
- Given ε and δ, choose

$$w = \lceil e / \varepsilon^2 \rceil \qquad d = \Theta(\log \delta^{-1})$$

- Create an array **`count`** of $w \times d$ counters.
- Choose hash functions h_i and s_i for each of the d rows.
- To ***increment***(x), add $s_i(x)$ to **`count`**$[i][h_i(x)]$ for each row i.
- To ***estimate***(x), return the median of $s_i(x) \cdot$ **`count`**$[i][h_i(x)]$ for each row i.

The Final Analysis

- With probability at least $1 - \delta$, all estimates are accurate to within a factor of $\varepsilon \| \boldsymbol{a} \|_2$.
- Space usage is $\Theta(w \times d)$, which we've seen to be $\Theta(\varepsilon^{-2} \cdot \log \delta^{-1})$.
- Updates and queries run in time $\Theta(\delta^{-1})$.
- Trades factor of ε^{-1} space for an accuracy guarantee relative to $\| \boldsymbol{a} \|_2$ versus $\| \boldsymbol{a} \|_1$.

In Practice

- These data structures have been and continue to be used in practice.
- These sketches and their variants have been used at Google and Yahoo! (or at least, there are papers coming from there about their usage).
- Many other sketches exist as well for estimating other quantities; they'd make for really interesting final project topics!

More to Explore

- A ***cardinality estimator*** is a data structure for estimating how many different elements have been seen in sublinear time and space. They're used extensively in database implementations.
- If instead of estimating $\boldsymbol{a}_i$ terms individually we want to estimate $\|\boldsymbol{a}\|_1$ or $\|\boldsymbol{a}_2\|$, we can use a ***frequency moment estimator***.
- You'll get to play around with at least one of these on Problem Set Five.

Some Concluding Notes

Randomized Data Structures

- You may have noticed that the final versions of these data structures are actually not all that complex – each just maintains a set of hash functions and some 2D tables.
- The analyses, on the other hand, are a lot more involved than what we saw for other data structures.
- This is common – randomized data structures often have simple descriptions and quite complex analyses.

The Strategy

- Typically, an analysis of a randomized data structure looks like this:
  - First, show that the data structure (or some random variable related to it), on expectation, performs well.
  - Second, use concentration inequalities (Markov, Chebyshev, Chernoff, or something else) to show that it's unlikely to deviate from expectation.
- The analysis often relies on properties of some underlying hash function. On Tuesday, we'll explore why this is so important.

Next Time

- ***Hashing Strategies***
  - There are a lot of hash tables out there. What do they look like?
- ***Linear Probing***
  - The original hashing strategy!
- ***Analyzing Linear Probing***
  - …is way, way more complicated than you probably would have thought. But it's beautiful! And a great way to learn about randomized data structures!