# $x$-Fast and $y$-Fast Tries

# Outline for Today

- ***Data Structures on Integers***
  - How can we speed up operations that work on integer data?
- ***x-Fast Tries***
  - Bit manipulation meets tries and hashing.
- ***y-Fast Tries***
  - Combining RMQ, strings, balanced trees, amortization, and randomization!

# Working with Integers

- Many practical problems involve working specifically with integer values.
  - ***CPU Scheduling:*** Each thread has some associated integer priority, and we need to maintain those priorities in sorted order.
  - ***Network Routing:*** Each computer has an associated IP address, and we need to figure out which connections are active.
  - ***ID Management:*** We need to store social security numbers, zip codes, phone numbers, credit card numbers, etc. and perform basic lookups and range searches on them.
- We've seen many general-purpose data structures for keeping things in order and looking things up.
- ***Question:*** Can we improve those data structures if we know in advance that we're working with integer data?

# Working with Integers

- Integers are interesting objects to work with:
  - Their values can directly be used as indices in lookup tables.
  - They can be treated as strings of bits, so we can use techniques from string processing.
  - They fit into machine words, so we can process the bits in parallel with individual word operations.
- The data structures we'll explore over the next few lectures will give you a sense of what sorts of techniques are possible with integer data.

# An Auxiliary Motive

- Integer data structures are also a great place to see just how much you've learned over the quarter!
- Today's data structures cover every single unit from the quarter (RMQ, strings, balanced trees, amortization, and randomization).
- I hope this gives you a chance to pause and reflect on just how far you've come!

# The Setup

# Our Machine Model

- We will assume we're working on a machine where memory is segmented into $\boldsymbol{w}$-bit words.
- We'll assume our integers are drawn from some set $\mathbf{[}\boldsymbol{U}\mathbf{]}$, where $\lg U = O(w)$.
  - In other words, we assume our integers fit into a constant number of machine words.
- We'll assume that the C integer operators work in constant time, and will not assume we have access to operators beyond them.

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# Ordered Dictionaries

# Ordered Dictionaries

- An ***ordered dictionary*** maintains a set $S$ drawn from an ordered universe $\mathscr{U}$ and supports these operations:
  - ***lookup***($x$), which returns whether $x \in S$;
  - ***insert***($x$), which adds $x$ to $S$;
  - ***delete***($x$), which removes $x$ from $S$;
  - ***max***() / ***min***(), which return the maximum or minimum element of $S$;
  - ***successor***($x$), which returns the smallest element of $S$ greater than $x$; and
  - ***predecessor***($x$), which returns the largest element of $S$ smaller than $x$.
- For context:

  Ordered Dictionary : BST :: Queue : Linked List

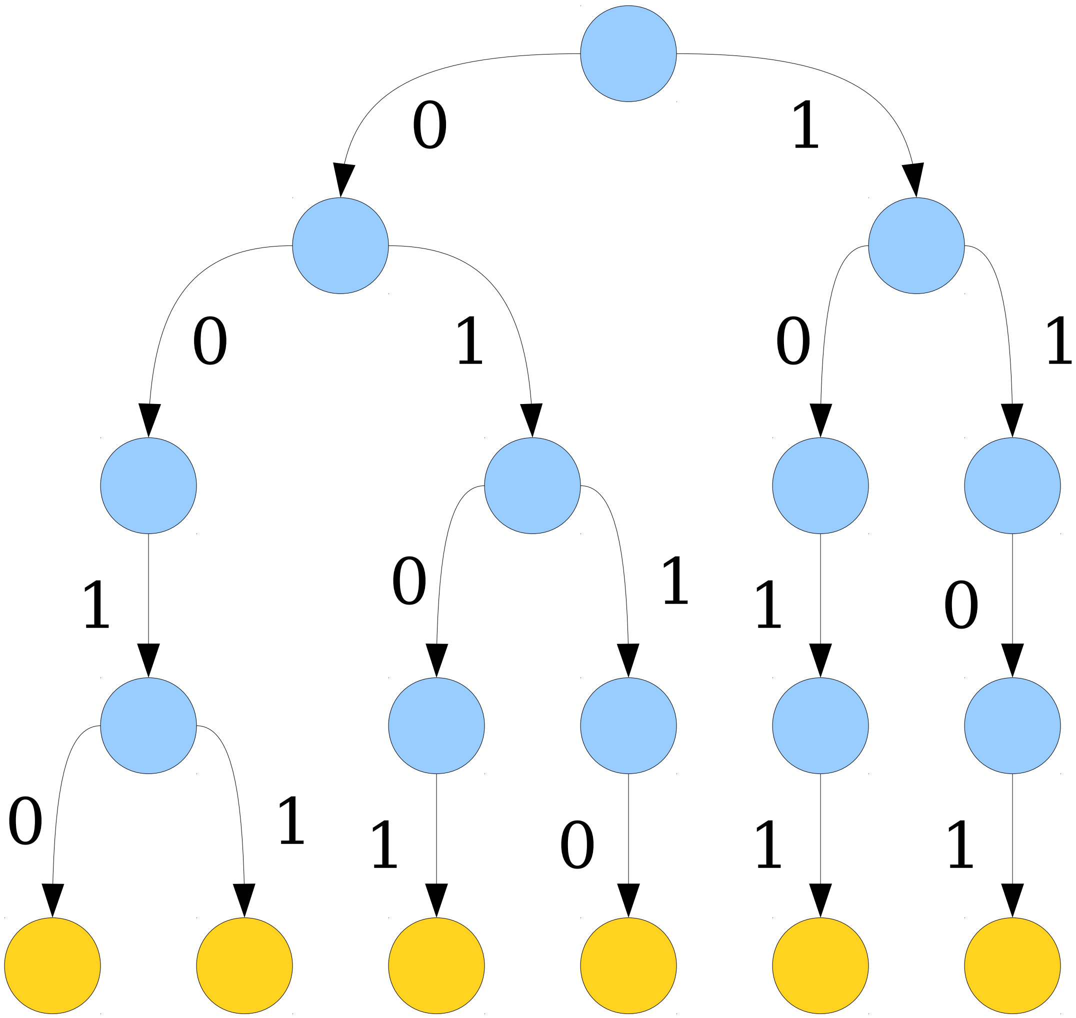
# Ordered Dictionaries

- Balanced BSTs support all ordered dictionary operations in time O(log $n$) each.
- Hash tables support insertion, lookups, and deletion in expected time O(1), but require time O($n$) for ***max***, ***min***, ***successor***, and ***predecessor***.
- ***Question:*** Can we improve upon these bounds if we know that we're working with integers drawn from [$U$]?

# A Start: ***Bitwise Tries***

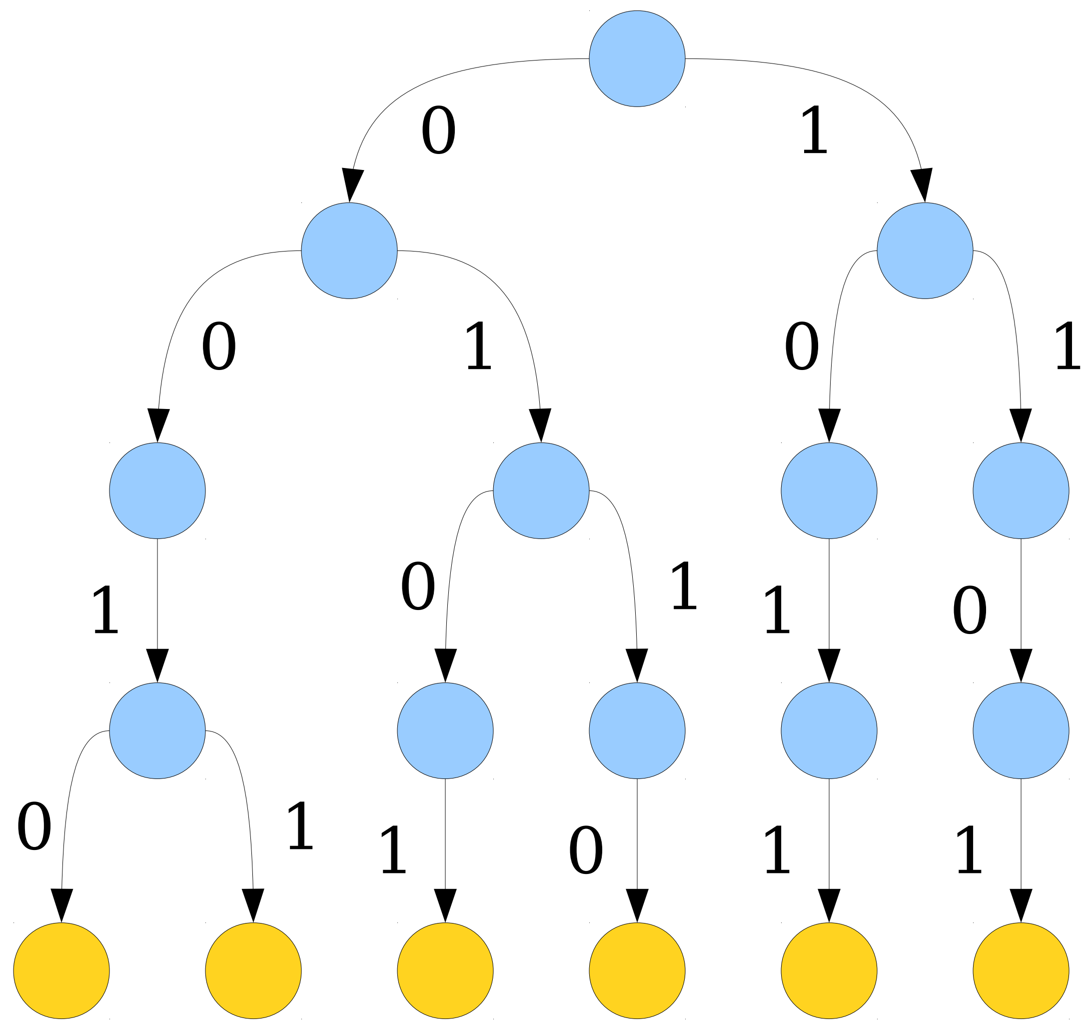
# Tries Revisited

- ***Recall:*** A trie is a simple data structure for storing strings.
- Integers can be thought of as strings of bits.
- ***Idea:*** Store integers in a ***bitwise trie***.



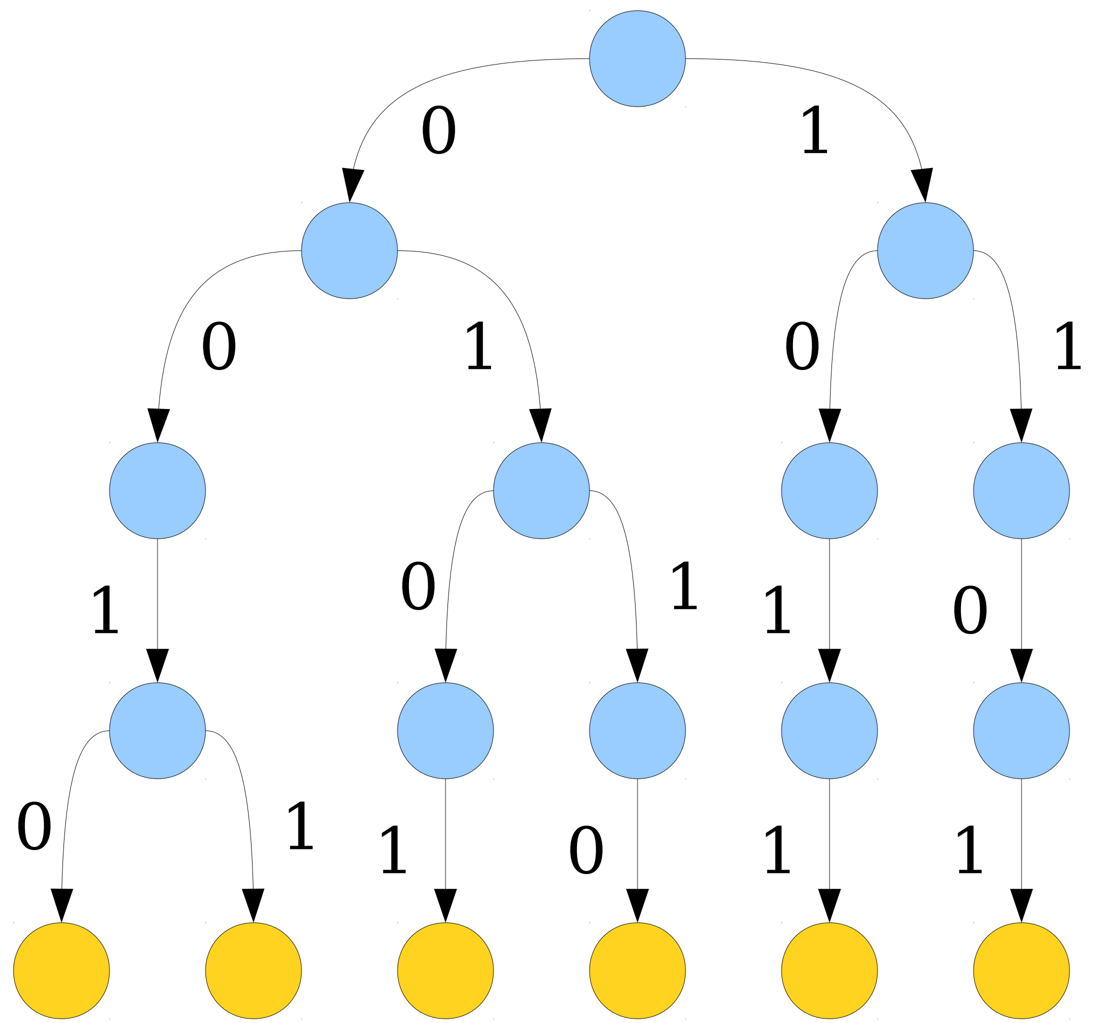
# Finding Successors

- To compute ***successor***($x$), do the following:
  - Search for $x$.
  - If $x$ is a leaf node, its successor is the next leaf.
  - If you don't find $x$, back up until you find a node with a 1 child not already followed, follow the 1, then take the cheapest path down.



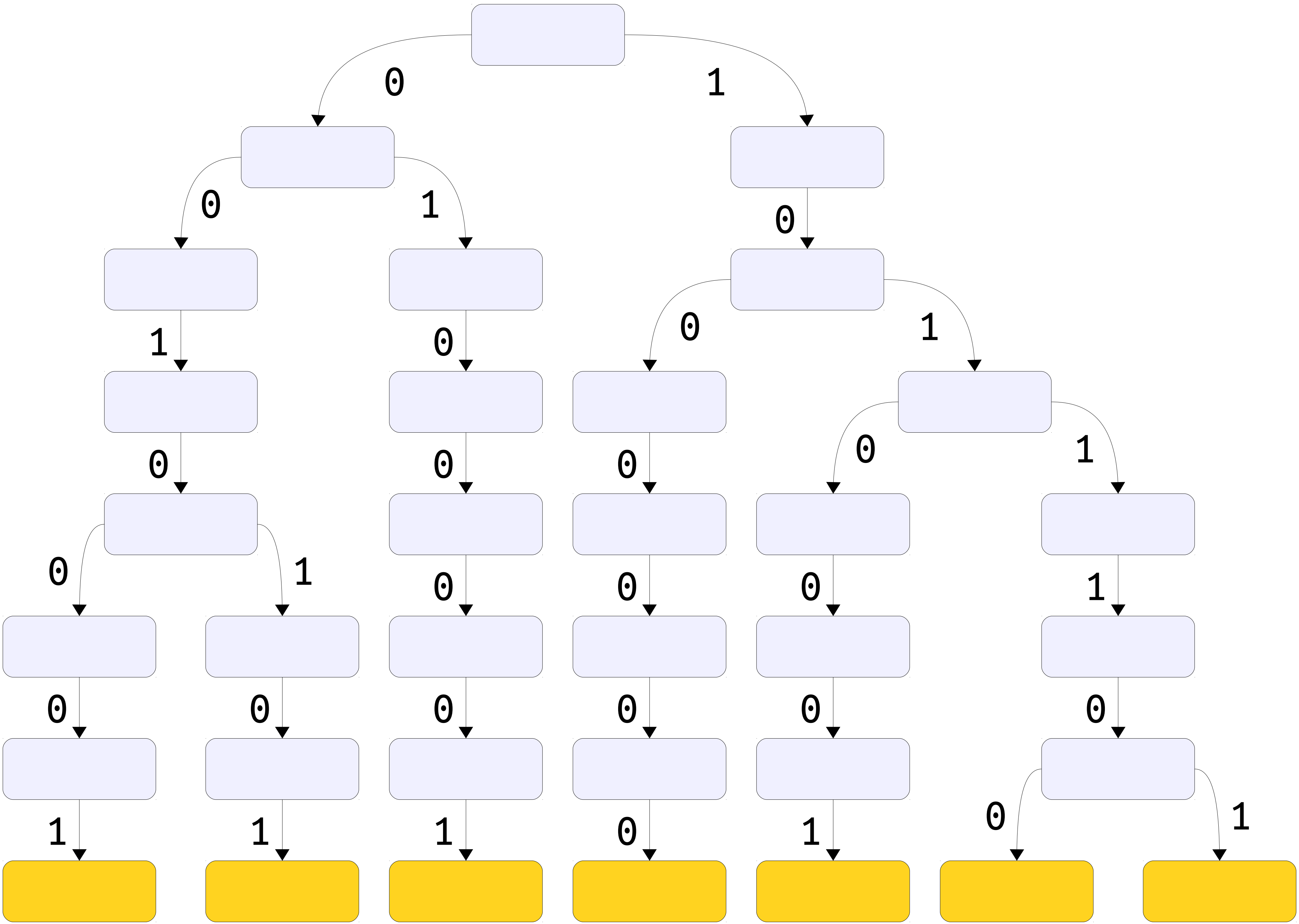
# Bitwise Trie Efficiency

- All operations on bitwise tries take time proportional to the number of bits in each number.
- Runtime for each operation: **$O(\log U)$**.
  - This is probably worse than $O(\log n)$.
- For each number stored, we need to store $\Theta(\log U)$ internal nodes.
- Space usage: **$O(n \log U)$**.
  - This is probably worse than a BST.
- ***Can we do better?***



# Speeding up Successors

- There are two independent pieces that contribute to the O(log $U$) runtime:
  - Need to walk down the trie following the bits of $x$, and there are $\Theta(\log U)$ of those.
  - From there, need to back up to a branching node where we can find the successor.
- Can we speed up those operations? Or at least work around them?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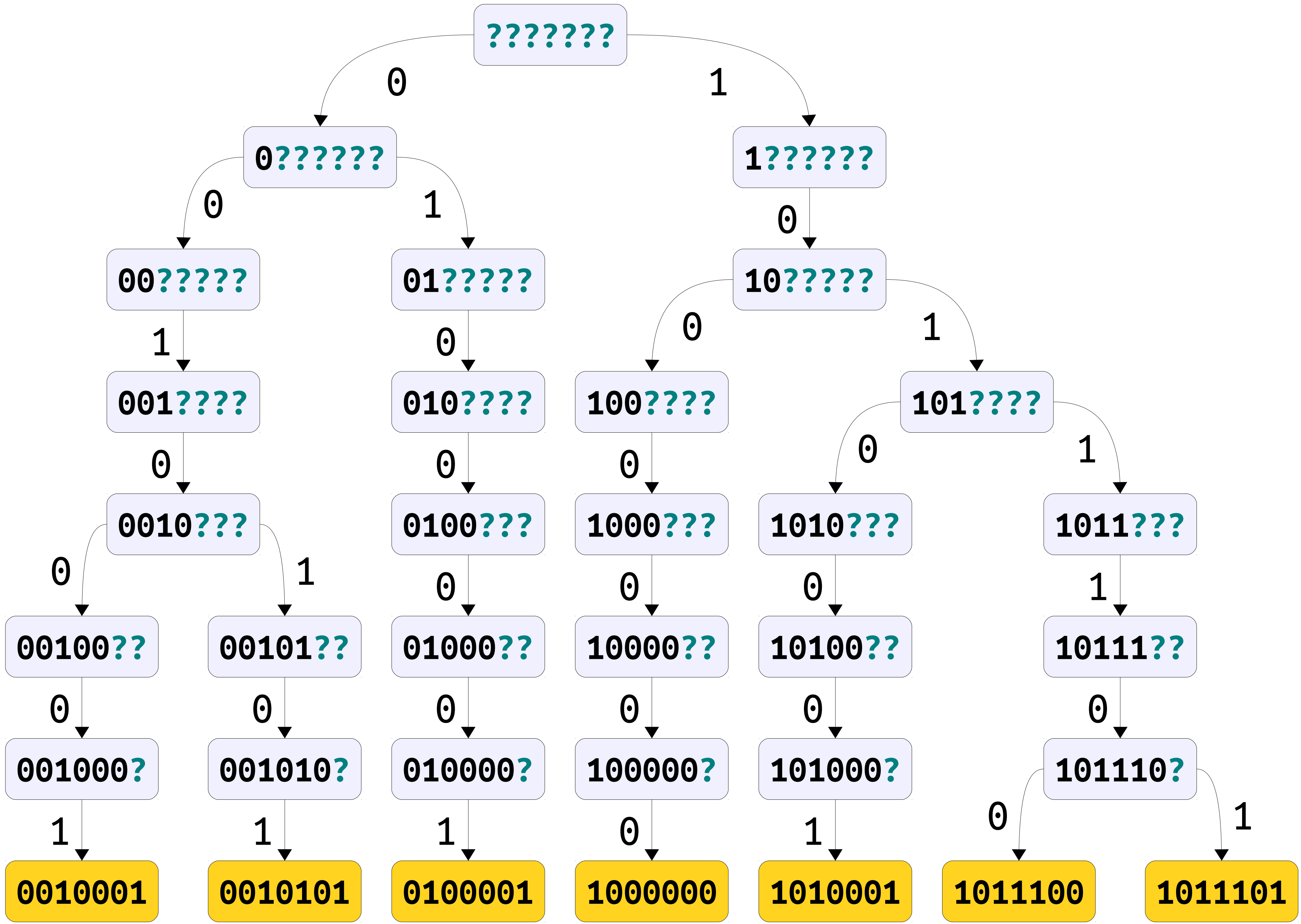
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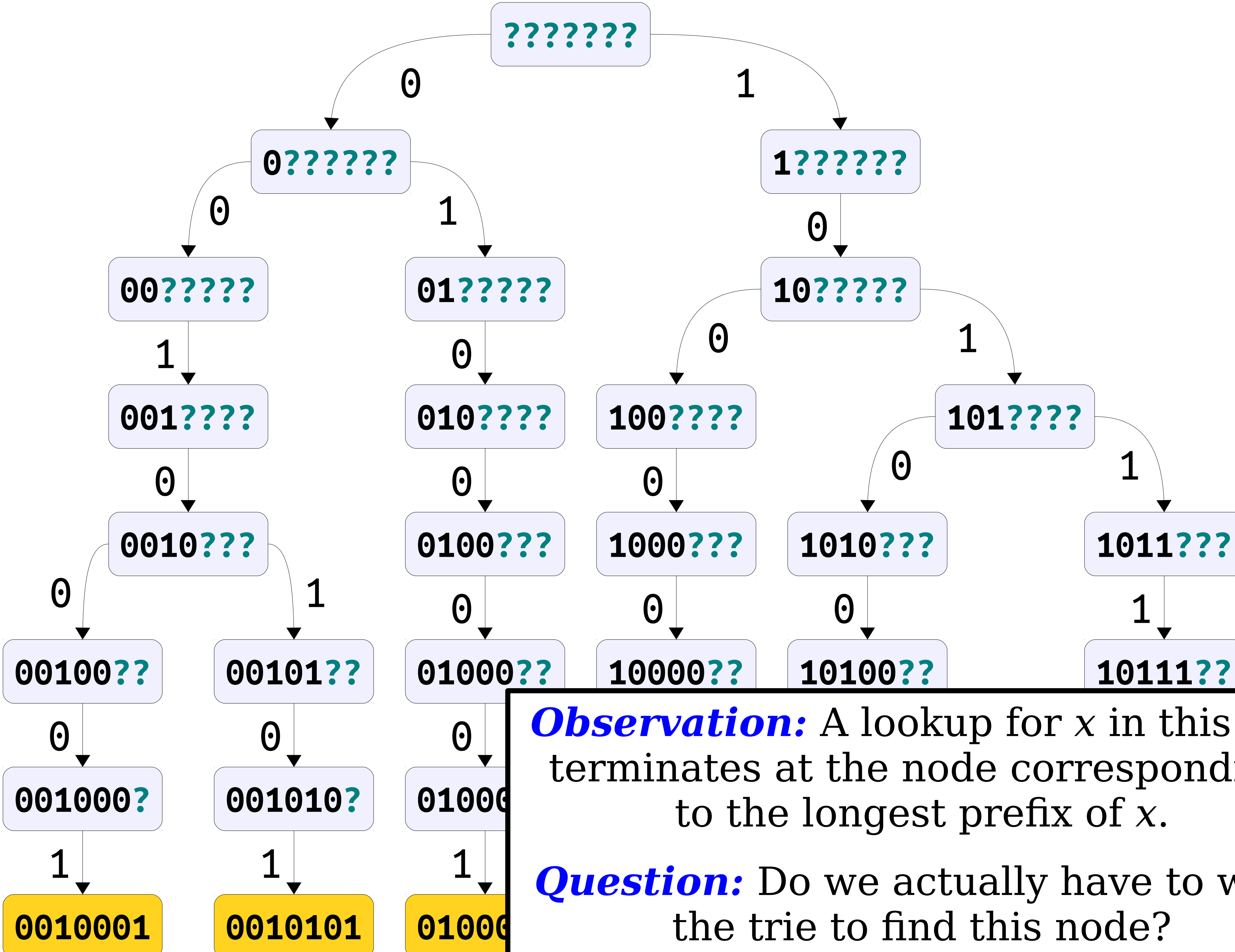
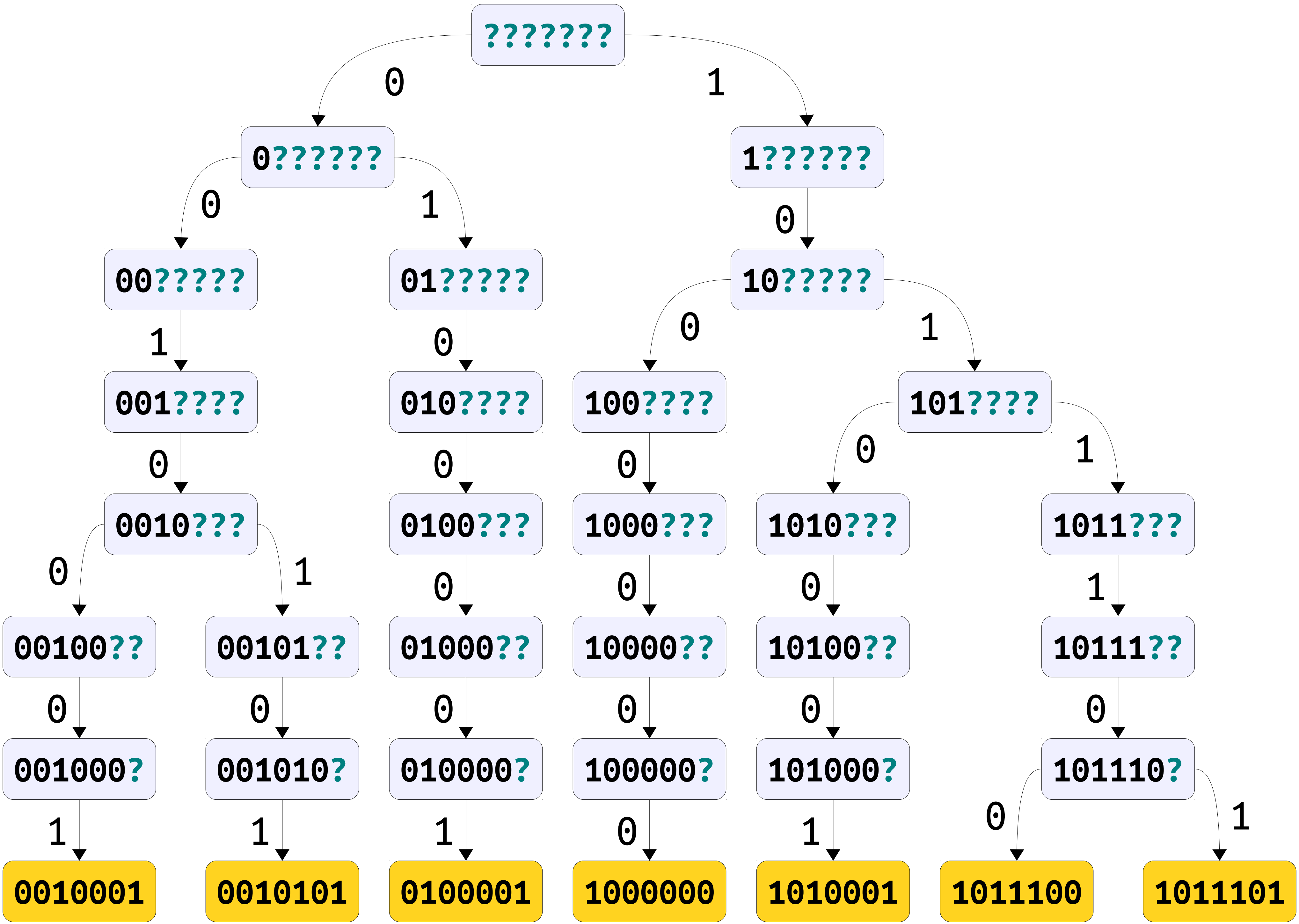



***Observation:*** A lookup for $x$ in this trie terminates at the node corresponding to the longest prefix of $x$.

***Question:*** Do we actually have to walk the trie to find this node?



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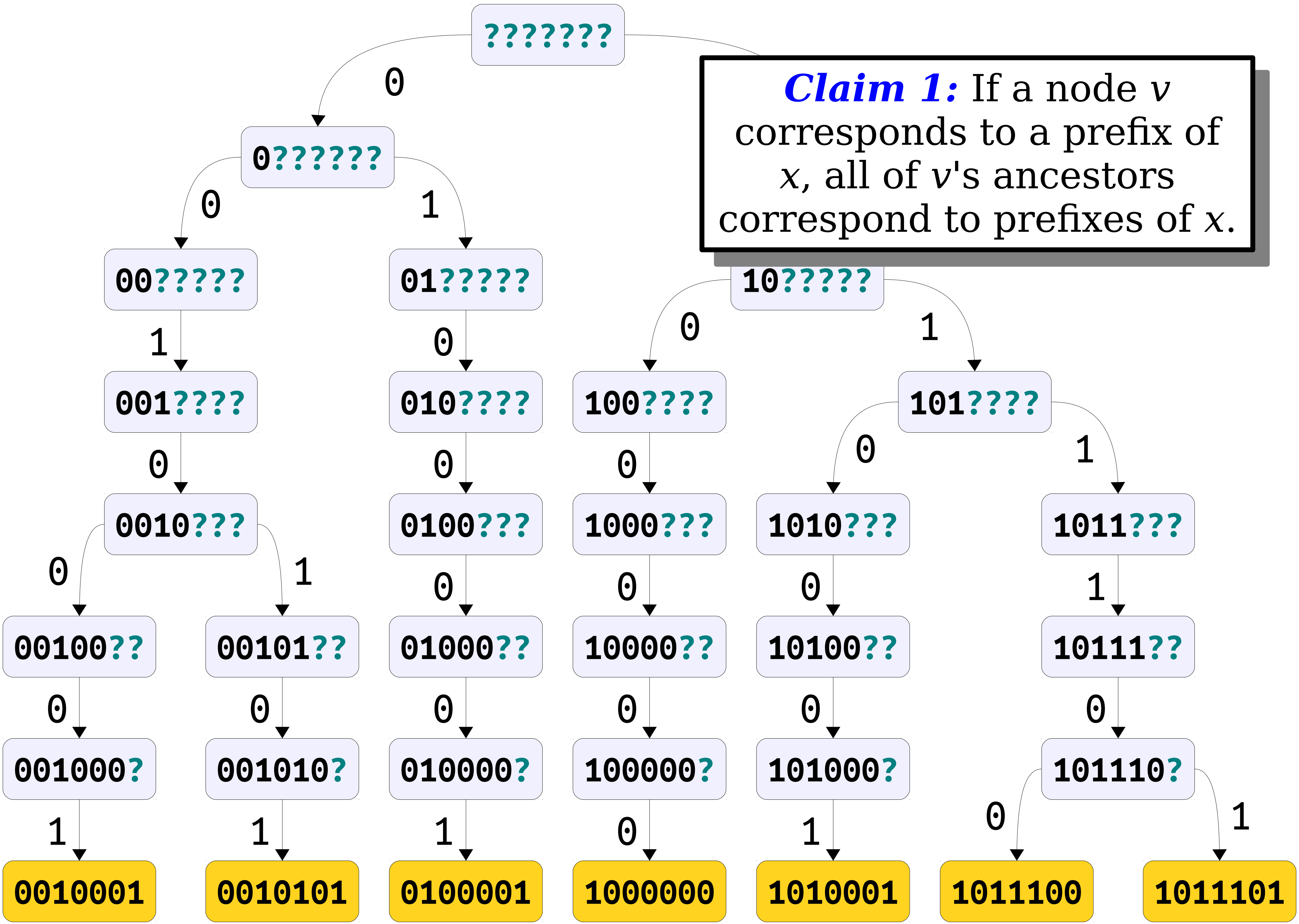
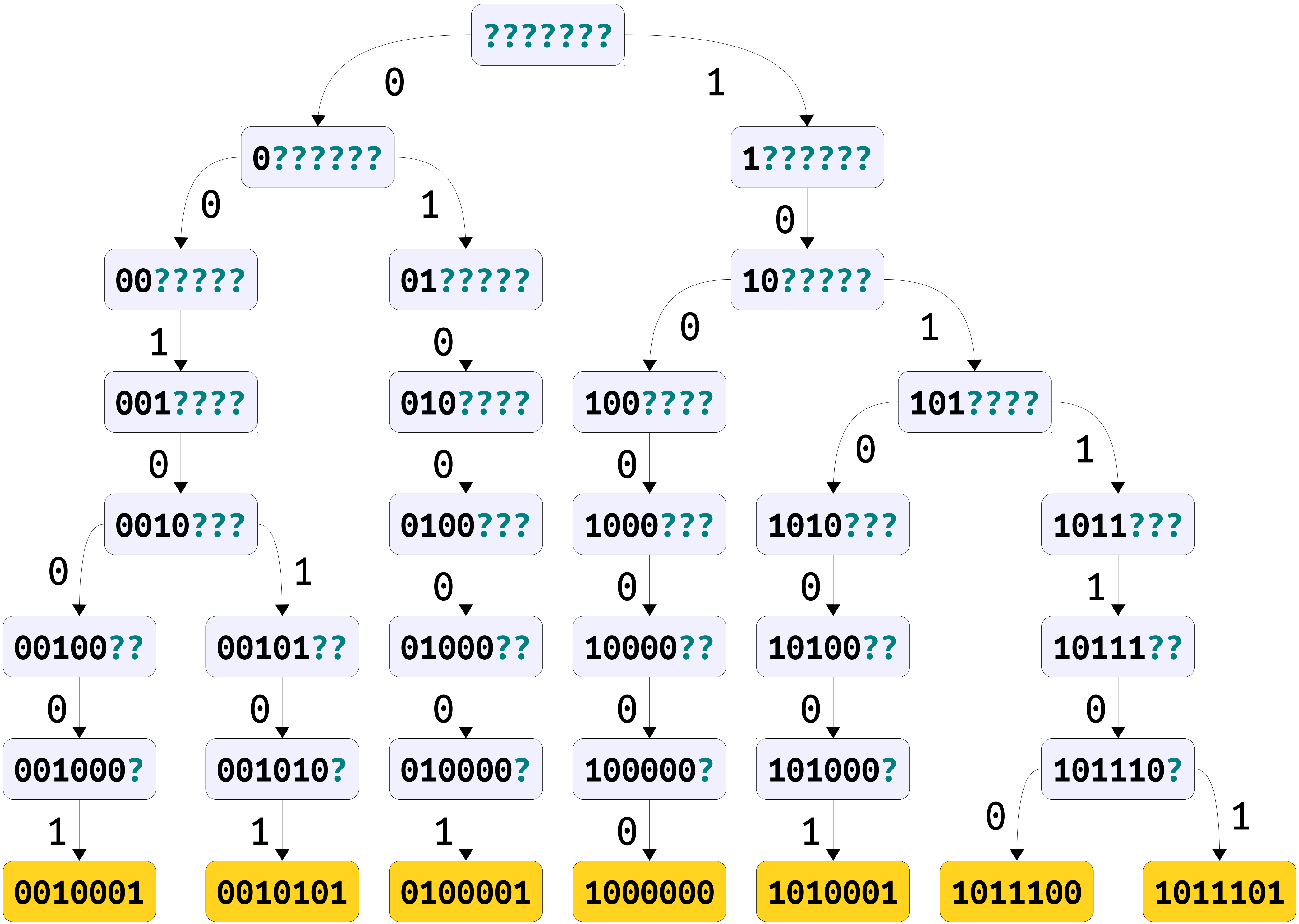



***Claim 1:*** If a node $v$ corresponds to a prefix of $x$, all of $v$'s ancestors correspond to prefixes of $x$.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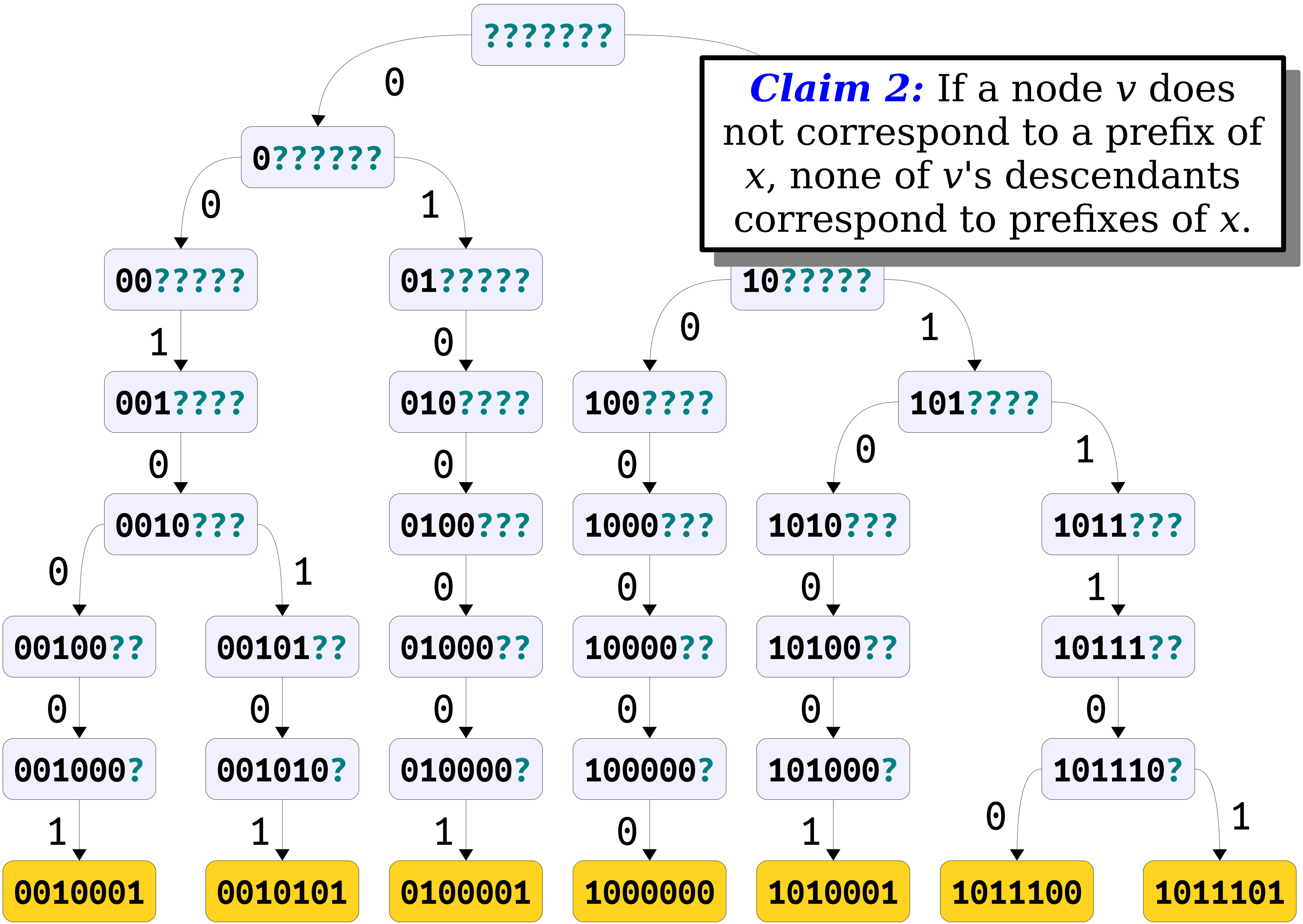

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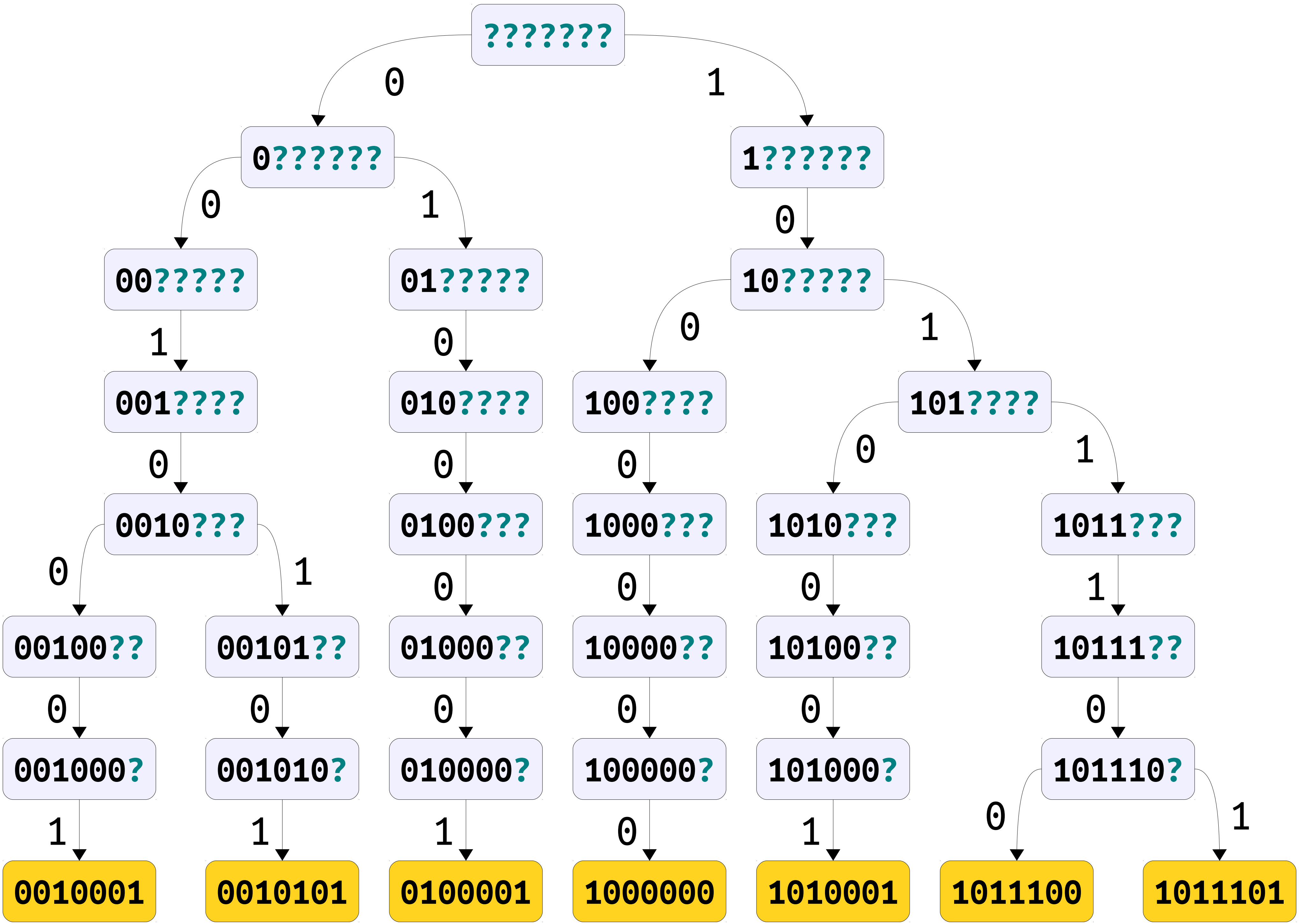
Claim 2: If a node v does not correspond to a prefix of x, none of v's descendants correspond to prefixes of x.



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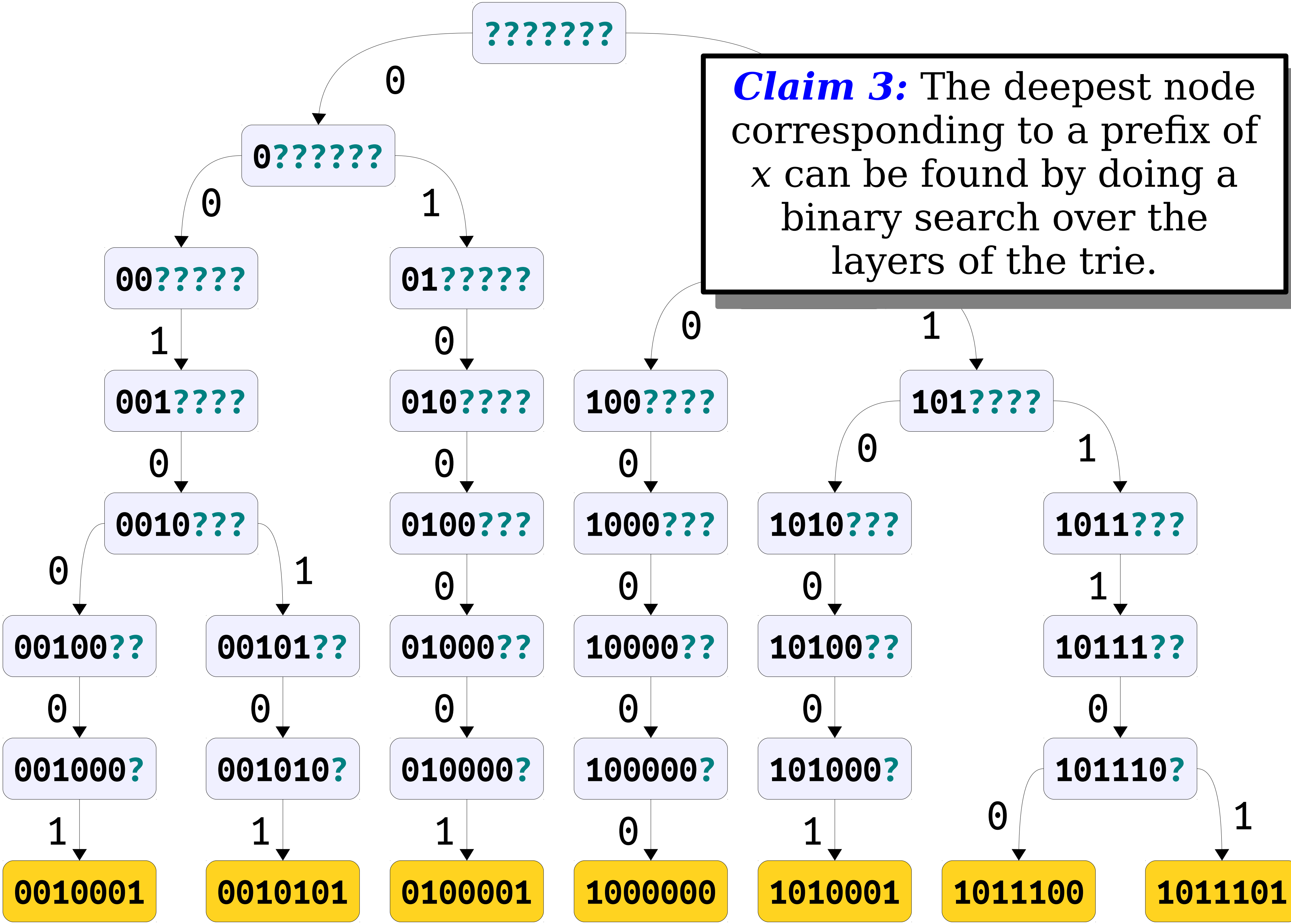
*Claim 3:* The deepest node corresponding to a prefix of $x$ can be found by doing a binary search over the layers of the trie.



# One Speedup

- ***Goal:*** Encode the trie so that we can do a binary search over its layers.
- ***One Solution:*** Store an array of cuckoo hash tables, one per layer of the trie, that holds all the nodes in that layer.
- Can now query, in worst-case time O(1), whether a node's prefix is present on a given layer.
- There are O(log *U*) layers in the trie.
- Binary search will take worst-case time **O(log log *U*)**.
- ***Nice side-effect:*** Queries are now worst-case O(1), since we can just check the hash table at the bottom layer.

# Performing the Binary Search

- This binary search assumes that, given a number $x$ and a length $k$, we can extract the first $k$ bits of $x$ in time O(1).
- Fortunately, we can do this!

$x$ `11011100 10111011 11000100 11010110 11110011 01111011 11110000 10001100`

# Performing the Binary Search

- This binary search assumes that, given a number $x$ and a length $k$, we can extract the first $k$ bits of $x$ in time O(1).
- Fortunately, we can do this!

| | |
|---|---|
| $x$ | 11011100 10111011 11000100 11010110 11110011 01111011 11110000 10001100 |
| *mask* | 11111111 11111111 11111111 11110000 00000000 00000000 00000000 00000000 |
| *prefix* | 11011100 10111011 11000100 11010000 00000000 00000000 00000000 00000000 |

```
uint64_t x      = /* … */;
uint64_t mask   = something magical;
uint64_t prefix = x & mask;
```

# Performing the Binary Search

- This binary search assumes that, given a number *x* and a length *k*, we can extract the first *k* bits of *x* in time O(1).
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| | |
|---|---|
| *x* | **11011100 10111011 11000100 1101**0110 11110011 01111011 11110000 10001100 |
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# Performing the Binary Search

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- Fortunately, we can do this!

| | |
|---|---|
| *x* | 11011100 10111011 11000100 11010110 11110011 01111011 11110000 10001100 |
| *mask* | 00000000 00000000 00000000 00010000 00000000 00000000 00000000 00000000 |
| *prefix* | 11011100 10111011 11000100 11010000 00000000 00000000 00000000 00000000 |

```
uint64_t x      = /* … */;
uint64_t mask   =  (uint64_t(1) << (64 - k));
uint64_t prefix = x & mask;
```

# Performing the Binary Search

- This binary search assumes that, given a number $x$ and a length $k$, we can extract the first $k$ bits of $x$ in time O(1).
- Fortunately, we can do this!

| | |
|---|---|
| $x$ | 11011100 10111011 11000100 11010110 11110011 01111011 11110000 10001100 |
| $mask$ | 11111111 11111111 11111111 11101111 11111111 11111111 11111111 11111111 |
| $prefix$ | 11011100 10111011 11000100 11010000 00000000 00000000 00000000 00000000 |

```
uint64_t x      = /* … */;
uint64_t mask   = ~(uint64_t(1) << (64 - k));
uint64_t prefix = x & mask;
```

# Performing the Binary Search

- This binary search assumes that, given a number *x* and a length *k*, we can extract the first *k* bits of *x* in time O(1).
- Fortunately, we can do this!

| | |
|---|---|
| *x* | 11011100 10111011 11000100 11010110 11110011 01111011 11110000 10001100 |
| *mask* | 11111111 11111111 11111111 11110000 00000000 00000000 00000000 00000000 |
| *prefix* | 11011100 10111011 11000100 11010000 00000000 00000000 00000000 00000000 |

```
-x = ~x + 1
```

Thanks, CS107!

```
uint64_t x      = /* … */;
uint64_t mask   = ~(uint64_t(1) << (64 - k)) + 1;
uint64_t prefix = x & mask;
```

# Performing the Binary Search

- This binary search assumes that, given a number $x$ and a length $k$, we can extract the first $k$ bits of $x$ in time O(1).
- Fortunately, we can do this!

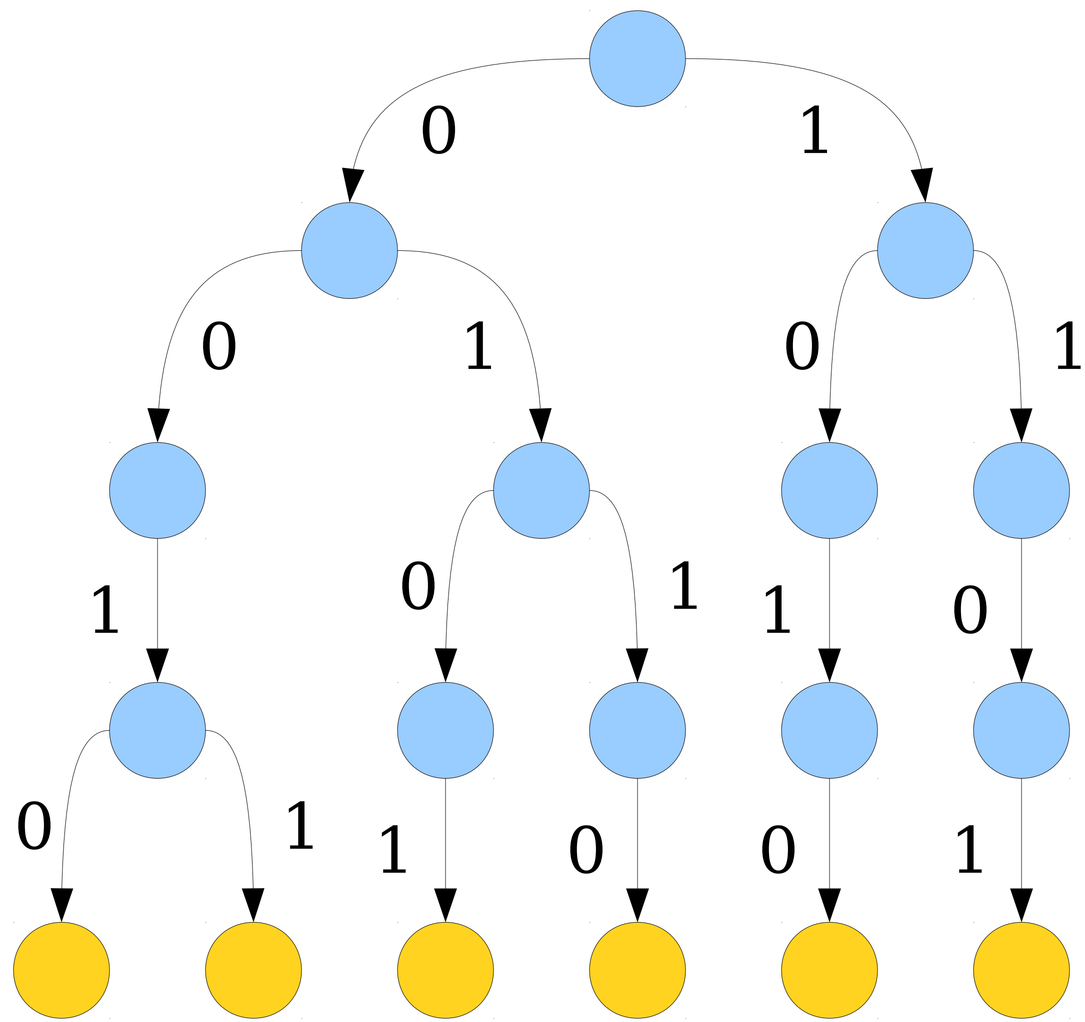
| | |
|---|---|
| $x$ | 11011100 10111011 11000100 11010110 11110011 01111011 11110000 10001100 |
| *mask* | 11111111 11111111 11111111 11110000 00000000 00000000 00000000 00000000 |
| *prefix* | 11011100 10111011 11000100 11010 |

There's an edge case to handle here for $k = 0$, but that's easily special-cased. Let me know if there's a way to avoid this!

```
uint64_t x      = /* … */;
uint64_t mask   = -(uint64_t(1) << (64 - k));
uint64_t prefix = x & mask;
```

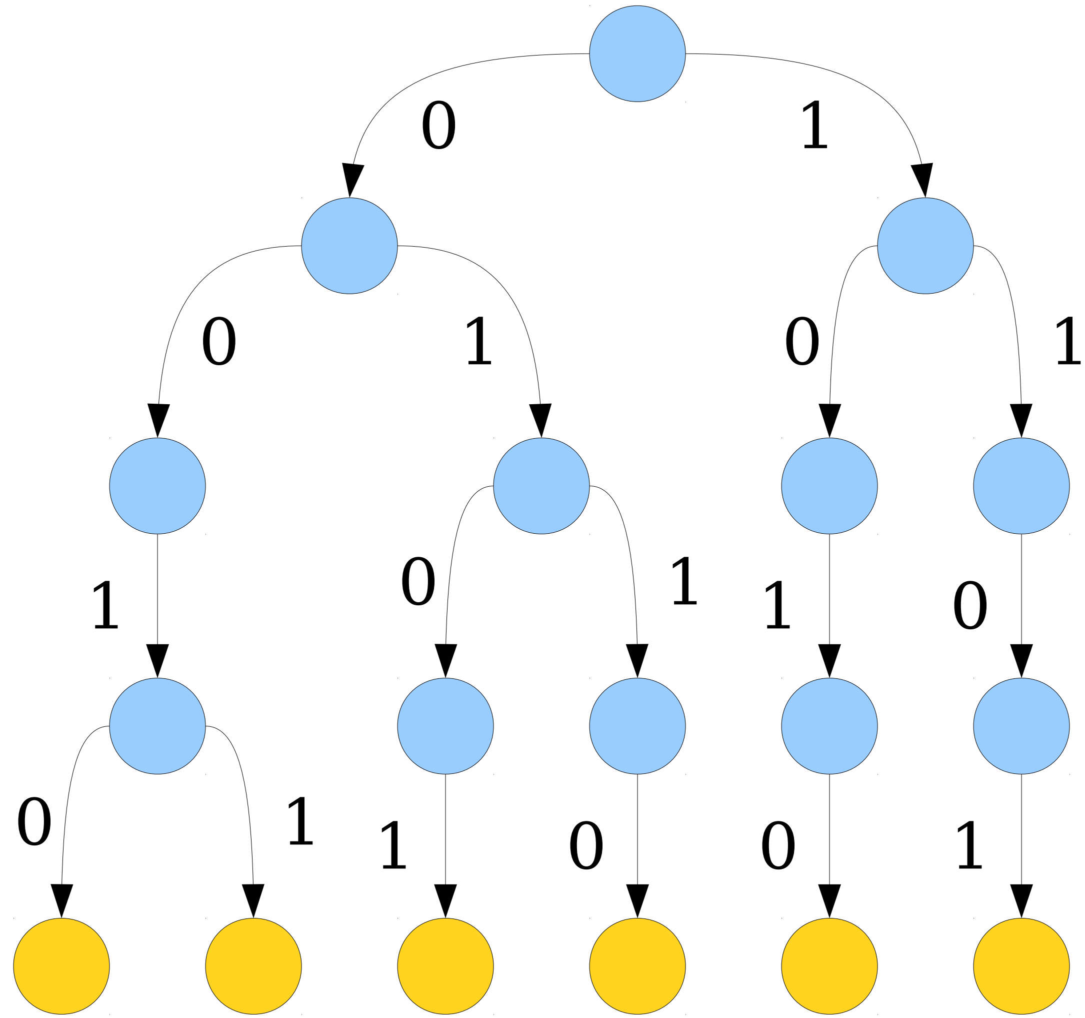
# Finding Successors

- We can now find the node where the successor search would initially arrive.
- At this point, we'd normally back up until we find a branching node where we can follow a 1 child pointer, then descend from there to the leaves.
- This will take time O(log $U$).
- ***Can we do better?***



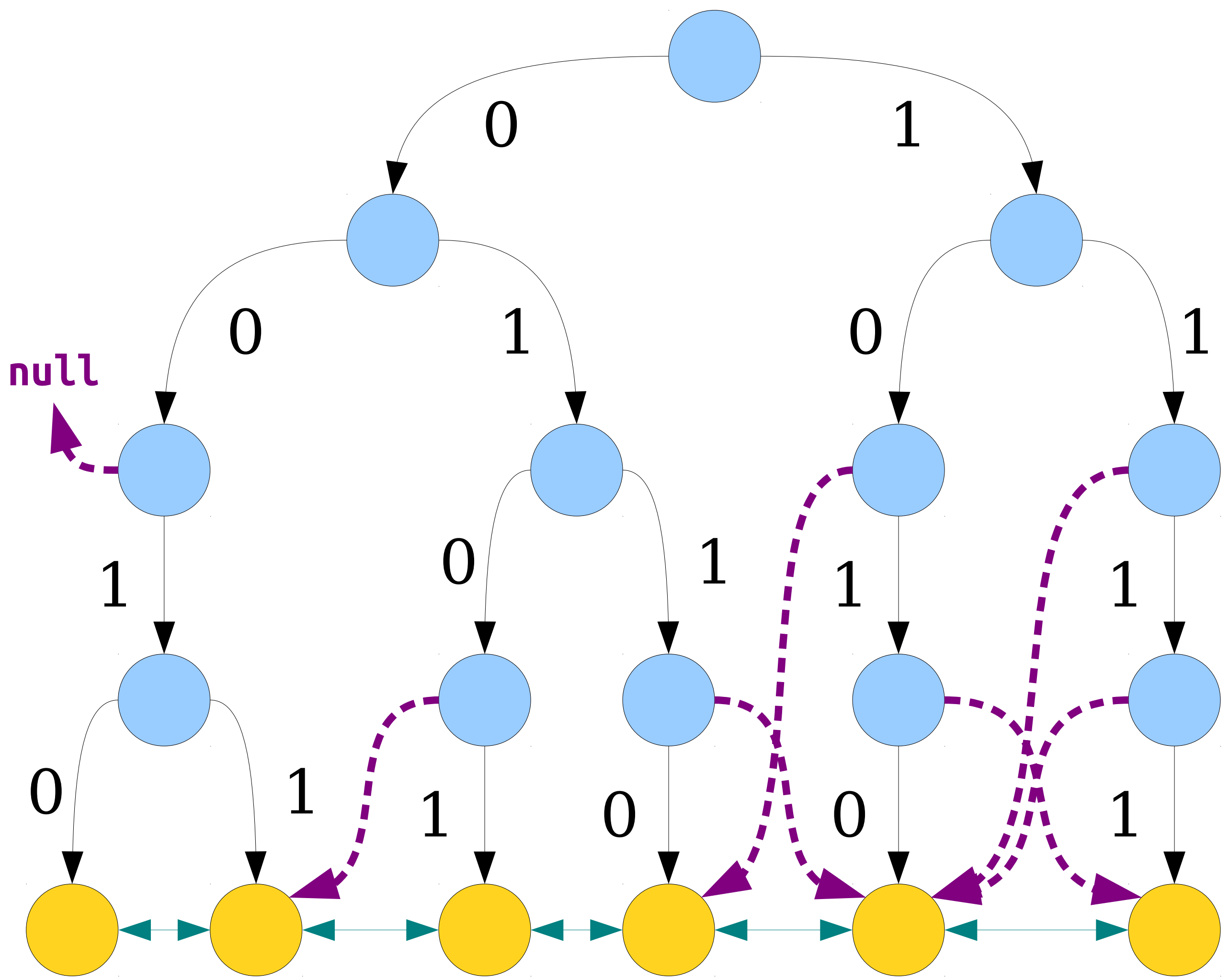
# Finding Successors

- ***Claim:*** If the binary search terminates at a node *v*, that node must have at most one child.
- If it doesn't, it has both a 0 child and a 1 child, so there's a longer prefix that can be matched.
- ***Idea:*** Steal the missing pointers and use them to speed up successor and predecessor searches.



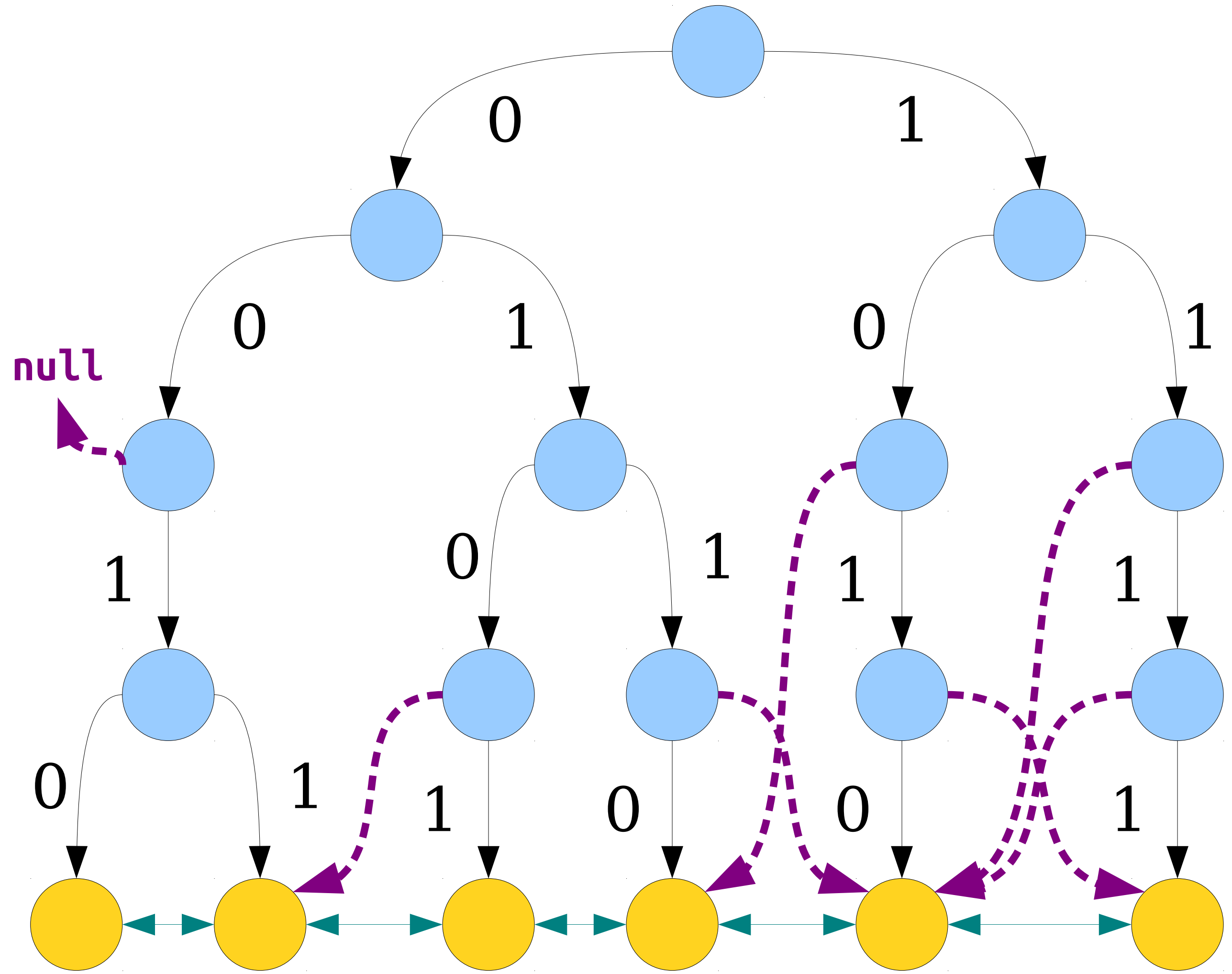
# Threaded Binary Tries

- A ***threaded binary trie*** is a binary tree where
  - each missing 0 pointer points to the inorder predecessor of the node and
  - each missing 1 points to the inorder successor of the node.
- Notice that the leaves end up in a doubly-linked list.



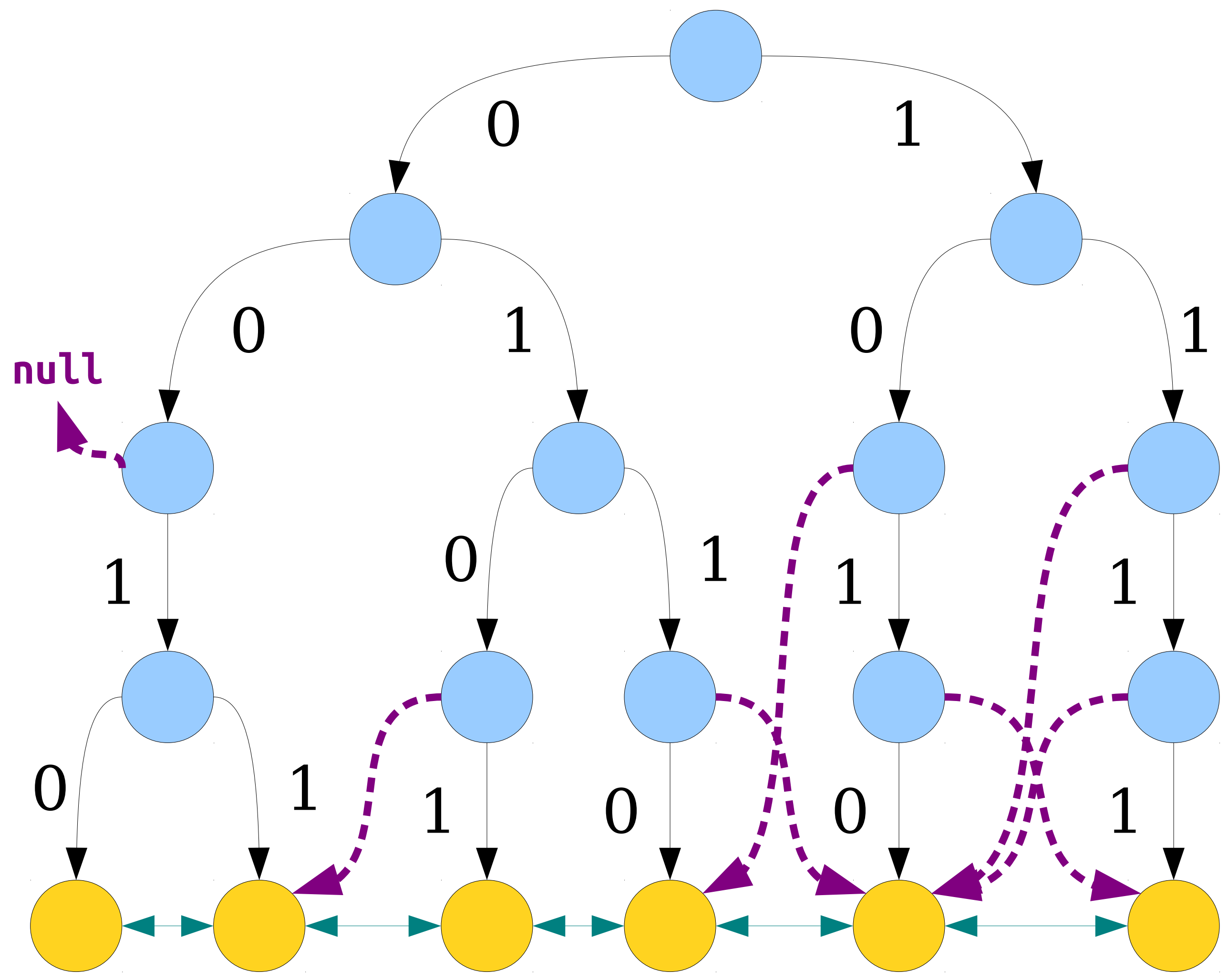
# x-Fast Tries

- An ***x-Fast Trie*** is a threaded binary trie with a cuckoo hash table at each level that stores the nodes at that level.
- Can do lookups in time O(1).



# *x*-Fast Tries

- ***Claim:*** Can determine ***successor***(*x*) in time O(log log *U*).
- Start by binary searching for the longest prefix of *x*.
- If that node has a missing 1 pointer, it points directly to the successor.
- Otherwise, it has a missing 0 pointer.
- If that pointer is null, return the minimum value (we can cache this.)
- Otherwise, follow it to a leaf, then follow the leaf's 1 pointer.

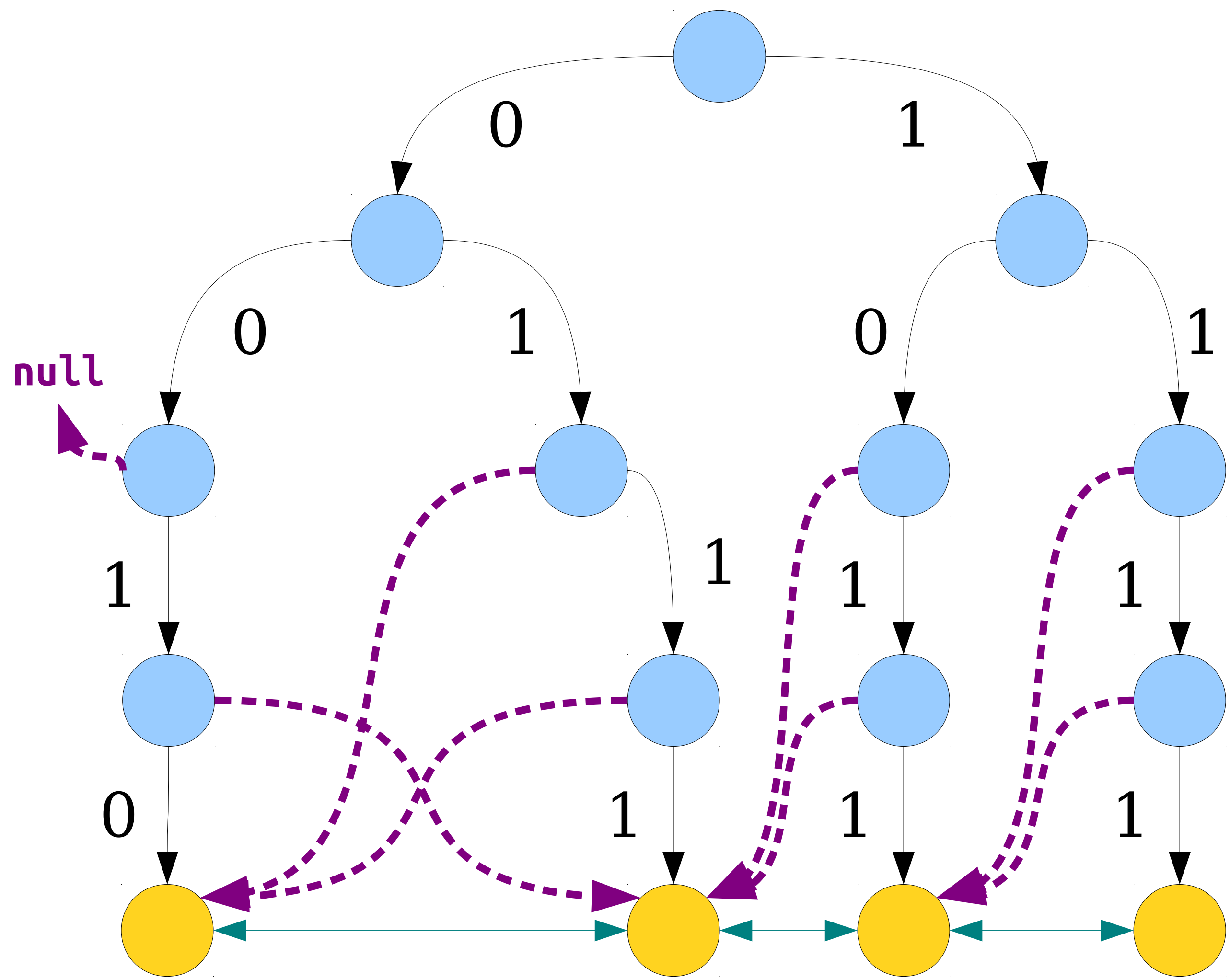


# *x*-Fast Trie Maintenance

- Based on what we've seen:
  - ***lookup*** takes worst-case time O(1).
  - ***successor*** and ***predecessor*** queries take worst-case time O(log log $U$).
  - ***min*** and ***max*** can be done in time O(1), assuming we cache those values.
- How efficiently can we support ***insert*** and ***delete***?

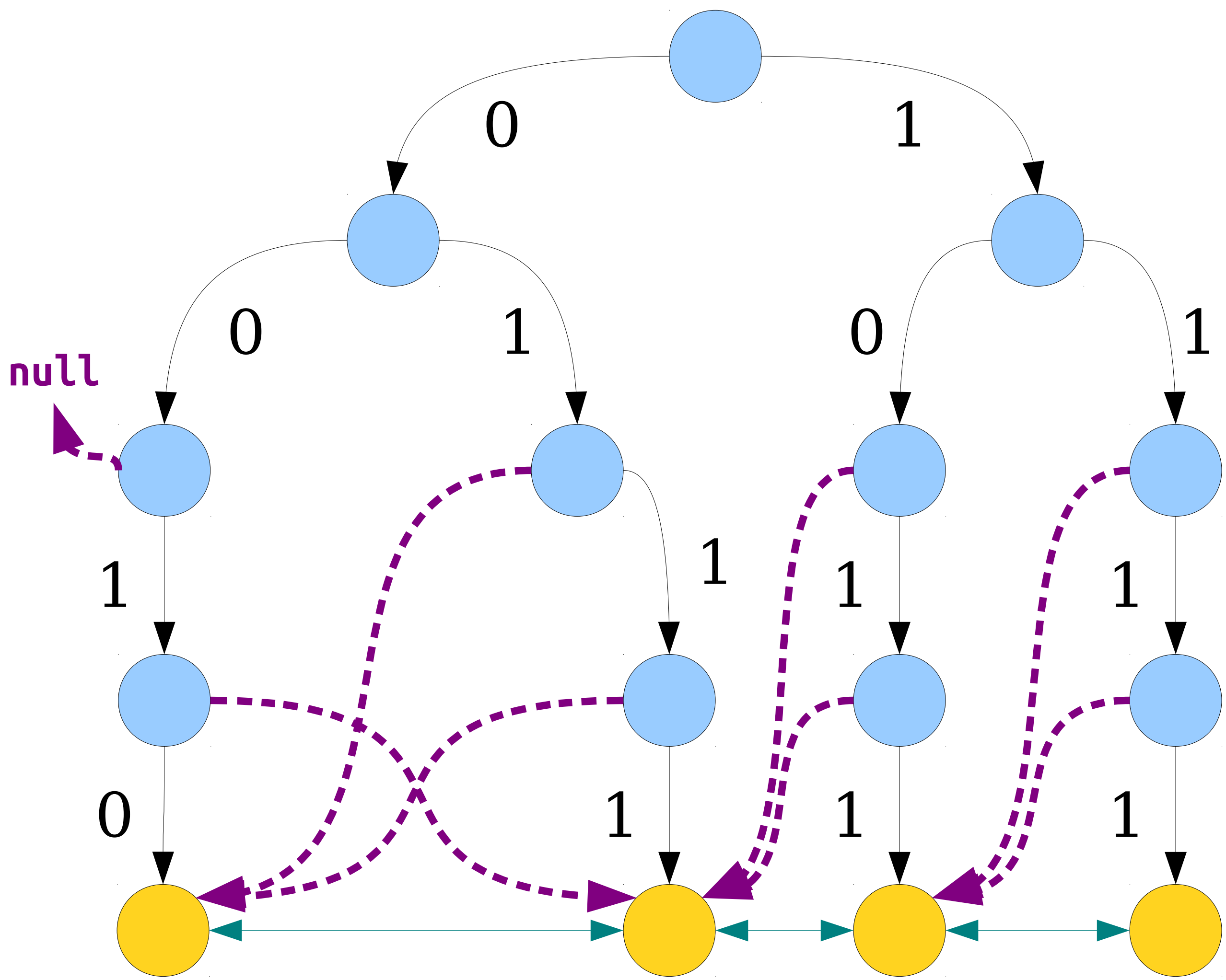
# $x$-Fast Tries

- If we ***insert***($x$), we need to
  - add some new nodes to the trie;
  - wire $x$ into the doubly-linked list of leaves; and
  - update the thread pointers to include $x$.
- Worst-case will be **Ω(log *U*)** due to the first and third steps.



# *x*-Fast Tries

- Here is an (amortized, expected) O(log *U*) time algorithm for ***insert***(*x*):
  - Find ***successor***(*x*).
  - Add *x* to the trie.
  - Using the successor from before, wire *x* into the linked list.
  - Walk up from *x*, its successor, and its predecessor and update threads.



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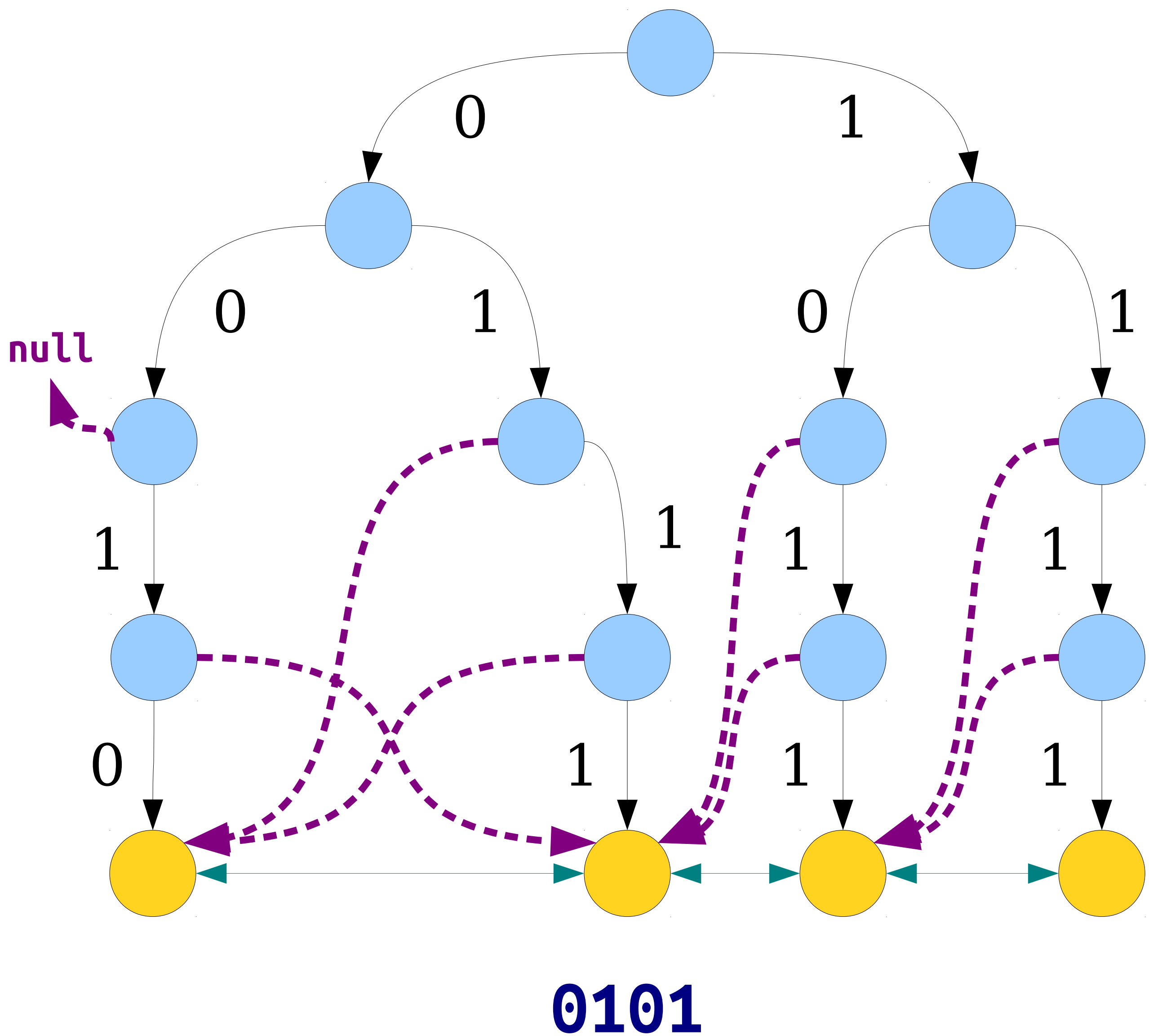

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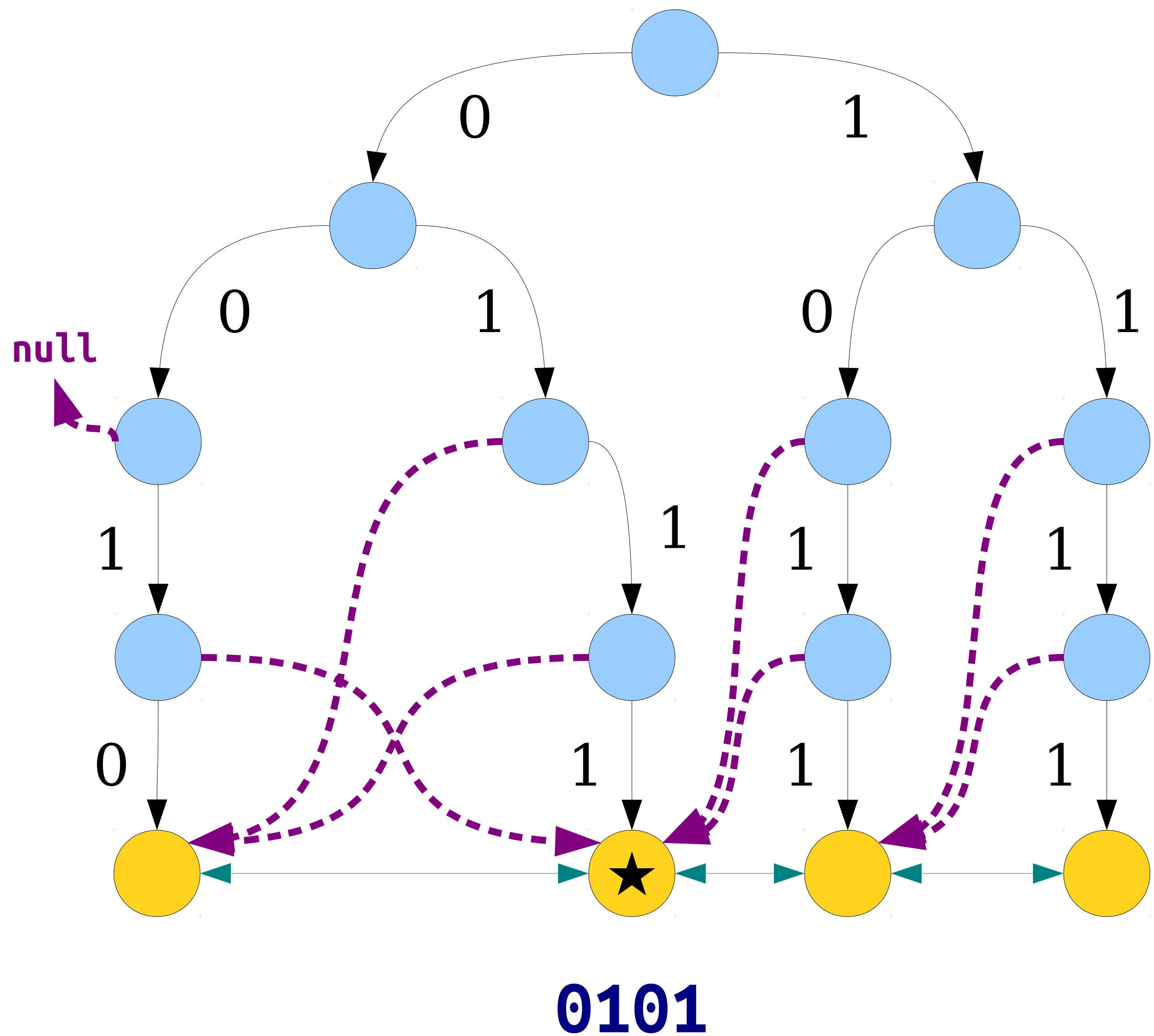

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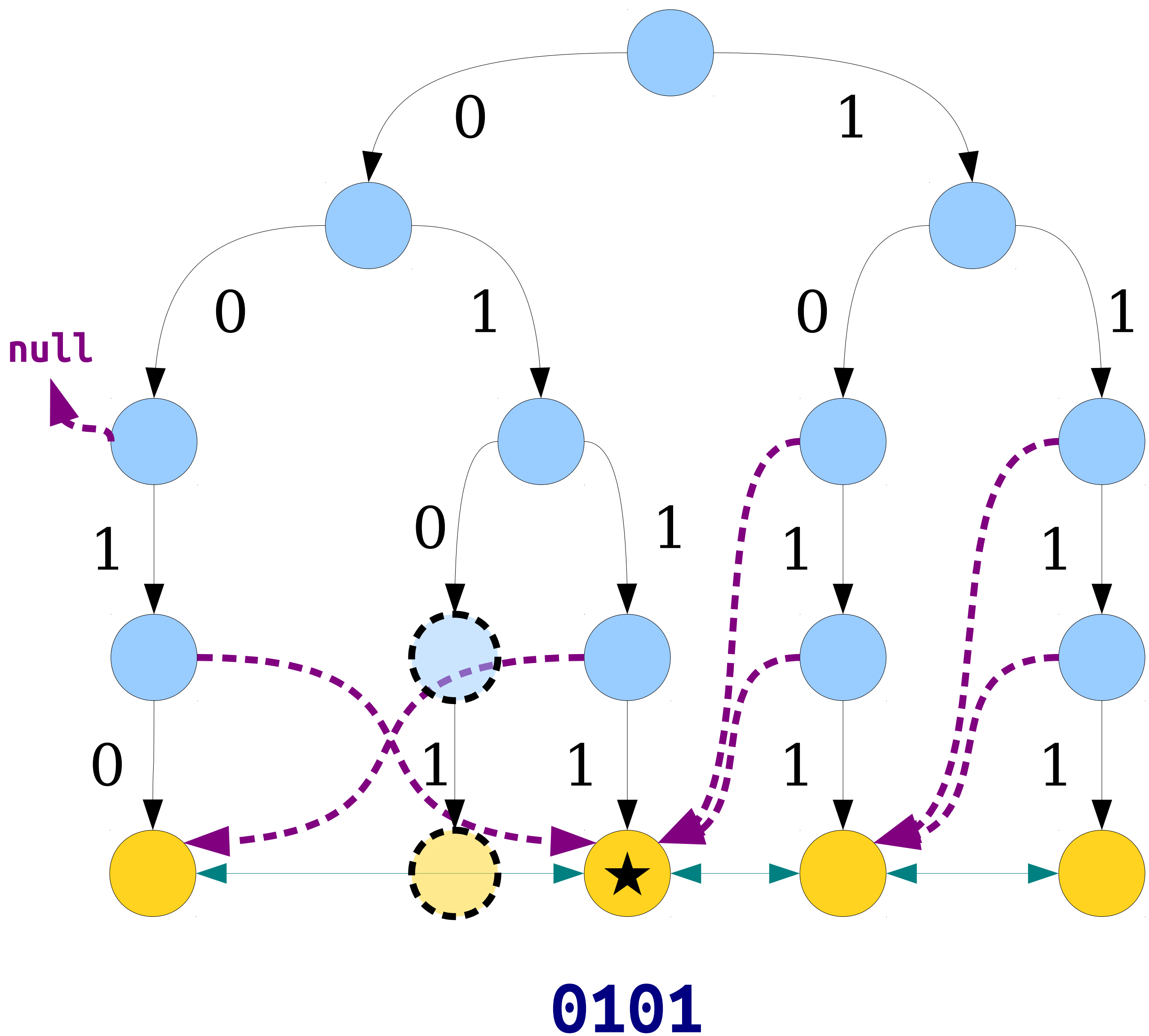

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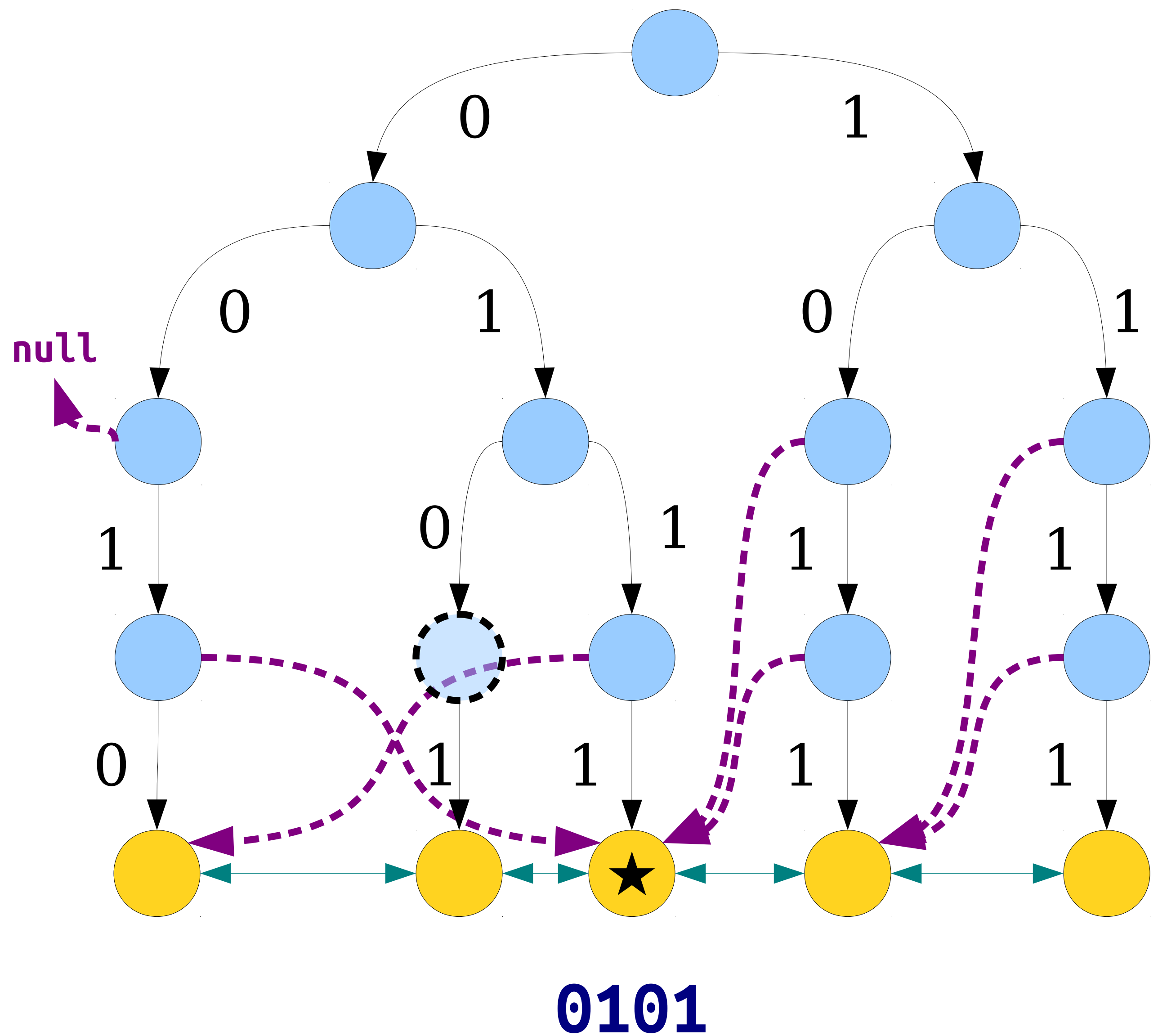

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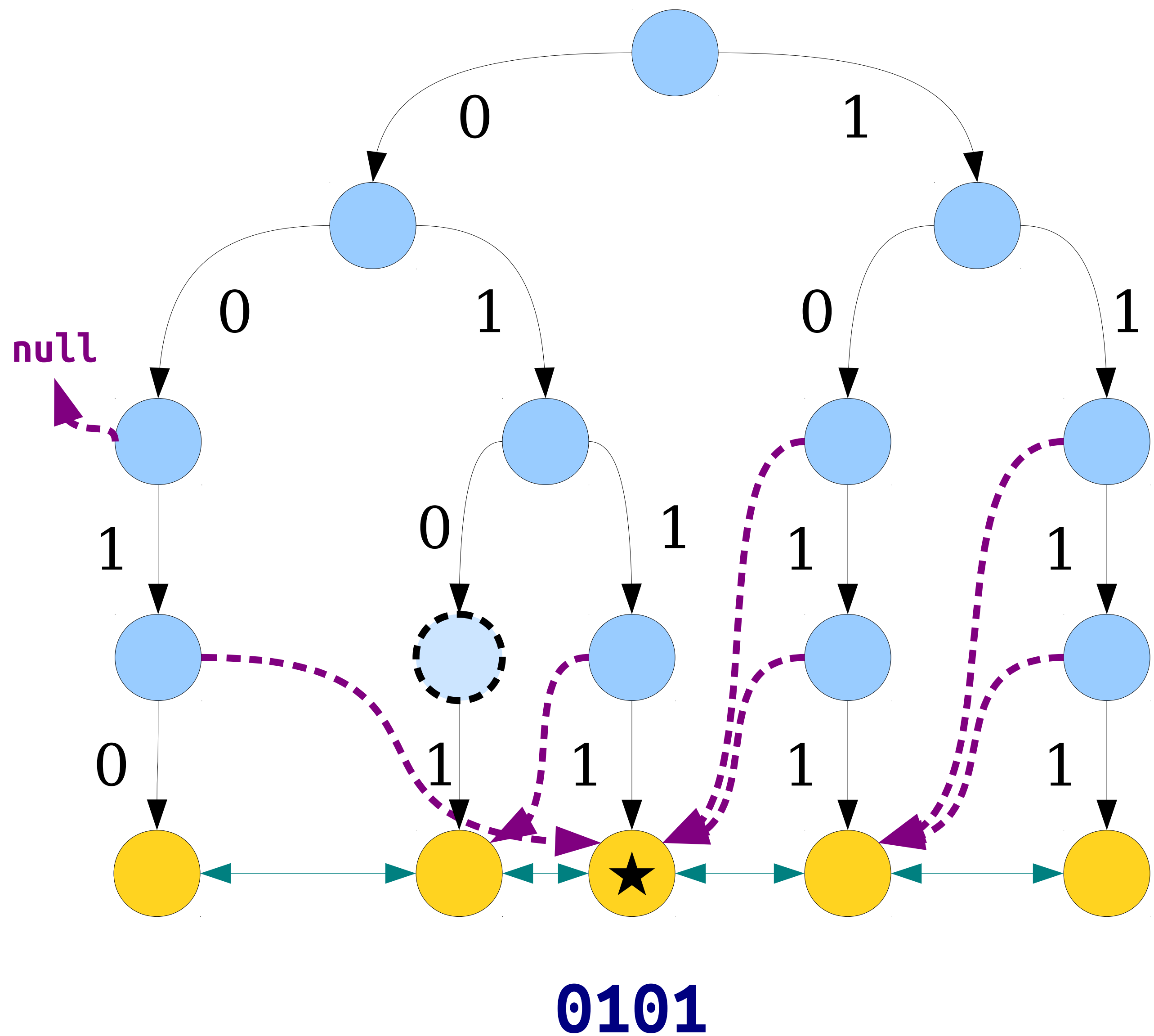

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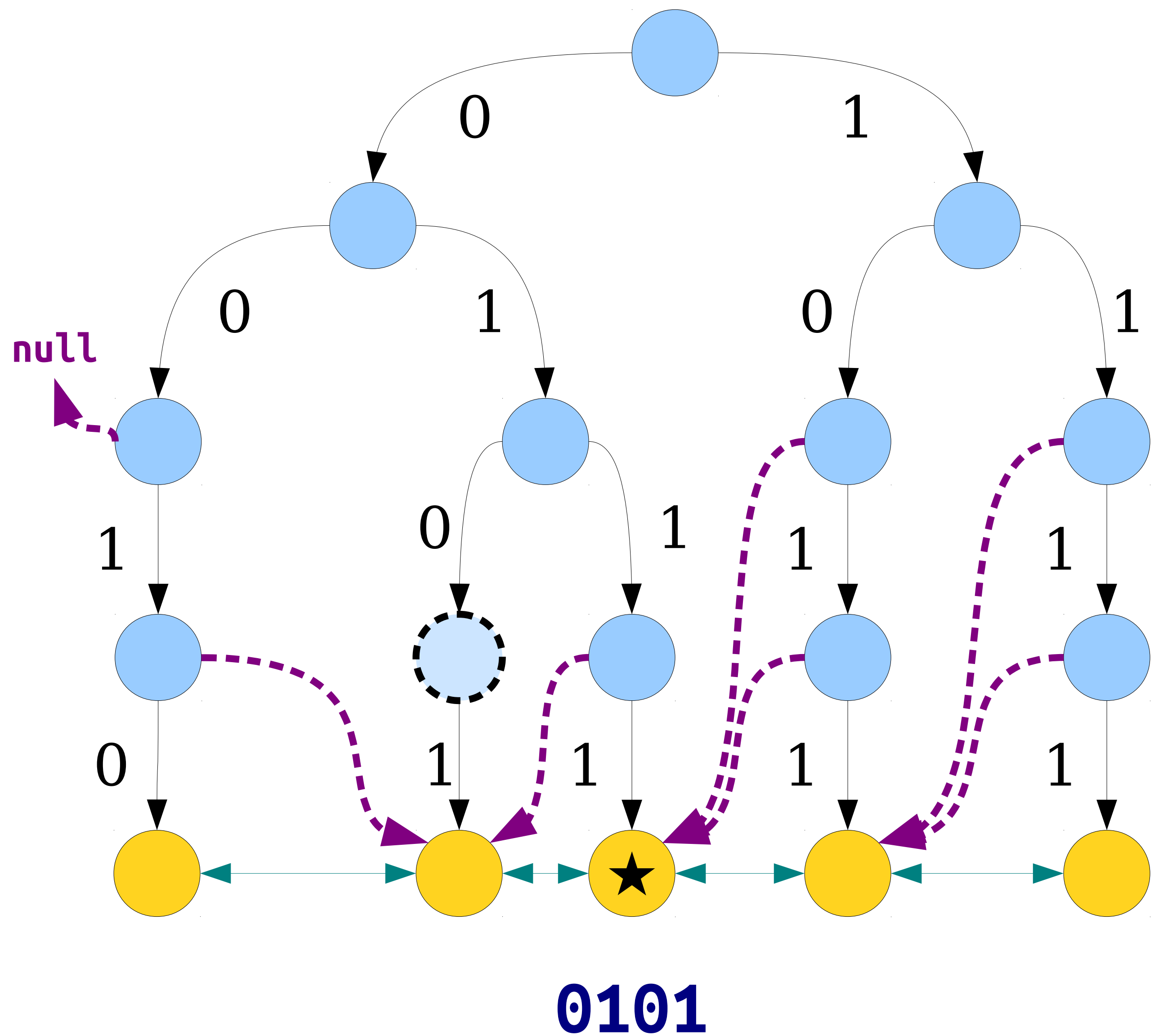

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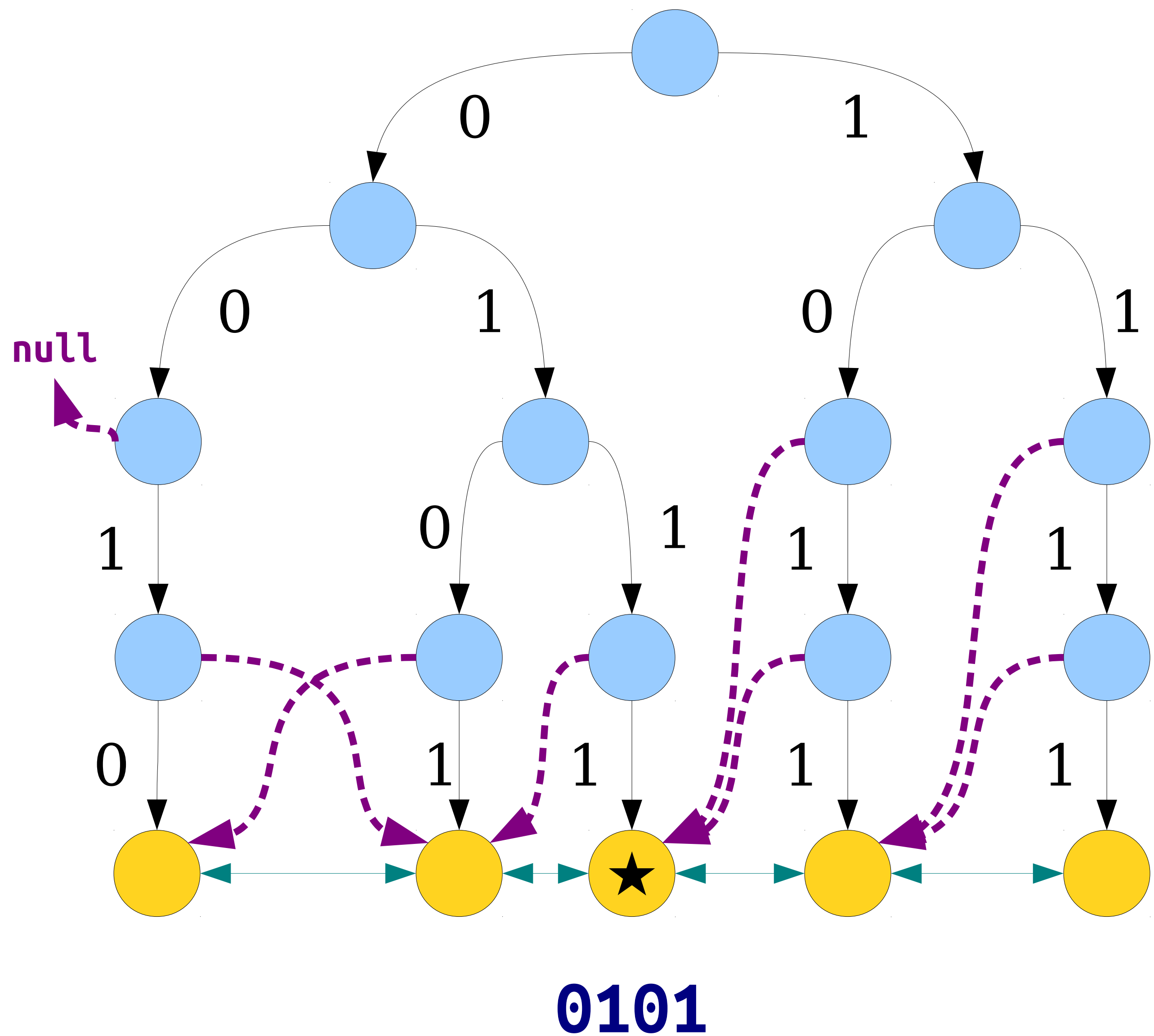
  - Using the successor from before, wire $x$ into the linked list.
  - Walk up from $x$, its successor, and its predecessor and update threads.



# Deletion

- To ***delete***($x$), we need to
  - Remove $x$ from the trie.
  - Splice $x$ out of its linked list.
  - Update thread pointers from $x$'s former predecessor and successor.
- Runs in expected, amortized time **O(log *U*)**.
- Full details are left as a proverbial Exercise to the Reader. ☺

# Space Usage

- How much space is required in an $x$-fast trie?
- Each leaf node contributes at most O(log $U$) nodes in the trie.
- Total space usage for hash tables is proportional to total number of trie nodes.
- Total space: **O($n$ log $U$)**.

# Where We Stand

- Right now, we have a reasonably fast data structure for storing a sorted set of integers.
- If we have a *static* set of integers that we want to make lots of queries on, this is pretty good as-is!
- As you'll see, though, we can make this even better with some kitchen sink techniques. 😃

$x$-Fast Trie:

- ***lookup***: O(1)
- ***insert***: O(log $U$)*
- ***delete***: O(log $U$)*
- ***max***: O(1)
- ***succ***: O(log log $U$)
- ***is-empty***: O(1)
- Space: O($n$ log $U$)

* Expected, amortized

Time-Out for Announcements!

# Midterm Logistics

- Our midterm will be held next Tuesday from 7:00PM – 10:00PM in ***Hewlett 200***.
- Exam is closed-book, closed-computer, and limited-note. You can bring a double-sided 8.5” × 11” sheet of notes with you to the exam.
- Topic coverage is material from PS1 – PS5. Topics from this week won’t be tested, but are an excellent review of the concepts.
- We've released a set of practice problems to help you prepare for the exam. They're up on the course website.
- Can't make the exam time? Have OAE accommodations? Let us know ASAP so that we can set up an alternate time.

# Final Project Presentations

- Final project presentations will run from ***Monday, June 4*** to ***Thursday, June 7***.
- Use this link to sign up for a time slot:

**http://www.slottr.com/cs166-2018**

- You can view the available time slots starting today. The form will be open from ***noon on Thursday, May 24*** until noon on Thursday, May 31. It's first-come, first-served.
- Presentations will be 15-20 minutes, plus five minutes for questions. Please arrive five minutes early to get set up.
- Presentations are open to the public, so feel free to stop by any of the presentations you're interested in.

Back to CS166!

# *y-Fast Tries*

# Where We Stand

- Right now, we have a reasonably fast data structure for storing a sorted set of integers.
- To make this really shine, we need to improve the highlighted costs.

$x$-Fast Trie:

- ***lookup***: O(1)
- ***insert***: **O(log *U*)***
- ***delete***: **O(log *U*)***
- ***max***: O(1)
- ***succ***: O(log log $U$)
- ***is-empty***: O(1)
- Space: **Θ(*n* log *U*)**

* Expected, amortized

# Shaving Off Logs

- We're essentially at a spot where we need to shave off a log factor from a couple of operations.
- ***Question:*** What techniques have we developed so far to do this?

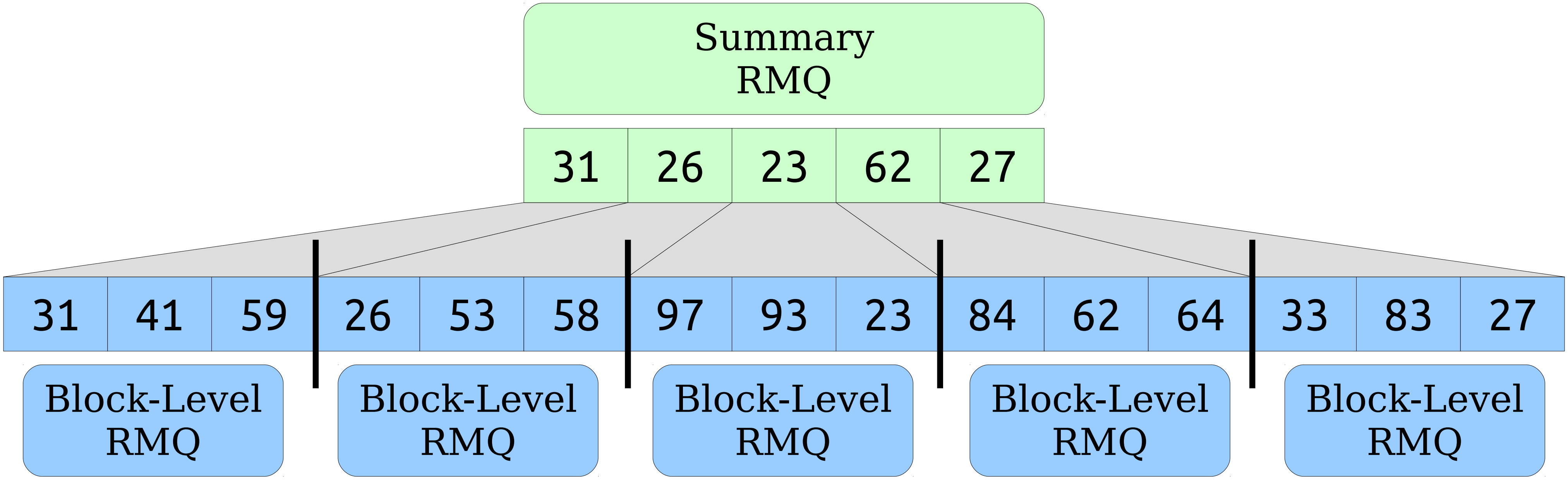
$x$-Fast Trie:

- ***lookup***: O(1)
- ***insert***: **O(log *U*)***
- ***delete***: **O(log *U*)***
- ***max***: O(1)
- ***succ***: O(log log *U*)
- ***is-empty***: O(1)
- Space: **Θ(*n* log *U*)**

* Expected, amortized

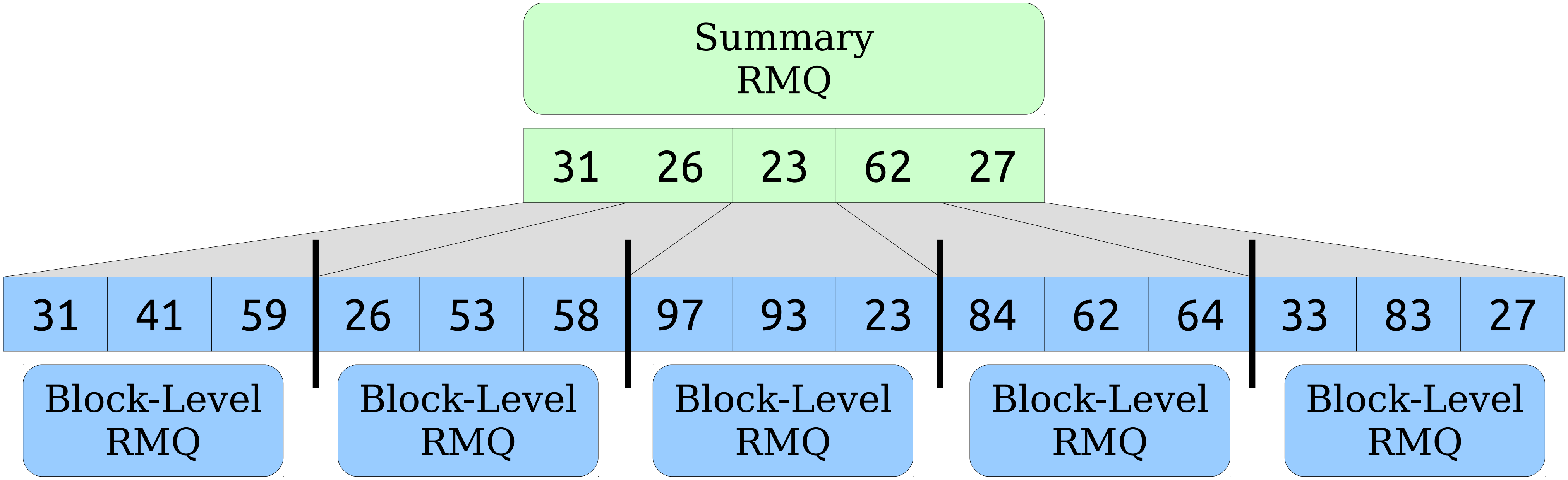
# Two-Level Structures

- Think back to the hybrid approach we used for solving RMQ.
- It consisted of a two-tiered structure:
  - A bunch of small, lower-level structures that each solve the problem in small cases.
  - A single, larger, top-level structure that helps aggregate those solutions together.



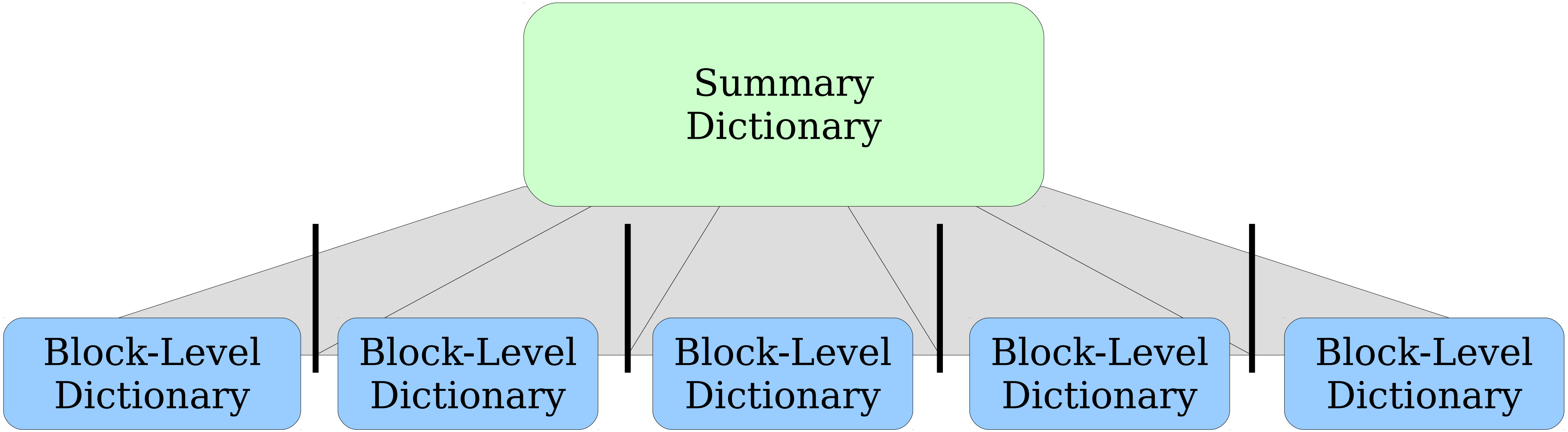
# Two-Level Structures

- One of the fastest RMQ hybrids in practice is the ⟨O($n$), O(log $n$)⟩ hybrid structure built with blocks of size Θ(log $n$) where
  - the summary structure is a ⟨O($n$ log $n$), O(1)⟩ sparse table, and
  - the block-level structures are ⟨O(1), O($n$)⟩ no-preprocessing RMQ structures.
- By breaking the input apart into blocks of size Θ(log $n$)
  - the summary structure only takes time O($n$) to build, and
  - the linear terms in the blocks become O(log $n$) terms.



# The Idea

- Build a two-level ordered dictionary out of existing ordered dictionaries.
- Split the keys apart into logarithmic-sized blocks.
- Build ordered dictionaries for each of the block-level dictionaries.
- Build a summary dictionary to aggregate the blocks together.



# The *y*-Fast Trie

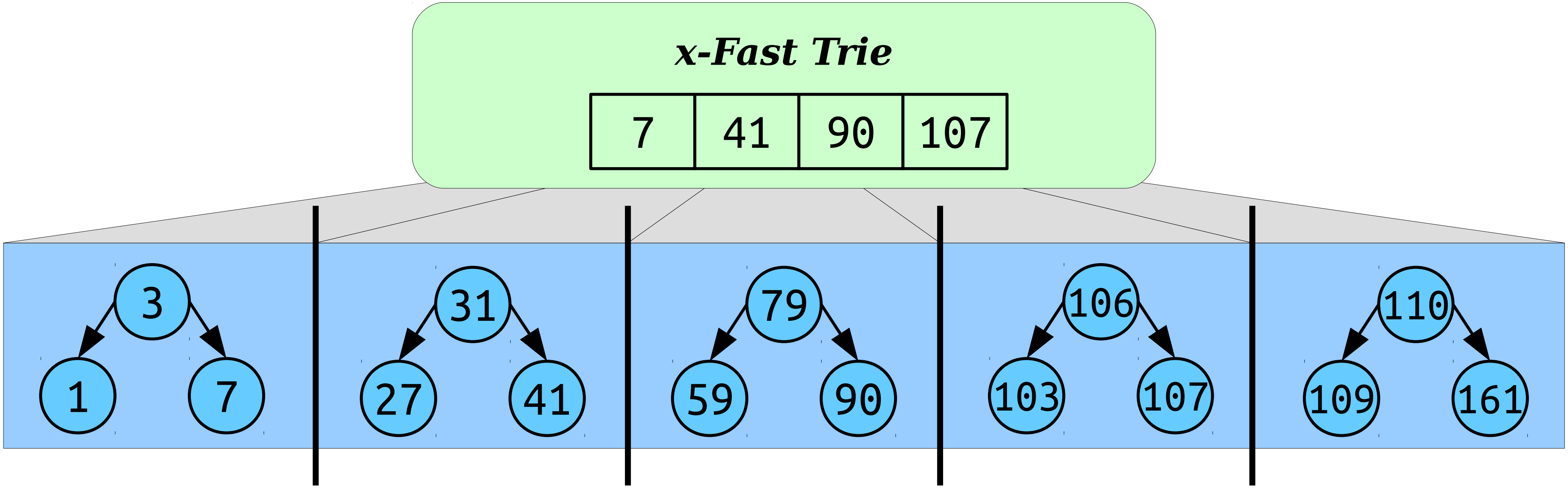
# The Setup

- For now, assume all keys are given to us in advance, in sorted order.
- Split the keys apart into blocks of size $\Theta(\log U)$ and store them in balanced BSTs.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 7 | 27 | 31 | 41 | 59 | 79 | 90 | 103 | 106 | 107 | 109 | 110 | 161 |

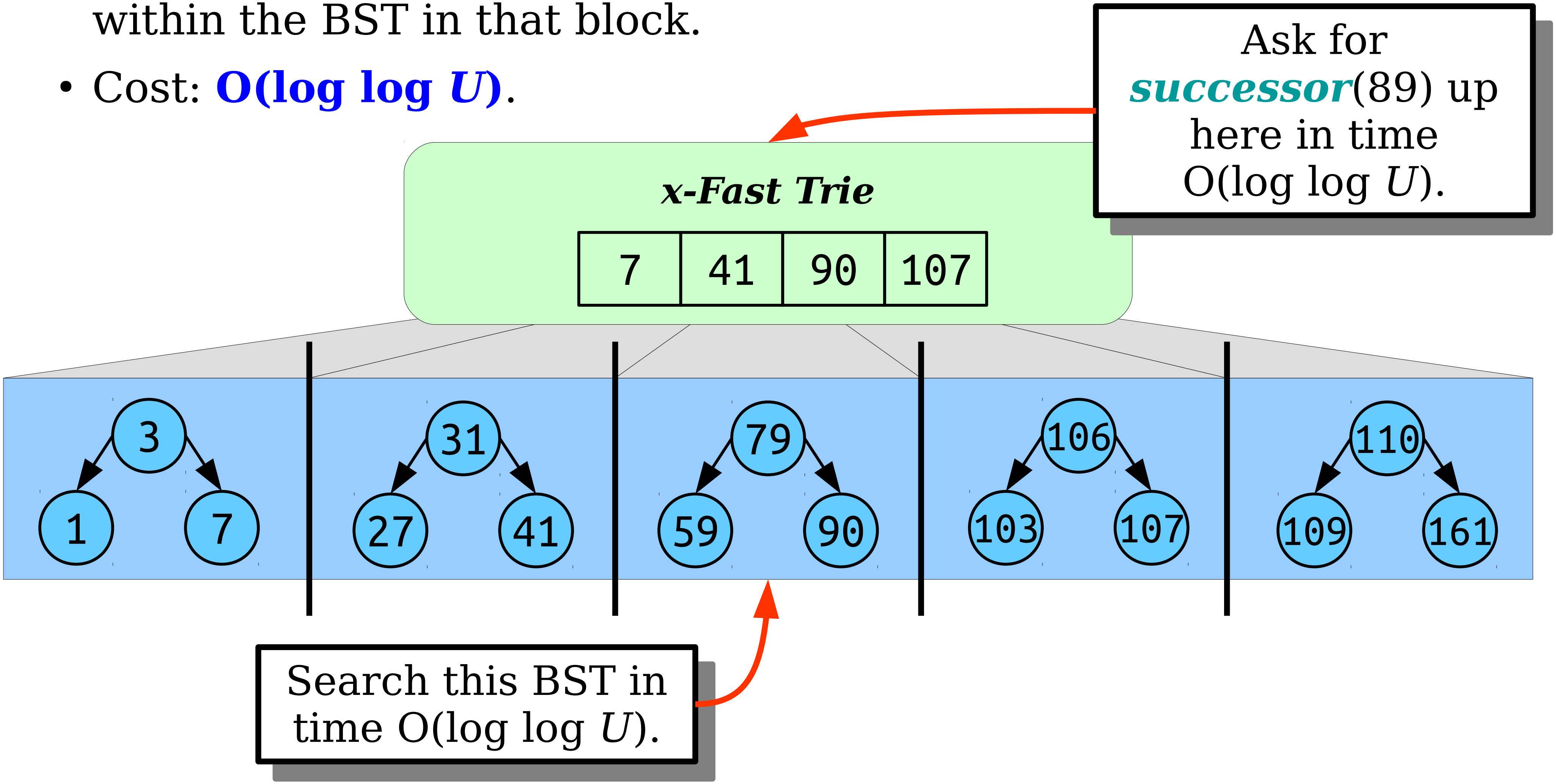
# The Setup

- For now, assume all keys are given to us in advance, in sorted order.
- Split the keys apart into blocks of size $\Theta(\log U)$ and store them in balanced BSTs.
- Create a summary *x*-fast trie that stores the maximum key from each block but the last.



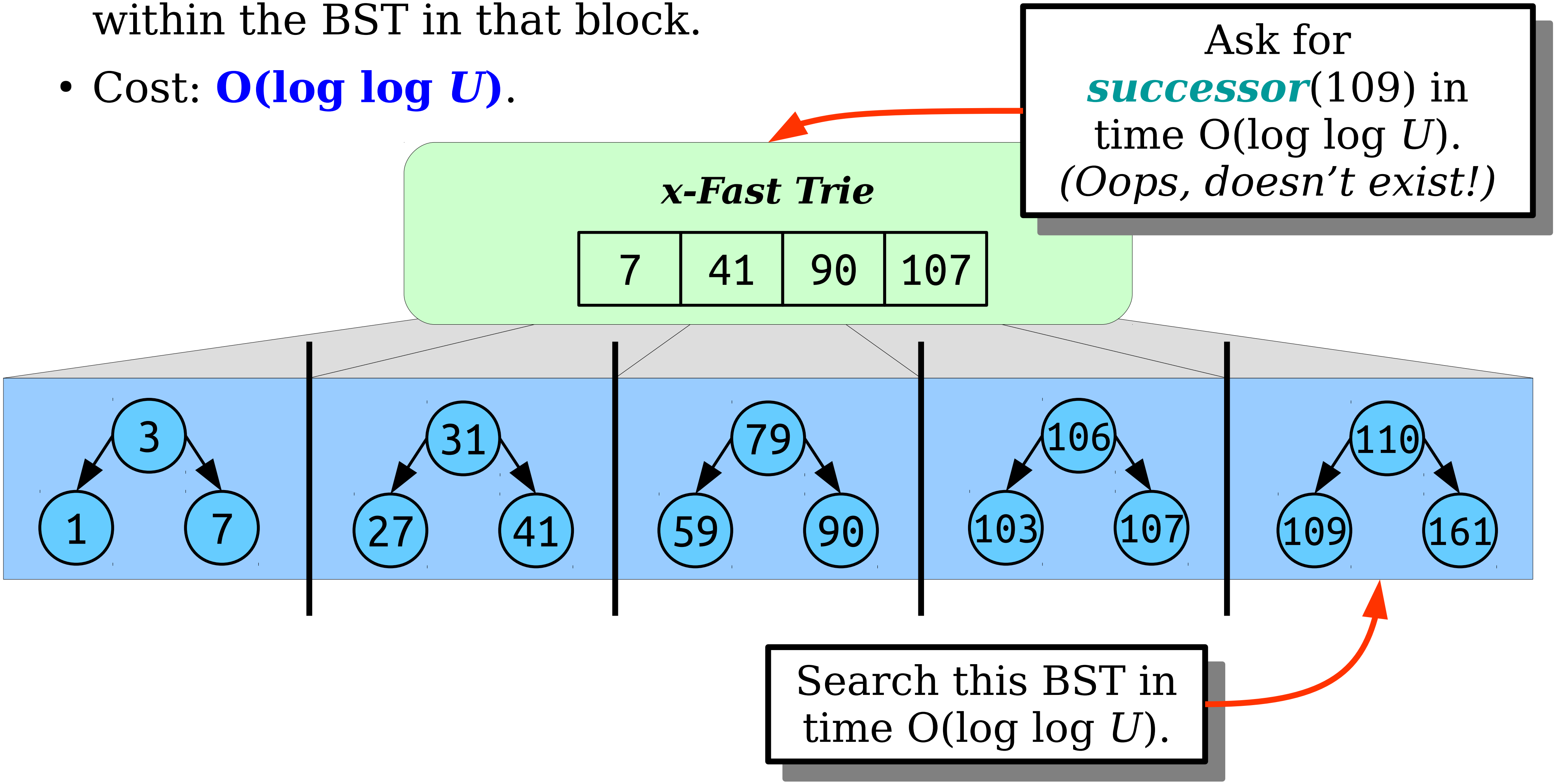
# Performing a Lookup

- Suppose we want to perform ***lookup***(90).
- ***Idea:*** figure out which block 90 would belong to, then search within the BST in that block.
- Cost: **O(log log *U*)**.



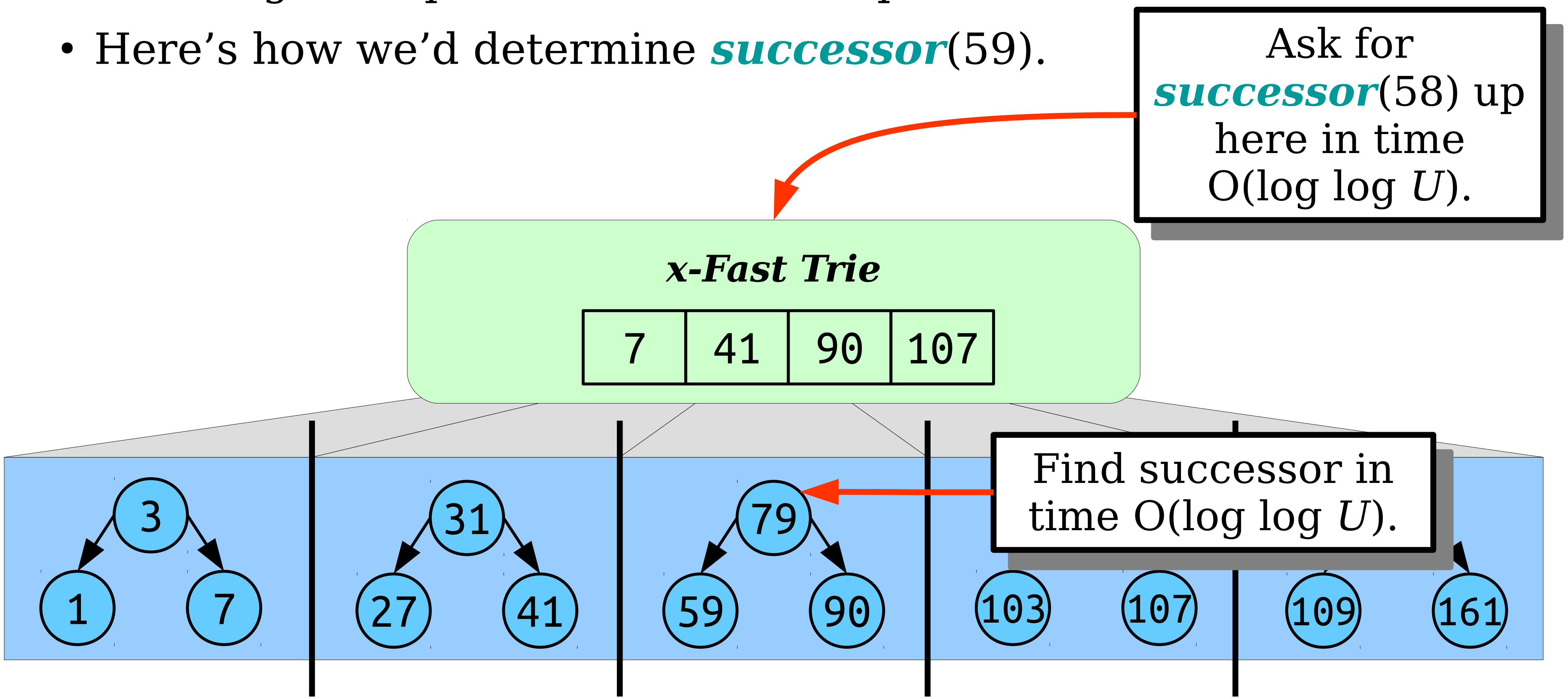
# Performing a Lookup

- Suppose we want to perform ***lookup***(110).
- ***Idea:*** figure out which block 109 would belong to, then search within the BST in that block.
- Cost: **O(log log *U*)**.



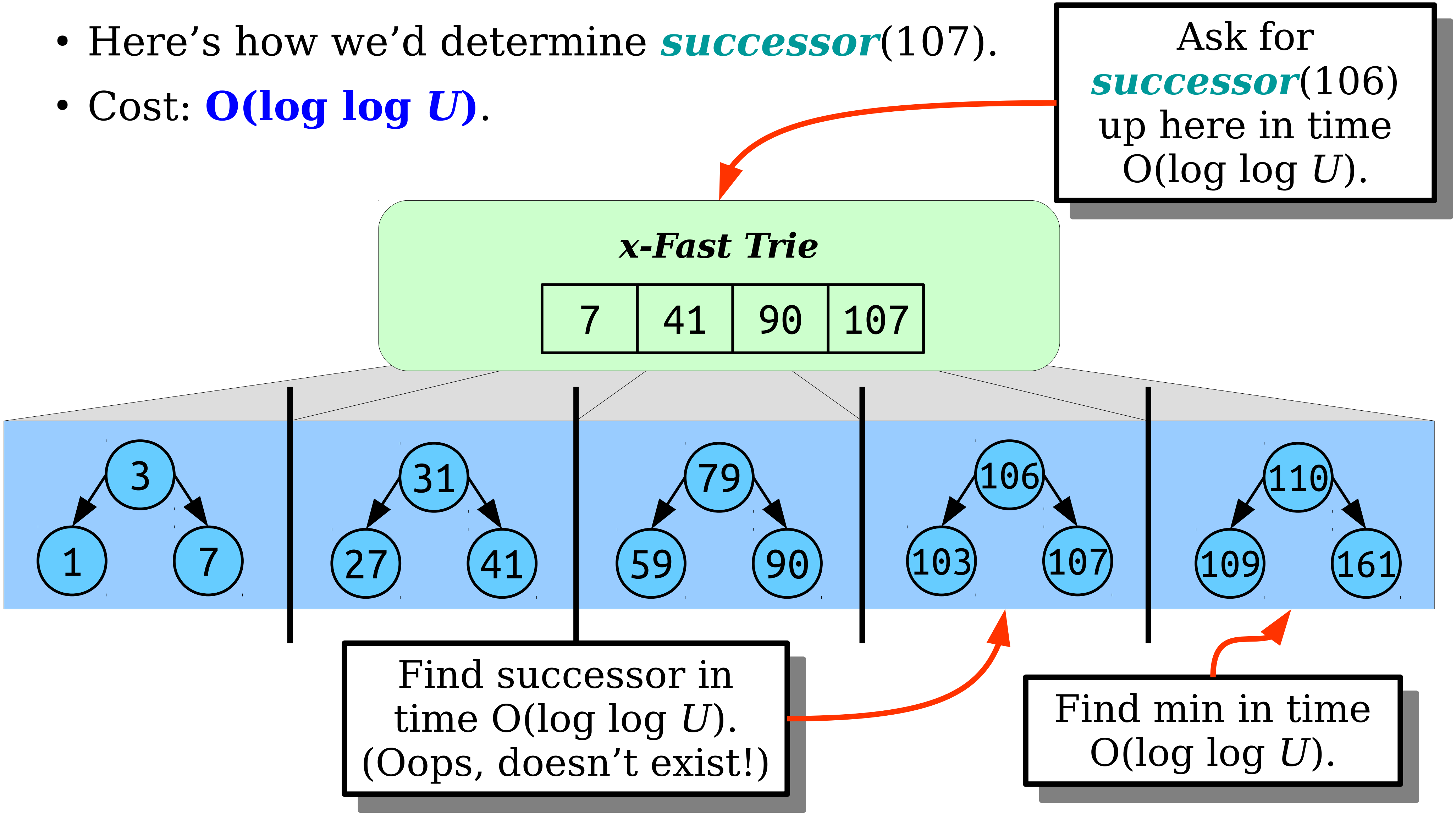
# Successor Queries

- How might we perform ***successor*** queries?
- Here's how we'd determine ***successor***(59).



# Successor Queries

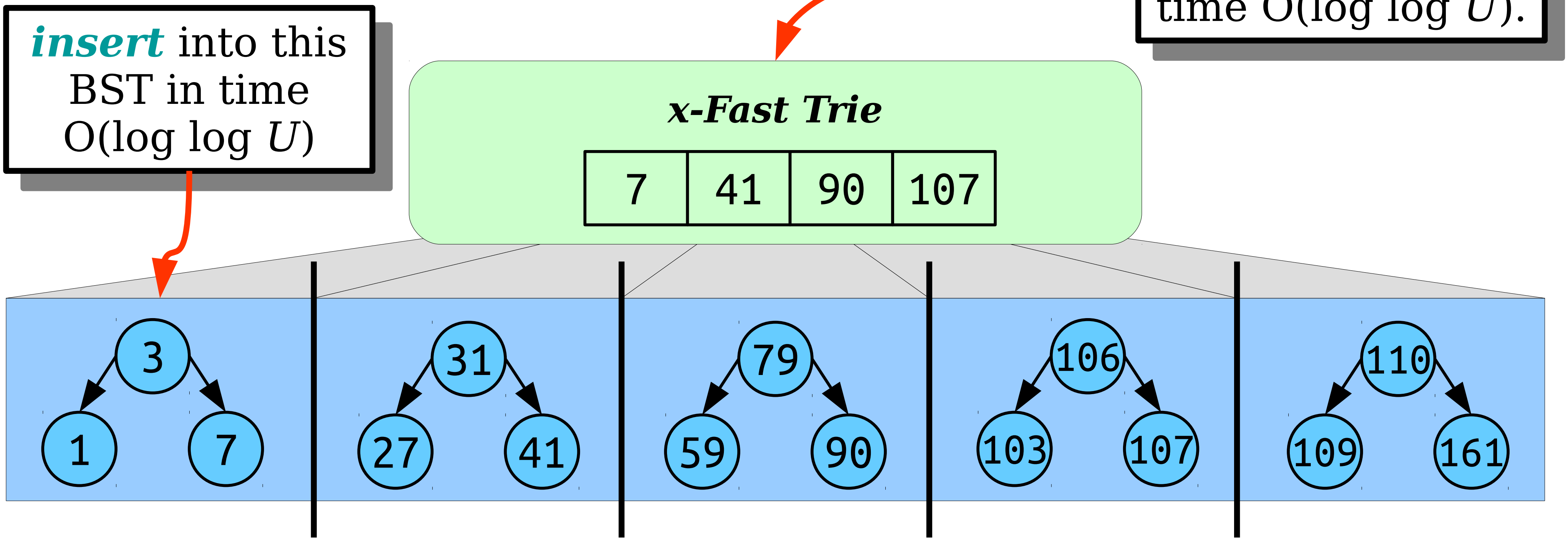
- How might we perform ***successor*** queries?
- Here's how we'd determine ***successor***(107).
- Cost: **O(log log *U*)**.



# Making Edits

- With a major caveat, insertions follow the same procedure as before.
- Here's how we'd ***insert***(6)

Ask for ***successor***(5) in time O(log log *U*).



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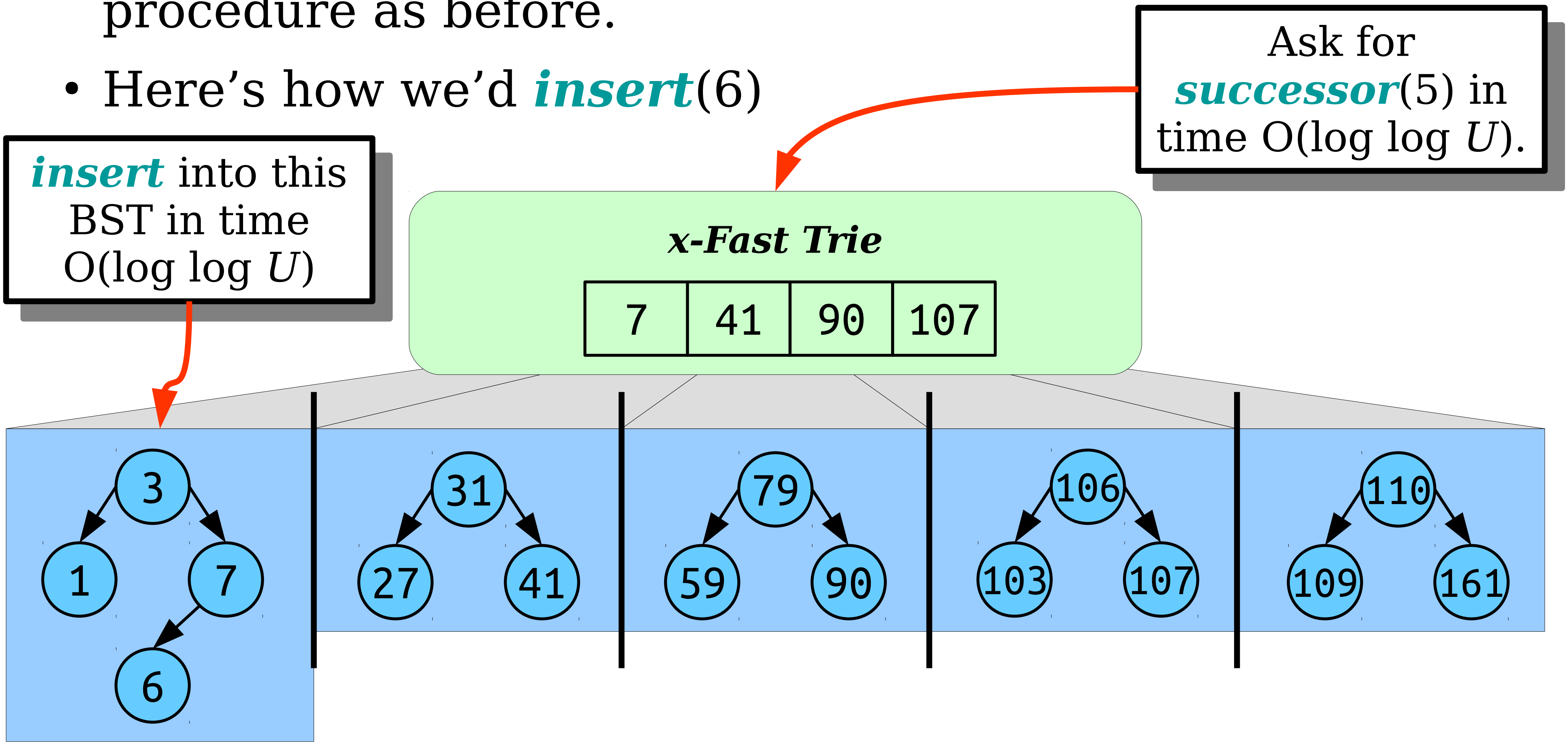
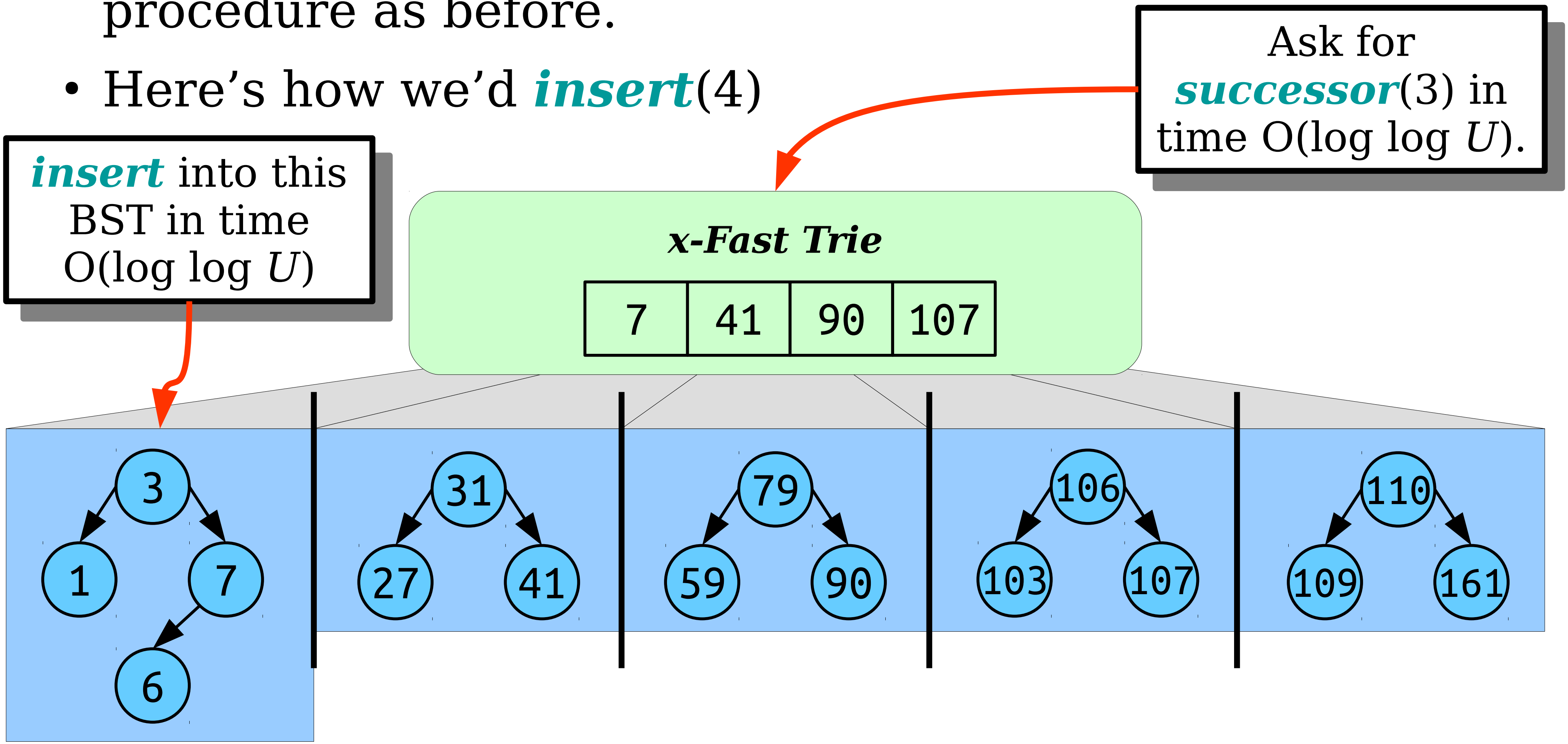


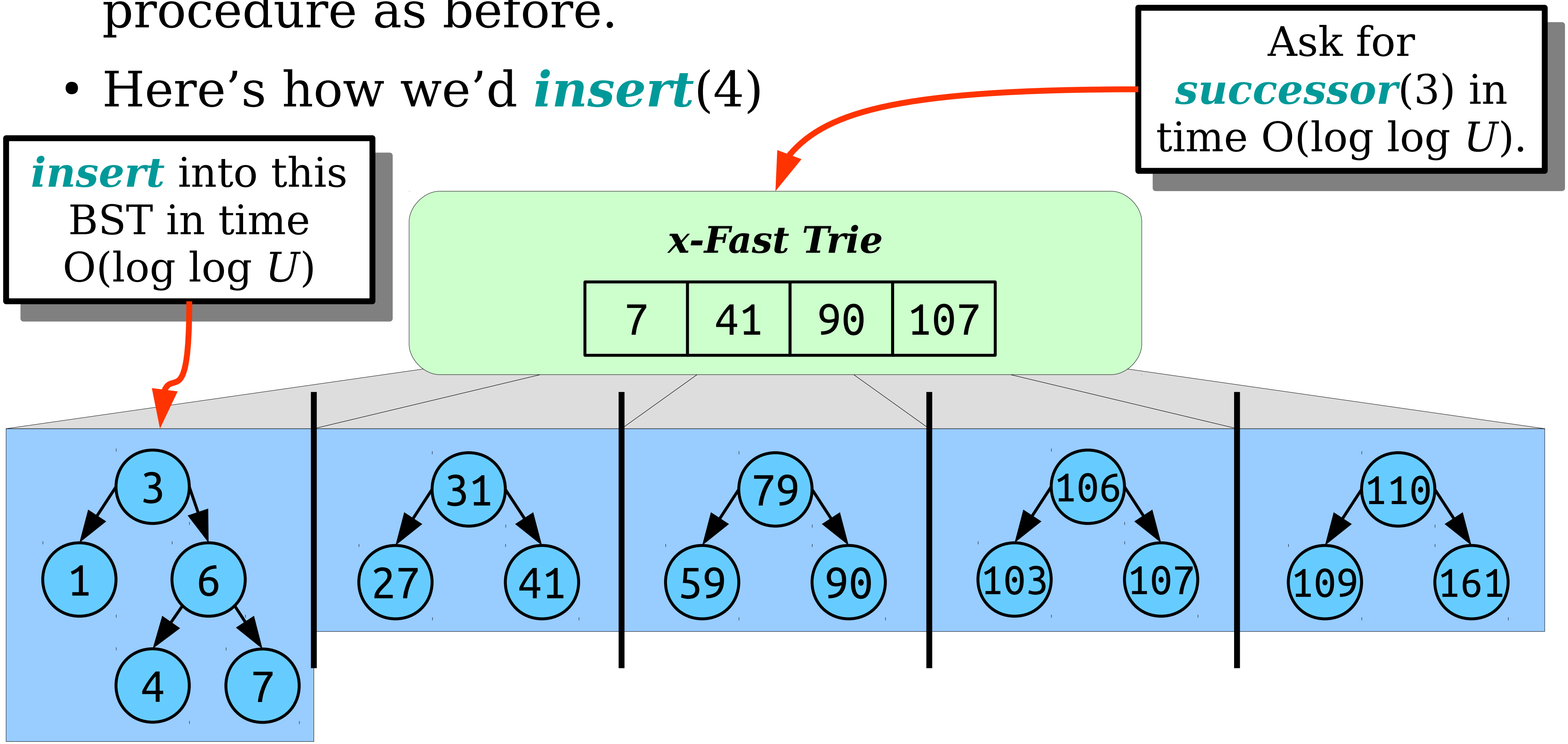
# Making Edits

- With a major caveat, insertions follow the same procedure as before.
- Here's how we'd ***insert***(4)



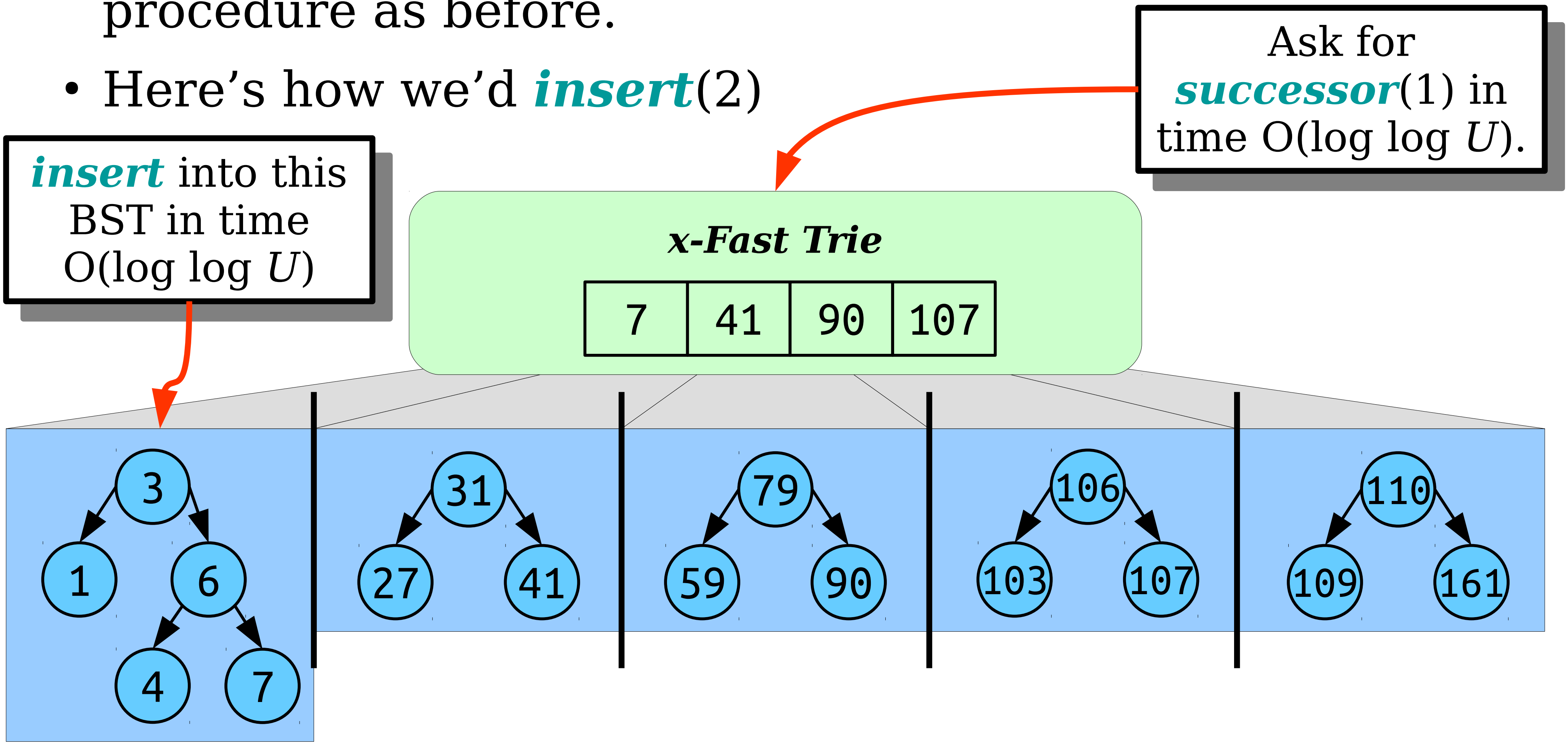
# Making Edits

- With a major caveat, insertions follow the same procedure as before.
- Here's how we'd ***insert***(4)



# Making Edits

- With a major caveat, insertions follow the same procedure as before.
- Here's how we'd ***insert***(2)

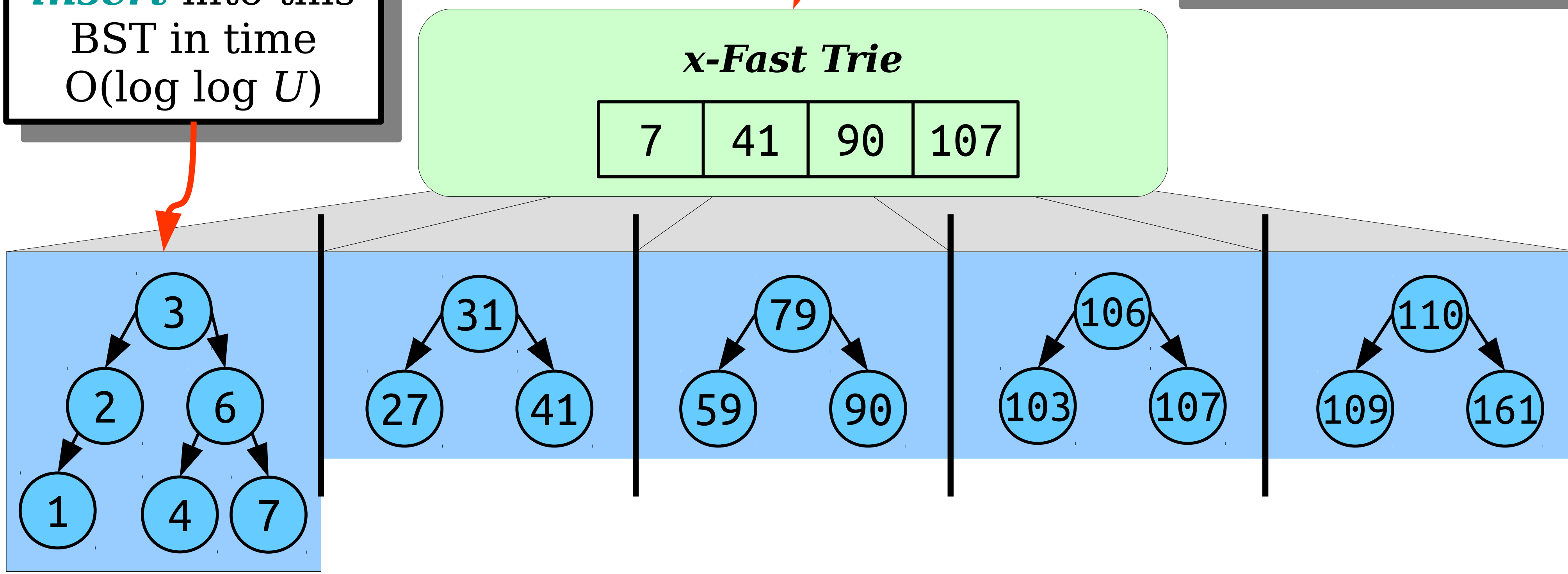


# Making Edits

- With a major caveat, insertions follow the same procedure as before.
- Here's how we'd ***insert***(2)

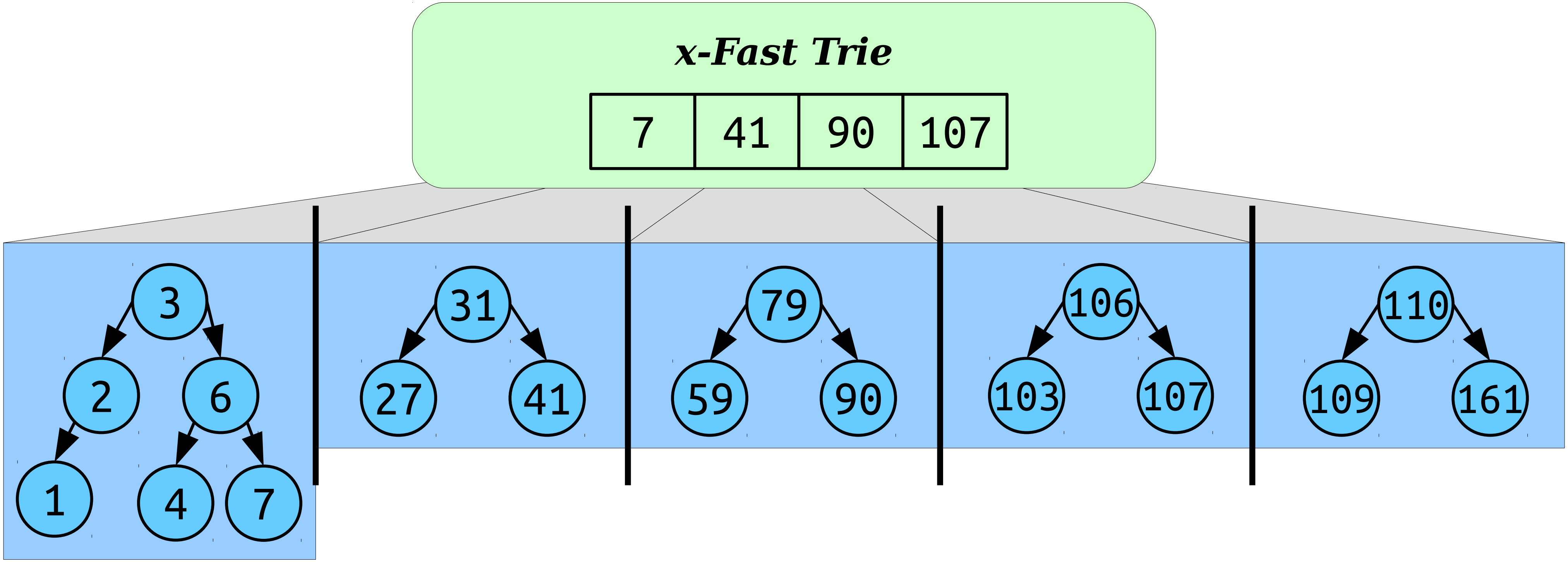
***insert*** into this BST in time O(log log *U*)

Ask for ***successor***(1) in time O(log log *U*).



# The Problem

- If our trees get too big, we may lose our O(log log $U$) time bound.
- ***Idea:*** Require each tree to have at most $2 \cdot \lg U$ elements. If it gets too big, split it and update the $x$-fast trie.



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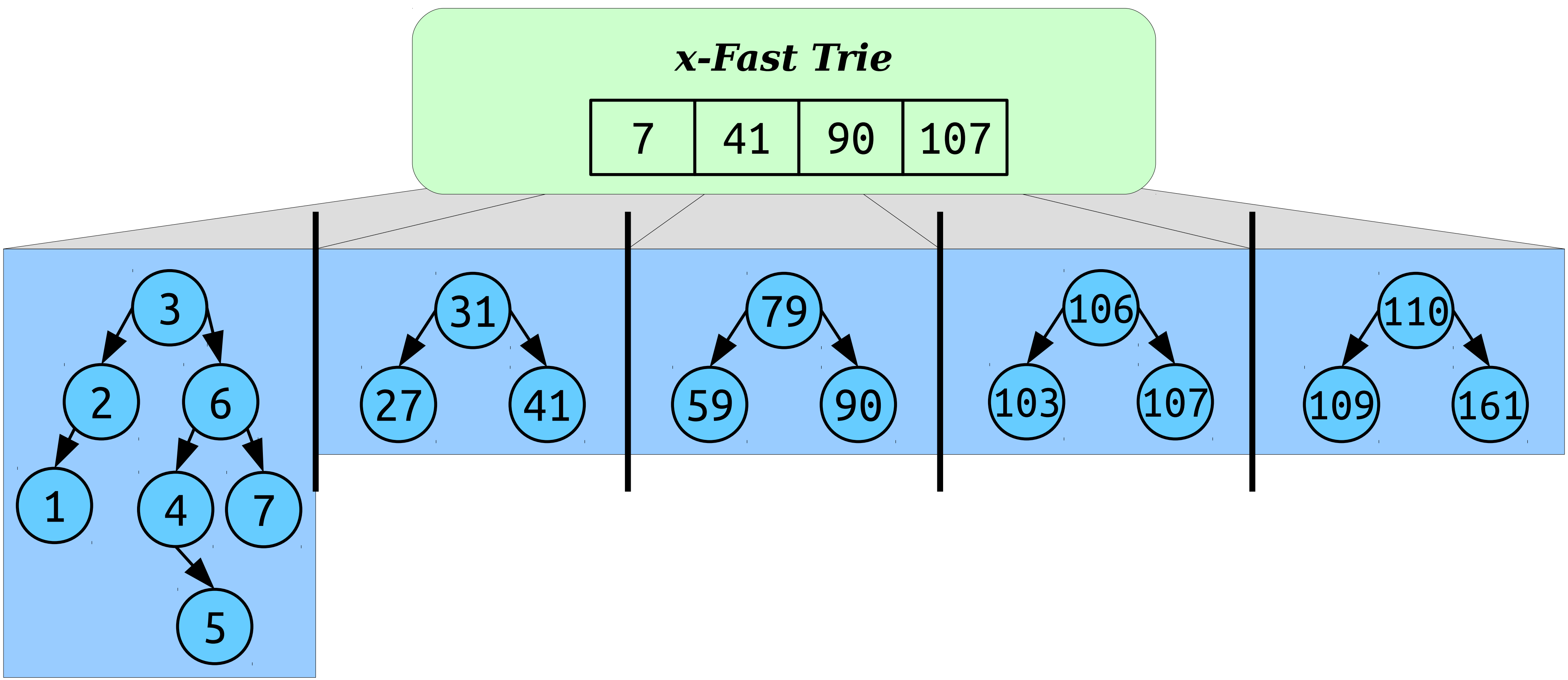

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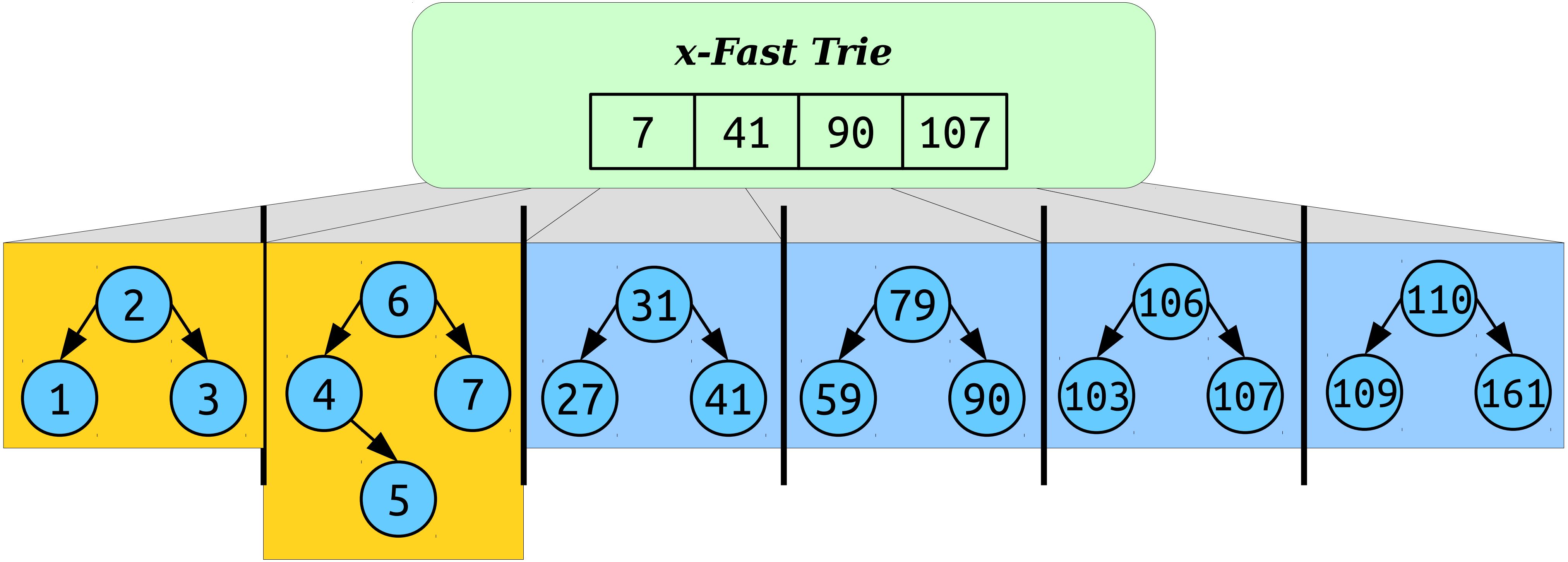

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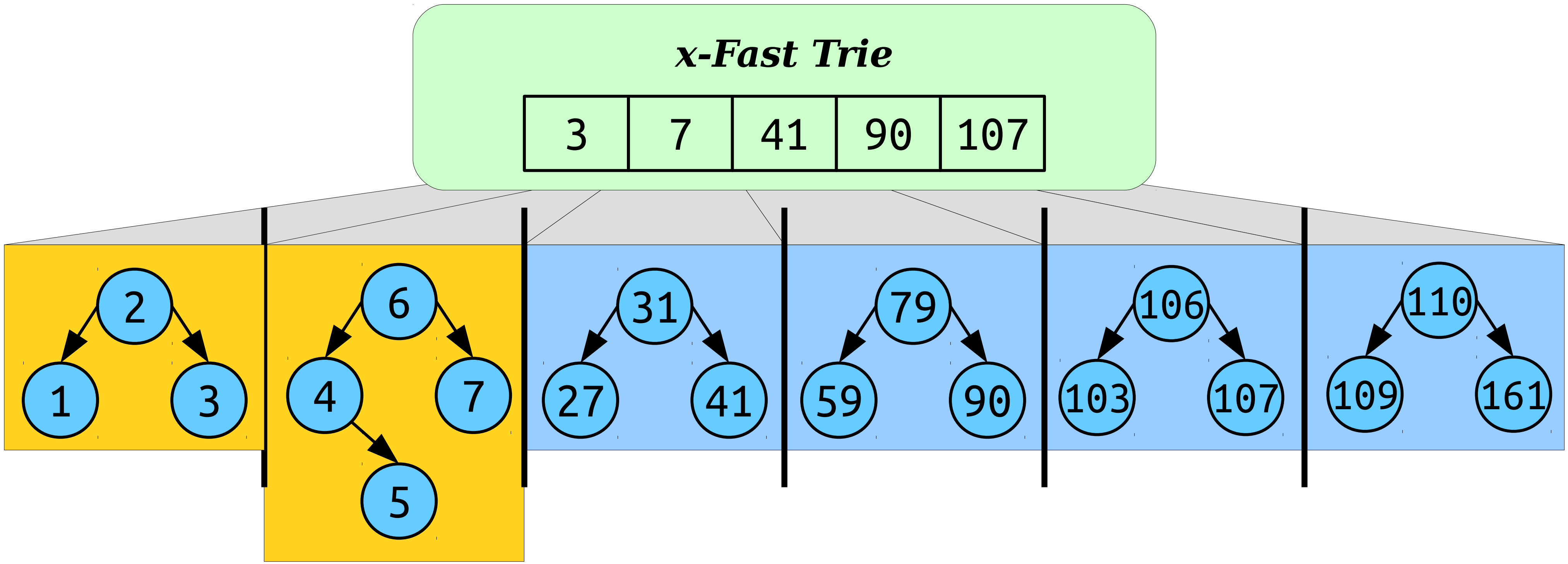

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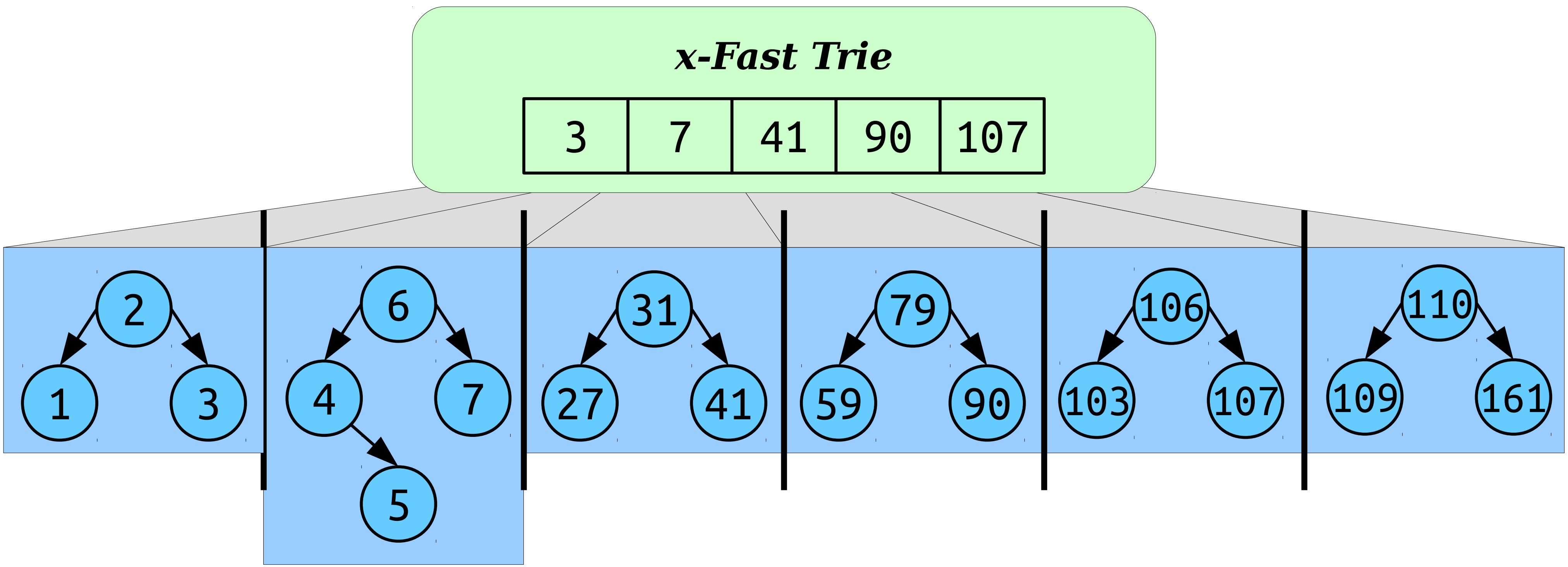

# Analyzing an Insertion

- If we perform an ***insert*** and don't end up doing a resize, the cost is O(log log $U$).
- If we perform an ***insert*** and *do* have to do a resize, the work done is
  - O(log log $U$) to ***split*** the binary search tree, and
  - O(log $U$) to insert into the $x$-fast trie.
- Total work: **O(log $U$)**.

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  - O(log $U$) to insert into the $x$-fast trie.
- Total work: **O(log $U$)**.

But this is uncommon! We only do this if a tree got way too big.

# An Amortized Analysis

- Whenever we do an insertion, place a credit on the newly-inserted element.
  - Cost of a "light" ***insert*** still O(log log $U$).
- If we have to split a tree, the tree size was above 2 lg $U$, so there must be lg $U$ credits on it (one for each element above lg $U$).
- The ***amortized*** cost of a "heavy" insert is then

$$O(\log \log U) + O(\log U) - \Theta(\log U) = \mathbf{O(\log \log U)}.$$

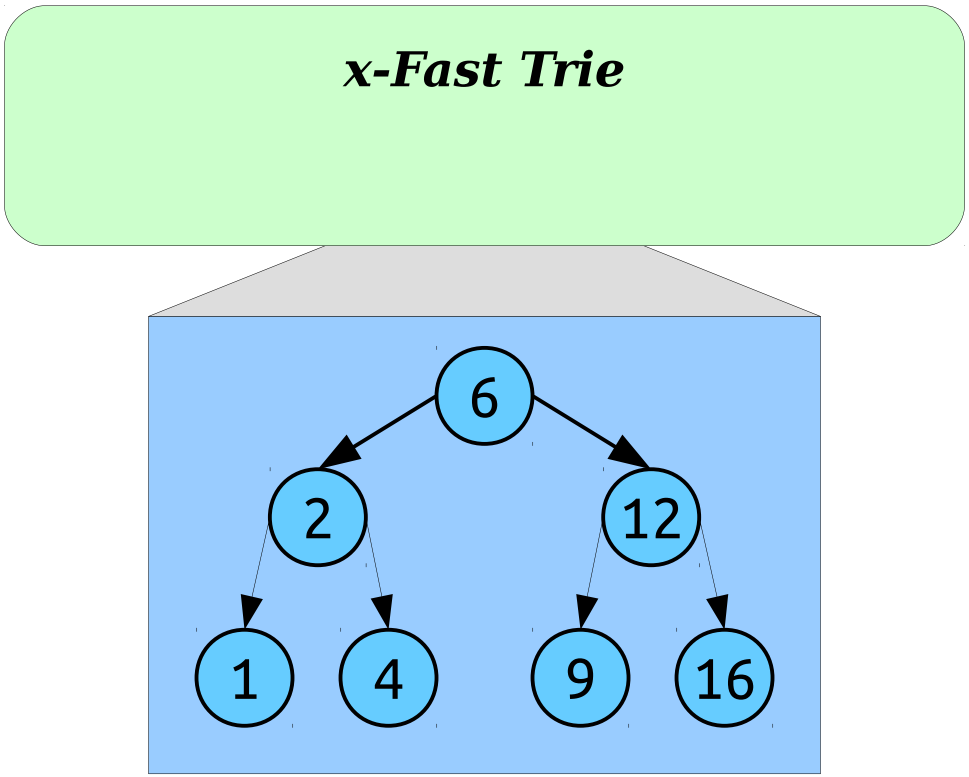
Cost of a regular insert, plus the tree split.

Cost of adding to the *x*-fast trie.

Credits spent.

# A Nice Side-Effect

- We can now abandon our assumption that we're given all the keys in sorted order in advance.
- Each insertion takes amortized time O(log log $U$), so we can build the structure up from scratch!



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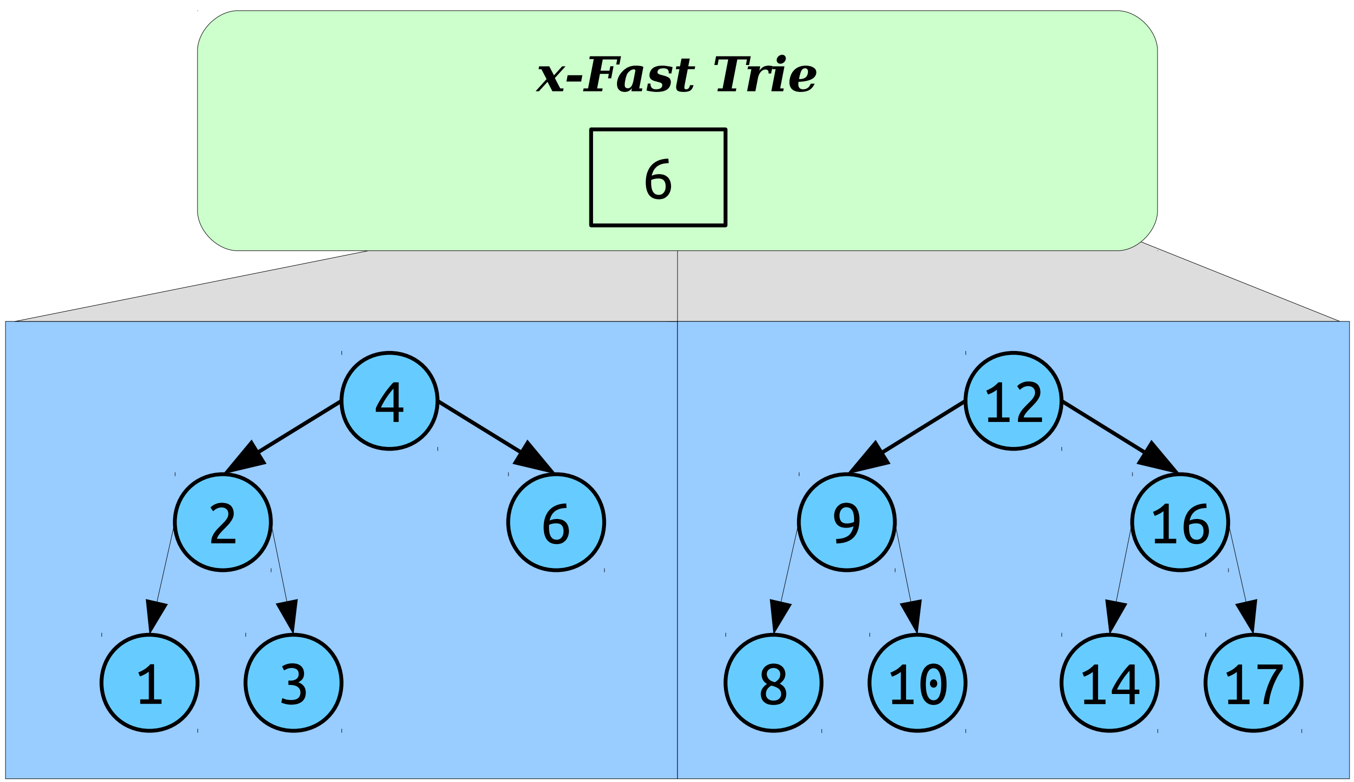

# A Nice Side-Effect

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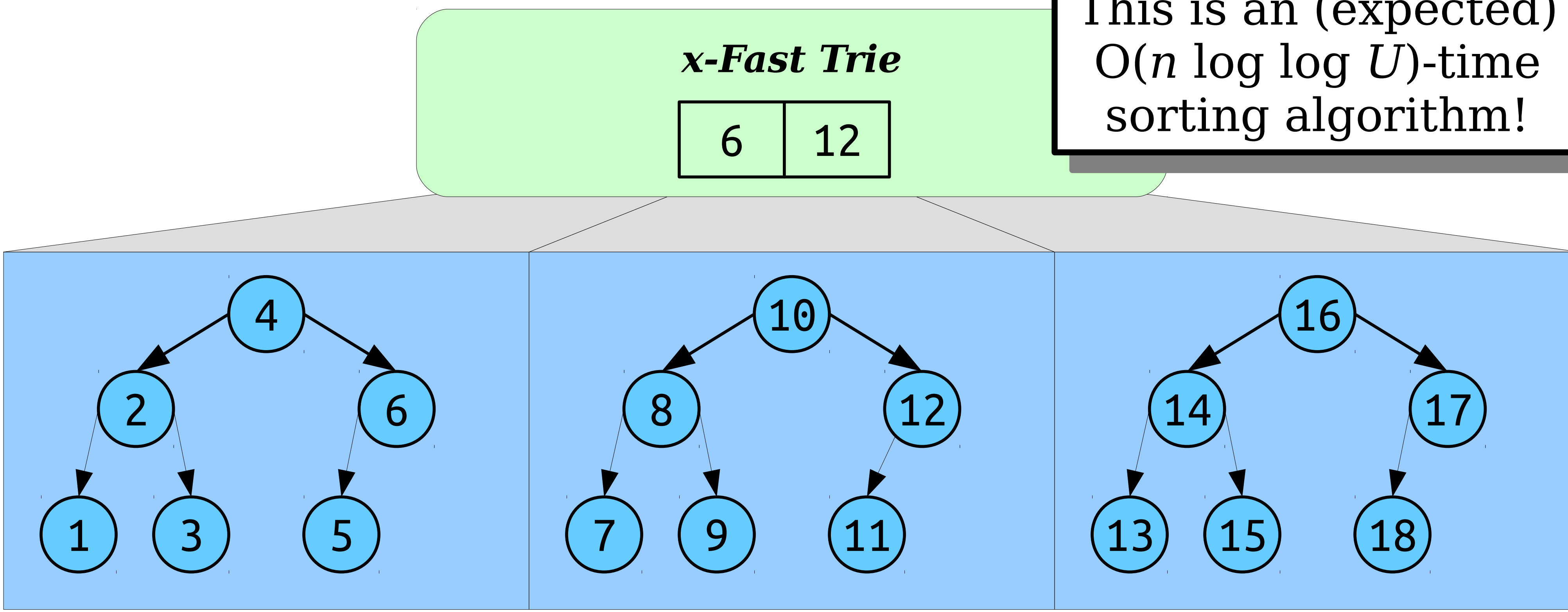
- Each insertion takes amortized time O(log log $U$), so we can build the structure up from scratch!

This is an (expected) O($n$ log log $U$)-time sorting algorithm!



# Making Edits

- With a major caveat, deletions follow the same procedure as insertions.
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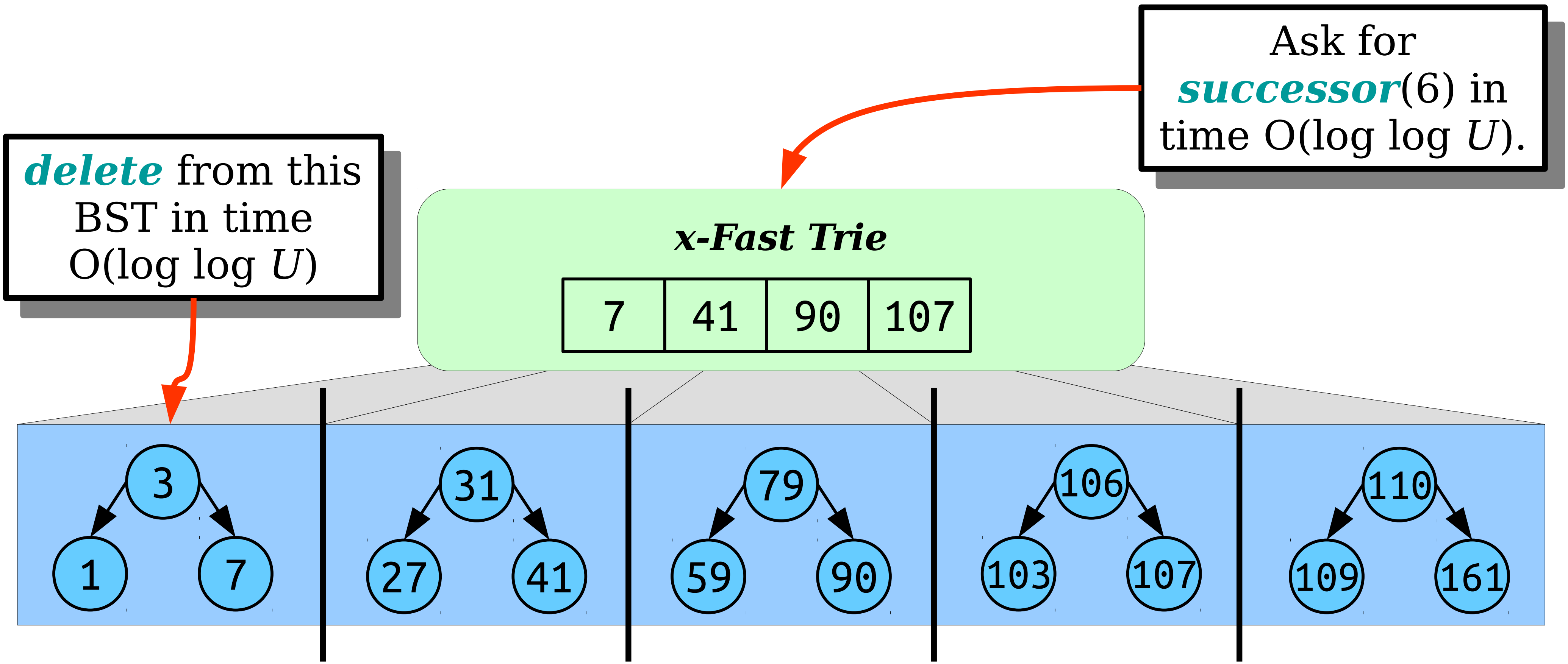
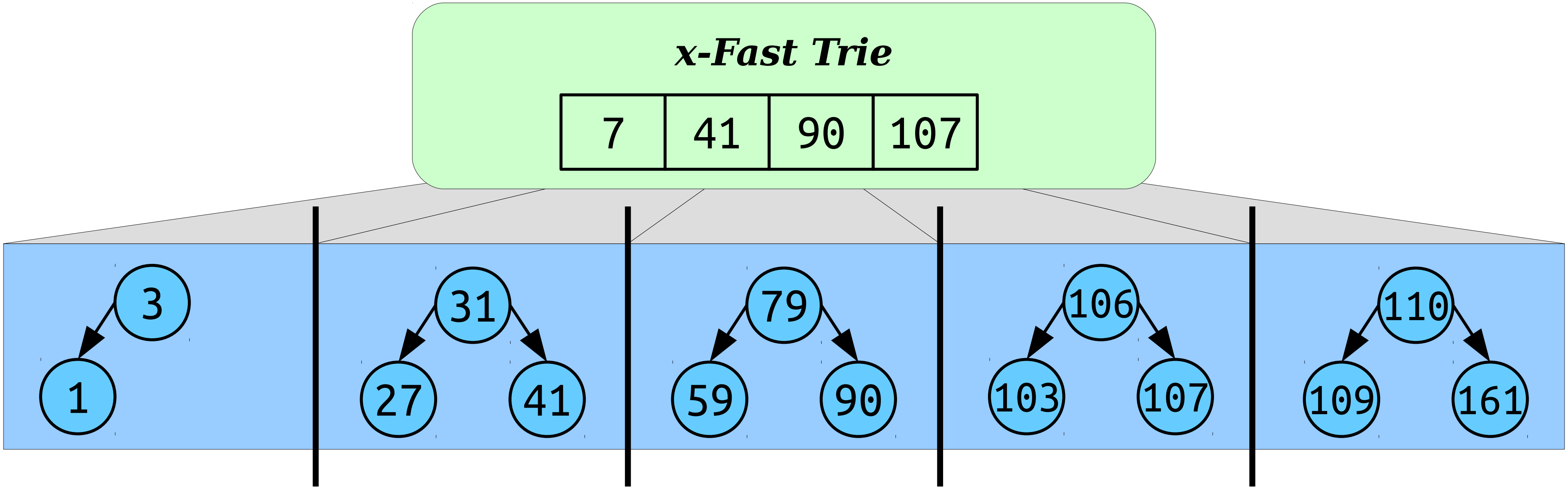


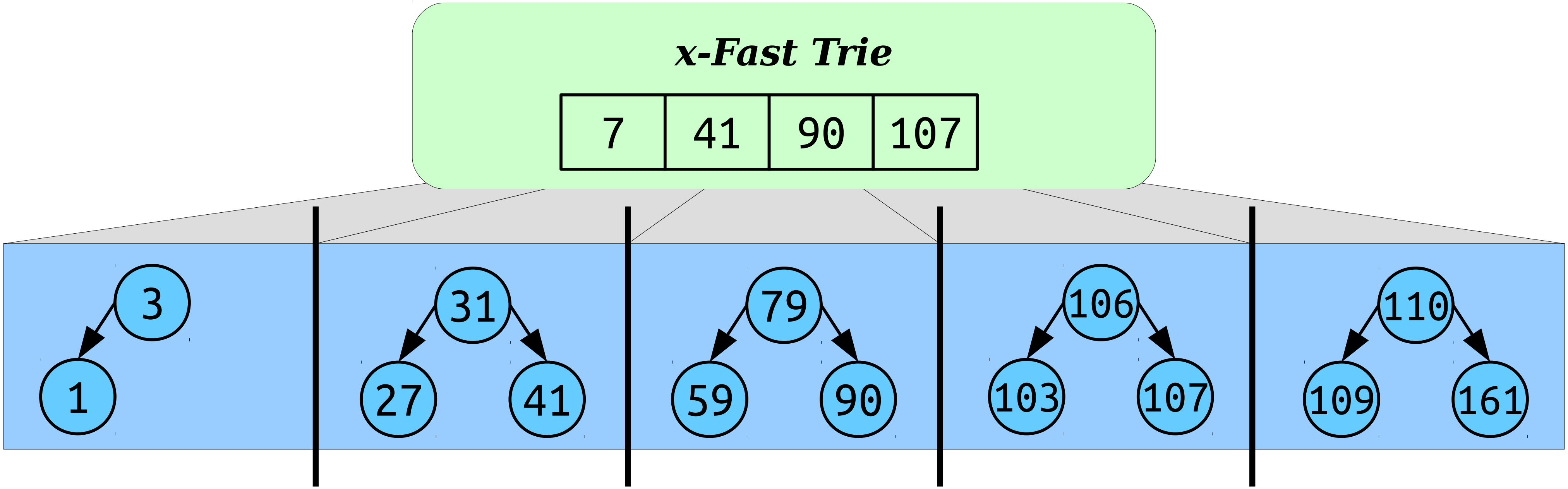
# Making Edits

- Our *x*-fast trie still holds 7, even though 7 is no longer present.
- That's not a problem – those keys just serve as "routing information" to tell us which BSTs to look at.
- ***Intuition:*** The *x*-fast trie keys act as partitions between BSTs. They don't need to actually be present in our data structure.



# Shrinking our Structure

- What happens if we remove all the elements from our structure without touching the *x*-fast trie?



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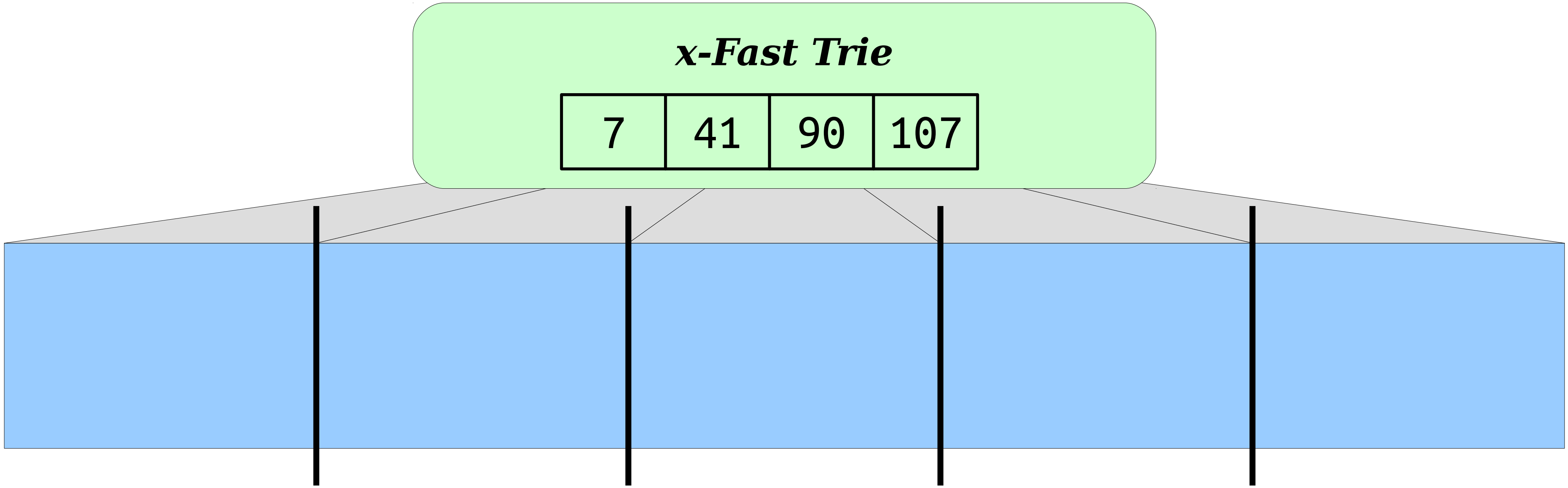
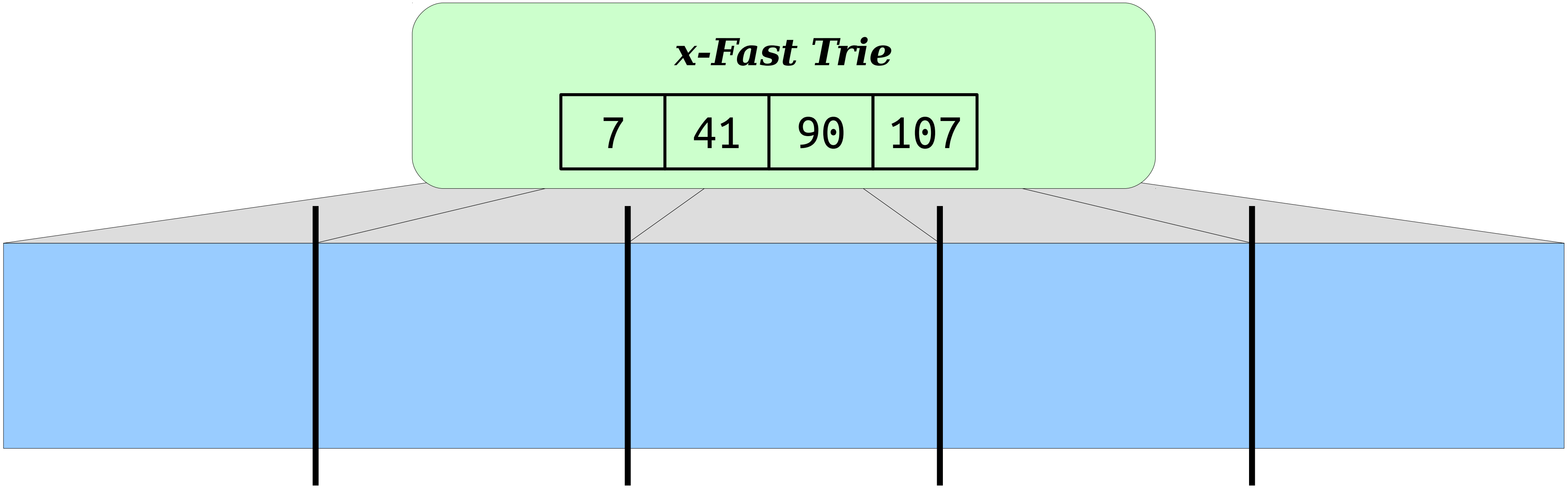


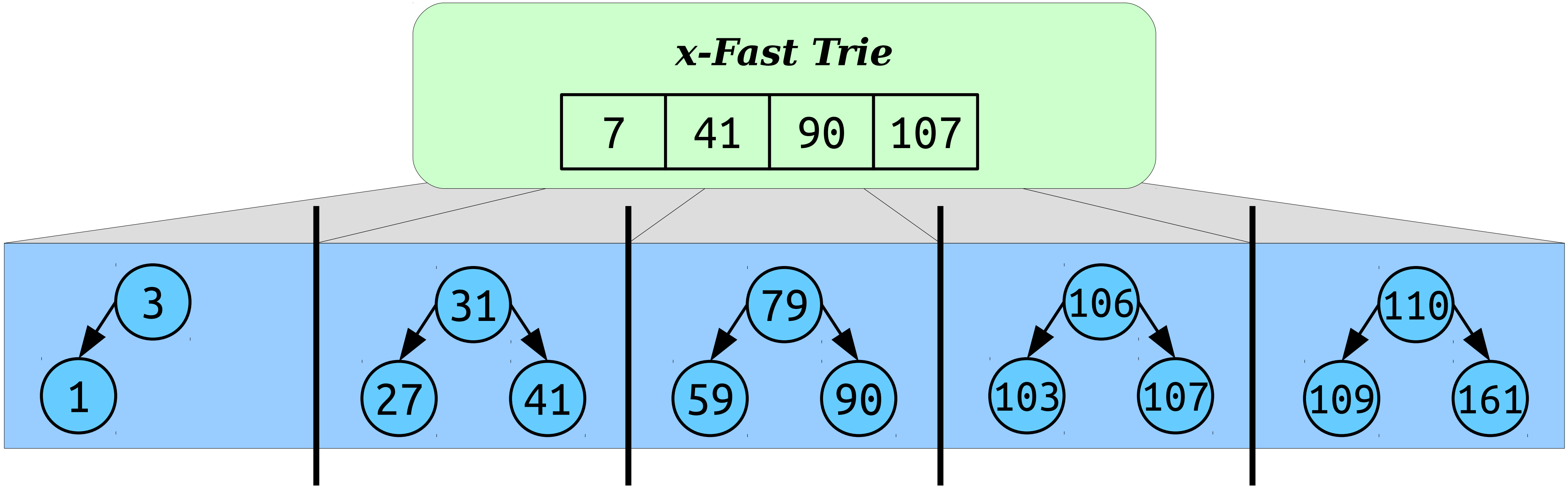
# Shrinking our Structure

- What happens if we remove all the elements from our structure without touching the *x*-fast trie?
- Each operation still takes time **O(log log *U*)**.
- But now our space usage depends on the maximum size we reached, not the current size!



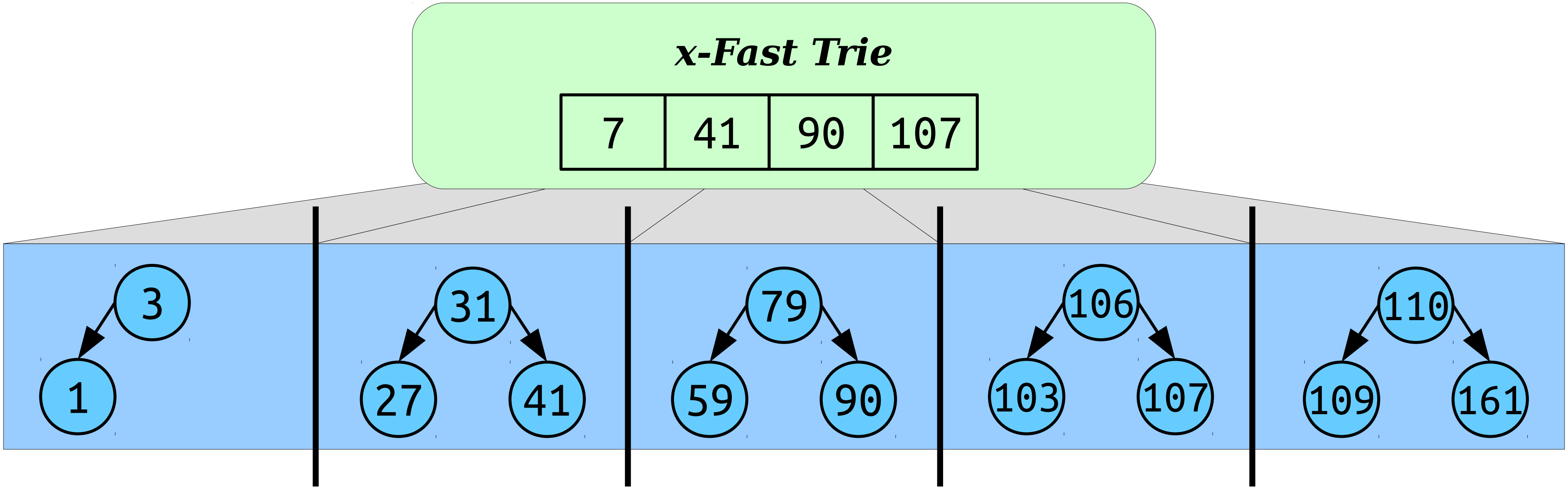
# Achieving a Balance

- If each tree has Θ(log $U$) elements in it, then our space usage is
  - Θ($n$) for all the trees, plus
  - Θ(($n$ / log $U$) log $U$) = Θ($n$) for the $x$-fast trie,
- This uses **Θ($n$)** total memory.



# Achieving a Balance

- ***Invariant:*** Require each tree to have between ½lg $U$ and 2 lg $U$ elements.
- If a tree gets too small, either
  - borrow lots of elements from a neighbor and update the $x$-fast trie, or
  - merge with a neighbor and update the $x$-fast trie.



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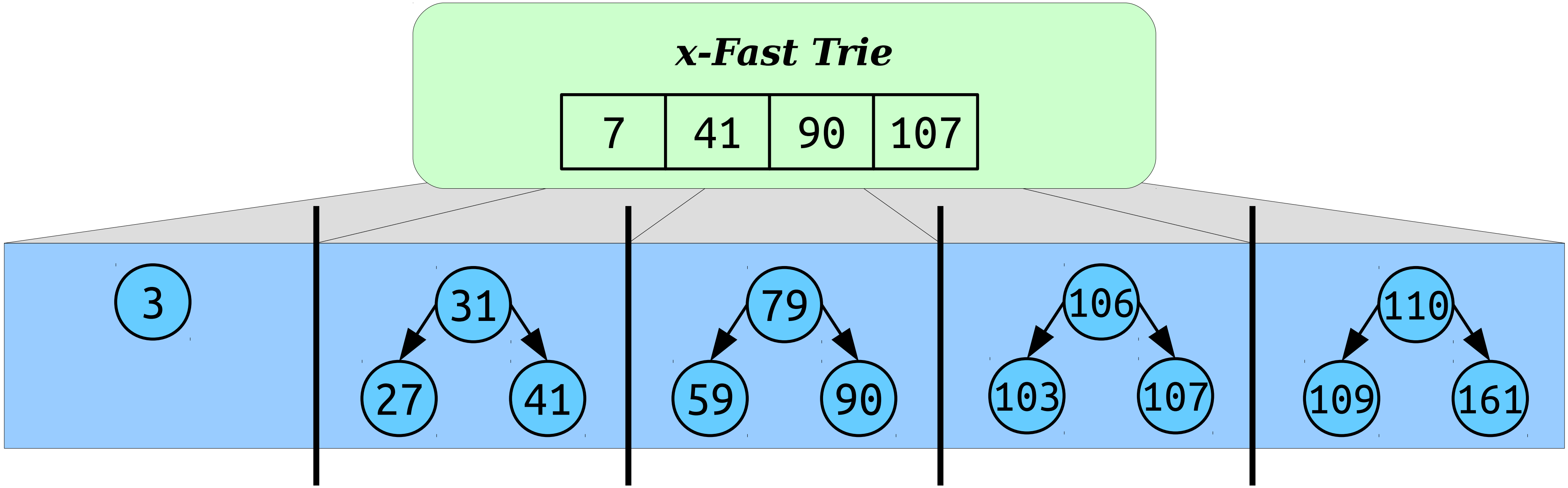

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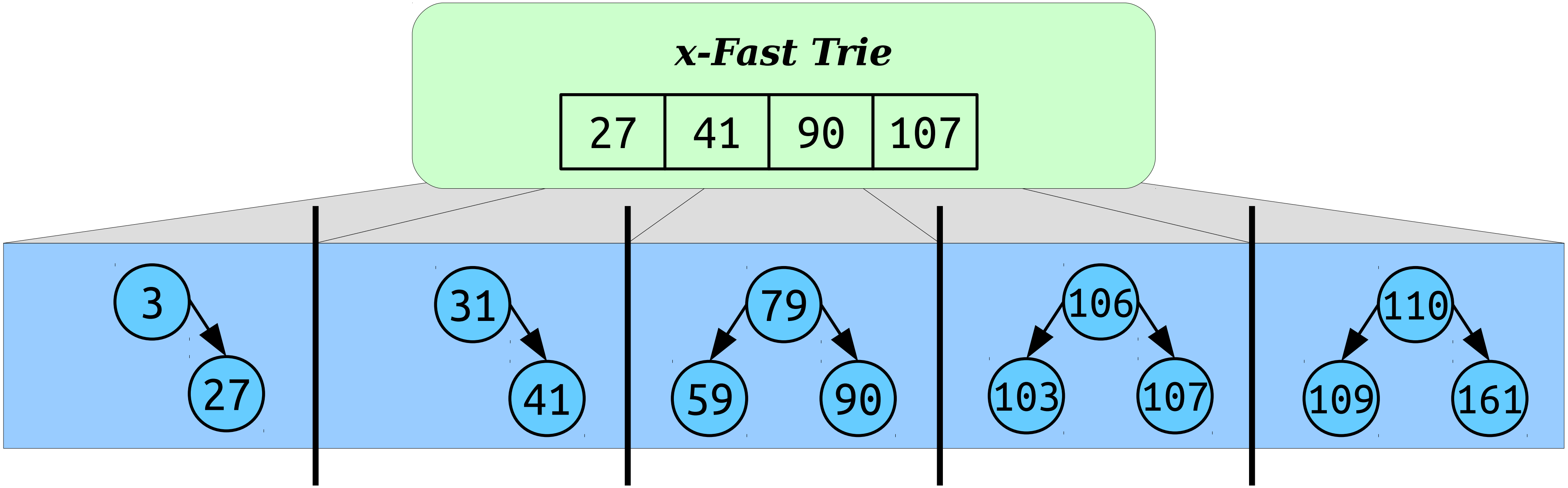

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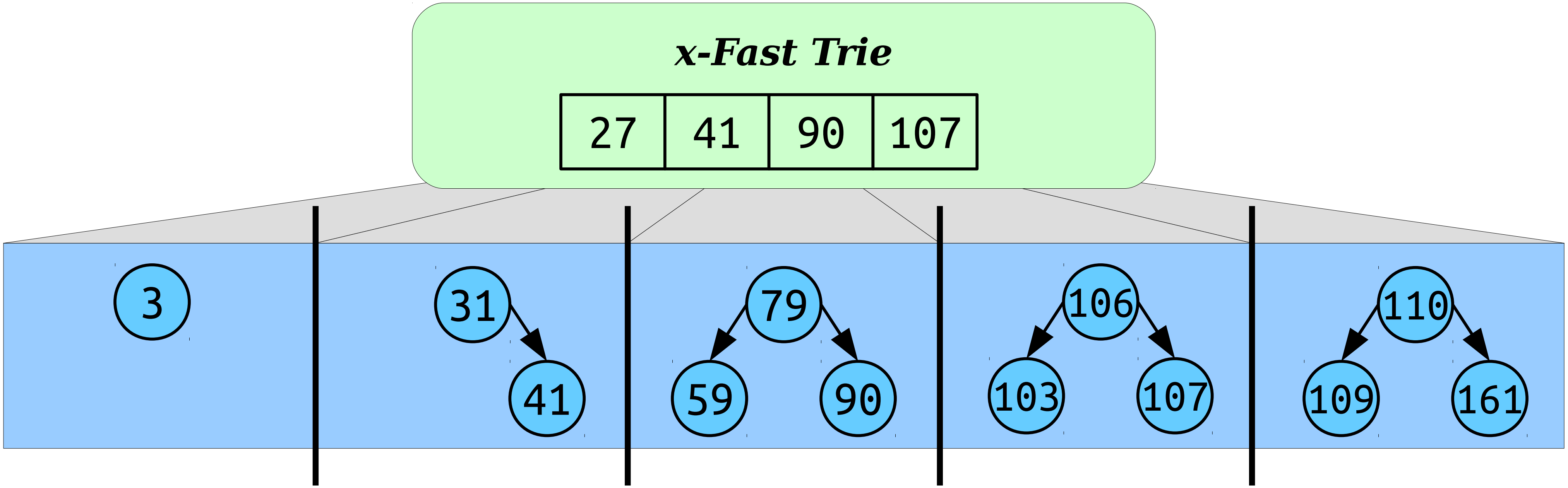

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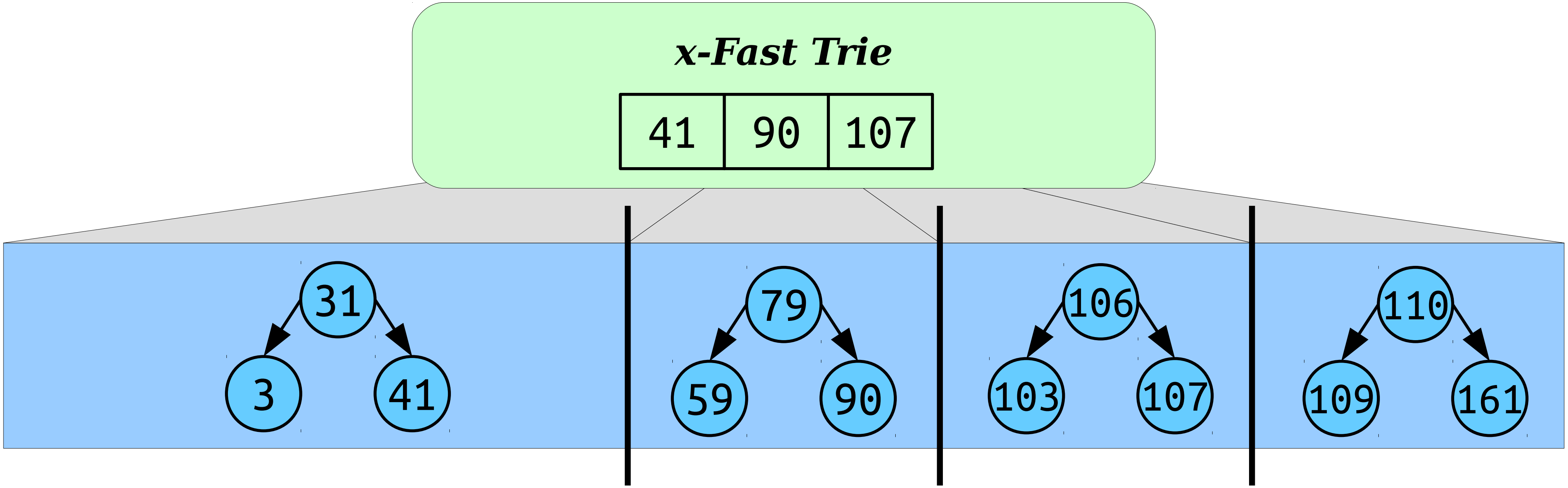

# What We've Seen

- Here's the final scorecard for the $y$-fast trie.
- Assuming $n = \omega(\log U)$, which it probably is, this is strictly better than a binary search tree.
- And it gives rise to an $O(n \log \log U)$-expected-time sorting algorithm!

The $y$-Fast Trie:

- ***lookup***: $O(\log \log U)$
- ***insert***: $O(\log \log U)^*$
- ***delete***: $O(\log \log U)^*$
- ***max***: $O(\log \log U)$
- ***succ***: $O(\log \log U)$
- ***is-empty***: $O(1)$
- Space: $\Theta(n)$

* Expected, amortized.

# What We Needed

- An *x*-fast trie requires ***tries*** and ***cuckoo hashing***.
- The *y*-fast trie requires amortized analysis and ***split/join*** on ***balanced BSTs***.
- *y*-fast tries also use the “blocking” technique from ***RMQ*** we used to shave off log factors.

# What's Missing

- There's still a little gap between where BSTs dominate and where *y*-fast tries take over.
  - Specifically, what if $n = O(\log U)$?
- Our solution still involves randomness.
  - We need that in the cuckoo hash tables at each level.
- ***Question:*** Can we build a solution with neither of these weaknesses?

# Next Time

- ***Word-Level Parallelism***
  - Parallel processing via addition, subtraction, and the like.
- ***Sardine Trees***
  - A fast ordered dictionary for truly tiny trees.