Image Completion Using Low Tensor Tree Rank and Total Variation Minimization

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Outline

- related work
  - Total variation
  - The state-of-art tensor completion with smooth constraint
- proposed optimization model
  - proposed algorithm
  - solution
- Simulation
  - Color image completion
- Conclusion

Tensor tree format

Specially , for the root D we have $A^{(D)} = U_D \in \mathbb{R}^{\mathcal{I}\times 1}(k_D = 1)$

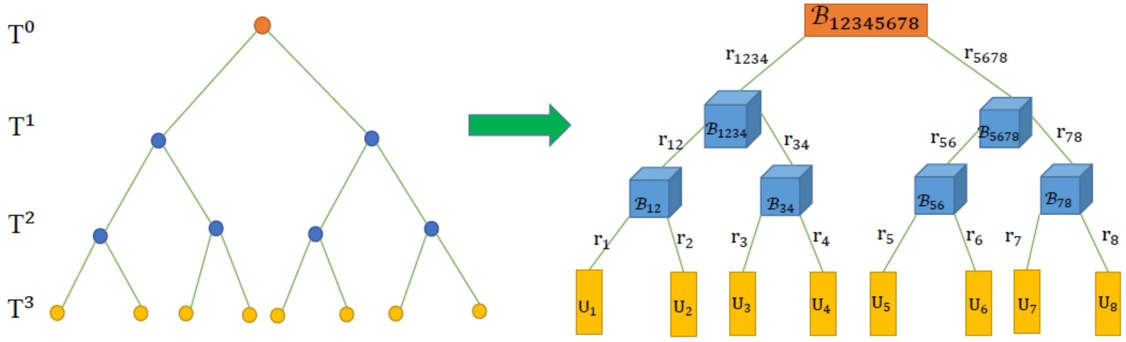
Storage:

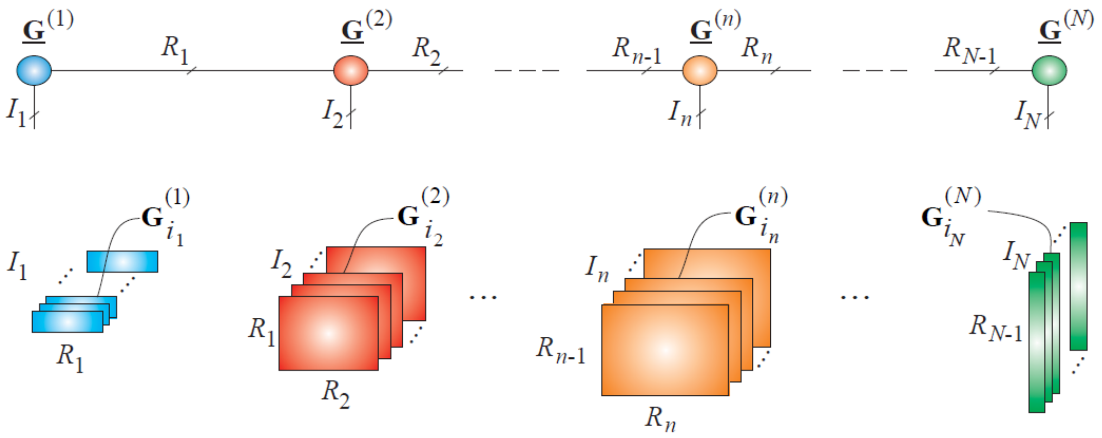
- Tensor tree format:

  $dnr + (d-2)r^3 + r^2.$

- Tensor train format:

  $(d-2)nr^2 + 2nr$





Total variation

$$\|\mathbf{X}\|_{\mathrm{TV\text{-}AS}} = \| \sqrt{|\nabla_h \mathbf{X}|^2 + |\nabla_v \mathbf{X}|^2}$$

$$\|\mathbf{X}\|_{\mathrm{TV\text{-}IS}} = \|\nabla_h \mathbf{X}\|_1 + \|\nabla_v \mathbf{X}\|_1$$

where $\|\mathbf{X}\|_{TV-AS}$ is the anisotropic TV
and $\|\mathbf{X}\|_{TV-IS}$ is the isotropic TV

The state-of-art tensor completion with smooth constraint

- Tucker based

$$\min_{\mathcal{X}}(1-\rho)\|\mathcal{X}\|_* + \rho\|\mathcal{X}\|_{\mathrm{TV}}$$
$$s.t.\ \nu_{min} \leq \mathcal{X} \leq \nu_{max},\ \|\mathcal{Z}_{\mathbb{O}} - \mathcal{X}_{\mathbb{O}}\|_F^2 \leq \delta.$$

- CP based

$$\min_{\mathcal{G},\mathbf{U}^{(1)},\ldots,\mathbf{U}^{(N)}} \frac{1}{2}\|\mathcal{X}-\mathcal{Y}\|_F^2 + \sum_{r=1}^{R}\frac{g_r^2}{2}\sum_{n=1}^{N}\rho^{(n)}\|\mathbf{L}^{(n)}\mathbf{u}_r^{(n)}\|_p^p,$$
$$s.t.\ \mathcal{Y} = \sum_{r=1}^{R} g_r \mathbf{u}_r^{(1)} \circ \mathbf{u}_r^{(2)} \circ \cdots \circ \mathbf{u}_r^{(n)},\ \mathcal{X}_{\mathbb{O}} = \mathcal{Z}_{\mathbb{O}},\ \mathcal{X}_{\bar{\mathbb{O}}} = \mathcal{Y}_{\bar{\mathbb{O}}},$$
$$\|\mathbf{u}_r^{(n)}\|_2 = 1,\ \forall r \in \{1,\ldots,R\},\ \forall n \in \{1,\ldots,N\}. \quad (11)$$

proposed algorithm

$$\min_{X} \ (1-\rho)\mathbf{w}^T\mathbf{k}(\mathcal{X}) + \rho CTV(\mathcal{X}), \ \ s.t. \ \mathcal{X}_{\mathbb{O}} = \mathcal{Z}_{\mathbb{O}},$$

$$\min_{\mathcal{X}} \quad (1-\rho)\sum_{q=1}^{Q} w_q \|\mathbf{X}^{(q)}\|_* + \rho\|\mathbf{D}(\mathcal{X})\|_{CTV}, \quad \text{s. t.} \ \ \mathcal{X}_{\mathbb{O}} = \mathcal{Z}_{\mathbb{O}},$$

$$\min_{\mathcal{X}} \quad (1-\rho)\sum_{q=1}^{Q} w_q \|\mathbf{X}^{(q)}\|_* + \rho\|\mathbf{L}\|_{CTV},$$
$$\text{s. t.} \ \ \mathcal{X}_{\mathbb{O}} = \mathcal{Z}_{\mathbb{O}}, \ \mathcal{M} = \mathcal{X}, \ \mathcal{Y} = \mathcal{M}, \ \mathbf{L} = \mathbf{D}(\mathcal{Y}).$$

Language funcrion

$$
\begin{aligned}
\min_{\mathcal{X},\mathcal{Y},\mathcal{M},\mathbf{L},\Lambda_1,\Lambda_2,\Lambda_3} \quad & (1-\rho)\sum_{q=1}^{Q} w_q \|\mathbf{M}^{(q)}\|_* + \rho\|\mathbf{L}\|_{CTV} \\
& -\langle \Lambda_1, \mathcal{M}-\mathcal{X}\rangle + \frac{\beta_1}{2}\|\mathcal{M}-\mathcal{X}\|_F^2 - \langle \Lambda_2, \mathcal{Y}-\mathcal{M}\rangle + \frac{\beta_2}{2}\|\mathcal{Y}-\mathcal{M}\|_F^2 \\
& -\langle \Lambda_3, \mathbf{L}-\mathbf{D}(\mathcal{Y})\rangle + \frac{\beta_3}{2}\|\mathbf{L}-\mathbf{D}(\mathcal{Y})\|_F^2, \ \text{s. t. } \mathcal{X}_{\mathbb{O}} = \mathcal{Z}_{\mathbb{O}}. \qquad (15)
\end{aligned}
$$

For the subproblem $\mathcal{M}$

$$\min_{\mathcal{M}} \quad (1-\rho)\sum_{q=1}^{Q} w_q \|\mathbf{M}^{(q)}\|_* - \langle \Lambda_1, \mathcal{M}-\mathcal{X}\rangle$$

$$+\frac{\beta_1}{2}\|\mathcal{M}-\mathcal{X}\|_F^2 - \langle \Lambda_2, \mathcal{Y}-\mathcal{M}\rangle + \frac{\beta_2}{2}\|\mathcal{Y}-\mathcal{M}\|_F^2.$$

$$\Downarrow$$

$$\min_{\mathcal{M}}(1-\rho)\sum_{q=1}^{Q} w_q\|\mathbf{M}^{(q)}\|_* + \frac{\beta_1+\beta_2}{2}\|\mathcal{M}-(\frac{\Lambda_1+\beta_1\mathcal{X}+\beta_2\mathcal{Y}-\Lambda_2}{\beta_1+\beta_2})\|_F^2.$$

$$\Downarrow$$

$$\min_{\mathcal{M}}\sum_{q=1}^{Q} \tau\|\mathbf{M}^{(q)}\|_* + \frac{1}{2}\|\mathbf{M}^{(q)} - \mathbf{S}^{(q)}\|_F^2.$$

Algorithm 1: The Updating of $\mathcal{M}$ from Leaves to Roots

Input :$\mathcal{X}$, τ
parfor $p = 1, \cdots, P$
  $[\mathbf{U}_p, k_p]$=SVT$_{\tau_p}(\mathbf{X}^{(p)})$
end for
  $\mathcal{C}_{H-1} = \mathcal{X} \times_1 \mathbf{U}_1 \times_2 \mathbf{U}_2 \cdots \times_P \mathbf{U}_P$
for $h = (H-1), \cdots, 0$ **do**
  parfor q_h= $1, \cdots, Q_h$ **do**
    if (the q_h-th node is an interior one)
    $[\hat{\mathbf{U}}_{q_h}, \hat{k}q_h]$=SVT$_{\tau_{q_h}}(\mathbf{C}^{(q_h)})$
      $\mathcal{B}_{q_h}$=reshape($\hat{\mathbf{U}}_{q_h}$, $\hat{k}_{q_h}$, $\hat{k}_{q_{h,1}}$,$\hat{k}_{q_{h,2}}$)
    end if
  end for
    $\mathcal{C}_{h-1} = \mathcal{C}_h \times_1 \hat{\mathbf{U}}_1 \times_2 \hat{\mathbf{U}}_2 \cdots \times_{(Q_h-P_h)} \hat{\mathbf{U}}_{(Q_h-P_h)}$
end for
$\mathcal{Y}$ can be constructed from $\mathcal{B}_q$ and $\mathbf{U}_p$
Output: $\mathcal{M}$

For the subproblem $\mathcal{Y}$

$$\min_{\mathcal{Y}} -\langle \Lambda_2, \mathcal{Y} - \mathcal{M} \rangle + \frac{\beta_2}{2} \|\mathcal{Y} - \mathcal{M}\|_F^2$$

$$-\langle \Lambda_3, \mathbf{L} - \mathbf{D}(\mathcal{Y}) \rangle + \frac{\beta_3}{2} \|\mathbf{L} - \mathbf{D}(\mathcal{Y})\|_F^2$$

$$\Downarrow$$

$$\mathcal{Y} = \frac{\mathbf{D}^*(\beta_3 \mathbf{L} - \Lambda_3) + \beta_2 \mathcal{M} + \Lambda_2}{\beta_2 + \beta_3 \mathbf{D}^* \mathbf{D}}$$

For the subproblem $\mathbf{L}$

$$\min_{\mathbf{L}} \rho||\mathbf{L}||_{CTV} - \langle \Lambda_3, \mathbf{L} - \mathbf{D}(\mathcal{Y}) \rangle + \frac{\beta_3}{2}||\mathbf{L} - \mathbf{D}(\mathcal{Y})||_F^2,$$

Isotropic TV

$$\min_{\mathbf{L}} \rho||\mathbf{L}||_{2,1} + \frac{\beta_3}{2}||\mathbf{L} - (\mathbf{D}(\mathcal{Y}) + \frac{\Lambda_3}{\beta_3})||_F^2.$$

$$shrink_\tau(x) = max(1 - \frac{\tau}{s}).*a,\ s = sqrt(a_h^2 + a_v^2)$$

Anisotropic TV

$$\min_{\mathbf{L}} \rho||\mathbf{L}||_1 + \frac{\beta_3}{2}||\mathbf{L} - (\mathbf{D}(\mathcal{Y}) + \frac{\Lambda_3}{\beta_3})||_F^2.$$

$$\mathrm{sth}_\tau(x) = \mathrm{sgn}(x)\max(|x| - \tau, 0).$$

For the subproblem $\mathcal{X}$ and dual variables

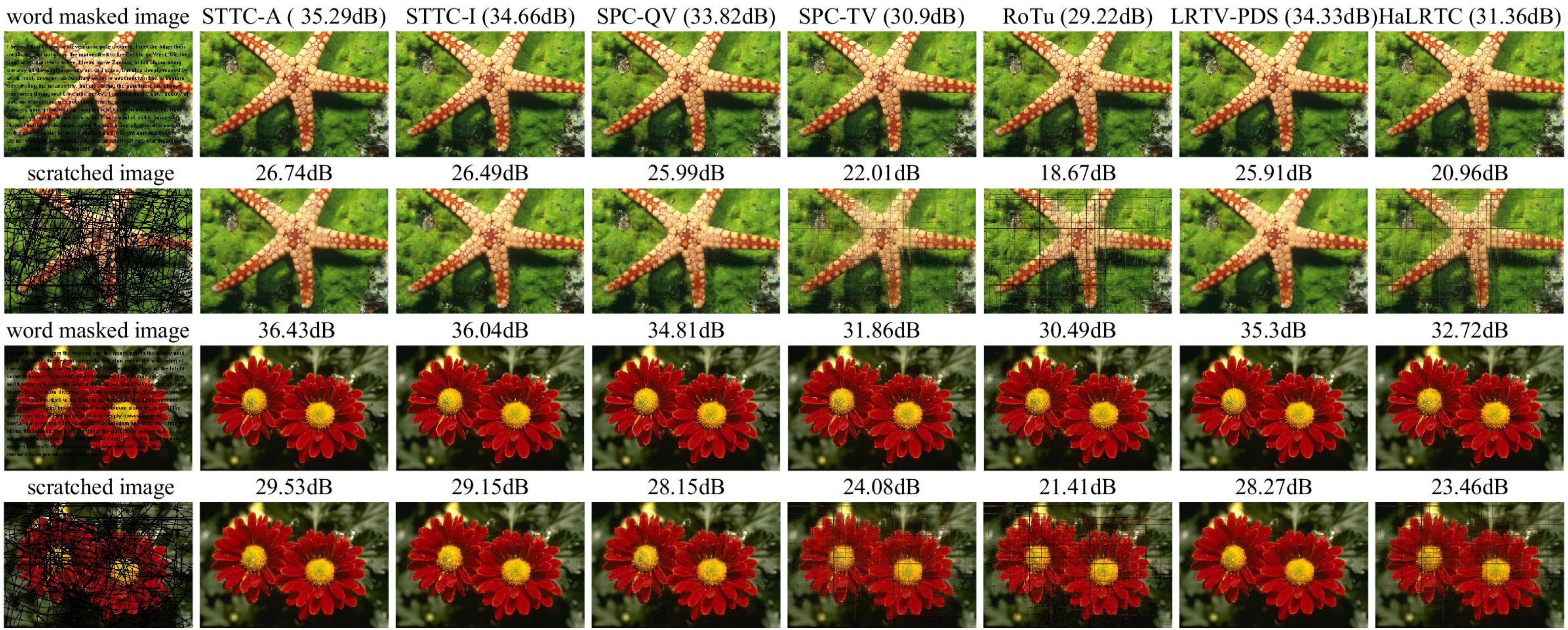
$$\mathcal{X}_{i_1,\cdots,i_d} = \begin{cases} (\mathcal{M} - \frac{\Lambda_1}{\beta_1})_{i_1,\cdots,i_d} & , \; i_1,\cdots,i_d \notin \mathbb{O} \\ \mathcal{Z}_{i_1,\cdots,i_d} & , \; i_1,\cdots,i_d \in \mathbb{O} \end{cases} .$$

$$\Lambda_1 = \Lambda_1 - \beta_1(\mathcal{M} - \mathcal{X})$$

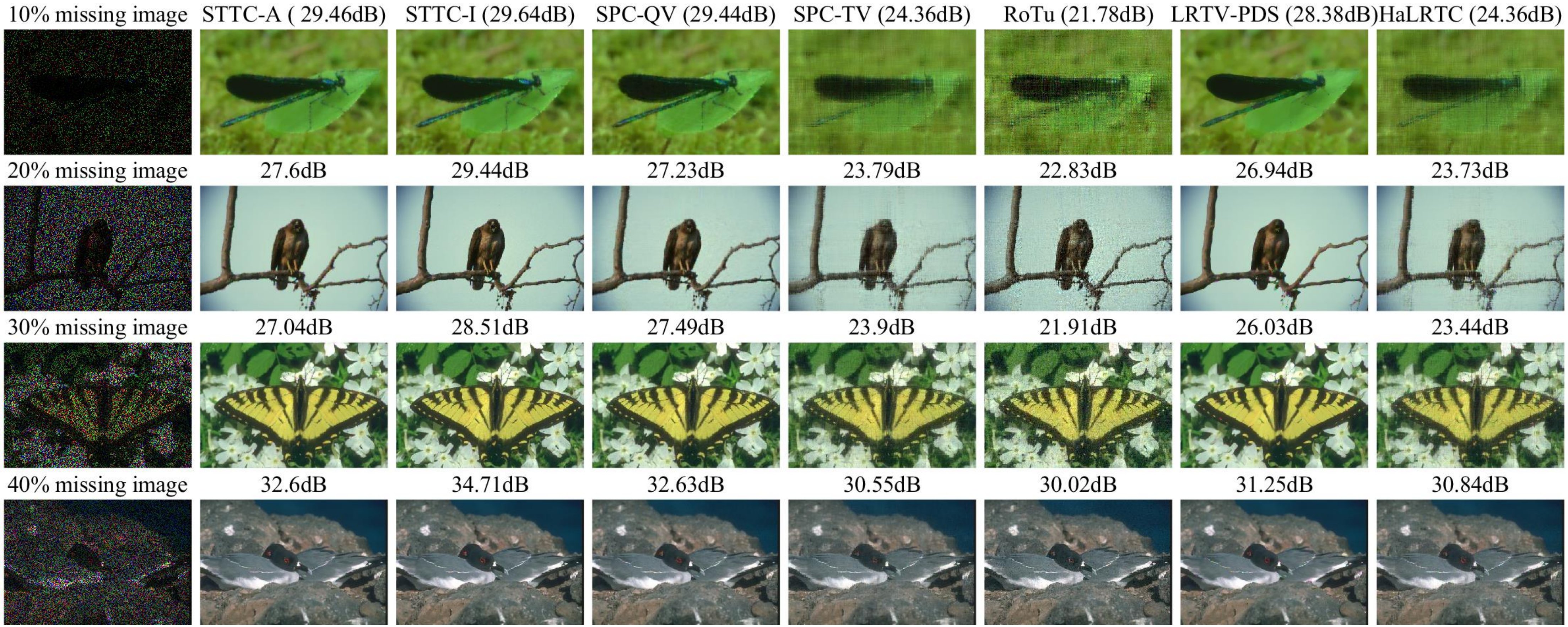
$$\Lambda_2 = \Lambda_2 - \beta_2(\mathcal{Y} - \mathcal{M})$$

$$\Lambda_3 = \Lambda_3 - \beta_3(\mathbf{L} - \mathbf{D}(\mathcal{Y}))$$

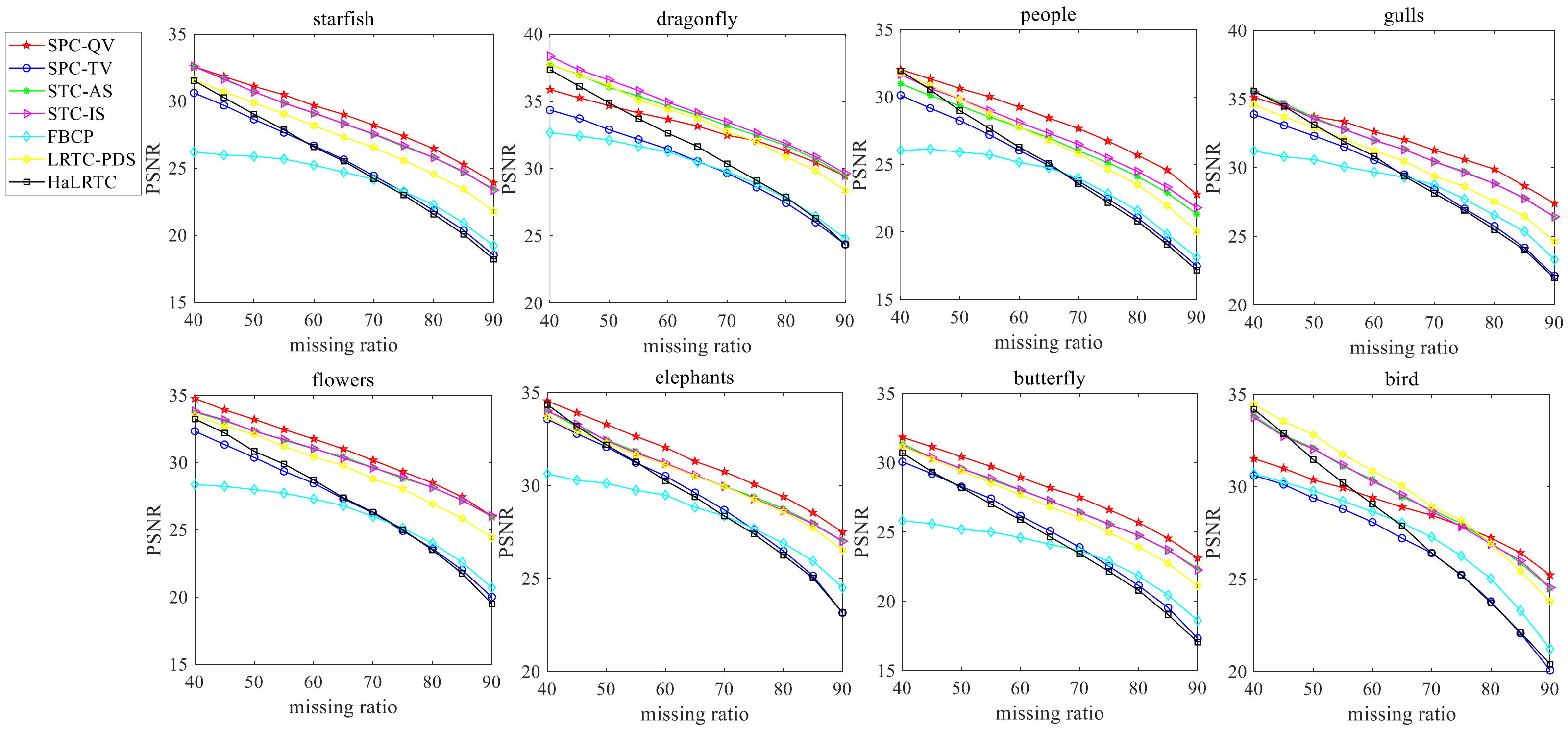
Color image completion



Color image completion

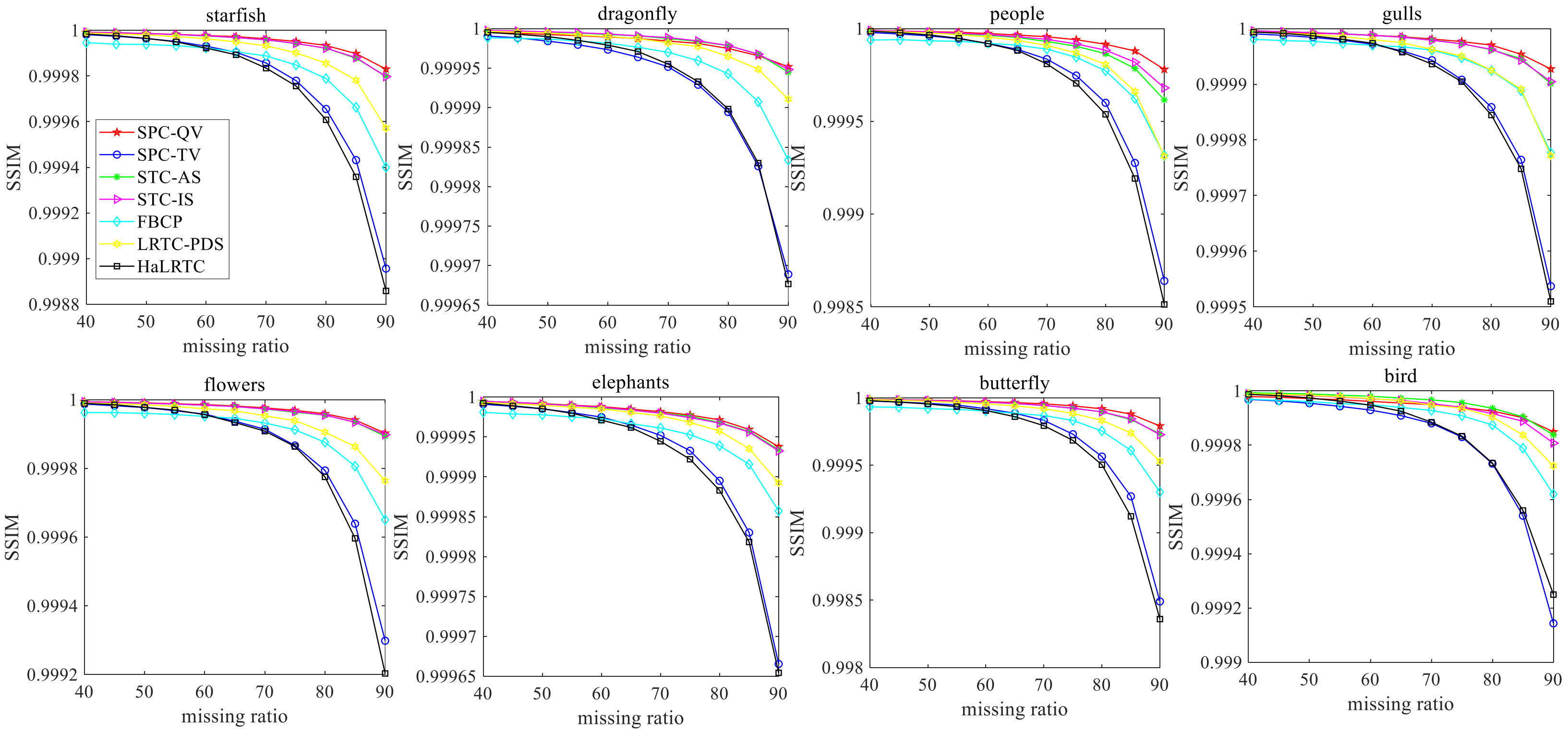


PSNR



SPC-QV
SPC-TV
STC-AS
STC-IS
FBCP
LRTC-PDS
HaLRTC
starfish
dragonfly
people
gulls
flowers
elephants
butterfly
bird
PSNR
missing ratio

SSIM



starfish
dragonfly
people
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bird
SPC-QV
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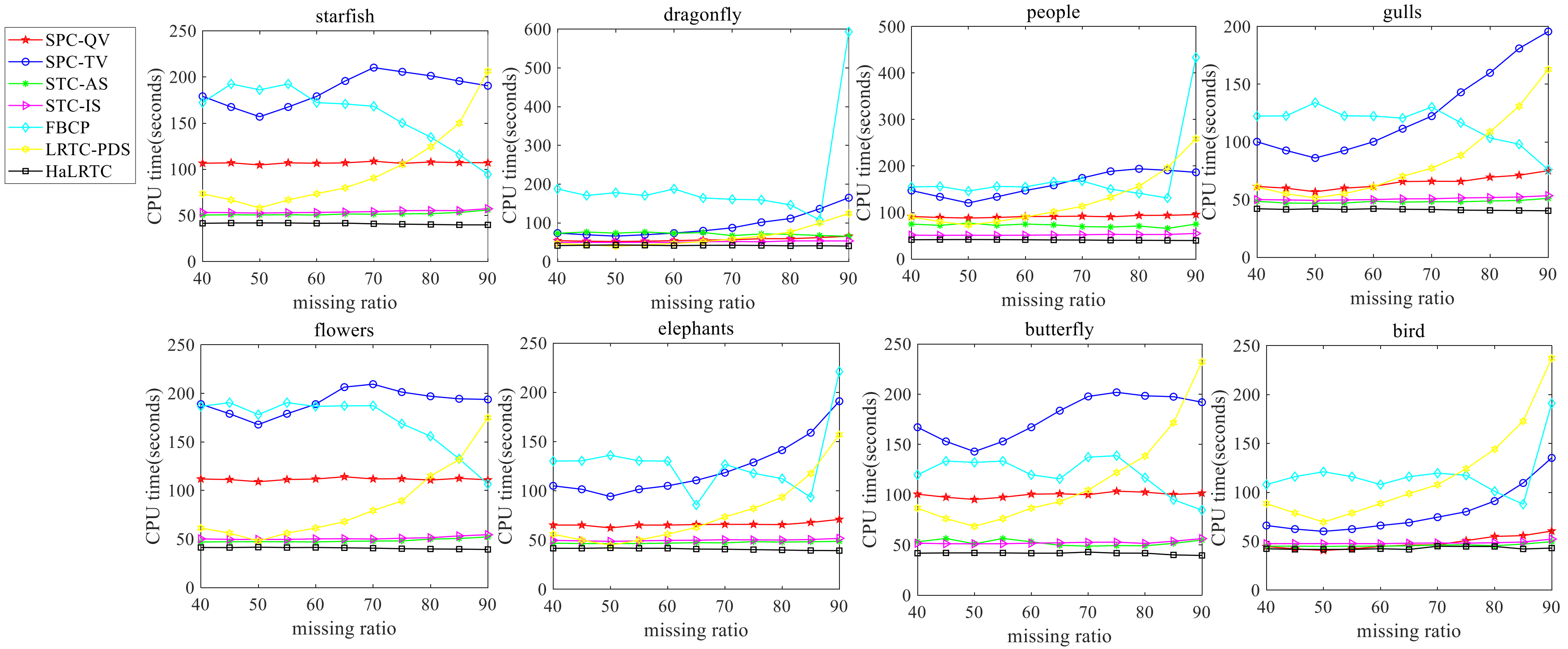
SSIM
missing ratio

CPU time



SPC-QV
SPC-TV
STC-AS
STC-IS
FBCP
LRTC-PDS
HaLRTC
starfish
dragonfly
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gulls
flowers
elephants
butterfly
bird
CPU time(seconds)
missing ratio

Thanks !