Research Statement

Deependra Singh
deeps@sas.upenn.edu

In my graduate work I have focused on the admissibility problem, posed by Schacher in [Sch68], that relates inverse Galois theory to division algebras. I have obtained a complete characterization over number fields in several cases, and partial results in other cases. In addition to continuing to investigate the case of global fields, I am also studying the case of semi-global fields.

1 Introduction

A division algebra over a field K is a finite dimensional associative K-algebra such that all of its non-zero elements have multiplicative inverses (for example, the algebra of quaternions over $\mathbb{R}$). The dimension of a division algebra as a K vector space is always a square [Pie82], and its *index* is the square-root of this dimension. If D is a K-division algebra of index n and center K then a subfield L of D containing K is maximal among all such subfields if and only if its degree over K is n [Pie82]. Such a subfield is called a *maximal subfield* of D. For example, the complex numbers are a maximal subfield of the division algebra of quaternions over $\mathbb{R}$.

Given a field K, the classical inverse Galois problem asks whether or not every finite group appears as the Galois group of some Galois extension of K. With the terminology in the previous paragraph, one can also ask which finite groups G are Galois groups of field extensions L/K such that L is a maximal subfield of a division algebra with center K. Such a group G is called *admissible over* K or K*-admissible*, and the field L is called K*-adequate.*

This connection between inverse Galois theory and division algebras was first explored by Schacher [Sch68]. Like the inverse Galois problem, the question remains open in general. But unlike the inverse Galois problem, the groups that occur in this fashion are generally quite restricted. For example, while every finite group is expected to be realized as a Galois group over $\mathbb{Q}$, a $\mathbb{Q}$-admissible group must be Sylow-metacyclic (a Sylow-metacyclic group is a group whose Sylow subgroups are metacyclic, and a metacyclic group is an extension of a cyclic group by a cyclic group). Nevertheless, every finite group *is* admissible over some number field [Sch68].

While the problem is open in general, including over $\mathbb{Q}$, some results are known. Sonn [Son83] proved the admissibility of solvable Sylow metacyclic groups over $\mathbb{Q}$. Many non-solvable groups with metacyclic Sylow subgroups have also been shown to be admissible over $\mathbb{Q}$ as well as over other classes of number fields, for example [FV87], [FF90], [SS92], [Fei04]. In [HHK11], groups that are admissible over function fields over complete discretely valued fields were characterized using patching techniques.

Over number fields, most results have focused on tamely ramimfied adequate extensions and Sylow metacyclic subgroups [Lie94; Nef13]. In fact, as I note below, a tamely ramified adequate extension must have Sylow metacyclic Galois group. My research concerns both tamely and wildly ramified adequate extensions over global fields, and in particular number fields. I have obtained a complete characterization of admissible groups in several cases, and partial results in others.

In section §2.1, I explain my results over general number fields, and in section §2.2, §2.3 I discuss my results over small degree number fields and abelian number fields, respectively. Finally, I discuss the case of global function fields in section §2.4. I have included extra hypotheses in stating several results where doing so would make the statements simpler. Therefore a number of these results extend to more general situations.

Since the Brauer group is intimately related to the division algebras over a field, it plays a key role in studying admissibility. In the next section I explain Schacher's observation that uses Brauer groups and class field theory to formulate the admissibility problem as a version of the inverse Galois problem with local conditions. In my proofs, I make essential use of the structure of absolute Galois group of local fields, results on embedding problems, and Neukirch's generalization of the Grunwald-Wang theorem.

2 Summary of research results

The equality of period and index over global fields [Pie82] implies that a G-Galois field extension L/K is adequate if and only if $H^2(G, L^*)$ has an element of order $[L : K]$. Using the exactness of $0 \to H^2(G, L^*) \to \bigoplus_{\mathfrak{p}} H^2(D_{\mathfrak{p}}, L^*_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$, Schacher [Sch68] obtained the following arithmetic criterion for the extension L/K to be K-adequate:

The field L is K-adequate if and only if for each rational prime $p \mid |G|$, there are two distinct places $\mathfrak{p}_1, \mathfrak{p}_2$ of K such that the decomposition group $D_{\mathfrak{p}_i}$ contains a p-Sylow subgroup of G.

This formulation makes the admissibility problem a version of inverse Galois problem with local conditions, a problem that is open in general, including for solvable groups. For example, while Shafarevich's construction shows that every solvable group can be realized as a Galois group over a number field, there is no known way to realize the given local extensions [SW98].

2.1 Admissibility over number fields

Starting with p-groups, I have shown

Theorem 1. *Let K be a finite Galois extension of $\mathbb{Q}$, and p an odd prime that is unramified and decomposes in K. Then a p-group G is admissible over K if and only if $d(G) \leq f_p + 1$ where f_p is the residue degree of p.*

Here $d(G)$ is the minimum number of generators of the p-group G, and we say that the prime p *decomposes* in K if the p-adic valuation on $\mathbb{Q}$ extends to at least two inequivalent valuations on K.

In the case that G is admissible and p does not decompose in K (i.e., the p-adic valuation on $\mathbb{Q}$ extends uniquely to K), one of the two primes in Schacher's criterion must have residue characteristic different from p. This forces G to be metacyclic, and the characterization in that case is already known [Lie94].

The proof for the above theorem uses a result of Neukirch [Neu79] that solves the inverse Galois problem with local conditions in this case, along with a result of Shafarevich on the p-part of the absolute Galois group of a local field that doesn't contain any p-th roots of unity. An important remark is that Theorem 1 generalizes to more general solvable groups as well as non-Galois number fields, but the statement is more technical.

Using tensor products of division algebras, one can show that

Lemma 2. *Over a global field, a nilpotent group G is admissible if and only if each of its Sylow subgroups are.*

An adaptation of Schacher's proof over $\mathbb{Q}$ leads to

Lemma 3. *All abelian metacyclic groups are admissible over every global field.*

These two lemmas, combined with the above theorem, give a complete characterization of admissible odd abelian groups with order coprime to the discriminant of the number field. If we denote by G_p the p-Sylow subgroup of an abelian group G, then my precise result is

Theorem 4. *Let K be a finite Galois extension of $\mathbb{Q}$, and G be an odd abelian group with $|G|$ coprime to the discriminant of K. Then G is admissible over K if and only if for each $p \mid |G|$ one of the following conditions holds:*

(i) prime p decomposes in K and $d(G_p) \leq f_p + 1$, or,

(ii) prime p does not decompose in K and $d(G_p) \leq 2$.

Furthermore, if the odd prime p ramifies in the number field K, I have obtained partial results. Once again, if p does not decompose in K, then every K-admissible p-group has to be metacyclic, and the result in that case is already known [Lie94].

In [Sch68], it was shown that if a p-group G is admissible over a degree n number field, then $d(G) \leq (n/2) + 2$. I have proved the following generalization, including an assertion in the converse direction:

Theorem 5. *Let K be a finite Galois extension of $\mathbb{Q}$, and p be an odd rational prime such that $\zeta_p \notin K$, and p decomposes in K. Let G be a p-group. Then*

(i) If $d(G) \leq (e_p f_p/2) + 1$ then G is admissible over K.

(ii) Moreover, if $\zeta_p \notin K_p$, then G is admissible over K if and only if $d(G) \leq e_p f_p + 1$.

Here e_p is the ramification degree and f_p is the residue degree of prime p. The field K_p is the completion of K at any prime over p (all such completions are isomorphic over $\mathbb{Q}_p$ since K is assumed to be Galois over $\mathbb{Q}$).

The proof of this theorem uses a result of Neukirch [Neu79] generalizing the Grunwald-Wang theorem, and the description of absolute Galois groups of local fields as Demuškin groups [NSW13]. Similar to Theorem 1, this result partially extends to more general solvable groups, as well as to non-Galois number fields.

2.2 Admissibility over number fields of small degree

Over $\mathbb{Q}$, it is known that a p-group is admissible if and only if it is metacyclic [Son83]. Over quadratic and cubic number fields, I have shown

Theorem 6. *Let K be a quadratic or cubic number field, and G be an odd p-group for $p \neq [K : \mathbb{Q}]$. Then G is K-admissible if and only if one of the following conditions holds:*

(i) prime p decomposes in K and $d(G) \leq 2$, or,

(ii) prime p does not decompose in K and G is metacyclic.

The proof for the case when p decomposes and $K/\mathbb{Q}$ is Galois follows from Theorem 1. For the case when p decomposes in K but K is not Galois over $\mathbb{Q}$, note that since $[K : \mathbb{Q}] < 4$, there cannot be two or more completions over p that have degree at least two. Consequently there is at most one completion that is not $\mathbb{Q}_p$. This forces a K-admissible p-group to have $d(G) \leq 2$. The proof for the converse direction in this case uses the structure of p-part of the absolute Galois group of local fields [Ser79], together with Neukirch's result on IGP (inverse Galois problem) with local conditions [Neu79].

For the case when p does not decompose in K, we know by Schacher's criterion that the group has to be metacyclic. Since metacyclic groups are quotients of semidirect product of cyclic groups by cyclic groups, and admissibility is closed under quotients [Sch68], it suffices to show that semidirect products of cyclic groups by cyclic groups are admissible over K. Using the structure of tamely ramified Galois extensions of local fields [Ser79], I show that there are two places (in fact, there are infinitely many) of K that realize such a semidirect product as a Galois group. Then I use a result on embedding problems [Neu73] to argue the existence of a global extension with the same Galois group that realizes these local extensions. Consequently, Schacher's criterion shows that these groups are admissible. Note that the formulation as an embedding problem is essential since Neukirch's result [Neu79], which was used in the proof of Theorem 5 above, does not apply if the global field K contains a p-th root of unity.

Note that Liedahl's results on metacyclic p-groups [Lie94], or Neftin's generalization of Liedahl's results [Nef13], can also be used to show the admissibility of metacyclic groups in this case. Furthermore, unlike the odd metacyclic p-groups, 2-metacyclic groups are not always admissible over quadratic number fields. For example, it is known that the dihedral group D_8 of order 8 is not admissible over $\mathbb{Q}(i)$ [Lie94]. Similarly, 3-groups are not always admissible over cubic number fields. For example, using Leidahl's results on admissibility of metacyclic p-groups [Lie94], I have shown that the non-abelian semi-direct product $\mathbb{Z}/9 \rtimes \mathbb{Z}/3$ is not admissible over $\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$.

More generally, I have proved the following

Proposition 7. *For p-prime, the non-abelian semi-direct product $\mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$ is not admissible over the unique degree p number field inside $\mathbb{Q}(\zeta_{p^2})$.*

I have also proved a result similar to Theorem 6 for prime degree number fields. More precisely,

Proposition 8. *Let K be a number field of prime degree q (not necessarily Galois). Let G be an odd p-group for $p \neq q$ and p does not decompose in K, then G is K-admissible if and only if G is metacyclic.*

In the case when $K/\mathbb{Q}$ is Galois, using Theorem 1 we can say more

Corollary 9. *Let K be Galois number field of prime degree q. Let G be an odd p-group for $p \neq q$ and p decomposes in K, then G is K-admissible if and only if $d(G) \leq 2$.*

I have also obtained partial results over degree 4 number fields using similar techniques. Once again, these results partially extend to general solvable groups.

2.3 Admissibility over abelian number fields

Using methods similar to the proof of Theorem 6, I have shown

Theorem 10. *Let K be an abelian number field with conductor m. An odd p-group with $p \nmid m$ is K-admissible if and only if one of the following conditions holds:*

(i) prime p decomposes in K and $d(G) \leq f_p + 1$, or,

(ii) prime p does not decompose in K and G is metacyclic.

Moreover, if $K = \mathbb{Q}(\zeta_{p^r})$, Theorem 10 leaves out only the case of p-groups. Since p does not decompose in K, any admissible p-group must be metacyclic [Sch68]. In the converse direction, it is known that metacyclic p-groups are admissible over $\mathbb{Q}(\zeta_p)$ [Lie94]. Using structure of tamely ramified extensions over local fields and a result of Liedahl [Lie94], I have shown that

Proposition 11. *If $\zeta_{p^2} \in K$ and p does not decompose in K then the non-abelian semi-direct product $\mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$ is not admissible over K.*

This generalizes the result that the dihedral group of order 8 is not admissible over $\mathbb{Q}(i)$.

2.4 Global function fields

Schacher showed that if a group G is admissible over a global function field K of characteristic $p > 0$, then the r-Sylow subgroups of G for $r \neq p$ must be metacyclic [Sch68]. A result of Saltman [Sal77] shows that every p-group is admissible over K. This result together with Lemma 3 yields

Proposition 12. *A finite abelian group G is admissible over a global function field of characteristic $p > 0$ if and only if r-Sylow subgroups for $r \neq p$ are metacyclic.*

3 Plan of future research

3.1 Admissibility of solvable groups over number fields

Let K be a number field. If the rational prime p does not decompose in K, then any admissible p-group must be metacyclic, and this case is fairly well understood [Lie94]. If instead the prime p decomposes in K then the class of admissible p-groups can be larger than metacyclic groups (Theorem 10, for example). In this case, if $\zeta_p \notin K$ then Theorem

1 and Theorem 5, and their more general versions over non-Galois number fields provide a complete characterization of admissible p-groups.

One of the directions in my current research is to study the remaining case, namely, when p decomposes in K and $\zeta_p \in K$. The main difficulty in this case is that the IGP (inverse Galois problem) with local conditions may have a negative answer (for example, Wang's counterexample to Grunwald's original result: there is no degree 8 cyclic extension L of $\mathbb{Q}$ such that its completion L_2 at 2 is the unique degree 8 unramified extension of $\mathbb{Q}_2$). Nevertheless, there are techniques that can yield partial results. These techniques rely on framing Schacher's criterion for admissibility as a series of embedding problems, as done in the proof of Theorem 6.

Going beyond p-groups, Lemma 2 reduces the case of nilpotent groups to p-groups. For general solvable groups, using Schacher's criterion and the techniques used in the case of p-groups, I have obtained sufficient conditions for a solvable group to be admissible in many cases. Once again, the case of $\zeta_p \in K$ remains open in general, and I am currently investigating it. Moreover, unlike the case of p-groups, the question of obtaining necessary conditions for a group to be admissible is more difficult. This is because, in this case, Schacher's criterion says that a p-Sylow subgroup of an admissible group is a *subgroup* of a decomposition group, and not necessarily the entire decomposition group. Consequently, the p-Sylow subgroups are Galois groups over *an extension of* $K_{\mathfrak{p}}$, and not necessarily over $K_{\mathfrak{p}}$.

3.2 Admissible groups and global field extensions

For a given number field, the class of admissible groups tend to be quite restricted. At the same time, every finite group is admissible over some number field [Sch68]. So it is a natural question how the class of admissible groups changes in extensions of number fields, and global fields in general.

For example, consider the extensions $\mathbb{Q} \subseteq K = \mathbb{Q}(\zeta_9) \subseteq L = \mathbb{Q}(\zeta_9, \sqrt{7})$, and let G be the non-abelian semidirect product $\mathbb{Z}/9 \rtimes \mathbb{Z}/3$. By Sonn's result, G is admissible over $\mathbb{Q}$, and Proposition 11 shows that G is *not* admissible over K. On the other hand, using results on embedding problems, it can be shown that G *is* admissible over L. There are two different things happening here. As we go from $\mathbb{Q}$ to K, roots of unity are adjoined, and that forces G to not be admissible. But when we go from K to L, the prime 3 decomposes in L, and that makes it possible for G to be admissible over L.

I am currently working on understanding this phenomenon and how adjoining roots of unity affects the class of admissible groups. So far the evidence suggests that adjoining roots of unity potentially shrinks the class of admissible groups. This is in line with the results of [HHK11], where in the case of semi-global fields, the authors show that presence of roots of unity forces the admissible group to be abelian (Corollary 3.4 in [HHK11]).

3.3 Admissibility over other arithmetically interesting fields

The question of admissibility can be asked over any field. In [HHK11], the authors considered the case of semi-global fields with algebraically closed residue fields. In [RS13], the authors obtained partial results for function fields of p-adic curves. My next research project is to investigate the case of semi-global fields with non-algebraically closed residue field. As in the case of global fields and semi-global fields with algebraically closed residue field, I expect roots of unity to play a crucial role in this case as well.

3.4 Admissibility of non-solvable groups over number fields

In addition to solvable groups, some classes of non-abelian simple groups have also been shown to be admissible over certain number fields [FV87; FF90; Fei04]. If K is a number field, and G is a group, then a common theme in the proofs to show admissibility of G over K is to find explicit polynomials over $K(t_1, \dots, t_n)$ for some n, and argue that it is possible to specialize t_i's to realize G as a Galois group while satisfying the local conditions in Schacher's criterion.

So far, only the Sylow metacyclic groups have been studied. I plan to investigate this theme further, particularly beyond the case of Sylow metacyclic groups. In addition, I also plan to study how realizing Galois groups through Galois representations of geometric objects (such as torsion points on an abelian variety) can be exploited to inform the admissibility problem.

References

[Fei04] Walter Feit. "$\mathrm{PSL}_2(11)$ is admissible for all number fields". In: *Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000).* Springer, Berlin, 2004, pp. 295–299. ISBN: 3-540-00475-0.

[FF90] Paul Feit and Walter Feit. "The K-admissibility of $\mathrm{SL}(2, 5)$". In: *Geom. Dedicata* 36.1 (1990), pp. 1–13.

[FV87] Walter Feit and Paul Vojta. "Examples of some $\mathbf{Q}$-admissible groups". In: *J. Number Theory* 26.2 (1987), pp. 210–226.

[HHK11] David Harbater, Julia Hartmann, and Daniel Krashen. "Patching subfields of division algebras". In: *Trans. Amer. Math. Soc.* 363.6 (2011), pp. 3335–3349.

[Lie94] Steven Liedahl. "Presentations of metacyclic p-groups with applications to K-admissibility questions". In: *J. Algebra* 169.3 (1994), pp. 965–983.

[Nef13] Danny Neftin. "Tamely ramified subfields of division algebras". In: *J. Algebra* 378 (2013), pp. 184–195.

[Neu73] Jürgen Neukirch. "Über das Einbettungsproblem der algebraischen Zahlentheorie". In: *Inventiones mathematicae* 21 (1973), pp. 59–116.

[Neu79] Jürgen Neukirch. "On solvable number fields". In: *Invent. Math.* 53.2 (1979), pp. 135–164.

[NSW13] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields.* Vol. 323. Springer Science & Business Media, 2013.

[Pie82] Richard S Pierce. *The associative algebra.* Springer, 1982.

[RS13] B Surendranath Reddy and Venapally Suresh. "Admissibility of groups over function fields of p-adic curves". In: *Advances in Mathematics* 237 (2013), pp. 316–330.

[Sal77] David J Saltman. "Splittings of cyclic p-algebras". In: *Proceedings of the American Mathematical Society* 62.2 (1977), pp. 223–228.

[Sch68] Murray M. Schacher. "Subfields of division rings. I". In: *J. Algebra* 9 (1968), pp. 451–477.

[Ser79] Jean-Pierre Serre. "Local Fields". In: *Graduate Texts in Mathematics* 67 (1979).

[Son83] Jack Sonn. "$\mathbf{Q}$-admissibility of solvable groups". In: *J. Algebra* 84.2 (1983), pp. 411–419.

[SS92] Murray Schacher and Jack Sonn. "K-admissibility of A_6 and A_7". In: *J. Algebra* 145.2 (1992), pp. 333–338.

[SW98] Alexander Schmidt and Kay Wingberg. "Safarevic's theorem on solvable groups as Galois groups". In: *arXiv preprint math/9809211* (1998).