RESEARCH STATEMENT

DEEPENDRA SINGH

In my graduate work I have focused on the admissibility problem, posed by Schacher in [Sch68], that relates inverse Galois theory to division algebras. I have obtained a complete characterization over number fields in several cases, and partial results in other cases. In addition to continuing to investigate the case of global fields, I am also studying the case of semi-global fields.

1. Introduction

A central division algebra over a field K is a finite dimensional associative K-algebra such that its center is K and all of its non-zero elements have multiplicative inverses (for example, the algebra of quaternions over $\mathbb{R}$). The dimension of a division algebra as a K vector space is always a square [Pie82], and its *index* is the square-root of this dimension. If D is a K-division algebra of index n and center K then a subfield L of D containing K is maximal among all such subfields if and only if its degree over K is n [Pie82]. Such a subfield is called a *maximal subfield* of D. For example, the complex numbers are a maximal subfield of the division algebra of quaternions over $\mathbb{R}$.

Given a field K, the classical inverse Galois problem asks whether or not every finite group appears as the Galois group of some Galois extension of K. With the terminology in the previous paragraph, one can also ask the following question.

Question 1. *Which finite groups G are Galois groups of field extensions L/K such that L is a maximal subfield of a central division algebra over K?*

Such a group G is called *admissible over K* or *K-admissible*, and the field L is called *K-adequate*. By the theory of division algebras, this is equivalent to asking whether there is a G-cross product central division algebra over K. This connection between inverse Galois theory and division algebras was first explored by Schacher [Sch68]. Like the inverse Galois problem, the question remains open in general. But unlike the inverse Galois problem, the groups that occur in this fashion are generally quite restricted. For example, while every finite group is expected to be realized as a Galois group over $\mathbb{Q}$, a $\mathbb{Q}$-admissible group must be Sylow-metacyclic (a Sylow-metacyclic group is a group whose Sylow subgroups are metacyclic, and a metacyclic group is an extension of a cyclic group by a cyclic group). Nevertheless, every finite group *is* admissible over some number field [Sch68].

While the problem is open in general, including over $\mathbb{Q}$, some results are known. Sonn [Son83] proved the admissibility of solvable Sylow metacyclic groups over $\mathbb{Q}$. Many non-solvable groups with metacyclic Sylow subgroups have also been shown to be admissible over $\mathbb{Q}$ as well as over other classes of number fields, for example [FV87], [FF90], [SS92], [Fei04]. In [HHK11], groups that are admissible over function fields over certain complete discretely valued fields were characterized using patching techniques.

Over number fields, most results have focused on tamely ramified adequate extensions and Sylow metacyclic subgroups [Lie94; Nef13]. In fact, as explained in §2.1, a tamely ramified adequate extension must have Sylow metacyclic Galois group. My research concerns both tamely and wildly ramified adequate extensions over global fields, and in particular over number fields. I have obtained a complete characterization of admissible p-groups in several cases, and partial results in others. Since not every non-solvable Sylow metacyclic group is known to be Galois over $\mathbb{Q}$, the problem of completely characterizing admissible groups over number fields remains out of reach at present.

In section §2.1, I explain my results over general number fields and in section §2.2 the results over special class of number fields, including Galois number fields and number fields of degree 2,3, and 4. Finally, I mention a result about global function fields in section §2.3. I have included extra hypotheses in stating several results where doing so would make the statements simpler. Therefore a number of these results extend to more general situations. See [Sin24] for more details about these results. Additionally, in section §2.4, I briefly explain my research work in computer science prior to starting my Ph.D. in mathematics [JAGSB20; SD18; PDS18].

Since the Brauer group is intimately related to the division algebras over a field, it plays a key role in studying admissibility. In the next section I explain Schacher's observation that uses Brauer groups and class field theory to formulate the admissibility problem as a version of the inverse Galois problem with local conditions. In my proofs, I make essential use of the structure of the Galois group of the maximal p-extension of a local field, results on embedding problems, and Neukirch's generalization of the Grunwald-Wang theorem.

2. Summary of research results

The equality of period and index for division algebras over global fields [Pie82] implies that a G-Galois field extension L/K is adequate if and only if $H^2(G, L^*)$ has an element of order $[L : K]$. Using the exactness of $0 \to H^2(G, L^*) \to \bigoplus_{\mathfrak{p}} H^2(D_{\mathfrak{p}}, L^*_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$, Schacher [Sch68] obtained the following arithmetic criterion for the extension L/K to be K-adequate:

A G-Galois field extension L/K is K-adequate if and only if for each rational prime p dividing the order of G, there are two distinct places $\mathfrak{p}_1, \mathfrak{p}_2$ of K such that the decomposition groups corresponding to these places in the field extension L/K contain a p-Sylow subgroup of G.

This formulation makes the admissibility problem a version of inverse Galois problem with local conditions, a problem that is open in general, including for solvable groups. For example, while Shafarevich's construction shows that every solvable group can be realized as a Galois group over a number field, there is no known way to realize the given local extensions [SW98].

2.1. **Admissibility over number fields.**

Schacher observed in [Sch68] that if K is a number field to which the p-adic valuation extends uniquely, then the p-Sylow subgroup of any K-admissible group is necessarily metacyclic. This follows at once from Schacher's criterion noted above. In particular, this is true for the field of rational numbers $\mathbb{Q}$, and so a $\mathbb{Q}$-admissible group must be Sylow-metacyclic.

In the converse direction, Sonn [Son83] proved that every solvable Sylow-metacyclic group is admissible over $\mathbb{Q}$. Admissibility of non-solvable Sylow-metacyclic groups over $\mathbb{Q}$ remains out of reach at present since there are examples of such groups that are not known to be even Galois over $\mathbb{Q}$ [CS81]. It is also known that not every solvable Sylow-metacyclic group is admissible over every number field (e.g., the dihedral group of order 8 is not admissible over $\mathbb{Q}(i)$ [Lie94]). In light of this background, a natural question is:

Can we classify the number fields K for which every solvable Sylow-metacyclic group is K-admissible?

In this direction, Liedahl [Lie94] proved a necessary and sufficient criterion for a *given* odd metacyclic p-group to be admissible over a given number field, and this criterion was later extended by Neftin [Nef13] to solvable Sylow metacyclic groups under the assumption that the adequate extension is tamely ramified. This criterion depends on whether the given group has a specific sort of presentation, and this presentation depends on the number field. Building on the previous work, I give a complete answer to the above question with the following theorem.

We say that a group G is *tamely admissible* over K if an adequate G-Galois extension L/K can be chosen to be tamely ramified over K.

Theorem 2. *Let K be a number field. Then*

(i) A solvable Sylow-metacyclic group is tamely admissible over K if and only if each of its Sylow subgroups are tamely admissible over K.
(ii) Every 2-metacyclic group is tamely admissible over K if and only if K does not contain $i, \sqrt{2}, \sqrt{-2}$.
(iii) Let p be any odd prime, and let α_p be a primitive element of the unique degree p-extension over $\mathbb{Q}$ in $\mathbb{Q}(\zeta_{p^2})/\mathbb{Q}$. Then every p-metacyclic group is tamely admissible over K if and only if $\alpha_p \notin K$.

As a result, we get the following corollary which generalizes Sonn's result about admissibility of solvable Sylow-metacyclic groups over $\mathbb{Q}$:

Corollary 3. *If $K/\mathbb{Q}$ is an abelian extension with square free conductor then every solvable Sylow-metacyclic group is admissible over K.*

If we ask the adequate extension to be tamely ramified then we get the following classification result about cyclotomic fields:

Corollary 4. *Every solvable Sylow-metacyclic group is tamely admissible over a cyclotomic field $\mathbb{Q}(\zeta_m)$ if and only if m is square free.*

Another corollary to Theorem 2 is that for a given number field K, as long as a solvable Sylow-metacyclic group is not divisible by certain primes (which belong to a finite list as in Corollary 5) then the group is K-admissible. More precisely,

Corollary 5. *Let K be a number field. If G is a solvable Sylow metacyclic group such that for each prime p dividing $|G|$, either p is unramified in K or $p \nmid [K : \mathbb{Q}]$, then G is K-admissible.*

Moreover, a corresponding adequate extension can be chosen to be tamely ramified over K.

Note that if we ask the adequate extension to be tamely ramified, then G is necessarily Sylow-metacyclic (by a similar argument as in Theorem 4.1 of [Sch68]). In that case the above result provides a complete characterization of admissible groups whose order is coprime to these finitely many primes.

As remarked above, in the cases not covered by the above proposition, there are indeed pairs (K, G), for K a number field and G a Sylow-metacyclic group, such that G is not admissible over K. In [Fei93] it was shown that the dihedral group of order 8 is not admissible over $\mathbb{Q}(i)$, and the generalized quaternion group of order 16 is not admissible over $\mathbb{Q}(\sqrt{-2})$. In addition, I also show that the semi-dihedral group of order 16 is not admissible over $\mathbb{Q}(\sqrt{2})$. Theorem 2 says that these are in some sense the first examples where admissibility of 2-metacyclic group goes wrong.

Inadmissibility of dihedral group of order 8 over $\mathbb{Q}(i)$ can be seen as a consequence of the following result. We say that a prime p *decomposes* in K if the p-adic valuation on $\mathbb{Q}$ extends to at least two inequivalent valuations on K.

Proposition 6. *Let K be an number field such that $\zeta_{p^n} \in K$ for $n \geq 0$, and p does not decompose in K. Let G be a finite group such that its p-Sylow subgroup is non-abelian of order $\leq p^{n+1}$. Then G is not admissible over K.*

Note that n needs to be at least 2 for this proposition to say something non-trivial since a group of order p or p^2 is necessarily abelian. This proposition provides evidence about how the presence of roots of unity forces admissible groups to be "more abelian". This is in line with the results of [HHK11] where the assumption of algebraically closed residue field for a semi-global field constrains the admissible groups to be abelian.

The above proposition shows that the class of tamely admissible groups may shrink as we go up in a field extension. For example, every metacyclic p-group is admissible over $\mathbb{Q}(\zeta_p)$ (for example by Corollary 4) but the (unique) non-abelian group $\mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$ is not admissible over $\mathbb{Q}(\zeta_{p^2})$ by Proposition 6. This is in contrast to the class of "wildly admissible" groups that may get larger in field extensions [NV13]. For a non-metacyclic p-group to be admissible, the prime p must decompose in the number field, and the corresponding adequate extension must be wildly ramified. Investigating this phenomenon further I show the the following:

Theorem 7. *Let K be a number field. Let p be an odd rational prime that decomposes in K, and is unramified in K. Let $p = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2} \dots \mathfrak{p}_k^{e_m}$ in K such that*

$$[K_{\mathfrak{p}_1} : \mathbb{Q}_p] \geq [K_{\mathfrak{p}_2} : \mathbb{Q}_p] \geq \cdots \geq [K_{\mathfrak{p}_m} : \mathbb{Q}_p].$$

Then a p-group G is K-admissible if and only if $d(G) \leq [K_{\mathfrak{p}_2} : \mathbb{Q}_p] + 1$.

Here $d(G)$ is the minimum number of generators of the p-group G.

The proof for the above theorem uses a result of Neukirch [Neu79] that solves the inverse Galois problem with local conditions in this case, along with a result of Shafarevich on the Galois group of the maximal p-extension of a local field that doesn't contain any p-th roots of unity.

Once again, an important remark is that away from a set of finitely many primes, the admissible p-groups are completely determined. Moreover, a solvable group G for which each prime p dividing the order of G satisfies the above criteria is K-admissible. For example, in case of Galois number fields, we get:

Proposition 8. *Let K be a Galois number field. Let G be an odd solvable group such that for each p dividing $|G|$,*

- *p decomposes in K.*
- *Either p is unramified in K, or $(p-1) \nmid [K : \mathbb{Q}]$.*

- $d(G_p) \leq [K_p : \mathbb{Q}_p] + 1$

Then G is K-admissible.

Here G_p is a p-Sylow subgroups of G (all such subgroups are conjugates and hence isomorphic), and K_p is a completion of K at a prime above p (all such completions are isomorphic over $\mathbb{Q}_p$ since K is assumed to be Galois over $\mathbb{Q}$).

The above theorem provides sufficient conditions for a group to be K-admissible. Unlike the case of rational numbers, the question of necessary conditions remains open for general number fields K once we go beyond p-groups and allow wildly ramified adequate extensions. But in some special cases the above conditions are also necessary. For example, in the case of nilpotent groups we can say more due to the following lemma which follows from taking the tensor products of appropriate division algebras:

Lemma 9. *A nilpotent group G is admissible over a global field if and only if all of its Sylow subgroups are.*

This leads to the following result:

Corollary 10. *Let K be a finite Galois extension of $\mathbb{Q}$, and G be an odd nilpotent group with $|G|$ coprime to the discriminant of K. Then G is admissible over K if and only if for each $p \mid |G|$ one of the following conditions holds:*

(i) prime p decomposes in K and $d(G_p) \leq f_p + 1$, or,

(ii) prime p does not decompose in K and G_p is metacyclic.

For a given number field K, the above results potentially leave out a finite set of primes for admissibility of p-groups. If such a prime p does not decompose in K then the Liedahl conditions [Lie94] provide a characterization. On the other hand, if such a prime p decomposes in K then I have obtained partial results. In [Sch68], it was shown that if a p-group G is admissible over a degree n Galois number field, then $d(G) \leq (n/2) + 2$. I have strengthened this result as follows:

Theorem 11. *Let K be a finite Galois extension of $\mathbb{Q}$, and p be an odd rational prime such that $\zeta_p \notin K$, and p decomposes in K. Let G be a p-group. Then*

- *If $\zeta_p \notin K_p$ then G is K-admissible if and only if $d(G) \leq [K_\mathfrak{p} : Q_p] + 1$.*
- *If $\zeta_p \in K_p$ then G is K-admissible if and only if G can be generated by $[K_\mathfrak{p} : Q_p] + 2$ many generators $x_1, x_2, \ldots, x_n$ satisfying the relation*

$$x_1^{p^s}[x_1, x_2][x_3, x_4] \ldots [x_{n-1}, x_n] = 1$$

where p^s is such that $\zeta_{p^s} \in K_p$ but $\zeta_{p^{s+1}} \notin K_p$.

Note that since we assumed p to be an odd prime, if $\zeta_p \in K_p$ then $n = [K_p : \mathbb{Q}_p]$ is divisible by $(p-1)$. In particular, it is an even number and the above description makes sense.

As a corollary, we get a result in the converse direction of [Sch68]:

Corollary 12. *Let K be a finite Galois extension of $\mathbb{Q}$, and p be an odd rational prime such that $\zeta_p \notin K$, and p decomposes in K. If G is a p-group with $d(G) \leq [K_\mathfrak{p} : \mathbb{Q}_p]/2 + 1$ then G is K-admissible.*

The proof of the above theorem uses a result of Neukirch [Neu79] generalizing the Grunwald-Wang theorem, and the description of the Galois group of maximal p-extension of local fields as Demuškin groups [NSW13], i.e., Poincaré groups of dimension 2. Presentation of these groups have a striking similarity to that of pro-p completion of fundamental groups of topological surfaces, and I am currently studying whether that analogy in the sense of arithmetic topology can be useful in providing an alternative description of admissible groups in this case. Similar to the Prop 8, this result partially extends to more general solvable groups, as well as to non-Galois number fields.

2.2. **Over special classes of number fields.** This section contains results after specializing to certain classes of number fields, such as Galois number fields, number fields of degree 2^n and odd degree over $\mathbb{Q}$, and finally the cyclotomic fields.

As a corollary to Theorem 2 and Theorem 7, for the Galois number fields we get the following result. Here f_p is the residue degree of prime p.

Corollary 13. *Let K be a Galois number field. An odd p-group with p coprime to the discriminant of $K|\mathbb{Q}$ is K-admissible if and only if one of the following conditions holds:*

(i) prime p decomposes in K and $d(G) \leq f_p + 1$, or,
(ii) prime p does not decompose in K and G is metacyclic.

Moreover, if $K = \mathbb{Q}(\zeta_{l^r})$ for l a prime, Corollary 13 leaves out only the case of l-groups. Since l does not decompose in K, any admissible l-group must be metacyclic [Sch68]. In the converse direction, while every metacyclic l-group is known to be admissible over $\mathbb{Q}(\zeta_l)$ [Lie94], it follows from Prop 6 that there are metacyclic l-groups that are not admissible over $\mathbb{Q}(\zeta_{l^r})$ for $r \geq 2$.

Similar to the previous corollary,

Corollary 14. *Let K be a Galois number field of degree 2^n, and G be an odd p-group. Then the following assertions hold:*

(i) If p does not decompose in K, then G is K-admissible if and only if G is metacyclic.
(ii) If p decomposes in K, and either $(p-1) \nmid [K_p : \mathbb{Q}_p]$ or p is unramified in K then G is K-admissible if and only if $d(G) \leq [K_p : \mathbb{Q}_p] + 1$.

Note that in order for $(p-1)$ to divide the local degree $[K_p : \mathbb{Q}_p]$, p must be a Fermat prime and smaller than or equal to $[K_p : \mathbb{Q}_p]/2$. At present, there are only 5 Fermat prime known and this list is conjectured to be exhaustive.

Since every quadratic extension is automatically Galois, we can use the above corollary. Moreover, there are no exceptional Fermat primes in that case, and so we get:

Corollary 15. *Let K be a quadratic number field, and G be an odd p-group. Then G is K-admissible if and only if one of the following conditions holds:*

(i) prime p decomposes in K and $d(G) \leq 2$, or,
(ii) prime p does not decompose in K and G is metacyclic.

The case of quartic number fields is more involved. First, the field can be non-Galois, and second, there is the possible Fermat prime 3 even if the field is Galois over $\mathbb{Q}$. When the field is non-Galois, I look at the various possible splittings of primes, and argue that the local field cannot contain p-th roots of unity. The result is

Proposition 16. *Let K be a quartic number field. Then a p-group G for $p \neq 2, 3$ is admissible over K if and only if one of the following two conditions hold:*

(i) p does not decompose in K, and G is metacyclic.
(ii) p decomposes in K, and $d(G) \leq \min_{\mathfrak{p}|p}([K_{\mathfrak{p}} : \mathbb{Q}_p]) + 1$.

Similar to Corollary 14, the Galois number fields of odd degree are another special class of number fields. My result is:

Corollary 17. *Let K be a Galois number field of odd degree, and G be an odd p-group. Then the following assertions hold:*

(i) If p does not decompose in K and either $p \nmid [K : \mathbb{Q}]$ or p is unramified in K, then G is K-admissible if and only if it is metacyclic.
(ii) If p decomposes in K, then G is K-admissible if and only if $d(G) \leq [K_p : \mathbb{Q}_p] + 1$.

Similar to the quartic case, arguing about the various splittings of prime p, I show the following result about cubic number fields.

Proposition 18. *Let K be a cubic number field, and G be a p-group for $p \neq 2, 3$. Then G is K-admissible if and only if one of the following conditions holds:*

(i) prime p decomposes in K and $d(G) \leq 2$, or,
(ii) prime p does not decompose in K and G is metacyclic.

Note that once again we cannot rely entirely on Corollary 17 since the number field K may not be Galois. The exceptional case of $p = 3$ in Corollary 18 and more generally the case of $p \mid [K : \mathbb{Q}]$ in Corollary 17 can have a more involved description of admissible p-groups. For example, using Liedahl's criterion about admissibility of metacyclic p-groups I show that

Proposition 19. *For p-prime, the non-abelian semi-direct product $\mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$ is not admissible over the unique degree p number field inside $\mathbb{Q}(\zeta_{p^2})$.*

In particular, $\mathbb{Z}/9 \rtimes \mathbb{Z}/3$ is not admissible over $\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$.

2.3. **Global function fields.** Let K be a global function field K of characteristic $p > 0$. Schacher showed that if a group G is admissible K then the l-Sylow subgroups of G for $l \neq p$ must be metacyclic [Sch68]. For the case when $l = p$, a result of Saltman [Sal77] implies that every p-group is admissible over K. As metacyclic abelian l-groups are always admissible over global fields, we get

Proposition 20. *A finite abelian group G is admissible over a global function field of characteristic $p > 0$ if and only if l-Sylow subgroups for $l \neq p$ are metacyclic.*

We also have an analogue of Proposition 6 for the case of global function fields.

Proposition 21. *Let K be an global function field of characteristic $p > 0$. Let $l \neq p$ be a rational prime such that $\zeta_{l^n} \in K$ for $n \geq 0$. Let G be a finite group such that its l-Sylow subgroup is non-abelian of order $\leq l^{n+1}$. Then G is not admissible over K.*

For example, as a consequence of this proposition, the dihedral group of order 8 is not admissible over global function fields of characteristic $p \equiv 1 \pmod 4$ (e.g., curves over $\mathbb{F}_5$). More generally, if the constant field $\mathbb{F}_q$ of the global function field K is such that $q \equiv 1 \pmod{l^2}$ for some prime l then the (unique) non-abelian group $\mathbb{Z}/l^2\mathbb{Z} \rtimes \mathbb{Z}/l\mathbb{Z}$ is not admissible over K.

2.4. **Other research works.** Before starting my doctoral training in mathematics, my work involved applying statistical methods to various problems. I will briefly describe here three of my projects that led to publications [JAGSB20; SD18; PDS18].

The first project concerns an important molecular biological process called RNA splicing. While various machine learning methods have been widely adopted in biomedical sciences for prediction tasks, some of the most successful methods (for e.g., deep learning methods) lack mechanisms to explain these predictions in terms of inputs. In this collaborative project, we designed several statistical methods that can be applied to general deep learning models to find partial explanations for various predictions, and used these methods to identify a new protein important to liver-specific RNA splicing [JAGSB20].

The other two projects involve using the user search data from an e-commerce search engine to inform product recommendations and inventory management. The first project involved designing methods inspired from information theory to improve the search recommendation framework, yielding significant revenue increase according to standard hypothesis testing methodology ("A/B testing") [SD18]. The other project involved devising probabilistic methods based on clustering and community detection techniques to utilize the user search data to identify subsets of product categories that are "similar" to each other [PDS18].

3. Plan of future research

3.1. **Admissibility of solvable groups over number fields.** For the question of admissibility of p-groups (and so also of nilpotent groups) over number fields, the remaining case that is not well understood is when $\zeta_p \in K$ and p decomposes in K. The main difficulty in this case is that the IGP (inverse Galois problem) with local conditions may have a negative answer (for example, Wang's counterexample to Grunwald's original result: there is no degree 8 cyclic extension L of $\mathbb{Q}$ such that its completion L_2 at 2 is the unique degree 8 unramified extension of $\mathbb{Q}_2$). Nevertheless, it is a natural question whether the class of admissible p-groups can be described in this case, partially if not completely. A promising approach that I am pursuing is to frame Schacher's criterion for admissibility as a series of embedding problems, as done in the proof of Theorem 2.

As remarked after Proposition 8, once we go beyond nilpotent groups, the question of necessary conditions for a group to be admissible is basically wide open. In this case Schacher's criterion says that a p-Sylow subgroup of an admissible group is a *subgroup* of a decomposition group, and not necessarily the full decomposition group. Consequently, the p-Sylow subgroups are Galois groups over *an extension of* $K_\mathfrak{p}$, and not necessarily over $K_\mathfrak{p}$ (here $\mathfrak{p} \mid p$).

3.2. **How does admissibility change in extension of number fields?** For a given number field, the class of admissible groups tend to be quite restricted. At the same time, every finite group is admissible over some number field [Sch68]. So a natural question is to understand how the class of admissible groups changes in extensions of number fields, and more generally in extension of global fields.

For example, consider the extensions $\mathbb{Q} \subseteq K = \mathbb{Q}(\zeta_9) \subseteq L = \mathbb{Q}(\zeta_9, \sqrt{7})$, and let G be the non-abelian semidirect product $\mathbb{Z}/9 \rtimes \mathbb{Z}/3$. By Sonn's result, G is admissible over $\mathbb{Q}$, and Proposition 6 shows that G is *not* admissible over K. On the other hand, using results on embedding problems, it can be shown that G *is* admissible over L. There are two different things happening here. As we go from $\mathbb{Q}$ to K, roots of unity are adjoined, and that forces

G to not be admissible. But when we go from K to L, the prime 3 decomposes in L, and that makes it possible for G to be admissible over L. I am working on understanding this phenomenon and how adjoining roots of unity affects the class of admissible groups.

A related question is whether admissibility determines the number field; namely, for two different number fields the class of admissible groups is also different. This was first investigated by Sonn [Son85]. and the question of whether admissibility determines the degree of the number field is currently open. The same question can also be asked over global function fields.

3.3. **Admissibility over other arithmetically interesting fields.** The question of admissibility can be asked over any field. In [HHK11], the authors considered the case of semi-global fields with algebraically closed residue fields. In [RS13], the authors obtained partial results for function fields of p-adic curves. My next research project is to investigate the case of semi-global fields with non-algebraically closed residue field. As in the case of global fields and semi-global fields with algebraically closed residue field, I expect roots of unity to play a crucial role in this case as well.

3.4. **Admissibility of non-solvable groups over number fields.** In addition to solvable groups, some classes of non-abelian simple groups have also been shown to be admissible over certain number fields [FV87; FF90; Fei04]. If K is a number field, and G is a group, then a common theme in the proofs to show admissibility of G over K is to find explicit polynomials over $K(t_1, \ldots, t_n)$ for some n, and argue that it is possible to specialize t_i's to realize G as a Galois group while satisfying the local conditions in Schacher's criterion.

I plan to investigate this theme further, particularly beyond the case of Sylow metacyclic groups. In addition, I also plan to study how realizing Galois groups through Galois representations of geometric objects (such as torsion points on an abelian variety) can be exploited to inform the admissibility problem.

References

[CS81] D. Chillag and J. Sonn. "Sylow-metacyclic groups and **Q**-admissibility". In: *Israel J. Math.* 40.3-4 (1981), pp. 307–323.

[Fei04] W. Feit. "$\mathrm{PSL}_2(11)$ is admissible for all number fields". In: *Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000)*. Springer, Berlin, 2004, pp. 295–299.

[Fei93] W. Feit. "The K-admissibility of $2A_6$ and $2A_7$". In: *Israel J. Math.* 82.1-3 (1993), pp. 141–156.

[FF90] P. Feit and W. Feit. "The K-admissibility of $\mathrm{SL}(2, 5)$". In: *Geom. Dedicata* 36.1 (1990), pp. 1–13.

[FV87] W. Feit and P. Vojta. "Examples of some **Q**-admissible groups". In: *J. Number Theory* 26.2 (1987), pp. 210–226.

[HHK11] D. Harbater, J. Hartmann, and D. Krashen. "Patching subfields of division algebras". In: *Trans. Amer. Math. Soc.* 363.6 (2011), pp. 3335–3349.

[JAGSB20] A. Jha, J. K. Aicher, M. R. Gazzara, D. Singh, and Y. Barash. "Enhanced integrated gradients: improving interpretability of deep learning models using splicing codes as a case study". In: *Genome biology* 21 (2020), pp. 1–22.

[Lie94] S. Liedahl. “Presentations of metacyclic p-groups with applications to K-admissibility questions”. In: *J. Algebra* 169.3 (1994), pp. 965–983.
[Nef13] D. Neftin. “Tamely ramified subfields of division algebras”. In: *J. Algebra* 378 (2013), pp. 184–195.
[Neu79] J. Neukirch. “On solvable number fields”. In: *Invent. Math.* 53.2 (1979), pp. 135–164.
[NSW13] J. Neukirch, A. Schmidt, and K. Wingberg. *Cohomology of number fields*. Vol. 323. Springer Science & Business Media, 2013.
[NV13] D. Neftin and U. Vishne. “Realizability and admissibility under extension of p-adic and number fields”. In: *Doc. Math.* 18 (2013), pp. 359–382.
[PDS18] N. Pradhan, V. Deolalikar, and D. Singh. “Islands of interest: Mining concentrations of user search intent over e-commerce product categories”. In: *2018 IEEE International Conference on Big Data (Big Data)*. IEEE. 2018, pp. 3717–3722.
[Pie82] R. S. Pierce. *The associative algebra*. Springer, 1982.
[RS13] B. S. Reddy and V. Suresh. “Admissibility of groups over function fields of p-adic curves”. In: *Advances in Mathematics* 237 (2013), pp. 316–330.
[Sal77] D. J. Saltman. “Splittings of cyclic p-algebras”. In: *Proceedings of the American Mathematical Society* 62.2 (1977), pp. 223–228.
[Sch68] M. M. Schacher. “Subfields of division rings. I”. In: *J. Algebra* 9 (1968), pp. 451–477.
[SD18] D. Singh and V. Deolalikar. “Automating relevance banding in e-commerce search using click model”. In: *2018 IEEE International Conference on Big Data (Big Data)*. IEEE. 2018, pp. 3793–3797.
[Sin24] D. Singh. *Admissible groups over global fields*. 2024 manuscript, available at https://deependra230.github.io/pdf/admissibility.pdf, 2024.
[Son83] J. Sonn. “$\mathbf{Q}$-admissibility of solvable groups”. In: *J. Algebra* 84.2 (1983), pp. 411–419.
[Son85] J. Sonn. “On equivalence of number fields”. In: *Israel J. Math.* 52.3 (1985), pp. 239–244.
[SS92] M. Schacher and J. Sonn. “K-admissibility of A_6 and A_7”. In: *J. Algebra* 145.2 (1992), pp. 333–338.
[SW98] A. Schmidt and K. Wingberg. “Safarevic’s theorem on solvable groups as Galois groups”. In: *arXiv preprint math/9809211* (1998).