ADMISSIBLE GROUPS OVER GLOBAL FIELDS

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Abstract. Given a field K, one may ask which finite groups are Galois groups of field extensions L/K such that L is a maximal subfield of a K-central division algebra. This connection between inverse Galois theory and division algebras was first explored by Schacher in 1960s. In this manuscript we consider this problem when K is a global field, and give a complete characterization of such groups in some cases, and partial results in other cases.

Introduction

A *central simple algebra* over a field K is a finite dimensional associative K-algebra such that its center is K and it has no non-trivial two-sided ideals. It is called a central division algebra if every non-zero element is a unit (for example, the algebra of quaternions over $\mathbb{R}$). The dimension of a central simple algebra as a K vector space is always a square [Pie82], and the square-root of this dimension is called its *index*. If D is a central K-division algebra of index n then a subfield L of D containing K is maximal among all such subfields if and only if its degree over K is n [Pie82]. Such a subfield is called a *maximal subfield* of D. For example, the complex numbers are a maximal subfield of the division algebra of quaternions over $\mathbb{R}$.

Given a field K, the classical inverse Galois problem asks whether or not every finite group appears as the Galois group of some Galois extension of K. With the terminology in the previous paragraph, one can also ask the following question.

Question 0.1. Which finite groups G are Galois groups of field extensions L/K such that L is a maximal subfield of a central division algebra over K?

Such a group G is called *admissible over* K or K*-admissible*, and the field L is called K*-adequate*. This connection between inverse Galois theory and division algebras was first explored by Schacher [Sch68]. In this paper we prove results about which groups are admissible over a given global field. See the next section for discussion concerning motivation for this problem.

Over number fields, most results have focused on tamely ramified adequate extensions and Sylow metacyclic subgroups [Lie94], [Nef13] (Sylow-metacyclic groups are those whose Sylow subgroups are metacyclic). Our results concern both tamely and wildly ramified adequate extensions. For tamely ramified adequate extensions, we extend Sonn's result [Son83] and characterize number fields over which every solvable Sylow-metacyclic group is

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tamely admissible (see Definition 2.5 for the term "tamely admissible"). More precisely, we show the following result in Theorem 2.15:

Theorem. *Let K be a number field. Then*

(i) A solvable Sylow-metacyclic group is tamely admissible over K if and only if each of its Sylow subgroups are tamely admissible over K.
(ii) Every 2-metacyclic group is tamely admissible over K if and only if K does not contain $i, \sqrt{2}, \sqrt{-2}$.
(iii) Let p be any odd prime, and let α_p be a primitive element of the unique degree p-extension over $\mathbb{Q}$ in $\mathbb{Q}(\zeta_{p^2})/\mathbb{Q}$. Then every p-metacyclic group is tamely admissible over K if and only if $\alpha_p \notin K$.

While a $\mathbb{Q}$-admissible group is necessarily Sylow-metacyclic (Theorem 4.1 of [Sch68]), it is also known that for any given finite group G there is some number field K over which G is admissible (Theorem 9.1 of [Sch68]). So a natural problem is to understand how admissible groups behave as we go to higher degree number fields. As we will see, if the admissible group is not Sylow-metacyclic then any corresponding adequate extension must be wildly ramified. We investigate this phenomenon and the following theorem describes a key result for admissibility of p-groups in this context (see Theorem 3.8):

Theorem. *Let K be a finite Galois extension of $\mathbb{Q}$, and p be an odd rational prime such that $\zeta_p \notin K$, and p decomposes in K. Let G be a p-group. Then*

- *If $\zeta_p \notin K_p$ then G is K-admissible if and only if $d(G) \leq [K_\mathfrak{p} : \mathbb{Q}_p] + 1$.*
- *If $\zeta_p \in K_p$ then G is K-admissible if and only if G can be generated by $[K_\mathfrak{p} : \mathbb{Q}_p] + 2$ many generators $x_1, x_2, \ldots, x_n$ satisfying the relation*

$$x_1^{p^s}[x_1, x_2][x_3, x_4] \ldots [x_{n-1}, x_n] = 1$$

where p^s is such that $\zeta_{p^s} \in K_p$ but $\zeta_{p^{s+1}} \notin K_p$.

Here $d(G)$ denotes the minimum number of generators of G.

The admissibility problem in the general case is open even in the case of p-groups. The key challenge seems to be to handle the case when $\zeta_p \in K$. But if we narrow our scope to special classes of number fields, more can be said. The following result characterizes the admissibility of odd p-groups over quadratic number fields (see Corollary 4.4 for this assertion, and Definition 2.4 for the term "decompose"):

Corollary. *Let K be a quadratic number field, and G be an odd p-group for some rational prime p. Then G is K-admissible if and only if one of the following conditions holds:*

(i) prime p decomposes in K and $d(G) \leq 2$, or,
(ii) prime p does not decompose in K and G is metacyclic.

Analogous results for number fields of degree 3 or 4 over $\mathbb{Q}$ appear in Propositions 4.8 and 4.6.

This paper is organized as follows. Section 1 provides additional motivation and context for the admissibility problem. In Section 2, we discuss admissibility of Sylow-metacyclic group over number fields, and characterize number fields for which solvable Sylow-metacyclic subgroups are admissible, extending Sonn's result. Section 3 goes beyond Sylow-metacyclic groups and studies how degree of the number field influences the class of admissible groups.

In Section 4, we specialize to special classes of number fields where we can make stronger statements, including Galois number fields, cyclotomic number fields, and number fields of degrees 2,3, and 4. Finally, Section 5 discusses the situation over global function fields. We sometimes include extra hypotheses in stating results where doing so would make the statements simpler, and indicate how the results extend to more general situations.

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1. Background and Motivation

The following observations provide motivation for studying the admissibility problem.

(i) Cross product algebras provide an explicit way to work with central simple algebras over a field. More specifically, each Brauer class $\alpha \in \mathrm{Br(K)}$ has a representative central simple algebra which is a G-cross product algebra over K for some finite group G, but a division algebra need not be a cross product algebra in general. On the other hand, essentially by definition, a finite group G is admissible over (a field) K if and only if there is a G-cross product division algebra over K.

(ii) Let K be a field such that $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$ for every Brauer class $\alpha \in \mathrm{Br}(K)$ (for example, a global field or a local field). Let L/K be a finite G-Galois extension with $n = [L : k]$. Then L is K-adequate if and only if the n-torsion abelian group $H^2(G, L^\star)$ has an element of order exactly equal to n (Proposition 2.1 of [Sch68]).

(iii) Let K be a field and let $f(x) \in K[x]$ be an irreducible polynomial. One may ask whether there exists a (finite dimensional) central division algebra over K containing a root α of $f(x)$. If $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$ for every $\alpha \in \mathrm{Br}(K)$, then such a division algebra exists if and only if the Galois closure of $K(\alpha)$ is a K-adequate extension (follows from Proposition 2.2 of [Sch68]). This was Schacher's motivation in the original paper to study the admissibility problem.

In light of Question 0.1, the admissibility problem can be thought of as a non-commutative version of the inverse Galois problem. In particular, a K-admissible finite group first needs to be a Galois group over K. Thus if K has no non-trivial Galois groups (e.g., K is separably closed), then no non-trivial group is admissible over K. Similarly, if $\mathrm{Br}(K) = 0$ then there are no non-trivial admissible groups over K since there are no non-trivial K-central division algebras. This is true in particular for C_1 fields (quasi-algebraically closed fields), including the following in addition to separably closed fields.

(i) finite fields;
(ii) function field of a smooth curve over an algebraically closed field, e.g., $\mathbb{C}(t)$;
(iii) a complete discretely valued field with an algebraically closed residue field, e.g., $\mathbb{C}((t))$;
(iv) maximal unramified extension of a complete discretely valued field with a perfect residue field, e.g., $\mathbb{Q}_p^{\mathrm{ur}}$.

Every finite group is known to be Galois over fields of type (ii) in the above list (by [Dou64] in char 0, and [Har84] in char $p > 0$). This shows that even if *every* finite group is Galois over a field K, there may not be any non-trivial groups admissible over K. On the other hand, if K is a local field, then every finite group which is Galois over K is also admissible over K. In fact, the following stronger statement is true.

Proposition 1.1. *If K is a local field, then every finite Galois extension L/K is K-adequate.*

To see this, note that since period equals index for local fields, L is K-adequate if and only if $H^2(G, L^\star)$ has an element of order $[L : K]$ (Proposition 2.1 of [Sch68]). But $H^2(G, L^\star)$ is cyclic of order $[L : K]$ for a local field K, and this completes the argument.

Like the inverse Galois problem, the admissibility problem remains open in general. But unlike the inverse Galois problem, the groups that occur in this fashion are often quite restricted. For example, while every finite group is expected to be realized as a Galois group over $\mathbb{Q}$, by Theorem 4.1 of [Sch68] a $\mathbb{Q}$-admissible group must be Sylow-metacyclic (a metacyclic group is an extension of a cyclic group by another cyclic group). On the other hand, every finite group *is* admissible over some number field [Sch68].

While the problem is open in general, including over $\mathbb{Q}$, some results are known. Sonn [Son83] proved the admissibility of solvable Sylow metacyclic groups over $\mathbb{Q}$. Many non-solvable groups with metacyclic Sylow subgroups have also been shown to be admissible over $\mathbb{Q}$ as well as over other classes of number fields, for example [FV87], [FF90], [SS92], [Fei04]. Since not every non-solvable Sylow metacyclic group is known to be even Galois over $\mathbb{Q}$ [CS81], the problem of completely characterizing admissible groups over number fields remains out of reach at present. In [HHK11], groups that are admissible over function fields over certain complete discretely valued fields were characterized using patching techniques.

Since the Brauer group is intimately related to division algebras over a field, it plays a key role in studying admissibility. Let K be a global field, and L/K be a G-Galois extension for some finite group G. By Proposition 2.1 of [Sch68], L is K-adequate if and only if $H^2(G, L^\star)$ has an element of order exactly equal to $[L : K]$. Using this observation and the exact sequence

$$0 \to H^2(G, L^*) \to \bigoplus_{\mathfrak{p}} H^2(D_{\mathfrak{p}}, L^*_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z} \to 0$$

from class field theory, Schacher [Sch68] obtained the following arithmetic criterion for the extension L/K to be K-adequate:

Criterion 1.2 (Schacher's Criterion)**.** The G-Galois field extension L/K is K-adequate if and only if for each rational prime p dividing the order of G, there are two distinct places $\mathfrak{p}_1, \mathfrak{p}_2$ of K such that the decomposition groups corresponding to these places in the field extension L/K contain a p-Sylow subgroup of G.

This formulation poses the admissibility problem over global fields as a refinement of the inverse Galois problem with extra local conditions, a problem that is open in general, including for solvable groups. For example, while Shafarevich's construction shows that every solvable group can be realized as a Galois group over any number field, there is no known way to realize the given local extensions [SW98]. Grunwald-Wang theorem (Theorem 5 of Chapter 10 in [AT68]) was the first result of this kind for cyclic Galois extensions, and the most far reaching result is Neukirch's generalization of the Grunwald-Wang theorem to solvable groups of order coprime to roots of unity in the global field (Theorem 9.5.9 of [NSW13]). We make extensive use of this result in addition to results on embedding problems.

Observe that if the group G is a p-group for some rational prime p then in Schacher's criterion above the decomposition groups corresponding to places $\mathfrak{p}_1, \mathfrak{p}_2$ need to be the whole group G. In this sense, the structure of the Galois group of the maximal p-extension of a local field yields important insights into the admissibility problem.

2. Sylow metacyclic groups

Schacher observed in [Sch68] that if K is a number field to which the p-adic valuation extends uniquely, then the p-Sylow subgroup of any K-admissible group is necessarily metacyclic. This follows at once from Schacher's criterion noted above. In particular, this is true for the field of rational numbers $\mathbb{Q}$, and so a $\mathbb{Q}$-admissible group must be Sylow-metacyclic.

In the converse direction, Sonn [Son83] proved that every solvable Sylow-metacyclic group is admissible over $\mathbb{Q}$. As noted above, there are examples of non-solvable Sylow metacyclic groups that are not even known to be Galois over $\mathbb{Q}$, so the converse direction is open for non-solvable groups. It is also known that not every solvable Sylow-metacyclic group is admissible over every number field (e.g., the dihedral group of order 8 is not admissible over $\mathbb{Q}(i)$ [Fei93]). In light of this background, a natural question is:

Can we classify the number fields K for which every solvable Sylow-metacyclic group is K-admissible?

In this direction, Liedahl proved a necessary and sufficient criterion for a *given* odd metacyclic p-group to be admissible over a given number field (Theorem 30 of [Lie94]), and this criterion was later extended by Neftin to solvable Sylow metacyclic groups under the assumption that the adequate extension is tamely ramified (Theorem 1.3 of [Nef13]). This criterion depends on whether the given group has a specific sort of presentation, and this presentation depends on the number field. Building on this work, we give a complete answer to the above question in this section in Theorem 2.15.

We first need some lemmas. The following lemma is a well-known result and follows from a group theory argument. We include a proof for completeness.

Lemma 2.1. *Every metacyclic group G is a quotient of a semidirect product G' of two cyclic groups. Moreover, if G is a p-group for some prime number p then G' can be chosen to be a p-group.*

Proof. Let G be a metacyclic group with presentation

$$\langle x, y \mid x^e = 1, y^f = x^i, yxy^{-1} = x^q \rangle.$$

Let r be the order of y in G. Since $x = y^r x y^{-r} = x^{q^r}$, we have $q^r \equiv 1 \bmod(e)$. This allows us to define the semidirect product $G' = \mathbb{Z}/e\mathbb{Z} \rtimes \mathbb{Z}/r\mathbb{Z}$ with presentation

$$\langle \tilde{x}, \tilde{y} \mid \tilde{x}^e = 1, \tilde{y}^r = 1, \tilde{y}\tilde{x}\tilde{y}^{-1} = \tilde{x}^q \rangle.$$

Mapping $\tilde{x} \to x, \tilde{y} \to y$ defines a surjective group homomorphism $G' \twoheadrightarrow G$.

It is clear from the construction that if G is a p-group for some rational prime p then so is G'. $\square$

The following result is a consequence of Proposition 2.2 of [Sch68].

Lemma 2.2. *Let K be a field such that per(α) = ind(α) for every $\alpha \in Br(K)$ (e.g., a global field). If G is admissible (tamely admissible) over K and $N \trianglelefteq G$ is a normal subgroup then G/N is admissible (tamely admissible, respectively) over K.*

The following Lemma shows that the presence of roots of unity constrains the tamely ramified admissible groups to be "more abelian".

Lemma 2.3. *Let k be a non-archimedean local field and l a prime different from the residue characteristic of k. If $\zeta_{l^n} \in k$ for some $n \geq 0$, then any Galois l-extension of degree dividing l^{n+1} is necessarily abelian.*

Proof. Let $k' \,|\, k$ be a G-Galois extension of degree d such that $d \,|\, l^{n+1}$. Let e be the ramification degree and f be the residue degree for this extension. We have $ef = n$.

Since the residue characteristic of k is different from l, the extension $k' \,|\, k$ is tamely ramified. If $e = l^{n+1}$ then the extension $k' \,|\, k$ is totally and tamely ramified, and therefore it is a cyclic extension (Corollary 1 to Proposition 4.1 in [Ser79]). So without loss of generality we can assume that $e \,|\, l^n$.

Let $m \,|\, k$ be the maximal unramified extension inside $k' \,|\, k$. Then $k' \,|\, m$ and $m \,|\, k$ are cyclic Galois extension of degrees e and f respectively. By Galois theory, there is an exact sequence of groups:

$$1 \to \mathbb{Z}/e\mathbb{Z} \to G \to \mathbb{Z}/f\mathbb{Z} \to 1$$

Here G is the Galois group of $k' \,|\, k$. Moreover, G has a presentation [CF67]:

$$< x, y \,|\, x^e = 1, y^f = x^i, yxy^{-1} = x^q >$$

where q is the number of elements in the residue field of k.

Since $\mathrm{char}(\bar{k}) \neq l$ and $\zeta_{l^n} \in k$, we must have $q \equiv 1 \pmod{l^n}$. As a result, $q \equiv 1 \pmod{e}$, and so $x^q = x$ in G. Therefore $yxy^{-1} = x$ in the above presentation, and so G is abelian. □

Note that n needs to be at least 2 for the above lemma to say something non-trivial since a group of order l or l^2 is necessarily abelian.

Definition 2.4. For a number field $K/\mathbb{Q}$, we say that a rational prime p *decomposes* in K if the p-adic valuation on $\mathbb{Q}$ extends to at least two inequivalent valuations on K.

Definition 2.5. We say that a group G is tamely admissible over K if an adequate G-Galois extension L/K can be chosen to be tamely ramified over K.

Proposition 2.6. *Let K be an number field and p be a prime number such that $\zeta_{p^n} \in K$ and p does not decompose in K. Let G be a finite group such that its p-Sylow subgroup is non-abelian of order $\leq p^{n+1}$. Then G is not admissible over K.*

Proof. If G were admissible over K, then by Schacher's criterion there will be two distinct places $\mathfrak{P}_1, \mathfrak{P}_2$ of K such that $K_{\mathfrak{P}_1}$ and $K_{\mathfrak{P}_2}$ admit Galois extensions with Galois groups containing a p-Sylow subgroup of G.

Since $p \in \mathbb{Q}$ does not decompose in K, one of these two places must have residue characteristic different from p. Without loss of generality, assume that $\mathfrak{P}_1$ is that place, and let $k = K_{\mathfrak{P}_1}$.

Let $l \,|\, k$ be a Galois extension of local fields such that the Galois group contains a p-Sylow subgroup H of G. Let m be the fixed field of P in this extension, and so $k' \,|\, m$ is a H-Galois extension.

Since $\zeta_{p^n} \in K \subset k \subseteq m$, and residue characteristic of m is different from p, this contradicts Lemma 2.3 since H is non-abelian. □

Corollary 2.7. *Let p be a rational prime number. Then the unique non-abelian group $\mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$ is not admissible over $\mathbb{Q}(\zeta_{p^n})$ for $n \geq 2$.*

Proof. Follows from the previous corollary since p does not decompose in $\mathbb{Q}(\zeta_{p^n})$, and the p-Sylow subgroup is the whole group. $\square$

Remark 2.8. This generalizes the observation in [Fei93] that the dihedral group D_8 of order 8 is not admissible over $Q(i)$. For example, the unique non-abelian group $\mathbb{Z}/9 \rtimes \mathbb{Z}/3$ is not admissible over $\mathbb{Q}(\zeta_9)$.

In fact, there is another field strictly contained in $\mathbb{Q}(\zeta_{p^2})$ over which the non-abelian group $\mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$ is not admissible. This is shown in Lemma 2.10 and requires the following lemma as an ingredient.

Lemma 2.9. *Let l/k be a finite G-Galois extension of local fields with G an p-group for a prime number p different from the residue characteristic of k. If the ramification index is p then G is abelian.*

Proof. Since any finite extension of an archimedean local field is abelian, we may (and do) assume that k is non-archimedean. Let m be the maximal unramified extension contained in l/k. By [CF67], G sits in an exact sequence

$$1 \to \mathrm{Gal}(l/m) = \mathbb{Z}/p\mathbb{Z} \to G \to \mathrm{Gal}(m/k) = \mathbb{Z}/\mathfrak{p}^a\mathbb{Z} \to 1,$$

and has a presentation

$$\langle x^p = 1, y^{p^a} = x^i, yxy^{-1} = x^q \rangle$$

for some appropriate i and q. Here x generates the inertia group, and y is a lift of the Frobenius automorphism.

Since q and p are coprime, we have $y^{p-1}xy^{p-1} = x^{q^{p-1}} = x$, i.e., y^{p-1} and x commute with each other. But $y' = y^{p-1}$ is another lift of the Frobenius. So x and y' generate G, and therefore G is abelian. $\square$

Lemma 2.10. *For any rational prime number l , the non-abelian semi-direct product $\mathbb{Z}/l^2 \rtimes \mathbb{Z}/l$ is not admissible over the unique degree p number field K inside $\mathbb{Q}(\zeta_l^2)$.*

Proof. If $G = \mathbb{Z}/l^2 \rtimes \mathbb{Z}/l$ were admissible over K, then by Schacher's criterion there would exist a G-Galois extension L/K such that over two places of K, the decomposition group would be the whole group G. We show that this is not possible.

Since l totally ramifies in K, one of these places must have residue characteristic different from l. Let $\mathfrak{p}$ be that place of K, and p be its residue characteristic. Let $k = K_{\mathfrak{p}}$, and k'/k be a G-Galois extension. Note that since $p \neq l$, the rational prime p is unramified in $K/\mathbb{Q}$.

Since G is non-abelian, $\mathfrak{p}$ must be non-archimedean. Since the residue characteristic of $\mathfrak{p}$ is different from l, k'/k is a tamely ramified extension. Since G is non-abelian, the ramification index cannot be l by Lemma 2.9. As a result, the only possibility for the ramification index is l^2.

Let m be the maximal unramified extension inside k'/k, and so k'/m is totally and tamely ramified extension of degree l^2. Therefore $\zeta_{l^2} \in m$. If q is the number of elements in the residue field of k, then this is the same thing as $q^l \equiv 1 \pmod{l^2}$.

We now look at the splitting behavior of rational primes $p \neq l$ in the extension $K/\mathbb{Q}$. Since $K/\mathbb{Q}$ is an abelian extension, this is determined by class field theory.

First consider the case when prime p splits in K. This happens if and only if the order of p in $(\mathbb{Z}/p^2)^*$ divides $(l-1)$. If $p \equiv 1 \pmod{l^2}$, then k already has ζ_{l^2}, and so by Lemma 2.3, this is not possible. Otherwise, $p^l \not\equiv 1 \pmod{l^2}$ and since $p = q$, we get $\zeta_{l^2} \notin m$. So this case is not possible either.

Finally, consider the case when prime p stays inert in K. In this case $q = p^l$, and so we have $q^l = p^{l^2} \equiv 1 \pmod{l^2}$. But $p^{l(l-1)} \equiv 1 \pmod{l^2}$, and therefore $p^l \equiv 1 \pmod{l^2}$. But this means that $q \equiv 1 \pmod{l^2}$, and hence $\zeta_{l^2} \in k$. But this contradicts Lemma 2.3. $\square$

The following lemma isolates a useful result whose main ideas are contained in the proof of Theorem 27 and 28 of [Lie94].

Lemma 2.11. *Let K be a number field and G be a metacyclic p-group for some prime number p. Then the following are equivalent.*

(i) G is tamely admissible over K.

(ii) There are infinitely many rational primes l that split completely in K and $\mathbb{Q}_l$ admits a G-Galois extension.

(iii) There is a non-archimedean places v of K with residue characteristic different from p such that the completion K_v admits a G-Galois extension.

Proof. (i) $\Longrightarrow$ (ii). Since G is tamely admissible over K, by Theorem 1.3 of [Nef13] G has a specific presentation as required in the hypothesis of Theorem 27 of [Lie94]. A careful reading of the proof of this theorem shows that there are infinitely many rational primes l that completely split in K with the property that $\mathbb{Q}_l$ admits a G-Galois extension.

(ii) $\Longrightarrow$ (iii) is clear.

(iii) $\Longrightarrow$ (i). By the proof of Theorem 28 in [Lie94] G has a presentation of the kind required in Theorem 1.3 of [Nef13] to assert the tame admissibility of G over K. $\square$

We are now in a position to classify the number fields for which every metacyclic p- group is tamely admissible. Starting with odd primes,

Proposition 2.12. *Let K be a number field, and p be an odd rational prime. The following are equivalent:*

(i) Every metacyclic p-group is tamely admissible over K.

(ii) The (unique) non-abelian group $\mathbb{Z}/p^2\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$ is tamely admissible over K.

(iii) Let $\mathbb{Q}(\alpha)$ be the unique degree p subfield of $\mathbb{Q}(\zeta_{p^2})$ for some primitive element α. Then $\alpha \notin K$ (equivalently, $K \cap \mathbb{Q}(\zeta_{p^2}) \subseteq \mathbb{Q}(\zeta_p)$).

Proof. (i) $\Longrightarrow$ (ii) is clear since $G = \mathbb{Z}/p^2\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$ is metacyclic.

(ii) $\Longrightarrow$ (iii). Let α be a primitive element as in the assertion (*iii*). For the sake of contradiction, assume that $\alpha \in K$. By Lemma 2.11, there exists a rational prime l that splits completely in K, and the local fields $\mathbb{Q}_l$ can realize $G = \mathbb{Z}/p^2\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$ as a Galois group. Since $\mathbb{Q}(\alpha) \subset K$, the prime l must split in $\mathbb{Q}(\alpha)$ as well, and therefore by Lemma 2.11 G is admissible over $\mathbb{Q}(\alpha)$. This contradicts Lemma 2.10. Therefore, $\alpha \notin K$.

(iii) $\Longrightarrow$ (i). Assume that $\alpha \notin K$ for a primitive element as in (*iii*). By Lemma 2.1 and 2.2, it suffices to show that every semidirect product of cyclic p-groups is admissible over K. So let

$$G = \mathbb{Z}/e\mathbb{Z} \rtimes \mathbb{Z}/f\mathbb{Z}$$

be a semi-direct product of cyclic p-groups with a presentation

$$\langle x, y \mid x^e = 1, y^f = 1, yxy^{-1} = x^q \rangle.$$

that corresponds to the group action

$$\varphi : \mathbb{Z}/f\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/e\mathbb{Z}) = \mathrm{Gal}(\mathbb{Q}(\zeta_e)/\mathbb{Q}).$$

Let $H = \mathrm{im}(\varphi)$, and consider the following diagram of field extensions.

$$\begin{tikzcd}
 & K(\zeta_e) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\zeta_e) \arrow[d, dash] \arrow[dr, dash] & & K \arrow[dl, dash] \\
M = \mathbb{Q}(\zeta_e)^H \arrow[dr, dash] & L = \mathbb{Q}(\zeta_e) \cap K \arrow[d, dash] & \\
 & \mathbb{Q} &
\end{tikzcd}$$

As a consequence of Theorem 1.3 of [Nef13], if $L \subseteq M$ then G is admissible over K, and a corresponding adequate extension can be chosen to be tamely ramified over K. So we now show that $L \subseteq M$ holds in this situation using basic Galois theory.

The extension $\mathbb{Q}(\zeta_e)/\mathbb{Q}$ is an extension of degree $(p-1)p^i$ for some $i \in \mathbb{N} \cup \{0\}$. Since H is a p-group (being the image of a p-group), its fixed field M must have degree $(p-1)p^j$ over $\mathbb{Q}$ (for some $j \leq i$).

On the other hand, since $\mathbb{Q}(\zeta_e)/\mathbb{Q}$ is a cyclic extension and the extension $\mathbb{Q}(\alpha) : \mathbb{Q}$ is of degree p, $\mathbb{Q}(\alpha) \in L$ if and only if $p \mid [L : \mathbb{Q}]$. As a consequence, $p \nmid L$ by the hypothesis that $\alpha \notin L$. Equivalently, $[L : \mathbb{Q}] \mid (p-1)$. This means $[L : \mathbb{Q}] \mid [M : \mathbb{Q}]$. Once again, since $\mathbb{Q}(\zeta_e)/\mathbb{Q}$ is a cyclic extension, this implies $L \subseteq M$ and we are done. □

For the even prime 2, the situation is a bit more involved, and we need to consider degree two extensions in $\mathbb{Q}(\zeta_8)$. To formulate the precise result, we recall some notation. Let Q_{16} be the generalized quaternion group of order 16 with presentation

$$\langle x, y \mid x^8 = 1, x^4 = y^2, yxy^{-1} = x^7 \rangle.$$

Let SD_{16} be the semi-dihedral group of order 16 with presentation

$$\langle x, y \mid x^8 = 1 = y^2, yxy^{-1} = x^3 \rangle.$$

We also need the following lemma.

Lemma 2.13. *Let K be a number field with $\sqrt{2} \in K$. Then SD_{16} is not tamely admissible over K. Moreover, if the 2-adic valuation on $\mathbb{Q}$ extends uniquely to K, then SD_{16} is not admissible over K (either tamely or wildly).*

Proof. By Schacher's criterion, it suffices to show that in both cases there are no places $\mathfrak{p}$ of K with residue characteristic different from 2 such that $k = K_{\mathfrak{p}}$ can realize $G = SD_{16}$ as a Galois group.

Suppose there were a G-Galois extension l/k with ramification index e and residue index f. Of course $e \neq 1, 16$ since G is not cyclic. By Lemma 2.9 e cannot be 2 since G is non-abelian. The group G has a unique cyclic normal group of order 4 but the quotient group is the Klein-four group, so $e = 4$ is not possible either. That means $e = 8$ and $f = 2$.

Let $\mathbb{F}_q$ be the residue field of $k = K_{\mathfrak{p}}$. Since $\sqrt{2} \in K$, $q \equiv \pm 1 \bmod 8$. If $q \equiv 1 \bmod 8$ then $\zeta_8 \in k$ and that contradicts Lemma 2.3. If $q \equiv -1 \bmod 8$ then G must have a presentation

$$\langle a, b \mid a^8 = 1, b^2 = a^i, bab^{-1} = a^{-1} \rangle$$

But a quick check shows that G has no two element g, h such that $g^8 = 1$, and $hgh^{-1} = g^{-1}$. □

Proposition 2.14. *For a number field K, the following are equivalent:*

(i) Every metacyclic 2-group is tamely admissible over K.
(ii) The groups Q_{16} and SD_{16} are tamely admissible over K.
(iii) K does not contain $\{i, \sqrt{2}, \sqrt{-2}\}$. (equivalently, $K \cap \mathbb{Q}(\zeta_8) = \mathbb{Q}$).

Proof. (i) $\Longrightarrow$ (ii). This follows because both Q_{16} and DS_{16} are metacyclic. Both of them have a cyclic normal subgroup of order 8 and the quotient by that subgroup is the cyclic group of order 2 (In fact, DS_{16} is a semidirect product $\mathbb{Z}/8\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$).

(ii) $\Longrightarrow$ (iii). By Theorem 3.2 of [Fei93], the number field K cannot contain i or $\sqrt{-2}$ since Q_{16} is admissible over K. By Lemma 2.13, K cannot contain $\sqrt{2}$ since DS_{16} is admissible over K.

(iii) $\Longrightarrow$ (i). The argument for this implication proceeds the same way as in the implication *(iii)* $\Longrightarrow$ *(i)* in Proposition 2.12. So let K, M, L as in that proof, and e be a power of 2. If $L \neq \mathbb{Q}$ then $L \cap \{i, \sqrt{2}, \sqrt{-2}\} \neq \emptyset$ since $\mathbb{Q}(i), \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{-2})$ are the only possible degree 2 extensions inside $\mathbb{Q}(\zeta_e)$ (e is a power of 2). Since by hypothesis K does not contain $\{i, \sqrt{2}, \sqrt{-2}\}$, either does L, and so $L = \mathbb{Q} \subseteq M$. □

By combining Proposition 2.12 and Proposition 2.14 we get the main result of this section as the following theorem. Note that part (i) of the theorem reduces the admissibility of a general solvable Sylow-metacyclic group to p-groups, and those cases are handled by Proposition 2.12 and Proposition 2.14.

Theorem 2.15. *Let K be a number field. Then*

(i) A solvable Sylow-metacyclic group is tamely admissible over K if and only if each of its Sylow subgroups are tamely admissible over K.
(ii) Every 2-metacyclic group is tamely admissible over K if and only if K does not contain $i, \sqrt{2}, \sqrt{-2}$.
(iii) Let p be any odd prime, and let α_p be a primitive element of the unique degree p-extension over $\mathbb{Q}$ in $\mathbb{Q}(\zeta_{p^2})/\mathbb{Q}$. Then every p-metacyclic group is tamely admissible over K if and only if $\alpha_p \notin K$.

Proof. Only part (i) is new, and it follows from Theorem 1.3 of [Nef13]. □

Corollary 2.16. *Let K be a number field and G be a metacyclic p-group for some prime number p. If either p is unramified in K, or p does not divide the degree $[K : \mathbb{Q}]$, then G is K-admissible.*

Furthermore, a corresponding adequate extension can be chosen to be tamely ramified over K.

Proof. Note that the odd prime p is totally ramified in $\mathbb{Q}(\alpha_p)/\mathbb{Q}$ for α_p as in Theorem 2.15. Similarly, $[\mathbb{Q}(\alpha_p) : \mathbb{Q}] = p$. Therefore with the hypothesis of this corollary, $\alpha_p \notin K$. A similar argument shows that $i, \sqrt{2}, \sqrt{-2} \notin K$ either. So the conclusion follows from Theorem 2.15. $\square$

Theorem 2.17. *Let K be a number field. If G is a solvable Sylow-metacyclic group such that for each prime p dividing the order of G, either p is unramified in K or p does not divide the degree $[K : \mathbb{Q}]$, then G is K-admissible.*

Furthermore, a corresponding adequate extension can be chosen to be tamely ramified over K.

Proof. By the previous corollary each p-Sylow subgroup of G is tamely admissible over K. The result then follows from Part (i) of Theorem 2.15. $\square$

We note a characterization of solvable Sylow-metacyclic groups in certain cases (with no restriction on the adequate extension being tamely ramified) as a corollary of Theorem 2.17:

Corollary 2.18. *Let K be a number field. If G is a solvable group such that for each prime p dividing the order of G, either p is unramified in K or p does not divide the degree $[K : \mathbb{Q}]$. Moreover for each p, the p-adic valuation on $\mathbb{Q}$ extends uniquely to K. Then G is K-admissible if and only if G is Sylow-metacyclic.*

Proof. Let G be a K-admissible group. Let p be a rational prime that divides $|G|$. Since the p-adic valuation extends uniquely to K, the p-Sylow subgroup of G must be metacyclic by Theorem 10.2 of [Sch68]. The converse direction follows from Theorem 2.17. $\square$

Corollary 2.19. *Let K be an abelian number field with square free conductor. Then every solvable Sylow-metacyclic group is admissible over K.*

Proof. With the hypothesis on the conductor, we have $K \subseteq \mathbb{Q}(\zeta_m)$ for a square free integer m. Therefore $K \cap \mathbb{Q}(\zeta_p^2) \subseteq K \cap \mathbb{Q}(\zeta_p)$ for every odd prime p, and $K \cap \mathbb{Q}(\zeta_8) = \mathbb{Q}$. So the hypothesis of part (ii) and (iii) of Theorem 2.15 are satisfied and the result follows. $\square$

Corollary 2.20. *Let $K = \mathbb{Q}(\zeta_m)$ be a cyclotomic field. Then every solvable Sylow-metacyclic group is tamely admissible over K if and only if m is square free.*

Proof. If m is square free then then by previous corollary every solvable Sylow-metacyclic group is tamely admissible over $\mathbb{Q}(\zeta_m)$. On the other hand, if p^2 divides m for some prime p then $\mathbb{Q}(\alpha_p) \subset \mathbb{Q}(\zeta_{p^2}) \subset K = \mathbb{Q}(\zeta_m)$ where α_p is as in Theorem 2.15. By Theorem 2.15 not every solvable Sylow-metacyclic group is tamely admissible over K. $\square$

3. Results over general number fields

Admissible groups are often characterized in terms of generators of the p-Sylow subgroups. To that end, if G is a p-group, let $d(G)$ denote the minimum number of generators of G. The number $d(G)$ is well-defined in this situation due to the Burnside basis theorem.

Notation 3.1. For a number field $K|\mathbb{Q}$, let Σ_K denote the set of places (inequivalent valuations) of K. If the extension $K|\mathbb{Q}$ is a Galois field extension, then e_p denotes the ramification degree of p, and f_p denotes the residue degree of p.

Theorem 3.2. *Let K be a number field. Let p be an odd rational prime that decomposes in K and such that $p = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\dots\mathfrak{p}_k^{e_m}$ in K with*

$$[K_{\mathfrak{p}_1} : \mathbb{Q}_p] \geq [K_{\mathfrak{p}_2} : \mathbb{Q}_p] \geq \cdots \geq [K_{\mathfrak{p}_m} : \mathbb{Q}_p].$$

If $\zeta_p \notin K_{\mathfrak{p}_i}$ for $i = 1, \dots, m$ then a p-group G is K-admissible if and only if

$$d(G) \leq [K_{\mathfrak{p}_2} : \mathbb{Q}_p] + 1.$$

Proof. Suppose that G is a K-admissible p-group, and L/K is a K-adequate G-Galois extension. If $G = \{1\}$ then the conclusion is true, so assume that $|G| > 2$ since p is an odd prime. By Schacher's criterion, there are at least two places of K such that the decomposition group at these places is the whole group G. Since $|G| > 2$, these places are necessarily non-archimedean. Let k_1, k_2 be the completion of K at any two such places, and let l_1/k_1 and l_2/k_2 be the local Galois extensions coming from the global extension L/K. Note that the valuation corresponding to k_1 (and k_2) might have more than one prolongation to L, but the completion of L over each of those prolongations will be isomorphic to l_1/k_1 (and l_2/k_2, respectively) since L/K is a Galois extension.

If one of these local fields, say k_1, has residue characteristic different from p then the extension l_1/k_1 is tamely ramified, and therefore G is a metacyclic group. In particular, $d(G) \leq 2$, and so the conclusion is true.

If both k_1 and k_2 have residue characteristic equal to p, then they are one of the fields $K_{\mathfrak{p}_i}$ for $i = 1, \dots, m$. Without loss of generality assume that $[k_1 : \mathbb{Q}_p] \geq [k_2 : \mathbb{Q}_p]$. Since $\zeta_p \notin k_2$, By a result of Shafarevich (Theorem 3 in II.§5, [Ser02]), the absolute Galois group of the maximal p-extension of k_2 is a free prop-p group on $[k_2 : \mathbb{Q}_p] + 1$ generators. Since G is a quotient of such a free pro-p group, $d(G) \leq [k_2 : \mathbb{Q}_p] + 1 \leq [K_{\mathfrak{P}_2} : \mathbb{Q}_p] + 1$. This proves one direction of the theorem.

For the other direction, let G be a p-group with $d(G) \leq [K_{\mathfrak{p}_2} : \mathbb{Q}_p] + 1$. Let $k_1 = K_{\mathfrak{p}_1}$ and $k_2 = K_{\mathfrak{p}_2}$. Once again, by Theorem 3 in II.§5 of [Ser02], there exist G-Galois extensions $l_1/k_1, l_2/k_2$ over the local fields. Since the local field k_1 does not have a primitive p-th root of unity, neither does the global field K. Therefore the hypothesis of Neukirch's generalization of Grunwald-Wang theorem (Corollary 2 in [Neu79]) are satisfied, and so there exists a G-Galois global extension L/K that has $l_1/k_1, l_2/k_2$ as completions. By Schacher's criterion, L/K is a K-adequate extension, and therefore G is K-admissible. □

Continuing with the notation in the previous Theorem:

Proposition 3.3. *Let K be a number field. Let p be an odd rational prime that decomposes in K and such that $p = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\dots\mathfrak{p}_k^{e_m}$ in K. The local fields $K_{\mathfrak{p}_i}$ for $i = 1, \dots, m$ do not contain a primitive p-th root of unity in each of the following situation, and therefore the conclusion of Theorem 3.2 is valid:*

(i) The prime p is unramified in $K/\mathbb{Q}$,
(ii) $(p-1) \nmid [K_{\mathfrak{p}_i} : \mathbb{Q}_p]$ for each $i = 1, \dots, m$,
(iii) $K/\mathbb{Q}$ is Galois, and $(p-1) \nmid [K : \mathbb{Q}]$.

Proof. Since p ramifies in $\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p$, if the prime p is unramified in $K/\mathbb{Q}$ then $\zeta_p \notin K_{\mathfrak{p}_i}$. This proves (i).

the fact that $[\mathbb{Q}_p(\zeta_p) : \mathbb{Q}_p] = p - 1$ shows (ii) and (iii) as well. □

Remark 3.4. It follows from this theorem that away from a set of finitely many primes (for example, the ramified primes), the admissible p-groups are completely determined. Moreover, a solvable group G for which each prime p dividing the order of G satisfies the above criteria is K-admissible. For example, in case of Galois number fields, we get the following Proposition 3.5.

For a finite group G, let G_p denote a p-Sylow subgroup of G (all such subgroups are conjugates of each other and hence isomorphic). For a number field K that is Galois over $\mathbb{Q}$, let $K-p$ denote the completion of K at a valuation extending the p-adic valuation (all such completions are isomorphic over $\mathbb{Q}_p$ since K is assumed to be Galois over $\mathbb{Q}$).

Proposition 3.5. *Let K be a Galois number field. Let G be an odd order group such that for each p dividing $|G|$,*

- *p decomposes in K.*
- *Either p is unramified in K, or $(p-1) \nmid [K:\mathbb{Q}]$.*
- *$d(G_p) \leq [K_p : \mathbb{Q}_p] + 1$*

Then G is K-admissible.

Proof. Let G be such a group. Since G has odd order, it is a solvable group. Let p bea rational prime such that $p \mid |G|$. Since p decomposes in K, there are at least two inequivalent prolongation (extension) of the p-adic valuation to K. Let k_1, k_2 be the completition of K at any of these two inequivalent extensions.

If p is unramified in K or $(p-1) \nmid [K:\mathbb{Q}]$ then as in Proposition 3.3, k_1, k_2 do not contain a primitive p-th root of unity. Therefore, by Theorem 3 in II.§5, [Ser02], the absolute Galois group of the maximal p-extension of $k_i, i = 1, 2$ is a free prop-p group on $[K_p : \mathbb{Q}_p] + 1$ generators. Since $d(G_p) \leq [K_p : \mathbb{Q}_p] + 1$, there exist G_p-Galois extensions of local fields l_1/k_1 and l_2/k_2. Also note that $\zeta_p \notin \mathbb{Q}$ since $\zeta_p \notin k_1$.

Similarly, for each $p \mid |G|$, we can get these G_p-Galois local extension over two distinct completions of K. Since $G_p \hookrightarrow G$, and $\zeta_p \notin K$, the hypotheses of Corollary 3 of [Neu79] are satisfied, and there is a G-Galois global extension L/K such that for each prime $p \mid |G|$, there are two distinct completions with decomposition group isomorphic to G_p. By Schacher's criterion, L/K is K-adequate and thus G is K-admissible. □

The above theorem provides sufficient conditions for a group to be K-admissible. Unlike the case of rational numbers, the question of necessary conditions remains open for general number fields K once we go beyond p-groups and allow wildly ramified adequate extensions. But in some special cases the above conditions are also necessary. For example, in the case of nilpotent groups we can say more due to the following lemma which follows from taking the tensor products of appropriate division algebras:

Lemma 3.6. *A nilpotent group G is admissible over a global field if and only if all of its Sylow subgroups are.*

This leads to the following result.

Corollary 3.7. *Let K be a finite Galois extension of $\mathbb{Q}$, and G be an odd nilpotent group with $|G|$ coprime to the discriminant of K. Then G is admissible over K if and only if for each $p \mid |G|$ one of the following two conditions holds:*

(i) prime p decomposes in K and $d(G_p) \leq f_p + 1$, or,

(ii) prime p does not decompose in K and G_p is metacyclic.

Proof. For a general nilpotent group, Lemma 3.6 reduces it to the case of p-groups. So assume that G is a p-group for some odd prime number p. The prime p is unramified in K by the hypothesis on $|G|$ being coprime to the discriminant of K. If p decomposes in K then by Prop 3.3 G is K-admissible if and only if $d(G) \leq f_p + 1$. If p does not decompose in K, then by Theorem 10.2 of [Sch68], G is metacyclic. Conversely, by Theorem 2.17, a metacyclic p-group is admissible over K. This proves the corollary for an odd p-group G. □

For a given number field K, the above results potentially leave out a finite set of primes for admissibility of p-groups. If such a prime p does not decompose in K then the Liedahl conditions [Lie94] provide a characterization of admissible p-groups. On the other hand, if such a prime p decomposes in K then we can still get some partial results. Theorem 10.1 of [Sch68] shows that if a p-group G is admissible over a Galois number field K with $[K : \mathbb{Q}] = n$, then $d(G) \leq (n/2) + 2$. The following result can be seen as a strengthening of it.

Theorem 3.8. *Let K be a finite Galois extension of $\mathbb{Q}$, and p be an odd rational prime such that $\zeta_p \notin K$, and p decomposes in K. Let G be a p-group. Then*

- *If $\zeta_p \notin K_p$ then G is K-admissible if and only if $d(G) \leq [K_{\mathfrak{p}} : \mathbb{Q}_p] + 1$.*
- *If $\zeta_p \in K_p$ then G is K-admissible if and only if G can be generated by $[K_{\mathfrak{p}} : \mathbb{Q}_p] + 2$ many generators $x_1, x_2, \ldots, x_n$ satisfying the relation*

$$x_1^{p^s}[x_1, x_2][x_3, x_4] \ldots [x_{n-1}, x_n] = 1$$

where p^s is such that $\zeta_{p^s} \in K_p$ but $\zeta_{p^{s+1}} \notin K_p$.

Proof. The case when $\zeta_p \notin K_p$ follows from Proposition 3.3. So assume that $\zeta_p \in K_p$, and let p^s be the largest power of p such that $\zeta_{p^s} \in K_p$. Since $[\mathbb{Q}(\zeta_p) : \mathbb{Q}_p] = p - 1$, we get that $n = [K_p : \mathbb{Q}_p] + 2$ is at least 4.

Let G be a K-admissible p-group. If G is metacyclic and generated by g_1 and g_2, then the free pro-p group F_n on n generators $x_1, \ldots x_n$ has a surjective map to G by $x_2 \mapsto g_1, x_4 \mapsto g_2$ and $x_i \mapsto 1, i \neq 2, 4$. Clearly, this map satisfies the relation $x_1^{p^s}[x_1, x_2][x_3, x_4] \ldots [x_{n-1}, x_n] = 1$.

Now consider the case when G is not metacyclic. Let L/K be a G-Galois K-adequate extension. By Schacher's criterion, there will be two distinct places of K for which the decomposition group coresponding to the adequate extension L/K will be the whole group G. Let k be the completion of K at one such place, and l/k be a corresponding G-Galois extension of local fields coming from the extension L/K (since L/K is Galois, all such local extensions over k will be k-isomorphic). Since tamely ramified Galois extensions of local fields have metacyclic Galois groups and G is not metacyclic, the residue characteristic of k must be p, i.e. $k \cong K_p$. Since $\zeta_p \in k$ by assumption, the absolute Galois group of the maximal p-extension of k is a Demuškin pro-p of rank $[k : \mathbb{Q}_p] + 2$ (Theorem 4 in [Ser02]). In particular, it's the free pro-p group on $[k : \mathbb{Q}_p] + 2$ generators $x_1, \ldots, x_n$ with one relation $x_1^{p^s}[x_1, x_2][x_3, x_4] \ldots [x_{n-1}, x_n] = 1$ (Theorem 7 of [Lab67]), and the result follows.

In the other direction, assume that G is finite p-group that can be generated by $[k : \mathbb{Q}_p] + 2$ many generators subject to the given relation. Since p decomposes in K, there are at least two distinct completions k_1, k_2 with residue characteristic p. Once again by Theorem 7 of [Lab67], there exist G-Galois local extensions l_1/k_1 and l_2/k_2. Since $\zeta_p \notin K$, Corollary 3 of

[Neu79] asserts the existence of a G-Galois global field extension L/K that has $l_i/k_i, i = 1, 2$ as completions. By Schacher's criterion this suffices to show that L/K is K-adequate and G is admissible over K. $\square$

Note that since we assumed p to be an odd prime, if $\zeta_p \in K_p$ then $n = [K_p : \mathbb{Q}_p]$ is divisible by $(p-1)$. In particular, it is an even number and the above description makes sense.

Adapting the proof of Theorem 3.8 we get a result in the converse direction of Theorem 10.1 of [Sch68]:

Proposition 3.9. *Let K be a finite Galois extension of $\mathbb{Q}$, and G be an odd order group such that for each prime p that divides $|G|$, the prime p decomposes in the number field K and K does not have a primitive p-th root of unity. If $d(G_p) \leq ([K_p : \mathbb{Q}_p]/2) + 1$ for each Sylow p-subgroup G_p then G is K-admissible.*

Proof. By Schacher's criterion, it suffices to construct a G-Galois field extension L/K such that for each prime p dividing $|G|$, there are two places of K for which the decomposition group is a p-Sylow subgroup of G. Since $\zeta_p \notin K$ for each prime p dividing $|G|$, the hypothesis of Neukirch's Corollary 3 in [Neu79] are satisfied. Therefore it suffices to show that for each prime p dividing the order of G, there are two (distinct) completions of K that admit G_p-Galois field extensions, where G_p is a p-Sylow subgroup of G.

So let p divide $|G|$, and since p decomposes in K, let k_1, k_2 be two distinct completions of K with respect to valuations extending the p-adic valuation. By hypothesis, $d(G_p) \leq ([k_i : \mathbb{Q}_p]/2) + 1$ for each $i = 1, 2$. If $zeta_p \notin k_1$ (and so $\zeta_p \notin k_2$ either since $k_1 \cong k_2$ over $\mathbb{Q}_p$) then by Theorem 3, the absolute Galois group of the maximal p-extension of $k_i, i = 1, 2$ is a free pro-p group on $[K_p : \mathbb{Q}_p] + 1$ generators. In particular, k_1, k_2 admit G_p-Galois field extensions.

On the other hand, if $\zeta_p \in k_1$ (and so also in k_2) then the absolute Galois group of the maximal p-extension of k_i has a presentation with $[K_p : \mathbb{Q}_p] + 2$ generators $x_1, \dots, x_n$ with one relation $x_1^{p^s}[x_1, x_2][x_3, x_4] \dots [x_{n-1}, x_n] = 1$. Sending each x_i for i odd number to 1 gives a surjection to a free pro-p group on $([K_p : \mathbb{Q}_p]/2) + 1$ generators, and thus there is a G_p-Galois extension of local fields over each k_1, k_2. This finishes the proof. $\square$

The proof of the above theorem uses a result of Neukirch [Neu79] generalizing the Grunwald-Wang theorem, and the description of the Galois group of maximal p-extension of local fields as Demuškin groups [NSW13], i.e., Poincaré groups of dimension 2. Presentation of these groups have a striking similarity to that of pro-p completion of fundamental groups of topological surfaces, and I am currently studying whether that analogy in the sense of arithmetic topology can be useful in providing an alternative description of admissible groups in this case. Similar to the Prop 3.5, this result partially extends to more general solvable groups, as well as to non-Galois number fields.

Remark 3.10. In the case that G is admissible and p does not decompose in K (i.e., the p-adic valuation on $\mathbb{Q}$ extends uniquely to K), one of the two places in Schacher's criterion must have residue characteristic different from p. This forces G to be metacyclic, and the characterization in that case is already known [Lie94].

4. Admissibility of p-groups over special classes of number fields

This section contains results about admissibility of p-groups after specializing to certain classes of number fields, such as Galois number fields, number fields of degree 2^n and odd degree over $\mathbb{Q}$, and finally the cyclotomic fields.

As a corollary to the Theorem 2.17 and Theorem 3.2 we get the following result for number fields that are Galois over $\mathbb{Q}$. Here f_p is the residue degree of prime p.

Corollary 4.1. *Let K be a Galois number field. An odd p-group with p coprime to the discriminant of $K|\mathbb{Q}$ is K-admissible if and only if one of the following conditions holds:*

(i) prime p decomposes in K and $d(G) \leq f_p + 1$, or,
(ii) prime p does not decompose in K and G is metacyclic.

A special class of Galois number fields are the cyclotomic number fields of type $K = \mathbb{Q}(\zeta_{l^r})$ for l a prime number. Since l is the only ramified prime in $K/\mathbb{Q}$, Corollary 4.1 leaves out only the case of l-groups. Since l does not decompose in K, any admissible l-group must be metacyclic by Theorem 10.2 of [Sch68]. As far as the K-admissibility of l-metacyclic group is concerned, it depends on the field. Every metacyclic l-group is known to be admissible over $\mathbb{Q}(\zeta_l)$, for example, by the discussion following Proposition 32 in [Lie94] or by Corollary 2.19. But it follows from Prop 2.7 that there are metacyclic l-groups that are not admissible over $\mathbb{Q}(\zeta_{l^r})$ for $r \geq 2$.

4.1. Number fields of degree 2^n.

Specializing further to number fields Galois over $\mathbb{Q}$ that have degree $[K : \mathbb{Q}]$ a power of 2, we have

Corollary 4.2. *Let K be a Galois number field of degree 2^n, and G be an odd p-group. Then the following assertions hold:*

(i) If p does not decompose in K, then G is K-admissible if and only if G is metacyclic.
(ii) If p decomposes in K, and either $(p-1) \nmid [K_p : \mathbb{Q}_p]$ or p is unramified in K then G is K-admissible if and only if $d(G) \leq [K_p : \mathbb{Q}_p] + 1$.

Proof. Consider first the case when p does not decompose in K. The prime p does not divide $[K : \mathbb{Q}]$ since p is odd, and so the result follows from Corollary 2.18.

The case when p decomposes in K follows from Proposition 3.3. □

Remark 4.3. Note that in order for $(p-1)$ to divide the local degree $[K_p : \mathbb{Q}_p]$ which is a power of 2, p must be a Fermat prime and smaller than or equal to $[K_p : \mathbb{Q}_p]/2$. At the time of writing this manuscript, only 5 Fermat prime are known (namely, $3, 5, 17, 257, 65537$) and this list is conjectured to be exhaustive.

Since every quadratic extension is automatically Galois, we can use the above corollary in that case. Moreover, there are no exceptional Fermat primes for quadratic extensions and so we get a complete characterization of admissible p-groups for odd primes p.

Corollary 4.4. *Let K be a quadratic number field, and G be an odd p-group for some rational prime p. Then G is K-admissible if and only if one of the following conditions holds:*

(i) prime p decomposes in K and $d(G) \leq 2$, or,
(ii) prime p does not decompose in K and G is metacyclic.

Proof. Apply Corollary 4.2 and observe that if a prime p splits in K then $[K_p : \mathbb{Q}_p] = 1$. □

Remark 4.5. The above corollary leaves out the case of 2-groups. We point out that there are examples of quadratic number field K and metacyclic 2-groups that are not admissible over K. For example, the dihedral group of order 8 is known to not be $\mathbb{Q}(i)$-admissible. (It follows from Corollary 2.7, for example)

The next group of number fields with degree a power of 2 are quartic number fields, and it is more involved than the quadratic case. First, the field can be non-Galois, and second, there is the possible Fermat prime 3 even if the field is Galois over $\mathbb{Q}$. When the field is non-Galois, our strategy is to look at the various possible splittings of primes, and argue that the local field cannot contain p-th roots of unity. The precise result is

Proposition 4.6. *Let K be a quartic number field. Then a p-group G for $p \neq 2, 3$ is admissible over K if and only if one of the following two conditions hold:*

(i) p does not decompose in K, and G is metacyclic.
(ii) p decomposes in K, and $d(G) \leq \min_{\mathfrak{p}|p}([K_{\mathfrak{p}} : \mathbb{Q}_p]) + 1$.

Proof. Note that the case when $K/\mathbb{Q}$ is Galois follows from Corollary 4.2 after observing the following two points. First, that the only exceptional Fermat prime in this case is 3, which is excluded from the statement. Second, if the prime p decomposes in K then $[K_{\mathfrak{p}} : \mathbb{Q}_p]$ is same for each $\mathfrak{p}$ extending the p-adic valuation.

The general case is proven with a similar argument as in the Corollary 4.2. The case when p does not decompose in K follows from Corollary 2.18 since p does not divide $4 = [K : \mathbb{Q}]$.

In the case that p decomposes in K, we argue that none of the completions at p contain ζ_p. Let $\mathfrak{p} \mid p$, and there are the following two subcases.

(1) $K_{\mathfrak{p}}/\mathbb{Q}_p$ is unramified. Since p is odd, and for odd primes $\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p$ is ramified. It follows that $K_{\mathfrak{p}}$ does not contain a primitive p-th root of unity.
(2) If $K_{\mathfrak{p}}/\mathbb{Q}_p$ is ramified then the ramification degree can only be 2 or 3 since we assumed that p decomposes in K and $[K : \mathbb{Q}] = 4$. If $\zeta_p \in K_{\mathfrak{P}}$ then the ramification degree must be at least four since $p \geq 5$ and $[\mathbb{Q}_p(\zeta_p) : \mathbb{Q}_p] = p - 1$. Therefore $\zeta_p \notin K_{\mathfrak{p}}$.

So the hypothesis of Theorem 3.2 is satisfied, and $\min_{\mathfrak{p}|p}([K_{\mathfrak{p}} : \mathbb{Q}_p])$ equals the second largest degree of local extensions as in Theorem 3.2. □

4.2. **Odd degree number fields.** Similar to Corollary 4.2, the Galois number fields of odd degree are another special class of number fields.

Theorem 4.7. *Let K be a Galois number field whose degree $[K : \mathbb{Q}]$ is an odd number, and G be an odd p-group. Then the following assertions hold:*

(i) If p does not decompose in K then G is K-admissible if and only if G has a Liedahl presentation for K.
 - *Moreover, if in addition $p \nmid [K : \mathbb{Q}]$ or p is unramified in K, then G is K-admissible if and only if it is metacyclic.*

(ii) If p decomposes in K, then G is K-admissible if and only if $d(G) \leq [K_p : \mathbb{Q}_p] + 1$.

Proof. The case when p does not decompose in K follows from [Lie94] and Corollary 2.18.

The case when p decomposes in K follows from Theorem 3.3 once we observe that $K/\mathbb{Q}$ is Galois and $(p-1) \nmid [K : \mathbb{Q}]$ since $(p-1)$ is even for an odd prime p whereas $[K : \mathbb{Q}]$ is odd by assumption. □

The first odd degree case is the case of cubic number fields. Arguing similar to the case of quartic number fields, we get

Proposition 4.8. *Let K be a cubic number field, and G be a p-group for $p \neq 2, 3$. Then G is K-admissible if and only if one of the following conditions holds:*

(i) prime p does not decompose in K and G is metacyclic.

(ii) prime p decomposes in K and $d(G) \leq 2$, or,

Proof. The case when $K/\mathbb{Q}$ is Galois follows from Theorem 4.7 since $p \neq 2, 3$, and if p decomposes in K then $K_p = \mathbb{Q}_p$.

Next consider the case when $K/\mathbb{Q}$ is not Galois. If p does not decompose in K then the result once again follows from Corollary 2.18. Finally, consider the case that p decomposes in K, and let k be any completion of K for a valuation extending the p-adic valuation on $\mathbb{Q}$. We have $[\mathbb{Q}_p(\zeta_p) : \mathbb{Q}_p] = p - 1 \geq 4$ since $p \geq 5$, and so $zeta_p \notin k$ since $[k : \mathbb{Q}_p] \leq 3$. Therefore, we can invoke Theorem 3.2. The result follows once we observe that because for a cubic number field the second biggest local degree is necessarily 1 in Theorem 3.2. □

The exceptional case of $p = 3$ in Proposition 4.8 and more generally the case of $p \mid [K : \mathbb{Q}]$ in Theorem 4.7 can have a more involved description of admissible p-groups. An example of this phenomenon is Lemma 2.10 where it was shown that the non-abelian semi-direct product $\mathbb{Z}/l^2 \rtimes \mathbb{Z}/l$ is not admissible over the unique degree l number field K inside $\mathbb{Q}(\zeta_l^2)$. In particular, $\mathbb{Z}/9 \rtimes \mathbb{Z}/3$ is not admissible over the cubic number field $\mathbb{Q}(\zeta_9 + \zeta_9^{-1})$.

5. Global Function Fields

Let K be a global function field K of characteristic $p > 0$. Schacher showed that if a group G is admissible K then the l-Sylow subgroups of G for $l \neq p$ must be metacyclic (Theorem 10.3 of [Sch68]). For the case when $l = p$, a result of Saltman (Theorem $1'$ of [Sal77]) implies that every p-group is admissible over K. As metacyclic abelian l-groups are always admissible over global fields, we get

Proposition 5.1. *A finite abelian group G is admissible over a global function field of characteristic $p > 0$ if and only if l-Sylow subgroups for $l \neq p$ are metacyclic.*

We also have an analogue of Proposition 2.6 for the case of global function fields.

Proposition 5.2. *Let K be an global function field of characteristic $p > 0$. Let $l \neq p$ be a rational prime such that $\zeta_{l^n} \in K$ for $n \geq 0$. Let G be a finite group such that its l-Sylow subgroup is non-abelian of order $\leq l^{n+1}$. Then G is not admissible over K.*

Proof. Assume that G were admissible over K, and L/K is a corresponding G-Galois K-adequate extension. By Schacher's criterion, there will be at least one place $\mathfrak{p}$ of K such that the decomposition group in L/K corresponding to $\mathfrak{p}$ will contain a l-Sylow subgroup H of G. By considering the fixed field of H in the local extension, we get a H-Galois extension of local fields l/k. Since K contains the l^nth roots of unity, so does k. But this contradicts Lemma 2.3. □

For example, as a consequence of this proposition, the dihedral group of order 8 is not admissible over global function fields of characteristic $p \equiv 1 \pmod 4$ (e.g., curves over $\mathbb{F}_5$). More generally, if the constant field $\mathbb{F}_q$ of the global function field K is such that

$q \equiv 1 \pmod{l^2}$ for some prime l then the (unique) non-abelian group $\mathbb{Z}/l^2\mathbb{Z} \rtimes \mathbb{Z}/l\mathbb{Z}$ is not admissible over K.

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