# 1 Background

The two mathematica notebooks in this folder carry out computations for examples of superminimal curves, which are example 1 and 3 in [2]. The motivation is to give examples of explicit solutions of [1, eq. (4.13)] for $\alpha_- \to 0$. For example 1, the computation is straightforward in Mathematica, as the solution is radially symmetric and makes use of the implemented Laplace operator. For example 3, the following computations are needed. However, the expression found for $\alpha_+$ is highly complicated and the Simplify command times out on my machine with the standard configuration. A 3D Plot of $\alpha_+^2$ can be produced though.

## 1.1 Computing the curvature of holomorphic curves

Let $\mathbb{C}^{n+1}$ be equipped with the standard Hermitian inner product $\langle v, w\rangle = \sum v_i \bar{w}_i$ and let $\gamma(v, w) = \|v\|^2\|w\|^2 - |\langle v, w\rangle|^2$.

**Lemma 1.1.** *Let $\phi\colon X \to \mathbb{CP}^n$ be a holomorphic curve of the form $[1, \varphi_1, \ldots, \varphi_n]$ for $\varphi_i$ meromorphic. Then, in a local chart, the induced metric on $X$ is given by*

$$u(z) = \frac{2}{|\phi|^4}\gamma_{\phi,\phi'}\mathrm{d}\bar{z}\mathrm{d}z. \tag{1.1}$$

*Where $\phi$ and $\phi'$ are regarded as vectors in $\mathbb{C}^{n+1}$. The Gauß curvature of the induced metric is equal to*

$$1 - \frac{\|\phi\|^4}{\gamma_1^2}\left(\frac{||\phi|^2\langle\phi'', \phi'\rangle - \langle\phi, \phi'\rangle\langle\phi'', \phi\rangle|^2}{\gamma_1} - \gamma_2\right)$$

*for $\gamma_1 = \gamma(\phi, \phi')$ and $\gamma_2 = \gamma(\phi, \phi'')$.*

*Proof.* The Kähler potential on the affine chart $\{Z_0 \neq 0\} \subset \mathbb{CP}^n$ is equal to $\rho = 2\log(1 + |Z_i|^2)$. Since $\phi$ is holomorphic, the induced Kähler potential on $X$ is $\rho_\varphi = \rho \circ \varphi$. In a local chart the induced metric on $X$ is $\frac{\partial^2}{\partial z\partial\bar{z}}\rho_\varphi \mathrm{d}\bar{z}\mathrm{d}z$. Using the identities $\frac{\partial}{\partial z}\langle f, g\rangle = \langle\frac{\partial}{\partial z}f, g\rangle$ for holomorphic functions $f, g$ and $\frac{\partial}{\partial\bar{z}}\bar{h} = \frac{\partial h}{\partial z}$ one computes

$$\frac{\partial^2}{\partial z\partial\bar{z}}\rho_\phi = 2\frac{\partial^2}{\partial z\partial\bar{z}}\log(|\phi|^2) = 2\frac{|\phi'|^2|\phi|^2 - |\langle\phi', \phi\rangle|^2}{|\phi|^4} = 2\frac{|\phi'|^2}{|\phi|^2}\gamma_{\phi,\phi'}.$$

From this, the Gauß curvature can be computed as

$$\frac{-1}{2u}\Delta_0(\log(u)) = 1 - \frac{1}{2u}\Delta_0(\log(\gamma_1)) = 1 - \frac{2}{\gamma_1 u}\left(\frac{1}{\gamma_1}\|\frac{\partial\gamma_1}{\partial z}\|^2 - \frac{1}{4}\Delta_0\gamma_1\right).$$

For $\Delta_0 = 4\frac{\partial^2}{\partial z\partial\bar{z}}$. The statement follows from the computation

$$\frac{\partial\gamma_1}{\partial z} = |\phi|^2\langle\phi'', \phi'\rangle - \langle\phi, \phi'\rangle\langle\phi'', \phi\rangle \tag{1.2}$$

$$\frac{1}{4}\Delta_0\gamma_1 = \gamma_2. \tag{1.3}$$

□

# References


[1] Benjamin Aslan. “Transverse $J$-holomorphic curves in nearly Kähler $\mathbb{CP}^3$”. In: *arXiv preprint 2101.03845* (2021).

[2] Thomas Friedrich. “On surfaces in four-spaces”. In: *Annals of Global Analysis and Geometry* 2.3 (1984), pp. 257–287.