Stability Control of Double Inverted Pendulum on a Cart using Full State Feedback with H infinity and H 2 Controllers

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Abstract

In this paper a full state feedback control of a double inverted pendulum on a cart (DIPC) are designed and compared. Modeling is based on Euler-Lagrange equations derived by specifying a Lagrangian, difference between kinetic and potential energy of the DIPC system. A full state feedback control with H infinity and H 2 is addressed. Two approaches are tested: open loop impulse response and a double inverted pendulum on a cart with full state feedback H infinity and H 2 controllers. Simulations reveal superior performance of the double inverted pendulum on a cart with full state feedback H infinity controller.

Keywords: Double inverted pendulum on a cart, Full state feedback H infinity controller, Full state feedback H 2 controller

1. Introduction

An inverted pendulum system is a highly nonlinear, unstable and natural timber of instabilities. All these features make it the system model of advanced control goal and typical experiment platform of trial control results. There are many types of the inverted pendulum designs presenting a variety of control challenges. The familiar types are the single inverted pendulum on a cart, the double inverted oscillator on a cart and a rotary inverted pendulum. The main concern is to balance a rod on a mobile platform that can claim in only two directions; left or right. The inverted pendulum is related to spaceship or missile guidance, where thrust is actuated at the bottom of a tall track. To control this unstable system, we have employed the full state feedback method. In this method, the full state feedback H infinity and H 2 controllers are used for the linear state space model and the calculated gain matrix have been obtained to stabilize a system. In the simulation part of this paper, graphical simulations for control task are given to show the effectiveness of the proposed full state feedback scheme.

2. Mathematical Modeling of the Double Inverted Pendulum on a Cart

The DIPC system design is shown in Figure 1 below. To derive the equations of motion, we have used the famous Lagrange equations:
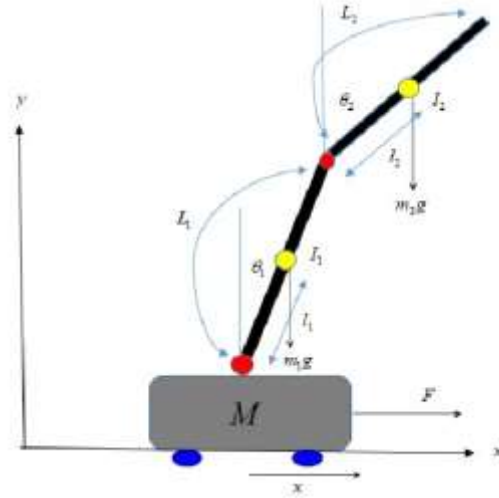
Figure 1 DIPC system design

The DIPC system design is shown in Figure 1 above. To derive the equations of motion, we have used the famous Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{g}}\right) - \frac{\partial L}{\partial g} = Q \qquad (1)$$

Where

$$L = T - V$$

Q forces (or moments) acting on the system
T kinetic energy of the system
V potential energy of the system
The kinetic energies of the systems are

$$T_1 = \frac{1}{2}M\dot{x}^2 \qquad (2)$$

$$T_2 = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}\left(m_1 l_1^2 + I_1\right)\dot{\theta}_1^2 + m_1 l_1 \dot{x}\dot{\theta}_1 \cos\theta_1 \qquad (3)$$

$$T_3 = \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2 L_1^2\dot{\theta}_1^2 + \frac{1}{2}\left(m_2 l_2^2 + I_2\right)\dot{\theta}_2^2$$
$$+ m_2 L_1 \dot{x}\dot{\theta}_1 \cos\theta_1 + m_2 l_2 \dot{x}\dot{\theta}_2 \cos\theta_2 + m_2 L_1 l_2 \dot{\theta}_1\dot{\theta}_2 \cos\left(\theta_1 - \theta_2\right) \qquad (4)$$

The potential energies become

$$V_1 = 0 \qquad (5)$$

$$V_2 = m_1 g l_1 \cos\theta_1 \qquad (6)$$

$$V_3 = m_2 g \left(L_1 \cos\theta_1 + l_2 \cos\theta_2\right) \qquad (7)$$

Thus the Lagrangian of the system is given by

$$L = T_1 + T_2 + T_3 - \left(V_1 + V_2 + V_3\right)$$

Differentiating equation (1) as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=F \qquad (8)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)-\frac{\partial L}{\partial \theta_1}=0 \qquad (9)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right)-\frac{\partial L}{\partial \theta_2}=0 \qquad (10)$$

$$\left(M+m_1+m_2\right)\ddot{x}+\left(m_1l_1+m_2L_1\right)\cos\left(\theta_1\right)\ddot{\theta}_1+m_2l_2\cos\left(\theta_2\right)\ddot{\theta}_2$$
$$-\left(m_1l_1+m_2L_1\right)\sin\left(\theta_1\right)\dot{\theta}_1^2-m_2l_2\sin\left(\theta_2\right)\dot{\theta}_2^2=F \qquad (11)$$

$$\left(m_1l_1+m_2L_1\right)\cos\left(\theta_1\right)\ddot{x}+\left(m_1l_1^2+m_2L_1^2+I_1\right)\ddot{\theta}_1+m_2L_1l_2\cos\left(\theta_1-\theta_2\right)\ddot{\theta}_2$$
$$+m_2L_1l_2\sin\left(\theta_1-\theta_2\right)\dot{\theta}_2^2-\left(m_1l_1+m_2L_1\right)g\sin\theta_1=0 \qquad (12)$$

$$m_2l_2\cos\left(\theta_2\right)\ddot{x}+m_2L_1l_2\cos\left(\theta_1-\theta_2\right)\ddot{\theta}_1+\left(m_2l_2^2+I_2\right)\ddot{\theta}_2$$
$$-m_2L_1l_2\sin\left(\theta_1-\theta_2\right)\dot{\theta}_1^2-m_2l_2g\sin\theta_2=0 \qquad (13)$$

2.1 Linearization of the System

Linearization of the system equations around certain equilibrium points have been done. In this paper, we linearize the system at the vertical unstable equilibrium by taking.

$$\theta_1 \approx \theta_2 \approx 0$$
$$\cos\theta_1 \approx \cos\theta_2 \approx 1$$
$$\sin\theta_1 \approx \theta_1$$
$$\sin\theta_2 \approx \theta_2$$

$$\theta_1 - \theta_2 \approx 0$$
$$\cos\left(\theta_1-\theta_2\right)\approx 1$$
$$\sin\left(\theta_1-\theta_2\right)\approx \theta_1-\theta_2$$
$$\dot{\theta}_1^2 \approx \dot{\theta}_2^2 \approx 0$$

After linearization Equation (11), Equation (12) and Equation (13) becomes

$$\left(M+m_1+m_2\right)\ddot{x}+\left(m_1l_1+m_2L_1\right)\ddot{\theta}_1+m_2l_2\ddot{\theta}_2=F \qquad (14)$$

$$\left(m_1l_1+m_2L_1\right)\ddot{x}+\left(m_1l_1^2+m_2L_1^2+I_1\right)\ddot{\theta}_1+m_2L_1l_2\ddot{\theta}_2-\left(m_1l_1+m_2L_1\right)g\theta_1=0 \qquad (15)$$

$$m_2l_2\ddot{x}+m_2L_1l_2\ddot{\theta}_1+\left(m_2l_2^2+I_2\right)\ddot{\theta}_2-m_2l_2g\,\theta_2=0 \qquad (16)$$

Or in matrix form

$$\begin{pmatrix} (M+m_1+m_2) & (m_1l_1+m_2L_1) & m_2l_2 \\ (m_1l_1+m_2L_1) & (m_1l_1^2+m_2L_1^2+I_1) & (m_2L_1l_2) \\ (m_2l_2) & (m_2L_1l_2) & (m_2l_2^2+I_2) \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(m_1l_1+m_2L_1)g & 0 \\ 0 & 0 & -m_2l_2g \end{pmatrix}\begin{pmatrix} x \\ \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} F \\ 0 \\ 0 \end{pmatrix} \quad (17)$$

The state space model equation for the system is

$$A = M^{-1}N, \ B = M^{-1}F$$

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (M+m_1+m_2) & (m_1l_1+m_2L_1) & m_2l_2 \\ 0 & 0 & 0 & (m_1l_1+m_2L_1) & (m_1l_1^2+m_2L_1^2+I_1) & (m_2L_1l_2) \\ 0 & 0 & 0 & (m_2l_2) & (m_2L_1l_2) & (m_2l_2^2+I_2) \end{pmatrix}, \ N = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (m_1l_1+m_2L_1)g & 0 & 0 & 0 & 0 \\ 0 & 0 & m_2l_2g & 0 & 0 & 0 \end{pmatrix}, \ F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} and \ x = \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

The parameters of the system are shown in Table 1 below

Table 1 System parameters

No	Parameter	Symbol	Value
1	Mass of the base	M	2 Kg
2	Mass of pendulum 1	m_1	1 Kg
3	Mass of pendulum 2	m_2	1.25 Kg
4	Length of pendulum 1	L_1	0.4 m
5	Length of pendulum 2	L_2	0.6 m
6	Center of Mass length of pendulum 1	l_1	0.2 m
7	Center of Mass length of pendulum 2	l_2	0.3 m
8	Moment of inertia of pendulum 1	I_1	0.0014 $Kg\,m^2$
9	Moment of inertia of pendulum 2	I_2	0.0375 $Kg\,m^2$
10	Gravity constant	g	9.8 m/s^2

The state space matrixes become

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -11.31 & 1.799 & 0 & 0 & 0 \\ 0 & 115.3 & -46.6 & 0 & 0 & 0 \\ 0 & -87 & 66.61 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.4636 \\ -1.649 \\ 0.4895 \end{bmatrix} \quad C = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

3. The Proposed Controllers Design
3.1 Full State Feedback $H\infty$ Controller Design

Consider Figure 2 and assume that

$$M = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ I & 0 & 0 \end{bmatrix} \qquad (18)$$
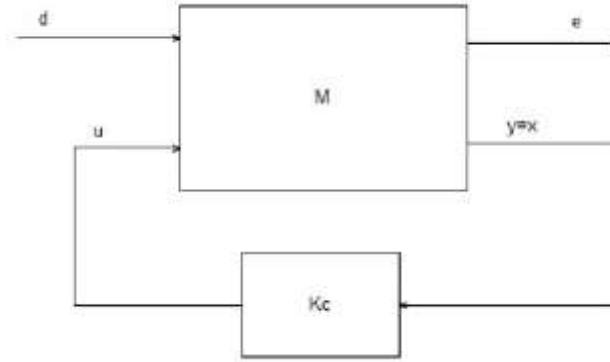
Figure 2 A full state feedback system.

From Equation (18),

$$\dot{x} = Ax + B_1 d(t) + B_2 u(t) \qquad (19)$$

$$e(t) = C_1 x(t) + D_{12} u(t) \qquad (20)$$

$$y(t) = x(t) \qquad (21)$$

The condition

$$\left\| T_{ed}(s) \right\|_{H\infty}^2 < \gamma \qquad (22)$$

Assume that the worst-case disturbance **d**(t) and the optimal control **u**(t) have the following structure

$$d(t) = K_d x(t) \quad and \quad u(t) = K_c x(t)$$

Then Kc becomes

$$K_c = -B_2^T P$$

Where P is

$$PA + A^T P + C_1^T C_1 - P\left(B_2 B_2^T - \frac{1}{\gamma^2} B_1 B_1^T \right) P = 0$$

The condition $\left\| T_{ed}(s) \right\|_{H\infty}^2 < \gamma$ is satisfied provided. For this system the full state feedback gain matrix becomes

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$$K_{H\infty} = \begin{bmatrix} 0.4472 & -287.4819 & 299.3632 & 2.2104 & -9.5959 & 37.5785 \end{bmatrix}$$

3.2 Full State Feedback H_2 Controller Design

Consider Figure 3 and assume that

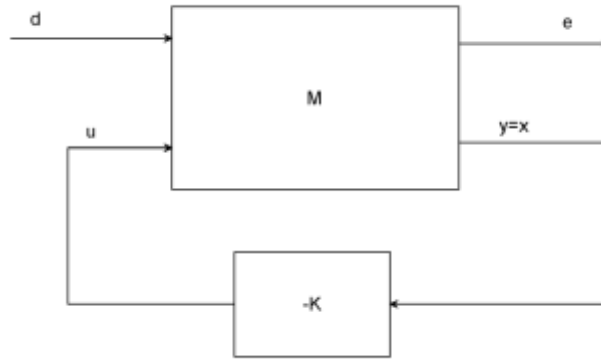$$M = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ I & 0 & 0 \end{bmatrix} \qquad (23)$$

Figure 3 A full state feedback system.

From Equation (23),

$$\dot{x} = Ax + B_1 d(t) + B_2 u(t) \qquad (24)$$

$$e(t) = C_1 x(t) + D_{12} u(t) \qquad (25)$$

$$y(t) = x(t) \qquad (26)$$

Assuming that d(t) is the white noise vector with unit intensity

$$\left\| T_{ed}(s) \right\|_{H_2}^2 = E\left(e^T(t)e(t) \right) \qquad (27)$$

Where

$$e^T e = x^T C_1^T C_1 x + 2x^T C_1^T D_{12} u + u^T D_{12}^T D_{12} u \qquad (28)$$

With Equation (24) and Equation (27), the minimization of $\left\| T_{ed}(s) \right\|_{H_2}$ is equivalent to the solution of the stochastic regulator problem. Setting

$$Q_f = C_1^T C_1, \quad N_f = C_1^T D_{12} \quad and \quad R_f = D_{12}^T D_{12} \qquad (29)$$

the optimal state feedback law is given by

6

$$u = -Kx \qquad (30)$$

Where

$$K = R_f^{-1} \left(PB_2 + N_f \right)^T \qquad (31)$$

And

$$P\left(A - B_2 R_f^{-1} N_f^T\right) + \left(A - B_2 R_f^{-1} N_f^T\right)^T P - PB_2 R_f^{-1} B_a^T P + Q_f - N_f R_f^{-1} N_f^T = 0 \qquad (32)$$

It should be noted that the gain K is independent of the matrix B1. For this system the full state feedback gain matrix becomes

$$K_{H_2} = \left(0.7906 \ \text{-}289.0233 \ 305.1824 \quad 3.0540 \quad \text{-}9.1705 \quad 38.5599 \right)$$

4. Result and Discussion
4.1 Controllability and Observability of the Pendulum

A system (state space representation) is controllable iff the controllable matrix C = [B AB A2B….An-1B] has rank n where n is the number of degrees of freedom of the system.

In our system, the controllable matrix C = [B AB A2B A3B A4B A5B] has rank 6 which the degree of freedom of the system. So, the system is controllable.

A system (state space representation) is Observable iff the Observable matrix D = [C CA CA2….CAn-1] T has a full rank n.

In our system, the Observable matrix D = [C CA CA2 CA3 CA4 CA5] T has a full rank of 6. So, the system is Observable.

4.2 Open Loop Impulse Response of the Double Inverted Pendulum

The open loop simulation for a 1 Nm impulse input of force for angular displacement 1 and 2 and for angular velocity 1 and 2 are shown in Figure 4, 5, 6 and 7 respectively.
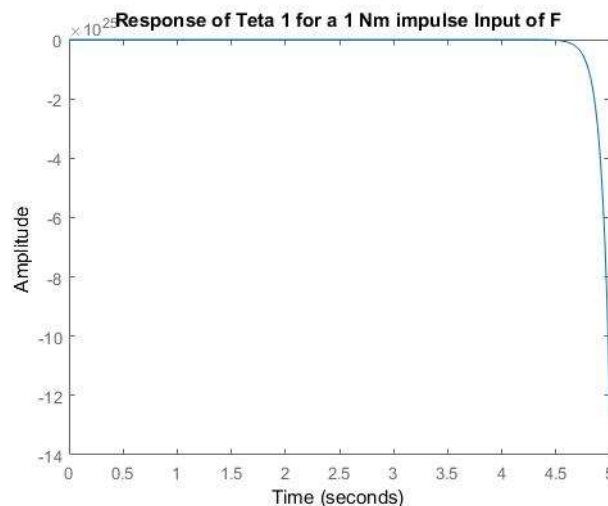

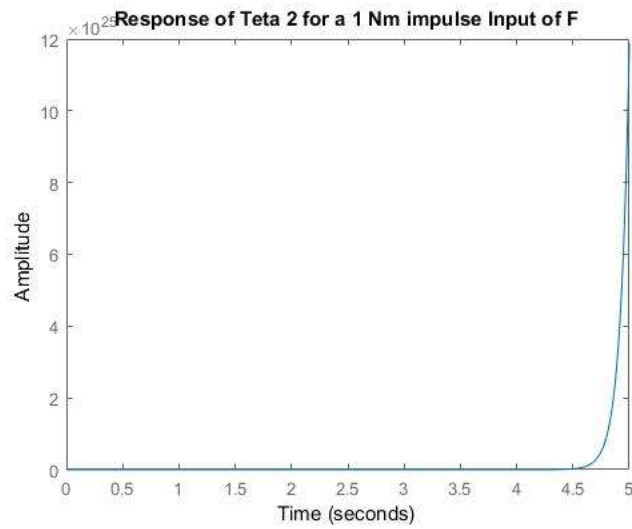Figure 4 Open loop impulse response of Teta 1

Figure 5 Open loop impulse response of Teta 2

The open loop angular displacement simulation results show that the double inverted pendulum angular displacements are not stable and indeed the system needs a feedback control system
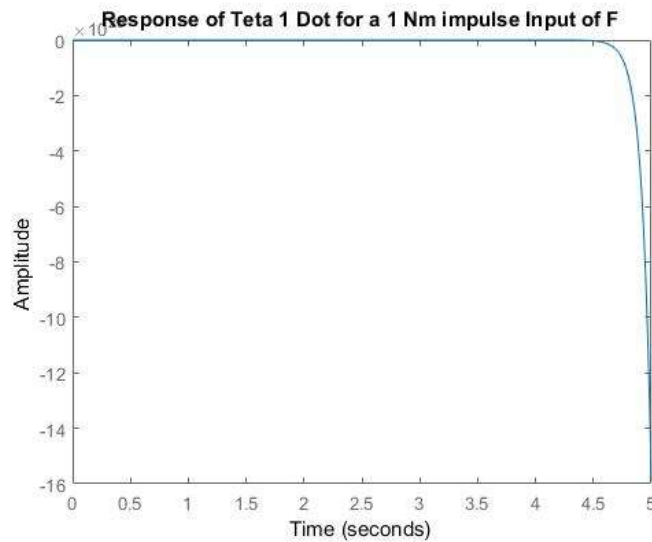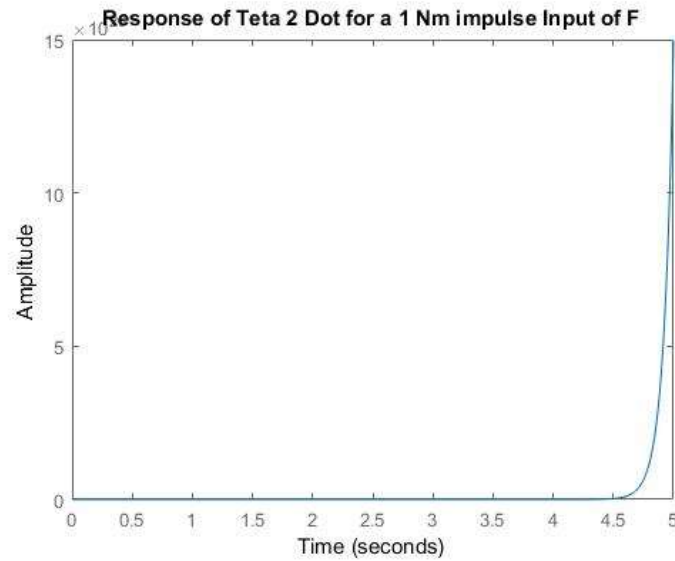

Figure 6 Open loop impulse response of Teta 1 dot

Figure 7 Open loop impulse response of Teta 2 dot

The open loop angular velocity simulation results show that the double inverted pendulum angular velocities are not stable and indeed the system needs a feedback control system

4.3 Comparison of the Double Inverted Pendulum with Full State Feedback $H\infty \& H_2$ Controllers for Impulse Input Signal

The comparison of the double inverted pendulum with Full State Feedback H infinity and H2 Controllers for a 1 N impulse input force for angular displacement 1 and 2 and for angular velocity 1 and 2 is shown in Figure 8, 9, 10 and 11 respectively.
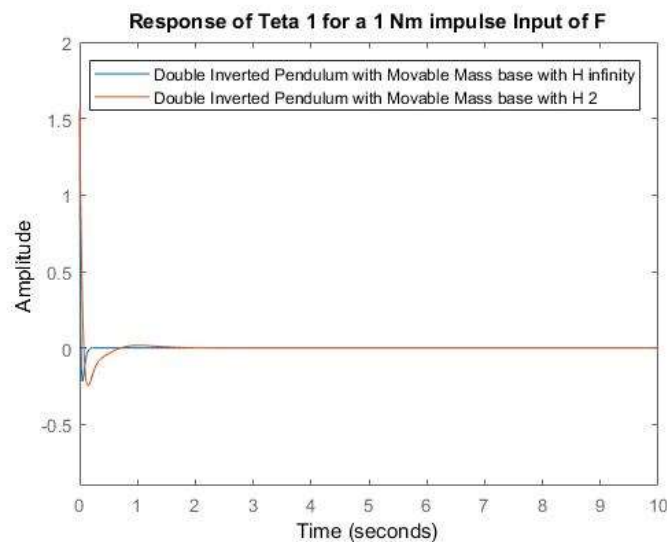

Figure 8 Impulse response the double inverted pendulum with Full State Feedback H infinity and H2 Controllers for Teta 1

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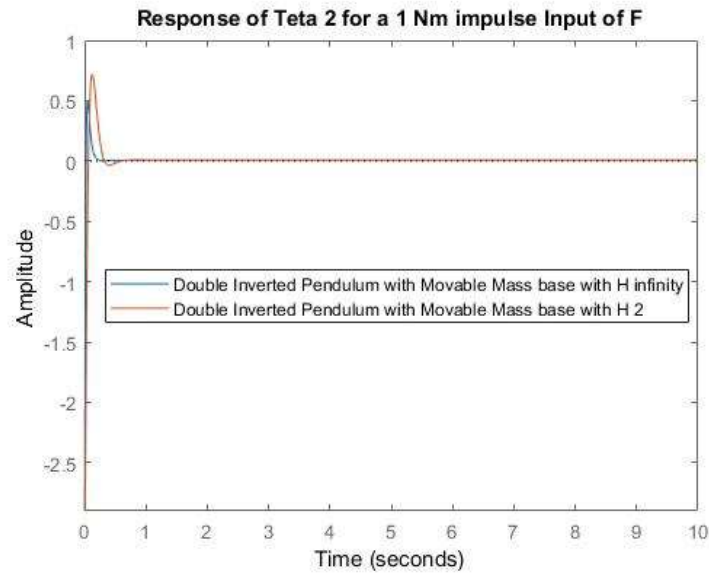Response of Teta 2 for a 1 Nm impulse Input of F

Figure 9 Impulse response the double inverted pendulum with Full State Feedback H infinity and H2 Controllers for Teta 2

The angular displacement simulation results shows that the double inverted pendulum with Full State Feedback H infinity controller improve the output of the system by minimizing the settling time and the percentage overshoot.

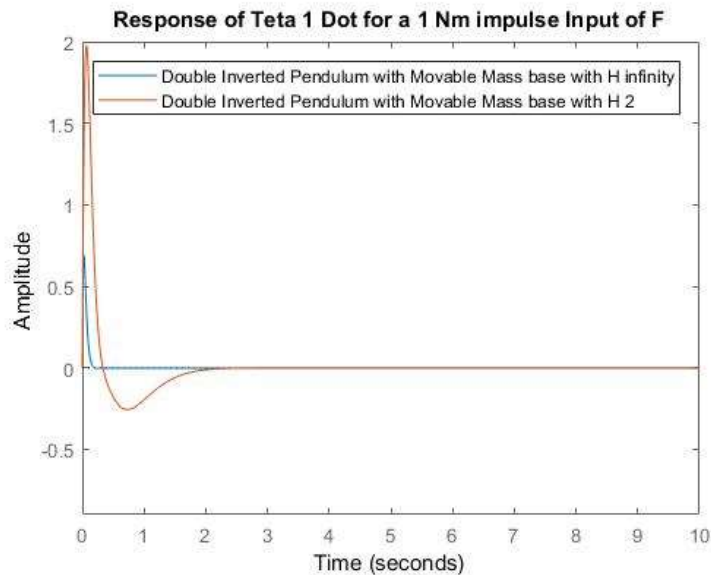Response of Teta 1 Dot for a 1 Nm impulse Input of F

Figure 10 Impulse response the double inverted pendulum with Full State Feedback H infinity and H2 Controllers for Teta 1 dot
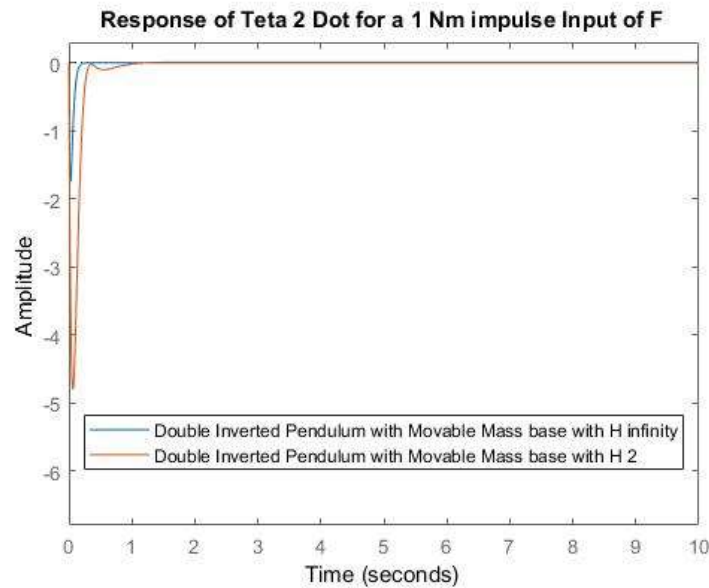
Figure 11 Impulse response the double inverted pendulum with Full State Feedback H infinity and H2 Controllers for Teta 2 dot

The angular velocity simulation results show that the double inverted pendulum with Full State Feedback H infinity controller improve the output of the system by minimizing the settling time and the percentage overshoot.

5. Conclusion

In this paper, the stability of the double inverted pendulum on a cart has been studied and analyzed using feedback control theory. The stability of the system for up rise position of the system have been simulated for the open loop and closed loop with the proposed controllers. The open loop response for the angular displacement and velocity of the two angles reveal that the double inverted pendulum is not stable and indeed the system needs a feedback control system. The closed loop response using the proposed controllers for the two angles position and velocity shows that the double inverted pendulum with full state feedback H infinity controller improve the output of the system by minimizing the settling time and the percentage overshoot. Finally, the simulation comparison results prove the effectiveness of the proposed controller full state feedback H infinity controller improves the stability of the system.

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