From MHD Induction Equation to the Basic Axisymmetric Dynamo Equations

We start from the magneto-hydrodynamic induction equation,

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} - \eta \, \vec{\nabla} \times \vec{B} \right) \tag{1}$$

Depending on the underlying theoretical model, we can split the large scale velocity $\vec{v}$ and the magnetic field $\vec{B}$ into their poloidal and toroidal components in a spherical polar coordinate system, as follows,

$$\vec{v}(r,\theta;t) = \vec{v_p} + \vec{v_t} = \left[v_r(r,\theta;t)\hat{r} + v_\theta(r,\theta;t)\hat{\theta} \right] + v_\phi(r,\theta;t)\hat{\phi} \tag{2}$$

$$\vec{B}(r,\theta;t) = \vec{B_p} + \vec{B_t} = \left[B_r(r,\theta;t)\hat{r} + B_\theta(r,\theta;t)\hat{\theta} \right] + B_\phi(r,\theta;t)\hat{\phi} \tag{3}$$

where,

$$v_\phi = (r \sin\theta) \; \Omega \tag{4}$$

$$\vec{B_p} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times A\hat{\phi} \tag{5}$$

Using equation (2) – (5) in equation (1),

$$\frac{\partial}{\partial t} \left[\left(\vec{\nabla} \times A\hat{\phi} \right) + B_\phi \hat{\phi} \right] = \vec{\nabla} \times \left[\left(\vec{v_p} + r\sin\theta \, \Omega \hat{\phi} \right) \times \left(\left(\vec{\nabla} \times A\hat{\phi} \right) + B_\phi \hat{\phi} \right) \right] - \; \vec{\nabla} \times \eta \left[\nabla \times \left(\left(\vec{\nabla} \times A\hat{\phi} \right) + B_\phi \hat{\phi} \right) \right] \tag{6}$$

Equation (6) can be decomposed in to a set of coupled *vector equations* for the poloidal and toroidal components of the magnetic fields respectively, as follows,

$$\frac{\partial}{\partial t} \left(\vec{\nabla} \times A\hat{\phi} \right) = \vec{\nabla} \times \left[\vec{v_p} \times \left(\vec{\nabla} \times A\hat{\phi} \right) \right] - \; \vec{\nabla} \times \eta \left[\vec{\nabla} \times \left(\vec{\nabla} \times A\hat{\phi} \right) \right] \tag{7}$$

$$\frac{\partial}{\partial t} \left(B_\phi \hat{\phi} \right) = \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right)_{(r,\theta)} - \; \vec{\nabla} \times \eta \left(\vec{\nabla} \times B_\phi \hat{\phi} \right) \tag{8}$$

Calculation for the poloidal components:

From equation (7),

$$\begin{aligned}
\frac{\partial A\hat{\phi}}{\partial t} &= \vec{v_p} \times \left(\vec{\nabla} \times A\hat{\phi} \right) - \eta \left[\nabla \times \left(\vec{\nabla} \times A\hat{\phi} \right) \right] \\
&= \nabla_A \left(\vec{v_p} \cdot A\hat{\phi} \right) - \left(\vec{v_p} \cdot \vec{\nabla} \right) A\hat{\phi} - \eta \left[\nabla \left(\vec{\nabla} \cdot A\hat{\phi} \right) - \vec{\nabla}^2 A\hat{\phi} \right] \\
&= \nabla_A \cancelto{0}{\left(\vec{v_p} \cdot A\hat{\phi} \right)} - \left(\vec{v_p} \cdot \vec{\nabla} \right) A\hat{\phi} - \eta \left[\nabla \cancelto{0}{\left(\frac{\partial A}{\partial \phi} \right)} - \vec{\nabla}^2 A\hat{\phi} \right] \\
\implies \frac{\partial A\hat{\phi}}{\partial t} &= \eta \, \vec{\nabla}^2 A\hat{\phi} - \left(\vec{v_p} \cdot \vec{\nabla} \right) A\hat{\phi} = \eta \left(\nabla^2 - \frac{1}{r^2 \sin^2\theta} \right) A\hat{\phi} - \left(\vec{v_p} \cdot \vec{\nabla} \right) A\hat{\phi} \\
\implies \frac{\partial A}{\partial t} &= \eta \left(\nabla^2 - \frac{1}{r^2 \sin^2\theta} \right) A - \left(\vec{v_p} \cdot \vec{\nabla} \right) A
\end{aligned} \tag{9}$$

Hence, the scalar equation for the poloidal component is,

$$\boxed{\frac{\partial A}{\partial t} = \eta \left(\nabla^2 - \frac{1}{r^2 \sin^2\theta} \right) A - \left(\vec{v_p} \cdot \vec{\nabla} \right) A}$$

where, in the last step the vector calculus identity for the **vector Laplacian operator** has been used:

$$\left(\nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial\left(A_\theta\sin\theta\right)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}\right)\hat{\mathbf{r}}$$
$$+\left(\nabla^2 A_\theta - \frac{A_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\varphi}{\partial\varphi}\right)\hat{\boldsymbol{\theta}}$$
$$+\left(\nabla^2 A_\varphi - \frac{A_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\varphi}\right)\hat{\boldsymbol{\varphi}}$$

Calculation for the toroidal component:

Using the vector calculus identity one can write,

$$\vec{\nabla}\times\eta\left(\vec{\nabla}\times B_\phi\hat{\phi}\right) = \eta\left[\vec{\nabla}\times\vec{\nabla}\times B_\phi\hat{\phi}\right] + \nabla\left[\eta(r)\right]\times\left(\vec{\nabla}\times B_\phi\hat{\phi}\right)$$

$$= \eta\left[\nabla\cancelto{0}{\left(\vec{\nabla}\cdot B_\phi\hat{\phi}\right)} - \vec{\nabla}^2 B_\phi\hat{\phi}\right] - \frac{1}{r\sin\theta}\frac{\partial\eta}{\partial r}\frac{\partial(r\sin\theta B_\phi)}{\partial r}\hat{\phi} \tag{10}$$
$$= -\eta\left(\nabla^2 - \frac{1}{r^2\sin^2\theta}\right)B_\phi\hat{\phi} - \frac{1}{r\sin\theta}\frac{\partial\eta}{\partial r}\frac{\partial(r\sin\theta B_\phi)}{\partial r}\hat{\phi}$$

And,

$$\vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)_{(r,\theta)} = \vec{\nabla}\times\left(\vec{v}_p\times B_\phi\hat{\phi} + v_\phi\hat{\phi}\times\vec{B}_p\right) \tag{11}$$

where,

$$\vec{\nabla}\times\left(\vec{v}_p\times B_\phi\hat{\phi}\right) = \frac{1}{r}\left[\frac{\partial}{\partial r}(-rv_rB_\phi) + \frac{\partial}{\partial\theta}(-v_\theta B_\phi)\right]\hat{\phi} \tag{12}$$

$$\vec{\nabla}\times\left(v_\phi\hat{\phi}\times\vec{B}_p\right) = v_\phi\hat{\phi}\left(\vec{\nabla}\cdot\vec{B}_p\right) - \vec{B}_p\left(\vec{\nabla}\cdot v_\phi\hat{\phi}\right) + \left(\vec{B}_p\cdot\vec{\nabla}\right)v_\phi\hat{\phi} + \left(v_\phi\hat{\phi}\cdot\vec{\nabla}\right)\vec{B}_p$$

$$= v_\phi\hat{\phi}\cancelto{0}{\left(\vec{\nabla}\cdot\vec{B}_p\right)} - \vec{B}_p\cancelto{0}{\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)} + \left(\vec{B}_p\cdot\vec{\nabla}\right)v_\phi\hat{\phi} + \cancelto{0}{\left(\frac{v_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)}\vec{B}_p \tag{13}$$

$$= \left(\vec{B}_p\cdot\vec{\nabla}\right)v_\phi\hat{\phi}$$

Using equations (10) – (13) in the equation (8),

$$\frac{\partial}{\partial t}\left(B_\phi\hat{\phi}\right) = \frac{1}{r}\left[\frac{\partial}{\partial r}(-rv_rB_\phi) + \frac{\partial}{\partial\theta}(-v_\theta B_\phi)\right]\hat{\phi} + \left(\vec{B}_p\cdot\vec{\nabla}\right)v_\phi\hat{\phi} - \eta\left(\nabla^2 - \frac{1}{r^2\sin^2\theta}\right)B_\phi\hat{\phi} - \frac{1}{r\sin\theta}\frac{\partial\eta}{\partial r}\frac{\partial(r\sin\theta B_\phi)}{\partial r}\hat{\phi} \tag{14}$$

Hence, the scalar equation for the toroidal component is:

$$\boxed{\frac{\partial B_\phi}{\partial t} = \left(\vec{B}_p\cdot\vec{\nabla}\right)v_\phi - \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_rB_\phi) + \frac{\partial}{\partial\theta}(v_\theta B_\phi)\right] + \eta\left(\nabla^2 - \frac{1}{r^2\sin^2\theta}\right)B_\phi + \frac{1}{r}\frac{\partial\eta}{\partial r}\frac{\partial(rB_\phi)}{\partial r}}$$

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