Assignment 1 Solution

1. We derived a linear relationship between pressure (P) and depth (z) for water at rest : $P(z) = P_0 - \rho g z$. Now, if the water is accelerating along x-axis and gravity, as usual, along z-axis, how will the pressure $P(x, z)$ expression be modified?

Solution 1.

Considering Euler equation for this inviscid, accelerated fluid with an acceleration $\vec{a}$ along the x-axis, we can write,

$$\rho\vec{a} = -\nabla P + \rho\vec{g}$$

Separating into vector components we obtain two partial differential equations,

$$\frac{\partial P(x,z)}{\partial z} = \rho\vec{g}\cdot\hat{k} = -\rho g \qquad \text{and,} \quad -\frac{\partial P(x,z)}{\partial x} = \rho\vec{a}\cdot\hat{i} = \rho a$$

Simultaneous solution of which gives,

$$P(x,z) = -\rho g z - \rho a x + \text{constant}$$

Imposing the boundary condition, $P(x,z)|_{x=0,z=0} = \mathcal{P}_0$, the final solution is obtained to be,

$$P(x,z) = \mathcal{P}_0 - \rho g z - \rho a x$$

2. In the Mid-sem exam following problem was given: If an object heavier than water is fully immersed in water, the net force exerted on the object by surrounding water is $-Mg$, where M is mass of water displaced by the object. How does this net force change when water is accelerating? Results from prob. 1 may help.

Solution 2.

By definition, the net force exerted on a body by a fluid that it is fully immersed in,

$$\vec{F} = -\oint_s P d\vec{s}$$

Here, a negative sign appears because the surface normals are anti-parallel to the direction of force exerted on the immersed object by the fluid. Applying Gauss's divergence theorem the previous equation can be restructured as,

$$\vec{F} = -\int_V \nabla P \; dV$$

Using results from previous solution,

$$\vec{F} = -\int_V \rho(\vec{g}-\vec{a})\; dV = -(\vec{g}-\vec{a})\int \rho dV = -M(\vec{g}-\vec{a})$$

Hence, in an accelerated fluid with acceleration $\vec{a}$, the net force on the immersed object is, $-M(\vec{g}-\vec{a})$, where M is the mass of the dispersed fluid.

3. Starting from the following energy conservation equation show that one can arrive at the later one,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2+\rho\epsilon\right)+\frac{\partial}{\partial x_i}\left[\left(\frac{1}{2}\rho u^2+\rho\epsilon+P\right)u_i\right]=\rho u_i F_i$$

$$\frac{\partial \epsilon}{\partial t}+u_i\frac{\partial \epsilon}{\partial x_i}+\frac{P}{\rho}\frac{\partial u_i}{\partial x_i}=0$$

Solution 3.

We start from the following equation,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}p u^2+\rho\epsilon\right)+\frac{\partial}{\partial x_i}\left[\left(\frac{1}{2}\rho u^2+p\epsilon+p\right)u_i\right]=\rho u_i F_i$$

Now, using the chain rule of derivative the above equation can be expanded as,

$$\begin{aligned}
&\underset{(A)}{\frac{u^2}{2}\frac{\partial \rho}{\partial t}}+\underset{(B)}{\rho u_i\frac{\partial u_i}{\partial t}}+\underset{(C)}{\epsilon\frac{\partial \rho}{\partial t}}+\underset{(D)}{\rho\frac{\partial \epsilon}{\partial t}}\\
&\quad+\underset{(E)}{\frac{1}{2}\rho u^2\frac{\partial u_i}{\partial x_i}}+\underset{(F)}{\frac{u^2 u_i}{2}\frac{\partial \rho}{\partial x_i}}+\underset{(G)}{\frac{1}{2}\rho u_i\frac{\partial}{\partial x_i}\left(u^2\right)}\\
&\qquad+\underset{(H)}{\rho\epsilon\frac{\partial u_i}{\partial x_i}}+\underset{(I)}{\epsilon u_i\frac{\partial \rho}{\partial x_i}}+\underset{(J)}{\rho u_i\frac{\partial \epsilon}{\partial x_i}}\\
&\qquad\quad+\underset{(K)}{u_i\frac{\partial p}{\partial x_i}}+\underset{(L)}{p\frac{\partial u_i}{\partial x_i}}-\underset{(M)}{\rho u_i F_i}=0
\end{aligned}$$

Now, $(A)+(E)+(F)$ give:

$$\frac{u^2}{2}\left[\frac{\partial \rho}{\partial t}+\rho\frac{\partial u_i}{\partial x_i}+u_i\frac{\partial \rho}{\partial x_i}\right]=\frac{u^2}{2}\times 0=0 \quad [\text{ Mass conservation}]$$

Similarly, $(C)+(H)+(I)$ give:

$$\epsilon\left[\frac{\partial \rho}{\partial t}+\rho\frac{\partial u_i}{\partial x_i}+u_i\frac{\partial \rho}{\partial x_i}\right]=\epsilon\times 0=0 \quad [\text{ Mass conservation}]$$

And, $(B)+(G)+(K)+(M)$ give:

$$\begin{aligned}
&\rho u_i\left[\frac{\partial u_i}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x_i}\left(u^2\right)+\frac{1}{p}\frac{\partial p}{\partial x_i}-F_i\right]\\
=&\rho u_i\left[\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_i}{\partial x_i}+\frac{1}{p}\frac{\partial p}{\partial x_i}-F_i\right]=0 \quad [\text{ Momentum conservation}]
\end{aligned}$$

Remaining terms $(D)+(J)+(L)$ give:

$$\begin{aligned}
&\rho\left[\frac{\partial \epsilon}{\partial t}+u_i\frac{\partial \epsilon}{\partial x_i}+\frac{p}{\rho}\frac{\partial u_i}{\partial x_i}\right]=0\\
\therefore\ &\frac{\partial \epsilon}{\partial t}+u_i\frac{\partial \epsilon}{\partial x_i}+\frac{p}{\rho}\frac{\partial u_i}{\partial x_i}=0
\end{aligned}$$

4. In many simulations, we use scaled units instead of real units and the underlying equations gets modified as per our re-definition of units.

   a) Consider the ideal, non-viscous hydrodynamics equations in conservative form (e.g., mass, momentum and energy density equations). Write these equations in spherical polar coordinates (r, θ, ϕ) and consider 1D flow along radial direction only (i.e., throw away $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \phi}$ terms.)

   b) Now, we wish to study the accretion problem onto a compact star of mass M by solving these equations. We measure lengths in units of GM/c^2, velocity in units of c. Also, we measure density in units of a reference density ρ_{ref} such that at certain radius R_{max}, the radial mass flux is equal to the accretion rate $\dot{m}$ g/sec. Write down the above 1D hydro-equations in this unit system. (Here, G is gravitational constant, c is speed of light, we use ideal gas and adiabatic equation of state $P = K\rho^\gamma$. Also, explicitly mention any assumption that you make while solving this problem.)

Solution 4.

a) For a one-dimensional inviscid fluid flow along radial direction in a spherical polar coordinate system, the Euler equations are,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho u_r\right) = 0$$

and,

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} = -\frac{1}{\rho}\frac{dP}{dr} + f_r$$

Here, f_r denotes the radial component of the body force.

b) We assume a **steady state condition** i.e., $\frac{\partial}{\partial t}$ of any dynamical quantity is zero. Therefore, the continuity equation can be written as,

$$4\pi r^2 \rho u_r = \text{constant} = \dot{\mathcal{M}}$$

where, $\dot{\mathcal{M}}$ is the mass accretion rate. Now, in the new unit system following equation is to be satisfied,

$$4\pi \left(\frac{R_{\text{max}}}{r_c}\right)^2 \left(\frac{\rho}{\rho_{\text{ref}}}\right) \left(\frac{u_r}{c}\right) = \dot{m}$$

Here, $r_c = GM/c^2$

5. Find out the solution of spherical accretion/wind using adiabatic condition. These are famous Bondi accretion and Parker wind solutions.

Solution 5.

Let us consider a spherically symmetric 1D, steady, inviscid, adiabatic, purely hydrodynamic flow around an object of mass M. The conservation equations and the equation of state are,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho v\right) = 0 \tag{5.1}$$

$$v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM}{r^2} \tag{5.2}$$

$$P = K\rho^\gamma \tag{5.3}$$

$$c_s^2 = \frac{\gamma P}{\rho} \tag{5.4}$$

Simplifying (5.1) we can write,

$$\frac{2}{r} + \frac{1}{\rho}\frac{\partial\rho}{\partial r} + \frac{1}{v}\frac{\partial v}{\partial r} = 0 \tag{5.5}$$

Taking derivative of (5.3) with respect to r, we can write,

$$\frac{dP}{dr} = K\gamma\rho^{\gamma-1}\left(\frac{\partial\rho}{\partial r}\right)$$

$$\therefore \frac{1}{\rho}\frac{dP}{dr} = K\rho^{\gamma-2}\left(\frac{\partial\rho}{\partial r}\right) = \frac{\gamma}{\rho}K\rho^{\gamma-1}\left(\frac{\partial\rho}{\partial r}\right) = \frac{c_s^2}{\rho}\left(\frac{\partial\rho}{\partial r}\right) \tag{5.6}$$

Using (5.6) in (5.2) to replace the pressure gradient term,

$$v\frac{\partial v}{\partial r} = -c_s^2\left(\frac{1}{\rho}\frac{\partial\rho}{\partial r}\right) - \frac{GM}{r^2} \tag{5.7}$$

Eliminating the density gradient term from (5.5) and (5.7),

$$v\frac{\partial v}{\partial r} = c_s^2\left(\frac{2}{r} + \frac{1}{v}\frac{\partial v}{\partial r}\right) - \frac{GM}{r^2}$$

$$\therefore \left(v^2 - c_s^2\right)\frac{1}{v}\frac{\partial v}{\partial r} = \frac{2}{r}c_s^2 - \frac{GM}{r^2}$$

Using the definition of the critical sonic radius, $r_c = GM/2c_s^2$, we can write the above equation as,

$$\left(v^2 - c_s^2\right)\frac{1}{v}\frac{\partial v}{\partial r} = \frac{2c_s^2}{r^2}\left(r - r_c\right) \tag{5.8}$$

This is a separable ODE, which can be integrated as follows,

$$\int\left(v - \frac{c_s^2}{v}\right)dv = \int\frac{2c_s^2}{r^2}\left(r - r_c\right)dr \tag{5.9}$$