

1D Compressible Gas Dynamics of a Perfect gas

Analytical Calculations: Part I

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Let us list down the basic thermodynamic properties of a perfect gas,

- (1) Equation of state: $p = \rho RT$ where, $R = \frac{k_B}{m}$
- (2) Internal energy: $\epsilon = C_V T = \left(\frac{R}{\gamma - 1}\right) T$
- (3) Entropy (per unit mass): $T dS = d\epsilon + p d\left(\frac{1}{\rho}\right)$

Using (2) in (3),

$$\begin{aligned} dS &= \left(\frac{R}{\gamma - 1}\right) \frac{dT}{T} + \frac{p}{T} \left(-\frac{1}{\rho^2}\right) d\rho \\ &= C_V \frac{dT}{T} - \frac{R}{\rho} d\rho \\ &= C_V \left[\frac{dp}{p} - \frac{d\rho}{\rho}\right] - R \frac{d\rho}{\rho} \quad (\text{using (1)}) \\ &= C_V \frac{dp}{p} - \frac{d\rho}{\rho} (C_V + R) \\ &= C_V \frac{dp}{p} - C_p \frac{d\rho}{\rho} \\ \therefore dS &= C_v \left[\frac{dp}{p} - \gamma \frac{d\rho}{\rho}\right] \end{aligned}$$

Integrating we can write,

$$\begin{aligned} \frac{S - S_0}{C_V} &= \ln p - \gamma \ln \rho = \ln \left(\frac{p}{\rho^\gamma}\right) \\ \therefore S &= C_V \ln \left(\frac{p}{\rho^\gamma}\right) + S_0 \end{aligned} \tag{4}$$

For an adiabatic process, $\frac{dS}{dt} = 0$

$$\therefore \frac{d}{dt} \left(\frac{p}{\rho^\gamma}\right) = 0 \tag{5}$$

Enthalpy (per unit mass),

$$W = \epsilon + \frac{p}{\rho} = \frac{R}{\gamma - 1} T + RT = \left(1 + \frac{1}{\gamma - 1}\right) RT$$

$$\therefore W = \frac{\gamma}{\gamma - 1} RT \quad (6)$$

Moreover,

$$dW = d\epsilon + p d\left(\frac{1}{\rho}\right) + \frac{1}{\rho} dp$$

$$\therefore dW = T dS + \frac{dp}{\rho} \quad (\text{using (3)})$$

Under adiabatic condition (i.e., $dS = 0$),

$$\int \frac{dp}{\rho} = W \quad (6a)$$

Hence, for a perfect gas under the adiabatic condition the Bernoulli's equation $\left[\frac{1}{2}v^2 + \int \frac{dp}{\rho} + \Phi = \text{constant}\right]$ reduces to,

$$\frac{1}{2}v^2 + \frac{\gamma}{\gamma - 1} RT + \Phi = \text{constant} \quad (7)$$

And, from (5),

$$\frac{1}{\rho^\gamma} \frac{dp}{dt} = \gamma p \frac{1}{\rho^{(\gamma+1)}} \frac{d\rho}{dt}$$

$$\therefore dp = \frac{\gamma p}{\rho} d\rho$$

$$\therefore c_s = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{\gamma p}{\rho}}$$

Here, c_s is the speed of sound propagation through this gas in adiabatic condition.

Small amplitude acoustic waves: Any process causing a pressure (or/and) density fluctuation (i.e. perturbation) in a gas (compressible medium) is expected to generate acoustic wave. Here, we consider homogeneous perfect gas of density ρ_0 and pressure p_0 in the absence of any external force. We show that linear perturbation in pressure $\rho_1(\vec{x}, t)$ and density $\rho_1(\vec{x}, t)$ will give rise to acoustic wave.

These perturbations will give rise to an additional velocity field $\vec{v}_1(\vec{x}, t)$. Then the continuity equation in this case can be written as,

$$\frac{\partial(\rho_0 + \rho_1)}{\partial t} + \vec{\nabla} \cdot [(\rho_0 + \rho_1)(\vec{v}_0 + \vec{v}_1)] = 0$$

$$\therefore \left\{ \frac{\partial \rho_0}{\partial t} + \vec{\nabla} \cdot (\rho_0 \vec{v}_0) \right\} + \frac{\partial \rho_1}{\partial t} + \vec{\nabla} \cdot (\rho_0 \vec{v}_1) + \vec{\nabla} \cdot (\rho_1 \vec{v}_0) + \underbrace{\vec{\nabla} \cdot (\rho_1 \vec{v}_1)}_{\text{linearization}} = 0$$

$$\therefore \frac{\partial \rho_1}{\partial t} + \underbrace{\vec{v}_1 \cdot \vec{\nabla} \rho_0}_{\text{zero; homogeneity}} + \vec{v}_0 \cdot \vec{\nabla} \rho_1 + \rho_0 (\vec{\nabla} \cdot \vec{v}_1) + \rho_1 (\vec{\nabla} \cdot \vec{v}_0) = 0$$

$$\therefore \frac{d}{dt}(\rho_0 \xi) + \rho_0 (\vec{\nabla} \cdot \vec{v}_1) + (\rho_0 \xi) \vec{\nabla} \cdot \vec{v}_0 = 0 \quad [\text{putting } \xi = \frac{\rho_1}{\rho_0}]$$

$$\therefore \rho_0 \frac{d\xi}{dt} + \xi \frac{d\rho_0}{dt} + \xi (\rho_0 \vec{\nabla} \cdot \vec{v}_0) + \rho_0 (\vec{\nabla} \cdot \vec{v}_1) = 0$$

$$\begin{aligned}
\therefore \xi \left[\frac{d\rho_0}{dt} + \rho_0 \vec{\nabla} \cdot \vec{v}_0 \right] + \rho_0 \frac{\partial \xi}{\partial t} + \rho_0 (\vec{\nabla} \cdot \vec{v}_1) &= 0 \\
\therefore \frac{d\xi}{dt} + (\vec{\nabla} \cdot \vec{v}_1) &= 0 \\
\therefore \frac{1}{\rho_0} \frac{d\rho_1}{dt} - \frac{\rho_1}{\rho_0^2} \frac{d\rho_0}{dt} + (\vec{\nabla} \cdot \vec{v}_1) &= 0 \\
\therefore \frac{d\rho_1}{dt} + \rho_0 (\vec{\nabla} \cdot \vec{v}_1) &= 0 \\
\therefore \frac{\partial \rho_1}{\partial t} + \rho_0 (\vec{\nabla} \cdot \vec{v}_1) + (\vec{v}_0 \cdot \vec{\nabla}) \rho_1 &= 0
\end{aligned}$$

Here, the non zero fluid velocity and the term associated with it (Compression) acts as the source in the continuity equation.

The Euler equation, under the same linear perturbation reduces to,

$$\begin{aligned}
(\rho_0 + \rho_1) \left[\frac{\partial \vec{v}_1}{\partial t} + (\vec{v}_1 \cdot \vec{\nabla}) \vec{v}_1 + (\vec{v}_0 \cdot \vec{\nabla}) \cdot \vec{v}_0 + (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_1 + (\vec{v}_1 \cdot \vec{\nabla}) \vec{v}_0 \right] &= -\nabla \rho_1 = -\nabla (c_s^2 \rho_1) \\
\therefore \rho_0 \frac{\partial \vec{v}_1}{\partial t} + \rho_1 (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_0 + \rho_0 (\vec{v}_1 \cdot \vec{\nabla}) \vec{v}_0 + \rho_0 (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_0 + \rho_0 (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_1 &= -c_s^2 \nabla \rho_1 \\
\therefore \vec{\nabla} \cdot \left[\rho_0 \frac{\partial \vec{v}_1}{\partial t} + \rho_1 (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_0 + \rho_0 (\vec{v}_1 \cdot \vec{\nabla}) \vec{v}_0 + \rho_0 (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_0 + \rho_0 (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_1 \right] &= -c_s^2 \nabla^2 \rho_1 \\
\therefore \frac{\partial}{\partial t} \left[\rho_0 (\vec{\nabla} \cdot \vec{v}_1) \right] + \vec{\nabla} \cdot [\] &= -c_s^2 \nabla^2 \rho_1 \\
\therefore -\frac{\partial^2 \rho_1}{\partial t^2} + \vec{\nabla} \cdot [\] &= -c_s^2 \nabla^2 \rho_1 \\
\therefore \square^2 \rho_1 - \text{source term} &= \vec{\nabla} \cdot [\]
\end{aligned}$$

Note: For a truly incompressible fluid, a very large excess pressure dP would produce a negligible change in density $d\rho$, thus, $c_s^2 = \sqrt{\frac{dP}{d\rho}} \rightarrow \infty$, hence the sound speed in such a medium is infinite.

Note: In an incompressible fluid system, any instability is wiped out instantaneously.