

# 1D Compressible Gas Dynamics of a Perfect gas

## Analytical Calculations: Part I

Teaching Assistant: Chitradeep Saha

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Let us list down the basic thermodynamic properties of a perfect gas,

- (1) Equation of state:  $p = \rho RT$  where,  $R = \frac{k_B}{m}$
- (2) Internal energy:  $\epsilon = C_V T = \left(\frac{R}{\gamma - 1}\right) T$
- (3) Entropy (per unit mass):  $T dS = d\epsilon + p d\left(\frac{1}{\rho}\right)$

Using (2) in (3),

$$\begin{aligned} dS &= \left(\frac{R}{\gamma - 1}\right) \frac{dT}{T} + \frac{p}{T} \left(-\frac{1}{\rho^2}\right) d\rho \\ &= C_V \frac{dT}{T} - \frac{R}{\rho} d\rho \\ &= C_V \left[\frac{dp}{p} - \frac{d\rho}{\rho}\right] - R \frac{d\rho}{\rho} \quad (\text{using (1)}) \\ &= C_V \frac{dp}{p} - \frac{d\rho}{\rho} (C_V + R) \\ &= c_v \frac{dp}{p} - C_p \frac{d\rho}{\rho} \\ \therefore ds &= c_v \left[\frac{dp}{p} - \gamma \frac{d\rho}{\rho}\right] \end{aligned}$$

Integrating we can write,

$$\begin{aligned} \frac{S - S_0}{C_V} &= \ln p - \gamma \ln \rho = \ln \left(\frac{p}{\rho^\gamma}\right) \\ \therefore S &= C_V \ln \left(\frac{p}{\rho^\gamma}\right) + S_0 \end{aligned} \tag{4}$$

For an adiabatic process,  $\frac{dS}{dt} = 0$

$$\therefore \frac{d}{dt} \left(\frac{p}{\rho^\gamma}\right) = 0 \tag{5}$$

Enthalpy (per unit mass),

$$W = \epsilon + \frac{p}{\rho} = \frac{R}{\gamma - 1} T + RT = \left(1 + \frac{1}{\gamma - 1}\right) RT$$

$$\therefore W = \frac{\gamma}{\gamma - 1} RT \quad (6)$$

Moreover,

$$dW = d\epsilon + p d\left(\frac{1}{\rho}\right) - \frac{1}{\rho} dp$$

$$\therefore dW = T dS - \frac{dp}{\rho} \quad (\text{using (3)})$$

For adiabatic flows (i.e.,  $ds = 0$ ),

$$\int \frac{dp}{\rho} = W \quad (6a)$$

Hence, for a perfect gas under the adiabatic condition the Bernoulli's equation  $\left[\frac{1}{2}v^2 + \int \frac{dp}{\rho} + \Phi = \text{constant}\right]$  reduces to,

$$\frac{1}{2}v^2 + \frac{\gamma}{\gamma - 1} RT + \Phi = \text{constant} \quad (7)$$

And, from (5),

$$\frac{1}{\rho\gamma} \frac{dp}{dt} = \gamma p \frac{1}{\rho(\gamma + 1)} \frac{d\rho}{dt}$$

$$\therefore dp = \frac{\gamma p}{\rho} d\rho$$

$$\therefore c_s = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{\gamma p}{\rho}}$$

Here,  $c_s$  is the speed of sound propagation through this gas in adiabatic condition.

**Small amplitude acoustic waves:** Any process causing a pressure (or/and) density fluctuation (i.e. perturbation) in a gas (compressible medium) is expected to generate acoustic wave. Here, we consider homogeneous perfect gas of density  $\rho_0$  and pressure  $p_0$  in the absence of any external force. We show that linear perturbation in pressure  $\rho_1(\vec{x}, t)$  and density  $\rho_1(\vec{x}, t)$  will give rise to acoustic wave.

These perturbations will give rise to an additional velocity field  $\vec{v}_1(\vec{x}, t)$ . Then the continuity equation in this case can be written as,

$$\frac{\partial(\rho_0 + \rho_1)}{\partial t} + \vec{\nabla} \cdot [(\rho_0 + \rho_1)(\vec{v}_0 + \vec{v}_1)] = 0$$

$$\therefore \left\{ \frac{\partial \rho_0}{\partial t} + \vec{\nabla} \cdot (\rho_0 \vec{v}_0) \right\} + \frac{\partial \rho_1}{\partial t} + \vec{\nabla} \cdot (\rho_0 \vec{v}_1) + \vec{\nabla} \cdot (\rho_1 \vec{v}_0) + \underbrace{\vec{\nabla} \cdot (\rho_1 \vec{v}_1)}_{\text{linearization}} = 0$$

$$\therefore \frac{\partial \rho_1}{\partial t} + \underbrace{\vec{v}_1 \cdot \vec{\nabla} \rho_0}_{\text{zero; homogeneity}} + \vec{v}_0 \cdot \vec{\nabla} \rho_1 + \rho_0 (\vec{\nabla} \cdot \vec{v}_1) + \rho_1 (\vec{\nabla} \cdot \vec{v}_0) = 0$$

$$\therefore \frac{d}{dt} (\rho_0 \xi) + \rho_0 (\vec{\nabla} \cdot \vec{v}_1) + (\rho_0 \xi) \vec{\nabla} \cdot \vec{v}_0 = 0 \quad [\text{putting } \xi = \frac{\rho_1}{\rho_0}]$$

$$\therefore \rho_0 \frac{d\xi}{dt} + \xi \frac{d\rho_0}{dt} + \xi (\rho_0 \vec{\nabla} \cdot \vec{v}_0) + \rho_0 (\vec{\nabla} \cdot \vec{v}_1) = 0$$

$$\begin{aligned}
&\therefore \xi \left[ \frac{d\rho_0}{dt} + \rho_0 \vec{\nabla} \cdot \vec{v}_0 \right] + \rho_0 \frac{\partial \xi}{\partial t} + \rho_0 (\vec{\nabla} \cdot \vec{v}_1) = 0 \\
&\therefore \frac{d\xi}{dt} + (\vec{\nabla} \cdot \vec{v}_1) = 0 \\
&\therefore \frac{1}{\rho_0} \frac{d\rho_1}{dt} - \frac{\rho_1}{\rho_0^2} \frac{d\rho_0}{dt} + (\vec{\nabla} \cdot \vec{v}_1) = 0 \\
&\therefore \frac{d\rho_1}{dt} + \rho_0 (\vec{\nabla} \cdot \vec{v}_1) = 0 \\
&\therefore \frac{\partial \rho_1}{\partial t} + \rho_0 (\vec{\nabla} \cdot \vec{v}_1) + (\vec{v}_0 \cdot \vec{\nabla}) \rho_1 = 0
\end{aligned}$$

Here, the non zero fluid velocity and the term associated with it (Compression) acts as the source in the continuity equation.

The Euler equation, under the same linear perturbation reduces to,

$$\begin{aligned}
&(\rho_0 + \rho_1) \left[ \frac{\partial \vec{v}_1}{\partial t} + (\vec{v}_1 \cdot \vec{\nabla}) \vec{v}_1 + (\vec{v}_0 \cdot \vec{\nabla}) \cdot \vec{v}_0 + (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_1 + (\vec{v}_1 \cdot \vec{\nabla}) \vec{v}_0 \right] = -\nabla \rho_1 = -\nabla (c_s^2 \rho_1) \\
&\therefore \rho_0 \frac{\partial \vec{v}_1}{\partial t} + \rho_1 (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_0 + \rho_0 (\vec{v}_1 \cdot \vec{\nabla}) \vec{v}_0 + \rho_0 (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_0 + \rho_0 (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_1 = -c_s^2 \nabla \rho_1 \\
&\therefore \vec{\nabla} \cdot \left[ \rho_0 \frac{\partial \vec{v}_1}{\partial t} + \rho_1 (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_0 + \rho_0 (\vec{v}_1 \cdot \vec{\nabla}) \vec{v}_0 + \rho_0 (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_0 + \rho_0 (\vec{v}_0 \cdot \vec{\nabla}) \vec{v}_1 \right] = -c_s^2 \nabla^2 \rho_1 \\
&\therefore \frac{\partial}{\partial t} \left[ \rho_0 (\vec{\nabla} \cdot \vec{v}_1) \right] + \vec{\nabla} \cdot [ \ ] = -c_s^2 \nabla^2 \rho_1 \\
&\therefore -\frac{\partial^2 \rho_1}{\partial t^2} + \vec{\nabla} \cdot [ \ ] = -c_s^2 \nabla^2 \rho_1 \\
&\therefore \square^2 \rho_1 - \text{source term} = \vec{\nabla} \cdot [ \ ]
\end{aligned}$$

**Note:** For a truly incompressible fluid, a very large excess pressure  $dP$  would produce a negligible change in density  $d\rho$ , thus,  $c_s^2 = \sqrt{\frac{dp}{d\rho}} \rightarrow \infty$ , hence the sound speed in such a medium is infinite.

**Note:** In an incompressible fluid system, any instability is wiped out instantaneously.