

1D Compressible Gas Dynamics of a Perfect gas

Analytical Calculations: Part I

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Let us list down the basic thermodynamic properties of a perfect gas,

- (1) Equation of state: $p = \rho RT$ where, $R = \frac{k_B}{m}$
- (2) Internal energy: $\epsilon = C_V T = \left(\frac{R}{\gamma - 1}\right) T$
- (3) Entropy (per unit mass): $T dS = d\epsilon + p d\left(\frac{1}{\rho}\right)$

From (1),

$$\log p = \log \rho + \log R + \log T$$

Taking differentials on both side,

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \quad (1a)$$

Using (2) in (3),

$$\begin{aligned} dS &= \left(\frac{R}{\gamma - 1}\right) \frac{dT}{T} + \frac{p}{T} \left(-\frac{1}{\rho^2}\right) d\rho \\ &= C_V \frac{dT}{T} - \frac{R}{\rho} d\rho \\ &= C_V \left[\frac{dp}{p} - \frac{d\rho}{\rho}\right] - R \frac{d\rho}{\rho} \quad (\text{using (1a)}) \\ &= C_V \frac{dp}{p} - \frac{d\rho}{\rho} (C_V + R) \quad (\text{using, } C_P - C_V = R) \\ &= C_V \frac{dp}{p} - C_P \frac{d\rho}{\rho} \\ \therefore dS &= C_V \left[\frac{dp}{p} - \gamma \frac{d\rho}{\rho}\right] \quad (\text{using, } C_P/C_V = \gamma) \end{aligned}$$

Integrating and using S_0 as the integration constant we can write,

$$\begin{aligned} \frac{S - S_0}{C_V} &= \ln p - \gamma \ln \rho = \ln \left(\frac{p}{\rho^\gamma}\right) \\ \therefore S &= C_V \ln \left(\frac{p}{\rho^\gamma}\right) + S_0 \end{aligned} \quad (4)$$

For an adiabatic process,

$$\frac{dS}{dt} = 0$$

Using (4) we can write,

$$\begin{aligned} \frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) &= 0 \\ \therefore \frac{1}{\rho^\gamma} \frac{dp}{dt} &= \frac{\gamma p}{\rho^{(\gamma+1)}} \frac{d\rho}{dt} \\ \therefore dp &= \frac{\gamma p}{\rho} d\rho \end{aligned} \tag{5}$$

$$\therefore \text{by definition of sound speed, } c_s = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{\gamma p}{\rho}}$$

Here, c_s is the speed of sound propagation through this gas under adiabatic condition.

Defining the speed of sound in non-relativistic regime

Consider the sound wave propagating at speed v through a pipe aligned with the x axis and with a cross-sectional area of A . In time interval dt it moves length $dx = vdt$. In steady state, the mass flow rate $\dot{m} = \rho v A$ must be the same at the two ends of the tube, therefore the mass flux $j = \rho v$ is constant and $v d\rho = -\rho dv$. As per Newton's second law, the pressure gradient force provides the acceleration:

$$\begin{aligned} \frac{dv}{dt} &= -\frac{1}{\rho} \frac{dP}{dx} \\ \therefore dP &= (-\rho dv) \frac{dx}{dt} = (v d\rho) v \\ \therefore v^2 &\equiv c_s^2 = \frac{dP}{d\rho} \end{aligned}$$

And therefore:

$$c_s = \sqrt{\left(\frac{\partial P}{\partial \rho} \right)}$$

Now, enthalpy (per unit mass),

$$\begin{aligned} W = \epsilon + \frac{p}{\rho} &= \frac{R}{\gamma - 1} T + RT = \left(1 + \frac{1}{\gamma - 1} \right) RT \\ \therefore W &= \frac{\gamma}{\gamma - 1} RT \end{aligned} \tag{6}$$

Moreover,

$$\begin{aligned} dW &= d\epsilon + p d \left(\frac{1}{\rho} \right) + \frac{1}{\rho} dp \\ \therefore dW &= T dS + \frac{dp}{\rho} \quad (\text{using (3)}) \end{aligned}$$

Under adiabatic condition (i.e., $dS = 0$),

$$\int \frac{dp}{\rho} = W \quad (6a)$$

Hence, for a perfect gas under the adiabatic condition the Bernoulli's equation $\left[\frac{1}{2}v^2 + \int \frac{dp}{\rho} + \Phi = \text{constant}\right]$ reduces to,

$$\frac{1}{2}v^2 + \frac{\gamma}{\gamma-1}RT + \Phi = \text{constant} \quad (7)$$

Small amplitude acoustic wave propagation in a homogeneous, perfect, compressible gas

Any process causing a pressure (or/and) density fluctuation (i.e. perturbation) in a gas (compressible medium) is expected to generate acoustic wave. Here, we consider homogeneous perfect gas of density ρ_0 and pressure p_0 **in the absence of any external force**. We show that linear perturbation in pressure $p_1(\vec{x}, t)$ and density $\rho_1(\vec{x}, t)$ will give rise to acoustic wave.

These perturbations will also give rise to an *additional* velocity field $\vec{v}_1(\vec{x}, t)$ on top of **previously existing velocity field** $\vec{v}_0(\vec{x}, t)$. Then the continuity equation in this case can be written as,

$$\begin{aligned} & \frac{\partial(\rho_0 + \rho_1)}{\partial t} + \vec{\nabla} \cdot [(\rho_0 + \rho_1)(\vec{v}_0 + \vec{v}_1)] = 0 \\ \therefore & \underbrace{\frac{\partial\rho_0}{\partial t} + \vec{\nabla} \cdot (\rho_0\vec{v}_0)}_{\text{cont. eqn.}} + \frac{\partial\rho_1}{\partial t} + \vec{\nabla} \cdot (\rho_0\vec{v}_1) + \vec{\nabla} \cdot (\rho_1\vec{v}_0) + \underbrace{\vec{\nabla} \cdot (\rho_1\vec{v}_1)}_{\text{linearization}} = 0 \\ \therefore & \frac{\partial\rho_1}{\partial t} + \underbrace{\vec{v}_1 \cdot \vec{\nabla}\rho_0}_{\text{zero; homogeneity}} + \vec{v}_0 \cdot \vec{\nabla}\rho_1 + \rho_0(\vec{\nabla} \cdot \vec{v}_1) + \rho_1(\vec{\nabla} \cdot \vec{v}_0) = 0 \\ \therefore & \frac{\partial}{\partial t}(\rho_0\xi) + \rho_0(\vec{\nabla} \cdot \vec{v}_1) + (\rho_0\xi)\vec{\nabla} \cdot \vec{v}_0 = 0 \quad [\text{putting } \xi = \frac{\rho_1}{\rho_0}] \\ \therefore & \rho_0 \frac{\partial\xi}{\partial t} + \xi \frac{\partial\rho_0}{\partial t} + \rho_0(\vec{\nabla} \cdot \vec{v}_1) + \xi(\rho_0\vec{\nabla} \cdot \vec{v}_0) = 0 \\ \therefore & \xi \underbrace{\left[\frac{\partial\rho_0}{\partial t} + \rho_0\vec{\nabla} \cdot \vec{v}_0\right]}_{\text{cont. eqn.}} + \rho_0 \frac{\partial\xi}{\partial t} + \rho_0(\vec{\nabla} \cdot \vec{v}_1) = 0 \\ \therefore & \frac{\partial\xi}{\partial t} + (\vec{\nabla} \cdot \vec{v}_1) = 0 \\ \therefore & \frac{1}{\rho_0} \frac{\partial\rho_1}{\partial t} - \frac{\rho_1}{\rho_0^2} \frac{\partial\rho_0}{\partial t} + (\vec{\nabla} \cdot \vec{v}_1) = 0 \\ \therefore & \frac{\partial\rho_1}{\partial t} - \frac{\rho_1}{\rho_0} \frac{\partial\rho_0}{\partial t} + \rho_0(\vec{\nabla} \cdot \vec{v}_1) = 0 \\ \therefore & \frac{\partial\rho_1}{\partial t} + \rho_0(\vec{\nabla} \cdot \vec{v}_1) + (\vec{\nabla} \cdot \vec{v}_0)\rho_1 = 0 \end{aligned}$$

In the last step, we use the continuity equation for corresponding unperturbed flow. **It is worth noting that the compression term associated with the non-zero fluid velocity acts as an additional source of mass flux in the continuity equation.**

The Euler equation, under the same linear perturbation reduces to,

$$(\rho_0 + \rho_1) \left[\frac{\partial \vec{v}_1}{\partial t} + \underbrace{\left(\vec{v}_1 \cdot \vec{\nabla} \right) \vec{v}_1}_{\text{linearization}} + \left(\vec{v}_0 \cdot \vec{\nabla} \right) \cdot \vec{v}_0 + \left(\vec{v}_0 \cdot \vec{\nabla} \right) \vec{v}_1 + \left(\vec{v}_1 \cdot \vec{\nabla} \right) \vec{v}_0 \right] = -\nabla p_1 = -\nabla (c_s^2 \rho_1)$$

$$\therefore \rho_0 \frac{\partial \vec{v}_1}{\partial t} + \rho_1 \left(\vec{v}_0 \cdot \vec{\nabla} \right) \vec{v}_0 + \rho_0 \left(\vec{v}_1 \cdot \vec{\nabla} \right) \vec{v}_0 + \rho_0 \left(\vec{v}_0 \cdot \vec{\nabla} \right) \vec{v}_0 + \rho_0 \left(\vec{v}_0 \cdot \vec{\nabla} \right) \vec{v}_1 = -c_s^2 \nabla \rho_1$$

Applying gradient operator on both sides,

$$\vec{\nabla} \cdot \left[\rho_0 \frac{\partial \vec{v}_1}{\partial t} + \rho_1 \left(\vec{v}_0 \cdot \vec{\nabla} \right) \vec{v}_0 + \rho_0 \left(\vec{v}_1 \cdot \vec{\nabla} \right) \vec{v}_0 + \rho_0 \left(\vec{v}_0 \cdot \vec{\nabla} \right) \vec{v}_0 + \rho_0 \left(\vec{v}_0 \cdot \vec{\nabla} \right) \vec{v}_1 \right] = -c_s^2 \nabla^2 \rho_1$$

$$\therefore \frac{\partial}{\partial t} \left[\rho_0 \left(\vec{\nabla} \cdot \vec{v}_1 \right) \right] + \vec{\nabla} \cdot \left[\rho_1 \left(\vec{v}_0 \cdot \vec{\nabla} \right) \vec{v}_0 + \rho_0 \left(\vec{v}_1 \cdot \vec{\nabla} \right) \vec{v}_0 + \rho_0 \left(\vec{v}_0 \cdot \vec{\nabla} \right) \vec{v}_0 + \rho_0 \left(\vec{v}_0 \cdot \vec{\nabla} \right) \vec{v}_1 \right] = -c_s^2 \nabla^2 \rho_1$$

$$\therefore -\frac{\partial^2 \rho_1}{\partial t^2} - \frac{\partial}{\partial t} \left(\vec{\nabla} \cdot \vec{v}_0 \right) \rho_1 + \vec{\nabla} \cdot [\text{terms containing } v_0] = -c_s^2 \nabla^2 \rho_1$$

$$\therefore \left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) \rho_1 = \vec{\nabla} \cdot [\text{terms containing } v_0] - \frac{\partial}{\partial t} \left(\vec{\nabla} \cdot \vec{v}_0 \right) \rho_1$$

$$\therefore \square^2 \rho_1 = \vec{\nabla} \cdot [\text{terms containing } v_0] - \frac{\partial}{\partial t} \left(\vec{\nabla} \cdot \vec{v}_0 \right) \rho_1$$

For $v_0 = 0$, above equation reduces to density perturbation wave equation,

$$\square^2 \rho_1 = 0$$

Concept: For a truly incompressible fluid, a very large excess pressure dP would produce a negligible change in density $d\rho$, thus, $c_s^2 = \sqrt{\frac{dp}{d\rho}} \rightarrow \infty$, hence the sound speed in such a medium is infinite.

Concept: In an incompressible fluid system, any instability is wiped out instantaneously.