1D Compressible Gas Dynamics of a Perfect gas Analytical Calculations: Part I

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Let us list down the basic thermodynamic properties of a perfect gas,

(1) Equation of state: $p = \rho R T$ where, $R = \frac{k_B}{m}$
(2) Internal energy: $\epsilon = C_V T = \left(\dfrac{R}{\gamma - 1}\right) T$
(3) Entropy (per unit mass): $T dS = d\epsilon + p\ d\left(\dfrac{1}{\rho}\right)$

Using (2) in (3),

$$
\begin{aligned}
dS &= \left(\frac{R}{\gamma-1}\right)\frac{dT}{T} + \frac{p}{T}\left(-\frac{1}{\rho^2}\right)d\rho \\
&= C_V\frac{dT}{T} - \frac{R}{\rho}d\rho \\
&= C_V\left[\frac{dp}{p} - \frac{d\rho}{\rho}\right] - R\frac{d\rho}{\rho} \qquad \text{(using (1))} \\
&= C_V\frac{dp}{p} - \frac{d\rho}{\rho}\left(C_V + R\right) \\
&= C_V\frac{dp}{p} - C_p\frac{d\rho}{\rho} \\
\therefore\ dS &= C_v\left[\frac{dp}{p} - \gamma\frac{d\rho}{\rho}\right]
\end{aligned}
$$

Integrating we can write,

$$\frac{S - S_0}{C_V} = \ln p - \gamma \ln \rho = \ln\left(\frac{p}{\rho^\gamma}\right)$$

$$\therefore S = C_V \ln\left(\frac{p}{\rho^\gamma}\right) + S_0 \tag{4}$$

For an adiabatic process, $\dfrac{dS}{dt} = 0$

$$\therefore \frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{5}$$

Enthalpy (per unit mass),

$$W = \epsilon + \frac{p}{\rho} = \frac{R}{\gamma-1}T + RT = \left(1 + \frac{1}{\gamma-1}\right)RT$$

$$\therefore W = \frac{\gamma}{\gamma - 1} RT \tag{6}$$

Moreover,

$$dW = d\epsilon + p\, d\left(\frac{1}{\rho}\right) - \frac{1}{\rho} dp$$

$$\therefore dW = TdS - \frac{dp}{\rho} \qquad \text{(using (3))}$$

For adiabatic flows (i.e., $ds = 0$),

$$\int \frac{dp}{\rho} = W \tag{6a}$$

Hence, for a perfect gas under the adiabatic condition the Bernoulli's equation $\left[\frac{1}{2}v^2 + \int \frac{dp}{\rho} + \Phi = \right.$ constant] reduces to,

$$\frac{1}{2}v^2 + \frac{\gamma}{\gamma - 1} RT + \Phi = \text{ constant} \tag{7}$$

And, from (5),

$$\frac{1}{\rho\gamma}\frac{dp}{dt} = \gamma p \frac{1}{\rho(\gamma + 1)}\frac{d\rho}{dt}$$

$$\therefore dp = \frac{\gamma p}{\rho} d\rho$$

$$\therefore\ c_s = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{\gamma p}{\rho}}$$

Here, c_s is the speed of sound propagation through this gas in adiabatic condition.

Small amplitude acoustic waves: Any process causing a pressure (or/and) density fluctuation (i.e. perturbation) in a gas (compressible medium) is expected to generate acoustic wave. Here, we consider homogeneous perfect gas of density ρ_0 and pressure p_0 in the absence of any external force. We show that linear perturbation in pressure $\rho_1(\vec{x}, t)$ and density $\rho_1(\vec{x}, t)$ will give rise to acoustic wave.

These perturbations will give rise to an additional velocity field $\overrightarrow{v_1}(\vec{x}, t)$. Then the continuity equation in this case can be written as,

$$\frac{\partial\left(\rho_0 + \rho_1\right)}{\partial t} + \vec{\nabla} \cdot \left[\left(\rho_0 + \rho_1\right)\left(\vec{v}_0 + \vec{v}_1\right)\right] = 0$$

$$\therefore \left\{\frac{\partial \rho_0}{\partial t} + \vec{\nabla} \cdot \left(\rho_0 \overrightarrow{v_0}\right)\right\} + \frac{\partial \rho_1}{\partial t} + \vec{\nabla} \cdot \left(\rho_0 \vec{v}_1\right) + \vec{\nabla} \cdot \left(\rho_1 \vec{v}_0\right) + \underbrace{\vec{\nabla} \cdot \cancel{\left(\rho_1 \overrightarrow{v_1}\right)}}_{\text{linearization}}{}^{0} = 0$$

$$\therefore\ \frac{\partial \rho_1}{\partial t} + \underbrace{\vec{v}_1 \cdot \vec{\nabla}\rho_0}_{\text{zero; homogeneity}} + \vec{v}_0 \cdot \vec{\nabla}\rho_1 + \rho_0\left(\vec{\nabla} \cdot \vec{v}_1\right) + \rho_1\left(\vec{\nabla} \cdot \vec{v}_0\right) = 0$$

$$\therefore\ \frac{d}{dt}\left(\rho_0 \xi\right) + \rho_0\left(\vec{\nabla} \cdot \vec{v}_1\right) + \left(\rho_0 \xi\right) \vec{\nabla} \cdot \vec{v}_0 = 0 \quad \left[\text{ putting } \xi = \frac{\rho_1}{\rho_0}\right]$$

$$\therefore\ \rho_0 \frac{d\xi}{dt} + \xi \frac{d\rho_0}{dt} + \xi\left(\rho_0 \vec{\nabla} \cdot \vec{v}_0\right) + \rho_0\left(\vec{\nabla} \cdot \overrightarrow{v_1}\right) = 0$$

$$\therefore\ \xi\left[\frac{d\rho_0}{dt}+\rho_0\vec{\nabla}\cdot\vec{v}_0\right]+\rho_0\frac{\partial\xi}{\partial t}+\rho_0\left(\vec{\nabla}\cdot\vec{v}_1\right)=0$$
$$\therefore\ \frac{d\xi}{dt}+\left(\vec{\nabla}\cdot\vec{v}_1\right)=0$$
$$\therefore\ \frac{1}{\rho_0}\frac{d\rho_1}{dt}-\frac{\rho_1}{\rho_0 2}\frac{d\rho_0}{dt}+\left(\vec{\nabla}\cdot\vec{v}_1\right)=0$$
$$\therefore\ \frac{d\rho_1}{dt}+\rho_0\left(\vec{\nabla}\cdot\vec{v}_1\right)=0$$
$$\therefore\ \frac{\partial\rho_1}{\partial t}+\rho_0\left(\vec{\nabla}\cdot\vec{v_1}\right)+\left(\vec{v_0}\cdot\vec{\nabla}\right)\rho_1=0$$

Here, the non zero fluid velocity and the term associated with it (Compression) acts as the source in the continuity equation.

The Euler equation, under the same linear perturbation reduces to,

$$(\rho_0+\rho_1)\left[\frac{\partial\vec{v}_1}{\partial t}+\left(\vec{v}_1\cdot\vec{\nabla}\right)v_1+\left(\vec{v}_0\cdot\vec{\nabla}\right)\cdot\vec{v}_0+\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_1+\left(\vec{v}_1\cdot\vec{\nabla}\right)\vec{v}_0\right]=-\nabla\rho_1=-\nabla\left(c_s^2\rho_1\right)$$
$$\therefore\ \rho_0\frac{\partial\vec{v}_1}{\partial t}+\rho_1\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\vec{v}_1\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_1=-c_s^2\nabla\rho_1$$
$$\therefore\ \vec{\nabla}\cdot\left[\rho_0\frac{\partial\vec{v}_1}{\partial t}+\rho_1\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\bar{V}_1\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_1\right]=-c_s^2\nabla^2\rho_1$$
$$\therefore\ \frac{\partial}{\partial t}\left[\rho_0\left(\vec{\nabla}\cdot\vec{v}_1\right)\right]+\vec{\nabla}\cdot[\quad]=-c_s^2\nabla^2\rho_1$$
$$\therefore\ -\frac{\partial^2\rho_1}{\partial t^2}\quad+\vec{\nabla}\cdot[\quad]=-c_s^2\nabla^2\rho_1$$
$$\therefore\ \Box^2\rho_1\text{ - source term }=\vec{\nabla}\cdot[\quad]$$

Note: For a truly incompressible fluid, a very large excess pressure dP would produce a negligible change in density $d\rho$, thus, $c_s^2=\sqrt{\frac{dp}{d\rho}}\rightarrow\infty$, hence the sound speed in such a medium is infinite.

Note: In an incompressible fluid system, any instability is wiped out instantaneously.