MidSem Exam – Spring 2024
SS4201: Fluid and Magneto-hydrodynamics
Model Answer

Answer Question 1 and any 4 out of the 5 remaining questions.

Note: Meaning of variables and symbols are same as we used in the class, unless stated otherwise. Clearly mention any assumption that you make while solving any problem.

Time : 90 min | Marks: 25

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1. a) Derive continuity equation starting from the below equation:

$$\frac{\partial}{\partial t}(n\langle\chi\rangle) + \frac{\partial}{\partial x_i}\left(n\left\langle u_i\chi\right\rangle\right) - \frac{n}{m}\left\langle F_i\frac{\partial\chi}{\partial u_i}\right\rangle = 0$$

where, χ is a conserved quantity in a binary collision.

Solution 1a.

For $\chi = m$, where m is the fluid mass, we have the following relations –

$$n\langle\chi\rangle = nm = \rho; \quad \frac{\partial m}{\partial x_i} = 0 \;\text{ and, }\; \langle u_i\chi\rangle = m\left\langle v_i + w_i\right\rangle = m\left(\langle v_i\rangle + \langle w_i\rangle\right) = mv_i$$

Utilizing these relations in the provided equation,

$$\tfrac{\partial}{\partial t}(nm) + \tfrac{\partial}{\partial x_i}\left(nm\nu_i\right) = 0 \qquad \therefore \tfrac{\partial\rho}{\partial t} + \tfrac{\partial}{\partial x_i}\left(\rho\nu_i\right) = 0 \quad \text{(Mass continuity equation for fluids)}$$

b) Lorentz force $\vec{F} = q(\vec{u}\times\vec{B})$ is velocity dependent. Show that even for such situation, $\left\langle \frac{\partial F_i}{\partial u_i}\chi\right\rangle = 0$

Solution 1b.

In vector notation we can write the i-th component of Lorentz force as,

$$F_i = q\left(u_jB_k - v_kB_j\right) \qquad \therefore \quad \frac{\partial F_i}{\partial u_i} = 0 \qquad \therefore \quad \left\langle\frac{\partial F_i}{\partial u_i}\chi\right\rangle = 0$$

c) Show that for an ideal, incompressible flow, the energy equation is redundant.

Solution 1c.

For an ideal flow the energy equation is,

$$\rho\frac{D\varepsilon}{Dt} + p\vec{\nabla}\cdot\vec{v} = 0$$

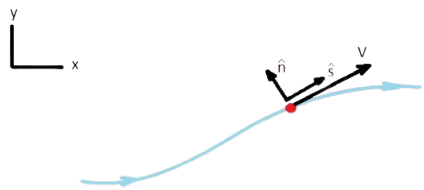
Again, for an incompressible flow, $\vec{\nabla}\cdot\vec{v} = 0$

$$\therefore \quad \frac{D\varepsilon}{Dt} = 0 \Rightarrow \varepsilon \text{ is conserved in space and time. So, the equation is redundant.}$$

d) With suitable Figure, explain the concept of streamline.

Solution 1d.

Streamlines are a family of curves whose tangent vectors constitute the velocity vector field of the flow. These show the direction in which a massless fluid element will travel at any point in time.



e) Show that viscous torque for thin accretion disk is $2\pi\nu\Sigma r^3\frac{d\Omega}{dr}$

Solution 1e.

Toque comes from as velocity shear

$$\frac{dv_\theta}{dr} = \Omega + r\frac{d\Omega}{dr} \qquad (\because v_\theta = r\Omega)$$

wherein, the term $r\frac{d\Omega}{dr}$ represents the viscous shear, i.e., $\nu\rho r\frac{d\Omega}{dr}$
By definition,

$$\frac{\text{Torque}}{\text{Area}} = r\times \text{ stress } = \nu\rho r^2\frac{d\Omega}{dr}$$

$$\therefore \text{ Torque } = \iint \nu\rho r^2\frac{d\Omega}{dr}rd\theta dz = \nu r^3\frac{d\Omega}{dr}\int d\theta\int\rho dz = \nu r^3\frac{d\Omega}{dr}\times 2\pi\times\sum$$

2. Internal energy ϵ is given as

$$\epsilon = \frac{1}{2}\left\langle w^2\right\rangle = \frac{1}{2}\frac{\int d^3uw^2f(\vec{x},\vec{u},t)}{\int d^3uf(\vec{x},\vec{u},t)}$$

where, total velocity $\vec{u} =$ average velocity $\vec{v}+$ random velocity $\vec{w}$ and f represents the distribution function. Assuming f to be Maxwell-Boltzmann distribution function as

$$f = n\left(\frac{m}{2\pi kT}\right)^{3/2}e^{-\frac{mw^2}{2kT}}$$

show that

$$\epsilon = \frac{3}{2}\frac{kT}{m}$$

Solution 2.

Since, the average velocity, v is constant $\therefore\, d^3u = d^3w = w^2\, dw\, \sin\theta\, d\theta\, d\phi$ (in spherical polar coordinate)

$$\epsilon = \frac{1}{2}\frac{\int d^3w\, w^2 n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mw^2}{2kT}}}{\int d^3w\, n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mw^2}{2kT}}}$$

$$\therefore\, \epsilon = \frac{1}{2}\frac{\int d^3w\, w^2 e^{-\frac{mw^2}{2kT}}}{\int d^3w\, e^{-\frac{mw^2}{2kT}}}$$

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$$\therefore\, \epsilon = \frac{1}{2}\frac{\int d\theta \int d\phi \int dw\, w^4 e^{-\frac{mw^2}{2kT}}}{\int d\theta \int d\phi \int dw\, w^2\, e^{-\frac{mw^2}{2kT}}}$$

$$\therefore\, \epsilon = \frac{1}{2}\frac{\left(\frac{2kT}{m}\right)^{5/2}\Gamma(\frac{5}{2})}{\left(\frac{2kT}{m}\right)^{3/2}\Gamma(\frac{3}{2})} = \frac{3}{2}\frac{kT}{m}$$

3. Navier-Stokes equation is given as follows:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} + \frac{1}{\rho}\vec{\nabla} P = \vec{F} + \nu \nabla^2 \vec{v}$$

Show that for high Reynolds number flow, the viscous part becomes unimportant if there is no solid boundary present in the flow.

Solution 3.

Let us do scaling as follows

$$\text{Let } u = x' L, \quad t = t' \frac{L}{V}, \quad \vec{v} = \vec{v}' V$$

$$\therefore \frac{\partial \vec{v}}{\partial t} = \frac{V}{L/V} \frac{\partial \vec{v}'}{\partial t'} = \frac{V^2}{L} \frac{\partial \vec{v}'}{\partial t'}$$

$$\text{and, } F = \frac{V^2}{L} F' \text{ (where, } F' \text{ is force per unit mass)}$$

$$\text{and, } (\vec{v} \cdot \nabla)\, \vec{v} = \frac{V^2}{L} (\vec{v}' \cdot \nabla')\, \vec{v}'$$

$$\text{and, } \nu' = \left(\frac{L}{V^2} \frac{1}{L^2} V \right) \nu$$

Using these defintions the modified Navier-Stokes equation becomes,

$$\frac{\partial \vec{v}'}{\partial t'} + (\vec{v}' \cdot \nabla')\, \vec{v}' + \frac{1}{\rho'} \vec{\nabla}' P' = F' + \left(\frac{L}{V^2} \frac{1}{L^2} V \right) \nu \nabla'^2 \vec{v}'$$

$$\therefore \ldots = \ldots + \frac{1}{R} \nabla'^2 \vec{v}'$$

when $R \to \infty$, $\frac{1}{R} \nabla'^2 \vec{v'} \to 0$ if $\nabla'^2 \vec{v'}$ remains finite (i.e., no solid boundary)

4. In Poiseuille's experiment, the kinematic viscosity ν of water is determined using the following formula:

$$\nu = \frac{\pi \Delta P}{8Ql} a^4$$

where, ΔP is the steadily maintained pressure across a pipe of radius a and length l and Q is the mass flux rate of water flowing through the pipe. Derive this formula from Navier-Stokes equation.

Solution 4.

In cylindrical coordinate the z-component of the Navier-Stokes equation for a steady flow can be written as,

$$-\frac{\Delta P}{\rho l} = \nu \frac{d}{dr}\left(r \frac{dv}{dr}\right)$$

Boundary conditions:

$$v = 0 \text{ at } r = a \quad \text{and, } \frac{dv}{dr} = 0 \text{ at } r = 0$$

Integrating we obtain,

$$v(r) = \frac{\Delta P}{4\rho\nu l}\left(a^2 - r^2\right)$$

Therefore, the mass flux rate through pipe

$$Q = \int \rho v \cdot 2\pi r dr = \frac{\pi \Delta P}{8\nu l} a^4$$

$$\therefore \quad \nu = \frac{\pi \Delta P}{8Ql} a^4$$

5. a) Show that the pressure in an incompressible fluid at rest increases linearly with depth.

b) Using this result, show that if an object heavier than water is fully immersed in water, the net force exerted on the object by surrounding water is $-Mg$, where M is the mass of the water displaced by the object.

Solution 5.

In the center of mass frame of an incompressible fluid at rest, the Navier-Stokes equation becomes,

$$\vec{\nabla} P = \rho \vec{F}$$

Along the z-direction,

$$\frac{dP}{dz} = -\rho g \Rightarrow P = P_0 - \rho g z \quad \text{(i.e. } P \text{ increases linearly with depth } -z\text{)}$$

For an object having vertical extent between the layers A and B,

$$\begin{aligned}
&P_A = P_0 - \rho g Z_A \\
&P_B = P_0 - \rho g Z_B \\
&\therefore P_A - P_B = -\rho g\left(z_A - z_B\right) = -\rho g l \\
&\therefore F = -\rho g l \times \text{ area } = -\rho V \cdot g = -Mg
\end{aligned}$$

6. Collisionless Boltzmann equation is given by

$$\frac{Df}{Dt} = 0$$

Argue that in presence of collisions, this equation gets modified as

$$\frac{Df}{Dt} = \int d^3u_1 \int d\Omega\, |\vec{u} - \vec{u}_1|\, \sigma(\Omega)\, (f' f_1' - f f_1)$$

Solution 6.

In presence of collisions,

$$\frac{Df}{Dt} d^3x\ d^3v = C_{\text{in}} - C_{\text{out}}$$

$$u + u_1 \to u' + u_1' : \text{ particles going out}$$

$$u' + u_1' \to u + u_1 : \text{ particles coming in}$$

For C_{out},

$$\delta n_c = n \cdot n_1 |u - u_1| \sigma d\Omega$$
$$n = f(\vec{x}, \vec{u}, t) d^3u$$
$$n_1 = f\left(\vec{x}, \vec{u}_1, t\right) d^3u_1$$
$$\therefore C_{\text{out}} = d^3u\, d^3x \int d^3u_1 \int d\Omega \int f f_1 |u - u_1| \sigma$$

For C_{in},

$$\delta n_c = n' \cdot n_1' |u' - u_1'| \sigma d\Omega$$

$$\therefore \delta n_c = f\left(\vec{x}_1, \vec{u}_1', t\right) d^3u' \cdot f\left(\vec{x}', \vec{u}_1', t\right) d^3u_1 |u_1' - u_1| \sigma d\Omega$$

Now,

$$|u - u_1| = |u' - u_1'|$$

$$\text{and,} \quad d^3u\, d^3u_1 = d^3u' d^3u_1'$$

$$\therefore C_{\text{in}} = d^3u\, d^3x \int d^3u_1 \int d\Omega \int f' f_1' |u - u_1| \sigma$$

$$\therefore \frac{Df}{Dt} = \int d^3u_1 \int d\Omega\, |\vec{u} - \vec{u}_1|\, \sigma(\Omega)\, (f' f_1' - f f_1)$$