1D Compressible Gas Dynamics of a Perfect Gas Analytical Calculations: Part I

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Let us list down the basic thermodynamic properties of a perfect gas,

(1) Equation of state: $p = \rho RT$ where, $R = \frac{k_B}{m}$
(2) Internal energy: $\epsilon = C_V T = \left(\frac{R}{\gamma - 1}\right) T$
(3) Entropy (per unit mass): $TdS = d\epsilon + p\ d\left(\frac{1}{\rho}\right)$

From (1),

$$\log p = \log \rho + \log R + \log T$$

Taking differentials on both side,

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \tag{1a}$$

Using (2) in (3),

$$\begin{aligned}
dS &= \left(\frac{R}{\gamma - 1}\right)\frac{dT}{T} + \frac{p}{T}\left(-\frac{1}{\rho^2}\right)d\rho \\
&= C_V\frac{dT}{T} - \frac{R}{\rho}d\rho \\
&= C_V\left[\frac{dp}{p} - \frac{d\rho}{\rho}\right] - R\frac{d\rho}{\rho} \qquad \text{(using (1a))} \\
&= C_V\frac{dp}{p} - \frac{d\rho}{\rho}\left(C_V + R\right) \qquad \text{(using, } C_P - C_V = R\text{)} \\
&= C_V\frac{dp}{p} - C_p\frac{d\rho}{\rho} \\
\therefore\ dS &= C_V\left[\frac{dp}{p} - \gamma\frac{d\rho}{\rho}\right] \qquad \text{(using, } C_P/C_V = \gamma\text{)}
\end{aligned}$$

Integrating and using S_0 as the integration constant we can write,

$$\frac{S - S_0}{C_V} = \ln p - \gamma \ln \rho = \ln\left(\frac{p}{\rho^\gamma}\right)$$

$$\therefore S = C_V \ln\left(\frac{p}{\rho^\gamma}\right) + S_0 \tag{4}$$

For an adiabatic process,

$$\frac{dS}{dt} = 0$$

Using (4) we can write,

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{5}$$

$$\therefore \ \frac{1}{\rho^\gamma}\frac{dp}{dt} = \frac{\gamma p}{\rho^{(\gamma+1)}}\frac{d\rho}{dt}$$

$$\therefore dp = \frac{\gamma p}{\rho} d\rho$$

$$\therefore \text{ by definition of sound speed, } c_s = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{\gamma p}{\rho}}$$

Here, c_s is the speed of sound propagation through this gas under adiabatic condition.

Defining the speed of sound in non-relativistic regime

Consider the sound wave propagating at speed v through a pipe aligned with the x axis and with a cross-sectional area of A. In time interval dt it moves length $dx = vdt$. In steady state, the mass flow rate $\dot{m} = \rho v A$ must be the same at the two ends of the tube, therefore the mass flux $j = \rho v$ is constant and $vd\rho = -\rho dv$. As per Newton's second law, the pressure gradient force provides the acceleration:

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{dP}{dx}$$

$$\therefore dP = (-\rho dv)\frac{dx}{dt} = (vd\rho)v$$

$$\therefore v^2 \equiv c_s^2 = \frac{dP}{d\rho}$$

And therefore:

$$c_s = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)}$$

Now, enthalpy (per unit mass),

$$W = \epsilon + \frac{p}{\rho} = \frac{R}{\gamma - 1}T + RT = \left(1 + \frac{1}{\gamma - 1}\right)RT$$

$$\therefore W = \frac{\gamma}{\gamma - 1}RT \tag{6}$$

Moreover,

$$dW = d\epsilon + p\,d\left(\frac{1}{\rho}\right) + \frac{1}{\rho}dp$$

$$\therefore dW = TdS + \frac{dp}{\rho} \quad \text{(using (3))}$$

Under adiabatic condition (i.e., $dS = 0$),

$$\int \frac{dp}{\rho} = W \tag{6a}$$

Hence, for a perfect gas under the adiabatic condition the Bernoulli's equation $\left[\frac{1}{2}v^2 + \int \frac{dp}{\rho} + \Phi = \text{constant}\right]$ reduces to,

$$\frac{1}{2}v^2 + \frac{\gamma}{\gamma-1}RT + \Phi = \text{ constant} \tag{7}$$

Small amplitude acoustic wave propagation in a homogeneous, perfect, compressible gas

Any process causing a pressure (or/and) density fluctuation (i.e. perturbation) in a gas (compressible medium) is expected to generate acoustic wave. Here, we consider homogeneous perfect gas of density ρ_0 and pressure p_0 **in the absence of any external force**. We show that linear perturbation in pressure $p_1(\vec{x},t)$ and density $\rho_1(\vec{x},t)$ will give rise to acoustic wave.

These perturbations will also give rise to an *additional* velocity field $\vec{v_1}(\vec{x},t)$ on top of **previously existing velocity field** $\vec{v_0}(\vec{x},t)$. Then the continuity equation in this case can be written as,

$$\frac{\partial\left(\rho_0+\rho_1\right)}{\partial t} + \vec{\nabla}\cdot\left[\left(\rho_0+\rho_1\right)\left(\vec{v}_0+\vec{v}_1\right)\right] = 0$$

$$\therefore \underbrace{\frac{\partial\rho_0}{\partial t} + \vec{\nabla}\cdot\left(\rho_0\vec{v_0}\right)}_{\text{cont. eqn.}} + \frac{\partial\rho_1}{\partial t} + \vec{\nabla}\cdot\left(\rho_0\vec{v}_1\right) + \vec{\nabla}\cdot\left(\rho_1\vec{v}_0\right) + \underbrace{\vec{\nabla}\cdot\cancelto{0}{\left(\rho_1\vec{v_1}\right)}}_{\text{linearization}} = 0$$

$$\therefore \frac{\partial\rho_1}{\partial t} + \underbrace{\vec{v}_1\cdot\vec{\nabla}\rho_0}_{\text{zero; homogeneity}} + \vec{v}_0\cdot\vec{\nabla}\rho_1 + \rho_0\left(\vec{\nabla}\cdot\vec{v}_1\right) + \rho_1\left(\vec{\nabla}\cdot\vec{v}_0\right) = 0$$

$$\therefore \frac{\partial}{\partial t}\left(\rho_0\xi\right) + \rho_0\left(\vec{\nabla}\cdot\vec{v}_1\right) + \left(\rho_0\xi\right)\vec{\nabla}\cdot\vec{v}_0 = 0 \quad \left[\text{ putting } \xi = \frac{\rho_1}{\rho_0}\right]$$

$$\therefore \rho_0\frac{\partial\xi}{\partial t} + \xi\frac{\partial\rho_0}{\partial t} + \rho_0\left(\vec{\nabla}\cdot\vec{v_1}\right) + \xi\left(\rho_0\vec{\nabla}\cdot\vec{v}_0\right) = 0$$

$$\therefore \underbrace{\xi\left[\frac{\partial\rho_0}{\partial t} + \rho_0\vec{\nabla}\cdot\vec{v}_0\right]}_{\text{cont. eqn.}} + \rho_0\frac{\partial\xi}{\partial t} + \rho_0\left(\vec{\nabla}\cdot\vec{v}_1\right) = 0$$

$$\therefore \frac{\partial\xi}{\partial t} + \left(\vec{\nabla}\cdot\vec{v}_1\right) = 0$$

$$\therefore \frac{1}{\rho_0}\frac{\partial\rho_1}{\partial t} - \frac{\rho_1}{\rho_0^2}\frac{\partial\rho_0}{\partial t} + \left(\vec{\nabla}\cdot\vec{v}_1\right) = 0$$

$$\therefore \frac{\partial\rho_1}{\partial t} - \frac{\rho_1}{\rho_0}\frac{\partial\rho_0}{\partial t} + \rho_0\left(\vec{\nabla}\cdot\vec{v}_1\right) = 0$$

$$\therefore \frac{\partial\rho_1}{\partial t} + \rho_0\left(\vec{\nabla}\cdot\vec{v_1}\right) + \left(\vec{\nabla}\cdot\vec{v_0}\right)\rho_1 = 0$$

In the last step, we use the continuity equation for corresponding unperturbed flow. **It is worth noting that the compression term associated with the non-zero fluid velocity acts as an additional source of mass flux in the continuity equation.**

The Euler equation, under the same linear perturbation reduces to,

$$(\rho_0+\rho_1)\left[\frac{\partial \vec{v}_1}{\partial t}+\underbrace{\cancelto{0}{\left(\vec{v}_1\cdot\vec{\nabla}\right)v_1}}_{\text{linearization}}+\left(\vec{v}_0\cdot\vec{\nabla}\right)\cdot\vec{v}_0+\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_1+\left(\vec{v}_1\cdot\vec{\nabla}\right)\vec{v}_0\right]=-\nabla p_1=-\nabla\left(c_s^2\rho_1\right)$$

$$\therefore\ \rho_0\frac{\partial \vec{v}_1}{\partial t}+\rho_1\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\vec{v}_1\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_1=-c_s^2\nabla\rho_1$$

Applying gradient operator on both sides,

$$\vec{\nabla}\cdot\left[\rho_0\frac{\partial \vec{v}_1}{\partial t}+\rho_1\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\bar{V}_1\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_1\right]=-c_s^2\nabla^2\rho_1$$

$$\therefore\ \frac{\partial}{\partial t}\left[\rho_0\left(\vec{\nabla}\cdot\vec{v}_1\right)\right]+\vec{\nabla}\cdot\left[\rho_1\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\bar{V}_1\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_0+\rho_0\left(\vec{v}_0\cdot\vec{\nabla}\right)\vec{v}_1\right]=-c_s^2\nabla^2\rho_1$$

$$\therefore\ -\frac{\partial^2\rho_1}{\partial t^2}-\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{v}_0\right)\rho_1\quad+\vec{\nabla}\cdot[\text{terms containing}\,v_0]=-c_s^2\nabla^2\rho_1$$

$$\therefore\ \left(\frac{\partial^2}{\partial t^2}-c_s^2\nabla^2\right)\rho_1=\vec{\nabla}\cdot[\text{terms containing}\,v_0]-\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{v}_0\right)\rho_1$$

$$\therefore\ \Box^2\rho_1=\vec{\nabla}\cdot[\text{terms containing}\,v_0]-\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{v}_0\right)\rho_1$$

For $v_0=0$, above equation reduces to density perturbation wave equation,

$$\Box^2\rho_1=0$$

Concept: For a truly incompressible fluid, a very large excess pressure dP would produce a negligible change in density $d\rho$, thus, $c_s^2=\sqrt{\frac{dp}{d\rho}}\to\infty$, hence the sound speed in such a medium is infinite.

Concept: In an incompressible fluid system, any instability is wiped out instantaneously.