Assignment 3 Solution

1. Consider a uniform magnetic field along Z-direction and a charged particle of mass m and charge q is making orbit on X-Y plane. Calculate the equation of motion of the charged particle.

Solution 1.

Let us consider a velocity vector $(v_x, v_y, 0)$ of the orbiting particle under the influence of the magnetic field $(0, 0, B_z)$. Component-wise equations of motion in this case would be,

$$m\dot{v}_x = qv_y B_z \tag{1.1}$$

$$m\dot{v}_y = -qv_x B_z \tag{1.2}$$

Differentiating (1.1) w.r.t time and using (1.2) we obtain,

$$\ddot{v}_x = -\left(\frac{qB_z}{m}\right)^2 v_x = -\Omega^2 v_x \quad \text{(say)} \tag{1.3}$$

The solution of (1.3) will be that of a linear harmonic oscillator,

$$v_x(t) = v_0 e^{i(\Omega t+\phi)} \tag{1.4}$$

Now, integrating (1.2) and using (1.4) we can write,

$$v_y(t) = -\frac{qB_z}{m}\int v_x(t)dt = -\Omega\frac{v_0}{i\Omega}e^{i(\Omega t+\phi)} = iv_0 e^{i(\Omega t+\phi)} \tag{1.5}$$

Integrating (1.4) and (1.5) further we obtain the following two equations, respectively,

$$x(t) = x_0 - \frac{iv_0}{\Omega}e^{i(\Omega t+\phi)} \tag{1.6}$$

$$y(t) = y_0 + \frac{v_0}{\Omega}e^{i(\Omega t+\phi)} \tag{1.7}$$

Combining (1.6) and (1.7) we get,

$$||x - x_0||^2 + ||y - y_0||^2 = \frac{2v_0^2}{\Omega^2} = \frac{v_\perp^2}{\Omega^2} \tag{1.8}$$

The above equation defines the orbital trajectory of the charged particle circling with a velocity $v_\perp$ in a plane perpendicular to the uniform external magnetostatic field $B_z\hat{z}$, with a radius $\dfrac{mv_\perp}{qB_z}$ around the center (x_0, y_0).

2. Consider an isothermal two-fluid plasma in a constant gravitational field g. Assuming ions to be singly ionized, write down the force balance equations for the ion and electron fluids. Show that they can be combined to give usual hydrostatic equation and an electric field $E = -\frac{m_i}{2e}g$ has to exist in atmosphere to prevent charge separation.

Solution 2.

The equation of motion of the electron fluid in an isothermal two-fluid plasma kept under a constant gravitational field g is,

$$m_e n \frac{\partial \vec{v_e}}{\partial t} = -\nabla p_e - ne\left(\vec{E} + \frac{\vec{v_e}}{c} \times \vec{B}\right) + ne\eta\vec{j} + m_e n\vec{g} \tag{2.1}$$

The equation of motion of a singly-ionized ion fluid in the same two-fluid plasma is,

$$m_i n \frac{\partial \vec{v_i}}{\partial t} = -\nabla p_i + ne\left(\vec{E} + \frac{\vec{v_i}}{c} \times \vec{B}\right) - ne\eta\vec{j} + m_i n\vec{g} \tag{2.2}$$

Under the hydrostatic equilibrium condition all flows are set to zero and thus the above equations reduce to,

$$0 = -\nabla p_e - ne\vec{E} + m_e n\vec{g} \tag{2.3}$$

$$0 = -\nabla p_i + ne\vec{E} + m_i n\vec{g} \tag{2.4}$$

Subtracting (2.3) from (2.4) we get,

$$\nabla(p_i - p_e) = 2ne\vec{E} + (m_i - m_e)n\vec{g} \tag{2.5}$$

Prevention of any charge separation implies that the electron and ion fluids can together be regarded as a single ionized fluid, i.e., $p_i = p_e$. Therefore, under no charge separation,

$$2ne\vec{E} = -(m_i - m_e)n\vec{g} \tag{2.6}$$

Under the approximation $m_i >> m_e$,

$$\vec{E} = -\frac{m_i}{2e}\vec{g} \tag{2.7}$$

3. Derive the energy conservation equation of MHD equations. (Look at Exercises 14.1 and 14.2 of ARC book for intermediate steps.)

Solution 3.

If we consider η to be isotropic, the induction equation is given by

$$\frac{\partial B}{\partial t} = \nabla \times v \times B - \eta\left(\nabla^2 B\right)$$

$$\frac{\partial B}{\partial t} = -B\left(\nabla \cdot v\right) - \left(v \cdot \nabla\right)B + \left(B \cdot \nabla\right)v - \eta\left(\nabla^2 B\right)$$

Consider the dot product of each term with $\frac{B}{4\pi}$ -

Rate of change of magnetic energy

$$\frac{\partial B}{\partial t} \cdot \frac{B}{4\pi} = \frac{1}{8\pi}\frac{\partial}{\partial t}\left(B.B\right) = \frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right)$$

Energy stored/released due to compressive flow

$$-B\left(\nabla \cdot v\right) \cdot \frac{B}{4\pi} = -\frac{1}{4\pi}\left(B \cdot B\right)\left(\nabla \cdot v\right) = -\frac{B^2}{4\pi}\frac{\partial v_i}{\partial x_i}$$

Advection energy

$$-\left(v\cdot\nabla\right)B\cdot\frac{B}{4\pi}=-\left(v_i\frac{\partial B}{\partial x_i}\right)\cdot\frac{B}{4\pi}=-\frac{v_i}{4\pi}\left(\frac{1}{2}\frac{\partial}{\partial x_i}(B\cdot B)\right)=-\frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right)v_i$$

Shearing/Induction energy stored

$$\left(B\cdot\nabla\right)v\cdot\frac{B}{4\pi}=\frac{1}{4\pi}\left(B\cdot\nabla\right)\left(v\cdot B\right)=\frac{1}{4\pi}\left(B_j\nabla_j\right)\left(v_iB_i\right)=\frac{B_iB_j}{4\pi}\frac{\partial v_i}{\partial x_j}$$

Dissipation energy = Lorentz energy + Ohmic dissipation

Since $j=\frac{c}{4\pi}\left(\nabla\times B\right)$, we get the following term,

$$-\frac{4\pi\eta}{c}\left(\nabla\times j\right)\cdot\frac{B}{4\pi}=-\frac{\eta}{c}\left(\nabla\cdot j\right)\cdot B=-\frac{\eta}{c}\left[\nabla\cdot(j\times B)+(\nabla\times B)\cdot j\right]=-\frac{\eta}{c}\left[\nabla\cdot(j\times B)+\frac{4\pi}{c}j^2\right]$$

Since, $\eta=\frac{c^2}{4\pi\sigma}$ we get,

$$-\frac{4\pi\eta}{c}\left(\nabla\times j\right)\cdot\frac{B}{4\pi}=-\frac{c}{4\pi\sigma}\left[\nabla\cdot(j\times B)\right]-\frac{j.j}{\sigma}$$

Combining all the terms together, we get

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right)=\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right)-\frac{B^2}{4\pi}\frac{\partial v_i}{\partial x_i}-\frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right)v_i+\frac{B_iB_j}{4\pi}\frac{\partial v_i}{\partial x_j}-\frac{c}{4\pi\sigma}\left[\nabla\cdot(j\times B)\right]-\frac{j.j}{\sigma}$$

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right)+\frac{\partial}{\partial x_i}\left(v_i\frac{B^2}{8\pi}\right)=-\mathcal{M}_{ij}\frac{\partial v_i}{\partial x_j}-\frac{j^2}{\sigma}-\frac{c}{4\pi\sigma}\nabla\cdot(j\times B) \tag{3.1}$$

Assuming infinite conductivity (i.e. zero resistivity) of the plasma medium, (3.1) reduces to,

$$\frac{\partial}{\partial t}\left(\frac{B^2}{8\pi}\right)=-\frac{\partial}{\partial x_i}\left(v_i\frac{B^2}{8\pi}\right)-\mathcal{M}_{ij}\frac{\partial v_i}{\partial x_j} \tag{3.2}$$

Now, in an inviscid, thermally insulated plasma medium under the external Lorentz force $\vec{F}_e$, the hydrodynamic energy conservation equation becomes,

$$\frac{\partial}{\partial t}\left(\rho\epsilon+\frac{1}{2}\rho v^2\right)=-\frac{\partial}{\partial x_i}\left[\rho v_i\left(\epsilon+\frac{p}{\rho}+\frac{1}{2}v^2\right)\right]+\rho\vec{v}\cdot\vec{F}_e \tag{3.3}$$

where,

$$\rho\vec{F}_e=\vec{j}\times\vec{B}=(\vec{\nabla}\times\vec{B})\times\vec{B}=\frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\delta_{ij}-\frac{B_iB_j}{4\pi}\right)=-\frac{\partial\mathcal{M}_{ij}}{\partial x_i} \tag{3.4}$$

Therefore, (3.3) reduces to,

$$\frac{\partial}{\partial t}\left(\rho\epsilon+\frac{1}{2}\rho v^2\right)=-\frac{\partial}{\partial x_i}\left[v_i\left(\rho w+\frac{1}{2}\rho v^2\right)\right]-v_i\frac{\partial\mathcal{M}_{ij}}{\partial x_i} \tag{3.5}$$

Adding (3.2) and (3.5), we obtain the energy conservation equation for this system,

$$\frac{\partial}{\partial t}\left(\rho\epsilon+\frac{1}{2}\rho v^2+\frac{B^2}{8\pi}\right)=-\frac{\partial}{\partial x_i}\left[v_i\left(\rho w+\frac{1}{2}\rho v^2+\frac{B^2}{8\pi}\right)+v_i\mathcal{M}_{ij}\right] \tag{3.6}$$

4. This problem provides an intuitive derivation of the Alfven speed. It is done by reminding us that string or rubber band with a tension T and mass per unit length μ carries a transverse oscillation with a speed $\sqrt{T/\mu}$. Plasmas that are threaded by magnetic field also experience a tension force. Of course, any compression of the plasma also causes a change in its pressure which will contribute as extra force. Since we only want the tension force, we will make sure that we do not make any volumetric change to the fluid that we consider in this problem.

   Thus, consider a cylinder of magnetized fluid (assume single fluid MHD) as in the Figure. Initially it has a cross sectional area A and length l. Let the cylinder be threaded by a longitudinal magnetic field with an initial value B. The cylinder be squeezed in a volume-preserving manner so that its cross-sectional area becomes $A - \delta A$ and its length becomes $l + \delta l$. Because magnetic flux is conserved, the magnetic field increases to $B + \delta B$.

   i. Asserting volume conservation and magnetic flux conservation, show that $\delta B/B = \delta l/l$ (keep only first order terms in δ).

   ii. The change in the magnetic energy density is given by $\Delta u_m = \left[(B + \delta B)^2 - B^2\right]/(8\pi)$. Let the tensional force provided by the magnetic field along the cylinder's axis be denoted by T. The work-energy theorem then asserts that the total change in the magnetic energy $\Delta u_m Al$ is equal to the work done by the tensional force provided by the magnetic field $T\delta l$. As a result, show that $T = B^2 A/(4\pi)$.

   iii. If the plasma has a density ρ, the linear mass density of the cylinder is given by $\mu = \rho A$. Now, given the tension force and the linear mass density, show that the Alfven wave speed in the plasma is given by $v_A = B/\sqrt{(4\pi\rho})$.

Solution 4.

i. Volume conservation under a first order approximation yields,

$$\begin{aligned}
& Al = (A - \delta A)(l + \delta l) \\
\therefore\quad & Al = Al + A\delta l - \delta A l - \delta A \delta l \approx Al + A\delta l - \delta A l \\
\therefore\quad & A\delta l = \delta A l \\
\therefore\quad & \frac{\delta A}{A} = \frac{\delta l}{l} \qquad (4.1)
\end{aligned}$$

Magnetic flux conservation under a first order approximation yields,

$$\begin{aligned}
& AB = (A - \delta A)(B + \delta B) \\
\therefore\quad & AB = AB + A\delta B - \delta A B - \delta A \delta l \approx AB + A\delta B - \delta A B \\
\therefore\quad & A\delta B = \delta A B \\
\therefore\quad & \frac{\delta A}{A} = \frac{\delta B}{B} \qquad (4.2)
\end{aligned}$$

Combining (4.1) and (4.2),

$$\frac{\delta B}{B} = \frac{\delta l}{l} \qquad (4.3)$$

ii. The work-energy theorem then asserts that the total change in the magnetic energy is equal to the work done by the tension force. Hence, using first order approximation,

$$
\begin{aligned}
& \Delta u_m A l = T \delta l \\
\therefore \quad & \frac{(B+\delta B)^2 - B^2}{8\pi} A l = T \delta l \\
\therefore \quad & 2B\ \delta B\ A\ l = 8\pi\ T\ \delta l \\
\therefore \quad & T = \frac{2}{8\pi}\ B\ A\ l\ \frac{\delta B}{\delta l} \\
\therefore \quad & T = \frac{2}{8\pi}\ B\ A\ l\ \frac{\delta B}{\delta l} \\
\therefore \quad & T = \frac{2}{8\pi}\ B\ A\ l\ \frac{B}{l} \quad \text{[using (4.3)]} \\
\therefore \quad & T = \frac{B^2 A}{4\pi} \qquad (4.4)
\end{aligned}
$$

iii. Using the definition of the Alfven speed, v_A,

$$
\begin{aligned}
& v_A = \sqrt{\frac{T}{\mu}} \\
\therefore \quad & v_A = \sqrt{\frac{\frac{B^2 A}{4\pi}}{\rho A}} \\
\therefore \quad & v_A = \frac{B}{\sqrt{4\pi\rho}} \qquad (4.5)
\end{aligned}
$$