1D Compressible Gas Dynamics of a Perfect Gas
Analytical Calculations: Part II

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Hydrodynamic Shock

We are essentially going to discuss the structure of large amplitude, normal shock waves in a gaseous medium without any viscous dissipation, conduction and radiation whatsoever.

Method of characteristics: It is a method by which one transforms (only) hyperbolic partial differential equations into ordinary differential equations along certain hyperbolic curves in the solution space known as the characteristic curves.

Let us consider the advection equation,

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0$$

The term $v\frac{\partial v}{\partial x}$ steepens the wavefront. We wish to find $v(x,t)$ in the $x-t$ plane. Let us consider the characteristic curve,

$$\frac{dx}{dt} = v$$

Along these curves,

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\frac{dx}{dt} \equiv \frac{dv}{dt} = 0$$

Then the hyperbolic PDE of advection can be reduced to a simple ODE. The solutions are straight lines in $x-t$ plane. Physically, the characteristic curve corresponds to the Lagrangian trajectory of fluid elements. The velocity v associated with a fluid element does not change as the element moves.

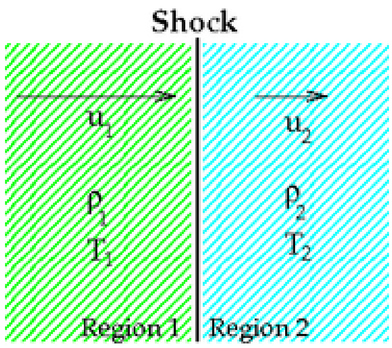

Figure 1: *Schematic of a hydrodynamic shock. Credit: Wiki.*

A shock wave is a region of small thickness over which different fluid dynamical variables (ρ, P, v, T) **change rapidly.** It is useful to regard the shock wave as a mathematical discontinuity across which different variables change rapidly.

In the rest frame of the shock front, density, pressure and velocity are ρ_1, p_1 and v_1 on one side and ρ_2, p_2, v_2 on the other side. From the frame of the undisturbed medium (1), the shock is moving into it with velocity -v_1.

Under steady condition, the mass flux, momentum flux and energy flux should be conserved from one side to the other:

- Conservation of mass flow rate:

$$\rho_1 v_1 A_1 = p_2 v_2 A_2 \Rightarrow \rho_1 v_1 = \rho_2 v_2 \,(\text{ if } A_1 = A_2) \tag{1}$$

- Conservation of momentum:

$$p_1 A_1 - p_2 A_2 = \overbrace{\underbrace{(\rho_2 A_2 v_2)}_{\text{mass flow rate}}\; v_2}^{\text{momentum flow rate}} - (\rho_1 A_1 v_1)\, v_1$$

$$\therefore\;\; p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 \quad (\text{if } A_1 = A_2) \tag{2}$$

- Conservation of energy (Bernoulli's theorem):

$$\frac{1}{2}v_1^2 + \frac{\gamma p_1}{(\gamma-1)\rho_1} = \frac{1}{2}v_2^2 + \frac{\gamma p_2}{(\gamma-1)\rho_2} \tag{3}$$

Rankine-Hugoniot relations

To find the pressure ratio:

From (2):

$$p_2 - p_1 = \rho_1 v_1^2 - \rho_2 v_2^2$$

$$\therefore\;\; \frac{p_2}{p_1} - 1 = \frac{\rho_1 v_1^2 - p_2 v_2^2}{p_1}$$

$$\therefore\;\; \frac{p_2}{p_1} = 1 + \gamma \cdot \frac{\rho_1 v_1^2}{\gamma p_1}\left[1 - \frac{\rho_2 v_2^2}{\rho_1 v_1^2}\right]$$

$$\therefore\;\; \frac{p_2}{\rho_1} = 1 + \frac{\gamma v_1^2}{\left(\frac{\gamma p_1}{\rho_1}\right)}\left[1 - \frac{\rho_2^2 v_2^2}{\rho_1^2 v_1^2} \times \frac{\rho_1}{\rho_2}\right]$$

$$\therefore\;\; \frac{p_2}{\rho_1} = 1 + \gamma \frac{v_1^2}{c_{s,1}^2}\left[1 - \frac{\rho_1}{p_2}\right] \quad (\text{using (1)})$$

$$\therefore\;\; \frac{p_2}{\rho_1} = 1 + \gamma \mathcal{M}_1^2\left[1 - \frac{\rho_1}{p_2}\right] \tag{4}$$

Here, $c_{s,1}$ is the sound speed in medium I. $\mathcal{M}$ is the sonic Mach Number associated with the flow in the same medium.

To find the density ratio:

From (3):

$$\frac{\gamma p_1}{(\gamma-1)\rho_1} - \frac{\gamma p_2}{(\gamma-1)\rho_2} = \frac{1}{2}\left(v_2^2 - {v_1}^2\right)$$

$$\therefore \frac{\gamma p_1}{\rho_1}\left[1 - \frac{p_2/p_1}{\rho_2/\rho_1}\right] = \frac{(\gamma-1){v_1}^2}{2}\left[\frac{{v_2}^2}{{v_1}^2} - 1\right]$$

$$\therefore \left[1 - \left(\frac{p_2}{p_1}\right)\cdot\left(\frac{\rho_1}{\rho_2}\right)\right] = \frac{(\gamma-1)}{2}\frac{v_1^2}{{c_{s,1}}^2}\left[\frac{\rho_1^2}{{\rho_2}^2} - 1\right] \quad \text{(using (1))}$$

$$\therefore\ 1 - \left[1 + \gamma\mathcal{M}_1^2\left(1 - \frac{\rho_1}{\rho_2}\right)\right]\cdot\left(\frac{\rho_1}{P_2}\right) = \frac{(\gamma-1)}{2}\mathcal{M}_1^2\left[\left(\frac{\rho_1}{\rho_2}\right)^2 - 1\right]$$

$$\therefore \left(\frac{\rho_1}{\rho_2}\right)^2\left[\gamma\mathcal{M}_1^2 - \frac{\gamma-1}{2}\mathcal{M}_1^2\right] + \left(\frac{\rho_1}{\rho_2}\right)\left[-1 - \gamma\mathcal{M}_1^2\right] + \left[1 + \frac{\gamma-1}{2}\mathcal{M}_1^2\right] = 0$$

$$\therefore \left(\frac{\rho_1}{\rho_2}\right)^2\left[\frac{\gamma+1}{2}\mathcal{M}_1^2\right] - \left(\frac{\rho_1}{\rho_2}\right)\left[\left(1 + \gamma\mathcal{M}_1^2\right)\right] + \left[1 + \frac{\gamma-1}{2}\mathcal{M}_1^2\right] = 0 \quad \text{(quadratic equation)}$$

$$\therefore \frac{\rho_1}{\rho_2} = \frac{\left(1+\gamma\mathcal{M}_1^2\right) \pm \left[\left(1+\gamma\mathcal{M}_1^2\right)^2 - 4\cdot\frac{\gamma+1}{2}\mathcal{M}_1^2\left(1 + \frac{\gamma-1}{2}\mathcal{M}_1^2\right)\right]^{1/2}}{(\gamma+1)\mathcal{M}_1^2}$$

$$\therefore \frac{\rho_1}{\rho_2} = \frac{\left(1+\gamma\mathcal{M}_1^2\right) \pm \left(\mathcal{M}_1^2 - 1\right)}{(\gamma-1)\mathcal{M}_1^2} = \begin{cases} \dfrac{\rho_1}{\rho_2} = 1, & \text{for '+' sign.} \\ \dfrac{\rho_1}{\rho_2} = \dfrac{2 + (\gamma-1)\mathcal{M}_1^2}{(\gamma+1)\mathcal{M}_1^2}, & \text{for '-' sign.} \end{cases} \tag{4}$$

The first case (i.e., for '+' sign) denotes the trivial condition where no shock is formed. Whereas, the density discontinuity at the shock front is expressed in the second case (i.e., for '-' sign). In fact, we can further show that,

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\left(\dfrac{2}{\mathcal{M}_1^2}\right) + \gamma - 1} > 1 \text{ for } \mathcal{M}_1 > 1 \text{ and increases as } \mathcal{M}_1 \text{ increases.}$$

$$= 1 \text{ for } \mathcal{M}_1 = 1 \text{ i.e., the shock disappears.}$$

Concept: Medium II behind a shock is always compressed (more dense) and a shock involving a stronger compression has to move faster.

Concept: In the frame of the undisturbed medium, as the shack advances, more and more material passes from the undisturbed medium through the shock wave and gets compressed to higher density and pressure. As more material of the undisturbed medium goes behind the shock, the position of the shock advances.

Concept: The density compression has a maximum limiting value which can be found when we put $\mathcal{M}_1 \to \infty$, giving $\dfrac{\rho_2}{\rho_1} \to \dfrac{\gamma+1}{\gamma-1} \approx \dfrac{5/3+1}{5/3-1} \approx 4$ (for monatomic gas)

If a disturbance tries to compress a monatomic gas to a factor 4 , that will give rise to a shock wave moving at infinite speed. Hence it is not possible to achieve a higher density compression beyond this limit.