Chapter 7: Inner Products

Ketan Rajawat

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Introduction

- We have seen

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- Example:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 4 + 10 + 18 = 32$$

Key Properties

- Commutativity: order does not matter $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

- Associativity with scalar multiplication

$$\langle a\mathbf{u}, b\mathbf{v} \rangle = ab\langle \mathbf{u}, \mathbf{v} \rangle$$

- Distributivity with vector addition

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$

Implication: bilinearity

- It follows that the inner product is *bilinear*

$$\langle \alpha\mathbf{u} + \beta\mathbf{y}, \mathbf{v}\rangle = \alpha\langle \mathbf{u}, \mathbf{v}\rangle + \beta\langle \mathbf{y}, \mathbf{v}\rangle$$
$$\langle \mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{y}\rangle = \alpha\langle \mathbf{u}, \mathbf{v}\rangle + \beta\langle \mathbf{u}, \mathbf{y}\rangle$$

- In other words,

linear with respect to *each* argument separately

Example 1

- Calculate $\langle(\mathbf{u}+\mathbf{v}),(\mathbf{x}+\mathbf{w})\rangle$

$$\langle(\mathbf{u}+\mathbf{v}),(\mathbf{x}+\mathbf{w})\rangle = \langle\mathbf{u},(\mathbf{x}+\mathbf{w})\rangle + \langle\mathbf{v},(\mathbf{x}+\mathbf{w})\rangle$$ distributivity

$$= (\langle\mathbf{u},\mathbf{x}\rangle + \langle\mathbf{u},\mathbf{w}\rangle) + (\langle\mathbf{v},\mathbf{x}\rangle+\langle\mathbf{v},\mathbf{w}\rangle)$$ distributivity

- Special case

$$\langle(\mathbf{u}+\mathbf{v}),(\mathbf{u}+\mathbf{v})\rangle$$ commutativity

$$= \langle\mathbf{u},\mathbf{u}\rangle + \langle\mathbf{u},\mathbf{v}\rangle + \langle\mathbf{v},\mathbf{u}\rangle + \langle\mathbf{v},\mathbf{v}\rangle$$

$$= \langle\mathbf{u},\mathbf{u}\rangle + 2\langle\mathbf{u},\mathbf{v}\rangle + \langle\mathbf{v},\mathbf{v}\rangle$$

Example 2

- Extracting the i^{th} element $\langle \mathbf{e}_i, \mathbf{v} \rangle = v_i$
- Sum of all elements $\langle \mathbf{1}, \mathbf{v} \rangle = \sum_i v_i$
- Sum of squares of all elements $\langle \mathbf{v}, \mathbf{v} \rangle = \sum_i {v_i}^2$

Example 3: Matrix-vector product

- New way of interpreting $\mathbf{u} = \mathbf{A}\mathbf{v}$

- Recall block multiplication $\mathbf{A} = \begin{bmatrix} \mathbf{a}_{1\cdot} \\ \mathbf{a}_{2\cdot} \\ \vdots \\ \mathbf{a}_{n\cdot} \end{bmatrix}$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{1\cdot} \\ \mathbf{a}_{2\cdot} \\ \vdots \\ \mathbf{a}_{n\cdot} \end{bmatrix} \mathbf{v}$$

- Hence, we have

$$u_j = \mathbf{a}_{j\cdot} \mathbf{v} = \langle \mathbf{a}_{j\cdot}^T, \mathbf{v} \rangle$$

Inner product of block vectors

- Should have compatible sizes

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_k \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \vdots \\ \mathbf{w}_k \end{bmatrix}$$

$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}_1, \mathbf{w}_1 \rangle + \langle \mathbf{v}_2, \mathbf{w}_2 \rangle + \cdots + \langle \mathbf{v}_k, \mathbf{w}_k \rangle$$

Vector-Matrix-Vector product

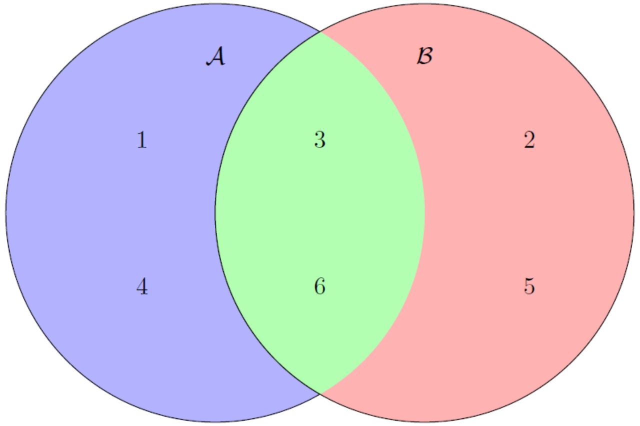
- Claim: $\langle \mathbf{y}, \mathbf{A}\mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{A}^T \mathbf{y} \rangle$
- Proof:
  - Associativity: $\langle \mathbf{y}, \mathbf{A}\mathbf{x} \rangle = \mathbf{y}^T \mathbf{A}\mathbf{x} = (\mathbf{y}^T \mathbf{A})\mathbf{x}$
  - Transpose of scalar = scalar: $= \mathbf{x}^T (\mathbf{y}^T \mathbf{A})^T$
  - Properties of transpose: $= \mathbf{x}^T \mathbf{A}^T \mathbf{y}$
  - Definition of inner product: $= \langle \mathbf{x}, \mathbf{A}^T \mathbf{y} \rangle$

Application 1

- Co-occurrence: number of common elements in sets $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{A} = \{1,3,4,6\} \qquad \mathcal{B} = \{2,3,5,6\}$$

$$\mathbf{v}_{\mathcal{A}} := \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_{\mathcal{B}} := \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$



$$\langle \mathbf{v}_{\mathcal{A}}, \mathbf{v}_{\mathcal{B}} \rangle = 1(0) + 0(1) + 1(1) + 1(0) + 0(1) + 1(1) = 2$$

Application 2: Polynomial Evaluation

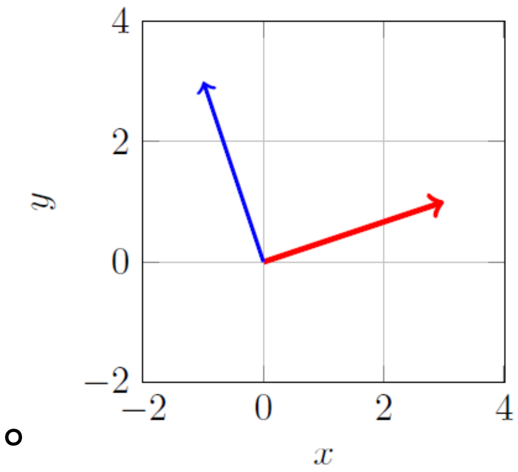
- Polynomial: $p(x) = c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1 x + c_0$

$$\mathbf{p} = \begin{bmatrix} c_{n-1} \\ c_{n-2} \\ \vdots \\ c_1 \\ c_0 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x^{n-1} \\ x^{n-2} \\ \vdots \\ x \\ 1 \end{bmatrix}$$

$$\langle \mathbf{p}, \mathbf{a} \rangle = \left\langle \begin{bmatrix} c_{n-1} \\ c_{n-2} \\ \vdots \\ c_1 \\ c_0 \end{bmatrix}, \begin{bmatrix} a^{n-1} \\ a^{n-2} \\ \vdots \\ a \\ 1 \end{bmatrix} \right\rangle$$

$$= c_{n-1}a^{n-1} + c_{n-2}a^{n-2} + \cdots + c_1 a + c_0$$

Orthogonal vectors



- Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v} = 0$

- 2D displacement vectors: orthogonal vectors are at 90°

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v} = [3 \quad 1] \begin{bmatrix} -1 \\ 3 \end{bmatrix} = (3 \cdot -1) + (1 \cdot 3) = 0$$

- Note: $\mathbf{0}$ is orthogonal to every vector

Complexity of Inner product calculation

- To calculate $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$
- Need n multiplications $+ n - 1$ additions
- Total $2n - 1$ flops

Matrix Inner Product

- Similar idea: sum of element-wise product $\langle \mathbf{A}, \mathbf{B} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ij}$

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \langle \mathbf{A}, \mathbf{B} \rangle &= (2 \cdot 1) + \big(4 \cdot (-1)\big) + (1 \cdot 0) + (3 \cdot 2) \\ &= 2 - 4 + 0 + 6 = 4 \end{aligned}$$

- Follows commutativity and bilinearity properties

$$\begin{aligned} \langle \mathbf{A}, \mathbf{B} \rangle &= \langle \mathbf{B}, \mathbf{A} \rangle \\ \langle \alpha \mathbf{A}, \mathbf{B} \rangle &= \alpha \langle \mathbf{A}, \mathbf{B} \rangle \\ \langle \mathbf{A} + \mathbf{C}, \mathbf{B} \rangle &= \langle \mathbf{A}, \mathbf{B} \rangle + \langle \mathbf{C}, \mathbf{B} \rangle \end{aligned}$$

Matrix Inner Product: other representations

- Trace representation

$$\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{tr}(\mathbf{A}^T \mathbf{B})$$

- Vectorized inner product form

$$\langle \mathbf{A}, \mathbf{B} \rangle = \langle \mathrm{vec}(\mathbf{A}), \mathrm{vec}(\mathbf{B}) \rangle$$

recall: $\mathrm{vec}(\quad)$ operator reshapes matrix into a column vector

Thank You

Next: Norm