Chapter 3: Vector Spaces

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Outline

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Fields

- So far, all numbers have been real numbers
- We can use any complex numbers also
- Field, denoted by $\mathbb{F}$, is a set
  - on which addition, subtraction, multiplication, division are well-defined

- Examples of $\mathbb{F}$
  - $\mathbb{R}$ (real numbers)
  - $\mathbb{C}$ (complex numbers)
- Set of n-dimensional vectors in $\mathbb{F}$ is denoted by $\mathbb{F}^n$

Vector space

- Any set $\mathbb{V}$ where linear combinations on $\mathbb{F}$ are well-defined

- To verify if $\mathbb{V}$ is a vector space, check
  1) $\mathbb{V}$ contains $\mathbf{0}$
  2) $\mathbb{V}$ is closed under addition,

     i.e., if $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{V}$ then $\mathbf{v}_1 + \mathbf{v}_2 \in \mathbb{V}$
  3) $\mathbb{V}$ is closed under scalar multiplication

     i.e., if $\mathbf{v} \in \mathbb{V}$ and $\alpha \in \mathbb{F}$, then $\alpha\mathbf{v} \in \mathbb{V}$

Closure under linear combinations

- Combining the properties, it follows that

for $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{F}$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{V}$

$$\boxed{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_m \mathbf{v}_m \in \mathbb{V}}$$

Example of Vector Spaces

- Canonical examples
  - $\mathbb{R}^n$ (set of n dimensional vectors of real numbers) on $\mathbb{F} = \mathbb{R}$
  - $\mathbb{R}^{m \times n}$ (set of $m \times n$ matrices) on $\mathbb{F} = \mathbb{R}$
  - $\mathbb{C}^n$ (set of n dimensional vectors of complex numbers) on $\mathbb{F} = \mathbb{C}$

- Counter example: $\mathbb{N}$ not a vector space over $\mathbb{R}$

Subspace

- The set $\mathbb{U} \subseteq \mathbb{V}$ is a subspace of $\mathbb{V}$ if $\mathbb{U}$ is also a vector space
- That is, $\mathbb{U}$ satisfies the three properties
  1) $\mathbb{U}$ contains $\mathbf{0}$
  2) $\mathbb{U}$ is closed under addition,

     i.e., if $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{U}$ then $\mathbf{u}_1 + \mathbf{u}_2 \in \mathbb{U}$
  3) $\mathbb{U}$ is closed under scalar multiplication

     i.e., if $\mathbf{u}_1 \in \mathbb{U}$ and $\alpha \in \mathbb{F}$, then $\alpha \mathbf{u}_1 \in \mathbb{U}$
- Or equivalently

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_m \mathbf{u}_m \in \mathbb{U}$$

Example 1

- Is $\mathbb{V} = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{C}^2 \middle| \, z_1 + iz_2 = 0 \right\}$ a subspace of $\mathbb{C}^2$

- Contains zero vector?
  - Since $0 + i(0) = 0$, therefore $\mathbf{0} \in \mathbb{V}$

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Closed under addition?

$$\left.\begin{array}{l} \text{Let } \mathbf{v} \in \mathbb{V} \Rightarrow v_1 + i(v_2) = 0 \\ \text{Let } \mathbf{w} \in \mathbb{V} \Rightarrow w_1 + i(w_2) = 0 \end{array}\right\} \Rightarrow v_1 + w_1 + i(v_2 + w_2) = 0$$

  - Therefore,

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} \in \mathbb{V}$$

Example 1 (continued)

- Is $\mathbb{V} = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{C}^2 \middle| \, z_1 + iz_2 = 0 \right\}$ a subspace of $\mathbb{C}^2$

- Closed under scalar multiplication?
  - Let $\mathbf{v} \in \mathbb{V} \Rightarrow v_1 + i(v_2) = 0 \Rightarrow c(v_1 + i(v_2)) = cv_1 + i(cv_2) = 0$
  - Hence, c$\mathbf{v} \in \mathbb{V}$

- Hence, $\mathbb{V}$ is a subspace of $\mathbb{C}^2$

Example 2

- Is $\mathcal{A} = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 + x_2 = b\}$ subspace of $\mathbb{R}^3$?

- Contains zero? No, unless $b = 0$
- Closed under addition?
  - Let $\mathbf{v} \in \mathcal{A} \Rightarrow v_1 + v_2 = b$
  - Let $\mathbf{w} \in \mathcal{A} \Rightarrow w_1 + w_2 = b$

$$\left.\begin{array}{l} v_1 + v_2 = b \\ w_1 + w_2 = b \end{array}\right\} \Rightarrow v_1 + v_2 + w_1 + w_2 = 2b$$

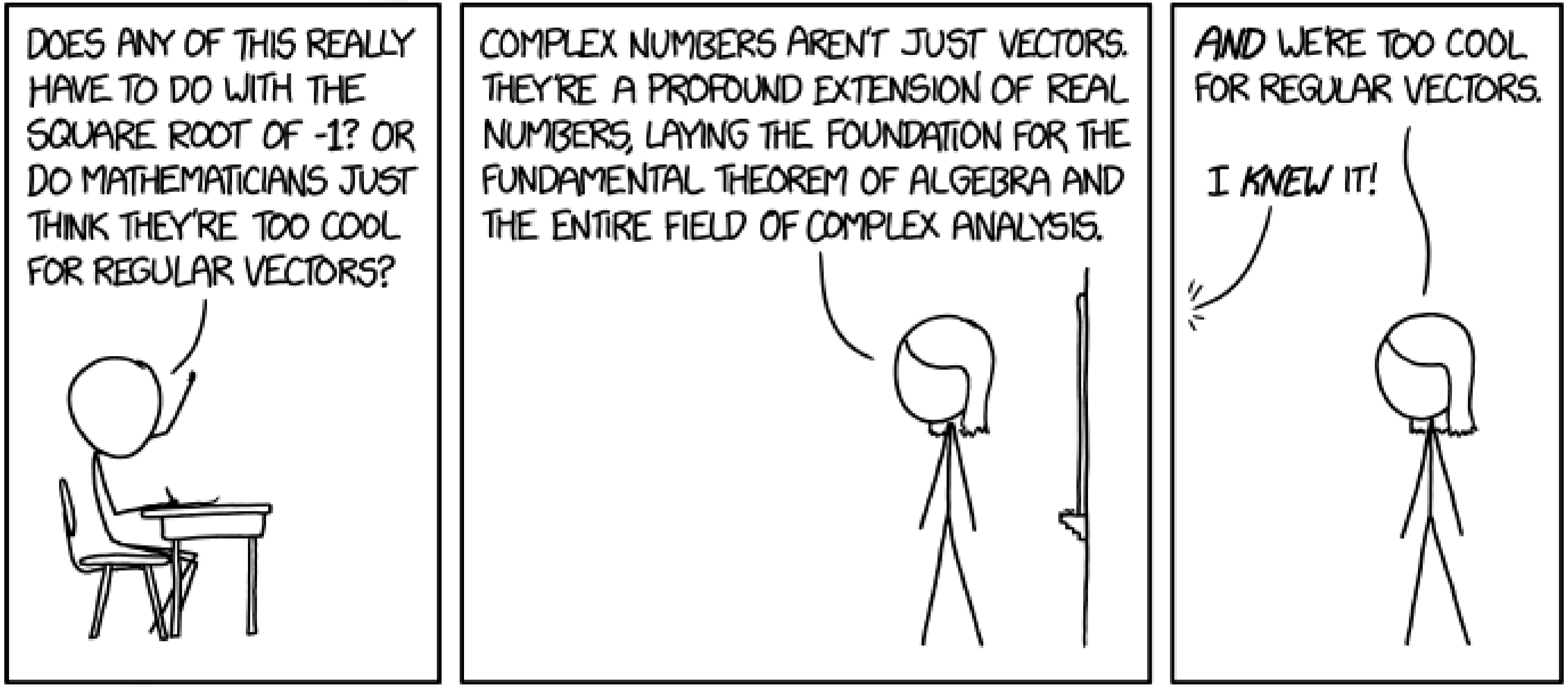
So, $\mathbf{v} + \mathbf{w} \notin \mathcal{A}$ unless $b = 0$

Example 2 (continued)

- Is $\mathcal{A} = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 + x_2 = b\}$ subspace of $\mathbb{R}^3$?

- Closed under scalar multiplication?
  - Let $\mathbf{v} \in \mathcal{A} \Rightarrow v_1 + v_2 = b \Rightarrow c(v_1 + v_2) = cv_1 + cv_2 = cb$
  - Hence, c$\mathbf{v} \notin \mathbb{V}$ unless $b = 0$

- So, $\mathcal{A}$ is not a subspace of $\mathbb{R}^3$ in general
  - But $\mathcal{A}$ is a subspace of $\mathbb{R}^3$ if $b = 0$



Thank You

Next: Matrices

https://xkcd.com/2028/