Chapter 9: Linear Functions

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Outline

- Function Notation
- Linear Functions and Superposition
- Linear Function Representation
- Vector-valued linear functions
- Projection Function
  - Orthogonal Projection Matrix

Function Notation

- Scalar-valued functions

$$f : \mathbb{R}^n \to \mathbb{R}$$

domain co-domain

$$\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^m$$

- Vector-valued functions

Example:

$$f(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x} \rangle$$

Linear Functions and Superposition

- Linear function or linear transformation or linear map $f : \mathbb{R}^n \to \mathbb{R}$

- Two key properties:
  - Additivity $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$
  - Homogeneity $f(\alpha\mathbf{u}) = \alpha f(\mathbf{u})$

- or, superposition

$$f(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2) = \alpha_1 f(\mathbf{v}_1) + \alpha_2 f(\mathbf{v}_2)$$

Example: Inner Product

- Consider $f(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x} \rangle = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$
- Then we have

$$
\begin{aligned}
f(\alpha\mathbf{x} + \beta\mathbf{y}) &= \langle \mathbf{a}, \alpha\mathbf{x} + \beta\mathbf{y} \rangle && \text{definition of } f \\
&= \langle \mathbf{a}, \alpha\mathbf{x} \rangle + \langle \mathbf{a}, \beta\mathbf{y} \rangle && \text{distributivity with vector addition} \\
&= \alpha\langle \mathbf{a}, \mathbf{x} \rangle + \beta\langle \mathbf{a}, \mathbf{y} \rangle && \text{associativity over scalar multiplication} \\
&= \alpha f(\mathbf{x}) + \beta f(\mathbf{y}) && \text{definition of } f
\end{aligned}
$$

Hence inner product (for given $\mathbf{a}$) satisfies superposition and is *linear*

Result: any linear function = inner product

- Let us consider linear $f : \mathbb{R}^n \rightarrow \mathbb{R}$ $\Rightarrow$ $$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$$ $$f(c\mathbf{x}) = cf(\mathbf{x})$$

- Recall standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$
- Can express any $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$

Result: any linear function = inner product

- Let us consider linear $f : \mathbb{R}^n \to \mathbb{R}$
- $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$
- Define $\mathbf{a} = \begin{bmatrix} f(\mathbf{e}_1) \\ f(\mathbf{e}_2) \\ \vdots \\ f(\mathbf{e}_n) \end{bmatrix}$

$$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$$
$$f(c\mathbf{x}) = cf(\mathbf{x})$$

$$\begin{aligned} f(\mathbf{x}) &= f(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) \\ &= x_1 f(\mathbf{e}_1) + x_2 f(\mathbf{e}_2) + \cdots + x_n f(\mathbf{e}_n) \\ &= \langle \mathbf{a}, \mathbf{x} \rangle \end{aligned}$$

Alternative way to calculate f

Representation of f as inner product is unique

- Proof by contraction
  - assume contrary, then show that it is not possible
- Suppose $f(\mathbf{x}) = \langle \mathbf{u}, \mathbf{x} \rangle = \langle \mathbf{v}, \mathbf{x} \rangle$
- Setting $\mathbf{x} = \mathbf{e}_i$

$$f(\mathbf{e}_i) = \langle \mathbf{u}, \mathbf{e}_i \rangle = u_i \quad \text{(from the first representation)}$$
$$= \langle \mathbf{v}, \mathbf{e}_i \rangle = v_i \quad \text{(from the second representation)}$$

- So, $u_i = v_i$ for all i: hence representation of f is unique

Reisz Representation Theorem: For any linear function $f: \mathbb{R}^n \to \mathbb{R}$, there exists unique $\mathbf{u}$ such that $f(\mathbf{v}) = \mathbf{u}^T \mathbf{v}$ for every $\mathbf{v} \in \mathbb{R}^n$

Vector-valued linear functions

n inputs $\quad$ m outputs

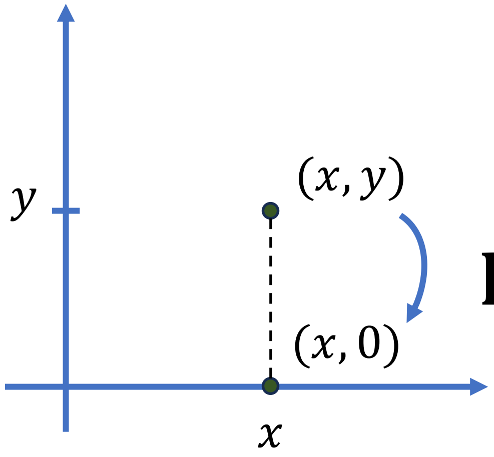
$$\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^m$$

- Satisfy superposition:

$$\mathbf{f}(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2) = \alpha_1 \mathbf{f}(\mathbf{v}_1) + \alpha_2 \mathbf{f}(\mathbf{v}_2)$$

- Example: projection onto x-axis

$$\mathbf{p}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$$



$$\mathbf{p}: \mathbb{R}^2 \to \mathbb{R}^2$$

Vector-valued linear function = $\mathbf{Ax}$

- Vector-valued function is a stack of scalar-valued functions

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix}$$

- If each element is linear, it must be inner product $f_i(\mathbf{x}) = \mathbf{a}_i^T\mathbf{x}$

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \mathbf{a}_1^T\mathbf{x} \\ \mathbf{a}_2^T\mathbf{x} \\ \vdots \\ \mathbf{a}_m^T\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}\mathbf{x} = \mathbf{Ax}$$

Example: scaling function

- Scale each coordinate by a positive scalar

$$\mathbf{S}(\mathbf{x}) = \mathrm{Diag}(\mathbf{s})\mathbf{x}$$

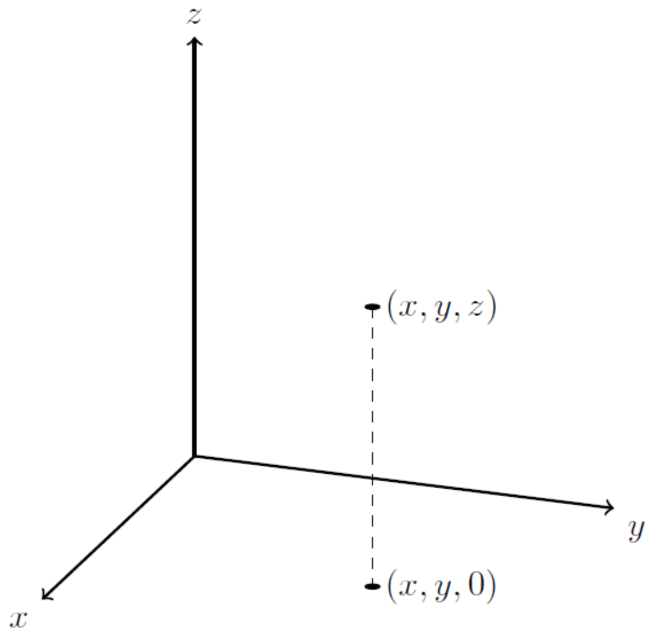
$$[\mathbf{S}(\mathbf{x})]_i = s_i x_i \qquad i \in \{1,2,\dots,n\}$$

Example: Projection in 3D

- Consider the *projection* function

$$\mathbf{p}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Projection Matrix $\mathbf{P}$



- Idempotent
- $\mathbf{P}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{P}$

Thank You

Next: Linear Systems