Chapter 4: Matrices

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Outline

Introduction

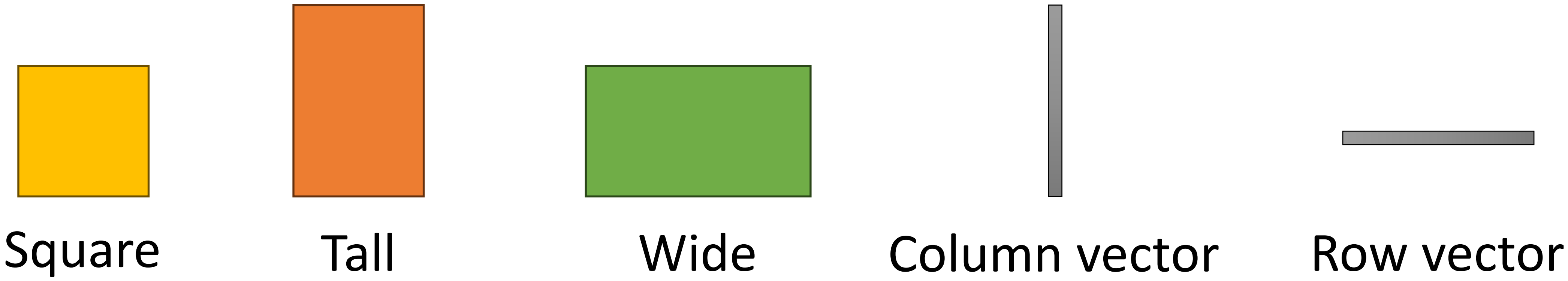
- Arrangement of numbers in rows and columns

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 5 & 2 & 1 & 4 \\ -3 & 0 & 2 & 6 \end{bmatrix} \quad A_{24}$$

3×4 matrix

- Entry in i^{th} row and j^{th} column denoted by A_{ij}

Matrix names by sizes



Rows and columns

columns $\mathbf{a}_i \in \mathbb{R}^m$

$$\mathbf{A} = \begin{array}{cc} \begin{array}{cc} \mathbf{a}_1 & \mathbf{a}_2 \end{array} & \\ \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 0 & 4 \end{bmatrix} & \begin{array}{c} \mathbf{a}_{1.} \\ \mathbf{a}_{2.} \\ \mathbf{a}_{3.} \end{array} \end{array}$$

rows $\mathbf{a}_{j.}^T \in \mathbb{R}^n$

$$\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \dots \mathbf{a}_n] = \begin{bmatrix} \mathbf{a}_{1.} \\ \vdots \\ \mathbf{a}_{m.} \end{bmatrix}$$

Block Matrices

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \end{array}\right]$$

- Sizes of blocks should be compatible

Submatrices

$$\mathbf{A}_{p:q,r:s}$$

$$\mathbf{A} = \begin{bmatrix} A_{11} & \cdots & A_{1r} & \cdots & A_{1s} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pr} & \cdots & A_{ps} & \cdots & A_{pn} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{q1} & \cdots & A_{qr} & \cdots & A_{qs} & \cdots & A_{qn} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mr} & \cdots & A_{ms} & \cdots & A_{mn} \end{bmatrix}$$

$$\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\quad \mathbf{A}_{:,2:3} \\ \mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \end{array}\right] \end{array}$$

$\mathbf{B} = \mathbf{A}_{1:2,1:3}$ $\quad$ $\mathbf{C} = \mathbf{A}_{1:2,4}$

$\mathbf{D} = \mathbf{A}_{3,1:3}$ $\quad$ $\mathbf{E} = A_{34}$

Interpretation 1: Table

Student	**Assignment 1**	**Assignment 2**	**Assignment 3**
Student 1	85	92	78
Student 2	76	80	88
Student 3	90	87	92
Student 4	82	78	84
Student 5	95	88	90

Region	**Product 1**	**Product 2**	**Product 3**
Region 1	100	200	150
Region 2	250	180	120
Region 3	180	150	220

Interpretation 2: Collection of vectors

- Each column/row may represent a meaningful vector
- Example 1: coordinates in 3D

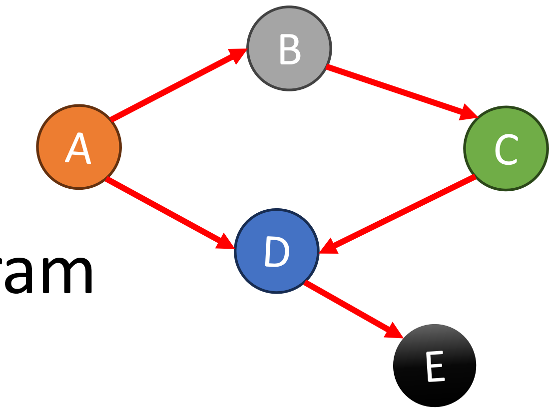
$$\mathbf{A} = \begin{bmatrix} x_1 & \boxed{x_2} & \cdots & x_T \\ y_1 & y_2 & \cdots & y_T \\ z_1 & z_2 & \cdots & z_T \end{bmatrix}$$

Coordinate at time $t = 2$

- Example 2: n images, each represented as a d-dimensional vector

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_n^T \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1d} \\ x_{21} & x_{22} & \cdots & x_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nd} \end{bmatrix}$$

Interpretation 3: Relationship graph



- Relationship $\mathcal{R}$ = set of ordered pairs
- Example: $(i, j) \in \mathcal{R}$, then person i follows j on instagram
- Consider $\mathcal{R} = \{(A, B), (A, D), (B, C), (C, D), (D, E)\}$
- Can represent $\mathcal{R}$ as a graph or a matrix

$$\mathbf{A}_{ij} = \begin{cases} 1 & (i, j) \in \mathcal{R} \\ 0 & (i, j) \notin \mathcal{R} \end{cases} \qquad A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Special Matrices: Zero & Identity Matrices

- All-zero matrix of size $m \times n$ denoted by $\mathbf{0}_{m \times n}$

- Identity matrix $\mathbf{I}_n$: ones along the diagonal, zeros elsewhere

$$(\mathbf{I}_n)_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- If subscript is omitted, infer size from context

$$\mathbf{I} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \dots \quad \mathbf{e}_n]$$

Special Matrices: Diagonal & Block Diagonal

$$\mathbf{D} = \mathrm{Diag}(\mathbf{d}) = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & d_n \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & 0 & 0 \\ 0 & \mathbf{B}_2 & 0 \\ 0 & 0 & \mathbf{B}_3 \end{bmatrix} = \mathrm{blkdiag}(\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)$$

Special Matrices: Triangular

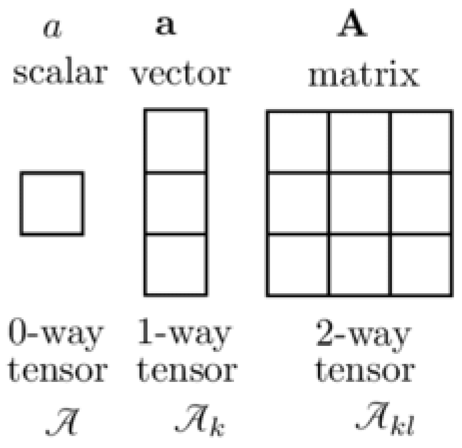
- Upper triangular

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

- Lower triangular

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

Tensors



- Vectors are 1D arrays
- Matrices are 2D arrays
- Similarly generalize to multi-dimensional arrays
- Elements in $\mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$
- Entry accessed as $\mathbf{A}_{i_1 i_2 \ldots i_d}$
- $\mathbf{A}_{:,:,\,i_3,\ldots,i_d}$ is an $n_1 \times n_2 \times 1 \times \cdots \times 1$ tensor
  - But can be interpreted as an $n_1 \times n_2$ matrix

Shulga, Dmytro & Morozov, Oleksii & Roth, Volker & Friedrich, Felix & Hunziker, Patrick. (2019).
Tensor B-Spline Numerical Methods for PDEs: a High-Performance Alternative to FEM.

Thank You

Next: Basic Matrix Operations