- Attempt all problems and submit 1 hour before the discussion session.
- You are free to discuss the problems with others. However, plagiarism will result in serious penalties, such as an F grade.

$\boxed{2}$ 1. Consider two rotation matrices: $\mathbf{A}$ and $\mathbf{B}$ given by

$$\mathbf{A} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \cos(\omega) & -\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{bmatrix}$$

Recall that multiplication of a displacement vector by a rotation matrix $\mathbf{A}$ results in another displacement vector which is rotated by θ. Verify that $\mathbf{AB} = \mathbf{BA}$ and show that $\mathbf{AB}$ is also a rotation matrix and rotates the displacement vector by an angle of $\theta + \omega$.

2. Show that the following function

$$f(\mathbf{x}) = (\|\mathbf{x} - \mathbf{c}\|^2 - \|\mathbf{c}\|^2) - (\|\mathbf{x} - \mathbf{d}\|^2 - \|\mathbf{d}\|^2)$$

for $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$, is linear and express it as an inner product $\langle \mathbf{a}, \mathbf{x} \rangle$ for some $\mathbf{a} \in \mathbb{R}^n$.

3. Solve the following problem:

$$\theta^\star = \arg\min_\theta \|\mathbf{p} - \theta\mathbf{u} - (1-\theta)\mathbf{v}\|^2 .$$

where $\mathbf{u}, \mathbf{v}, \mathbf{p} \in \mathbb{R}^n$. For a given $\mathbf{p}$, the optimization problem seeks to find the nearest point that lies on the line passing through $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u} \neq \mathbf{v}$. You may note that the line passing through two points $\mathbf{u}$ and $\mathbf{v}$ is the set of vectors

$$\mathcal{L} = \{\theta\mathbf{u} + (1-\theta)\mathbf{v} \mid \theta \in \mathbb{R}\}$$

Therefore, the goal is to find θ for which the distance of the point $\mathbf{p}$ from the point $\theta\mathbf{u}+(1-\theta)\mathbf{v}$ is as small as possible.

4. Let us consider the problem of approximating a non-zero vector as a multiple of another one. We have already seen that the solution to the problem

$$\mathbf{x}^\star = \arg\min_x \|\mathbf{a}x - \mathbf{b}\|$$

is given by $\mathbf{x}^\star = \frac{\mathbf{a}^\intercal \mathbf{b}}{\mathbf{a}^\intercal \mathbf{a}}$. For non-zero $\mathbf{a}, \mathbf{b}$, Show that

$$\|\mathbf{a}x^\star - \mathbf{b}\|^2 = \|\mathbf{b}\|^2 \sin^2(\theta)$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

5. Consider the matrix:

$$\mathbf{A} = \alpha \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

Find the constant such that the matrix is orthogonal.