Chapter 1: Vectors

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Outline

Introduction

Block vectors and subvectors

All-zero and all-one vectors

Standard unit vectors

Vector applications

- Location and displacement
- Color, Image, Video
- Word count, one-hot coding

Introduction

- Vector is an ordered finite list of numbers

$$\mathbf{v} = \begin{bmatrix} 1.2 \\ 2.1 \\ -1.8 \end{bmatrix}$$ Components/elements/entries

- Order is important

- Related term: array meaning “arrangement”
  (mostly used in context of programming)

Column vs. Row vectors

- Column vector

$$\mathbf{v} = \begin{bmatrix} 1.2 \\ 2.1 \\ -1.8 \end{bmatrix}$$

- Row vector

$$\mathbf{v} = \begin{bmatrix} 1.2 & 2.1 & -1.8 \end{bmatrix}$$

- By default: consider column vectors only

Elements of a vector

- Symbols to denote elements

$$\mathbf{v} = \begin{bmatrix} 1.2 \\ 2.1 \\ -1.8 \end{bmatrix} \qquad v_2$$

- Similarly, i^{th} element is denoted by v_i
- These are called scalars, mostly real-valued, i.e., belonging to $\mathbb{R}$
- Dimension: number of elements (e.g. 3 in the above example)
- For a n-dimensional vector, we say

$$\boxed{\mathbf{v} \in \mathbb{R}^n}$$

Block or stacked vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \mathbf{d} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \end{bmatrix}$$

Subvectors

$j - i + 1$ elementts

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \\ a_k \end{bmatrix} \qquad \mathbf{a}_{i:j} = \begin{bmatrix} a_i \\ a_{i+1} \\ \vdots \\ a_j \end{bmatrix}$$

$$\mathbf{w} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \mathbf{w}_{2:4}$$

Zero vector

- All-zero vector of dimension n denoted by $\mathbf{0}_n$

$$\mathbf{0}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Additive identity

$$\mathbf{v} + \mathbf{0}_n = \mathbf{v}$$

- Subscript omitted

$$\mathbf{v} + \mathbf{0}$$

(only vectors of same size can be added)

Standard unit vectors

- In 3D, standard unit vectors are

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Also called standard basis vectors, coordinate vectors, etc.
- For n-dimensional case, standard unit vectors are $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$

$$(\mathbf{e}_{\mathrm{i}})_j = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \quad \text{for all } 1 \le i, j \le n$$

All-one vector

- Vector of all ones $\mathbf{1}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

- Similarly, all–one vector of size n denoted by $\mathbf{1}_n$

Application 1: Location

- Position of a point in space
  - relative to a reference point/coordinate system

- 2D space: coordinates (x, y) represented by position vector

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Application 2: Displacement

- Change of position from one point to another point

- Initial position: $\mathbf{r}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

Displacement: $\mathbf{\Delta r} = \mathbf{r}_2 - \mathbf{r}_1 = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$

- Final position: $\mathbf{r}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

Location vs. displacement

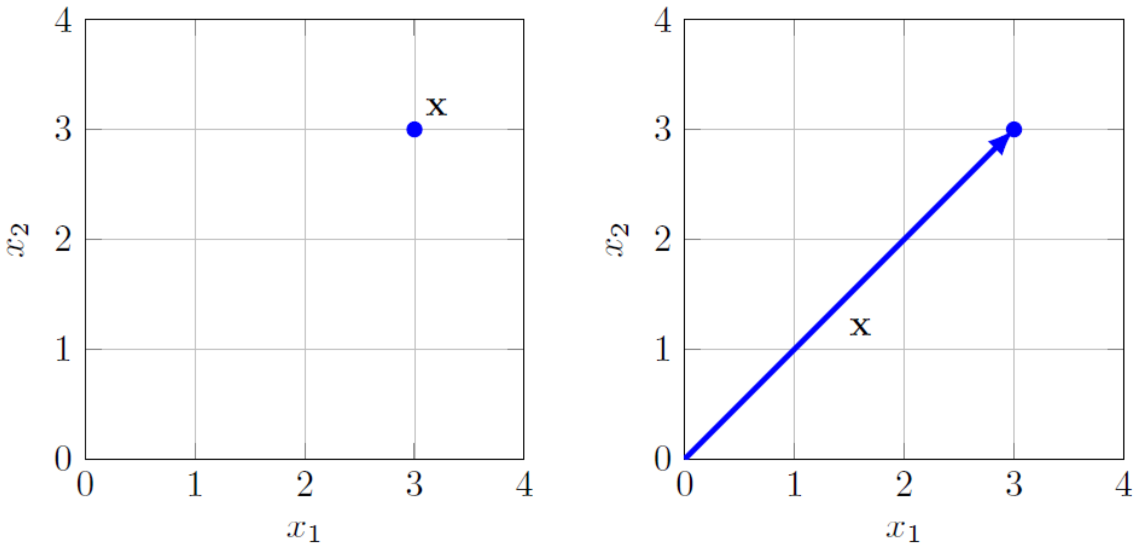


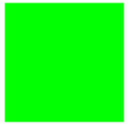
Figure 1.1: Left: Vector $\mathbf{x}$ specifies the position (shown as a dot) with coordinates $(3, 3)$. Vector $\mathbf{x}$ represents a displacement from $(0, 0)$ by 3 units along x_1 and 3 units along x_2.

- Location vector = displacement from origin

Application 3: Color

- RGB model: represents each color by a vector of size 3

RGB Vector: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$



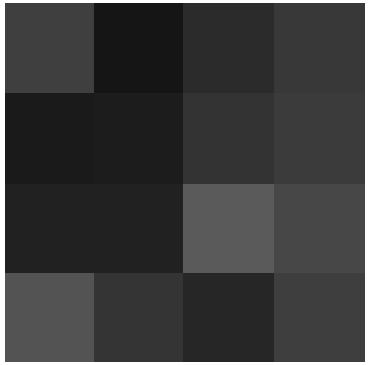
RGB Vector: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

RGB Vector: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Application 4: Images and Videos

- Monochrome (aka grayscale) image
  - Vector of pixel intensities
  - Arranged sequentially
  - Image read from left-to-right or top-to-bottom



- Size of vector determined by resolution
- Example: image with 800 pixels width and 600 pixels height
- Corresponding vector will have 480000 elements, one per pixel
- Color images: 3 pixel intensities per pixel, arrange sequentially
- Video: stacked color image vectors (one color image vector per frame)

Application 5: Word counts

Text A: “I like apple and banana”

Text B: “She prefers banana and orange”

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Dictionary
apple
banana
orange

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Allows mathematical operations like measuring similarity between documents

Application 6: One-hot coding

- For categorical data (e.g. class labels, attributes, features)
- Each category represented as a standard unit vector
- Example: 3 categories

$$\text{Apple: } [1 \quad 0 \quad 0]$$

$$\text{Banana: } [0 \quad 1 \quad 0]$$

$$\text{Orange: } [0 \quad 0 \quad 1]$$

Thank You

Next: Vector Operations