Chapter 5:
Basic Matrix Operations

Ketan Rajawat

Outline

Reshaping Matrices

Transpose and Addition

- Matrix Transpose, Reshaping
- Matrix Addition

Scaling a Matrix

Vector Space of Matrices

Matrix-Vector Multiplication

- Applications
- Properties

Complexity of Matrix Operations

Storing Matrices in Computer Memory

- In computers, matrices are usually stored as $1\mathrm{D}$ arrays

C/C++
Python

$$\begin{bmatrix}1\\2\\3\\4\\5\\6\end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$$

$$\begin{bmatrix}1\\4\\2\\5\\3\\6\end{bmatrix}$$

Fortran
MATLAB
$\mathrm{vec}(\mathbf{A})$

- Other data structures may also be used,
e.g., for sparse matrices

Reshaping Matrices

- $m \times n$ matrix can be reshaped into $p \times q$ matrix provided $mn = pq$

- Programming languages provide different types of reshape commands

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

- Process:
  - $m \times n$ matrix stored as an mn −sized vector
  - Read it as a $p \times q$ matrix
- Result depends on the ordering

(row-major or column-major)

Matrix Transpose

- Flips the matrix over its main diagonal

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow \mathbf{A}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

- Transpose of the transpose of matrix is the matrix itself $(\mathbf{A}^T)^T = \mathbf{A}$
- Similarly for block matrices

$$\mathbf{A} = [\mathbf{B} \quad \mathbf{C}] \Rightarrow \mathbf{A}^T = \begin{bmatrix} \mathbf{B}^T \\ \mathbf{C}^T \end{bmatrix}$$

- Symmetric matrix is same as its transpose $\mathbf{A} = \mathbf{A}^T$
- That is, $A_{ij} = A_{ji}$ for all $1 \leq i, j \leq n$

$$\mathbf{A} = \begin{bmatrix} 2 & 5 & 7 \\ 5 & 1 & 9 \\ 7 & 9 & 4 \end{bmatrix}$$

Matrix addition

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

- Properties
  - Commutativity $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
  - Associativity $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
  - Additive Identity $\mathbf{A} + \mathbf{0} = \mathbf{A}$
  - Transpose of Sum $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

Scaling a Matrix

- $\alpha\mathbf{A}$ amounts to multiplying each entry of $\mathbf{A}$ with the scalar α
- Properties:
  - Commutativity $\alpha\mathbf{A} = \mathbf{A}\alpha$
  - Scaling with zero $0\mathbf{A} = \mathbf{0}$
  - Compatibility with
    - Scalar multiplication $(\alpha\beta)\mathbf{A} = \alpha(\beta\mathbf{A})$
    - Scalar addition $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$

Vector Space of Matrices

- The space of $m \times n$ matrices is a vector space
  - Contains the zero matrix
  - Closed under addition
  - Closed under scalar multiplication

- Concept of linear combinations

can be similarly defined

Matrix-vector multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

$$\mathbf{A} \qquad \mathbf{v} \quad = \qquad \mathbf{Av}$$

- Linear combination of columns of $\mathbf{A}$

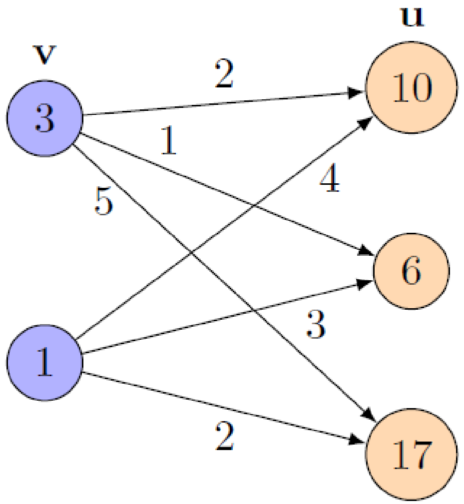
$$[\mathbf{a_1} \quad \mathbf{a_2} \quad \dots \quad \mathbf{a}_n]\,\mathbf{v} \;=\; [v_1\mathbf{a_1} + v_2\mathbf{a_2} + \cdots + v_n\mathbf{a}_n]$$

Example

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 5 & 2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\mathbf{u} = \mathbf{A}\mathbf{v} = \begin{bmatrix} 2(3) + 4(1) \\ 1(3) + 3(1) \\ 5(3) + 2(1) \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ 17 \end{bmatrix}$$

Neural Network



Applications

- Multiplication with standard unit vectors
  - Extracting columns: $\mathbf{A}\mathbf{e}_i = \mathbf{a}_i$
  - Extracting rows: $\mathbf{A}^T\mathbf{e}_j = \mathbf{a}_{j.}^T$
- Sums of rows/columns
  - Sum of columns: $\mathbf{A}\mathbf{1}$
  - Sum of rows: $\mathbf{A}^T\mathbf{1}$

Applications (continued)

- Multiplication with difference matrix gives forward differences

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} v_2 - v_1 \\ v_3 - v_2 \\ v_4 - v_3 \\ v_5 - v_4 \end{bmatrix}$$

- Multiplication with cumulative sum matrix gives cumulative sum

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_1 + v_2 \\ v_1 + v_2 + v_3 \\ v_1 + v_2 + v_3 + v_4 \\ v_1 + v_2 + v_3 + v_4 + v_5 \end{bmatrix}$$

Properties of Matrix-Vector Multiplication

- Distribution across vector addition $\mathbf{A}(\mathbf{v}+\mathbf{w}) = \mathbf{A}\mathbf{v} + \mathbf{A}\mathbf{w}$
- Distribution across matrix addition $(\mathbf{A}+\mathbf{B})\mathbf{v} = \mathbf{A}\mathbf{v} + \mathbf{B}\mathbf{v}$
- Multiplication with scalar $(\alpha\mathbf{A})\mathbf{v} = \alpha(\mathbf{A}\mathbf{v}) = \mathbf{A}(\alpha\mathbf{v})$

Complexity of matrix operations

- Matrix-vector multiplication: each element requires n multiplications and $n-1$ additions, so total flops $= n(2n-1) \approx 2n^2$

Operation	**Flop count**
Transpose	0
Scaling	n^2
Addition	n^2
Matrix $-$ vector multiplication	$2n^2$

Thank You

Next: Matrix Multiplication