EE951: Linear Algebra **eMasters**

Chapter 3: Vector Spaces

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3.1 Fields

So far we have been dealing with the field of real numbers only. In simpler terms, a *field* is a set on which addition, subtraction, multiplication, and division are well-defined in a consistent manner. The set of real numbers is the most commonly used field. Other examples include the field of rational numbers and the field of complex numbers.

We will denote a field by $\mathbb{F}$, whose specific examples may be real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. For instance, vectors and matrices in $\mathbb{F}$ are denoted by $\mathbb{F}^n$ and $\mathbb{F}^{m\times n}$, respectively.

A counter-example is the set of colors, expressed in RGB format. Such a set is not a field because we cannot generally define mathematical operations such as addition, subtraction, multiplication, and division for colors in a consistent manner.

3.2 Vector Space

A vector space is a set of objects where the concept of linear combinations is well-defined. In a vector space, any linear combination of objects within the set produces another object that also belongs to the same set. Formally, a vector space $\mathbb{V}$ is a set of elements, called vectors, and two operations: addition and scalar multiplication.

A vector space has three key properties:

1. $\mathbb{V}$ contains zero, i.e., $0 \in \mathbb{V}$
2. $\mathbb{V}$ is closed under addition, i.e., if $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{V}$, then $\mathbf{v}_1 + \mathbf{v}_2 \in \mathbb{V}$
3. $\mathbb{V}$ is closed under scalar multiplication, i.e., if $\mathbf{v}_1 \in \mathbb{V}$ and $\alpha \in \mathbb{F}$, then $\alpha\mathbf{v}_1 \in \mathbb{V}$

A consequence of these properties is that vector spaces are closed over linear combinations. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be elements of a vector space $\mathbb{V}$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{F}$, then we have that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_m\mathbf{v}_m \in \mathbb{V}. \tag{3.1}$$

The definition is trivial with $m = 1$ if the vector space contains only a single element. Canonical examples include $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ over real numbers, since linear combinations of vectors (matrices) are themselves vectors (matrices).

We will mostly be dealing with vector spaces over real numbers. However, when considering complex vectors $\mathbb{C}^n$ for instance, we also allow $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$.

In the general case, the addition and scalar multiplication operations must be defined to follow certain axioms, such as commutativity, existence of additive inverse, etc. We need not detail these axioms here because when dealing with vector spaces over $\mathbb{R}$ or $\mathbb{C}$, the usual definitions of addition and scalar multiplication suffice and satisfy all the required axioms.

Let us discuss another example and a counter-example of vector field over $\mathbb{R}$ next.

1. The most basic example of a vector space is the simplest one: 0. This vector space consists of just the zero vector. In this case, vector addition and scalar multiplication are very straightforward.
2. The set of natural numbers $\mathbb{N}$ is *not* a vector space over $\mathbb{R}$, since linear combination of several natural numbers with real-valued weights will generally not be a natural number.

3.3 Subspace

The set $\mathbb{U} \subseteq \mathbb{V}$ is called a subspace of $\mathbb{V}$ if $\mathbb{U}$ is also a vector space, i.e., any linear combination of objects within $\mathbb{U}$ also belong to $\mathbb{U}$. A subspace is also sometimes referred to as linear subspace. From the definition, we can see that the set $\mathbb{U} \subseteq \mathbb{V}$ is a subspace if for all $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \mathbb{U}$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$, it holds that

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \ldots + \alpha_m\mathbf{u}_m \in \mathbb{U}. \tag{3.2}$$

Trivially, $\{0\}$ is the smallest possible subspace of $\mathbb{V}$, while $\mathbb{V}$ is the largest possible subspace of $\mathbb{V}$.

Example 3.1. Consider the set $\mathbb{V}$ defined as follows:

$$\mathbb{V} = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{C}^2 \mid z_1 + iz_2 = 0 \right\}$$

To show that $\mathbb{V}$ is a subspace of $\mathbb{C}^2$, we need to demonstrate the three properties of a subspace: that it contains the zero vector, it is closed under addition, and it is closed under scalar multiplication.

1. The zero vector is $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since $0 + i(0) = 0$, we can see that $\mathbf{0} \in \mathbb{V}$.
2. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ be two arbitrary vectors in $\mathbb{V}$. Since $\mathbf{v} \in \mathbb{V}$, we have $v_1 + iv_2 = 0$. Similarly, since $\mathbf{w} \in \mathbb{V}$, we have $w_1 + iw_2 = 0$. The sum of these vectors is given by $\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$. Using the fact that $v_1 + iv_2 = 0$ and $w_1 + iw_2 = 0$, we can see that $(v_1 + w_1) + i(v_2 + w_2) =$

$0 + i(0) = 0$. Hence,

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} \in \mathbb{V} \tag{3.3}$$

3. Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be an arbitrary vector in $\mathbb{V}$, and let c be an arbitrary scalar from $\mathbb{C}$. Since $\mathbf{v} \in \mathbb{V}$, we have $v_1 + iv_2 = 0$. The scalar multiple of this vector is given by $c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$. Therefore, we have that $cv_1 + i(cv_2) = c(v_1 + iv_2) = c(0) = 0$.

Since $\mathbb{V}$ satisfies all three properties of a subspace, we can conclude that $\mathbb{V}$ is a subspace of $\mathbb{C}^2$.

Example 3.2. Is the set $\mathcal{A} = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 + x_2 = b\}$ a subspace of $\mathbb{R}^3$? Here, we can see that the zero element does not belong to $\mathcal{A}$ unless $b = 0$. Hence for $b \neq 0$, $\mathcal{A}$ is not a vector space and therefore not a subspace of $\mathbb{R}^3$. For the case when $b = 0$ however, it can be seen that $\mathcal{A}$ is indeed a subspace of $\mathbb{R}^3$. Specifically, we can check the three properties. First, we see that $\mathbf{0}_3 \in \mathcal{A}$. Consider two vectors $\mathbf{x}, \mathbf{y} \in \mathcal{A}$ such that $\mathbf{z} = \mathbf{x} + \mathbf{y}$. Then it holds that

$$z_1 + z_2 = x_1 + x_2 + y_1 + y_2 = 0 \tag{3.4}$$

and hence $\mathbf{z} \in \mathcal{A}$. Finally, if $\mathbf{x} \in \mathcal{A}$, it is easy to see that $\alpha\mathbf{x} \in \mathcal{A}$. We remark that if $b \neq 0$, none of the three properties hold.