Chapter 8: Norms

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Outline

Introduction

- Euclidean Norm or 2-norm

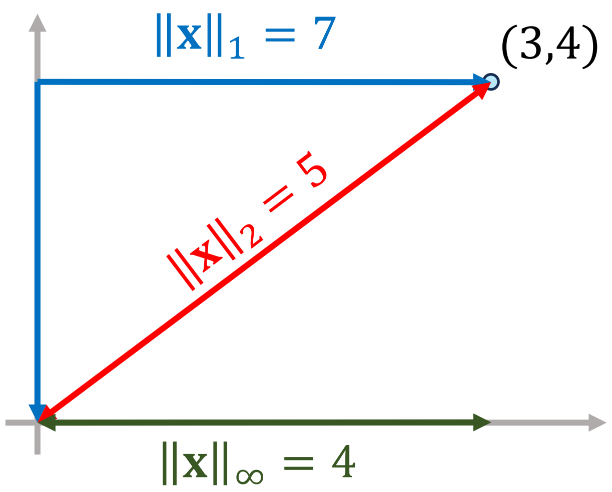
$$||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

- Measure of “size,” “length,” or “magnitude”
- Example: $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ then,

$$\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

- Distance from origin to the coordinates represented by $\mathbf{v}$

Other Norms: 1- and ∞-norms



- 1-norm, Manhattan, or taxicab norm

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

- ∞-norm, Chebyshev, or maximum norm

$$\|\mathbf{x}\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$$

Matrix Norm

- Frobenius Norm

$$\|\mathbf{A}\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} \left|A_{ij}\right|^2}$$

- Recall: reshaping matrix as vector using vec()

$$\|\mathbf{A}\|_F = \|\mathrm{vec}(\mathbf{A})\|_2$$

Frobenius norm-squared

$$\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] \quad = \begin{bmatrix} \mathbf{a}_{1\cdot} \\ \mathbf{a}_{2\cdot} \\ \vdots \\ \mathbf{a}_{m\cdot} \end{bmatrix}$$

$$\|\mathbf{A}\|_F^2 = \|\mathbf{a}_1\|^2 + \|\mathbf{a}_2\|^2 + \cdots \|\mathbf{a}_n\|^2$$

$$= \left\|\mathbf{a}_{1\cdot}^T\right\|^2 + \left\|\mathbf{a}_{2\cdot}^T\right\|^2 + \cdots + \|\mathbf{a}_{m\cdot}^T\|^2$$

Norms over inner-product spaces

- Norms may or may not be defined using inner products
- When norm is defined using an inner product

$$||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

- Examples: Euclidean norm, Frobenius norm

$$||\mathbf{v}||_2 = \sqrt{\mathbf{v}^T \mathbf{v}} \qquad ||\mathbf{A}||_F = \sqrt{\mathrm{tr}(\mathbf{A}^T \mathbf{A})}$$

- But Manhattan and Chebyshev norms

cannot be defined like this

Properties of Norm

Norm is any real-valued function that satisfies:

1. Non-negativity: $||\mathbf{x}|| \geq 0$
2. Homogeneity: for any scalar c : $||c\mathbf{x}|| = |c|\,||\mathbf{x}||$
3. Definiteness: $||\mathbf{x}|| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$
4. Triangle inequality: $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$

Aside: "if and only if" or "iff"
Notation: $A \Leftrightarrow B$
Meaning: EITHER both statements (A and B) are true OR both statements are false
Example: $x = 1$ if and only if $x + 1 = 2$
To prove: Two key steps:

1. Assume A, prove B
2. Assume B, prove A

Proof for Euclidean Norm $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

1. Non-negativity: since $x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$, its square root is also non-negative
2. Homogeneity: $\|\alpha\mathbf{v}\| = \sqrt{(\alpha\mathbf{v})^T(\alpha\mathbf{v})} \qquad = \sqrt{\alpha^2\mathbf{v}^T\mathbf{v}}$

$$= |\alpha|\sqrt{\mathbf{v}^T\mathbf{v}} \qquad = |\alpha|\|\mathbf{v}\|$$

3a. If $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$,

then each term must be 0

($\|\mathbf{x}\|_2 = 0 \Rightarrow \mathbf{x} = \mathbf{0}$)

3b. If $\mathbf{x} = \mathbf{0}$,

then $\|\mathbf{x}\|_2 = 0$: $\mathbf{x} = \mathbf{0} \Rightarrow \|\mathbf{x}\|_2 = 0$

$$\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$$

Norm of sum

$$\|\mathbf{u}+\mathbf{v}\| = \sqrt{\langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle}$$ Definition of norm

$$= \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle}$$ Distributivity

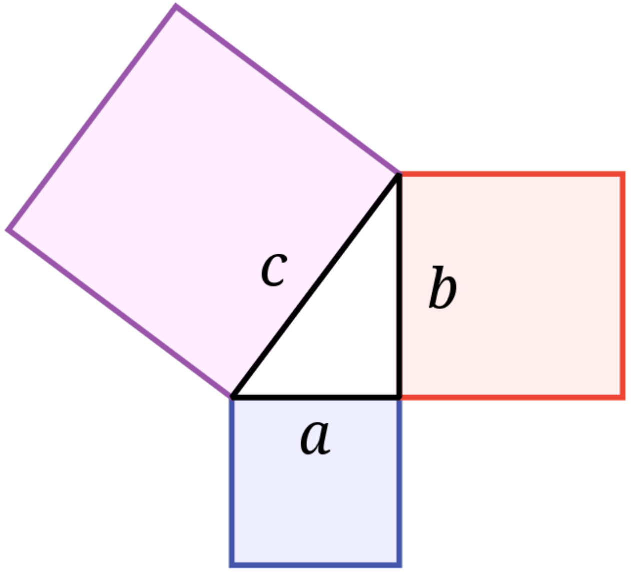
$$= \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle + 2\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle}$$ Commutativity

$$= \sqrt{\|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2}$$ Definition of norm

Pythagorean Theorem

- If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then

$$\|\mathbf{u}\|_2^2 + \|\mathbf{v}\|_2^2 = \|\mathbf{u} + \mathbf{v}\|_2^2$$



since $\mathbf{u}^T\mathbf{v} = 0$

Orthogonal Decomposition

- Write $\mathbf{u}$ as a sum of
  - scalar multiple of $\mathbf{v} \neq \mathbf{0}$, and
  - Another vector $\mathbf{w}$ such that $\mathbf{w} \perp \mathbf{v}$
- Let $$\mathbf{u} = \alpha\mathbf{v} + \underbrace{(\mathbf{u} - \alpha\mathbf{v})}_{\mathbf{w}}$$
- Since $\mathbf{w} \perp \mathbf{v}$: $$\mathbf{v}^T\mathbf{w} = \mathbf{v}^T(\mathbf{u} - \alpha\mathbf{v}) = 0$$
- $\Rightarrow \alpha = \dfrac{\mathbf{u}^T\mathbf{v}}{\mathbf{v}^T\mathbf{v}} = \dfrac{\mathbf{u}^T\mathbf{v}}{\|\mathbf{v}\|^2}$

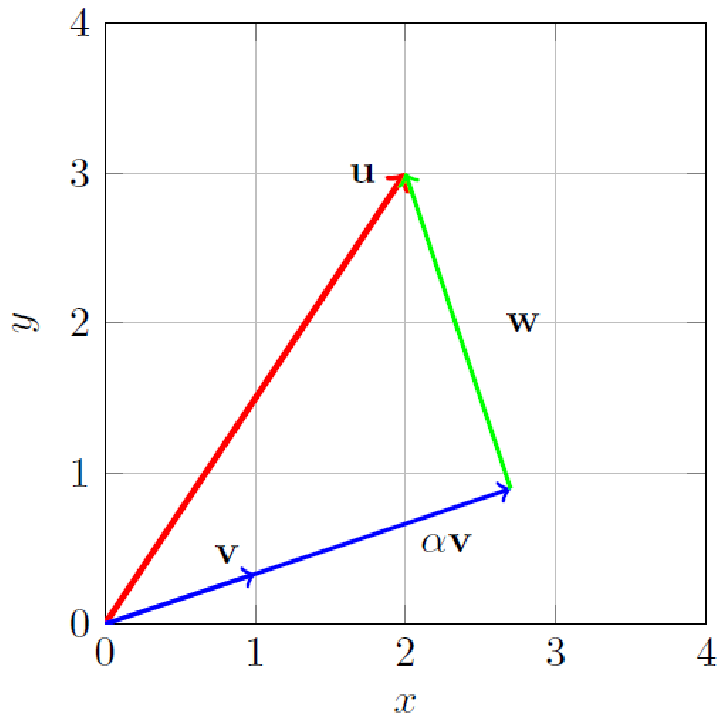
$$\mathbf{u} = \frac{\mathbf{u}^T\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} + \left(\mathbf{u} - \frac{\mathbf{u}^T\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\right)$$

Example: Orthogonal Decomposition

$$\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\alpha = \frac{\mathbf{u}^T \mathbf{v}}{||\mathbf{v}||^2} = 2.7$$

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1/3 \end{bmatrix}$$



$$\mathbf{u} = 2.7\mathbf{v} + (\mathbf{u} - 2.7\mathbf{v}) = \begin{bmatrix} 2.7 \\ 0.9 \end{bmatrix} + \begin{bmatrix} -0.7 \\ 2.1 \end{bmatrix}$$

Cauchy-Schwarz Inequality

$$|\mathbf{u}^T\mathbf{v}| \leq ||\mathbf{u}||\,||\mathbf{v}||$$
$$-||\mathbf{u}||\,||\mathbf{v}|| \leq \mathbf{u}^T\mathbf{v} \leq ||\mathbf{u}||\,||\mathbf{v}||$$

- Proof: suppose $\mathbf{v} \neq \mathbf{0}$ (otherwise inequality is trivially true)
- Apply orthogonal decomposition: $\mathbf{u} = \dfrac{\mathbf{u}^T\mathbf{v}}{||\mathbf{v}||^2}\mathbf{v} + \mathbf{w}$
- Since $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, from Pythagorean:

$$||\mathbf{u}||^2 = \left\|\frac{\mathbf{u}^T\mathbf{v}}{||\mathbf{v}||^2}\mathbf{v}\right\|^2 + ||\mathbf{w}||^2$$
$$= \frac{|\mathbf{u}^T\mathbf{v}|^2}{||\mathbf{v}||^2} + \underbrace{||\mathbf{w}||^2}_{\geq 0}$$
$$\geq \frac{|\mathbf{u}^T\mathbf{v}|^2}{||\mathbf{v}||^2}$$

Becomes equality when $\mathbf{w} = \mathbf{0}$

$$\mathbf{w} = 0 \Leftrightarrow \mathbf{u} = \frac{\mathbf{u}^T\mathbf{v}}{||\mathbf{v}||^2}\mathbf{v}$$

Triangle Inequality

- Recall $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}^T\mathbf{v}$ Norm of Sum Expression

$$\leq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\|$$ Cauchy-Schwarz Inequality

$$= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$$ Collect into square

$$\boxed{\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|}$$

- Becomes equality when $\mathbf{u}^T\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|$
- i.e., when $\mathbf{u}$ is scalar multiple of $\mathbf{v}$

Angle Between Vectors

- We define

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{u}||\,||\mathbf{v}||}$$

- Recall Cauchy-Schwarz inequality $-||\mathbf{u}||\,||\mathbf{v}|| \leq \mathbf{u}^T\mathbf{v} \leq ||\mathbf{u}||\,||\mathbf{v}||$

- $\theta = 0$ parallel vectors
- $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ orthogonal vectors

Norm of stacked vectors

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_m \end{bmatrix}$$

$$\|\mathbf{x}\| = \sqrt{\|\mathbf{x}_1\|^2 + \|\mathbf{x}_2\|^2 + \cdots + \|\mathbf{x}_m\|^2}$$

$$\left\| \begin{bmatrix} \underbrace{\mathbf{x}_1}_{\mathbb{R}^{m_1}} \\ \underbrace{\mathbf{x}_2}_{\mathbb{R}^{m_2}} \end{bmatrix} \right\| = \left\| \underbrace{\begin{bmatrix} \|\mathbf{x}_1\| \\ \|\mathbf{x}_2\| \end{bmatrix}}_{\text{vector with 2 entries}} \right\|$$

Orthonormal Vectors

$$\mathbf{v}_i^T \mathbf{v}_j = 0 \qquad i \neq j$$
$$\|\mathbf{v}_i\| = 1 \qquad 1 \leq i \leq m$$

or

$$\mathbf{v}_i^T \mathbf{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Complexity of calculating norm

- To calculate $\|\mathbf{x}\| = \sqrt{\mathbf{x}_1^2 + \mathbf{x}_2^2 + \cdots + \mathbf{x}_n^2}$
  - $n - 1$ additions
  - n multiplications
  - 1 square root (≈ 6 flops)
- Total $2n$ flops
- Matrix norm requires $2n^2$ flops

Thank You

Next: Linear Functions