Point to discuss: is $\underline{a}^T \underline{b} = \underline{b}^T \underline{a}$?

for $\underline{a}, \underline{b} \in \mathbb{R}^n$

note that $a^T b = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

and also $b^T a = b_1 a_1 + b_2 a_2 + \dots + b_n a_n$

since $a_1 b_1 = b_1 a_1$ (scalar product is commutative)

$\Rightarrow \quad b^T a = a_1 b_1 + a_2 b_2 + \dots + a_n b_n = a^T b$

Hence $\underline{a}^T \underline{b} = \underline{b}^T \underline{a} = \langle a, b \rangle$

another notation

Q1 outer product xy^T Note: $(xy^T)^T = yx^T$

we can take $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

note that, in general, $n \neq m$

size of xy^T : $n \times m$

size of yx^T : $m \times n$

not same generally

In fact: $xy^T \neq yx^T$

we can always disprove an equality using a counterexample couter example : take $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $y = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

$$xy^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 8 & 10 \end{bmatrix}$$

not equal

$$yx^T = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 4 & 8 \\ 5 & 10 \end{bmatrix}$$

Recall: $(xy^T)^T = yx^T$

when can $xy^T = yx^T$?

clearly $m = n$

suppose $\underline{x}, y \in \mathbb{R}^n$

then $[xy^T]_{ij} = x_i y_j$

when would these be equal ?

also $[yx^T]_{ij} = y_i x_j$

let us say $x_i y_j = y_i x_j \ \forall \ i, j$

Equivalently: $\frac{x_i}{y_i} = \frac{x_j}{y_j} \quad \forall\ i,j$

In other words $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3} = \dots = \frac{x_n}{y_n}$

(recall $x, y \in \mathbb{R}^n$)

suppose that $x_i / y_i = c \quad \forall\ i$

then $x_i = c y_i \quad \forall\ i$

or equivalently $\underline{x} = c\, y$

($\underline{x}$ & y are scalar multiples of each other)

So xy^T symmetric when $\underline{x} = c y$ for some $\underline{c} \in \mathbb{R}$

Q2

$$A \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

downward circular shift

what about $A^2 \underline{x}$?

$$A^2\underline{x} = A(A\underline{x})$$

recall $Ax = \begin{bmatrix} x_4 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$A(Ax) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_4 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_1 \\ x_2 \end{bmatrix}$$

downward circular shift by 2

$$A^3x = A(A(A\underline{x})) = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{bmatrix}$$

(following pattern)

and finally $A^4x = A(A(A(Ax)))$

$$= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underline{x}$$

so $A^4\underline{x} = \underline{x}$

likewise $A^5x = A(A^4x) = Ax$

$$A^6x = A^2x$$

generally consider A^nx for $n = 4k+j$

$$A^{4k+j}\underline{x} = A^{j}\underline{x} \qquad j = 0,1,2,3$$

(downward circular shift by j)

Q3 $\quad \|\underline{u}+\underline{v}\|^2 + \|\underline{u}-\underline{v}\|^2$

$$= (\underline{u}+\underline{v})^T(\underline{u}+\underline{v}) + (\underline{u}-\underline{v})^T(\underline{u}-\underline{v})$$

$$= u^Tu + \cancel{\underline{v}^T\underline{u}} + \cancel{\underline{u}^T\underline{v}} + \underline{v}^T\underline{v} + u^Tu - \cancel{v^Tu} - \cancel{u^Tv} + v^Tv$$

$$= 2(u^Tu + v^Tv)$$

$$= 2(\|u\|^2 + \|v\|^2)$$

Q4 $\quad \|\alpha\underline{u}+\beta\underline{v}\|^2 - \|\alpha\underline{v}+\beta\underline{u}\|^2$

$$= (\alpha\underline{u}+\beta\underline{v})^T(\alpha\underline{u}+\beta\underline{v}) - (\alpha\underline{v}+\beta\underline{u})^T(\alpha\underline{v}+\beta\underline{u})$$

$$= \alpha^2\underline{u}^T\underline{u} + \cancel{\alpha\beta v^Tu} + \cancel{\alpha\beta u^Tv} + \beta^2 v^Tv$$

$$- \alpha^2 v^Tv - \cancel{\alpha\beta u^Tv} - \cancel{\alpha\beta v^Tu} - \beta^2 u^Tu$$

$$= \alpha^2(u^Tu - v^Tv) + \beta^2(v^Tv - u^Tu)$$

$$= \underbrace{(\alpha^2-\beta^2)(\|u\|^2 - \|v\|^2)}$$

Note: this is zero when $\alpha^2 = \beta^2$

But can this be zero for all α, β ?

can be zero regardless of α, β when

$$\|\underline{u}\| = \|\underline{v}\|$$

Hence $\quad \|\alpha \underline{u} + \beta \underline{v}\| = \|\alpha \underline{v} + \beta \underline{u}\| \quad \forall \quad \alpha, \beta$

if & only if $\quad \|u\| = \|v\|$

Note : $\|u\|^2 - \|v\|^2 = \underbrace{(\|u\| - \|v\|)}_{\text{zero when } \|u\| = \|v\|} \underbrace{(\|u\| + \|v\|)}_{\text{zero when } \underline{u} = \underline{v} = \underline{0}}$

combining

zero when $\|\underline{u}\| = \|\underline{v}\|$

Q5 Find $\underline{u}, \underline{v} \in \mathbb{R}^2$

$$\underline{u} = \alpha \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad - \quad (1)$$

Some scalar (unknown)

$$v^T \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0 \quad - \quad (2)$$

& $\quad \underline{u} + \underline{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad - \quad (3)$

Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

then $v_1 + 3v_2 = 0$ (from ②)

also $u + v = \begin{bmatrix} \alpha \\ 3\alpha \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ (from ①)

$= \begin{bmatrix} \alpha + v_1 \\ 3\alpha + v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (from ③)

so we have

$$\boxed{\begin{aligned} v_1 + 3v_2 &= 0 \\ \alpha + v_1 \quad &= 1 \\ 3\alpha \quad + v_2 &= 2 \end{aligned}} \Rightarrow v_1 = 1 - \alpha$$

$$1 - \alpha + 3v_2 = 0 \qquad \times 3$$

$$3\alpha + v_2 = 2 \qquad +$$

$$3 + 10v_2 = 2 \Rightarrow v_2 = -1/10$$

$\alpha = 7/10 \qquad v_1 = 3/10$

$\underline{u} = \frac{7}{10}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \underline{v} = \frac{1}{10}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Q6 $\|\underline{u}-\underline{v}\| \geq \|u\| - \|v\|$

start from triangle inequality

$$\|a+b\| \leq \|a\| + \|b\|$$

$$\Leftrightarrow \quad \|a\| \geq \|a+b\| - \|b\|$$

comparing, we can set

$$\left.\begin{array}{l} \underline{a} = \underline{u} - \underline{v} \\ b = \underline{v} \end{array}\right] \Leftrightarrow a+b = \underline{u}$$

or $\|u\| = \|u-v+v\| \leq \|u-v\| + \|v\|$

$$\Rightarrow \quad \|u-v\| \geq \|u\| - \|v\|$$

Other points to discuss:

how is $tr(A^TA) = \|A\|_F^2$

let us check on example $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^T A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}$$

$$\|A\|_F^2 = \left\| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\|_F^2 = a^2+b^2+c^2+d^2$$

$$= \operatorname{trace}(A^T A)$$