

Chapter 8: Norms

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Outline

Introduction

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Introduction

- Euclidean Norm or 2-norm

$$||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

- Measure of “size,” “length,” or “magnitude”

- Example: $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ then,

$$||\mathbf{v}|| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

- Distance from origin to the coordinates
represented by \mathbf{v}

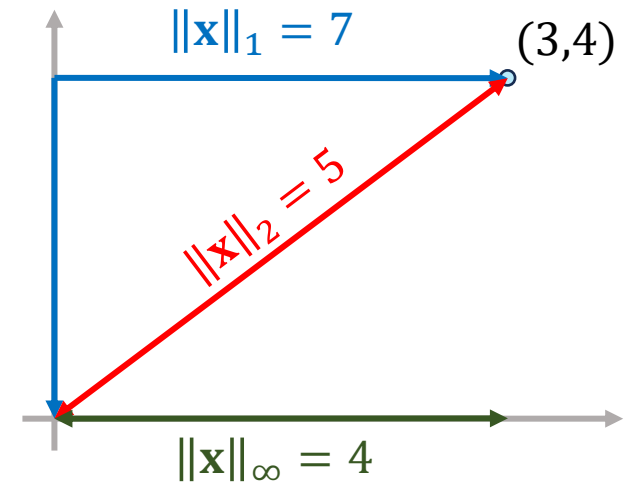
Other Norms: 1- and ∞ -norms

- 1-norm, Manhattan, or taxicab norm

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

- ∞ -norm, Chebyshev, or maximum norm

$$\|\mathbf{x}\|_\infty = \max(|x_1|, |x_2|, \dots, |x_n|)$$



Matrix Norm

- Frobenius Norm

$$\|\mathbf{A}\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |A_{ij}|^2}$$

- Recall: reshaping matrix as vector using `vec()`

$$\|\mathbf{A}\|_F = \|\text{vec}(\mathbf{A})\|_2$$

Frobenius norm-squared

$$\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = \begin{bmatrix} \mathbf{a}_{1\cdot} \\ \mathbf{a}_{2\cdot} \\ \vdots \\ \mathbf{a}_{m\cdot} \end{bmatrix}$$

$$\|\mathbf{A}\|_F^2 = \|\mathbf{a}_1\|^2 + \|\mathbf{a}_2\|^2 + \dots + \|\mathbf{a}_n\|^2$$

$$= \|\mathbf{a}_{1\cdot}^T\|^2 + \|\mathbf{a}_{2\cdot}^T\|^2 + \dots + \|\mathbf{a}_{m\cdot}^T\|^2$$

Norms over inner-product spaces

- Norms may or may not be defined using inner products
- When norm is defined using an inner product

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

- Examples: Euclidean norm, Frobenius norm

$$\|\mathbf{v}\|_2 = \sqrt{\mathbf{v}^T \mathbf{v}} \qquad \|\mathbf{A}\|_F = \sqrt{\text{tr}(\mathbf{A}^T \mathbf{A})}$$

- But Manhattan and Chebyshev norms cannot be defined like this

Properties of Norm

Norm is any real-valued function that satisfies:

1. Non-negativity: $\|\mathbf{x}\| \geq 0$
2. Homogeneity: for any scalar c : $\|c\mathbf{x}\| = |c|\|\mathbf{x}\|$
3. Definiteness: $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$
4. Triangle inequality: $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

Aside: “if and only if” or “iff”

Notation: $A \Leftrightarrow B$

Meaning: EITHER both statements (A and B) are true OR both statements are false

Example: $x = 1$ if and only if $x + 1 = 2$

To prove: Two key steps:

1. Assume A , prove B
2. Assume B , prove A

Proof for Euclidean Norm $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^T \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

1. Non-negativity: since $x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$, its square root is also non-negative

2. Homogeneity: $\|\alpha \mathbf{v}\| = \sqrt{(\alpha \mathbf{v})^T (\alpha \mathbf{v})} = \sqrt{\alpha^2 \mathbf{v}^T \mathbf{v}}$
 $= |\alpha| \sqrt{\mathbf{v}^T \mathbf{v}} = |\alpha| \|\mathbf{v}\|$

3a. If $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$,
then each term must be 0
($\|\mathbf{x}\|_2 = 0 \Rightarrow \mathbf{x} = \mathbf{0}$)

3b. If $\mathbf{x} = \mathbf{0}$,
then $\|\mathbf{x}\|_2 = 0$: $\mathbf{x} = \mathbf{0} \Rightarrow \|\mathbf{x}\|_2 = 0$

$$\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$$

Norm of sum

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{\langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle}$$

Definition of norm

$$= \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle}$$

Distributivity

$$= \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle}$$

Commutativity

$$= \sqrt{\|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2}$$

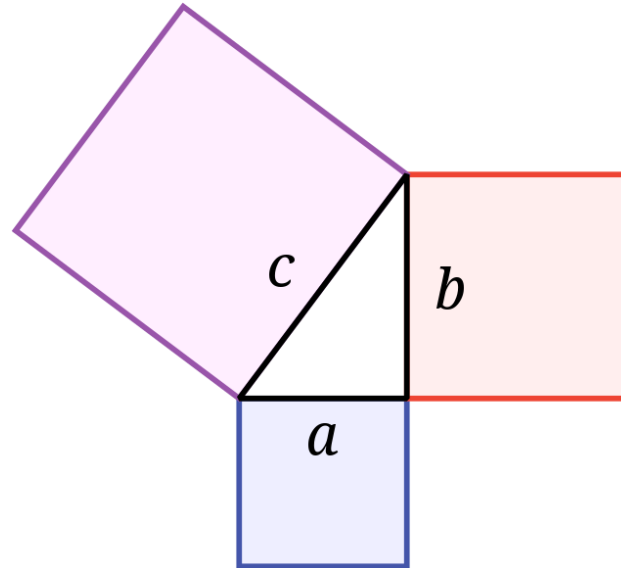
Definition of norm

Pythagorean Theorem

- If \mathbf{u} and \mathbf{v} are orthogonal, then

$$\|\mathbf{u}\|_2^2 + \|\mathbf{v}\|_2^2 = \|\mathbf{u} + \mathbf{v}\|_2^2$$

since $\mathbf{u}^T \mathbf{v} = 0$



Orthogonal Decomposition

- Write \mathbf{u} as a sum of
 - scalar multiple of $\mathbf{v} \neq \mathbf{0}$, and
 - Another vector \mathbf{w} such that $\mathbf{w} \perp \mathbf{v}$

- Let

$$\mathbf{u} = \alpha \mathbf{v} + \underbrace{(\mathbf{u} - \alpha \mathbf{v})}_{\mathbf{w}}$$

- Since $\mathbf{w} \perp \mathbf{v}$: $\mathbf{v}^T \mathbf{w} = \mathbf{v}^T (\mathbf{u} - \alpha \mathbf{v}) = 0$

- $\Rightarrow \alpha = \frac{\mathbf{u}^T \mathbf{v}}{\mathbf{v}^T \mathbf{v}} = \frac{\mathbf{u}^T \mathbf{v}}{\|\mathbf{v}\|^2}$

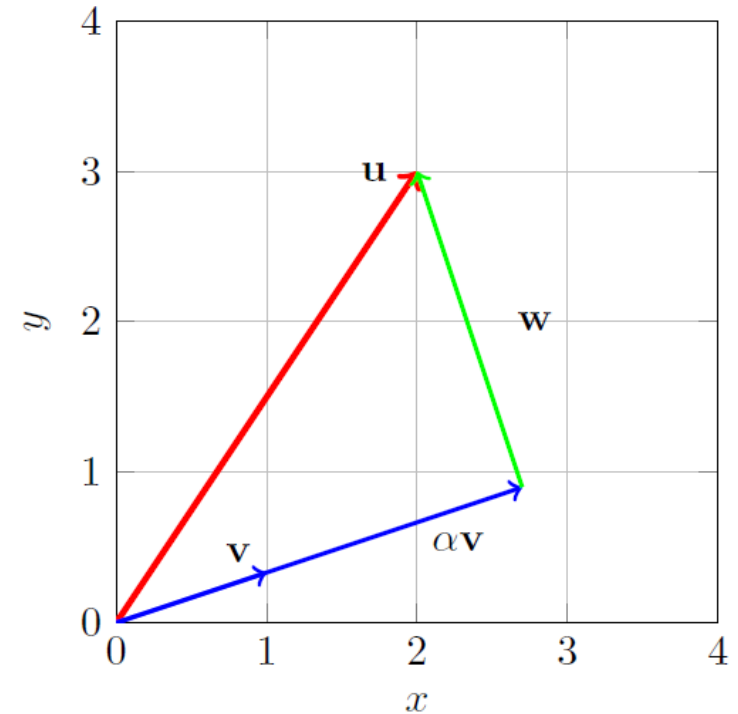
$$\mathbf{u} = \frac{\mathbf{u}^T \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} + \left(\mathbf{u} - \frac{\mathbf{u}^T \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} \right)$$

Example: Orthogonal Decomposition

$$\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1/3 \end{bmatrix}$$

$$\alpha = \frac{\mathbf{u}^T \mathbf{v}}{\|\mathbf{v}\|^2} = 2.7$$



$$\mathbf{u} = 2.7\mathbf{v} + (\mathbf{u} - 2.7\mathbf{v}) = \begin{bmatrix} 2.7 \\ 0.9 \end{bmatrix} + \begin{bmatrix} -0.7 \\ 2.1 \end{bmatrix}$$

Cauchy-Schwarz Inequality

$$|\mathbf{u}^T \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$
$$-\|\mathbf{u}\| \|\mathbf{v}\| \leq \mathbf{u}^T \mathbf{v} \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

- Proof: suppose $\mathbf{v} \neq \mathbf{0}$ (otherwise inequality is trivially true)
- Apply orthogonal decomposition: $\mathbf{u} = \frac{\mathbf{u}^T \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} + \mathbf{w}$
- Since \mathbf{u} and \mathbf{v} are orthogonal, from Pythagorean:

$$\begin{aligned}\|\mathbf{u}\|^2 &= \left\| \frac{\mathbf{u}^T \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} \right\|^2 + \|\mathbf{w}\|^2 \\ &= \frac{|\mathbf{u}^T \mathbf{v}|^2}{\|\mathbf{v}\|^2} + \|\mathbf{w}\|^2 \\ &\geq \frac{|\mathbf{u}^T \mathbf{v}|^2}{\|\mathbf{v}\|^2} \geq 0\end{aligned}$$

Becomes equality when $\mathbf{w} = \mathbf{0}$

$$\mathbf{w} = \mathbf{0} \Leftrightarrow \mathbf{u} = \frac{\mathbf{u}^T \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}$$

Triangle Inequality

- Recall $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}^T \mathbf{v}$ Norm of Sum Expression
 $\leq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\|$ Cauchy-Schwarz Inequality
 $= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$ Collect into square

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

- Becomes equality when $\mathbf{u}^T \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|$
- i.e., when \mathbf{u} is scalar multiple of \mathbf{v}

Angle Between Vectors

- We define

$$\cos \theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

- Recall Cauchy-Schwarz inequality $-\|\mathbf{u}\| \|\mathbf{v}\| \leq \mathbf{u}^T \mathbf{v} \leq \|\mathbf{u}\| \|\mathbf{v}\|$
- $\theta = 0$ parallel vectors
- $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ orthogonal vectors

Norm of stacked vectors

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_m \end{bmatrix}$$

$$\|\mathbf{x}\| = \sqrt{\|\mathbf{x}_1\|^2 + \|\mathbf{x}_2\|^2 + \cdots + \|\mathbf{x}_m\|^2}$$

The diagram illustrates the relationship between the norm of a stacked vector and the norm of a vector of its individual norms. On the left, a vector $\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$ is shown, with blue arrows pointing to it from the labels \mathbb{R}^{m_1} and \mathbb{R}^{m_2} . This vector is enclosed in double vertical bars $\|\cdot\|$. This is followed by an equals sign and another expression where the same vector is enclosed in double vertical bars, but each component \mathbf{x}_1 and \mathbf{x}_2 is itself enclosed in double vertical bars, representing their individual norms. A blue arrow points from the text "vector with 2 entries" to the inner vector $\begin{bmatrix} \|\mathbf{x}_1\| \\ \|\mathbf{x}_2\| \end{bmatrix}$.

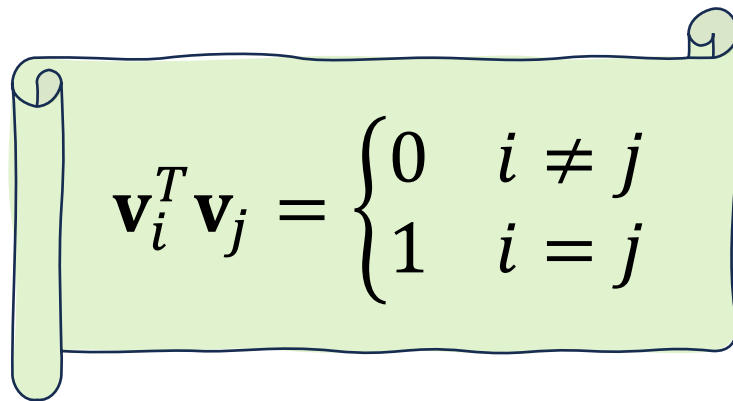
$$\begin{array}{l} \mathbb{R}^{m_1} \\ \mathbb{R}^{m_2} \end{array} \left\| \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \|\mathbf{x}_1\| \\ \|\mathbf{x}_2\| \end{bmatrix} \right\|$$

↑
vector with 2 entries

Orthonormal Vectors

$$\begin{aligned}\mathbf{v}_i^T \mathbf{v}_j &= 0 & i \neq j \\ \|\mathbf{v}_i\| &= 1 & 1 \leq i \leq m\end{aligned}$$

or


$$\mathbf{v}_i^T \mathbf{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Complexity of calculating norm

- To calculate $\|\mathbf{x}\| = \sqrt{\mathbf{x}_1^2 + \mathbf{x}_2^2 + \cdots + \mathbf{x}_n^2}$
 - $n - 1$ additions
 - n multiplications
 - 1 square root (≈ 6 flops)
- Total $2n$ flops
- Matrix norm requires $2n^2$ flops

Thank You

Next: Linear Functions