Math Bootcamp AMPBA

Lecture 2

Oct 2020

Abhishek Rishabh

Linear Independence and Rank of a Matrix

Linear Independence

- Two vectors x_1, x_2 are linearly independent, if $ax_1 + bx_2 = 0$
  - a, b are real numbers
  - Vectors x_1, x_2, $x_3 \ldots x_n$ are linearly independent if $\sum a_i x_i = 0$ implies $a_i = 0$
  - There are no non-trivial linear combinations which equals to 0

Linear Independence: Example

- Are the vectors $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ linearly independent?

- Independence if $a_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} -4 \\ 6 \\ 5 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

- If it has **only a trivial solution a1 = a2 = a3 = 0**

Rank of a matrix

- For a (m x n) matrix X,
  - (x_1, x_2, x_3 … x_n) – column vectors of length 'm'
  - Similarly, it has 'm' row vectors with length 'n'
- For a (m x n) matrix X,
  - Rank of a matrix can be defined as maximum number of linearly independent rows (or) maximum number of linearly independent columns, **whichever is lower**
- Rank of the matrix X if m < n is **$\rho(X) \leq m$**

Example 1: Rank of a matrix

- Find the Rank of the following matrix:

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Example 1: Rank of a matrix (1/2)

$X = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Rank of this matrix (2 x 3), $\rho(X) \leq 2$

$\Rightarrow a_1 \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} + a_2 \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} = 0$ (Check for row vectors' independence)

$\Rightarrow a_1 + 2a_2 = 0$ - eq(1)

$\Rightarrow 3a_1 + 4a_2 = 0$ - eq(2)

$\Rightarrow 5a_1 + 6a_2 = 0$ - eq(3)

Example 1: Rank of a matrix (2/2)

$\Rightarrow$ eq(2) – 3 eq(1) gives $a_2 = 0$

$\Rightarrow$ Substituting a2 = 0 in eq(1) gives $a_1 = 0$

$\Rightarrow$ Hence, the row vectors are linearly independent.

$\Rightarrow$ $\rho(X) = 2$

Example 2: Rank of a matrix

- Find the Rank of the following matrix:

$$\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$$

Example 2: Rank of a matrix (1/2)

X = $\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$

Rank of this matrix (2 x 2), $\rho(X) \leq 2$

$\Rightarrow a_1 [4 \quad 6] + a_2 [6 \quad 9] = 0$ (Check for row vectors' independence)

$\Rightarrow 4a_1 + 6a_2 = 0$ - eq(1)

$\Rightarrow 6a_1 + 9a_2 = 0$ - eq(2)

Example 2: Rank of a matrix (2/2)

$\Rightarrow$ Solving eq(1), eq(2) gives

$\Rightarrow$ a_1 = -3, a_2 = 2 (Not a trivial solution)

$\Rightarrow$ Hence, the vectors are **linearly dependent**

$\Rightarrow$ $\rho(X) = 1$

Example 3: Rank of a matrix

- Find the Rank of the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 5 & 6 & -1 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$

Example 3: Rank of a matrix (1/2)

$$X = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 5 & 6 & -1 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$

Rank of this matrix (3 x 4), ρ(X) ≤ 3

$\Rightarrow$ But, $[1 \quad 2 \quad 3 \quad 2] + [4 \quad 5 \quad 6 \quad -1] = [5 \quad 7 \quad 9 \quad 1]$

$\Rightarrow$ These rows are linearly dependant

Example 3: Rank of a matrix (1/2)

$$X = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 5 & 6 & -1 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$

$\Rightarrow$ Also, R1 and R2 are independent, i.e. cannot be expressed as a real multiple of one another

$\Rightarrow$ Hence, ρ(X) = 2

Can you find the Rank of X^T (Transpose of X)?

Example 4: Rank of a matrix

- Find the Rank of the following matrix:

$$\begin{bmatrix} 1 & 5 & 6 \\ 2 & 6 & 8 \\ 7 & 1 & 8 \end{bmatrix}$$

Example 4: Rank of a matrix (1/2)

$$X = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 6 & 8 \\ 7 & 1 & 8 \end{bmatrix}$$

Rank of this matrix (3 x 3), $\rho(X) \leq 3$ &

$\rho(X) \geq 2$ (Column1 & Column2 are independent)

$\Rightarrow a_1 \begin{bmatrix} 1 & 5 & 6 \end{bmatrix} + a_2 \begin{bmatrix} 2 & 6 & 8 \end{bmatrix} + a_3 \begin{bmatrix} 7 & 1 & 8 \end{bmatrix} = 0$

$\Rightarrow a_1 + 5a_2 + 6a_3 = 0$ - eq(1)

$\Rightarrow 2a_1 + 6a_2 + 8a_3 = 0$ - eq(2)

$\Rightarrow 7a_1 + 1a_2 + 8a_3 = 0$ - eq(3)

Example 4: Rank of a matrix (1/2)

$\Rightarrow$ Using $a_1 = -5a_2 - 6a_3$ in eq(2) and eq(3)

$\Rightarrow$ $a_2 + a_3 = 0$

$\Rightarrow$ (a1, a2, a3) = (1, 1, -1) (a non-trivial solution)

$\Rightarrow$ Hence, $\rho(X) = 2$

Full Rank

- A (m x n) Matrix X is said to be of **full rank** if
  - **ρ(X) = min(m, n)**

$X = \begin{bmatrix} 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix}$ is a 2x3 matrix and ρ(X) = 2

- X is a matrix of Full Rank

Rank of a Diagonal Matrix

- Rank of a diagonal Matrix D
  - Is equal to Number of **non-zero diagonal elements** in D. How?

$$X = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}; Y = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (Special\ case\ of\ Diagonal\ matrix\ - Identity\ Matrix)$$

Find Ranks of X, Y and I?

Rank of a Square Matrix

- In the case of **square n x n** Matrix Y is said to be full rank if
  - det(Y) ≠ 0, i.e. **ρ(Y) = n**
  - Why?

N = $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$; M = $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$

Rank of a Square Matrix

- N = $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$; M = $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$
- Det(N) = ad – bc = 2*3 – 1*6 = 0
- Det(M) = ad – bc = 2*5 – 1*3 = 7 ≠ 0
- M is a matrix of Full Rank 2

Rank Inequalities: Sum

- $\rho(X + Y) \leq \rho(X) + \rho(Y)$
  - Consider $\rho(X) = r$, $\rho(Y) = s$
  - X can be expressed as $\sum^r a_i x_i = 0$, x_i – independent columns of X
  - Y can be expressed as $\sum^s b_i y_i = 0$, y_i – independent columns of Y
  - $(X + Y)$ can be written as linear combination of $(r + s)$ vectors x_i & y_i
  - Hence, $\rho(X + Y) \leq \rho(X) + \rho(Y)$

Rank Inequalities : Difference

- Similarly, for $\rho(X - Y)$
- $\rho(X + (-Y))$
- But $\rho(Y) = \rho(-Y)$

$\Rightarrow \rho(X + (-Y)) \leq \rho(X) + \rho(-Y)$

$\Rightarrow \boldsymbol{\rho(X + (-Y)) \leq \rho(X) + \rho(Y)}$

Rank Inequalities : Example(1/2)

- N = $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$; M = $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$
- ρ(N) = 1, ρ(M) = 2
- M + N = $\begin{bmatrix} 4 & 9 \\ 2 & 8 \end{bmatrix}$

Rank Inequalities : Example(2/2)

- M + N = $\begin{bmatrix} 4 & 9 \\ 2 & 8 \end{bmatrix}$
- ρ(M+N) = 2, as Det(M+N) = 4*8 – 2*9 = 14 ≠ 0
- ρ(M) + ρ(N) = 2 + 1 ≤ 2 i.e. ρ(M + N)

Rank Inequalities: Rank Products

- $\rho(XY) \leq \min(\rho(X), \rho(Y))$ and if $\rho(Y) = r$
- Y can be expressed as $\sum^r b_i y_i = 0$, y_i – independent columns of Y
- Let $Z = XY$
- Column $Z_j = Xy_i$

Rank Inequalities: Rank Products(..cont)

- This implies, Z_j can be written as linear combination of $z_1, z_2 \ldots z_r$
- $\rho(Z) \leq r = \rho(Y)$
- Similarly, it can be proved that $\rho(X) \leq \rho(Z)$
- Hence, $\rho(XY = Z) \leq \min(\rho(X), \rho(Y))$

Rank Inequalities : Example(1/2)

- N = $\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$; M = $\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$
- ρ(N) = 2, ρ(M) = 1
- N * M = $\begin{bmatrix} 19 & 19 \\ 18 & 18 \end{bmatrix}$

Rank Inequalities : Example(2/2)

- Rank of N * M = $\begin{bmatrix} 19 & 19 \\ 18 & 18 \end{bmatrix}$ is 1
- $\rho(NM) \leq \min(\rho(N), \rho(M))$
- $1 \leq \min(2, 1) = 1$

Rank Inequalities

Product with orthogonal matrix:

- If C is an orthogonal matrix then $\rho(AC) = \rho(A)$ because
  - $\rho(A) = \rho(ACC^T) \leq \rho(AC) \leq \rho(A)$

Submatrices:

- If A_{ij} is a submatrix of A, then $\rho(A_{ij}) \leq \rho(A)$

Rank in Statistics

- Rank can be used as a crucial tool in Statistics in the form of analysis
  - In Simple linear Models, $y = X\beta + \varepsilon$
  - In Multivariate analysis, as few statistical techniques depend on the X having a full Rank
  - More details will be on these will be covered as we move along in the course

R Exercise

Summary

Determinant

Determinant of a Matrix

- For every (n x n) Matrix $A = (a_{ij})$, det(A) can be calculated using a_{ij}
- It is denoted as det(A) or $|A|$

Determinant of a matrix – Why?

- To calculate the inverse of a square matrix
- To calculate the eigenanalyses of a matrix
- In multivariate analysis for transformations

Determinant of a matrix – How?

- In a (2 x 2) matrix, for A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

- det(A) = $a_{11}*a_{22} - a_{12}*a_{21}$

Determinant of a matrix – Example

- Find the determinant of the following matrices

$$A = \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}$$

Determinant of a matrix – Example

$$A = \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$$

- det(A) = 2*(-3) – 1*4 = -10

$$B = \begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}$$

- det(B) = 6*6 – 4*9 = 0

Determinant of a matrix – How?

- In a (3 x 3) matrix, for A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

- det(A) = $a_{11} * \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} * \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} * \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Determinant of a matrix – Example

- Find the determinant of the following matrices

$$A = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 3 & -3 \\ 4 & 6 & 7 \end{bmatrix}$$

Determinant of a matrix – Example

$$A = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 3 & -3 \\ 4 & 6 & 7 \end{bmatrix}$$

$$\det(A) = 2^* \begin{vmatrix} 3 & -3 \\ 6 & 7 \end{vmatrix} - 2* \begin{vmatrix} 1 & -3 \\ 4 & 7 \end{vmatrix} + 4^* \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

= 2*(3*7 –(-3)*6) – 2*(1*7 – (-3)*4) + 4*(1*6 – 3*4)

= 2*(39) – 2*(19) + 4*(-6)

= 78 – 38 -24

=16

Co-factors in matrix

- Cofactor is a number that you get by eliminating the row and column of a element and finding the determinant of the rest.
- Cofactor is always preceded by a ‘+’ or ‘-’ depending on the position of the element

Co-factors in matrix

- In a (m x n) matrix A, the co-factor of the element a_{ij}

$$\mathbf{c_{ij} = (-1)^{i+j} * det(M_{ij})}$$

- M_{ij} is the(m – 1 x n – 1) matrix formed from the rest of the elements after removing the Row ‘i’, Column ‘j’
- M_{ij} is called **Minor** of the element ‘a_{ij}’

Co-factor in matrix - Example

$$A = \begin{bmatrix} 2 & 5 & -1 \\ 0 & 3 & 4 \\ 1 & -2 & -5 \end{bmatrix}$$

Cofactor of $a_{32} = c_{32} = (-1)^{3+2} * \det(M_{11}) = (-1)^5 \begin{vmatrix} 2 & -1 \\ 0 & 4 \end{vmatrix}$

$= -1*(8 - 0)$

$= -8$

Adjoint or Adjugate matrix

- Adjoint matrix A is a square matrix formed by the **transpose** of a co-factor matrix of A
- It is denoted as **Adj A**

Adjoint matrix - Example

Find the adjoint of the matrix A.

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

Adjoint matrix - Example

Calculating co-factors of A

Co-factor of a11 = $(-1)^2\begin{vmatrix} -2 & 0 \\ 2 & -1 \end{vmatrix} = 2$

Cofactor of a12 = $(-1)^3\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} = 2$

Cofactor of a13 = $(-1)^4\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} = 6$

Adjoint matrix - Example

Calculating co-factors of A

Co-factor of a21 = $(-1)^3 \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1$

Cofactor of a22 = $(-1)^4 \begin{vmatrix} 3 & -1 \\ 1 & -1 \end{vmatrix}$ = -2

Cofactor of a23 = $(-1)^5 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$ = -5

Adjoint matrix - Example

Calculating co-factors of A

Co-factor of a31 = $(-1)^4\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -2$

Cofactor of a32 = $(-1)^5\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}$ = -2

Cofactor of a33 = $(-1)^6\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}$ = -8

Adjoint matrix - Example

Cofactor matrix of A, $C_{ij} = \begin{bmatrix} 2 & 2 & 6 \\ -1 & -2 & -5 \\ -2 & -2 & -8 \end{bmatrix}$

Adjoint of the matrix A, Adj A = $(C_{ij})^T = \begin{bmatrix} 2 & -1 & -2 \\ 2 & -2 & -2 \\ 6 & -5 & -8 \end{bmatrix}$

Determinant of a (n x n) matrix

- Co-factors can be used to find the determinants of a (n x n) matrix
- In a (n x n) matrix, for A, for $1 \leq i \leq n$
- $\det(A) = \sum_j^n (a_{ij} * c_{ij}) = a_{i1} * c_{i1} + a_{i2} * c_{i2} + \dots + a_{in} * c_{in}$

Properties of Determinents

- In a 2 x 2 matrix, A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$
  - If (a11, a21) and (a12, a22) are co-ordinates of point in a place,

  det(A) is the area of the parallelogram by (a11, a21)’ and (a12, a22)’

- Similarly, what will det of the 3 x 3 matrix will represent?

Properties of Determinants

- In a 2 x 2 matrix, A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$
  - If (a11, a21) and (a12, a22) are co-ordinates of point in a place,

  det(A) is the Area of the parallelogram by (a11, a21)’ and (a12, a22)’
- Similarly, what will det of the 3 x 3 matrix will represent?
  - Volume of the parallelopiped formed by the ‘3’ columns of A

Properties of Determinants: Row & Column Operations

- If we multiply a row or a column with a scalar 'λ', the determinant is also multiplied by λ

A = $\begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$, det(A) = -6 –(4) = -10

Properties of Determinants: Row & Column Operations

A = $\begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$, det(A) = -6 –(4) = -10

C1 is multiplied by scalar ‘2’ to form matrix B

B = $\begin{bmatrix} 4 & 4 \\ 2 & -3 \end{bmatrix}$, det(B) = -12 – 8 = -20 = **2*(-10)**

Properties of Determinants

- $\det(A) = \det(A^T)$

$A = \begin{bmatrix} 5 & 2 \\ 3 & -4 \end{bmatrix}$, det(A) = -20 –(6) = -26

$A^T = \begin{bmatrix} 5 & 3 \\ 2 & -4 \end{bmatrix}$, det($A^T$) = -20 –(6) = -26

Properties of Determinants

- If we multiply a n x n matrix A with scalar ‘λ’, the determinant of the new matrix is λ^n *det(A)

$A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$, det(A) = 12 – 10 = 2

$3*A = \begin{bmatrix} 9 & 15 \\ 6 & 12 \end{bmatrix}$, det($A^T$) = 108 – 90 = 18 = 3^2*2 = 3^2*det(A)

Properties of Determinants

- If a complete row or column has zero as elements in matrix A, det(A) = 0

A = $\begin{bmatrix} 0 & 5 \\ 0 & 4 \end{bmatrix}$, det(A) = 0 – 0 = 0

Properties of Determinants

- If A has two identical rows or columns, then det(A) = 0

A = $\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$, det(A) = 6 – 6 = 0

Properties of Determinants

- If A is n x n matrix and $\rho(A) < n$, then $\det(A) = 0$
- Why?

Properties of Determinants

- If A is n x n matrix and $\rho(A) < n$, then $\det(A) = 0$
- Why?
- A column or row can be written as a linear combination of another column or row

Properties of Determinants

- A column or row can be written as a linear combination of another column or row

A = $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$, det(A) = 12 – 12 = 0

Properties of Determinants

A = $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$

Rank of this matrix (2 x 2), $\rho(X) \leq 2$

$\Rightarrow a_1 \begin{bmatrix} 2 & 4 \end{bmatrix} + a_2 \begin{bmatrix} 3 & 6 \end{bmatrix} = 0$ (Check for row vectors' independence)

$\Rightarrow 2a_1 + 3a_2 = 0$ - eq(1)

$\Rightarrow 4a_1 + 6a_2 = 0$ - eq(2)

Properties of Determinants

$\Rightarrow$ Solving eq(1), eq(2) gives

$\Rightarrow a_1 = -3, a_2 = 2$ (Not a trivial solution)

$\Rightarrow$ Hence, the vectors are **linearly dependent**

$\Rightarrow \rho(X) = 1$

Properties of Determinants

- Determinant of the diagonal matrix D with d_1, d_2, $d_3 \ldots d_n$
- $\det(D) = d_1 * d_2 * d_3 \ldots * d_n$

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

$\det(D) = d_1 * d_2 * d_3$

Properties of Determinants

- Determinant of the diagonal matrix D with $d_1, d_2, d_3 \ldots d_n$
- $\det(D) = d_1 * d_2 * d_3 \ldots * d_n$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\det(A) = 2 * \begin{vmatrix} 3 & 0 \\ 0 & 7 \end{vmatrix} - 0 * \begin{vmatrix} 0 & 0 \\ 0 & 7 \end{vmatrix} + 0 * \begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix}$$

$$= 2*(3*7 - 0*0) - 0*(0*7 - 0*0) + 0*(0*0 - 3*0)$$

$$= 2*3*7 = 42$$

Properties of Determinants

- Determinant of the triangular matrix T(upper or lower) with t_1, t_2, $t_3 \ldots t_n$ as the diagonal elements
- det(T) = $t_1 * t_2 * t_3 \ldots * t_n$
- T = $\begin{bmatrix} t_1 & a & b \\ 0 & t_2 & c \\ 0 & 0 & t_3 \end{bmatrix}$

det(T) = $t_1 * t_2 * t_3$

Properties of Determinants

- Determinant of the triangular matrix T with t_1, t_2, $t_3 \ldots t_n$ are the diagonal elements

$$T = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\det(A) = 2^* \begin{vmatrix} 3 & 5 \\ 0 & 7 \end{vmatrix} - 3 * \begin{vmatrix} 0 & 5 \\ 0 & 7 \end{vmatrix} + 4^* \begin{vmatrix} 0 & 3 \\ 0 & 0 \end{vmatrix}$$

$$= 2*(3*7 - 5*0) - 3*(0*7 - 0*5) + 4*(0*0 - 3*0)$$

$$= 2*3*7 = 42$$

Orthogonal Matrix

- A n x n matrix A is said to be orthogonal if
  - $AA^T = I_n$
- Since we already know that $\det(A) = \det(A^T)$ and $\det(I_n) = 1$
- $\det(A) = +1$ or -1

Orthogonal Matrix

- $\det(A) = +1$ or -1
- If $\det(A) = 1$, it is called a Rotation matrix
- If $\det(A) = -1$, it is called a Reflection matrix

Block Matrix or Partitioned Matrix

- A block matrix is a matrix that is interpreted as having been broken into sections – Blocks or Submatrices
- It can be visualised as a matrix broken into a collection of Smaller Matrices

Block Matrix - Example

$$P = \begin{bmatrix} 2 & 2 & 3 & 4 \\ 1 & 3 & 0 & 5 \\ 8 & 1 & 3 & 4 \\ 4 & 6 & 5 & 7 \end{bmatrix}$$

- It can be broken into four (2 x 2) matrices

Block Matrix - Example

$$P_{11} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \qquad P_{12} = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$$

$$P_{21} = \begin{bmatrix} 8 & 1 \\ 4 & 6 \end{bmatrix} \qquad P_{22} = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

R Exercise

Summary