Solved Problems

1. Two fair dice are rolled. Let X equal the product of the 2 dice. Compute P{X = i} for i = 1,...,36.

Solution:

p(1) = 1/36 p(2) = 2/36 p(3) = 2/36 p(4) = 3/36
p(5) = 2/36 p(6) = 4/36 p(7) = 0 p(8) = 2/36
p(9) = 1/36 p(10) = 2/36 p(11) = 0 p(12) = 4/36
p(15) = 2/36 p(16) = 1/36 p(18) = 2/36 p(20) = 2/36
p(24) = 2/36 p(25) = 1/36 p(30) = 2/36 p(36) = 1/36

2. Two balls are chosen randomly from an urn containing 8 white, 4 black, and 2 orange balls. Sup- pose that we win \$2 for each black ball selected and we lose \$1 for each white ball selected. Let X denote our winnings. What are the possible values of X, and what are the probabilities associated with each value?

Solution:

First construct the sample space of all possible draws. We must draw 2 balls at random from this url:

$$S = \{(W,W),(W,B),(W,O),(B,W),(B,O),(B,B),(O,W),(O,B),(O,O)\}$$

Also, we need to assign values of X for each $s \in S$.

s	X
(W,W)	-2
(W,B)	$+1$
(W,O)	-1
(B,W)	$+1$
(B,O)	$+2$
(B,B)	$+4$
(O,W)	-1
(O,B)	$+2$
(O,O)	0

Now we must compute the probabilities for each value of X.

$X = +1$ when we draw a white ball and black ball, or when we draw a black ball and a white ball. We assume sampling without replacement:

$$P(X=1) = \frac{8}{14}\cdot\frac{4}{13} + \frac{4}{14}\cdot\frac{8}{13} = \boxed{0.3516}$$

$X = -1$ when we draw a white ball and orange ball, or an orange ball and white ball.

$$P(X=-1) = \frac{8}{14}\cdot\frac{2}{13} + \frac{2}{14}\cdot\frac{8}{13} = \boxed{0.176}$$

$X = +2$ when we draw a black ball followed by an orange ball, or an orange ball followed by a black ball.

$$P(X=+2) = \frac{4}{14}\cdot\frac{2}{13} + \frac{2}{14}\cdot\frac{4}{13} = \boxed{0.088}$$

$X = -2$ when we draw two white balls.

$$P(X=-2) = \frac{8}{14}\cdot\frac{7}{13} = \boxed{0.308}$$

$X = +4$ when two black balls are drawn.

$$P(X = +4) = \frac{4}{14} \cdot \frac{3}{13} = \boxed{0.066}$$

$X = 0$ when two orange balls are drawn.

$$P(X = 0) = \frac{2}{14} \cdot \frac{1}{13} = \boxed{0.011}$$

Then the probability distribution is given by

x	P(X = x)
-2	0.308
-1	0.176
0	0.011
$+1$	0.3516
$+2$	0.088
$+4$	0.066

3. A gambling book recommends the following "winning strategy" for the game of roulette: Bet \$1 on red. If red appears (which has probability 18/38), then 38 take the \$1 profit and quit. If red does not appear and you lose this bet (which has probability 20/38 of occurring), make additional \$1 bets on red on each of the next two spins of the roulette wheel and then quit. Let X denote your winnings when you quit.

a) Find $P\{X > 0\}$

b) Are you convinced that the strategy is indeed a "winning" strategy?

c) Find E[X]

Solution:

a) $P\{X > 0\} = P\{\text{win first bet}\} + P\{\text{lose, win, win}\} = 18/38 + (20/38)(18/38)^2 \approx .5918$

b) No, because if the gambler wins then he or she wins \$1. However, a loss would either be \$1 or \$3

c) $E[X] = 1[18/38 + (20/38)(18/38)^2] - [(20/38)2(20/38)(18/38)] - 3(20/38)^3 \approx -.108$

4. You have \$1000, and a certain commodity presently sells for \$2 per ounce. Suppose that after one week the commodity will sell for either \$1 or \$4 an ounce, with these two possibilities being equally likely. If your objective is to maximize the expected amount of money that you possess at the end of the week, what strategy should you employ?

Solution:

Use all your money to buy 500 ounces of the commodity and then sell after one week. The expected amount of money you will get is E[money] = (1/2)*500 + (1/2)*2000 = 1250

5. Two coins are to be flipped. The first coin will land on heads with probability .6, the second with probability .7. Assume that the results of the flips are independent, and let X equal the total number of heads that result.

a) Find $P\{X = 1\}$

b) Determine E[X]

Solution:

a)

$$\begin{aligned} P\{X = 1\} &= P(\text{1st heads and 2nd tails } \cup \text{ 1st tails and 2nd heads}) \\ &= P(\text{1st heads and 2nd tails}) + P(\text{1st tails and 2nd heads}) \\ &\qquad \text{since the two events are disjoint} \\ &= P(\text{1st heads})P(\text{2nd tails}) + P(\text{1st tails})P(\text{2nd heads}) \\ &\qquad \text{since the coin flips are independent} \\ &= .6 \cdot .3 + .4 \cdot .7 \\ &= 0.46 \end{aligned}$$

b)

First we calculate

$$\begin{aligned} P\{X=0\} &= P(\text{1st tails and 2nd tails}) \\ &= P(\text{1st tails})P(\text{2nd tails}) \\ &= .4 \cdot .3 \\ &= 0.12 \end{aligned}$$

and

$$\begin{aligned} P\{X=2\} &= P(\text{1st heads and 2nd heads}) \\ &= P(\text{1st heads})P(\text{2nd heads}) \\ &= .6 \cdot .7 \\ &= 0.42 \end{aligned}$$

Hence

$$\begin{aligned} E[X] &= \sum_x xP\{X=x\} \\ &= 0 \cdot 0.12 + 1 \cdot 0.46 + 2 \cdot 0.42 \\ &= 1.3. \end{aligned}$$

6. A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win \$1.10; if they are different colors, then you win –\$1.00 (that is, you lose \$1.00.) Calculate

a) the expected value of the amount you win

b) the variance of the amount you win

Solution:

If X is the amount that you win, then

$P\{X = 1.10\} = 4/9 = 1 - P\{X = -1\}$

$E[X] = (1.1)4/9 - 5/9 = -.6/9 \approx = -.067$

$\mathrm{Var}(X) = (1.1)^2(4/9) + 5/9 - (.6/9)^2 \approx 1.089$

7. Four buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus.

a) Which of E[X] or E[Y] do you think is larger? Why?

b) Compute E[X] and E[Y]

c) Compute Var(X) and Var(Y)

Solution:

a) In the first sampling method, we draw one of 148 students at random. Naturally, the probability of choosing a student from the fullest bus is higher than the probability of drawing a student from the other buses. Thus, the fuller buses are weighted higher than the other buses, so we would expect E(X) to be larger. In the second method, one of the four bus drivers is randomly selected. Each bus driver is equally likely to be chosen! Each bus is weighted the sum, therefore, $E(Y) \leq E(X)$. Equality holds when each bus contains the same number of students.

Analogy: Consider calculating your GPA two different ways. Suppose your grades this quarter are as follows: A, C, C. The class that you earned an A in is worth 5 units and the classes you earned the Cs in are worth 4 units. Computing your GPA using the standard method of weighting the grades by units would weight the higher grade heavier than each of the two Cs. This calculation is similar to the calculation of E(X). The GPA in this case is 3.00.

Now suppose we ignore the course units and recompute the GPA such that each grade is weighted the same. Then the GPA is 2.67. This calculation is similar to E(Y). In this particular case, we see that the weighted method is higher.

b)

First we need the probability distributions for X and Y.

X corresponds to the method of drawing a student at random. The probability that a student comes from bus $i, i = \{A, B, C, D\}$ is simply the number of students on that bus divided by the total number of students.

$x =$	$P(X = x)$
40	$\frac{40}{148}$
33	$\frac{33}{148}$
25	$\frac{25}{148}$
50	$\frac{50}{148}$

Then, $E(X) = 40 \cdot \frac{40}{148} + 33 \cdot \frac{33}{148} + 25 \cdot \frac{25}{148} + 50 \cdot \frac{50}{148} = \boxed{39.28}$.

Y corresponds to the method of drawing a bus driver at random. The probability that a bus driver comes from a particular bus is constant, $\frac{1}{4}$.

$x =$	$P(X = x)$
40	$\frac{1}{4}$
33	$\frac{1}{4}$
25	$\frac{1}{4}$
50	$\frac{1}{4}$

Then, $E(Y) = 40 \cdot \frac{1}{4} + 33 \cdot \frac{1}{4} + 25 \cdot \frac{1}{4} + 50 \cdot \frac{1}{4} = \boxed{37}$.

c)

$E[X^2] = [(40)^3 + (33)^3 + (25)^3 + (50)^3]/148 \approx 1625.4$
$\mathrm{Var}(X = E[X^2] - (E[X])^2 \approx 82.2$
$E[Y^2] = = [(40)^2 + (33)^2 + (25)^2 + (50)^2]/4 = 1453.5$
$\mathrm{Var}(Y) = 84.5$

8. A ball is drawn from an urn containing 3 white and 3 black balls. After the ball is drawn, it is then replaced and another ball is drawn. This goes on indefinitely. What is the probability that of the first 4 balls drawn, exactly 2 are white?

Solution:

First, note that we have a finite number of trials, $n = 4$ (although the game goes on forever, we are only concerned with the first 4 balls.) Each trial is a Bernoulli trial - that is, each trial has only one of two outcomes: white and not white. Define a success as the event that a white ball is drawn. Then the probability of success p is $p = \frac{1}{2}$. Since each ball is replaced after it is drawn, we have sampling with replacement, and thus independence.

Since we are dealing with a finite number of independent Bernoulli trials with a constant probability of success p, we use the *binomial* distribution.

Let X be the number of white balls (successes) that appear in $n = 4$ trials. Then we want to find $P(X = 2)$.

Recall that

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Then,

$$\begin{aligned} P(X=2) &= \binom{4}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{4-2} \\ &= \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 \\ &= \frac{6}{2^4} \\ &= \frac{3}{8} \\ &= \boxed{0.375} \end{aligned}$$

9. Consider a roulette wheel consisting of 38 numbers 1 through 36, 0, and double 0. If Smith always bets that the outcome will be one of the numbers 1 through 12, what is the probability that Smith will lose his first 5 bets?

Solution:

Since Smith always bets on the numbers 1 through 12, which occupy 12 spaces on the wheel, the probability of success (winning) is p = 12/38.
The first five bets form a finite set of n = 5 trials. Each spin of the roulette wheel is independent, and the probability of success p is constant. We use the binomial model. We could also solve this problem intuitively by using independence.

Let W denote the event that Smith wins his bet. Using independence, the probability that Smith loses his first 5 bets is,

$$P(W'W'W'W'W') = \left(\frac{26}{38}\right)^5 \approx \boxed{0.15}$$

Using the binomial, let X be the number of bets that Smith wins. Then,

$$\begin{aligned} P(X=0) &= \binom{5}{0}\left(\frac{12}{38}\right)^0\left(\frac{26}{38}\right)^5 \\ &= \left(\frac{26}{38}\right)^5 \\ &\approx \boxed{0.15} \end{aligned}$$

10. The monthly worldwide average number of air- plane crashes of commercial airlines is 3.5. What is the probability that there will be
a) at least 2 such accidents in the next month;
b) at most 1 accident in the next month?
Explain your reasoning!

Solution:

First we need to decide which probability model to use. All we know is the mean, which does not tell us much. However, the *context* of the problem implies which model to use. An airline crash is very rare, thus it has a small probability of occurring. We use the **Poisson** distribution for this situation.

For the Poisson distribution, recall that

$$P(X=i) = \frac{e^{-\lambda}\lambda^i}{i!}$$

Since the mean is 3.5, and the mean is λ, we know that $\lambda = 3.5$.

(a) *at least 2 such accidents in the next month?*

$$\begin{aligned} P(X \geq 2) &= 1 - P(X=1) - P(X=0) \\ &= 1 - \frac{e^{-3.5}(3.5)^1}{1!} - \frac{e^{-3.5}(3.5)^0}{0!} \\ &= 1 - 3.5e^{-3.5} - e^{-3.5} \\ &= 1 - 4.5e^{-3.5} \\ &= \boxed{0.8641} \end{aligned}$$

(b) *at most 1 accident in the next month?*

$$\begin{aligned} P(X \leq 1) &= P(X=0) + P(X=1) \\ &= \frac{e^{-3.5}(3.5)^0}{0!} + \frac{e^{-3.5}(3.5)^1}{1!} \\ &= 4.5e^{-3.5} \\ &= \boxed{0.1359} \end{aligned}$$

11. Approximately 80,000 marriages took place in the state of New York last year. Estimate the probability that, for at least one of these couples,
a) both partners were born on April 30
b) both partners celebrated their birthday on the same day of the year.

Solution:

(a) *both partners were born on April 30.*

The probability that both partners share the same birthday can be found by independence:

$$P(\text{Same Birthday, } \textit{April 30}) = \frac{1}{365} \cdot \frac{1}{365} = 7.51 \times 10^{-6}$$

which is a *tiny* probability, so we use the **Poisson** approximation to the binomial.

The number of couples that share the same birthday, April 30, is Poisson with mean $\lambda = np = 80,000 \cdot 7.51 \times 10^{-6} \approx 0.6$.

Then, let X be the number of couples that share April 30 as their birthday.

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - \frac{e^{-0.6}(-0.6)^0}{0!} \\ &= 1 - e^{-0.6} \\ &= \boxed{0.4512} \end{aligned}$$

(b) *both partners celebrated their birthday on the same day of the year.*

This problem is different! It is a generalization of part (a). Now want to find the probability that both partners celebrated their birthday on the same day of the year, but not just April 30. The computation is similar, however, we note that we can choose from one of 365 days of the year for both partners,

$$P(\text{Same Birthday}) = 365 \cdot \frac{1}{365} \cdot \frac{1}{365} = \frac{1}{365}$$

Again, this probability is tiny so we use the **Poisson** with mean $\lambda = np = 80,000 \cdot \frac{1}{365} = 219.2$.

Then, let X be the number of couples that share the same birthday.

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - \frac{e^{-219.2}(219.2)^0}{0!} \\ &= 1 - e^{-219.2} \\ &\approx \boxed{1} \end{aligned}$$

12. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 AM, whereas trains headed to destination B arrive at 15-minute intervals starting at 7:05 AM.

a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 AM and then gets on the first train that arrives, what proportion of time does he or she go to destination A?

b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10 AM?

Solution:

a) For this random person to randomly wander onto the train going to destination A, he/she must arrive right after the train to B leaves, but before the next train to A leaves. This means the person must arrive some time between 7:05-7:15, or between 7:20-7:30 or between 7:35-7:45 or between 7:50-8:00. The person must arrive during one of these 10 minute windows, and there are 4 of these non-overlapping windows. Thus, the probability that this random person will arrive at location A is, P(A) = 40/60 = 2/3

b) All of the 10 minute windows in the previous part still apply, except for the first. To randomly jump onto a train headed to location A, the person must arrive between 7:10-7:15, 7:20-7:30, 7:35-7:45, 7:50-8:00 or between 8:10-8:15. So we have a total of 3 windows of 10 minutes each, and 2 windows of 5 minutes each. There are a total of 40 minutes (still) during which a person could arrive and end up at location A. Therefore, P(A) = 40/60 = 2/3

13. If X is a normal random variable with parameters $\mu = 10$and $\sigma^2 = 36$, compute

a) P{X > 5}

b) P{4<X<16}

c) P{X < 8}
d) P{X < 20}
e) P{X > 16}

Solution:

Whenever μ and σ are given, we know we are working with the normal distribution. The PDF for the normal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The trick to this problem is that we are going to use $\mathbf{\Phi}$ to represent the CDF of the normal distribution so no integration is necessary. That is,

$$P(Z \leq z) = \mathbf{\Phi}(z)$$

The value at which we compute $\mathbf{\Phi}$ is called a *z-score* and,

$$z = \frac{x - \mu}{\sigma}$$

The value $\mathbf{\Phi}(z)$ can be found by using a standard normal distribution z-table, using a calculator with the Normal CDF or by integration.

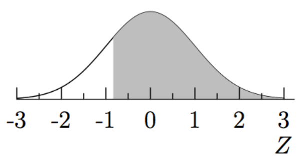
Also, remember that for continuous distributions, $P(Z \leq z) = P(Z < z)$ because $P(Z = z) = 0$. I may use these interchangeably.

(a) $P(X > 5)$

Recall that $P(X > 5) = 1 - P(X \leq 5)$ and $P(Z \leq z) = \mathbf{\Phi}(z)$.

Then,

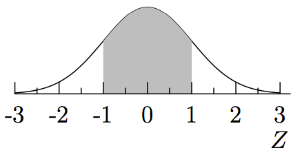
$$\begin{aligned} P(X > 5) &= 1 - P(X \leq 5) \\ &= 1 - \mathbf{\Phi}\left(\frac{5 - 10}{6}\right) \\ &= 1 - \mathbf{\Phi}\left(-\frac{5}{6}\right) \\ &\approx 1 - 0.202 \\ &= \boxed{0.798} \end{aligned}$$



(b) $P(4 < X < 16)$

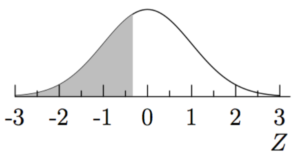
$$\begin{aligned}
P(4 < X < 16) &= P(X < 16) - P(X < 4) \\
&= \mathbf{\Phi}\left(\frac{16-10}{6}\right) - \mathbf{\Phi}\left(\frac{4-10}{6}\right) \\
&= \mathbf{\Phi}(1) - \mathbf{\Phi}(-1) \\
&\approx 0.841 - 0.159 \\
&= \boxed{0.683}
\end{aligned}$$

If you have taken introductory Statistics, you may know that this particular problem can also be solved using the Empirical Rule. By the Empirical Rule, we know that 68% of the data lies within one standard deviation of the mean, 95% lies within two standard deviations and 99.7% lies within three standard deviations.



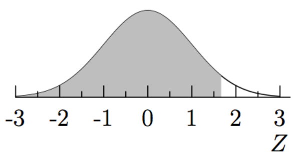
(c) $P(X < 8)$

$$\begin{aligned}
P(X < 8) &= \mathbf{\Phi}\left(\frac{8-10}{6}\right) \\
&= \mathbf{\Phi}\left(-\frac{1}{3}\right) \\
&= \boxed{0.369}
\end{aligned}$$



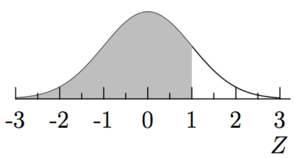
(d) $P(X < 20)$

$$\begin{aligned}
P(X < 20) &= \mathbf{\Phi}\left(\frac{20-10}{6}\right) \\
&= \mathbf{\Phi}\left(\frac{5}{3}\right) \\
&\approx \boxed{0.952}
\end{aligned}$$



(e) $P(X > 16)$

$$\begin{aligned}
P(X > 16) &= 1 - \mathbf{\Phi}\left(\frac{16-10}{6}\right) \\
&= 1 - \mathbf{\Phi}(1) \\
&\approx \boxed{0.159}
\end{aligned}$$



14. Suppose that X is a normal random variable with mean 5. If P{X > 9} = .2, approximately what is Var(X)?

Solution:

$.2 = P\left\{\frac{X-5}{\sigma} > \frac{9-5}{\sigma}\right\} = P\{Z > 4/\sigma\}$ where Z is a standard normal. But from the normal table $P\{Z < .84\} \approx .80$ and so

$.84 \approx 4/\sigma$ or $\sigma \approx 4.76$

That is, the variance is approximately $(4.76)^2 = 22.66$.

15. Let X be a normal random variable with mean 12 and variance 4. Find the value of c such that P{X > c} = 0.10

Solution:

Letting Z = (X – 12)/2 then Z is a standard normal.
Now, 0.10 = P{Z > (c – 12)/2}. But, P{Z < 1.28} = .90 and so (c–12)/2=1.28 or c=14.56

16. Each item produced by a certain manufacturer is, independently, of acceptable quality with probability .95. Approximate the probability that at most 10 of the next 150 items produced are unacceptable.

Solution:

Let X denote the number of unacceptable items among the next 150 produced. Since X is a binomial random variable with mean 150(.05) = 7.5 and variance 150(.05)(.95) = 7.125, we obtain that, for a standard normal random variable Z.

$$\begin{aligned} P\{X \le 10\} &= P\{X \le 10.5\} \\ &= P\left\{\frac{X-7.5}{\sqrt{7.125}} \le \frac{10.5-7.5}{\sqrt{7.125}}\right\} \\ &\approx P\{Z \le 1.1239\} \\ &= .8695 \end{aligned}$$

Unsolved Problems

1. Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares her number with that of player 3, and so on. Let X denote the number of times player 1 is a winner. Find P{X = i},i = 0,1,2,3,4.

2. You have \$1000, and a certain commodity presently sells for \$2 per ounce. Suppose that after one week the commodity will sell for either \$1 or \$4 an ounce, with these two possibilities being equally likely. If your objective is to maximize the expected amount of the commodity that you possess at the end of the week, what strategy should you employ?

3. If E[X] = 1 and Var(X) = 5, find
a) $E[(2 + X)^2]$
b) Var(4 + 3X)

4. A man claims to have extrasensory perception. As a test, a fair coin is flipped 10 times and the man is asked to predict the outcome in advance. He gets 7 out of 10 correct. What is the probability that he would have done at least this well if he had no ESP?

5. If you buy a lottery ticket in 50 lotteries, in each of which your chance of winning a prize is 1/100 , what is the (approximate) probability that you will win a prize
a) at least once?
b) exactly once?
c) at least twice?

6. You arrive at a bus stop at 10 o'clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.
a) What is the probability that you will have to wait longer than 10 minutes?
b) If at 10:15 the bus has not yet arrived, what is the probability that you will have to what at least an additional 10 minutes?

7. The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that, starting with this year, it will take over 10 years before a year occurs having a rainfall of over 50 inches? What assumptions are you making?