Linear Programming: Applications

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LP Applications

Frosty distributors: Data

- 15 vegetables in cartons of size 1.25 cubic feet are procured every week.
- Rarely any inventory is left over at the end of the week.
- Warehouse space available is 18000 cubic feet.
- Credit available per week is $30000.
- Forecast is expressed in terms of a minimum and maximum anticipated sales quantity.
- The minimum quantity is based on a contractual agreement.
- The maximum quantity represents a conservative estimate of the sales potential per week.

Write down a model that maximizes Frosty's week profit.

Vegetable (x_i)	Cost or Credit (\$)	Given Price (\$)	Cartons Min	Cartons Max	Profit per Carton (\$)
Whipped potatoes (x_1)	2.15	2.27	300	1500	p_1 =0.12
Creamed corn (x_2)	2.20	2.48	400	2000	p_2 =0.28
Black-eyed peas (x_3)	2.40	2.70	250	900	p_3 =0.30
Artichokes (x_4)	4.80	5.20	0	150	p_4 =0.40
Carrots (x_5)	2.60	2.92	300	1200	p_5 =0.32
Succotash (x_6)	2.30	2.48	200	800	p_6 =0.18
Okra (x_7)	2.35	2.20	150	600	p_7 =-0.15
Cauliflower (x_8)	2.85	3.13	100	300	p_8 =0.28
Green peas (x_9)	2.25	2.48	750	3500	p_9 =0.23
Spinach (x_{10})	2.10	2.27	400	2000	p_{10} =0.17
Lima beans (x_{11})	2.80	3.13	500	3300	p_{11} =0.33
Brussels sprouts (x_{12})	3.00	3.18	100	500	p_{12} =0.18
Green beans (x_{13})	2.60	2.92	500	3200	p_{13} =0.32
Squash (x_{14})	2.50	2.70	100	500	p_{14} =0.20
Broccoli (x_{15})	2.90	3.13	400	2500	p_{15} =0.23

Table: Frosty Distributor's data.

The LP formulation

- What are the decision variables?
  - How many decision variables in the problem?
- What is the objective function?
- What are the constraints?

Frosty Distributors: LP solution

Frosty Distributors

Vegetable	*Cost*	*Given Price*	*Data Min*	*Max*	*Objective (Profit)*	*Decisions (Cartons)*	*Constraints Credit*	*Space*
Whipped potatoes	2.15	2.27	300	1500	0.12	300.0	2.15	1.25
Creamed corn	2.20	2.48	400	2000	0.28	2000.0	2.20	1.25
Black-eyed peas	2.40	2.70	250	900	0.30	900.0	2.40	1.25
Artichokes	4.80	5.20	0	150	0.40	0.0	4.80	1.25
Carrots	2.60	2.92	300	1200	0.32	1200.0	2.60	1.25
Succotash	2.30	2.48	200	800	0.18	200.0	2.30	1.25
Okra	2.35	2.20	150	600	-0.15	150.0	2.35	1.25
Cauliflower	2.85	3.13	100	300	0.28	100.0	2.85	1.25
Green peas	2.25	2.48	750	3500	0.23	750.0	2.25	1.25
Spinach	2.10	2.27	400	2000	0.17	400.0	2.10	1.25
Lima beans	2.80	3.13	500	3300	0.33	2150.0	2.80	1.25
Brussels sprouts	3.00	3.18	100	500	0.18	100.0	3.00	1.25
Green beans	2.60	2.92	500	3200	0.32	3200.0	2.60	1.25
Squash	2.50	2.70	100	500	0.20	100.0	2.50	1.25
Broccoli	2.90	3.13	400	2500	0.23	400.0	2.90	1.25
				Obj. Fn. =	3395.50	*LHS*	30000	14938
							<=	<=
						RHS	30000	18000

Frosty Distributors: Making sense of the optimal solution

- Which constraints are binding at optimality?
- What is the shadow price of the credit constraint?
  - What does it signify?
    - How much weekly interest should you be willing to pay for additional credit?
  - How can you compute the shadow price? Is there a pattern in the optimal solution?
  - Why are the allowable increase and decrease 3220 and 4620 respectively?
- Can you develop a policy to manage procurement decisions based on the optimal pattern?
- What would be the cost implication if you were to procure additional squash cartons?
  - Why is the reduced cost of squash (x_{14}) -0.0946?
  - Does it have an associated range of validity?

Frosty Distributors: Priority list

Frosty Distributors **Priority List**

Credit at min 10940

Vegetable	*Cost*	*Given Price*	*Data Min*	*Max*	*Objective (Profit)*	*Decisions (Amount)*	*Constraints Credit*	*Space*	Profit/Cost	Cumulative Credit
Creamed corn	2.20	2.48	400	2000	0.28	2000	2.20	1.25	0.1273	14460
Black-eyed peas	2.40	2.70	250	900	0.30	900	2.40	1.25	0.1250	16020
Carrots	2.60	2.92	300	1200	0.32	1200	2.60	1.25	0.1231	18360
Green beans	2.60	2.92	500	3200	0.32	3200	2.60	1.25	0.1231	25380
Lima beans	2.80	3.13	500	3300	0.33	2150	2.80	1.25	0.1179	30000
Green peas	2.25	2.48	750	3500	0.23	750	2.25	1.25	0.1022	
Cauliflower	2.85	3.13	100	300	0.28	100	2.85	1.25	0.0982	
Artichokes	4.80	5.20	0	150	0.40	0	4.80	1.25	0.0833	
Spinach	2.10	2.27	400	2000	0.17	400	2.10	1.25	0.0810	
Squash	2.50	2.70	100	500	0.20	100	2.50	1.25	0.0800	
Broccoli	2.90	3.13	400	2500	0.23	400	2.90	1.25	0.0793	
Succotash	2.30	2.48	200	800	0.18	200	2.30	1.25	0.0783	
Brussels sprouts	3.00	3.18	100	500	0.18	100	3.00	1.25	0.0600	
Whipped potatoes	2.15	2.27	300	1500	0.12	300	2.15	1.25	0.0558	
Okra	2.35	2.20	150	600	-0.15	150	2.35	1.25	-0.0638	
				Obj. Fn. =	3395.50	*LHS*	30000	14938		
							<=	<=		
						RHS	30000	18000		

Frosty Distributors: Sensitivity report

Microsoft Excel 8.0d Sensitivity Report

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
G5	Whipped potatoes (Amount)	300.0	-0.1334	0.12	0.1334	1E+30
G6	Creamed corn (Amount)	2000.0	0.0207	0.28	1E+30	0.0207
G7	Black-eyed peas (Amount)	900.0	0.0171	0.30	1E+30	0.0171
G8	Artichokes (Amount)	0.0	-0.1657	0.40	0.1657	1E+30
G9	Carrots (Amount)	1200.0	0.0136	0.32	1E+30	0.0136
G10	Succotash (Amount)	200.0	-0.0911	0.18	0.0911	1E+30
G11	Okra (Amount)	150.0	-0.4270	-0.15	0.4270	1E+30
G12	Cauliflower (Amount)	100.0	-0.0559	0.28	0.0559	1E+30
G13	Green peas (Amount)	750.0	-0.0352	0.23	0.0352	1E+30
G14	Spinach (Amount)	400.0	-0.0775	0.17	0.0775	1E+30
G15	Lima beans (Amount)	2150.0	0.0000	0.33	0.0146	0.0438
G16	Brussels sprouts (Amount)	100.0	-0.1736	0.18	0.1736	1E+30
G17	Green beans (Amount)	3200.0	0.0136	0.32	1E+30	0.0136
G18	Squash (Amount)	100.0	-0.0946	0.20	0.0946	1E+30
G19	Broccoli (Amount)	400.0	-0.1118	0.23	0.1118	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
H20	Credit	30000	0.1179	30000	3220	4620.0
I20	Space	14938	0.0000	18000	1E+30	3062.5

TelecomOptics' transportation problem data

Supply City ↓	Demand city Production and Transportation cost per 1000 units						*Capacity (000's)* ↓
	Atlanta (A)	Boston (B)	Chicago (C)	Denver (D)	Omaha (O)	Portland (P)	
Baltimore (L)	1675	400	685	1630	1160	3800	*18*
Cheyenne (H)	1460	1940	970	100	495	1200	*30*
Salt Lake (S)	1925	2400	100	500	950	800	*22*
Memphis (M)	380	1355	543	1045	665	2321	*24*
Wichita (W)	922	1646	700	508	311	1797	*31*
Demand (000's)	*20*	*16*	*28*	*12*	*14*	*22*	

Table: TelecomOptics' production and shipping costs.

The management is debating how to serve all the six markets at the least possible total production and transportation cost. Formulate a model and recommend a least total cost shipping solution.

The LP formulation

- What are the decision variables?
  - How many decision variables in the problem?
- What is the objective function?
- What are the constraints?

LP solution

	Atlanta (A)	Boston (B)	Chicago (C)	Denver (D)	Omaha (O)	Portland (P)	*Supply (000's)*	
Baltimore (L)	1,675	400	685	1,630	1,160	3,800	18	
Cheyenne (H)	1,460	1,940	970	100	495	1,200	30	
Salt Lake (S)	1,925	2,400	100	500	950	800	22	
Memphis (M)	380	1,355	543	1,045	665	2,321	24	
Wichita (W)	922	1,646	700	508	311	1,797	31	Total demand
Demand (000's)	20	16	28	12	14	22		112
						Total supply	125	

	Atlanta (A)	Boston (B)	Chicago (C)	Denver (D)	Omaha (O)	Portland (P)	*Supply (000's)*
Baltimore (L)	0	16	2	0	0	0	18
Cheyenne (H)	0	0	0	12	0	18	30
Salt Lake (S)	0	0	18	0	0	4	22
Memphis (M)	20	0	4	0	0	0	24
Wichita (W)	0	0	4	0	14	0	18
Demand (000's)	20	16	28	12	14	22	

Total cost (000's $)	52,496.0

Making sense of the optimal solution

- What do you observe about the number of routes chosen at optimality?
- Which constraints are binding at optimality?
- Which constraints are binding?
  - Is there any discernible optimal pattern in the solution?
- How should the management change the shipping routes if the demand at Boston went up by 1000 units?
  - By how much will the cost change?
  - How long can we keep increasing the demand at Boston before we would have to reoptimize?
- By how much would the optimal cost change if we must ship 1000 units from Cheyenne to Boston? How would the shipping routes adjust themselves?
- What would be the change in the optimal cost and shipping routes if demands at Boston and Chicago went up simultaneously by 1000 units each?

LP sensitivity report: Shadow prices

Constraints

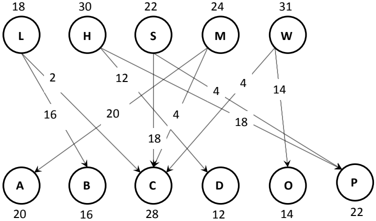
Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C19	Demand (000's) Atlanta (A)	20	537	20	4	4
D19	Demand (000's) Boston (B)	16	415	16	2	4
E19	Demand (000's) Chicago (C)	28	700	28	13	4
F19	Demand (000's) Denver (D)	12	300	12	13	4
G19	Demand (000's) Omaha (O)	14	311	14	13	14
H19	Demand (000's) Portland (P)	22	1400	22	13	4
I14	Baltimore (L) Supply (000's)	18	-15	18	4	2
I15	Cheyenne (H) Supply (000's)	30	-200	30	4	13
I16	Salt Lake (S) Supply (000's)	22	-600	22	4	13
I17	Memphis (M) Supply (000's)	24	-157	24	4	4
I18	Wichita (W) Supply (000's)	18	0	31	1E+30	13

Reduced costs

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
C14	Baltimore (L) Atlanta (A)	0	1153	1675	1E+30	1153
D14	Baltimore (L) Boston (B)	16	0	400	1097	415
E14	Baltimore (L) Chicago (C)	2	0	685	15	1097
F14	Baltimore (L) Denver (D)	0	1345	1630	1E+30	1345
G14	Baltimore (L) Omaha (O)	0	864	1160	1E+30	864
H14	Baltimore (L) Portland (P)	0	2415	3800	1E+30	2415
C15	Cheyenne (H) Atlanta (A)	0	1123	1460	1E+30	1123
D15	Cheyenne (H) Boston (B)	0	1725	1940	1E+30	1725
E15	Cheyenne (H) Chicago (C)	0	470	970	1E+30	470
F15	Cheyenne (H) Denver (D)	12	0	100	208	300
G15	Cheyenne (H) Omaha (O)	0	384	495	1E+30	384
H15	Cheyenne (H) Portland (P)	18	0	1200	200	208
C16	Salt Lake (S) Atlanta (A)	0	1988	1925	1E+30	1988
D16	Salt Lake (S) Boston (B)	0	2585	2400	1E+30	2585
E16	Salt Lake (S) Chicago (C)	18	0	100	200	208
F16	Salt Lake (S) Denver (D)	0	800	500	1E+30	800
G16	Salt Lake (S) Omaha (O)	0	1239	950	1E+30	1239
H16	Salt Lake (S) Portland (P)	4	0	800	208	200
C17	Memphis (M) Atlanta (A)	20	0	380	385	537
D17	Memphis (M) Boston (B)	0	1097	1355	1E+30	1097
E17	Memphis (M) Chicago (C)	4	0	543	157	385
F17	Memphis (M) Denver (D)	0	902	1045	1E+30	902
G17	Memphis (M) Omaha (O)	0	511	665	1E+30	511
H17	Memphis (M) Portland (P)	0	1078	2321	1E+30	1078
C18	Wichita (W) Atlanta (A)	0	385	922	1E+30	385
D18	Wichita (W) Boston (B)	0	1231	1646	1E+30	1231
E18	Wichita (W) Chicago (C)	4	0	700	208	15
F18	Wichita (W) Denver (D)	0	208	508	1E+30	208
G18	Wichita (W) Omaha (O)	14	0	311	384	311
H18	Wichita (W) Portland (P)	0	397	1797	1E+30	397

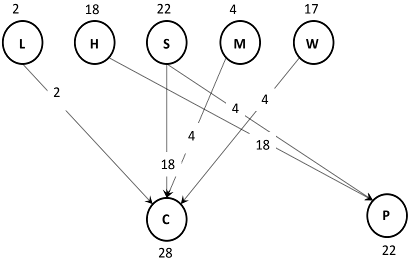
Finding a pattern in the optimal solution



	Atlanta (A)	Boston (B)	Chicago (C)	Denver (D)	Omaha (O)	Portland (P)	*Supply (000's)*	*Excess*
Baltimore (L)	0	16	2	0	0	0	18	0
Cheyenne (H)	0	0	0	12	0	18	30	0
Salt Lake (S)	0	0	18	0	0	4	22	0
Memphis (M)	20	0	4	0	0	0	24	0
Wichita (W)	0	0	4	0	14	0	18	13
Demand (000's)	20	16	28	12	14	22		

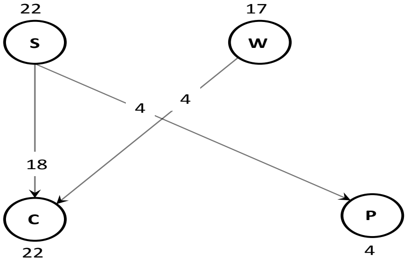
Finding a pattern in the optimal solution

	Chicago (C)	Portland (P)	*Supply (000's)*	*Excess*
Baltimore (L)	2	0	2	0
Cheyenne (H)	0	18	18	0
Salt Lake (S)	18	4	22	0
Memphis (M)	4	0	4	0
Wichita (W)	4	0	4	13
Demand (000's)	28	22		



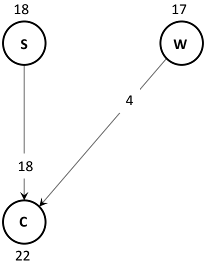
Finding a pattern in the optimal solution

	Chicago (C)	Portland (P)	*Supply (000's)*	*Excess*
Salt Lake (S)	18	4	22	0
Wichita (W)	4	0	4	13
Demand (000's)	22	4		



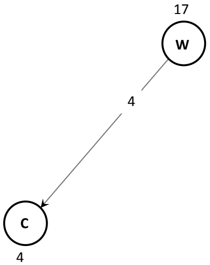
Finding a pattern in the optimal solution

	Chicago (C)	*Supply (000's)*	*Excess*
Salt Lake (S)	18	18	0
Wichita (W)	4	4	13
Demand (000's)	22		



Finding a pattern in the optimal solution

	Chicago (C)	*Supply (000's)*	*Excess*
Wichita (W)	4	4	13
Demand (000's)	4		



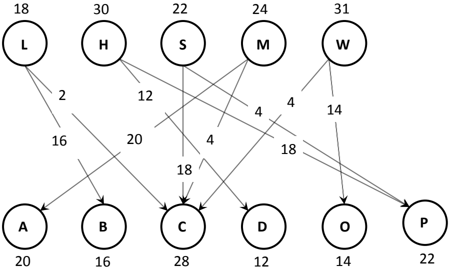
Optimal shipping pattern

Supply locations	Priority order		Demand locations	Priority order
L→	B, C		A←	M
H→	D, P		B←	L
S→	P, C		C←	L/M, S, W
M→	A, C		D←	H
W→	O, C		O←	W
			P←	H, S

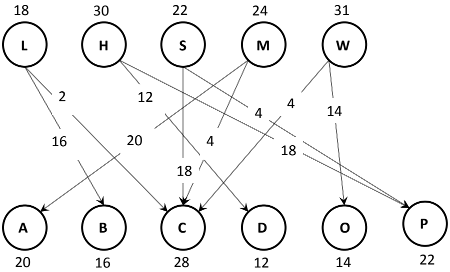
Understanding the optimal pattern

1. List the best routes and critical sources.
2. Identify a high priority demand—one that is met from a **unique source**—and allocate the demand to this route. Then remove this destination from consideration. Repeat if possible.
3. Identify a **critical source with only one best route unallocated** and allocate the remaining supply to this route. Then remove this source from consideration.
4. Repeat the previous two steps using remaining demands and remaining supplies each time, until all shipments are accounted for.

Using the pattern



L	B, C	**A**	M
H	D, P	**B**	L
S	P, C	**C**	L/M, S, W
M	A, C	**D**	H
W	O, C	**O**	W
		P	H, S



	Atlanta (A)	Boston (B)	Chicago (C)	Denver (D)	Omaha (O)	Portland (P)	*Supply (000's)*
Baltimore (L)	1675	400	685	1630	1160	3800	*18*
Cheyenne (H)	1460	1940	970	100	495	1200	*30*
Salt Lake (S)	1925	2400	100	500	950	800	*22*
Memphis (M)	380	1355	543	1045	665	2321	*24*
Wichita (W)	922	1646	700	508	311	1797	*31*
Demand (000's)	*20*	*16*	*28*	*12*	*14*	*22*	

Understanding the optimal pattern

This retrospective description of the optimal solution (which is assumed to be nondegenerate) has two important features:

1. It is *complete*, that is, we can specify the entire shipment schedule.
2. It is *unambiguous*, that is, the description leads to just one schedule.

Anyone who constructs the solution using these steps should reach the same result. The significance of this *pattern* is that it holds not just for the specific problem that we solved, but also for other problems that are very similar but with some of the parameters slightly altered.

Appendix section: Formulations

Frosty distributors: LP Formulation

Problem parameters:
Let S_i represent the space occupied by a carton of the i^{th} vegetable, $i = 1, \ldots, 15$. In our problem $S_i = 1.25$ cubic feet for all $i = 1, \ldots, 15$. Let C_i represent the credit (cost) required to procure a carton of the i^{th} vegetable, $i = 1, \ldots, 15$. Let L_i represent the minimum number of cartons of the i^{th} vegetable that must be procured and U_i represent the maximum number of cartons of the i^{th} vegetable that may be procured. Let p_i represent the profitability ($=$ price$_i$ - cost$_i$) of i^{th} vegetable.

Decision variables:
Let x_i represent the number of cartons of the i^{th} vegetable procured, $i = 1, \ldots, 15$. These are our decision variables.

Frosty distributors: LP Formulation

We can now express our LP model as follows:

$$\begin{aligned}
\max \quad & \sum_{i=1}^{15} p_i \, x_i \\
\text{s.t.} \quad & \sum_{i=1}^{15} S_i \, x_i \leq 18000 \qquad \text{(Space constraint)}, \\
& \sum_{i=1}^{15} C_i \, x_i \leq 30000 \qquad \text{(Credit limit constraint)}, \\
& x_i \geq L_i \quad \text{for all } i = 1, \ldots, 15 \quad \text{(Minimum \# of cartons)}, \\
& x_i \leq U_i \quad \text{for all } i = 1, \ldots, 15 \quad \text{(Maximum \# of cartons)}.
\end{aligned}$$

Frosty Distributor's LP formulation

$$
\begin{aligned}
\max \quad & 0.12x_1 + 0.28x_2 + 0.3x_3 + 0.4x_4 + 0.32x_5 + 0.18x_6 \\
& - 0.15x_7 + 0.28x_8 + 0.23x_9 + 0.17x_{10} + 0.33x_{11} + 0.18x_{12} \\
& + 0.32x_{13} + 0.2x_{14} + 0.23x_{15} \\
\text{s.t.} \quad & 2.15x_1 + 2.2x_2 + 2.4x_3 + 4.8x_4 + 2.6x_5 + 2.3x_6 \\
& + 2.35x_7 + 2.85x_8 + 2.25x_9 + 2.1x_{10} + 2.8x_{11} \\
& + 3x_{12} + 2.6x_{13} + 2.5x_{14} + 2.9x_{15} \leq 30000 && : \text{Credit limit,} \\
& 1.25 \times (x_1 + x_2 + \cdots + x_{15}) \leq 18000 && : \text{Warehouse space,} \\
& x_1 \geq 300,\ x_2 \geq 400,\ x_3 \geq 250, \ldots,\ x_{15} \geq 400 && : \text{Minimum limit,} \\
& x_1 \leq 1500,\ x_2 \leq 2000,\ x_3 \leq 900, \ldots,\ x_{15} \leq 2500 && : \text{Maximum limit.}
\end{aligned}
$$

TelecomOptics: LP Formulation

Problem parameters:
Let S_i represent the supply available (capacity available) at the i^{th} supply location. Notice there are 5 supply locations, $\{L, H, S, M, W\}$. Similarly, let D_j be the demand required to be met at the j^{th} demand destination. Notice that there are 6 demand destinations, $\{A, B, C, D, O, P\}$. Let c_{ij} represent the cost of producing and shipping 1000 units from supply location i to demand destination j. For example, the production and shipping cost from Baltimore (L) to Chicago (C) is $c_{LC} = 685$. There are $5 \times 6 = 30$ such combinations (or shipping routes i, j).

Decision variables:
Let x_{ij} represent the number of units, thousands, shipped from supply location i to demand location j. In other words, x_{ij} represents the units shipped on shipping route i, j. Clearly there are 30 such nonnegative decision variables.

TelecomOptics: LP Formulation

The LP formulation can be written down as follows:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{5} \sum_{j=1}^{6} c_{ij} x_{ij} & & \\
\text{s.t.} \quad & \sum_{j=1}^{6} x_{ij} \le S_i & \text{for all } i = L, H, S, M, W & \quad : \text{Supply constraints,} \\
& \sum_{i=1}^{5} x_{ij} \ge D_j & \text{for all } j = A, B, C, D, O, P & \quad : \text{Demand constraints,} \\
& x_{ij} \ge 0 & \text{for all } i = L, H, S, M, W \text{ and } j = A, B, C, D, O, P. &
\end{aligned}
$$

TelecomOptics: LP Formulation

The LP formulation can be written down as follows:

$$
\begin{array}{lll}
\min & 1675x_{LA} + 400x_{LB} + 685x_{LC} + 1630x_{LD} + 1160x_{LO} + 3800x_{LP} & \\
 & + 1460x_{HA} + \cdots + 1200x_{HP} + 1925x_{SA} + \cdots + 800x_{SP} & \\
 & + 380x_{MA} + \cdots + 2321x_{MP} + 922x_{WA} + \cdots + 1797x_{WP} & \\
\text{s.t.} & x_{LA} + x_{LB} + x_{LC} + x_{LD} + x_{LO} + x_{LP} \leq 18 & : \text{Supply constraints,} \\
 & x_{HA} + x_{HB} + x_{HC} + x_{HD} + x_{HO} + x_{HP} \leq 30, & \\
 & x_{SA} + x_{SB} + x_{SC} + x_{SD} + x_{SO} + x_{SP} \leq 22, & \\
 & x_{MA} + x_{MB} + x_{MC} + x_{MD} + x_{MO} + x_{MP} \leq 24, & \\
 & x_{WA} + x_{WB} + x_{WC} + x_{WD} + x_{WO} + x_{WP} \leq 31, & \\
 & x_{LA} + x_{HA} + x_{SA} + x_{MA} + x_{WA} \geq 20 & : \text{Demand constraints,} \\
 & x_{LB} + x_{HB} + x_{SB} + x_{MB} + x_{WB} \geq 16, & \\
 & x_{LC} + x_{HC} + x_{SC} + x_{MC} + x_{WC} \geq 28, & \\
 & x_{LD} + x_{HD} + x_{SD} + x_{MD} + x_{WD} \geq 12, & \\
 & x_{LO} + x_{HO} + x_{SO} + x_{MO} + x_{WO} \geq 14, & \\
 & x_{LP} + x_{HP} + x_{SP} + x_{MP} + x_{WP} \geq 22, & \\
 & x_{ij} \geq 0 \quad \text{for all } i = L, H, S, M, W \text{ and } j = A, B, C, D, O, P. &
\end{array}
$$