Modeling Business Logic: Using Binary Variables

Vishwakant Malladi

AMPBA Optimization
The Indian School of Business

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Integer/Binary models: A brief introduction

Why integer programming?

Advantages of restricting variables to take on integer values:

1. More realistic (economic indivisibility).
2. More flexibility (use of binary variables, logical constraints).

Disadvantages:

1. More difficult to model.
2. Can be much more difficult to solve than LPs.

Binary integer variables

Binary integer variables can take values of 0 or 1. Typically, a decision variable, say x_j, which is binary is represented in a formulation in any one of the following ways:

- $x_j \in \{0, 1\}$, or
- $0 \leq x_j \leq 1$ and integer.

Binary variables are very useful in modeling several business situations. Particularly,

- Logical constraints, e.g., if-then-else, go-no/go decisions,
- Application areas include supply-chain optimization models (for example, transportation, facility location), financial models (for example, budget models, auctions), marketing models (for example, consumer choice), and many more.

The integer programming (IP) mystery

Some integer programs are easy (we can solve problems with millions of variables)

- Can you think of some you have seen earlier?

Some integer programs are notoriously hard (even 100 variables can be extremely challenging)

- It takes expertise and experience to know which is which
  - Solving large IPs is an active area of research

Why are IPs sometimes extremely hard to solve?

Representing an integer program (2-D problem)

Consider the following example:

$$
\begin{array}{lll}
(\mathrm{P}) & \max & 3x_1 + 4x_2 \\
& \text{s.t.} & 5x_1 + 8x_2 \leq 24, \\
& & x_1,\ x_2 \geq 0 \quad \underline{\text{and integer}}.
\end{array}
$$

What is the optimal solution this problem?

Representation of the feasible region

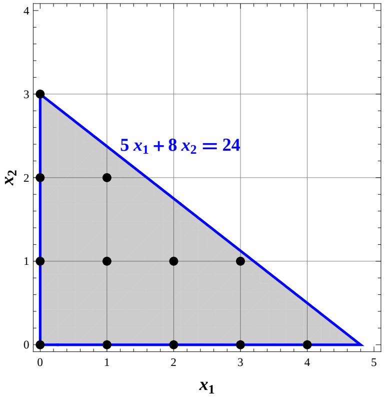


Figure: Integer feasible region for (P).

Finding the optimal solution via LP relaxation

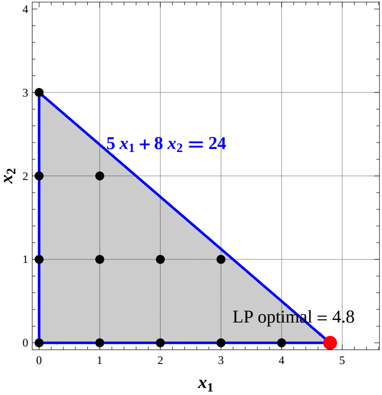
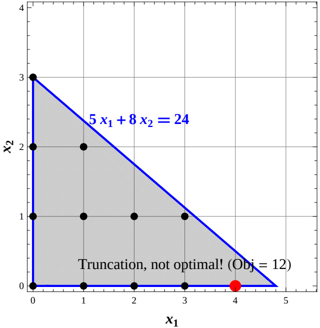
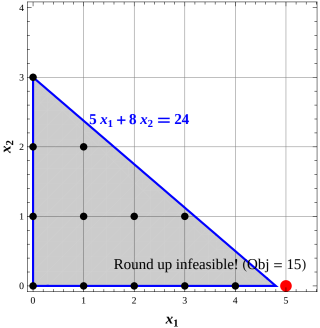


Figure: LP solution to (P) – It's not integral valued.

Truncating and rounding-up may not help



(a) Truncating (not optimal).



(b) Rounding-up (infeasible).

Figure: Finding an integer solution by truncating or rounding-up a LP solution may not work.

Finding the optimal solution via LP relaxation

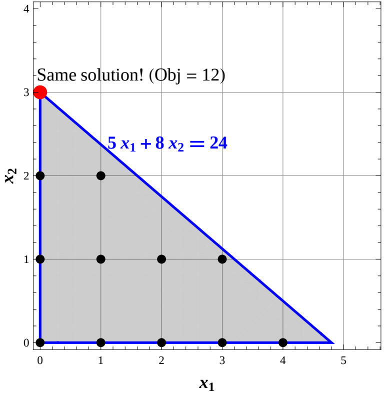


Figure: Had the truncated solution been optimal, the LP would have found it at another corner point! That's why it is not optimal.

The IP optimal solution

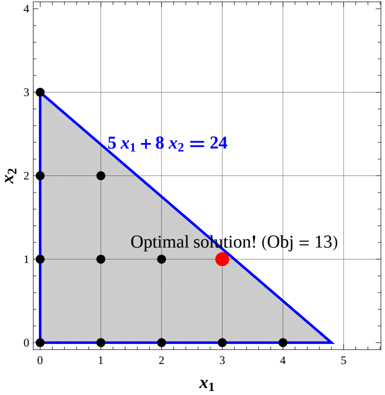


Figure: Unfortunately, the IP optimal solution is not at a "corner" point of the original LP feasible region. It is much harder to find.

Representing logical conditions in optimization models

Simple logical conditions

Consider a business situation that has two options. For a more concrete example, consider a firm deciding to locate warehouses at two sites, Hyderabad (H) and Mumbai (M). To deal with various logical (business) conditions involving locating warehouses at H and/or M, let us define two binary decision variables as follows:

$$x_1 = \begin{cases} 1 & : \text{ if location H is chosen,} \\ 0 & : \text{ otherwise.} \end{cases} \quad \text{and} \quad x_2 = \begin{cases} 1 & : \text{ if location M is chosen,} \\ 0 & : \text{ otherwise.} \end{cases}$$

Simple logical conditions

Let us consider the potential logical conditions linking H and M, their binary representations, and the corresponding constraints.

Logical condition	Logical constraint	How/Why does the constraint work?
The firm must locate at H <u>and</u> M.	$x_1 + x_2 = 2$	Notice both the binary decision variables are forced to take a value of 1, implying both locations will be chosen.
The firm must locate <u>at least</u> at one of the locations (<u>or</u>).	$x_1 + x_2 \geq 1$	Notice at least one of the binary decision variables is forced to take a value of 1, implying either H, or M, or both are chosen.
The firm must not locate at H.	$x_1 = 0$	H will never be chosen.

Simple logical conditions

Let us consider the potential logical conditions linking H and M, their binary representations, and the corresponding constraints.

Logical condition	Logical constraint	How/Why does the constraint work?
Choose exactly one option between H and M (xor).	$x_1 + x_2 = 1$	The only way to satisfy the constraint is if one of the variables equals 1 and the other is 0. Implying either we locate of H or at M, but not both.
If the firm locates at H then it must locate at M (implication).	$x_1 \leq x_2$	If $x_1 = 1$, i.e., H is chosen, then the constraint forces x_2 to also take a value of 1, implying the firm must locate at M too. However, if $x_1 = 0$ then the constraint does not impose any condition on x_2.

Simple logical conditions

Let us consider the potential logical conditions linking H and M, their binary representations, and the corresponding constraints.

Logical condition	Logical constraint	How/Why does the constraint work?
Whatever is done at H must be done at M (equivalence).	$x_1 = x_2$	If H is not chosen, i.e., $x_1 = 0$ then the constraint forces $x_2 = 0$, i.e., M is not chosen either. Similarly, if H is chosen then the constraint forces M to be chosen too.

An example

Consider the following problem. StockCo is considering investing in possible 6 stocks. The cash required for each investment, as well as the NPV of the investment, is given in the table below. The cash available for the investments is $14,000. Stockco wants to maximize its NPV. What is the optimal strategy? Note that, an investment can be selected or not, i.e., one cannot select a fraction of an investment.

Investment (i)	1	2	3	4	5	6
Cash required (000's $)	5	7	4	3	4	6
NPV added (000's $)	16	22	12	8	11	19

The Stockco example

Let us define a binary decision variable x_i that denotes if Stockco invests in stock i or not. That is,

$$x_i = \begin{cases} 1 & : \text{ if Stockco invests in stock } i, \\ 0 & : \text{ otherwise.} \end{cases}$$

Stockco's optimization problem, of maximizing the NPV subject to a budget constraint, can be written as follows:

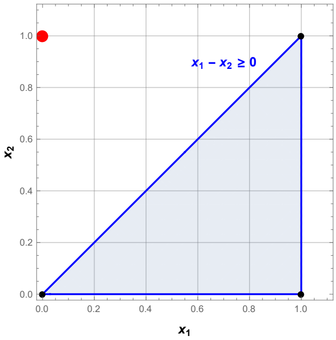
$$\begin{array}{lll} \max & 16x_1 + 22x_2 + 12x_3 + 8x_4 + 11x_5 + 19x_6 & \\ \text{s.t.} & 5x_1 + 7x_2 + 4x_3 + 3x_4 + 4x_5 + 6x_6 \leq 14, & \text{(Budget constraint)} \\ & x_1, x_2, \ldots, x_6 \in \{0,\, 1\}\,. & \end{array}$$

The Stockco example

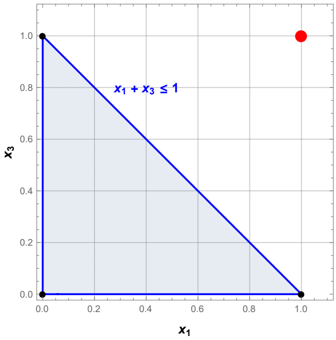
Develop the following constraints:

1. Exactly 3 stocks are selected.
2. If stock 2 is selected, then so is stock 1.
3. If stock 1 is selected, then stock 3 is not selected.
4. Either stock 4 is selected or stock 5 is selected, but not both.

Geometric representation of the constraints



(a) $x_1 - x_2 \geq 0$.



(b) $x_1 + x_3 \leq 1$.

Figure: Cuts generated for the Stockco model. How do they prune integer points?

Geometric representation of the cuts

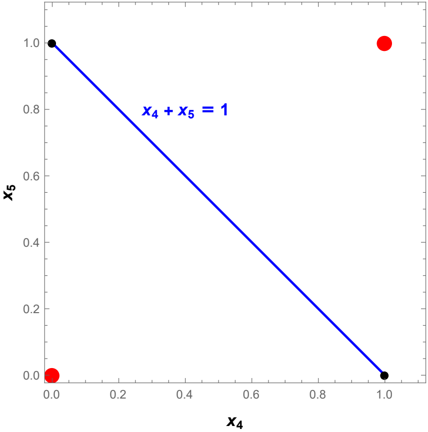


Figure: $x_4 + x_5 = 1$.

Extending our understanding of building logical constraints

Let us go back to our earlier example of building simple logical constraints (the warehouse location constraints). Suppose there are more than 2 locations and we represent the decision of locating a warehouse at location i with a binary variable x_i. Consider the following logical business conditions.

Logical condition	Logical constraint
At least k warehouses must be built	?
Must build warehouse 3 if both 1 and 2 are built	?
Must build warehouse 3 if either 1 or 2 is built	?
Must build 3 if either 1 or 2 is built but not both	?
Cannot build warehouse 3 if both 1 and 2 are built	?

Classifying objects based on information revealed

Knights, knaves, and werewolves

The information we have:

- Knights always tell the truth and knaves always lie
- Of the three inhabitants you interview, exactly one of them is a werewolf.
- Each of three inhabitants interviewed is either a knight or a knave, and could be a werewolf too.

The statements made:

A: I am a werewolf.

B: I am a werewolf.

C: At most one of us is a knight.

Knights, knaves, and werewolves

- What are our decision variables?
  - How many decision variables do we have?
- What are the constraints?
- What is the objective?
  - More importantly, when can we guarantee that a person is not a werewolf?

Capacitated facility location problem

The capacitated facility location problem

- This problem is a frequently encountered problem in supply chain management while designing "facility networks".
- Some times it is also referred to as the "fixed charge problem".
  - Instances of the fixed charge problem are seen in several applications including product line selection, portfolio design, and others.

- Let us consider the same set up we discussed earlier while studying the transportation model. However, now we have an additional challenge:
  - It costs us to keep a supply (production) facility open. This cost is independent of the the demand that will be served from the supply facility – hence it is a fixed cost. Of course, we can only serve any demand from a supply location if it is open, else we neither incur the fixed cost nor any operating costs.

TelecomOptics data

Supply City ↓	Demand city Production and Transportation cost per 1000 units						*Capa-city (000's) ↓*	*Fixed cost ($) (000's) ↓*
	Atlanta (A)	Boston (B)	Chicago (C)	Denver (D)	Omaha (O)	Portland (P)		
Baltimore (L)	1675	400	685	1630	1160	2800	*18*	*7650*
Cheyenne (H)	1460	1940	970	100	495	1200	24	*3500*
Salt Lake (S)	1925	2400	1425	500	950	800	*27*	*5000*
Memphis (M)	380	1355	543	1045	665	2321	*22*	*4100*
Wichita (W)	922	1646	700	508	311	1797	*31*	*2200*
Demand (000's)	*10*	*8*	*14*	*6*	*7*	*11*		

Table: TelecomOptics' production, shipping costs and plant location fixed costs.

The capacitated facility location model formulation

- What are our decision variables?
  - How many decision variables do we have?
  - What is the type of each decision variable?
- What are the constraints?
- What is the objective?

The decision variables

Decision variables:
Let i represent the i^{th} supply location. That is $i = L, H, S, M, W$. Similarly, let j represent the j^{th} demand destination, i.e., $j = A, B, C, D, O, P$. There are two types of decisions:

1. Whether to keep facility i open or not. This by definition is binary. Since we have 5 supply locations, we have 5 such binary decision variables. Let us denote, of each $i = L, H, S, M, W$

$$y_i = \begin{cases} 1 & : \text{ if facility } i \text{ is kept open,} \\ 0 & : \text{ otherwise.} \end{cases}$$

   as the binary decision variable.

2. How much to supply from supply location i to demand center j. Just as in the transportation problem we studied earlier, we have $5 \times 6 = 30$ such decision variables. They are of the continuous type. Let us denote

$$\begin{aligned} x_{ij} = &\text{ The amount shipped from supply location} \\ &\ i \text{ to demand center } j, \end{aligned}$$

   for each combination of i, j.

Constraints

Demand constraints:

Just as with the transportation problem we need to ensure that the demand is met at each demand center. We use similar constraints in this formulation too. Suppose, the demand at location j is D_j. We model the demand constraints as follows: $\sum_i x_{ij} \geq D_j$ for all $j = A, B, C, D, O, P.$.

That is,

$$
\begin{aligned}
x_{LA} + x_{HA} + x_{SA} + x_{MA} + x_{WA} &\geq 10, \\
x_{LB} + x_{HB} + x_{SB} + x_{MB} + x_{WB} &\geq 8, \\
x_{LC} + x_{HC} + x_{SC} + x_{MC} + x_{WC} &\geq 14, \\
x_{LD} + x_{HD} + x_{SD} + x_{MD} + x_{WD} &\geq 6, \\
x_{LO} + x_{HO} + x_{SO} + x_{MO} + x_{WO} &\geq 7, \\
x_{LP} + x_{HP} + x_{SP} + x_{MP} + x_{WP} &\geq 11.
\end{aligned} \tag{1}
$$

Constraints

Supply constraints:

We have to model the following constraint written in "English": If a supply location is open then it can ship at most up to its capacity, that is, If $y_i = 1$ then amount supplied from i should be at most capacity of i. However, if $y_i = 0$ then nothing should be supplied from i. Suppose, the supply available at i is S_i. We model the supply constraints as follows: $\sum_j x_{ij} \leq S_i \times y_i$for all $i = L, H, S, M, W$. That is,

$$\begin{aligned}
& x_{LA} + x_{LB} + x_{LC} + x_{LD} + x_{LO} + x_{LP} \leq 18\, \boldsymbol{y_L}, \\
& x_{HA} + x_{HB} + x_{HC} + x_{HD} + x_{HO} + x_{HP} \leq 24\, \boldsymbol{y_H}, \\
& x_{SA} + x_{SB} + x_{SC} + x_{SD} + x_{SO} + x_{SP} \leq 27\, \boldsymbol{y_S}, \\
& x_{MA} + x_{MB} + x_{MC} + x_{MD} + x_{MO} + x_{MP} \leq 22\, \boldsymbol{y_M}, \\
& x_{WA} + x_{WB} + x_{WC} + x_{WD} + x_{WO} + x_{WP} \leq 31\, \boldsymbol{y_W}.
\end{aligned} \tag{2}$$

Constraints

Non negativity and binary variable constraints:
To ensure that we only ship non-negative amounts we impose the non-negativity constraints on the x_{ij} variables, i.e.,

$$x_{ij} \geq 0 \qquad \text{for all } i = L, H, S, M, W \text{ and } j = A, B, C, D, O, P. \tag{3}$$

Additionally, we need to impose that

$$y_i \in \{0, 1\} \qquad \text{for all } i = L, H, S, M, W, \tag{4}$$

to model our open/closure decisions.

The objective function

The objective function:

The goal is to minimize total cost. Notice it is combination of the fixed cost incurred by keeping a supply locations open and the operating cost of meeting the demand of the demand centers from the supply locations that are open.

Operating costs:

Operating costs = (the cost per unit of shipping from i to j)$\times$ (number of units shipped from i to j), summed over all 30 combinations of i, $i = L, H, S, M, W$ and j, $j = A, B, C, D, O, P$. That is,

$$\begin{aligned}\text{Operating cost} = \sum_i \sum_j c_{ij} x_{ij} &= 1675x_{LA} + \cdots + 3800x_{LP} \\ &+ 1460x_{HA} + \cdots + 1200x_{HP} \\ &+ 1925x_{SA} + \cdots + 800x_{SP} \\ &+ 380x_{MA} + \cdots + 2321x_{MP} \\ &+ 922x_{WA} + \cdots + 1797x_{WP}\end{aligned}$$

Fixed costs:

Fixed costs are incurred by deciding to keep a facility open and are independent of the transaction volume. Thus,

$$\text{Fixed costs} = \sum_i f_i y_i = 7650y_L + 3500y_H + 5000y_S + 4100y_M + 2200y_W.$$

The capacitated facility location model (CFLM)

The entire CFLM model can now be written as

$$\text{(CFLM)} \quad \min \quad \sum_i f_i y_i + \sum_i \sum_j c_{ij} x_{ij}$$

subject to constraint sets (1), (2), (3), and (4).

Solver solution

	Atlanta (A)	Boston (B)	Chicago (C)	Denver (D)	Omaha (O)	Portland (P)	*Supply (000's)*
Baltimore (L)	1,675	400	685	1,630	1,160	3,800	18
Cheyenne (H)	1,460	1,940	970	100	495	1,200	24
Salt Lake (S)	1,925	2,400	1,425	500	950	800	27
Memphis (M)	380	1,355	543	1,045	665	2,321	22
Wichita (W)	922	1,646	700	508	311	1,797	31
Demand (000's)	10	8	14	6	7	11	

	Atlanta (A)	Boston (B)	Chicago (C)	Denver (D)	Omaha (O)	Portland (P)	Open/ Closed
Baltimore (L)	0	8	2	0	0	0	1
Cheyenne (H)	0	0	0	6	7	11	1
Salt Lake (S)	0	0	0	0	0	0	0
Memphis (M)	10	0	12	0	0	0	1
Wichita (W)	0	0	0	0	0	0	0
Demand (000's)	10	8	14	6	7	11	

Total cost (000's $) 47,401.0

The single sourcing formulation

Suppose we want to model a single sourcing solution, i.e., when a demand center can receive its shipment only from one single supply location. However a supply location can ship to multiple demand locations, as long as it ships the entire demand of a demand center.

- How do our decision variables change?
  - Notice we no longer must decide how much to ship from i to j because if we ship, we must ship the entire demand D_j at j. So our transportation decision change to if we ship or not (binary in nature).
  -
- How do the constraints change?
- And, the objective?

The single sourcing formulation

Decision variables:

Shipping decisions: In this case the variables x_{ij} are re-defined as follows:

$$x_{ij} = \begin{cases} 1 & : \text{ if supply location } i \text{ ships to demand location } j, \\ 0 & : \text{ otherwise.} \end{cases}$$

where $i = L, H, S, M, W$ and $j = A, B, C, D, O, P$.
Notice, ff the demand to be met at demand center j , $j = A, B, C, D, O, P$ is denoted by D_j, then the amount shipped from i to j can be represented as $D_j \cdot x_{ij}$. If x_{ij} takes a value of 1, the shipment amount is D_j else it is 0.
Just as in the earlier case we define

$$y_i = \begin{cases} 1 & : \text{ if facility } i \text{ is kept open,} \\ 0 & : \text{ otherwise.} \end{cases}$$

where $i = L, H, S, M, W$.

The single sourcing constraints

Demand constraints:

Given the redefined variables x_{ij} we now define the demand constraints as follows. Every demand center must source its demand from exactly one supply location, that is $\sum_i x_{ij} = 1$ for all $j = A, B, C, D, O, P$.

Specifically,

$$\begin{aligned}
x_{LA} + x_{HA} + x_{SA} + x_{MA} + x_{WA} &= 1, \\
x_{LB} + x_{HB} + x_{SB} + x_{MB} + x_{WB} &= 1, \\
x_{LC} + x_{HC} + x_{SC} + x_{MC} + x_{WC} &= 1, \\
x_{LD} + x_{HD} + x_{SD} + x_{MD} + x_{WD} &= 1, \\
x_{LO} + x_{HO} + x_{SO} + x_{MO} + x_{WO} &= 1, \\
x_{LP} + x_{HP} + x_{SP} + x_{MP} + x_{WP} &= 1.
\end{aligned} \tag{5}$$

The single sourcing constraints

Supply constraints:

Given the redefined variables x_{ij} we now define the supply constraints as follows. $\sum_j D_j x_{ij} \leq S_i \times y_i$ for all $i = L, H, S, M, W$. Specifically,

$$
\begin{aligned}
10x_{LA} + 8x_{LB} + 14x_{LC} + 6x_{LD} + 7x_{LO} + 11x_{LP} &\leq 18\,\boldsymbol{y_L}, \qquad (6)\\
10x_{HA} + 8x_{HB} + 14x_{HC} + 6x_{HD} + 7x_{HO} + 11x_{HP} &\leq 24\,\boldsymbol{y_H},\\
10x_{SA} + 8x_{SB} + 14x_{SC} + 6x_{SD} + 7x_{SO} + 11x_{SP} &\leq 27\,\boldsymbol{y_S},\\
10x_{MA} + 8x_{MB} + 14x_{MC} + 6x_{MD} + 7x_{MO} + 11x_{MP} &\leq 22\,\boldsymbol{y_M},\\
10x_{WA} + 8x_{WB} + 14x_{WC} + 6x_{WD} + 7x_{WO} + 11x_{WP} &\leq 31\,\boldsymbol{y_W}.
\end{aligned}
$$

Constraints

The binary variable constraints:
We need to impose the following constraints to model our binary choices.

$$\begin{aligned} y_i &\in \{0, 1\} \qquad \text{for all } i = L, H, S, M, W, \\ x_{ij} &\in \{0, 1\} \qquad \text{for all } i = L, H, S, M, W, \quad \text{and} \quad j = A, B, C, D, O, P. \end{aligned} \tag{7}$$

The objective function

The objective function:
Just as before the goal is to minimize total cost. Notice it is combination of the fixed cost incurred by keeping a supply locations open and the operating cost of meeting the demand of the demand centers from the supply locations that are open.
Operating and fixed costs:
Operating costs = (Demand at j)× (the cost per unit of shipping from i to j)× (if i ships to j), summed over all 30 combinations of i, $i = L, H, S, M, W$ and j, $j = A, B, C, D, O, P$. That is,

$$
\begin{aligned}
\text{Operating cost} = \sum_j D_j \sum_i (c_{ij} x_{ij}) &= 10 \times (1675x_{LA} + 1460x_{HA} + \cdots + 922x_{WA}) \\
&+ 8 \times (400x_{LB} + \cdots + 1646x_{WB}) \\
&+ 14 \times (685x_{LC} + \cdots + 700x_{WC}) \\
&+ 6 \times (1630x_{LD} + \cdots + 508x_{WD}) \\
&+ 7 \times (1160x_{LO} + \cdots + 311x_{WO}) \\
&+ 11 \times (2800x_{LP} + \cdots + 1797x_{WP})
\end{aligned}
$$

$$
\text{Fixed costs} = \sum_i f_i y_i = 7650y_L + 3500y_H + 5000y_S + 4100y_M + 2200y_W.
$$

The single sourcing capacitated facility location model (CFLM-SS)

The entire CFLM model can now be written as

$$\text{(CFLM-SS)} \quad \min \qquad \sum_i f_i y_i + \sum_j D_j \sum_i (c_{ij} x_{ij})$$

subject to constraint sets (5), (6), and (7).

Single sourcing solver solution

	Atlanta (A)	Boston (B)	Chicago (C)	Denver (D)	Omaha (O)	Portland (P)	*Supply (000's)*	*Fixed cost (000's)*
Baltimore (L)	1,675	400	685	1,630	1,160	3,800	18	7650
Cheyenne (H)	1,460	1,940	970	100	495	1,200	24	3500
Salt Lake (S)	1,925	2,400	1,425	500	950	800	27	5000
Memphis (M)	380	1,355	543	1,045	665	2,321	22	4100
Wichita (W)	922	1,646	700	508	311	1,797	31	2200
Demand (000's)	10	8	14	6	7	11		

	Atlanta (A)	Boston (B)	Chicago (C)	Denver (D)	Omaha (O)	Portland (P)	Open/ Closed	*Supply (000's)*
Baltimore (L)	0	0	0	0	0	0	0	0
Cheyenne (H)	0	0	0	0	0	0	0	0
Salt Lake (S)	0	0	0	1	0	1	1	17
Memphis (M)	1	1	0	0	0	0	1	18
Wichita (W)	0	0	1	0	1	0	1	21
Demand (000's)	1	1	1	1	1	1		
LHS Demand Ctr	1	1	1	1	1	1		
RHS Demand Ctr	1	1	1	1	1	1		

Total cost (000's $)	49,717.0

Piecewise linear functions

Modeling piecewise linear functions

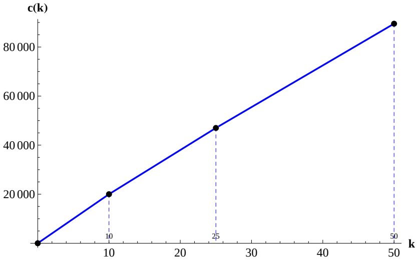
Our goal is to build a simple costing model, using a set of linear constraints, that computes the cost of a commodity exactly. Suppose, we want to procure k units of the commodity and the marginal cost of procurement is as follows:

$$\text{Marginal cost} = \begin{cases} \$2,000 \text{ per unit} & : \text{ for the first 10 units,} \\ \$1,800 \text{ per unit} & : \text{ for the next 15 units,} \\ \$1,700 \text{ per unit} & : \text{ for the remaining 25 units.} \end{cases}$$

We can procure at most 50 units. For example, if we procure $k = 12$ units of the commodity, the total cost would evaluate to

$$\begin{aligned} c(12) &= (2,000 \times 10) + (1,800 \times 2) \\ &= 20,000 + 3,600 \\ &= 23,600. \end{aligned}$$

The marginal cost curve



Our goal is to write a set of constraints that models the following non-linear cost function

$c(k) = 2000 \min\{k, 10\} + 1800 \min\{\max\{k-10, 0\}, 15\} + 1700 \min\{\max\{k-25, 0\}, 25\},$

where

$$k = x_1 + x_2 + x_3$$

and x_1, x_2, and x_3 are the number of units we procure at the three marginal costs of 2000, 1800, and 1700 respectively.

Constraints

Let x_1, x_2, and x_3 be our decision variables, i.e., the number of units of the commodity we procure at 2000, 1800, and 1700 respectively. Clearly, we must have

$$0 \leq x_1 \leq 10, \qquad 0 \leq x_2 \leq 15, \qquad 0 \leq x_3 \leq 25.$$

Now we wish to model the following statements:

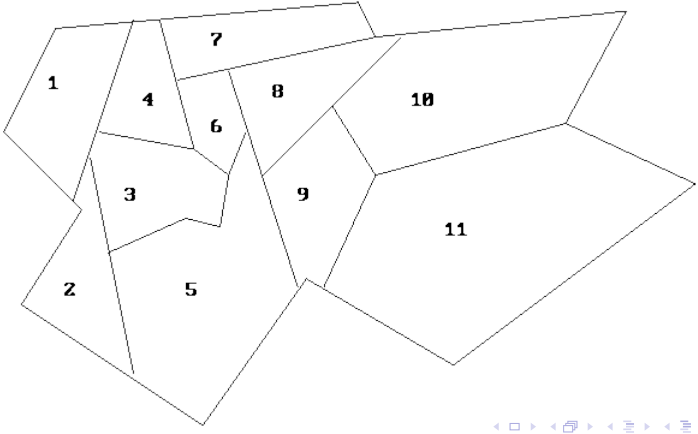
1. If $x_1 < 10$ then $x_2 = 0$, and
2. If $x_2 < 15$ then $x_3 = 0$.

How should we model these statements?

Set covering formulation

Where to locate fire stations?

To illustrate this model, consider the following location problem: A city is reviewing the location of its fire stations. The city is made up of a number of neighborhoods, as illustrated in the figure below.



Where to locate fire stations?

A fire station can be placed in any neighborhood. It is able to service both its neighborhood and any adjacent neighborhood (any neighborhood with a non-zero border with its home neighborhood). The objective is to minimize the number of fire stations used.

- What are the decision variables?
    - How many such decision variables?
- What are our constraints?
- What is the objective function?

Can you think of applications of this model in other settings?

Decision variables and cover constraints

Decision variables:

Let i denote a city neighborhood. There are 11 such neighborhoods, that is, $i = 1, 2, \ldots, 11$. We define a binary variable as follows:

$$x_i = \begin{cases} 1 & : \text{ if we locate a fire station in neighborhood } i, \\ 0 & : \text{ otherwise.} \end{cases}$$

Clearly, there are 11 such binary decision variables.

Constraints:

Notice that if we locate a fire station in neighborhood i not only does it service neighborhood i, but it also services all neighborhoods that are "adjacent" (share a common boundary) with i. For example, consider neighborhood $i = 1$. Its adjacent neighborhoods are 2, 3, and 4. Similarly, the adjacent neighborhoods of $i = 2$ are 1, 3, and 5. Let N_i denote the set of adjacent neighborhoods of i including itself. For example, $N_1 = \{1, 2, 3, 4\}$ and $N_2 = \{1, 2, 3, 5\}$. Clearly there are 11 such sets, $N_1, \ldots, N_{11}$. We need to ensure that at least one neighborhood is chosen from every such set to ensure that those adjacent neighborhoods (the ones in every set) are covered (serviced). That is

$$\sum_{j \in N_i} x_j \geq 1 \qquad \text{for each } i = 1, \ldots, 11.$$

These constraints are called *cover constraints*.

Objective and formulation

The objective is to minimize the number of fire stations located given by

$$\sum_{i=1}^{11} x_i.$$

Specifically, we can write these constraints as

$$\begin{array}{lccccccccccccc}
\min & x_1 & +x_2 & +x_3 & +x_4 & +x_5 & +x_6 & +x_7 & +x_8 & +x_9 & +x_{10} & +x_{11} & & \\
\text{s.t.} & x_1 & +x_2 & +x_3 & +x_4 & & & & & & & & \geq & 1 \\
2: & x_1 & +x_2 & +x_3 & & +x_5 & & & & & & & \geq & 1 \\
3: & x_1 & +x_2 & +x_3 & +x_4 & +x_5 & +x_6 & & & & & & \geq & 1 \\
4: & x_1 & & +x_3 & +x_4 & & +x_6 & +x_7 & & & & & \geq & 1 \\
5: & & x_2 & +x_3 & & +x_5 & +x_6 & & +x_8 & +x_9 & & & \geq & 1 \\
6: & & & x_3 & +x_4 & +x_5 & +x_6 & +x_7 & +x_8 & & & & \geq & 1 \\
7: & & & & x_4 & & +x_6 & +x_7 & +x_8 & & & & \geq & 1 \\
8: & & & & & x_5 & +x_6 & +x_7 & +x_8 & +x_9 & +x_{10} & & \geq & 1 \\
9: & & & & & x_5 & & & +x_8 & +x_9 & +x_{10} & +x_{11} & \geq & 1 \\
10: & & & & & & & & x_8 & +x_9 & +x_{10} & +x_{11} & \geq & 1 \\
11: & & & & & & & & & x_9 & +x_{10} & +x_{11} & \geq & 1
\end{array}$$

$$x_i \in \{0, 1\} \qquad \text{for all } i = 1, \ldots, 11.$$

Objective and formulation

One optimal solution to this problem is $x_3 = x_8 = x_9 = 1$ and the rest equal to 0.

This is an example of the set covering problem. The *set covering* problem is characterized by having binary variables, $\geq$ constraints each with a right hand side of 1, and having simply sums of variables as constraints. In general, the objective function can have any coefficients, though here it is of a particularly simple form.

Appendix

Stockco logical constraints

Logical condition	Logical constraint
Exactly 3 stocks are selected	$\sum_{i=1}^{6} x_i = 3$
If stock 2 is selected, then so is stock 1	$x_2 \leq x_1$
If stock 1 is selected then stock 3 is not selected	$x_1 + x_3 \leq 1$
Either stock 4 is selected or stock 5, but not both	$x_4 + x_5 = 1$

Extending our understanding of building logical constraints

Going back to the warehouse location problem discussed during building simple logical constraints. Suppose there are more than 2 locations and we represent the decision of locating a warehouse at location i with a binary variable x_i. Consider the following logical business conditions.

Logical condition	Logical constraint
At least k warehouses must be built	$\sum_{i=1}^{n} x_i \geq k$
Must build warehouse 3 if both 1 and 2 are built	$x_1 + x_2 \leq 1 + x_3$
Must build warehouse 3 if either 1 or 2 is built	$x_1 \leq x_3$ and $x_2 \leq x_3$
Must build 3 if either 1 or 2 is built but not both	$x_1 + x_3 \geq x_2$ and $x_2 + x_3 \geq x_1$
Cannot build warehouse 3 if both 1 and 2 are built	$x_1 + x_2 \leq 2 - x_3$

Knights, knaves, and werewolves formulation

Decision variables:

Let i index over the three inhabitants (basically, represent the i^{th} inhabitant) who were interviewed, i.e., $i = A, B, C$. Let us define our binary decision variables as follows:

$$x_i = \begin{cases} 1 & : \text{ if inhabitant } i \text{ is a knight,} \\ 0 & : \text{ if inhabitant } i \text{ is a knave.} \end{cases} \qquad y_i = \begin{cases} 1 & : \text{ if inhabitant } i \text{ is a werewolf,} \\ 0 & : \text{ if inhabitant } i \text{ is not a werewolf.} \end{cases}$$

There are 6 binary decision variables, i.e., x_A, x_B, x_C and y_A, y_B, y_C.

Constraints:

Inhabitant A said "I am werewolf". If A is a knight, the statement made must be true. In "English" this implies

$$\text{If } x_A = 1 \text{ then } y_A = 1, \text{ else if } x_A = 0 \text{ then } y_A = 0.$$

Notice this is an "equivalence" relationship between x_A and y_A. Thus, the constraint we develop is

$$x_A = y_A \text{ or } x_A - y_A = 0. \tag{8}$$

Knights, knaves, and werewolves formulation

Constraints:

A similar argument can be made for inhabitant B. The corresponding constraint is

$$x_B - y_B = 0. \tag{9}$$

Inhabitant C says "At most one of us is a knight". This can be expressed as

$$x_A + x_B + x_C \leq 1.$$

Now, to develop our constraints, let us express the constraint in "English". If C is a knight, then the statement he made must be true, i.e., if $x_C = 1$ then $x_A + x_B + x_C \leq 1$. Equivalently,

$$\text{If } x_C = 1 \text{ then } x_A + x_B \leq 0. \tag{10}$$

Of course, if $x_C = 0$ then the statement is false, which can be expressed as if $x_C = 0$ then $x_A + x_B + x_C > 1$. Which further implies, if $x_C = 0$ then $x_A + x_B \geq 2$ (because of the "strict" inequality). Equivalently,

$$\text{If } x_C = 0 \text{ then } x_A + x_B \geq 2. \tag{11}$$

Knights, knaves, and werewolves formulation

Constraints:

The constraints that model statements (10) and (11) respectively, can be written as follows:

$$x_A + x_B \leq 2\,(1 - x_C) \text{or}\, x_A + x_B + 2x_C \leq 2, \tag{12}$$

$$x_A + x_B \geq 2\,(1 - x_C) \text{or}\, x_A + x_B + 2x_C \geq 2. \tag{13}$$

It is important to realize that when the condition in (10) does not hold, i.e. $x_C \neq 1$, then constraint (12) is true trivially, i.e. it becomes redundant. That's because, $x_A$and x_B are binary variables and their sum can never exceed 2. This is important to note while developing constraints for such "if ... then ..." type of statements. Similarly (13) is redundant when the condition in (11) doesn't hold ($x_C \neq 0$). Both these constraints in tandem model C's statement.

An astute reader will observe that constraints (12) and (13) together imply

$$x_A + x_B + 2x_C = 2. \tag{14}$$

Knights, knaves, and werewolves formulation

Constraints:

What is also known is that exactly one of A, B, and C is a werewolf. This constraint can be modeled as

$$y_A + y_B + y_C = 1. \tag{15}$$

Objective function:

Notice that all we need is a classification – **which is given by the feasible region (defined by the constraints)**. We can use any objective. However, for example if we wanted find if there was a classification in which inhabitant A was a werewolf we could use

$$\max \quad y_A,$$

as an objective.

On the other hand, if we wanted to find if there existed a classification where A is a knight, we could use

$$\max \quad x_A.$$

Knights, knaves, and werewolves formulation

The binary variable formulation:

$$
\begin{aligned}
\max \quad & y_A \\
\text{s.t.} \quad & x_A - y_A = 0, \\
& x_A - y_A = 0, \\
& x_A + x_B + 2x_C = 2, \\
& y_A + y_B + y_C = 1, \\
& x_A,\, x_B,\, x_C,\, y_A,\, y_B,\, y_C \in \{0, 1\}\,.
\end{aligned}
$$

Notice that, if at optimality $y_A^\star = 0$, then we are guaranteed that there is no classification (with the current set of information that we have) in which A can be a werewolf. You can befriend him if you are confident about the information you have!

Constraints for piecewise linear functions

We know

$$0 \leq x_1 \leq 10, \qquad 0 \leq x_2 \leq 15, \qquad 0 \leq x_3 \leq 25.$$

To model "If $x_1 < 10$ then $x_2 = 0$" let us define a binary variable w_1 such that

$$w_1 = \begin{cases} 0 & : \text{ if } x_1 < 10, \\ 0/1 & : \text{ otherwise.} \end{cases}$$

Now we break the original "if ... then" statement into two "if ... then" statements using w_2. That is

$$\text{(a) If } x_1 < 10 \text{ then } w_1 = 0 \qquad \text{and} \qquad \text{(b) If } w_1 = 0 \text{ then } x_2 = 0.$$

(a) can be modeled as

$$w_1 \leq \frac{x_1}{10}, \tag{16}$$

and (b) can be modeled as

$$x_2 \leq 15 w_1. \tag{17}$$

Together, constraints (16) and (17) model the "If $x_1 < 10$ then $x_2 = 0$".

Constraints for piecewise linear functions

We know

$$0 \leq x_1 \leq 10, \qquad 0 \leq x_2 \leq 15, \qquad 0 \leq x_3 \leq 25.$$

To model "If $x_2 < 15$ then $x_3 = 0$" let us define a binary variable w_2 such that

$$w_2 = \begin{cases} 0 & : \text{ if } x_2 < 15, \\ 0/1 & : \text{ otherwise.} \end{cases}$$

Now we break the original "if ... then" statement into two "if ... then" statements using w_2. That is

(a) If $x_2 < 15$ then $w_2 = 0$ and (b) If $w_2 = 0$ then $x_3 = 0$.

(a) can be modeled as

$$w_2 \leq \frac{x_2}{15}, \tag{18}$$

and (b) can be modeled as

$$x_3 \leq 25 w_2. \tag{19}$$

Together, constraints (18) and (19) model the "If $x_2 < 15$ then $x_3 = 0$". Further, notice that constraints (16), (17), (18) and (19) automatically imply that "If $x_1 < 10$ then $x_3 = 0$".

Modeling piecewise linear concave functions

The linear costing constraints can be written as follows:

$$
\begin{aligned}
&10w_1 - x_1 \leq 0,\\
&15w_2 - x_2 \leq 0,\\
&x_2 - 15w_1 \leq 0,\\
&x_3 - 25w_2 \leq 0,\\
&0 \leq x_1 \leq 10, \qquad 0 \leq x_2 \leq 15, \qquad 0 \leq x_3 \leq 25,\\
&w_1,\ w_2 \in \{0,\ 1\}.
\end{aligned}
$$

The cost of $k = x_1 + x_2 + x_3$ commodity units can then be computed as

$$2000x_1 + 1800x_2 + 1700x_3.$$