Linear Optimization: Basic Concepts, Geometry, and Sensitivity Analysis

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Topics

Topics

Linear Programs

- Using Excel Solver to solve the LP model
- Building insights through LP geometry

Sensitivity analysis

Appendix

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Linear Programs

- Using Excel Solver to solve the LP model
- Building insights through LP geometry

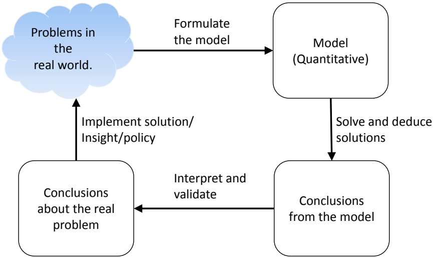
Sensitivity analysis

Appendix

Building models

- Tools for decision-making
  - Model helps select a policy / action to follow

  Decision-maker implements the policy
- Sources of insight
  - Models illuminate the relationship among many / complex variables
  - Decision-maker's understanding of the relationship informs his / her judgment
- Training tools
  - Decision-maker selects a policy / action
  - Model describes the outcome of following the selected policy / action
- Descriptors of philosophical concepts



Abstracting from a real world problem.

Optimization models/math programming/simulation

There are several types of mathematical models and solution procedures. A broad classification (not necessarily complete) is given below.

	Strategy evaluation	*Strategy generation*
Certainty	Deterministic simulation Econometric models Systems of simultaneous equations Input-output models	Linear programming Network models Integer and mixed-integer programming Nonlinear programming Control theory
Uncertainty	Monte Carlo simulation Econometric models Stochastic processes Queueing theory Reliability theory	Decision theory Dynamic programming Inventory theory Stochastic programming Stochastic control theory

Statistics and subjective assessment are used in all models to determine values for parameters of the models and limits on the alternatives.

Optimization models/math programming/simulation

The classification presented in the previous slide is not rigid, since strategy evaluation models are used for improving decisions by trying different alternatives until one is determined that appears "best." The important distinction of the proposed classification is that, for strategy evaluation models, the user must first choose and construct the alternative and then evaluate it with the aid of the model. For strategy generation models, the alternative is not completely determined by the user; rather, the class of alternatives is determined by establishing constraints on the decisions, and then an algorithmic procedure is used to automatically generate the "best" alternative within that class. The horizontal classification should be clear, and is introduced because the inclusion of uncertainty (or not) generally makes a substantial difference in the type and complexity of the techniques that are employed. Problems involving uncertainty are inherently more difficult to formulate well and to solve efficiently.

For a complete taxonomy please visit:
`https://neos-guide.org/content/optimization-taxonomy`

For a description of several real-world applications please visit:

`http://www.hsor.org/case_studies.cfm` or

`http://pubsonline.informs.org/journal/inte`

Deterministic, Constrained, Linear Optimization

Example: The custom molder's problem

Problem data:

- 100 cases of six-ounce juice glasses require six production hours.
- 100 cases of ten-ounce fancy cocktail glasses require five production hours
- 60 hours of production capacity available per week.
- Effective storage capacity of 15,000 cubic feet is available per week.
  - A case of six-ounce juice glasses requires 10 cubic feet of storage space,
  - A case of ten-ounce cocktail glasses requires 20 cubic feet.
- The contribution of the six-ounce glasses is $5.00 per case.
- The contribution of ten-ounce cocktail glasses is $4.50 per case.
- Customer available will not accept more than 800 cases per week of six-ounce glasses.
- There is no limit on the amount that can be sold of ten-ounce glasses.

How many cases of each type of glass should be produced per week in order to maximize the total contribution?

Building a linear optimization model

- What are the decision variables?
- What is the objective function?
- What are the constraints?

What makes an optimization model linear?

1. Proportionality:
   1.1 Contribution from x_1 is proportional to x_1 : max $\boxed{500}x_1 + \cdots$
   1.2 Consumption of production capacity (by x_1) is proportional to x_1 : $\boxed{6}x_1 + \cdots$
2. Additivity:
   2.1 For example, contribution to the objective function from x_1 is independent of x_2.
3. Divisibility:
   3.1 Each decision variable is allowed to take fractional values.
4. Certainty of data:
   4.1 Coefficients in the constraints and objective function are not "random variables". They are known with certainty.

Summary

1. Every deterministic, constrained, optimization problem must have three components.
   1.1 A set of decision variables,
   1.2 An Objective function,
   1.3 A set of constraints.
2. The type and number of decision variables, the objective function, and the set of constraints are determined from the business context.
3. If the objective function and constraints satisfy the conditions of proportionality, additivity, and divisibility, the model is linear program (LP).

Using Excel Solver for linear optimization

The Excel Solver model

Decision variables:

x1 Number of cases of six-ounce juice glasses produced per week (100 of cases per week)
x2 Number of cases of ten-ounce fancy cocktail glasses produced per week (100 of cases per week)

Problem Formulation

Objective: max 500 x1 + 450 x2

Constraints:
Production capacity: 6 x1 + 5 x2 <= 60
Storage capacity: 10 x1 + 20 x2 <= 150
Demand: x1 <= 8

VARS	x1	x2			Z
	0.00	0.00			
PROFIT	500	450			0
C1	6	5	0.000	<=	60
C2	10	20	0.000	<=	150
C3	1		0.000	<=	8

Objectivefunction:
SUMPRODUCT(G33:H33,G34:H

RHSvaluesof the constraints

LHSof the production constraint:
SUMPRODUCT(G36:H36,$G$33:$H$33)

LHSof the production constraint:
SUMPRODUCT(G36:H36,$G$33:$H$33)

LHSof the production constraint:
SUMPRODUCT(G36:H36,$G$33:$H$33)

Figure: The custom molder's LP problem in Excel solver.

Refer to the CustomMolder_LP.xls

The geometry of linear programming

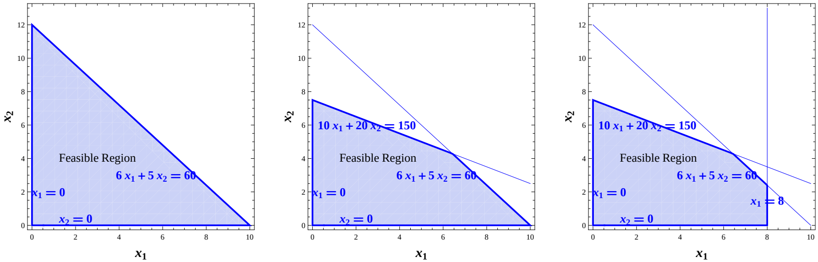
The LP feasible region

- The number of decision variables determines the problem space's dimensionality.
  - Two-variable problems can be plotted on a 2-D graph. Pick an axis for each variable.
  - The insights we will develop using a 2-D problem readily carry over to n-dimensional problems (only visualization remains a challenge).
- Note:
  - The set of constraints determine the feasible region.
  - The feasible region of the problem is the intersection of the feasible regions (also called as half-planes) of individual constraints (also called as hyper-planes).
    - If F_i is the feasible region of the i^{th} constraint and F is the feasible of the the entire problem (all constraints together), then $F = \bigcap_i F_i$.
  - The objective function has no role to play in determining the feasible region.

The LP feasible region (bounded)

Recollect the custom molder's problem:

$$\max \quad 500x_1 + 450x_2, \quad \text{s.t.} \quad 6x_1 + 5x_2 \le 60, \quad 10x_1 + 20x_2 \le 150, \quad x_1 \le 8, \quad x_1, x_2 \ge 0.$$



(a) Feasible region of the production and non-negative constraints.

(b) Feasible region of the storage and production constraint.

(c) The entire feasible region.

Figure: Bounded feasible region in 2-D.

Infeasibility and unboundedness

- Sometimes the constraint set can lead to an *infeasible* or *unbounded* feasible region.
  - An infeasible region implies the constraints are "contradictory" and hence the intersection set is empty.
  - An unbounded feasible region may mean that the solution could go off to $-\infty$ or $+\infty$ if the objective function "improves" in the direction in which the feasible region is unbounded.
- You may have to check your constraints to take care of such issues.

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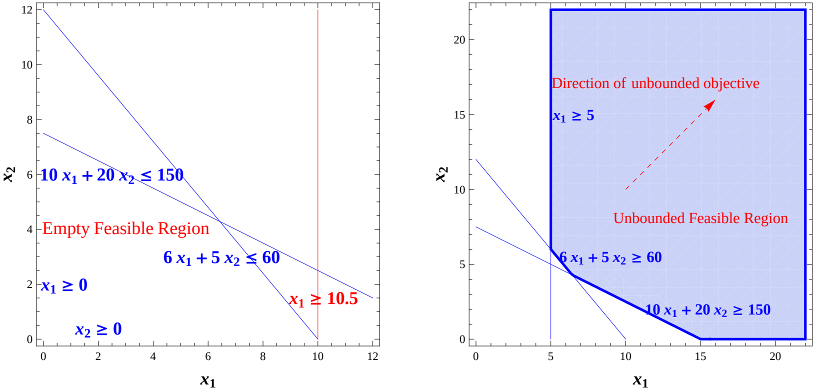
Infeasibility and unboundedness



(a) Feasible region is empty (Infeasible).   (b) Feasible region is unbounded.

Figure: Infeasible and unbounded feasible regions in 2-D.

In search of the optimal solution

LP feasible regions can either be unbounded, infeasible, or bounded. But, what about the optimal solution? Here's one way to find it in 2-D. We draw an "iso-profit" line over the feasible region and slide it "parallely" to itself in an "improving" direction, i.e., such that the objective value is "improving" (increasing in the case of the glass manufacturer).

Consider, our objective function $500x_1 + 450x_2$. Let us begin by drawing the first "iso-profit" line as $500x_1 + 450x_2 = \alpha$ (the dashed red line in the figure), where $\alpha = 1000$ and superimpose it over our feasible region. Notice that the intersection of this line with the feasible region provides all those production decisions that would result in a profit of exactly \$1000. As we increase α the profit increases (and the iso-profit line moves parallely to itself over the feasible region). This is the "improving direction" denoted by the dashed magenta colored line in the figure.

In search of the optimal solution

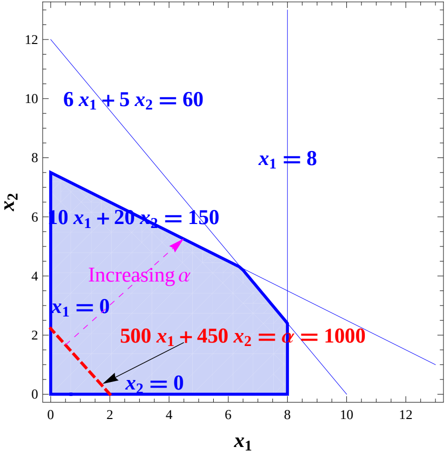


Figure: Drawing iso-profit lines over the feasible region.

Finding an optimal solution

An optimal solution is always found at corner point (vertex), if one exists.
A vertex or corner point is defined by intersecting constraints ("hyperplanes" defined by the constraints).

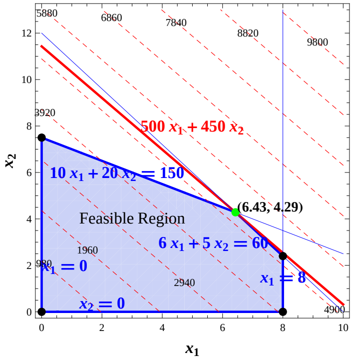


Figure: Sweeping the iso-profit line across the feasible region until it is about to exit. An optimal solution exists at a corner point (vertex).

A unique optimal solution

If the optimal vertex is uniquely determined by a set of intersecting constraints and the optimal solution exists at that unique vertex (corner point) then we have a unique optimal solution to our problem.

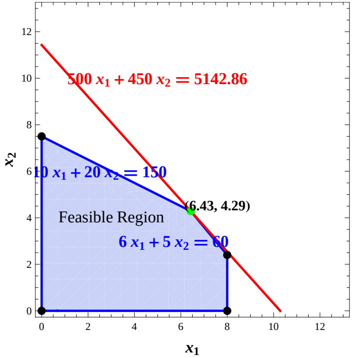


Figure: A unique optimal solution exists at a single corner point (vertex).

The case of multiple optimal solutions

Even with multiple optimal solutions, one will always exist at a corner point. Multiple optimal solutions are typically found when the objective function hyperplane is parallel to some “face” of the feasible region.

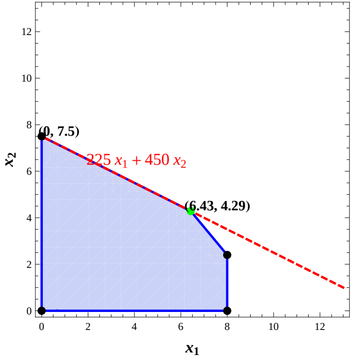


Figure: Multiple optimal solutions along a face (includes corner points).

Binding and nonbinding constraints

A constraint is binding if it passes through the optimal vertex, and non-binding if it does not pass through the optimal vertex. In our example the production constraint $6x_1 + 5x_2 \leq 60$ and storage capacity constraint $10x_1 + 20x_2 \leq 150$ are binding a optimality. In other words, if we substitute the optimal solution (production decisions) $x_1^\star = 6.43$ and $x_2^\star = 4.29$ in these constraints, they are satisfied at equality, i.e., $6 \times 6.43 + 5 \times 4.29 = 60$ and $10 \times 6.43 + 20 \times 4.29 = 150$. Equivalently, the optimal production decision $(x_1^\star, x_2^\star) = (6.43, 4.29)$ is the solution to the set of simultaneous equations $6x_1 + 5x_2 = 60$ and $10x_1 + 20x_2 = 150$. Sometime, this is referred to as: *at optimality the left-hand-side (lhs)* $=$ *right-hand-side (rhs) for a binding constraint*.

In economic terms, a binding constraint implies that the resource corresponding to that constraint is exhausted at optimality.

Slack and surplus for nonbinding constraints

A slack variable for non-binding "$\leq$" constraint is defined to be the difference between its rhs and the value of the lhs of the constraint evaluated at the optimal vertex (in the n-dimensional space). Consider our problem and the nonbinding demand constraint, $x_1 \leq 8$. At optimality note that $x_1^\star = 6.43$. Thus, the slack on this constraint,

$$s = \text{rhs} - \text{lhs evaluated at the optimal vertex} = 8 - (1 \times 6.43) = 1.57.$$

Similarly, the surplus associated with a non-binding "$\geq$" constraint is the extra value that may be reduced from the constraint's lhs function before the constraint becomes binding, i.e., the left-hand-side equals the rhs. In our problem we do not have constraint with a surplus. The formal definition of the surplus variable of a i^{th} "$\geq$" constraint is:

$$\text{surplus}_i = \sum_{j=1}^{n} a_{ij} x_j^\star - b_i,$$

where a_{ij} is the coefficient associated with the j^{th} decision variable in the i^{th} constraint.

Summary

- The number of decision variables determines the problem space's dimensionality.
  - Two-variable problems can be plotted on a 2-D graph – pick an axis for each variable.
- The constraints define the set of feasible solutions – each inequality defines a feasible half-plane.
  - The problem's feasible region is the intersection of the half-planes.
  - Vertices of the feasible region are determined (defined) by a subset of intersecting constraints (hyperplanes).
- A LP feasible region can be bounded, unbounded (in the direction in which the objective value improves), or infeasible.

Summary

- If an optimal solution exists it can always be found at a vertex (corner point). The objective function determines which vertex is optimal.
  - The constraints determining the optimal vertex are called binding constraints (no slack or surplus at optimality). Those that don't are called nonbinding constraints (they either have a slack or a surplus at optimality).
  - An optimal solution may be unique or there could be multiple optimal solutions.
  - In 2-D the finding an optimal solution is easy: (1) Calculate the vertices of the feasible region by solving the simultaneous equations of the constraints defining the vertex, (2) Calculate the objective value solution corresponding to each candidate vertex, and (3) Compare objective function values, to find the optimal solution and optimal vertex.

Excel answer report

Target Cell (Max)

Cell	Name	Original Value	Final Value
$K$34	PROFIT Z	0	5142.86

Adjustable Cells

Cell	Name	Original Value	Final Value
$G$33	Objective: max 500 x1 + 450 x2 X1	0.00	6.43
$H$33	Objective: max 500 x1 + 450 x2 X2	0.00	4.29

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$I$36	Production	60.000	$I$36<=$K$36	Binding	0
$I$37	Storage	150.000	$I$37<=$K$37	Binding	0
$I$38	Demand	6.429	$I$38<=$K$38	Not Binding	1.571428571

Figure: Answer report.

Understanding shadow prices, reduced costs, and sensitivity to perturbing the objective function

Sensitivity analysis

The idea of sensitivity analysis involves making one change at a time and studying the impact on the optimal solution.

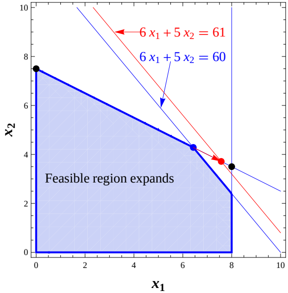
Suppose we want to answer the following questions:

- What is the maximum amount the custom molder should be willing to invest each week to increase the production time by one hour?
- How much should the custom molder charge a friend if he were to in-source jobs from her?

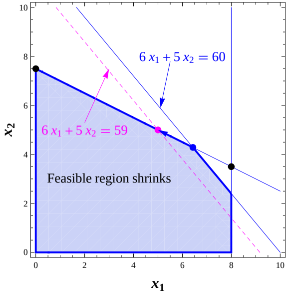
This is akin to changing the rhs of the production constraint by one unit. Keeping everything else the same, what would the impact on the optimal objective value?

Changing the rhs of a constraint

"Relaxing" ("Tightening") a constraint increases (decreases) the feasible and "improves" ("worsens") the optimal objective value. value.



(a) Feasible region expands (relaxing a constraint).



(b) Feasible region shrinks (tightening a constraint).

Figure: Changing the rhs of the production constraint. The optimal vertex "slides" along the other constraint (storage capacity constraint).

Changing the rhs of a constraint

The shadow price associated with a particular constraint is the change in the optimal value of the objective function per unit increase in the rhs value for the constraint, all other problem data remaining unchanged. Now, suppose we want to compute the shadow price of the production constraint. Let b_1 denote the rhs of the production constraint. Currently, $b_1 = 60$. Notice that the optimal objective value is 5142.86 when $b_1 = 60$. Let $Z^\star(b_1)$ denote the optimal objective value as a function of the rhs, i.e., b_1. Thus, $Z^\star(60) = 5142.86$.Keeping all other values in the LP unchanged, change b_1 to 61 and recompute the new optimal objective, $Z^\star(61) = 5221.43$. The shadow price of the production constraint is computed as follows:

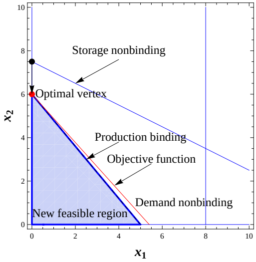
$$\begin{aligned}\text{Shadow price of the production constraint} &= \frac{Z^\star(61) - Z^\star(60)}{61 - 60} \\ &= \frac{5221.43 - 5142.86}{1} \\ &= 78.57.\end{aligned}$$

The range of a shadow price

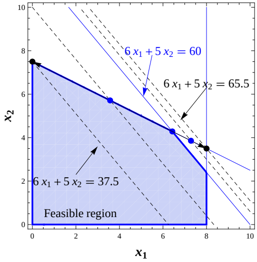
Shadow prices are only locally accurate. That is, shadow prices remain constant over a particular range of the rhs. The range is defined by interval over which the set of binding constraints does not change.

Specifically, the particular shadow price holds only within an allowable range of changes to the constraints rhs; outside of this allowable range the shadow price may change. This allowable range is composed of two components. The allowable increase is the amount by which the rhs may be increased before the shadow price can change; similarly, the allowable decrease is the corresponding reduction that may be applied to the rhs before a change in the shadow price can take place (whether this increase or decrease corresponds to a tightening or a relaxation of the constraint depends on the direction of the constraints inequality.)

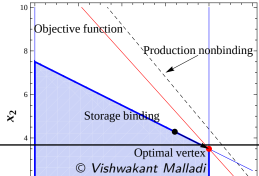
Determining the range of a shadow price



(a) Decreasing the rhs beyond the range.



(b) The range of the rhs for which shadow price remains constant.



The shadow price of a constraint

Notice that the shadow price is the rate at which the optimal objective changes with respect to the rhs of a particular constraint, all else remaining the same. It **should not** be interpreted as the absolute change in the optimal objective value. One possible economic interpretation of a shadow price is the imputed marginal value of the resource associated with that constraint.

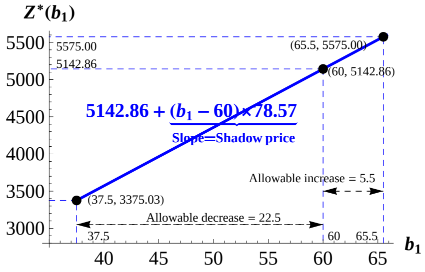


Figure: The optimal objective value as function of the rhs of the production constraint (Notice that the shadow price is the slope of the line segment).

Summary: Shadow price of a constraint

1. Shadow price of a constraint – defines the price you are willing to pay to "relax" the rhs by one unit, or the price you must be paid to "tighten" the rhs by one unit.
2. The shadow price of nonbinding constraint is always 0.
   2.1 Nonbinding constraints do not define the optimal vertex, i.e., they don't intersect at the optimal vertex.
   2.2 Changing the rhs of such constraints (or resource availability associated with these constraints) may not have any impact on the business until such constraints become binding.
3. A binding constraint has a non-zero shadow price.
   3.1 Binding constraints define the optimal vertex, i.e., they intersect at the optimal vertex.
   3.2 Changing the resource availability corresponding to these constraints will affect business decisions. One can choose how to invest in these resources by prioritizing their marginal values (shadow prices).
4. Shadow prices are valid over ranges of the rhs of the constraint for which the set of binding constraints remain the same.

Summary: Shadow price of a constraint

Slack (or surplus) on a constraint $\times$ shadow price of the constraint $= 0$.

The optimal objective value = Product of the rhs value of a constraint
$\times$ the shadow price of the constraint,
summed over all the constraints, i.e.,

$$\sum_{j=1}^{n} c_j x_j^\star = \sum_{i=1}^{m} b_i y_i^\star,$$

where $y_i^\star$ is the shadow price of the i^{th} constraint at optimality, b_i is the value of the rhs of constraint i, c_j is the objective coefficient of the j^{th} decision variable, and $x_j^\star$ is the optimal value of the j^{th} decision variable. For the custom molder's problem,

$$\sum_{j=1}^{n} c_j x_j^\star = (500 \times 6.429) + (450 \times 4.285) = 5142.8.$$

$$\sum_{i=1}^{m} b_i y_i^\star = (60 \times 78.571) + (150 \times 2.857) = 5142.8.$$

Excel sensitivity report

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$G$33	Objective: max 500 x1 + 450 x2 X1	6.43	0.00	500	40	275
$H$33	Objective: max 500 x1 + 450 x2 X2	4.29	0.00	450	550	33.33

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$I$36	Production	60.000	78.571	60	5.5	22.5
$I$37	Storage	150.000	2.857	150	90	22
$I$38	Demand	6.429	0.000	8	1E+30	1.57

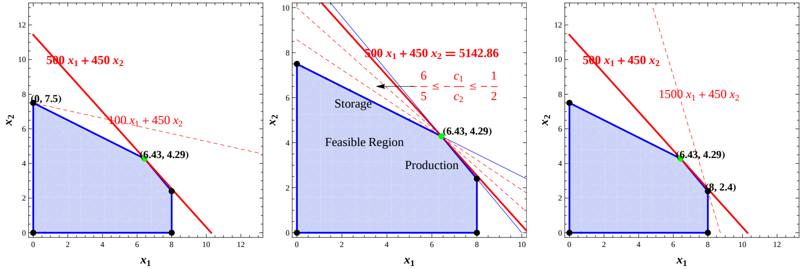
Figure: Sensitivity report (shadow prices and validity range).

Changing the objective coefficient of a variable

Suppose, we wanted to study the impact of changing the objective coefficient of six-ounce glasses, denoted by c_1. For example, what if the Marketing department were to re-estimate the contribution margin from six-ounce glasses after we have already computed the optimal mix? Note that currently the value of $c_1 = 500$. By changing c_1 to any other value, we are essentially changing the "slope" (gradient) of the objective function.

1. How long will the current optimal production decision $(x_1^\star, x_2^\star) = (6.43, 4.29)$ continue to remain optimal?
   1.1 What is the range for the values of c_1?
2. What happens when the slope (or the value of c_1) is changed beyond the range? Where does the optimal solution move?

Changing the objective coefficient of a variable



(a) When the slope is increased beyond the range.

(b) When the slope is within the range.

(c) When the slope is reduced beyond the range.

Figure: As long as the "slope" of the objective is within the slopes of the binding constraints the optimal objective solution remains the same.

$$\text{Slope capacity constraint} \le \text{Slope Obj} \le \text{Slope production constraint} \iff -\frac{6}{5} \le -\frac{c_1}{450} \le -\frac{1}{2}.$$

Summary: Perturbing the objective function

Note the following.

1. Perturbing the objective coefficient of a decision variable affects the slope (gradient) of the objective function.
   1.1 As long as the slope (gradient) of the perturbed objective function is "within" the the cone defined by the slopes (gradients) of the binding constraints the current optimal solution (defined by the optimal vertex) will continue to remain optimal.
   1.2 Every objective coefficient has an allowable increase and decrease associated with it such that if the corresponding objective function is changed within this range, the current optimal solution (vertex) will continue to remain optimal, i.e., the set of binding constraints will not change in this range.
2. If the objective coefficient of a decision variable is changed beyond the range, then the optimal vertex moves to an "adjacent" vertex defined by a different set of binding constraints (at least one constraint will be different from the original set).

Excel sensitivity report

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$G$33	Objective: max 500 x1 + 450 x2 X1	6.43	0.00	500	40	275
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Figure: Sensitivity report (changing the objective coefficient and its validity range).

Introducing a new product (variable)

Suppose we want to evaluate the option of introducing a new product in our mix. Say champagne glasses. Assume that

- The production time is 8 hours for 100 cases.
- The storage capacity is 1000 cubic feet per hundred cases.
- There is no limit on champagne glass demand, and
- The profit contribution is \$6.00 per case.

1. What is the inherent tradeoff in making such a decision?
2. Should we produce champagne glasses?

Introducing a new product (variable)

A naive approach to make this decision would be to introduce another variable x_3 in our LP model that represents the number of 100s of cases of Champagne glasses to produce. The new model is as follows:

$$\begin{aligned}
& \max && 500x_1 + 450x_2 + 600x_3 \\
\text{s.t.} \quad & \text{Production:} && 6x_1 + 5x_2 + 8x_3 \leq 60, \\
& \text{Storage:} && 10x_1 + 20x_2 + 10x_3 \leq 150, \\
& \text{Demand:} && x_1 \leq 8, \\
& \text{Non-negativity:} && x_1,\ x_2,\ x_3 \geq 0.
\end{aligned}$$

The marginal opportunity cost

What is the opportunity cost of producing champagne glasses?
With no additional production and storage resources available, the opportunity cost is due to diverting production and storage capacity (the binding constraints) from six-ounce and ten-ounce glasses. Thus, the implicit marginal opportunity cost (OC) to produce ε units (100s of cases) of champagne glasses is

$$\begin{aligned} \text{OC} &= \varepsilon \times [8 \times (\text{Shadow price of production}) + 10 \times (\text{Shadow price of storage})], \\ &= \varepsilon \times [8 \times 78.57 + 10 \times 2.86] = 657.16\,\varepsilon. \end{aligned}$$

In general, suppose we have m constraints (resources) and the shadow price associated with the i^{th} constraint is $y_i^\star$, $i = 1, \ldots, m$. Further suppose, decision variable x_j uses (consumes) a_{ij} amounts of resource i. Then the opportunity cost of producing a unit of product j is given by

$$\text{OC}_j = (a_{1j} \times y_1^\star) + (a_{2j} \times y_2^\star) + \cdots + (a_{mj} \times y_m^\star) = \sum_{i=1}^{m} a_{ij} y_i^\star.$$

Computing the opportunity cost

Consider the custom molder's LP problem. Suppose we wish to compute the opportunity cost of producing one unit of x_3 (i.e., champagne glasses). Here $j = 3$ and $m = 3$.

$$
\begin{array}{llllllllll}
 & & & & & \downarrow & & & & \text{SP at} \\
\text{max} & 500x_1 & + & 450x_2 & + & \boxed{600x_3} & & & & \text{optimality}\downarrow \\
\text{Production:} & 6x_1 & + & 5x_2 & + & \boxed{8}\,x & \leq & 60 & : & y_1^\star = \boxed{78.57} \\
\text{Storage:} & 10x_1 & + & 20x_2 & + & \boxed{10}\,x_3 & \leq & 150 & : & y_2^\star = \boxed{2.86} \\
\text{Demand:} & x_1 & & & & & \leq & 8 & : & y_3^\star = \boxed{0} \\
\text{Non-negativity:} & x_1 & & & & & \geq & 0 & & \\
 & & & x_2 & & & \geq & 0 & & \\
 & & & & & x_3 & \geq & 0 & &
\end{array}
$$

$$
\begin{aligned}
OC_3 = \sum_{i=1}^{m} a_{ij} y_i^\star &= (a_{13} \times y_1^\star) + (a_{23} \times y_2^\star) + (a_{33} \times y_3^\star), \\
&= (8 \times 78.57) + (10 \times 2.86) + (0 \times 0) = 657.16.
\end{aligned}
$$

Understanding the tradeoff

Notice that, while there is an opportunity cost associated with producing champagne glasses, the custom molder can makes \$600 per 100s of cases of champagne glasses produced. That's his tradeoff. Thus, the net contribution margin after producing ε units of 100s of cases of champagne glasses is

$$\begin{aligned}\text{Net marginal contribution} &= 600\varepsilon - [\mathrm{OC}_3\varepsilon]\,,\\ &= \varepsilon \times [600 - 657.16]\,,\\ &= -57.16\varepsilon.\end{aligned}$$

Notice that (i) 600 is the objective coefficient of x_3 (decision variable corresponding of champagne glasses), and (ii) the value of ε doesn't matter in deciding whether to produce or not, as long as it is nonnegative. The net contribution is always negative for any production amount. Thus, it is not profitable to produce champagne at this point in time. In fact, we can simply use the value of $\varepsilon = 1$ for all our computation (since we are only interested in the sign of the net contribution).

Pricing out a decision variable and its reduced cost

This operation of subtracting the OC from the objective coefficient is called pricing out a decision variable. In general for any decision variable j in the LP, with objective coefficient c_j and opportunity cost OC_jwe can price it out as $c_j - \mathrm{OC}_j$.

The value of the net marginal contribution is called the reduced cost. In other words, the operation of determining the reduced cost of a variable, j, from the shadow prices of the constraints and the objective function is referred to as pricing out the variable. Algebraically, the reduced cost $\bar{c}_j$ of the j^{th} decision variable in our LP is computed as (m constraints with shadow prices $y_1^\star, \ldots, y_m^\star$):

$$\begin{aligned}\bar{c}_j &= c_j - \mathrm{OC}_j,\\ &= c_j - [a_{1j} \times y_1^\star + a_{2j} \times y_2^\star + \cdots + a_{mj} \times y_m^\star]\\ &= c_j - \sum_{i=1}^{m} a_{ij} y_i^\star.\end{aligned}$$

Reduced costs exist for every decision variable in the problem.

Interpreting the reduced cost

The reduced cost of a decision variable indicates how much more "attractive" (i.e., higher in profit or lower in cost) its coefficient in the objective function must be before this variable is worth using. In our champagne glass example, unless the objective coefficient (profit per 100s of cases) of champagne glasses (which currently is at 600) improves (increases) by $57.16 producing champagne glasses is not a viable proposition for the custom molder.

Alternately, one could view the reduced cost of a variable as the shadow price of the non-negativity constraint associated with that variable, i.e., the corresponding change in the objective function per unit increase in the lower bound of the variable.

Basic and nonbasic variables at optimality

As noted earlier, each decision variable has a reduced cost associated with it. At optimality on a few decision variables, corresponding to the optimal vertex, take a non-zero value. These variables are called basic variables. The remaining decision variable are held at zero and they are called nonbasic variables. Note

1. At optimality all basic variables have a zero reduced cost (MR = MC).
   1.1 For example, in the custom molder's problem, the reduced cost of x_1 (six-ounce glasses) is zero at optimality. Notice, in this case $c_1 = 500,\ a_{11} = 6,\ a_{21} = 10,\ a_{31} = 1$.

$$\begin{aligned}\bar{c}_1 &= c_1 - \left(a_{11}y_1^\star + a_{21}y_2^\star + a_{31}y_3^\star\right), \\ &= 500 - (6 \times 78.57 + 10 \times 2.86 + 1 \times 0), \\ &= 0.\end{aligned}$$

2. All nonbasic variable have a nonzero reduced cost. For example, variable x_3 (champagne glasses).

Summary: Reduced costs

Note the following.

1. Every decision variable has a reduced cost associated with it, which essentially tells us by how much the corresponding objective function must change to make this variable "attractive", i.e., it'll enter the optimal solution with a nonzero value.

2. Basic variables, at optimality, have a zero reduced cost and nonbasic variables have a nonzero reduced cost.

3. The following is true at optimality:

$$\text{Reduced cost a variable} \times \text{the optimal value of the variable} = 0.$$

Excel sensitivity report

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$G$33	Objective: max 500 x1 + 450 x2 + 600 x3 X1	6.43	0.00	500	40	36.36
$H$33	Objective: max 500 x1 + 450 x2 + 600 x3 X2	4.29	0.00	450	200	33.33
$I$33	Objective: max 500 x1 + 450 x2 + 600 x3 X3	0.00	-57.14	600	57.14	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$J$36	Production	60.000	78.571	60	5.5	22.5
$J$37	Storage	150.000	2.857	150	90	22
$J$38	Demand	6.429	0.000	8	1E+30	1.57

Figure: Sensitivity report (changing the objective coefficient and its validity range).

Marginal value function of a constraint

Let us express the optimal objective value $\left(Z^\star\left(b_1\right)\right)$ as a function of the rhs value of the production constraint $\left(b_1\right)$. Note, the current value of $b_1 = 60$. We keep all other values the same, vary b_1 and plot $Z^\star\left(b_1\right)$ vs. b_1.

$$\begin{array}{lcccccc}
\boxed{Z^\star\left(b_1\right)} = \max & 500x_1 & + & 450x_2 & & \\
\text{Production} \rightarrow & 6x_1 & + & 5x_2 & \leq & \boxed{b_1 = 60} \\
\text{Storage:} & 10x_1 & + & 20x_2 & \leq & 150 \\
\text{Demand:} & x_1 & & & \leq & 8 \\
\text{Non-negativity:} & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}$$

Marginal value function of the production constraint

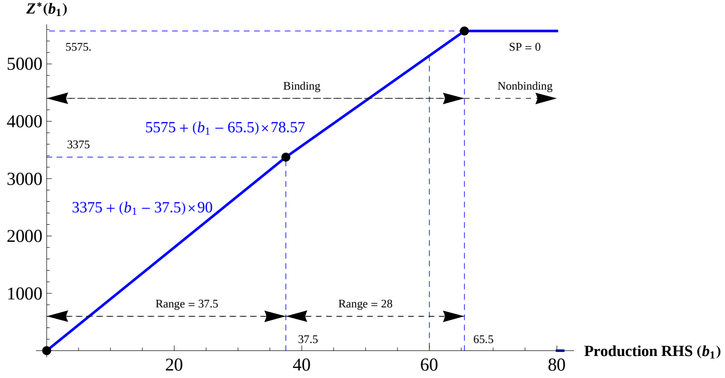

Figure: Piecewise linear marginal value curve of the production constraint.

What happens at the "inflection points"? Notice the slopes (shadow prices) to the right and left of the inflection points are different.

Quick note on detecting "degenerate" solutions

Suppose we are solving an LP where the available production capacity is exactly 37.5 units. In this case Excel Solver may come back with one of the following two solutions:

- At optimality the shadow price of the production constraint is 90 and the allowable increase on the rhs is 0.
- Or, at optimality the shadow price of the production constraint is 78.57 and the allowable decrease on the rhs is 0.

Both solutions are technically correct – however, interpreting them becomes a challenge because either the allowable increase or decrease is zero. Such a solution is called a degenerate solution - where one of the two values, allowable increase or allowable decrease, associated with the shadow price of a constraint is zero. In such a case at least one (or more) basic variable(s) is(are) zero valued at optimality (at optimality nonbasic variables are held at zero).

Quick note about detection of "degenerate" solutions

There are techniques to fix degeneracy in LPs so that determining managerial implications becomes possible. We will not study those in this course. All we need to worry about in this course is detection of a degenerate optimal solution. In your class notes (on LMS) I have shown what does degeneracy mean geometrically – only for those who may be interested a bit further.

Marginal cost function of a variable

Let us express the optimal objective value $(Z^\star(c_1))$ as a function of the objective coefficient of six-ounce glasses(x_1). Note, the current optimal value of $x_1 = 6.43$ 100s of cases. We keep all other values the same, vary c_1 and plot $Z^\star(c_1)$ vs. c_1.

$$
\begin{array}{lcccl}
\boxed{Z^\star(c_1)} = \max & \boxed{c_1 = 500}\,x_1 & + & 450x_2 & \\
\text{Production:} & 6x_1 & + & 5x_2 & \leq 60 \\
\text{Storage:} & 10x_1 & + & 20x_2 & \leq 150 \\
\text{Demand:} & x_1 & & & \leq 8 \\
\text{Non-negativity:} & x_1 & & & \geq 0 \\
 & & & x_2 & \geq 0
\end{array}
$$

Marginal cost function corresponding to variable x_1

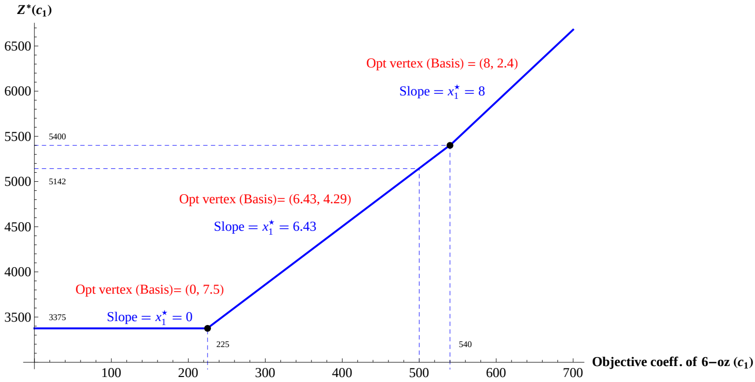

Figure: Piecewise linear marginal objective coefficient curve for six-ounce glasses.

What happens at the “inflection points”? What do the slopes represent?

Summary of sensitivity analysis

- Sensitivity analysis definitions
  - Shadow price, allowable increase and decrease
  - Objective function coefficients, allowable increase and decrease
  - Reduced cost
- Uses of sensitivity analysis
  - Testing the "robustness" of a model's solution
  - Answering "what if" questions concerning resources and products
  - Understanding the marginal values of resources and products
- Shadow prices exhibit decreasing returns to scale (always)
  - Restrict constraint RHS past allowable limit and shadow price increases
  - Relax constraint RHS past allowable limit and shadow price decreases

Formulations and solutions to problems discussed in class

Custom Molder's LP formulation

The LP formulation for the custom molder's problem is as follows. Let x_1 denote the number of 6-oz glasses (in 100s of cases) produced and x_2 denote the number 10-oz glasses (again, in 100s of cases) produced.

$$
\begin{array}{rll}
\max & 500x_1 + 450x_2 & \\
\text{s.t.} & 6x_1 + 5x_2 \leq 60 & : \text{Production constraint,} \\
 & 10x_1 + 20x_2 \leq 150 & : \text{Storage capacity constraint,} \\
 & x_1 \leq 8 & : \text{Demand for 6-oz glasses,} \\
 & x_1,\ x_2 \geq 0 & : \text{Non negativity constraint.}
\end{array}
$$