Homework: Linear and Integer optimization

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1 LP Modeling

1.1 Restaurant staffing

Consider a restaurant that is open 7 days a week. Based on past experience, the number of workers needed on a particular day is given as follows:

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Number	14	13	15	16	19	18	11

Every worker works five consecutive days and then takes off two days, repeating this pattern indefinitely. Our goal is to minimize the number of workers that staff the restaurant. Define your variables, constraints, and objective function clearly. Develop a Solver model and solve for the optimal staffing plan.

1.2 Managing a portfolio

We are going to manage an investment portfolio over a 6-year time horizon. We begin with $1,000,000, and at various times we can invest in one or more of the following:

(a) Savings account X, annual yield 5%,

(b) Security Y, 2-year maturity, total yield 12% if bought now, 11% thereafter,

(c) Security Z, 3-year maturity, total yield 18%, and

(d) Security W, 4-year maturity, total yield 24%.

To keep things simple we will assume that each security can be bought in any denomination. We can make savings deposits or withdrawals anytime. We can buy Security Y any year but year 3. We can buy Security Z anytime after the first year. Security W, now available, is a one-time opportunity. Write down a LP model to maximize the final investment yield. Assume all investments must mature on or before year 6 and you cannot sell securities in between. Define your decision variables and constraints clearly.

1.3 Riskless profit (Does arbitrage exist?)

A <u>European call option</u> is a contract with the following conditions: At a prescribed time in the future, known as the <u>expiration date</u>, the holder of the option has the right, but not the obligation to purchase a prescribed asset, known as the underlying asset/security, for a prescribed amount, known as the <u>strike price or exercise price</u>. For example, suppose an investor purchases a call option on stock XYZ with a $50 strike price. At expiration, say a month from the time of purchase, the spot price of stock XYZ is $75. In this case, the owner of the call option has the right to purchase the stock at $50 and exercises the option, making $25, or ($75 - $50), per share. However, in this scenario, if the spot price of stock XYZ is $30 at expiration, it does not make sense to exercise the option to purchase the stock at $50 when the same stock could be purchased in the spot market for $30. In this case, the payoff is $0. <u>Note the payoff and profit are different</u>. To calculate the profit from the option, the cost of the contract must be subtracted from the payoff. In this sense, the most an investor in the option can lose is the premium price paid for the option. In general, if S is the spot price of stock XYZ on the expiration date and K is the strike price of the European call option, then the call option payoff $= \max\{0, (S - K)\}$ and the profit $=$ payoff-option price.

Consider the following problem: You have $20,000 to invest. Stock XYZ sells at $20 per share today. A European call option to buy 100 shares of stock XYZ at $15 exactly six months from today sells for $1000. You can also raise additional funds which can be immediately invested, if desired, by *selling call options* with the above characteristics. In addition, a 6-month riskless zero-coupon bond with $100 face value (the amount you will make at the end of 6 months) sells for $90. You have decided to limit the number of call options that you *buy or sell* to at most 50.

You consider three scenarios for the price of stock XYZ six months from today: the price will be the same as today, the price will go up to $40, or drop to $12. Your best estimate is that <u>each of these scenarios is equally likely</u>.

1. Formulate and solve (in Solver) a linear program to determine the portfolio of stocks, bonds, and options that <u>maximizes expected profit</u>.

   (a) What happens to profitability if the price of XYZ goes to $40 per stock?

2. Suppose you want a profit of at least $2000 in any of the three scenarios. Write and solve (in Solver) a linear program that will maximize your expected profit under this additional constraint.

   (a) How does the solution compare to the earlier case described in (1).

3. <u>Riskless profit</u> is defined as the largest possible profit that a portfolio is guaranteed to earn, no matter which scenario occurs. Formulate and solve (in Solver) the model to determine the portfolio that maximizes riskless profit for the above three scenarios?

2 Interpreting the sensitivity report

Your friend's diet requires that all the food your friend eats come from one of the four "basic food groups" (chocolate cake, ice cream, soda, and cheesecake). At present, the following four foods are available for

consumption: brownies, chocolate ice cream, cola, and pineapple cheesecake. Each brownie costs $50, each scoop of chocolate ice cream costs $20, each bottle of cola costs $30, and each piece of pineapple cheesecake costs $80. Each day, your friend must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. The nutritional content per unit of each food is shown in Table (1) below. The variable names are indicated in parenthesis.

Item	Calories	Chocolate (oz)	Sugar (oz)	Fat (oz)
Brownie (BR)	400	3	2	2
Chocolate ice cream scoop (IX)	200	2	2	4
Cola (bottle) (COLA)	150	0	4	1
Pineapple cheese cake piece (PC)	500	0	4	5

Table 1: Nutritional content per unit of food item.

After solving the LP formulation the sensitivity report is given in Figure (2).

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B3	BR	0	27.5	50	1E+30	27.5
C3	IC	3	0	20	18.33333333	5
D3	COLA	1	0	30	10	30
E3	PC	0	50	80	1E+30	50

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
F6	Calories	750	0	500	250	1E+30
F7	Chocolate	6	2.5	6	4	2.857142857
F8	Sugar	10	7.5	10	1E+30	4
F9	Fat	13	0	8	5	1E+30

Figure 2: The sensitivity report.

Answer the following questions:

1. Formulate (need not input into Solver) a linear programming model that can be used to satisfy my daily nutritional requirements at minimum cost. Use the same variable and constraint names as indicated in the sensitivity report.

2. If a brownie costs $30, what would be the new optimal solution to the problem? How much will our costs change?

3. If a bottle of cola costs $35, what would be the new optimal solution to the problem?

4. If a bottle of cola cost $45, would the new optimal solution be the same or different? Can you calculate the new optimal solution easily just from looking at the sensitivity reports? Why or why not?

5. If at least 8 oz of chocolate were required, what would be the cost of the optimal diet?

6. If at least 600 calories were required, what would be the cost of the optimal diet?

7. If at least 9 oz of sugar were required, what would be the cost of the optimal diet?

8. What would the price of pineapple cheesecake have to be before it would be optimal to eat some cheesecake?

9. What would the cost of the optimal diet be if your friend had to eat at least one brownie?

10. If 10 oz of fat were required, would the optimal solution to the problem change? Why or why not?

11. Suppose your friend could also eat Cad bury chocolate eggs. Each egg costs $45 and contains 250 calories, 5 oz of chocolate, 5 oz of sugar, and 2 oz of fat. Should you add Cad bury eggs to your friend's diet? Why or why not?

3   Modeling Business Logic

Model the following situations exactly using binary/integer variables and linear constraints:

1. Suppose a consumer derives U_A units of utility from product A and U_B units from product B. Further suppose $U_A \neq U_B$ and $U_A > 0$ and $U_B > 0$. Model the following consumer choice constraint using binary variables: If both products are shown (presented) to the consumer, the consumer will not choose the product with lower utility (notice that the consumer may choose none of the products.)

2. A group of friends are browsing through the local video store, trying to decide which movies to rent. The friends, all ISB students, would like to plan their movie-watching schedule using integer programming. Write exactly one binary variable constraint to model the following statement: "If we rent both Bahubali II (B) and Dangal (D), then we can rent at most one of Drishyam (R), The Lunch Box (L), and Bhaag Milkha Bhaag (M)."

3. Suppose a broker must choose to invest in four investments 1, 2, 3, and 4. Let x_1, x_2, x_3, and x_4 denote the binary variables if she chooses the particular investment or not. Suppose she has the following constraints: If she invests in 3 or 4 or both then she must invest in exactly one of 1 or 2. Otherwise, if she invests in neither (of 3 or 4) then there are no constraints on investing in 1 or 2.