Math Bootcamp AMPBA

Lecture 3

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Inverse of a Matrix

Inverse of a matrix

- The inverse of a square matrix A, denoted as A^{-1} , such that
  - $AA^{-1} = I$
- A square matrix has an inverse if $\det(A) \neq 0$

Singular Matrix

- A square matrix is a singular matrix if it is not inversible
  - $\det(A) = 0$
- A square matrix is a non-singular matrix if it is inversible
  - $\det(A) \neq 0$

- What is the rank of a non-singular (n x n) matrix?

Singular Matrix

- A square matrix is a singular matrix if it is not inversible
  - $\det(A) = 0$
- A square matrix is a non-singular matrix if it is inversible
  - $\det(A) \neq 0$

- What is the rank of a non-singular (n x n) matrix?
  - Ans: n

Inverse of a (2 x 2) matrix – How?

- In a (2 x 2) matrix, for A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$
- det(A) = $a_{11}*a_{22} - a_{12}*a_{21}$
- Inverse of A, A^{-1} = 1/(det(A))* $\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

Inverse of a (2 x 2) matrix – Example

- Find the inverse of the matrix, A = $\begin{bmatrix} 2 & 3 \\ 7 & 5 \end{bmatrix}$?

- det(A) = a_{11}*a_{22} – a_{12}*a_{21} = 2*5 – 3*7 = -11

- Inverse of A, A^{-1} = 1/(det(A))* $\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

  = 1/(-11) * $\begin{bmatrix} 5 & -3 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} -5/11 & 3/11 \\ 7/11 & -2/11 \end{bmatrix}$

Inverse of a (3 x 3) matrix – How?

- In a (3 x 3) matrix, for A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

- In general, for an n x n matrix,

Inverse of A, A^{-1} = 1/(det(A))* $(Adjoint\ Matrix\ of\ A)$

Inverse of a (3 x 3) matrix – Example

- Find the inverse of the matrix, A = $\begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$?

- det(A) = 3*$\begin{vmatrix} -2 & 0 \\ 2 & -1 \end{vmatrix}$ $-\ 1*\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix}$ + (-1)*$\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix}$

  = 3(2 – 0) – 1(-2 – 0) -1(4 + 2) = 2

- Inverse of A, A^{-1} = 1/(det(A))* $[adj\ (A)]$

  = 1/(-11) * $\begin{bmatrix} 5 & -3 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} -5/11 & 3/11 \\ 7/11 & -2/11 \end{bmatrix}$

Inverse of a (3 x 3) matrix – Example

- Adj A = $(C_{ij})^T$ = $\begin{bmatrix} 2 & -1 & -2 \\ 2 & -2 & -2 \\ 6 & -5 & -8 \end{bmatrix}$
- Inverse of A, A^{-1} = 1/(det(A))* $[adj\ (A)]$

$$A^{-1} = (½)^* \begin{bmatrix} 2 & -1 & -2 \\ 2 & -2 & -2 \\ 6 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & -1 \\ 1 & -1 & -1 \\ 3 & -5/2 & -4 \end{bmatrix}$$

Inverse of a Matrix - Properties

- If Matrix A has an inverse B and also an inverse C, then
  - B = C

Inverse of a Matrix - Properties

Proof:

- Given, $AB = I$ & $CA = I$
- $C = C*I$

  $= C*(AB)$

  $= (C*A)*B = I*B$

  $= B$

So every square matrix A has an **unique Inverse**

Inverse of a Matrix - Properties

- Determinant of an inverse A^{-1}
  - $\det(A^{-1}) = \det(A)^{-1}$

Proof:

- I = AA-1
- $|I| = 1 = |AA^{-1}| = |A|\,|A^{-1}|$
- $\Rightarrow |A^{-1}| = |A|^{-1}$

Inverse of a Matrix - Properties

- Inverse of the transpose matrix is equal to transpose of the inverse matrix
  - $(A^T)^{-1} = (A^{-1})^T$

Inverse of a Matrix - Properties

- If a matrix is orthogonal, then $A^{-1} = A^T$

Proof:

- $AA^T = A^TA = I$ (By definition of orthogonal Matrix)
- Also, $AA^{-1} = I$ (By definition of Inverse Matrix)
- This implies,
    - $\mathbf{A^T = A^{-1}}$ in case of an **orthogonal matrix**

Inverse of a Matrix - Properties

- Inverse of Product of two square matrices A, B
  - $(AB)^{-1} = B^{-1}A^{-1}$

Inverse of a Matrix - Properties

Proof:

- $(AB)B^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$
- Similarly, $B^{-1}A^{-1}(AB) = B^{-1}IB = B^{-1}B = I$
- Hence,
  - $\mathbf{(AB)^{-1} = B^{-1}A^{-1}}$

Applications of Inverse of a matrix

- Consider a system of linear equations, $y = Ax$
- There exists a unique solution for $x = A^{-1}y$, only if
  - A is inversible i.e. A is a non-singular matrix

Applications of Inverse of a matrix

- In a simple linear model, $y = X\beta + \varepsilon$
    - Where y, ε are $n \times 1$ vectors,
    - X is a $n \times p$ design model matrix, β is $p \times 1$ vector of parameters

- In these cases, Inverses can be used in evaluating and finding maximum likelihood estimator of β, the vector of parameters

Eigen Values

Eigen Values

- If S is an n x n matrix, then the eigen values of S are roots of characteristic equation S,
    - $|S - \lambda I_n| = 0$

Eigen Values - Example

$S = \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$, for eigen values $|S - \lambda I_n| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 9 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1 - \lambda)^2 - 36 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 35 = 0$$

$$\Rightarrow (\lambda - 7)(\lambda + 5) = 0 \Rightarrow \lambda = 7, -5$$

Eigen Vectors

- If $A = S - \lambda I_n$, where λ denotes the eigen value of S
  - det(A) = 0, A is a singular matrix
  - A can be written as linear combination of rows/matrices
  - So there can be a vector x, $(x_1, x_2, x_3, \ldots x_n)'$ such that
    - $Ax = 0$ (Not all values of vector X are 0)
    - Why?

Eigen Vectors

- If $A = S - \lambda I_n$, where λ denotes the eigen value of S
  - det(A) = 0, A is a singular matrix
  - A can be written as linear combination of rows/matrices
  - So there can be a vector x, $(x_1, x_2, x_3, \ldots x_n)'$ such that
    - Ax = 0 (Not all values of vector X are 0)
    - Why?
    - Ans: Non-trivial solution exists, when det of a matrix = 0

Eigen Vectors

$$Ax = Sx – \lambda x = 0$$

$$\Rightarrow Sx = \lambda x \text{ (Eigen Equation)}$$

- x is called the Eigen Vector
- λ is called the Eigen Value
- The pair(x, λ) is called Eigen Pair
- For each value of λ, you get a corresponding eigen vector

Eigen Vector - Example

- For the previous example, S = $\begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$, find the Eigen Vectors?

Eigen Vector - Example

- S = $\begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$, λ = 7, -5
- For λ = 7, $\begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$ = 7* $\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$

=> $x_{11} + 4x_{12} = 7x_{11}$

$\Rightarrow 9x_{11} + x_{12} = 7x_{12}$

$\Rightarrow x_{11} = 2/3$ & $x_{12} = 1$

Eigen Vector - Example

- For λ = -5, $\begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$ = -5* $\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$

=> $x_{11} + 4x_{12} = -5x_{11}$

$\Rightarrow 9x_{11} + x_{12} = -5x_{12}$

$\Rightarrow x_{11} = -2/3$ & $x_{12} = 1$

Eigen Vector - Example

- For S = $\begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$
- The Eigen values = 7, -5
- The Eigen vectors = $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$

Properties of Eigenanalyses

- If x is an eigen vector of S, any scalar multiple kx is also an eigen vector
  - $S(kx) = \lambda kx$

Properties of Eigenanalyses

- If λ_i, λ_j are the distinct eigen values of S with eigen vectors x_i, x_j
    - Then x_i, x_j are distinct vectors

Properties of Eigenanalyses

Proof:

- Say $x_i = x_j = x$, we have $Sx = \lambda_i x$ & $Sx = \lambda_j x$

$$\Rightarrow (\lambda_i - \lambda_j)x = 0$$

- Since $\lambda_i \neq \lambda_j$; $x = 0$ contradicts x being a eigen vector
- Hence, $\mathbf{x_i \neq x_j}$

Properties of Eigenanalyses

- If x_i, x_j are the distinct eigenvectors of S with same eigen value λ,

linear combination of x_i and x_j is also an eigen vector of S

Properties of Eigenanalyses

Proof:

- Since, Sxi = λxi & Sxj = λxj ; Consider a, b are real scalars

  => $Sax_i = \lambda ax_i$ (Multiply with a) & $Sbx_j = \lambda bx_j$ (Multiply with b)

- Adding both equations,
- $S(ax_i + ax_j) = \lambda(ax_i + ax_j)$

Properties of Eigenanalyses

- If S is a symmetric matrix, eigen vectors corresponding to distinct eigen values are orthogonal, i.e. $x_i'x_j = 0$

Properties of Eigenanalyses

- $S = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, gives us, $\lambda = 1, 3$

- For $\lambda = 1$, $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 1 * \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$

=> $2x_{11} + 1x_{12} = x_{11}$

$\Rightarrow 1x_{11} + 2x_{12} = x_{12}$

$\Rightarrow x_{11} = -1$ & $x_{12} = 1$

Properties of Eigenanalyses

- For λ = 3, $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$ = 3* $\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$

=> $2x_{11} + 1x_{12} = 3x_{11}$

$\Rightarrow 1x_{11} + 2x_{12} = 3x_{12}$

$\Rightarrow x_{11} = 1$ & $x_{12} = 1$

$\Rightarrow \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ = [-1 + 1] = [0]

Properties of Eigenanalyses

- If x and λ are eigen pair of a non-singular matrix S, then
  - x is an eigen vector of S^{-1} with λ^{-1} as eigen value

Properties of Eigenanalyses

Proof:

- We have $Sx = \lambda x$

Multiplying both sides with S^{-1}

$S^{-1}Sx = S^{-1}\lambda x \qquad \Rightarrow \lambda^{-1}x = S^{-1}x$

$\Rightarrow S^{-1}x = \lambda^{-1}x$

- Hence, x is an eigen vector of $\mathbf{S^{-1}}$ with $\boldsymbol{\lambda^{-1}}$ as eigen value

Properties of Eigenanalyses - Example

Find the Eigen values and vectors for S & S^{-1} , where

$$S = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

Properties of Eigenanalyses - Example

For $S = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, for eigen values $|S - \lambda I_n| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3 - \lambda)(1 - \lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 1 = 0$$

$$\Rightarrow \lambda = 2 + \sqrt{3},\ 2 - \sqrt{3}$$

Properties of Eigenanalyses - Example

- $S = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, $\lambda = 2 + \sqrt{3}, 2 - \sqrt{3}$
- For $\lambda = 2+ \sqrt{3}$, $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = (2 + \sqrt{3})^* \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$

=> $x_{11} + x_{12} = (2+ \sqrt{3})x_{11}$

$\Rightarrow 2x_{11} + 3x_{12} = (2+ \sqrt{3})x_{12}$

$\Rightarrow x_{11} = (\sqrt{3} - 1)/2$ & $x_{12} = 1$

Properties of Eigenanalyses - Example

- S = $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, λ = 2 + √3, 2 - √3

- For λ = 2- √3 , $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$ = (2 - √3)* $\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$

=> $x_{11} + x_{12} = (2 - \sqrt{3})x_{11}$

$\Rightarrow 2x_{11} + 3x_{12} = (2 - \sqrt{3})x_{12}$

$\Rightarrow x_{11} = -(\sqrt{3} + 1)/2$ & $x_{12} = 1$

Properties of Eigenanalyses - Example

- For S = $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$
- The Eigen values = 2 + √3, 2 - √3
- The Eigen vectors = $\begin{bmatrix} -(\sqrt{3} + 1)/2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} (\sqrt{3} - 1)/2 \\ 1 \end{bmatrix}$

Properties of Eigenanalyses - Example

- For S = $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

Finding λ^{-1}

1. $1/(2+\sqrt{3}) = (2-\sqrt{3})/((2+\sqrt{3})*(2-\sqrt{3}))$

   $= (2-\sqrt{3})/(2^2-(\sqrt{3})^2) = (2-\sqrt{3})$

2. $1/(2-\sqrt{3}) = (2+\sqrt{3})/((2+\sqrt{3})*(2-\sqrt{3}))$

   $= (2+\sqrt{3})/(2^2-(\sqrt{3})^2) = (2+\sqrt{3})$

Properties of Eigenanalyses - Example

- For S = $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$
- det(S) = 1*3 – 1*2 = 1
- Inverse of S, S^{-1} = 1/(det(S))* $\begin{bmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{bmatrix}$

$S^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

Properties of Eigenanalyses - Example

For $S^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$, for eigen values $|S - \lambda I_n| = 0$

$$=> \begin{vmatrix} 3-\lambda & -1 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$=> (3 - \lambda)(1 - \lambda) - 2 = 0$$

$$=> \lambda^2 - 4\lambda + 1 = 0$$

$$=> \lambda = 2 + \sqrt{3}, 2 - \sqrt{3}$$

Properties of Eigenanalyses - Example

- $S^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$, $\lambda = 2 - \sqrt{3}, 2 + \sqrt{3}$
- For $\lambda = 2 + \sqrt{3}$, $\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = (2 + \sqrt{3})^* \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$

=> $3x_{11} - x_{12} = (2 + \sqrt{3})x_{11}$

$\Rightarrow -2x_{11} + x_{12} = (2 + \sqrt{3})x_{12}$

$\Rightarrow x_{11} = -(\sqrt{3} + 1)/2$ & $x_{12} = 1$

Properties of Eigenanalyses - Example

- $S^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$, $\lambda = 2 + \sqrt{3}, 2 - \sqrt{3}$

- For $\lambda = 2 - \sqrt{3}$, $\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = (2 - \sqrt{3})^* \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$

=> $3x_{11} - x_{12} = (2 - \sqrt{3})x_{11}$

$\Rightarrow -2x_{11} + x_{12} = (2 - \sqrt{3})x_{12}$

$\Rightarrow x_{11} = (\sqrt{3} - 1)/2$ & $x_{12} = 1$

Properties of Eigenanalyses - Example

- For $S^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$
- The Eigen values = $2 + \sqrt{3}$, $2 - \sqrt{3}$
- The Eigen vectors = $\begin{bmatrix} -(\sqrt{3}+1)/2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} (\sqrt{3}-1)/2 \\ 1 \end{bmatrix}$
- Clearly, S^{-1} has eigen value of λ^{-1} with x as eigen vector

Properties of Eigenanalyses

- If x and λ are eigen pair of a non-singular matrix S, then
  - x is an eigen vector of S^k with λ^k as eigen value

Properties of Eigenanalyses

Proof:

- We have $Sx = \lambda x$

Multiplying both sides with S^{k-1}

$S^k x = S^{k-1}(Sx) = S^{k-1}(\lambda x) = S^{k-2}(S\lambda x) = S^{k-2}(\lambda^2 x) \ldots = \lambda^k x$

- Hence, x is an eigen vector of $\mathbf{S}^k$ with $\boldsymbol{\lambda}^k$ as eigen value

Properties of Eigenanalyses - Example

Find the Eigen values and vectors for S & S^2 , where

$$S= \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

Properties of Eigenanalyses - Example

As showed in the previous example,

- The Eigen values = $2 + \sqrt{3}$, $2 - \sqrt{3}$
- The Eigen vectors = $\begin{bmatrix} -(\sqrt{3} + 1)/2 \\ 1 \end{bmatrix}, \begin{bmatrix} (\sqrt{3} - 1)/2 \\ 1 \end{bmatrix}$

Properties of Eigenanalyses - Example

- For S = $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

Finding λ^2

1. $(2+\sqrt{3})^2 = (2^2 + (\sqrt{3})^2 + 2*2*\sqrt{3})$

$= 7 + 4\sqrt{3}$

2. $(2-\sqrt{3})^2 = (2^2 + (\sqrt{3})^2 - 2*2*\sqrt{3})$

$= 7 - 4\sqrt{3}$

Properties of Eigenanalyses - Example

$S^2 = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+2 & 1+3 \\ 2+6 & 2+9 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 8 & 11 \end{bmatrix},$

Properties of Eigenanalyses - Example

For $S^2 = \begin{bmatrix} 3 & 4 \\ 8 & 11 \end{bmatrix}$, for eigen values $|S - \lambda I_n| = 0$

$$=> \begin{vmatrix} 3-\lambda & 4 \\ 8 & 11-\lambda \end{vmatrix} = 0$$

$$=> (3 - \lambda)(11 - \lambda) - 32 = 0$$

$$=> \lambda^2 - 14\lambda + 1 = 0$$

$$=> \lambda = 7 + 4\sqrt{3},\ 7 - 4\sqrt{3}$$

Properties of Eigenanalyses - Example

- $S^2 = \begin{bmatrix} 3 & 4 \\ 8 & 11 \end{bmatrix}$, λ = 7 + 4√3, 7 - 4√3
- For λ = 7 + 4√3 , $\begin{bmatrix} 3 & 4 \\ 8 & 11 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$ = (7 + 4√3)* $\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$

=> $3x_{11} + 4x_{12} = (7 + 4\sqrt{3})x_{11}$

$\Rightarrow 8x_{11} + 11x_{12} = (7 + 4\sqrt{3})x_{12}$

$\Rightarrow x_{11} = (\sqrt{3} - 1)/2$ & $x_{12} = 1$

Properties of Eigenanalyses - Example

- $S^2 = \begin{bmatrix} 3 & 4 \\ 8 & 11 \end{bmatrix}$, $\lambda = 7 + 4\sqrt{3},\ 7 - 4\sqrt{3}$
- For $\lambda = 7 - 4\sqrt{3}$, $\begin{bmatrix} 3 & 4 \\ 8 & 11 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = (7 - 4\sqrt{3})^* \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$

=> $3x_{11} + 4x_{12} = (7 - 4\sqrt{3})x_{11}$

$\Rightarrow 8x_{11} + 11x_{12} = (7 - 4\sqrt{3})x_{12}$

$\Rightarrow x_{11} = -(\sqrt{3} + 1)/2$ & $x_{12} = 1$

Properties of Eigenanalyses - Example

For S^2

- The Eigen values = $7 + 4\sqrt{3}$, $7 - 4\sqrt{3}$
- The Eigen vectors = $\begin{bmatrix} -(\sqrt{3}+1)/2 \\ 1 \end{bmatrix}, \begin{bmatrix} (\sqrt{3}-1)/2 \\ 1 \end{bmatrix}$
- Clearly, S^2 has eigen value of λ^2 with x as eigen vector

Properties of Eigenanalyses

- If x and λ are eigen pair of a non-singular matrix S, then
  - x and $(a - b\lambda)$ is an eigen pair of $aI_n - bS$

Properties of Eigenanalyses - Example

- Let's take S = $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, b = 1, a = 2

- This implies, S* = aI_n – bS = 2*$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ - 1* $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

S* = $\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}$

Properties of Eigenanalyses - Example

For $S^* = \begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}$, for eigen values $|S - \lambda I_n| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ -2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1 - \lambda)(1 - \lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 - 3 = 0$$

$$\Rightarrow \lambda = \sqrt{3}, -\sqrt{3}$$

Properties of Eigenanalyses - Example

- $S^* = \begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}$, $\lambda = \sqrt{3}, -\sqrt{3}$

- For $\lambda = \sqrt{3}$, $\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = (\sqrt{3})^* \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$

=> $x_{11} - x_{12} = (\sqrt{3})x_{11}$

$\Rightarrow$ $-2x_{11} - x_{12} = (\sqrt{3})x_{12}$

$\Rightarrow$ $x_{11} = -(\sqrt{3} + 1)/2$ & $x_{12} = 1$

Properties of Eigenanalyses - Example

- $S^* = \begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}$, $\lambda = \sqrt{3}, -\sqrt{3}$

- For $\lambda = -\sqrt{3}$, $\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = (-\sqrt{3})^* \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$

=> $x_{11} - x_{12} = (-\sqrt{3})x_{11}$

$\Rightarrow -2x_{11} - x_{12} = (-\sqrt{3})x_{12}$

$\Rightarrow x_{11} = (\sqrt{3} - 1)/2$ & $x_{12} = 1$

Properties of Eigenanalyses - Example

- For $S^* = \begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}$
- The Eigen values = $\sqrt{3}$, $-\sqrt{3}$
- The Eigen vectors = $\begin{bmatrix} -(\sqrt{3}+1)/2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} (\sqrt{3}-1)/2 \\ 1 \end{bmatrix}$
- Clearly, S^* has eigen value of $(a - b\lambda)$ with x as eigen vector

Properties of Eigenanalyses

- If S is a symmetric matrix, there exists an orthogonal matrix T and diagonal matrix D, such that
  - **T'ST = D**

- Rank of a Symmetric matrix (S) is equal to Number of **non-zero** eigen values

Properties of Eigenanalyses

- Let us take an example of S = $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, T = (1/√2)* $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
- T’ = (1/√2)* $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ (Check if TT’ = I)

Properties of Eigenanalyses

- Let us take an example of S = $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, T = (1/√2)* $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
- T'ST = (1/√2)* $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$* $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$* (1/√2)* $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

  = (1/2)* $\begin{bmatrix} 1+2 & 2+1 \\ -1+2 & -2+1 \end{bmatrix}$ * $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ = $\begin{bmatrix} 3 & 3 \\ 1 & -1 \end{bmatrix}$* (1/2)* $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

  = (1/2)* $\begin{bmatrix} 3+3 & -3+3 \\ 1-1 & -1-1 \end{bmatrix}$ = $\begin{bmatrix} \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{bmatrix}$ **= D**

Properties of Eigenanalyses - Example

For S= $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, for eigen values $|S - \lambda I_n| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1 - \lambda)(1 - \lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$$

$\Rightarrow \lambda = -1, 3$ **(Two non-zero λ)**

Properties of Eigenanalyses - Example

Rank of a square matrix, when det S = $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \neq 0$ (-3, here) is always n, i.e. 2

Hence, Rank of a Symmetric matrix (S) is equal to Number of **non-zero** eigen values if **T'ST = D**

Properties of Eigenanalyses

- If D is a diagonal or triangular matrix, then the eigen values are the diagonal elements

Properties – Eigenanalyses

$S = \begin{bmatrix} 3 & 1 \\ 0 & -3 \end{bmatrix}$, for eigen values $|S - \lambda I_n| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ 0 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3 - \lambda)(3 + \lambda) - 0 = 0$$

$$\Rightarrow (\lambda - 3)(\lambda + 3) = 0 \Rightarrow \boldsymbol{\lambda = 3, -3}$$

The diagonal elements are the eigen values of triangular matrix

Positive Definite Matrices

- A matrix A is said to be positive definite if all the eigen values of the matrix are positive.

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Properties – Positive Definite Matrix

A = $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, for eigen values $|S - \lambda I_n| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2 - \lambda)^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0 \Rightarrow \boldsymbol{\lambda = 3, 1}$$

The matrix A is a positive definite matrix

Positive Definite Matrices

- A matrix A is said to be positive definite if all the eigen values of the matrix are positive.

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Matrix Exponential

- If X is a n x n square matrix, then the exponential of X is

$$e^X = 1 + \frac{X}{1!} + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots = \sum_r^{\infty} \frac{X^r}{r!}$$

Matrix Exponential

- If X has distinct eigen values, then $X = T\Lambda T^{-1}$
  - Λ – Diagonal matrix with eigen values of X

$$\exp(X) = \sum_{r=1}^{\infty} \frac{X^r}{r!} = T \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{pmatrix} T^{-1}.$$