Math Bootcamp AMPBA

Lecture 1

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Today’s agenda

- About this course.
- How to get most out of this course?
- Student interaction – to set expectations and understanding starting knowledge.
- Vectors and Matrices.

Introduction

- This course introduces you to linear algebra and matrix algebra.
- This will help you to understand the math behind most of the econometrics, machine and deep learning models.
- Ideas/Concepts covered in these series of lectures are mathematical in nature.
- I will provide some applications/real world connections of these ideas.

How to get most out of this course?

- Ideas grasping through writing.
- Practice.
- Asking questions on LMS/email.
- Knowing that, this is a starting point. You might not be able to appreciate all the ideas now. It’s okay, it takes time.

Student Interaction

- 20 mins session.
- What do students know? Covering extremes. (10-15 mins)
- R session. (5-10 mins)

What is a vector?

- A vector is collection of numbers.

- A vector x of order p (or dimension p) is a column of p numbers:

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}.$$

Vector

- We can refer x as column vector.
- The transpose of x, $x^T = (x_1, x_2, \ldots., x_p)$ is a row vector.
- Addition and subtraction of vectors of the same order is performed element by element

$$\boldsymbol{x} + \boldsymbol{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_p + y_p \end{pmatrix}.$$

Vector

- It is not possible to add or subtract vectors which are not conformable, i.e., which do not have the same dimension or order.

- Scalar multiplication of a vector is element by element:

$$\lambda \boldsymbol{x} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_p \end{pmatrix}$$

Vector

- Two vectors x and y are equal if they are of the same order and each pair of corresponding elements is equal, i.e., $x_i = y_i$
- A vector with all elements 0 is denoted by ***0***
- A vector e_i with i^{th} element 1 and all others 0 is the i^{th} unit vector.
- A vector with all elements 1 is and is denoted by ***Ip***

Vector

- Sum vector. $X^T\boldsymbol{l}_\mathbf{p} = \sum_{i=1}^{p} x_i$
- x+y = y+x (commutativity) and (x+y)+z = x+(y+z) (associativity).
- Inner Product x^Ty
- Example: $x=(1,2,3)^T$ and $y=(4,5,6)^T$. Find x^Ty

Vector

- Calculate $x^T x$, where x is a p dimensional vector?

Summary

- Vector definition.
- Row v/s column vector.
- Vector dimension.
- Vector Operations.
- Types of vectors.
- Inner product.

Matrices

- An mxn matrix X is a rectangular array of scalar numbers:

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix}.$$

- This matrix has m rows and n columns; it is an mn matrix (m by n matrix), it is a matrix of order mn. X has dimensions m and n.

Matrices

- The numbers x_{ij} are the components (or elements) of X.
- A matrix is a square matrix if m = n, i.e.,if it has the same number of rows and columns, i.e.,a nxn matrix is a square matrix.
- The transpose of the mxn matrix X is the nxm matrix X^T

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix}.$$

Matrices

- A square matrix X is **symmetric** if $X^T = X$. Two matrices X and Y are **equal** if they are of the same order and each pair of corresponding elements are equal, i.e.,$x_{ij} = y_{ij}$, for i = 1;2; … ;m and j = 1;2; … ;n.

Matrices

- A square matrix with all elements not on the diagonal equal to 0 is a diagonal matrix, i.e.,if x_{ij} = 0 for all i!= j
- A diagonal matrix with all diagonal elements 1 (and all others not on the diagonal 0) is denoted by $\mathbf{I_n}$, called Identity Matrix.

$$\mathbf{I_n} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Matrices

- The trace of a square matrix is the sum of all the diagonal elements of the matrix, i.e.,trace (X) = tr(X) = tr(x_{ij}) =$\sum_{i=1}^{n} x_{ii}$. Note that tr($\mathbf{I}_n$) = n.

Matrix Arithmetic

- **X+Y =** $x_{ij} + y_{ij}$

- X + Y = Y + X (commutativity) and (X + Y) + Z = X + (Y + Z) (associativity).

- Scalar multiplication of a matrix is element by element:

$$\lambda \boldsymbol{X} = \lambda(x_{ij}) = (\lambda x_{ij})$$

Matrix Arithmetic

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Matrix Multiplication

- If A and B are matrices then we can multiply A by B (to get AB) only if the number of columns of A equals the number of rows of B.
- If C is mxn and D is pxq then the product CD can only be defined if n = p, in which case C and D are said to be **conformable**. If C and D are such that CD is not defined then they are **non-conformable**.

Matrix Multiplication

- Example

If $\boldsymbol{U} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \boldsymbol{V} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ then $\boldsymbol{U}$ is 2×2 and $\boldsymbol{V}$ is 2×2 so $\boldsymbol{UV}$ is $2 \times 2 \times 2 \times 2 \equiv 2 \times 2$ and $\boldsymbol{VU}$ is $2 \times 2 \times 2 \times 2 \equiv 2 \times 2$.

Matrix Multiplication

$$
\begin{aligned}
\boldsymbol{UV} &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \\
&= \begin{pmatrix} 1 \times 5 + 2 \times 7 & 1 \times 6 + 2 \times 8 \\ 3 \times 5 + 4 \times 7 & 3 \times 6 + 4 \times 8 \end{pmatrix} \\
&= \begin{pmatrix} 5 + 14 & 6 + 16 \\ 15 + 28 & 18 + 32 \end{pmatrix} \\
&= \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}
\end{aligned}
$$

Matrix Multiplication

- Now calculate **VU.**
- Is **UV=VU ?**

Matrix Multiplication

If $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

Matrix Multiplication

$$\begin{aligned}
\boldsymbol{AB} &= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 1\times 1+2\times 3+3\times 5 & 1\times 2+2\times 4+3\times 6 \\ 4\times 1+5\times 3+6\times 5 & 4\times 2+5\times 4+6\times 6 \end{pmatrix} = \begin{pmatrix} 22 & 28 \\ 49 & 64 \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{BA} &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 1\times 1+2\times 4 & 1\times 2+2\times 5 & 1\times 3+2\times 6 \\ 3\times 1+4\times 4 & 3\times 2+4\times 5 & 3\times 3+4\times 6 \\ 5\times 1+6\times 4 & 5\times 2+6\times 5 & 5\times 3+6\times 6 \end{pmatrix} = \begin{pmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \\ 29 & 40 & 51 \end{pmatrix}.
\end{aligned}$$

Matrix Multiplication

- Is the following true?

- $\mathbf{(A+B)^2=A^2+B^2+2AB}$

- If false, when will it be true?

Transpose and Trace of Sums and Products

If $\boldsymbol{A}+\boldsymbol{B}$ is defined (i.e., $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same orders) then $(\boldsymbol{A}+\boldsymbol{B})' = \boldsymbol{A}'+\boldsymbol{B}'$ and $\operatorname{tr}(\boldsymbol{A}+\boldsymbol{B})$=$\operatorname{tr}(\boldsymbol{A})$+$\operatorname{tr}(\boldsymbol{B})$. If $\boldsymbol{A}$ is $m\times n$ and $\boldsymbol{B}$ is $n\times p$ then $\boldsymbol{AB}$ is $m\times n\times n\times p \equiv m\times p$ so $(\boldsymbol{AB})'$ is $p\times m$. It is easy to show that $(\boldsymbol{AB})' = \boldsymbol{B}'\boldsymbol{A}'$:

$$(\boldsymbol{AB})' = ((a_{ij})(b_{jk}))' = \left(\sum_{j=1}^{n} a_{ij}b_{jk}\right)' = \left(\sum_{j=1}^{n} a_{kj}b_{ji}\right) = (b_{jk})'(a_{ij})' = \boldsymbol{B}'\boldsymbol{A}'.$$

Note that $\boldsymbol{A}'$ is $n\times m$ and $\boldsymbol{B}'$ is $p\times n$ so the product $\boldsymbol{A}'\boldsymbol{B}'$ is not defined (unless $p=m$) but $\boldsymbol{B}'\boldsymbol{A}'$ is defined.

Clearly $\operatorname{tr}(\boldsymbol{A}) = \operatorname{tr}(\boldsymbol{A}')$ and if $\boldsymbol{A}$ and $\boldsymbol{B}$ are $m\times n$ and $n\times m$ matrices (so that both $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are defined) then $\operatorname{tr}(\boldsymbol{AB}) = \operatorname{tr}(\boldsymbol{BA})$ because $\operatorname{tr}(\boldsymbol{AB}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}b_{ij} = \operatorname{tr}(\boldsymbol{BA})$.

If λ is any scalar then $(\lambda\boldsymbol{A})' = \lambda\boldsymbol{A}'$.

Symmetric and skew-symmetric matrices

A square $n \times n$ matrix $\boldsymbol{A} = (a_{ij})$ is ***symmetric*** if $\boldsymbol{A}' = \boldsymbol{A}$, i.e., $a_{ij} = a_{ji}$ for all i, j. A square matrix $\boldsymbol{B} = (b_{ij})$ is ***skew-symmetric*** if $\boldsymbol{B}' = -\boldsymbol{B}$, i.e.,$b_{ij} = -b_{ji}$ for all i, j. It is easy to see that all skew-symmetric matrices have zero elements on the diagonals.

Symmetric and skew-symmetric matrices

Any square matrix $\boldsymbol{X}$ can be expressed as $\boldsymbol{X} = \frac{1}{2}(\boldsymbol{X}+\boldsymbol{X}') + \frac{1}{2}(\boldsymbol{X}-\boldsymbol{X}')$. Since $(\boldsymbol{X}+\boldsymbol{X}')' = (\boldsymbol{X}+\boldsymbol{X}')$ and $(\boldsymbol{X}-\boldsymbol{X}')' = -(\boldsymbol{X}-\boldsymbol{X}')$ we have that any square matrix can be expressed as the sum of a *symmetric part* and a *skew-symmetric part*.

Symmetric and skew-symmetric matrices

- Are A and B symmetric?

- Is **AB** symmetric?

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Orthogonal

- A pxp square matrix A is orthogonal if $A'A = AA' = \mathbf{I}_p$
- If A and B are both orthogonal and both pxp then AB is orthogonal. **Why?**
- An orthogonal matrix whose elements are all +1 or -1 is known as a **Hadamard matrix**.

Orthogonal

- Are A and B orthogonal matrices?

$$\boldsymbol{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Other names

- A pxp matrix is **normal** if AA' = A'A
- A pxp matrix A is **idempotent** if $A^2 = A$
- A matrix A != 0 is said to be **nilpotent** if $A^2 = 0$
- A nxn matrix A such that $A^2 = \mathbf{I_n}$ is said to be **unipotent**.

- Example: A=$\begin{bmatrix} 1 & x \\ 0 & -1 \end{bmatrix}$, which of the above is A?

R exercises