Composite Function:

$f(x) = x^2 \quad g(x) = \sin(x)$

$(f \circ g)(x) = f(g(x)) = (g(x))^2$
$= (\sin(x))^2 = \sin^2 x$

$x \in D_g = R$ Range of $g(a) = [-1, 1] =$ Domain of (x) $f(g(x)$

Domain of g

$(g \circ f)(x) = \sin^2 x$

$\sin(x)^2$ and $\sin^2 x$ are not similar because
$\sin x^2 = \sin(x^2)$
$\sin^2 x = (\sin x)^2$

Question:

$f(x) = \sqrt{3x^2 - 2x + 1}$

$h(x) = 3x^2 - 2x + 1$

$g(u) = \sqrt{u}$

$g(h(x)) = \sqrt{3x^2 - 2x + 1}$

Question:

$f(x) = e^{\sqrt{(x+1)^2}}$

$h(x) = (x+1)$

$g(u) = u^2$

$k(v) = \sqrt{v}$

$e(m) = e^m$

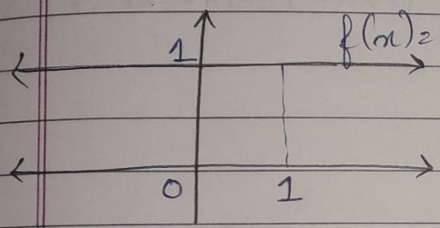
$e(k(g(h(x)))) = e^{\sqrt{(x+1)^2}}$

Question:

$$f(x) = \sin(\sqrt{x+1})$$
$$h(u) = \sqrt{u}$$
$$g(x) = x+1$$
$$i(v) = \sin(v)$$

$$i(h(g(x))) = \sin(\sqrt{x+1})$$

Limits:



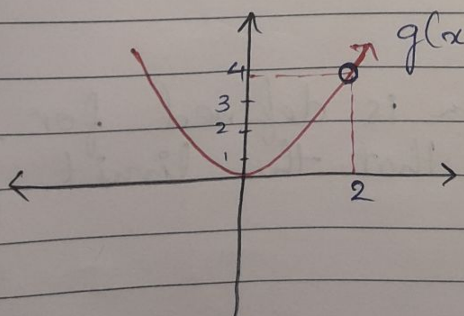
$f(x) = 1, \forall x \in R$

$$\lim_{x\to 1} f(x) = 1.$$

$$\lim_{x\to 1} f(x) = \frac{x-1}{x-1} = \text{not defined}$$

e.g.

$$g(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$$



$g(x) = x^2$

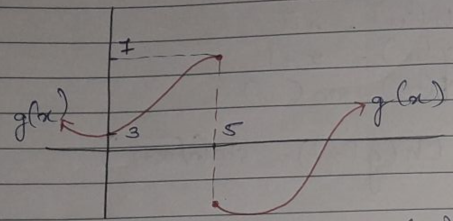
$$\lim_{x\to 2} g(x) = 4.$$

→ it will be not 2, but near to 2 from left and right.

x	$g(x)$	x	$g(x)$
1.9	3.6	2.1	4.41
1.99	3.96	2.01	4.04
1.999	3.996	2.001	4.004

an $x = 2$, $g(x)$ is defined that is 1.

e.g.



$\lim_{x \to 0} g(x) = 3$ $\quad$ $\lim_{x \to 5^+} g(x) = -2$.

$\lim_{x \to 5^-} g(x) = 7$ $\quad$ Here, limit doesn't exist.

Characteristic :

1) It is possible for the function value to be different from limit value.

2) Just because the function is undefined for some x value doesn't mean the limit doesn't exist.

3) Even if the function is defined for some x values doesn't mean that the limit necessarily exist.

e.g. $f(x) = \dfrac{x-2}{x^2-4}$

$\lim_{x \to 2^-} = \dfrac{1.999 - 2}{(1.999)^2 - 4} = -2.501$

$\lim_{x \to 2^+} = \dfrac{2.001 - 2}{(2.001)^2 - 4} = -2.498$

x	$f(x)$
1.9	
1.99	
1.999	-2.501
2.001	-2.498
2.01	
2.1	

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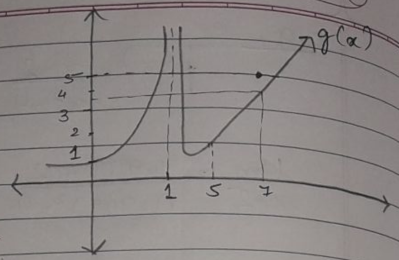
e.g.

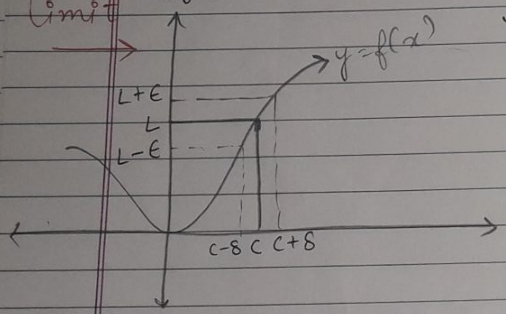
$\lim_{x\to 1} g(x) = \text{unbounded}$ 1 pe ∞ aya kyoki graph me koi line touch nahi ho raha

$\lim_{x\to 5} g(x) = 1$ 5 par x-axis par 1 touch ho raha he

$\lim_{x\to 7} g(x) = 3.5$ 7 x-axis par 3.5 par touch ho raha he.



Limit



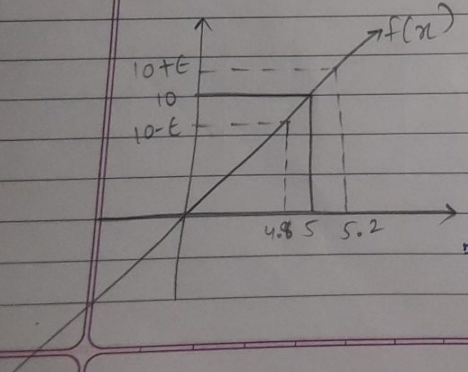
If $\lim_{x\to c} f(x) = L$ exist

Given $\epsilon > 0$

we can find $\delta > 0$ such that

$|x-c| < \delta \Rightarrow |f(x) - L| < \epsilon$

e.g. $f(x) = \begin{cases} 2x, & x \neq 5 \\ 5, & x = 5 \end{cases}$



$\epsilon = 0.5$, $\delta = 0.2$

$\lim_{x\to 5} f(x) = 10$

$\Rightarrow |f(x) - L| < 0.5$

$f(4.82) = 2(4.82) = 9.64$

$\Rightarrow |f(x) - L| = |9.64 - 10|$

$= 0.36 < \epsilon$ i.e. 0.5

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Properties of limits:

Given $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$

1. $\lim_{x\to c} [f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
   $= L + M$

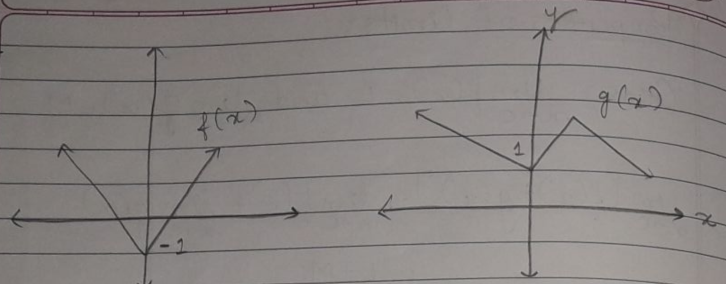
2. $\lim_{x\to c} [f(x) - g(x)] = L - M$

3. $\lim_{x\to c} [f(x) \cdot g(x)] = L \cdot M$

4. $\lim_{x\to c} \left[\dfrac{f(x)}{g(x)}\right] = L/M$ where $g(x) \neq 0$

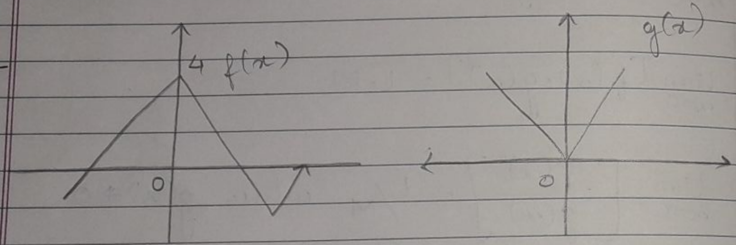
5. $\lim_{x\to c} k \cdot f(x) = k \cdot \lim_{x\to c} f(x) = k \cdot L$

Q.



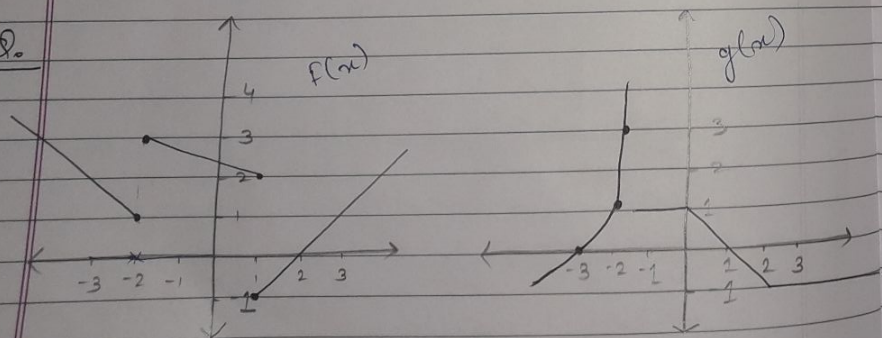
$$\lim_{x\to 0} f(x).g(x) = -1\times 1 = -1$$

Q.



$$\lim_{x\to 0}\left[\frac{f(x)}{g(x)}\right] = \frac{4}{0} = \text{does not exist.}$$

Q.



$\Rightarrow \lim_{x\to -2} [f(x)+g(x)] =$

$\lim_{x\to 2^-} f(x) = 1$ $\quad\lim_{x\to 2^+} f(x) = 3$

$\lim_{x\to 2^-} g(x) = 3$ $\quad\lim_{x\to 2^+} g(x) = 1$

$\therefore \lim_{x\to -2} [f(x)+g(x)] = 4$

$\Rightarrow \lim_{x\to 1} [f(x)+g(x)]$

$\lim_{x\to 1^-} f(x) = 2$ $\quad\lim_{x\to 1^+} f(x) = 1$

$\lim_{x\to 1^-} g(x) = 0$ $\quad\lim_{x\to 1^+} g(x) = 0$

$\therefore \lim_{x\to 1} [f(x)+g(x)] =$ not defined

$\Rightarrow \lim_{x\to 1} [f(x).g(x)] = 0$

Que: $\lim_{x\to -1} 6x^2+5x-1$

Ans: $6(-1)^2+5(-1)-1$

$6-5-1$

0

Que: $\lim_{x\to 1} \frac{x}{\ln x}$

Ans: $\frac{1}{\ln 1}$ = Not defined.

Que: $f(x) = \begin{cases} \frac{x^2+2}{x-1} & 0<x<4 \\ \sqrt{x} & x>4 \end{cases}$

Ans: $\lim_{x\to 4} f(x) \Rightarrow \lim_{x\to 4^-} \frac{x+2}{x-1} = \frac{3.9999+2}{3.9999-1} = 2$

here because in que both x are not 4 so we will take nearby number $\Rightarrow \lim_{x\to 4^+} \sqrt{x} = \sqrt{4.0001} = 2$

$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x+2}{x-1} = \frac{2+2}{2-1} = 4$

because given that $0<x<4$

If x value vary, then we will check LHL & RHL.

Q. $g(x) = \begin{cases} \sin(x+1) & x<-1 \\ 2^x & -1\le x<5 \end{cases}$

$\lim_{x\to -1} g(x) \Rightarrow \lim_{x\to -1^+} g(x) = 2^x = 2^{-1} = \frac{1}{2} = 0.5$

$\lim_{x\to -1^-} g(x) = \sin(-1+1) = \sin 0 = 0$

$\lim_{x\to -1^+} \neq \lim_{x\to -1^-}$ ∴ not defined

$\lim_{x\to 0} g(x) = 2^x = 2^0 = 1$

Q. $f(x) = \frac{|x-3|}{(x-3)}$

$\lim_{x\to 3} \frac{|3-3|}{(3-3)} = \frac{0}{0}$ = undefined → function.

Q. $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$

$\lim_{x\to 2} \frac{(2)^2+2-6}{2-2} = \frac{0}{0}$ = not defined → fun^n

$\lim_{x\to 2} \frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}} = x+3 = 2+3 = 5$ → limit exist.

Q. $\lim_{x\to -1} \frac{x+1}{\sqrt{x+5}-2}$

$\lim_{x\to -1} \frac{-1+1}{\sqrt{-1+5}-2} = \frac{0}{0}$ = undefined

$\lim_{x\to -1} \frac{x+1}{\sqrt{x+5}-2} \times \frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} \Rightarrow \frac{(x+1)(\sqrt{x+5}+2)}{x+5-4}$

$\Rightarrow \frac{\cancel{x+1}\,\sqrt{x+5}+2}{\cancel{x+1}} \Rightarrow \sqrt{-1+5}+2$

$\Rightarrow \sqrt{4}+2 \Rightarrow 2+2 = 4$

limit exist ↑

Continuity:

The function f is said to be continuous at point c iff

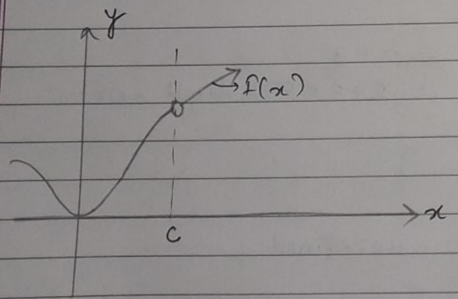
1. function is defined at point c.
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

→ Any function $f(x)$ is said to be continuous if we draw the graph of that function without lifting pen.

Types of discontinuity:

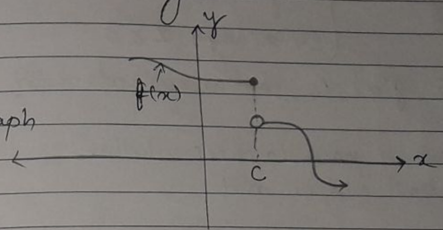
1. Point discontinuity

at particular point the graph will break and it will not continuous.



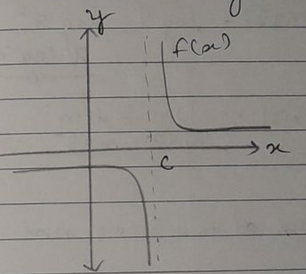
2. Jump discontinuity:

There will be jump on the points of graph and it is not continuous.



3. Asymptotic discontinuity:

In this the point are coming close but they are not close



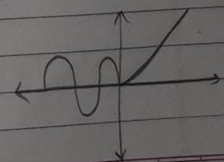
e.g. $g(x) = \begin{cases} \sin x & x < 0 \\ x^2 & x \geq 0 \end{cases}$

1. $\sin 0 = 0$
   $x^2 = (0)^2 = 0$
   funⁿ defined

2. $\lim_{x \to 0^-} \sin 0 = 0$
   $\lim_{x \to 0^+} (0)^2 = 0$
   Limit exists

3. $\lim_{x \to 0} \sin(0) = f(0)$

∴ It is continuity.



Que: $g(x) = \begin{cases} \log(3x) & 0 < x < 3 \\ (4-x)\log 9 & x \geq 3 \end{cases}$

Is $g(x)$ is continuous at $x=3$?

Ans: First we have to check whether limit exist or not.

$\lim_{x \to 3^+} = (4-3)\log 9 = \log 9 = \log 3^2 = \underline{2\log 3}$.

$\lim_{x \to 3^-} = \log(3(3)) = \log 9 = \log 3^2 = \underline{2\log 3}$

$\therefore$ Limit exist $\because \lim_{x \to 3^+} = \lim_{x \to 3^-}$

Now, continuous at $x=3$ or not

$g(3) = \lim_{x \to 3} (4-3)\log 9 = \log 9 = \log 3^2 = \underline{2\log 3}$

$\therefore$ It is continuous at $x=3$

Que: $f(x) = \begin{cases} \ln(x) & 0 \leq x \leq 2 \\ x^2 \ln(x) & x > 2 \end{cases}$

Is $f(x)$ continuous at $x=2$?

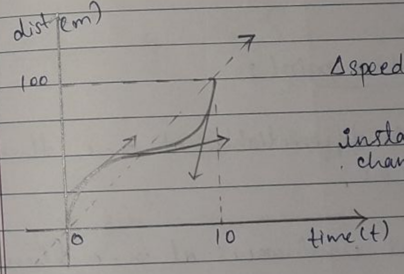
Ans: $\lim_{x \to 2^+} = (2)^2 \ln 2 = 4\ln 2 = 2.77$ $\quad \lim_{x \to 2^-} \ln(2) = 0.693$

$\therefore$ limit doesn't exist.

$\lim_{x \to 2} \ln 2 = 0.693$

$\therefore$ It is not continuous/discontinuous

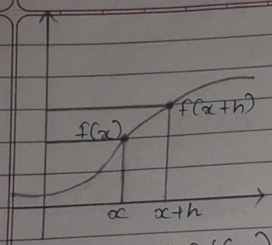
Differentiation



$\Delta \text{speed} = \frac{\Delta \text{dist}}{\Delta \text{time}} = \frac{100}{10} = 10$ m/s.

instantaneous change in speed $= \frac{f(10) - f(0)}{10 - 0} = \frac{100}{10} = 10$ m/s.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-x}$$



$\rightarrow f'(x)$

$$\rightarrow f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

For any pt at any curve = slope of tangent at that point

Differentiability at a point:

- If a function f is differentiable at $x=c$ then f is continuous at $x=c$.
- If a function f is not continuous at $x=c$ then f is not differentiable at $x=c$.

e.g. $f(x)=\begin{cases} x^2, & x<3 \\ 6x-9, & x\geq 3\end{cases}$

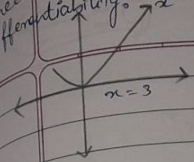
sol^n: $\lim_{x\to 3^+} 6x-9 = 6(3)-9 = 18-9 = 9$

$\lim_{x\to 3^-} (3)^2 = 9$

$\lim_{x\to 3} 6(3)-9 = 18-9 = 9$

$\therefore \lim_{x\to 3^+} = \lim_{x\to 3^-} = \lim_{x\to 3}$

Check differentiability: x^2



$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$$

$$\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$$

$$\lim_{h\to 0}\frac{\cancel{9}+6h+h^2-\cancel{9}}{h} \quad [\because f(x)=x^2]$$

$$\lim_{h\to 0}\frac{6h+h^2}{h} = \lim_{h\to 0}\frac{\cancel{h}(6+h)}{\cancel{h}}$$

$$= \underline{\underline{6}}.$$

If in place of 3 we keep x then,

$$\frac{f(x+h)-f(x)}{h} = \frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$$

$$= \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x$$

e.g. $g(x)=\begin{cases} x-1 & x<1 \\ (x-1)^2 & x\geq 1\end{cases}$

Differentiable at $x=1$?

$\lim_{x\to 1^-} g(x) = x-1 = 1-1 = 0$ $\qquad \lim_{x\to 1} g(x) = (x-1)^2 = 0$

$\lim_{x\to 1^+} g(x) = (1-1)^2 = (0)^2 = 0$

$$\lim_{x\to 1^-}\frac{g(x)-g(1)}{x-1} = \frac{x-1-0}{x-1} = 1$$

$$\lim_{x\to 1^+}\frac{g(x)-g(1)}{x-1} = \frac{x^2-2x+1-0}{x-1} = \frac{(x-1)\cancel{(x-1)}-0}{\cancel{(x-1)}}$$

$$= x-1 = 1-1 = 0$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \frac{x^n + {}^nC_1 x^{n-1}\Delta x + {}^nC_2 x^{n-2}(\Delta x)^2 + \dots + {}^nC_n(\Delta x)^n - x^n}{\Delta x}$$

$$\Rightarrow \lim_{x\to 0} = {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2}\Delta x + \dots + {}^nC_n(\Delta x)^{n-1}$$

$$\therefore \frac{d}{dx}x^n = nx^{n-1}$$

Q. $f(x) = x^3 + x$

$f'(x) = 3x^2 + 1$

$f''(x) = 6x.$

Q. $f(x) = x^2(x+3)$

$= x^2 + (x+3)2x$

$f'(x) = x^2 + 2x^2 + 6x$

$f''(x) = 4x + 2x + 6$

Q. $f(x) = 6x^2 - 4x$

$f'(x) = 12x - 4$

$f''(x) = 12$

Q. $f(x) = \frac{x+a}{x+b}$

$\frac{(x+b) - (x+a)}{(x+b)^2}$

$f'(x) = \frac{(b-a)}{(x+b)^2}$

$f''(x) = (b-a)\frac{d}{dx}(x+b)^{-2}$

$= (b-a)(-2)(x+b)^{-3}$

$f''(x) = \frac{-2(b-a)}{(x+b)^3}$

Rules:

① $\frac{d}{dx}k = 0$ ② $\frac{d}{dx}(k(u(x))) = k.\frac{du}{dx}$

③ $\frac{d}{dx}(u^n) = n.u^{n-1}.\frac{du}{dx}$ ④ $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

⑤ $\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$ ⑥ $\frac{d}{dx}(u.v) = u\frac{dv}{dx} + v\frac{du}{dx}$

⑦ $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Q. $f(x) = \sin(x^2)$

$f'(x) = \cos(x^2).2x$

$f''(x) = \cos x^2(2) + 2x(-\sin x^2)(2x)$

Q. $f(x) = x\sin^2 x$

$x.\frac{d}{dx}\sin^2(x) + \sin^2 x\frac{dx}{dx}$

$f'(x) = x\,2\sin x\cos x + \sin^2 x$

$f''(x) = \underset{u}{\sin x}\ \frac{(\underset{u.v}{2x\cos x} + \sin x)}{v}$

$= \sin(x)[-2x\sin x + 2\cos x + \cos x] + (2x\cos x + \sin x)\sin x$

Q. $f(x) = \tan(2x+3)$

$f'(x) = \sec^2(2x+3)$

$f''(x) =$

Q. $f(x) = \sin(\cos(x^2))$

$= \cos(\cos x^2) \frac{d}{dx} \cos x^2$

$= \cos(\cos x^2)(-\sin x^2) \frac{d}{dx} x^2$

$= \cos(\cos x^2) \cdot (-\sin x^2) \cdot 2x$

Q. Find $\frac{dy}{dx}$ if $x = a\cos\theta$ $y = a\sin\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\cos\theta}{-a\sin\theta}$

$= -\cot\theta$

Q. $\frac{dy}{dx}$ if $x = at^2$, $y = 2at$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cancel{a}}{2at} = \frac{1}{t}$

Q. $f(x) = \cos^{-1}(\sin(x))$

$= \cos^{-1}(\cos(\pi/2 - x))$ or $\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

$f'(x) = \pi/2 - x$

$f'(x) = -1$

Q. differentiate $\underset{u(x)}{\sin^2 x}$ w.r.t $\underset{v(x)}{e^{\cos x}}$

$\frac{u(x)}{v(x)} = \frac{du/dx}{dv/dx} = \frac{2\sin x \cos x}{e^{\cos x}(-\sin x)}$

$= \frac{-2\cos x}{e^{\cos x}}$

$f'(x)$ is rate of change of $f(x)$.

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example:

$f(x) = x^3+3x^2-9x+7.$

$f'(x) = 3x^2+6x-9$

$= x^2+2x-3$

$= x^2-x+3x-3$

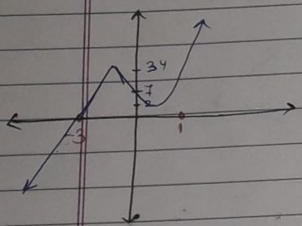
$= x(x-1)+3(x-1)$

$= (x-1)(x+3)$

$= x=1$ & $x=-3$

-3 se pehle increase and 0 that decrease and then vapas 2 se increase

$f'(-4)=15$ & $f'(0)=-9$ $f'(2)=15.$



$f(x)$ is increasing in $x<-3$ & $x>1$

decreasing in $-3<x<1$

example: $f(x) = x^6-3x^5$

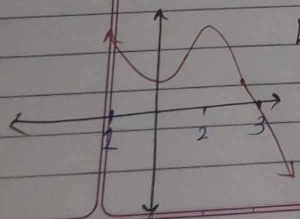
$f'(x) = 6x^5-15x^4<0$

$= 3x^4(2x-5)$

$= 2x-5<0$

$\Rightarrow x<5/2$

example: $h(x) = -x^3+3x^2+9$



$h'(x) = -3x^2+6x$

$-3x^2+6x=0$

$-3x(x-2)=0$

$x-2=0 \quad -3x=0$

$x=2 \quad x=0$

$h'(-1)=-9$

$h'(1)=3$

$h'(3)=-9$

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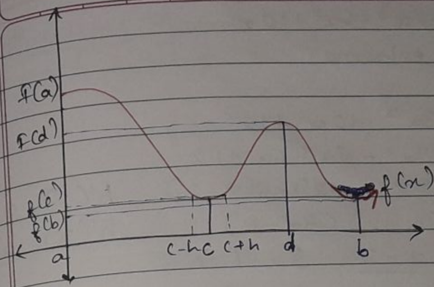


$f(x): [a,b]$, $f(a) \to max$

$f(b) \to min$

$[c,d] \in (a,b)$

c is relative minima

→ $f(d)$ is local maxima if $f(d) \geq f(x)$ for all $x \in (d+h, d-h)$ for $h>0$

→ $f(c)$ local min if $f(c) \leq f(x)$ $\forall$ $x \in (c-h, c+h)$ for $h>0$.

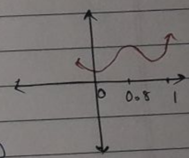
example: $g(x) = x^4-x^5$

$g'(x) = 4x^3-5x^4$

$= x^3(4-5x)=0$

$\Rightarrow x^3=0 \quad 4-5x=0$

$\Rightarrow x=0 \quad \Rightarrow x=+\frac{4}{5}$



$g(-1) = (-1)^4-(-1)^5$

$= 1-(-1) = 1+1 = 2$

$g\left(\frac{4}{5}\right) = \left(\frac{4}{5}\right)^4 - \left(\frac{4}{5}\right)^5$

$= 0.4096 - 0.1677$

$= 0.2419$

$g(0) = (0)^4-(0)^5 = 0$

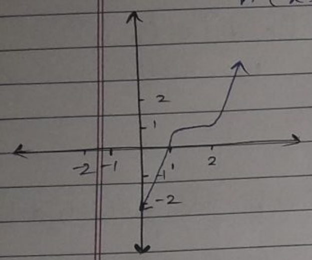
$g(1) = (1)^4-(1)^5 = 0$

$g'(-1) = -4-5=-9$

$g'(1) = 4-5=-1$

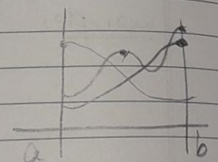
extrema → rate of change is zero.

Question: $h(x) = x^3 - 6x^2 + 12x$
Find extrema.



$h'(x) = 3x^2 - 12x + 12$
$= 3x^2 - 6x - 6x + 12$
$= 3x(x-2) - 6(x-2)$ $h'(1) = 3$
$= (x-2)(3x-6)$ $h'(3) = 3$
$= \underline{x = 2}$ $3x - 6 = 0$
$3x = 6$
$x = \cancel{6}/\cancel{3}\,2$

Question: Maxima of $f(x) = 8\ln x - x^2$ on interval $[1,4]$
$f'(x) = \frac{8}{x} - 2x$



$8 - 2x^2 = 0$
$4 = x^2$
$x = \pm 2$
$f(1) = 8\ln 1 - (1)^2 = -1$

$f(4) = 8\ln(4) - 16 = -4.9096$

$\underline{f(2) \simeq 0.8}$ $\therefore$ maximum at $f(2)$

humne yaha f(x) kyo value rakha and pehle ~~k~~ queme kyo humne f'(x) me rakha

Question: Maxima of $f(x) = x^3 - 3x^2 + 12$ on interval $x \in [-2, 4]$.
$f(x) = x^3 - 3x^2 + 12$
$f'(x) = 3x^2 - 6x$
$= 3x^2 - 6x$
$= 3x(x-2)$ $f(-2) = 8$
$\Rightarrow (x-2) = 0$ $3x = 0$ $f(4) = 28$
$x = 2$ $x = 0$ $f(0) = 12$
$f(2) = 8$

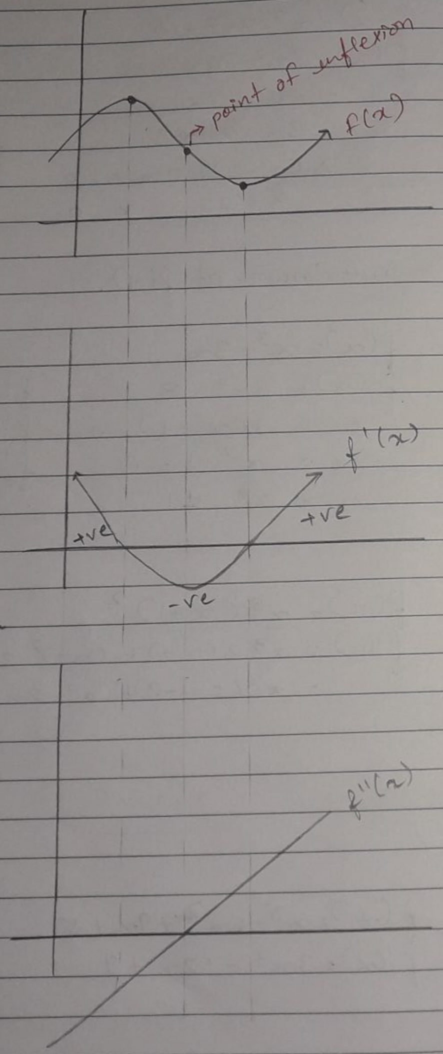
$\therefore$ maximum at $f(4)$.

Question. Find extrema $f(x) = x^3 - 3x$
$f'(x) = 3x^2 - 3$ $f'(-2) = -2$
$= 3(x^2 - 1) = 0$ $f'(0) = 0$
$\Rightarrow x^2 - 1 = 0$ $f'(2) = 2$
$\Rightarrow x = \pm 1$

$f(x) = x^3(x-1)^2$
$f'(x) = x^3\, 2(x-1) + (x-1)^2\, 3x^2$
$= x^3(2x-2) + (x^2 - 2x + 1) \cdot 3x^2$

$f(x) = x^3 - 6x^2 + 9x + 5$
$f'(x) = 3x^2 - 12x + 9$

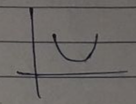
Shape of function:



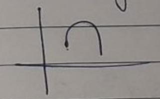
$f''(x) > 0$
$f(x) =$ minima

$f''(x) < 0$
$f(x) =$ maxima

- Given $f(x)$ it is concave up on an interval I if all the tangents to the curve on I are below the graph of $f(x)$.



- Given $f(x)$ it is concave down on an interval I if all the tangent of the curve are above the I.



- A point $x = c$ is called an infexion point if the function is continous at that point and the concavity of graph change at that point.

Suppose that $x=c$ is critical point of $f(x)$ such that $f'(c)=0$ and $f''(x)$ is continuous around the region $x=c$ then,

① $f''(c)<0$ then $x=c$ is relative maximum.
② $f''(c)>0$ then $x=c$ is relative minimum.
③ $f''(c)=0$ then $x=c$ can be relative maximum or minimum or neither of this.

Questions:

$$h(x)=3x^5-5x^3+3$$
$$h'(x)=15x^4-15x^2$$
$$=15x^2(x^2-1)$$
$$\Rightarrow 15x^2=0 \quad x^2-1=0$$
$$\underline{x=0} \quad (x+1)(x-1)=0$$
$$\underline{x=-1} \quad \underline{x=1}$$

$$h''(x)=60x^3-30x$$
$$h''(0)=0$$
$$h''(-1)=-30<0$$
$\therefore h(-1)$ is local max.
$$h''(1)=30>0$$
$\therefore h(1)$ is local min.

Que: Find two positive number whose sum of twice the first and seven times the second is 600 and whose product is maximum.

Ans: Let x and y,

$$2x+7y=600 \quad —①$$

$x.y=max$

$$x=\frac{600-7y}{2}$$

$$\left(\frac{600-7y}{2}\right)y=f(y) \quad f'(y)=0$$

$$f(y)=300y-3.5y^2$$
$$f'(y)=300-7y=0$$

$$\underline{y=300/7}$$

$$x=\frac{600-7\left(\frac{300}{7}\right)}{2}$$
$$=\frac{300}{2}=150$$

$$x.y=\frac{\cancel{300}^{150}}{\cancel{2}}\cdot\frac{300}{7}$$

$$\underline{x.y=600=max}$$

Que: Let x and y be two positive no. whose sum is 175 and $(x+3)(x+4)$ is maximum. Determine x and y.

→ $x+y=175$

Que: Find two positive no. whose product is 250 and sum of first and four times second is ~~max~~ minimum.

Ans: $x \cdot y = 250$

$x+4y = \text{min}$

$x+4y = f(x,y)$

$\frac{250}{y} + 4y = f(y)$

$f'(y) = 4 - \frac{250}{y^2} = 0$

$4y^2 - 250 = 0$

$y^2 = \frac{250}{4}$

$y = \pm\sqrt{62.5}$

$f''(y) = \frac{250 \times 2}{y^3}$ $\therefore f''(+\sqrt{62.5}) > 0$

$\therefore f(\sqrt{62.5})$ will be min.

$\therefore f''(-\sqrt{62.5}) < 0$

$\therefore f(\sqrt{62.5})$ will be max.

Que: Find a +ve no. ↗ the sum of number and its reciprocal is minimum.

Ans: Assume, the +ve no. is x.

$f(x) = x + \frac{1}{x}$

~~$f'(x) = \frac{x^2+1}{x}$~~

$f'(x) = 1 - \frac{1}{x^2} = 0$

$= \frac{x^2-1}{x^2} = 0$

$x^2 = 1$

$x = \pm 1 \Rightarrow x = 1$

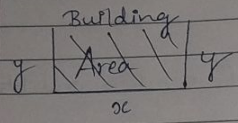
$f''(x) = 2/x^2$

$f''(1) = 2 > 0$ (min)

$f''(-1) = -2 < 0$ (max).

Que. We need to enclose a rectangle field with a fence. We have 500 feet of fencing material and a building on one side of the field. Determine the dimension of the field enclose the largest ~~one~~ area.

Ans:



$y + x + y$

$2y + x = 500 \qquad f(x,y) = x \cdot y$

$x = 500 - 2y$

put in $f(x,y)$ and change to $f(y)$

$f(y) = (500 - 2y)\, y$

$f(y) = 500y - 2y^2$

$f'(y) = 500 - 4y$

$\therefore 500 - 4y = 0$

$\therefore y = \frac{500}{4}$ 125

$\therefore y = 125$

put in x

$x = 500 - 2(125)$

$= 500 - 250$

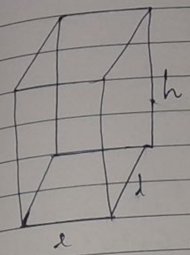
$x = 250$

$f(x,y) = x \cdot y$

$= (250)(125)$

Que. We want to construct a box with a square base and we have only 10 meter square of material. Determine the maximum body that box can hold.

Ans:



$2(lb + bh + lh) =$ surface area of cuboid

$2(l^2 + lh + lh) =$ surface area of cuboid

$\text{Area} = 2l^2 + 4lh$

$\text{Volume} = l^2 h$

$\text{Area} = 2l^2 + 4lh = 10$

$\Rightarrow h = \frac{10 - 2l^2}{4l} = \frac{5 - l^2}{2l}$

$\text{Volume} = l^{\cancel{2}} \left(\frac{5 - l^2}{2\cancel{l}} \right)$

$f(l) = \frac{5l - l^3}{2}$

$f'(l) = 5 - 3l^2 = 0$

$f'(l) \Rightarrow -3l^2 = -5$

$l^2 = 5/3$

$l = \pm\sqrt{5/3} \quad \therefore l = \sqrt{5/3}$

$h = \frac{5 - l^2}{2l} = \frac{5 - (\sqrt{5/3})^2}{2(\sqrt{5/3})}$

$= \frac{5 - 5/3}{2\sqrt{5/3}} = \frac{15 - 5/3}{2\sqrt{5/3}} = \frac{\cancel{10}5/3}{\cancel{2}\sqrt{5/3}}$

$= \frac{\sqrt{5}\cancel{5}\cdot\cancel{\sqrt{3}}}{\sqrt{3}\cdot\cancel{3}\sqrt{3}}$

$= \sqrt{5}/\sqrt{3}$

Now

$V = l^2 h$

$= (\sqrt{5/3})^2 \frac{\sqrt{5}}{\sqrt{3}}$

$= \frac{5}{3} \cdot \frac{\sqrt{5}}{\sqrt{3}} = \frac{5\sqrt{5}}{3\sqrt{3}}$

A function $f(x,y)$ is said to be continuous a point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

example:

$$f(x,y) = 2x^2y^3$$

$$\frac{\partial f}{\partial x} = 4xy^3 \qquad \frac{\partial f}{\partial y} = 6x^2y^2$$

$\frac{\partial f}{\partial x}$ means derivative w.r.t x

example:

$$f(x,y) = x^4 + 6\sqrt{y} - 10$$

$$\frac{\partial f}{\partial x} = 4x^3 + 0 - 0 = 4x^3$$

$$\frac{\partial f}{\partial y} = 0 + 3y^{-1} - 0 = 3y^{-1}$$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 \qquad \frac{\partial^2 f}{\partial y^2} = -3$$

Q. $h(s,t) = t^7(\ln(s^2)) + \frac{9}{t^3} - s^{7/4}$

$$\frac{\partial h}{\partial s} = t^7 \cdot \frac{1}{s^2} \cdot 2s + 0 - \frac{7}{4}s^{3/4}$$

$$= \frac{2\cancel{s}t^7}{s^{\cancel{2}}} - \frac{7}{4}s^{3/4}$$

$$\boxed{\frac{9}{t^3} \Rightarrow 9t^{-3} \Rightarrow 9(-3t^{-3-1}) \Rightarrow 9(-3t^{-4})}$$

$$\frac{\partial h}{\partial t} = 7t^6(\ln(s^2)) - \frac{27}{t^4}$$

$\frac{3}{4}-1 \quad \frac{3-4}{4} = -\frac{1}{4} \quad \frac{7}{4}\left(\frac{3}{4}\right)$

$$\frac{\partial^2 h}{\partial s^2} = 2t^7 - \frac{21}{16}s^{-1/4}$$

$$\frac{\partial^2 h}{\partial t^2} = 42t^5(\ln(s^2)) + \frac{108}{t^5}$$

Q. $f(x,y,z) = x^2y - 10y^2z^3 + 43x - 7\ln(4y)$

$$\frac{\partial f}{\partial x} = 2xy - 0 + 43 - 0 = 2xy + 43$$

$$\frac{\partial f}{\partial y} = x^2 - 20yz^3 + 0 - 7 \cdot \frac{1}{\cancel{4}y} \cdot \cancel{4}$$

$$= x^2 - 20yz^3 - \frac{7}{y}$$

$$\frac{\partial f}{\partial z} = 0 - 30y^2z^2 + 0 - 0 = -30y^2z^2$$

Partial derivative

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

example: $f(x,y) = \sin(x)y^2$

$\frac{\partial f}{\partial x}$ → $\cos x\, y^2$ $\qquad$ $\frac{\partial f}{\partial y}$ → $\sin x\, 2y$

$\frac{\partial f}{\partial x}$ $\qquad$ $\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial x}$ → $-\sin x\, y^2$ $\qquad$ $\frac{\partial f}{\partial y}\cdot\frac{\partial f}{\partial x}$ → $\cos x\, 2y$ $\qquad$ $\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$ → $\cos x\, 2y$ $\qquad$ $\frac{\partial f}{\partial y}\cdot\frac{\partial f}{\partial y}$ → $2\sin x$

$$\frac{\partial f}{\partial y}\cdot\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}\times\frac{\partial f}{\partial x}.$$

example: $f(x,y) = 2x\ln(y)+3$

$$\frac{\partial f}{\partial x} = 2\ln y. \qquad \frac{\partial f}{\partial y} = 2x/y$$

$$\frac{\partial f}{\partial y}\cdot\frac{\partial f}{\partial x} = 2\cdot 1/y \qquad \frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y} = \frac{2}{y}$$

Gradient.

Vector of all 1^{st} order partial derivatives of a scalar valued function. Represented as $\nabla f(x,y)$

$\uparrow$ nabla

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x}(x,y) \\ \\ \frac{\partial f}{\partial y}(x,y) \end{bmatrix}$$

In general for $f(x_1, x_2, \ldots, x_n)$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

example: $\nabla f(x,y) = 2x\ln(y)+3$

$$\nabla f(x,y) = \begin{bmatrix} 2\ln(y) \\ \\ \frac{2x}{y}. \end{bmatrix}$$

example: Write gradient of $f(x,y) = x^2(\sin y)$

$$\nabla f(x,y) = \begin{bmatrix} 2x(\sin y) \\ \\ x^2 \cos y \end{bmatrix}$$

Directional derivative

The rate of change of $f(x,y)$ in the direction of unit vector is called the directional derivative and is denoted by

$$D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{f(x+ah, y+ah) - f(x,y)}{h}$$

example:

$f(x,y) = x^2 y \quad \vec{u} = (a,b).$

$$\nabla_{\vec{u}} f(x,y) = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 2xy \\ x^2 \end{bmatrix}$$

$$= 2axy + bx^2$$

$$= a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$$

if $\vec{u} = (2,1)$

$$= (2 \cdot 2xy + 1x^2)_{(1,1)} = 4 + 1 = 5$$

example: Find the direction derivative $f(x,y) = x\cos y$ in the unit vector $\vec{v} = (2,1)$

$$\nabla_{\vec{u}} f(x,y) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} \cos y \\ -x \sin y \end{bmatrix}$$

$$= 2\cos y - x \sin y$$

at point $(1,1)$

$$= 2\cos(1) - 1\sin(1)$$

example: $f(x,y,z) = x^2 z + y^3 z^2 - xyz \quad \vec{v} = \langle -1, 0, 3 \rangle$

point $p = (1,1,1)$

$$f(x,y,z) = \begin{bmatrix} 2xz - yz \\ 3y^2 z^2 - xz \\ x^2 + 2y^3 z - xy \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2xz - yz \\ 3y^2 z^2 - xz \\ x^2 + 2y^3 z - xy \end{bmatrix}$$

$= -2xz + yz + 3x^2 + 6y^3z - 3xy$

at point $(1,1,1)$

$\nabla_u f(x,y,z)_{(1,1,1)} = -2(1)(1) + (1)(1) + 3(1)^2 + 6(1)^3(1) - 3(1)(1)$
$= -2 + 1 + \cancel{3} + 6 - \cancel{3} = 5$

Hessian

Hessian is a square matrix of second order partial derivative.

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

example:

$$f(x,y) = \begin{bmatrix} -\sin(x)y^2 & \cos(x).2y \\ \cos(x)2y & \sin(x)(2) \end{bmatrix}$$

In general,

$f(x_1, x_2, ----, x_n)$

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Jacobian

$$f = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$$

$$\text{Jacobian } f = \begin{bmatrix} \partial f_1/\partial x & \partial f_2/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix}$$

Definition: In a vector calculus, Jacobian matrix of a vector valued function of several variables is the matrix of all its partial derivatives.

Que:

$$\vec{f} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} x + \sin y \\ y + \sin x \end{bmatrix}$$

$$J_{\vec{f}} = \begin{bmatrix} 1 & \cos(y) \\ \cos(x) & 1 \end{bmatrix}$$

Que:

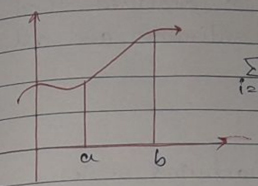
$$\vec{f} = \begin{bmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{bmatrix} = \begin{bmatrix} x + 3y^2 - z^3 \\ 4x^2yz \\ 2z^2 - xy \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{bmatrix}$$

at point(1, 1, 1)

$$= \begin{bmatrix} 1 & 6 & -3 \\ 8 & 4 & 4 \\ -1 & -1 & 4 \end{bmatrix}$$

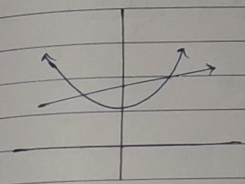
Integration.



$\sum_{i=x} f(x_i)\Delta x_i$

area $= \lim_{n\to\infty} f(x)\,dx$

Que:



$y = 2x^2+10$
$y = 4x+16$

$2x^2+10 = 4x+16$
$2x^2-4x-6=0$
$x^2-2x-3=0$
$x^2+x-3x-3=0$
$x(x+1)-3(x+1)=0$
$\underline{x=-1} \quad \underline{x=3}$

$$\int_{-1}^{3} \left[(4x+16) - (2x^2+10) \right] dx$$

$$\int_{-1}^{3} (4x-2x^2+6)\,dx = \left[\frac{\cancel{4}^{2}x^2}{\cancel{2}} - \frac{2x^3}{3} + 6x \right]_{-1}^{3}$$

$$\left[\cancel{18} - \cancel{18} + 18 \right] - \left[2 + \frac{2}{3} - 6 \right] = 18 - [-10]$$

$$= 18 + 10 = \underline{28}$$

Taylor Series

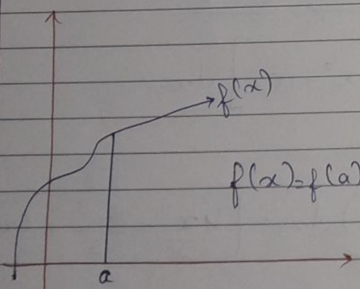
→ It gives the expansion of a function $f(x)$ in the neighbourhood of the point 'a' provided that in the neighbourhood function is continuous and all its derivative exist.

At $x=0$ it becomes Maclurin's series

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^n(0)x^n}{n!}$$

$$\therefore f(x) = \sum_{n=0}^{\infty} \frac{f^n(0).x^n}{n!}$$

And for any other value of x is remains Taylor series.



$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^n(a)(x-a)^n}{n!}$$

$f(a) = f(x)$

$$\text{Taylor series} = f(x) = \sum_{n=0}^{\infty} \frac{f^n(a).(x-a)^n}{n!}$$

Que: $f(x) = e^x$ at $x=0$

$$e^x = e^0 + e^x.x + \frac{e^x.x^2}{2!} + \frac{e^x.x^3}{3!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

In general, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Que: $f(x) = \sin x$

$f'(x) = \cos x \quad f''(x) = -\sin x \quad f'''(x) = -\cos x$

$$f(x) = \sin(0) + \cos x(x) - \sin x\left(\frac{x^2}{2!}\right) - \cos x\left(\frac{x^3}{3!}\right) + \frac{\sin x(x^4)}{4!}$$

$$f(x) = 0 + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}(a)^{2n+1}}{(2n+1)!}$$

→ $\cos(a) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$