Linear Equations
Unit 3

Linear Algebra and its Applications , David C. Lay

Linear Algebra and its Applications , Gilbert Strang

Applied Numerical Analysis, Gerald and Wheatly

Linear equation

A linear equation in the variables $x_1, x_2, \dots \dots ., x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \; a_3x_3 + \cdots \dots . + a_nx_n = b$$

where b and the coefficients of $x_1, x_2, \dots \dots ., x_n$ are real or complex numbers

Eg. $7x_1 + 5x_2 - 12x_3 = 4.5$

System of linear equations

A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables $x_1, x_2, \dots \dots, x_n$

- $a_{11}x_1 + a_{12}x_2 + \dots \dots + a_{1n}x_n = b_1$
- $a_{21}x_1 + a_{22}x_2 + \dots \dots + a_{2n}x_n = b_2$
- $a_{m1}x_1 + a_{m2}x_2 + \dots \dots + a_{mn}x_n = b_m$

Homogeneous linear equations

A system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \; \dots \dots . + a_{1n}x_n \; = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \; \dots \dots . + a_{2n}x_n \; = b_2$$
$$a_{m1}x_1 + a_{m2}x_2 + \; \dots \dots . + a_{mn}x_n \; = b_m$$

is called homogeneous if $b_1 = \; b_2 \; = \cdots = \; b_m$ = 0 and

non-homogeneous, otherwise.

Eg.

$$x_1 - \; x_2 + \qquad x_3 = 0$$
$$x_1 \qquad\quad - \quad 4x_3 = 0$$

Solution of the system

- A solution of the system is a list $\{s_1, s_2, \dots \dots., s_n\}$ of numbers that makes each equation a true statement when the values $s_1, s_2, \dots \dots., s_n$ are substituted for $x_1, x_2, \dots \dots., x_n$ respectively.

Eg.,

$$\begin{aligned} x_1 - x_2 + x_3 &= 8 \\ x_1 - 4x_3 &= 7 \end{aligned}$$

- {11, 4, 1} is a solution of the above equations because, when these values are substituted for $x_1, x_2, \dots \dots., x_n$, respectively, the equations simplify to 8 = 8 and 7 = 7

Solution of the system

- The set of all possible solutions is called the solution set of the linear system.
- Two linear systems are called equivalent if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

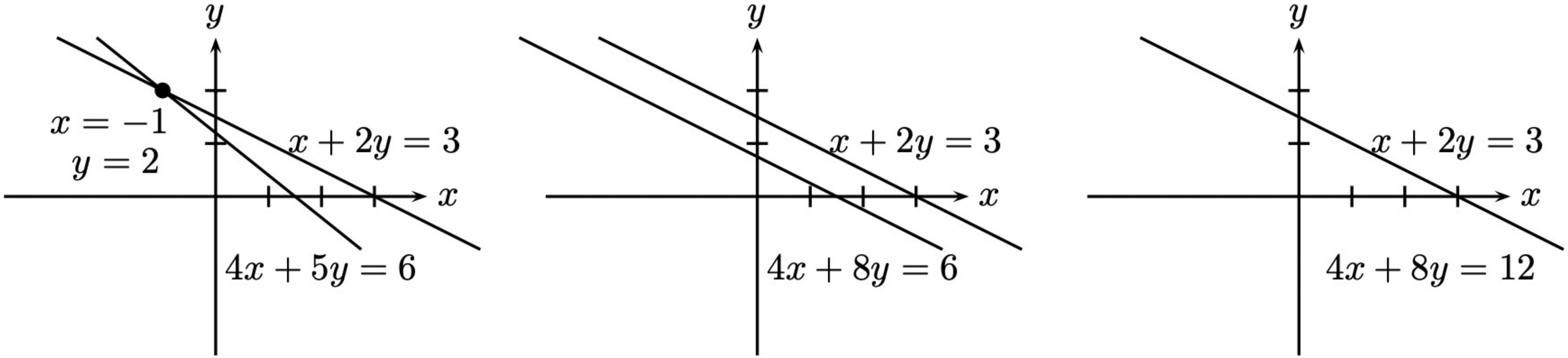
- A system of linear equations has

1. no solution
2. exactly one solution
3. infinitely many solutions.

Solution of the system

- A system of linear equations is said to be consistent if it has either one solution or infinitely many solutions;
- a system is inconsistent if it has no solution.

Solution of the system



One solution $(x, y) = (-1, 2)$ **Parallel: No solution** **Whole line of solutions**

Figure 1.1: The example has one solution. Singular cases have none or too many.

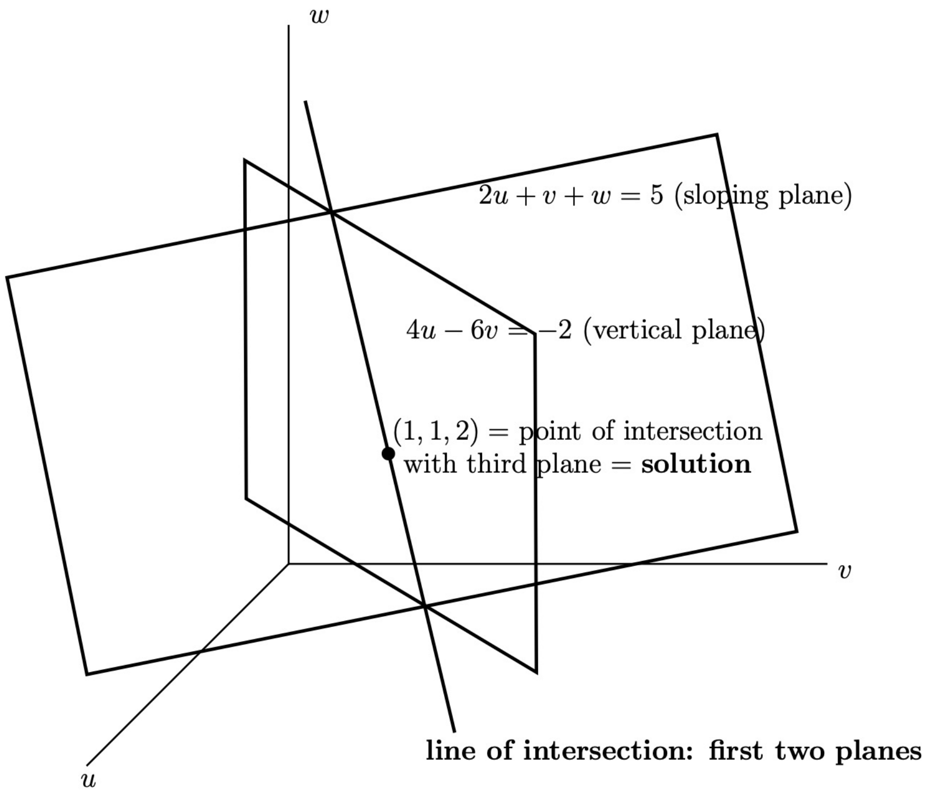


Figure 1.3: The row picture: three intersecting planes from three linear equations.

Cramer’s Rule

Let the system be given by

$a_{11}x_1 + a_{12}x_2 = b_1$ - (1)
$a_{21}x_1 + a_{22}x_2 = b_2$ -(2)

To solve for x_1
Multiply first equation by a_{22} and second by a_{12} we get

$$a_{22}a_{11}x_1 + a_{22}a_{12}x_2 = a_{22}b_1$$
$$a_{12}a_{21}x_1 + a_{12}a_{22}x_2 = a_{12}b_2$$

Cramer's Rule

Therefore, we get

$$(a_{22}a_{11} - a_{12}a_{21})x_1 = a_{22}b_1 - a_{12}b_2$$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{22}a_{11} - a_{12}a_{21}}$$

$$x_1 = \frac{D_{x1}}{D} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

D_{x1} :Determinant of the numerator in the solution of x_1

If we are solving for x_1, the column 1 is replaced with constants

Cramer’s Rule

To solve for x_2

Multiply first equation by a_{21} and second by a_{11} we get

$$a_{21}a_{11}x_1 + a_{21}a_{12}x_2 = a_{21}b_1$$
$$a_{11}a_{21}x_1 + a_{11}a_{22}x_2 = a_{11}b_2$$

Cramer's Rule

Therefore, we get

$(a_{21}a_{12} - a_{11}a_{22})x_2 \;=\; a_{21}b_1 - \; a_{11}b_2$

$$x_2 \;=\; \frac{a_{21}b_1 - \; a_{11}b_2}{a_{21}a_{12} - a_{11}a_{22}}$$

$$x_2 = \frac{D_{x2}}{D} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

D_{x2} :determinant of the numerator in the solution of x_2

If we are solving for x_2, the column 2 is replaced with constants

Cramer's Rule

For the system of equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

If $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \; and \; x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Cramer's Rule

$eg.$

$12x_1 + 3x_2 = 15$

$2x_1 - 3x_2 = 13$

$$x_1 = \frac{\begin{vmatrix} 15 & 3 \\ 13 & -3 \end{vmatrix}}{\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}} = \frac{-45 - 39}{-36 - 6} = \frac{-84}{-42} = 2$$

$$x_2 = \frac{\begin{vmatrix} 12 & 15 \\ 2 & 13 \end{vmatrix}}{\begin{vmatrix} 12 & 3 \\ 2 & -3 \end{vmatrix}} = \frac{156 - 30}{-36 - 6} = \frac{126}{-42} = -3$$

Cramer’s Rule

Solving for 3 equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\mathrm{D} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Cramer's Rule

$$\mathrm{D}\,x_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$= b_1 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} b_2 & a_{23} \\ b_3 & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} b_2 & a_{22} \\ b_3 & a_{32} \end{vmatrix}$$

$\mathrm{D}\,x_1$ is determinant of the numerator in the solution of x_1

Cramer’s Rule

$$\mathrm{D}\, x_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} b_2 & a_{23} \\ b_3 & a_{33} \end{vmatrix} - b_1 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & b_2 \\ a_{31} & b_3 \end{vmatrix}$$

$\mathrm{D}\, x_2$ is determinant of the numerator in the solution of x_2

Cramer's Rule

$$\mathrm{D}\,x_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & b_2 \\ a_{32} & b_3 \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & b_2 \\ a_{31} & b_3 \end{vmatrix} + b_1\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$\mathrm{D}\,x_3$ is determinant of the numerator in the solution of x_3

$$x_1 = \frac{\mathrm{D}\,x_1}{\mathrm{D}}, x_2 = \frac{\mathrm{D}\,x_2}{\mathrm{D}}, x_3 = \frac{\mathrm{D}\,x_3}{\mathrm{D}}$$

Cramer's Rule

Eg.

$$\begin{aligned} x_1 &+ x_2 - x_3 = 6 \\ 3\,x_1 &- 2x_2 + x_3 = -5 \\ x_1 &+ 3x_2 - 2x_3 = 14 \end{aligned}$$

$$\mathrm{D} = \begin{vmatrix} 1 & 1 & -1 \\ 3 & -2 & 1 \\ 1 & 3 & -2 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} - 1 \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix}$$

=1(4-3) – (-6-1) – (9+2) = 1+7-11 = -3

Cramer's Rule

$$\mathrm{D}x1 = \begin{vmatrix} 6 & 1 & -1 \\ -5 & -2 & 1 \\ 14 & 3 & -2 \end{vmatrix}$$

$$= 6\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} - 1\begin{vmatrix} -5 & 1 \\ 14 & -2 \end{vmatrix} - 1\begin{vmatrix} -5 & -2 \\ 14 & 3 \end{vmatrix}$$

= 6(4-3) – (10-14) – (-15+28)

= 6+4-13 = -3

Cramer's Rule

$$\mathrm{D}x2 = \begin{vmatrix} 1 & 6 & -1 \\ 3 & -5 & 1 \\ 1 & 14 & -2 \end{vmatrix}$$

= $1 \begin{vmatrix} -5 & 1 \\ 14 & -2 \end{vmatrix} - 6 \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} - 1 \begin{vmatrix} 3 & -5 \\ 1 & 14 \end{vmatrix}$

= 1(10-14)- 6(-6-1) – (42+5)

= -4+ 42 - 47= -9

Cramer's Rule

$$\mathrm{D}x3 = \begin{vmatrix} 1 & 1 & 6 \\ 3 & -2 & -5 \\ 1 & 3 & 14 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -2 & -5 \\ 3 & 14 \end{vmatrix} - 1 \begin{vmatrix} 3 & -5 \\ 1 & 14 \end{vmatrix} + 6 \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix}$$

=1(-28+15) – (42+5) + 6(9 + 2)

= -13 – 47 +66 = 6

Cramer's Rule

$$x_1 = \frac{\mathrm{D}\,x_1}{\mathrm{D}}\,, x_2 = \frac{\mathrm{D}\,x_2}{\mathrm{D}}\,, x_3 = \frac{\mathrm{D}\,x_3}{\mathrm{D}}$$

$$x_1 = \frac{-3}{-3}\,, x_2 = \frac{-9}{-3}\,, x_3 = \frac{6}{-3}$$

$$x_1 = 1, \qquad x_2 = 3, \qquad x_3 = -2$$

Cramer’s Rule

Eg2.

$$2x_1 - 3x_2 + x_3 = 1$$
$$3\,x_1 + x_2 - x_3 = 2$$
$$x_1 - x_2 - x_3 = 1$$

Cramer’s Rule

$$\mathrm{D} = \begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix}$$

= -14

$\mathrm{D}x1 = \begin{vmatrix} 1 & -3 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix}$ = -8

Cramer’s Rule

$$\mathrm{D}x2 = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 1$$

$$\mathrm{D}x3 = \begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = 5$$

Cramer's Rule

$$x_1 = \frac{\mathrm{D}\,x_1}{\mathrm{D}}\ , x_2 = \frac{\mathrm{D}\,x_2}{\mathrm{D}}\ , x_3 = \frac{\mathrm{D}\,x_3}{\mathrm{D}}$$

$$x_1 = \frac{-8}{-14}\ , x_2 = \frac{1}{-14}\ , x_3 = \frac{5}{-14}$$

$$x_1 = \frac{4}{7}, \qquad x_2 = -1/14, \qquad x_3 = -5/14$$

Gauss Elimination Process

We now start with solving a systems of linear equations. The idea is to manipulate the rows of the augmented matrix in place of the linear equations themselves. Since, multiplying a matrix on the left corresponds to row operations, we left multiply by certain matrices to the augmented matrix so that the final matrix is in row echelon form . The process of obtaining the row echelon form of a matrix is called the Gauss Elimination method.

The general Gaussian elimination procedure is applied to the linear systems:

$$R_1 : a_{11}x_1 + a_{12}x_2 + \ \ldots \ldots . + a_{1n}x_n = b_1$$
$$R_2 : a_{21}x_1 + a_{22}x_2 + \ \ldots \ldots . + a_{2n}x_n = b_2$$
$$R_n : a_{n1}x_1 + a_{n2}x_2 + \ \ldots \ldots . + a_{nn}x_n = b_n$$

Form the augmented matrix from the system of equations

The unknowns are eliminated to obtain an upper-triangular matrix.

Gauss Elimination Process

$R_1 : a_{11}x_1 + a_{12}x_2 + \ \ldots\ldots . + a_{1n}x_n = b_1$

$R_2: a_{21}x_1 + a_{22}x_2 + \ \ldots\ldots . + a_{2n}x_n = b_2$

To eliminate x_1 from R_2 , we multiply R_1 by (-a_{21}/a_{11}) and obtain

$$-a_{21}x_1 - a_{12}\left(\frac{a_{21}}{a_{11}}\right)x_2 - \ \ldots\ldots . - a_{1n}\left(\frac{a_{21}}{a_{11}}\right)x_n = -b_1\left(\frac{a_{21}}{a_{11}}\right)$$

Adding the above equation to R_2 we obtain

$$\left(a_{22} - \ a_{12}\frac{a_{21}}{a_{11}}\right)x_2 - \left(a_{23} - a_{13}\frac{a_{21}}{a_{11}}\right)x_3 \ldots\ldots . - \left(a_{2n} - a_{1n}\frac{a_{21}}{a_{11}}\right)x_n$$
$$= b_2 - b_1\left(\frac{a_{21}}{a_{11}}\right)$$

Gauss Elimination Process

R_2 can be rewritten as

$$R_2 : a'_{22}x_2 + a'_{23}x_3 + \cdots \ldots . + a'_{2n}x_n = b'_2$$

Where $a'_{22} = \left(a_{22} - a_{12}\frac{a_{21}}{a_{11}}\right)$ and so on.

Gauss Elimination Process

In a similar fashion, we can eliminate x_1 from the remaining equations and after eliminating x_1 from the last row Rn, we obtain the system

$$R_1 : a_{11}x_1 + a_{12}x_2 + \; a_{13}x_3 + \cdots \ldots . + a_{1n}x_n \; = b_1$$

$$R_2 : \qquad a'_{22}x_2 + \; a'_{23}x_3 + \cdots \ldots . + a'_{2n}x_n \; = b'_2$$

$$R_n : \qquad a'_{n2}x_2 + \; a'_{n3}x_3 + \cdots \ldots . + a'_{nn}x_n \; = b'_n$$

Gauss Elimination Process

In the process of obtaining the above system, we have multiplied the first row by (-a_{21}/a_{11}), i.e. we have divided it bya_{11} which is therefore assumed to be nonzero. For this reason, the first row R_1 in is called the pivot equation, and a_{11} is called the pivot or pivotal element. The method obviously fails if a_{11}= 0.

Gauss Elimination Process

Similarly, we eliminate the variables will be obtain the upper-triangular matrix in the form:

$$R_1 : a_{11}x_1 + a_{12}x_2 + \ a_{13}x_3 + \cdots \ldots . + a_{1n}x_n \ = b_1$$

$$R_2 : \qquad a'_{22}x_2 + \ a'_{23}x_3 + \cdots \ldots . + a'_{2n}x_n \ = b'_2$$

$$R_3 : \qquad a''_{33}x_3 + \cdots \ldots . + a''_{3n}x_n \ = b''_3$$

$$R_n : \qquad a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

where $a_{nn}^{(n-1)}$ indicates the element a_{nn} has changed (n-1) times.

Gauss Elimination Process

From R_n : $a_{nn}^{(n-1)} x_n = b_n^{(n-1)}$

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

This is then substituted in the $R_{(n-1)}$ to obtain x_{n-1} and the process is repeated to compute the other unknowns. We have therefore first computed x_n $then$ $x_{n-1}, \ldots \ldots$ x_2, x_1 in that order. Due to this reason, the process is called back substitution.

Gauss Elimination Process

$$
\begin{aligned}
x_2 + x_3 &= 2 \\
2x_1 \quad\quad + 3x_3 &= 5 \\
x_1 + x_2 + x_3 &= 3.
\end{aligned}
$$

The augmented matrix can be written as

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 2 & 0 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

Interchange R_2 and R_1 to get

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

Gauss Elimination Process

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

Replace R_3 by $R_3 - {}^1/_2 R_1$ to get

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 1-({}^1/_2)2 & 1-({}^1/_2)0 & 1-({}^1/_2)3 & 3-({}^1/_2)5 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -{}^1/_2 & {}^1/_2 \end{array}\right]$$

Gauss Elimination Process

Replace R_3 by $R_3 - R_2$ to get

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0-0 & 1-1 & -(1/2)-1 & (1/2)-2 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -3/2 & -3/2 \end{array}\right]$$

The matrix is in row echelon form. Using the last row we get $x_3 = 1$
Second row of the matrix gives us $x_2 + x_3 = 2$ So, $x_2 = 1$
First row gives us $2x_1 + 3x_3 = 5$ So $x_1 = 1$

Gauss Elimination Process

Eg2.

$$x_1 + 3x_2 + 5x_3 = 14$$
$$2x_1 - x_2 - 3x_3 = 3$$
$$4x_1 + 5x_2 - x_3 = 7$$

Gauss Elimination Process

The augmented matrix can be written as

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right]$$

Replace R_2 by $R_2 - 2R_1$ and R_3 by $R_3 - 4R_1$ to get

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 2-2 & -1-2(3) & -3-2(5) & 3-2(14) \\ 4-4 & 5-4(3) & -1-4(5) & 7-4(14) \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right]$$

Gauss Elimination Process

Since all the elements in R_2 and R_3 are negative, we multiply throughout by -1

Replace R_2 by (-1)R_2 and R_3 by (-1)R_3 to get

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & 7 & 13 & 25 \\ 0 & 7 & 21 & 49 \end{array}\right]$$

Gauss Elimination Process

Replace R_3 by R_3 - R_2 to get

$$= \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & 7 & 13 & 25 \\ 0 & 0 & 8 & 24 \end{array}\right]$$

Now back substitution gives us

$$\begin{aligned} x_1 + 3x_2 \; + \; 5x_3 &= 14 \\ 7x_2 \; + \; 13x_3 &= 25 \\ 8x_3 &= 24 \end{aligned}$$

$$x_1 = 5, \qquad x_2 = -2, \qquad x_3 = 3$$