Gauss Seidel

Eg3.

$$
\begin{aligned}
3x_1 + 7x_2 + 13x_3 &= 76 \\
12x_1 + 3x_2 - 5x_3 &= 1 \\
x_1 + 5x_2 + 3x_3 &= 28
\end{aligned}
$$

Take (1,0,1) as initial values

Gauss Seidel

$$3x_1 + 7x_2 + 13x_3 = 76$$
$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$

Check for diagonal dominance

$$|a_{11}| = 3 < |a_{12}| + |a_{13}| = 7 + 13 = 20$$
$$|a_{22}| = 3 < |a_{21}| + |a_{23}| = 12 + 5 = 17$$
$$|a_{33}| = 3 < |a_{31}| + |a_{32}| = 1 + 5 = 6$$

Gauss Seidel

Re arranging the equations as gives diagonal dominance

$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$3x_1 + 7x_2 + 13x_3 = 76$$

Gauss Seidel

We rewrite the equations as

$$x_1 = \frac{1}{12}(1 - 3x_2 + 5x_3)$$

$$x_2 = \frac{1}{5}(28 - x_1 - 3x_3)$$

$$x_3 = \frac{1}{13}(76 - 3x_1 - 7x_2)$$

Take initial approx. as $x_1 = 1, x_2 = 0, x_3 = 1$

Gauss Seidel

- $|\epsilon_a| = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| * 100$

	x_1	$\|\epsilon_a\|$	x_2	$\|\epsilon_a\|$	x_3	$\|\epsilon_a\|$
1	0.5	100	4.9		3.0923	
2	0.14679	240%	3.7153	31.88%	3.8118	18.87%
3	0.74275	80%	3.1644	17.4%	3.9708	4.004%
4	0.94675	21%	3.0281	4.49%	3.9971	0.657%
5	0.99177	4.53%	3.0034	0.83%	4.0001	0.074%
6	0.99919	0.74%	3.0001	0.108%	4.0001	0.001%

The solution is close to the exact solution $x_1 = 1,\ x_2 = 3,\ x_3 = 4$

Gauss –Jordan Elimination Process

The Gauss-Jordan method is a variation of the Gauss Elimination method. In this method, the augmented coefficient matrix is transformed by row operations such that the coefficient matrix reduces to the Identity matrix. The solution of the system is then directly obtained as the reduced augmented column of the transformed augmented matrix. The process of obtaining the row reduced echelon form of a matrix is called the Gauss-Jordan Elimination method.

The Gaussian-Jordan elimination procedure is applied to the linear systems:

$$R_1 : a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$R_2 : a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$R_3 : a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Gauss –Jordan Elimination Process

Form the augmented matrix from the system of equations

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_3 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

We assume that a_{11} is non-zero. If a_{11} =0, we can interchange rows so that a_{11} is non-zero in the resulting system.

Gauss –Jordan Elimination Process

The first step is to divide the first row by a_{11}

- $\begin{bmatrix} a_{11} & a_{12} & a_{13} & | & b_1 \\ a_{21} & a_{22} & a_{23} & | & b_2 \\ a_{31} & a_{32} & a_{33} & | & b_3 \end{bmatrix} \xrightarrow{\frac{R_1}{a_{11}}} \begin{bmatrix} 1 & a'_{12} & a'_{13} & | & b'_1 \\ a_{21} & a_{22} & a_{23} & | & b_2 \\ a_{31} & a_{32} & a_{33} & | & b_3 \end{bmatrix}$

$$a'_{12} = {}^{a_{12}}/_{a_{11}}\,, a'_{13} = {}^{a_{13}}/_{a_{11}}\,, b'_1 = {}^{b_1}/_{a_{11}}$$

Gauss –Jordan Elimination Process

Eliminate x_1 from 2nd and 3rd equation by row operations of

- Replace R_2 by multiplying the reduced first row by a_{21} and subtracting from the second and
- Replace R_3 by multiplying the reduced first row by a_{31} and subtracting from the third row.

$$\left[\begin{array}{ccc|c} 1 & a'_{12} & a'_{13} & b'_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \xrightarrow{R_2 - R_1 a_{21}, R_3 - R_1 a_{31}} \left[\begin{array}{ccc|c} 1 & a'_{12} & a'_{13} & b'_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right]$$

$$a'_{22} = a_{22} - a_{21}a'_{12}, and\ so\ on\ , b'_2 = b_2 - a_{21}b'_1\ and\ so\ on$$

Gauss –Jordan Elimination Process

- Now considering a'_{22} as the non-zero pivot, we first divide the second row by a'_{22}
- $\left[\begin{array}{ccc|c} 1 & a'_{12} & a'_{13} & b'_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right] \xrightarrow{R_2/a'_{22}} \left[\begin{array}{ccc|c} 1 & a'_{12} & a'_{13} & b'_1 \\ 0 & 1 & a''_{23} & b''_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right]$

Where $a''_{23} = {a'_{23}}/{a'_{22}}, b''_2 = {b'_2}/{a'_{22}}$

Gauss –Jordan Elimination Process

- Replace R_1 by Multiplying the reduced second row by a'_{12} and subtract it from the first row and
- Replace R_3 by multiplying the reduced second row by a'_{32} and subtract it from the third row

$$\left[\begin{array}{ccc|c} 1 & a'_{12} & a'_{13} & b'_1 \\ 0 & 1 & a''_{23} & b''_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right] \xrightarrow{R_1 - R_2 a'_{12}, R_3 - R_2 a'_{32}} \left[\begin{array}{ccc|c} 1 & 0 & a''_{13} & b''_1 \\ 0 & 1 & a''_{23} & b''_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right]$$

$$a''_{13} = a'_{13} - a'_{12} a''_{23}, and\ so\ on\ , b''_1 = b'_1 - a'_{12} b''_2\ and\ so\ on$$

Gauss –Jordan Elimination Process

- Now considering a''_{33} as the non-zero pivot, we first divide the third row bya''_{33}

- $$\left[\begin{array}{ccc|c} 1 & 0 & a''_{13} & b''_1 \\ 0 & 1 & a''_{23} & b''_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right] \xrightarrow{R_3/a''_{33}} \left[\begin{array}{ccc|c} 1 & 0 & a''_{13} & b''_1 \\ 0 & 1 & a''_{23} & b''_2 \\ 0 & 0 & 1 & b'''_3 \end{array}\right]$$

Where $b'''_3 = {b''_3}/{a''_{33}}$

Gauss –Jordan Elimination Process

Replace R_1 by Multiplying the reduced third row by a''_{13} and subtract it from the first row and

Replace R_2 by multiply the reduced third row by a''_{23} and subtract it from the second row

$$\left[\begin{array}{ccc|c} 1 & 0 & a''_{13} & b''_1 \\ 0 & 1 & a''_{23} & b''_2 \\ 0 & 0 & 1 & b'''_3 \end{array}\right] \xrightarrow{R_1 - R_3 a''_{13}, R_2 - R_3 a''_{23}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & b'''_1 \\ 0 & 1 & 0 & b'''_2 \\ 0 & 0 & 1 & b''_3 \end{array}\right]$$

$$b'''_1 = b''_1 - a''_{13} b'''_3 \; and \; b'''_2 = b''_2 - a''_{23} b'''_3$$

Gauss –Jordan Elimination Process

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1''' \\ 0 & 1 & 0 & b_2''' \\ 0 & 0 & 1 & b_3'' \end{array}\right]$$

Finally the solution of the system is given by the reduced augmented column

i.e. $x_1 = b_1''', x_2 = b_2''', x_3 = b_3'''$

Gauss –Jordan Elimination Process

The advantage of using Gauss Jordan method is that it **involves no labour of back substitution**. Back substitution has to be done while solving linear equations formed during solving the problem.

Difference between gaussian elimination and Gauss Jordan elimination. The difference between Gaussian elimination and the Gaussian Jordan elimination is that **one produces a matrix in row echelon form while the other produces a matrix in row reduced echelon form.**

Gauss –Jordan Elimination Process

- Three possible outcomes
- Unique Solution: If the reduced row echelon form has no free variables, then it looks like:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

and there is a unique solution, namely, $x_1 = b_1$, $x_2 = b_2$, $x_3 = b_3$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

A unique solution exists

$x_1 = 5$, $x_2 = 2$, $x_3 = 4$

Gauss –Jordan Elimination Process

- Infinite Solutions (dependent system): If the reduced row echelon form has free variables, then there are an infinite number of solutions. The parameter assigned to any one free variable can take on an infinite number of values.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The equations are x_1 +2 x_3 =3 and x_2 -3 $x_3 = 4$

Solve for x_1 and x_2

x_1 = 3 - 2 x_3 and x_2 = 4+ $3x_3$

Thus, the solution is

(3 - 2 x_3 , 4+ 3x_3 , x_3)

Gauss –Jordan Elimination Process

- No Solution (inconsistent system)
- If the reduced row echelon form has a row of the form [0,0,…,0,b] then the system of linear equations has no solution.
- Eg.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

Here we get 0 = 4 which is false. Hence, there is no solution for the system of equations

Gauss –Jordan Elimination Process

Eg.

$$2x_1 + 2x_2 + 4x_3 = 18$$
$$x_1 + 3x_2 + 2x_3 = 13$$
$$3x_1 + x_2 + 3x_3 = 14$$

The augmented matrix can be written as

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 18 \\ 1 & 3 & 2 & 13 \\ 3 & 1 & 3 & 14 \end{array}\right]$$

Gauss –Jordan Elimination Process

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 18 \\ 1 & 3 & 2 & 13 \\ 3 & 1 & 3 & 14 \end{array}\right]$$

The first step is to divide the first row by 2 gives us

Replace R_1 by (1/2) R_1

- $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 1 & 3 & 2 & 13 \\ 3 & 1 & 3 & 14 \end{array}\right]$

Gauss –Jordan Elimination Process

- $\begin{bmatrix} 1 & 1 & 2 & | & 9 \\ 1 & 3 & 2 & | & 3 \\ 3 & 1 & 3 & | & 14 \end{bmatrix}$
- Subtract the reduced first row from the second row and also multiply the first row by 3 and then subtract from the third, gives us

Replace R_2 by $R_2 - R_1$ and R_3 by $R_3 - 3R_1$ to get

- $\begin{bmatrix} 1 & 1 & 2 & | & 9 \\ 0 & 2 & 0 & | & 4 \\ 0 & -2 & -3 & | & -13 \end{bmatrix}$

Gauss –Jordan Elimination Process

- $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & 0 & 4 \\ 0 & -2 & -3 & -13 \end{array}\right]$

The second step is to divide the second row by 2 gives us

Replace R_2 by (1/2) R_2

- $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & -2 & -3 & -13 \end{array}\right]$

Gauss –Jordan Elimination Process

- $\begin{bmatrix} 1 & 1 & 2 & \vert & 9 \\ 0 & 1 & 0 & \vert & 2 \\ 0 & -2 & -3 & \vert & -13 \end{bmatrix}$
- We reduce the second column to [0,1,0] by row operations

Replace R_1 by $R_1 - R_2$ and R_3 by $R_3 + 2R_2$ to get

- $\begin{bmatrix} 1 & 0 & 2 & \vert & 7 \\ 0 & 1 & 0 & \vert & 2 \\ 0 & 0 & -3 & \vert & -9 \end{bmatrix}$

Gauss –Jordan Elimination Process

- $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & -9 \end{array}\right]$
- The third step is to divide the third row by -3 gives us

Replace R_3 by (-1/3) R_2

- $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Gauss –Jordan Elimination Process

- $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$
- Subtract from the first row, elements of the third row multiplied by 2

Replace R_1 by $R_1 - 2R_3$

- $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$
- The solution is $x_1 = 1,\ x_2 = 2,\ x_3 = 3$