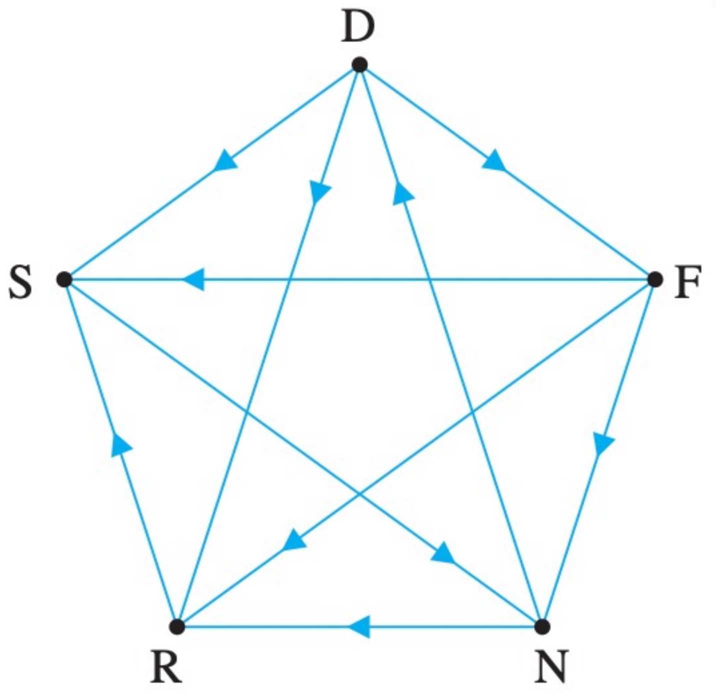
Matrix Algorithms in the Graph Theory

- Five tennis players (Djokovic, Federer, Nadal, Roddick, and Safin) compete in a round-robin tournament in which each player plays every other player once. The digraph in Figure summarizes the results. A directed edge from vertex *i* to vertex *j* means that player *i* defeated player *j*. (A digraph in which there is exactly one directed edge between every pair of vertices is called a ***tournament***.)



Matrix Algorithms in the Graph Theory

- The adjacency matrix for the digraph is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

where the order of the vertices (and hence the rows and columns of *A*) is determined alphabetically. Thus, Federer corresponds to row 2 and column 2, for example.

Matrix Algorithms in the Graph Theory

- Suppose we wish to rank the five players, based on the results of their matches. One way to do this might be to count the number of wins for each player. Observe that the number of wins each player had is just the sum of the entries in the corresponding row; equivalently, the vector containing all the row sums is given by the product $A\mathbf{j}$, where

$$\mathbf{j} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

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$$A\mathbf{j} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

which produces the following ranking:

- First: Djokovic, Federer (tie)
- Second: Nadal
- Third: Roddick, Safin (tie)

Matrix Algorithms in the Graph Theory

- Are the players who tied in this ranking equally strong?
- Djokovic might argue that since he defeated Federer, he deserves first place. Roddick would use the same type of argument to break the tie with Safin. However, Safin could argue that he has two “indirect” victories because he beat Nadal, who defeated *two* others; furthermore, he might note that Roddick has only *one* indirect victory (over Safin, who then defeated Nadal).

Matrix Algorithms in the Graph Theory

- Since in a group of ties there may not be a player who defeated all the others in the group, the notion of indirect wins seems more useful. Moreover, an indirect victory corresponds to a 2-path in the digraph, so we can use the square of the adjacency matrix.

Matrix Algorithms in the Graph Theory

- To compute both wins and indirect wins for each player, we need the row sums of the matrix $A + A^2$, which are given by

$$(A + A^2)\mathbf{j} = \left(\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Matrix Algorithms in the Graph Theory

$$= \begin{bmatrix} 0 & 1 & 2 & 2 & 3 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 6 \\ 2 \\ 3 \end{bmatrix}$$

Thus, we would rank the players as follows: Djokovic, Federer, Nadal, Safin, Roddick. Unfortunately, this approach is not guaranteed to break all ties.

Example 1

Use powers of adjacency matrices to determine the number of paths of the specified length between the given vertices.

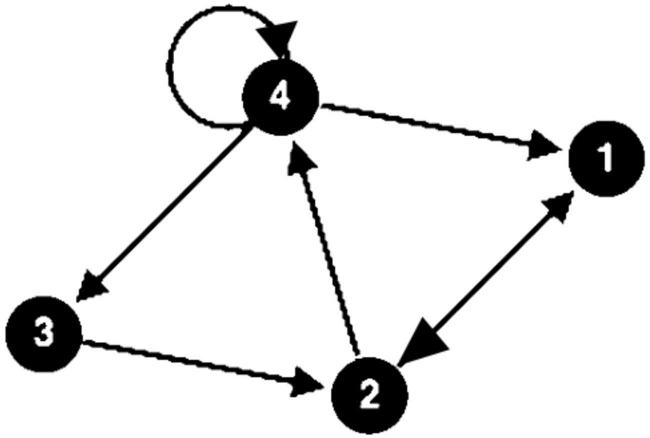
- length 2, v_1 and v_3
- Length 3, v_4 and v_1

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Solution 1

- $A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ = $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}$

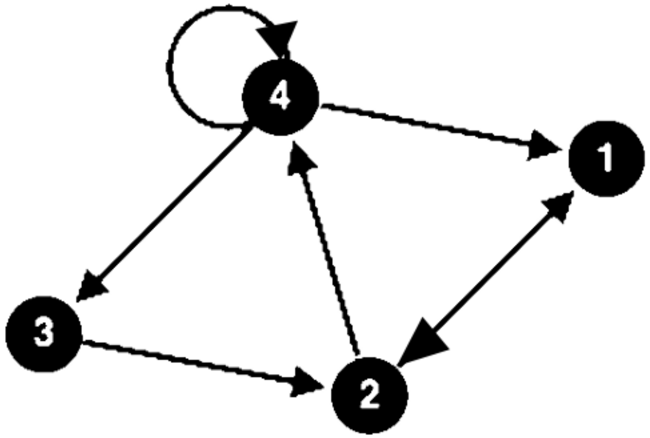
No of paths of length 2, v_1 and v_3 = 0



Solution 1

- $A^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 3 \end{bmatrix}$

No of paths of length 3, v_4 and v_1 = 3



Examples

Use powers of adjacency matrices to determine the number of paths of the specified length between the given vertices.

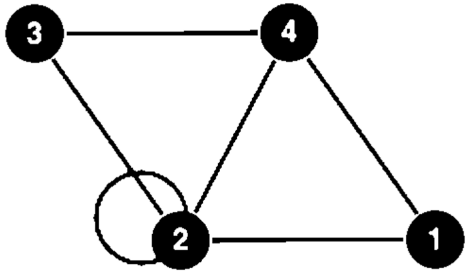
2. No of paths of length 2, v_2 to v_4
   No of paths of length 2, v_1 to v_2 for matrix A = $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$
   No of paths of Length 3, v_1 to v_3

3. No of paths of length 2, v_1 and v_2
   No of paths of length 2, v_4 and v_5 for matrix A = $\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$
   No of paths of Length 4, v_2 to v_3

4. No of paths of length 3, v_4 to v_1
   No of paths of length 3, v_5 to v_3 for matrix A = $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$
   No of paths of Length 4, v_1 to v_4

Solution 2

- $A^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$



$= \begin{bmatrix} 2 & 2 & 2 & 1 \\ 2 & 4 & 2 & 3 \\ 2 & 2 & 2 & 1 \\ 1 & 3 & 1 & 3 \end{bmatrix}$

No of paths of length 2, v_2 to v_4 = 3

No of paths of length 2, v_1 to v_2 = 2

Solution 2

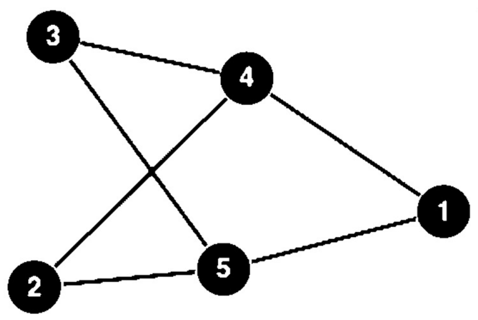
- $A^3 = \begin{bmatrix} 2 & 2 & 2 & 1 \\ 2 & 4 & 2 & 3 \\ 2 & 2 & 2 & 1 \\ 1 & 3 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 7 & 3 & 6 \\ 7 & 11 & 7 & 8 \\ 3 & 7 & 3 & 6 \\ 6 & 8 & 6 & 5 \end{bmatrix}$$

No of paths of length 3, v_1 to v_3 = 3

Solution 3

- $A^2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$



- = $\begin{bmatrix} 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 \end{bmatrix}$

No of paths of length 2, v_1 and v_2 = 2

No of paths of length 2, v_4 and v_5 = 3

Solution 3

- $A^4 = \begin{bmatrix} 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 \end{bmatrix} . \begin{bmatrix} 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 12 & 12 & 12 & 0 & 0 \\ 12 & 12 & 12 & 0 & 0 \\ 12 & 12 & 12 & 0 & 0 \\ 0 & 0 & 0 & 18 & 18 \\ 0 & 0 & 0 & 18 & 18 \end{bmatrix}$$

No of paths of length 4, v_2 and v_3 = 12

Solution 4

- $A^3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 2 & 3 & 0 & 3 & 3 \\ 3 & 3 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 \end{bmatrix}$

No of paths of length 3, v_4 to v_1 = 3

No of paths of length 3, v_5 to v_3 = 1

Solution 4

- $A^4 = \begin{bmatrix} 3 & 2 & 1 & 2 & 2 \\ 3 & 3 & 1 & 1 & 1 \\ 6 & 5 & 3 & 3 & 2 \\ 3 & 4 & 1 & 4 & 4 \\ 2 & 2 & 1 & 2 & 3 \end{bmatrix}$

No of paths of length 4, v_1 to v_4 = 2