Chapter 4

Co-ordinate Geometry

4.1 Introduction

René Descartes

In school you have been studying geometry which is known as the *Euclidian Geometry*. It was based on certain concepts and axioms.

René Descartes (1596–1650), French Mathematician and Philosopher first published his book 'La Geometric' in 1637 in which for the first time he used algebra in the study of geometry. He represented points in the plane by ordered pairs of real numbers, called Cartesian coordinates (named after his name Decartes) and represented lines and curves by algebraic equations. This branch of mathematics in which methods of algebra are used to solve geometrical problems is known as **Algebraic Geometry** or **Analytic Geometry** or **Cartesian Geometry** or **Co-ordinate Geometry**.

4.2 Cartesian Co-ordinate System

Let X'OX and Y'OY be two mutually perpendicular lines in the plane of the paper intersecting at O. We call the point O, the origin and the line X'OX is called X–axis and the line Y'OY is called Y–axis. These axes divides the plane

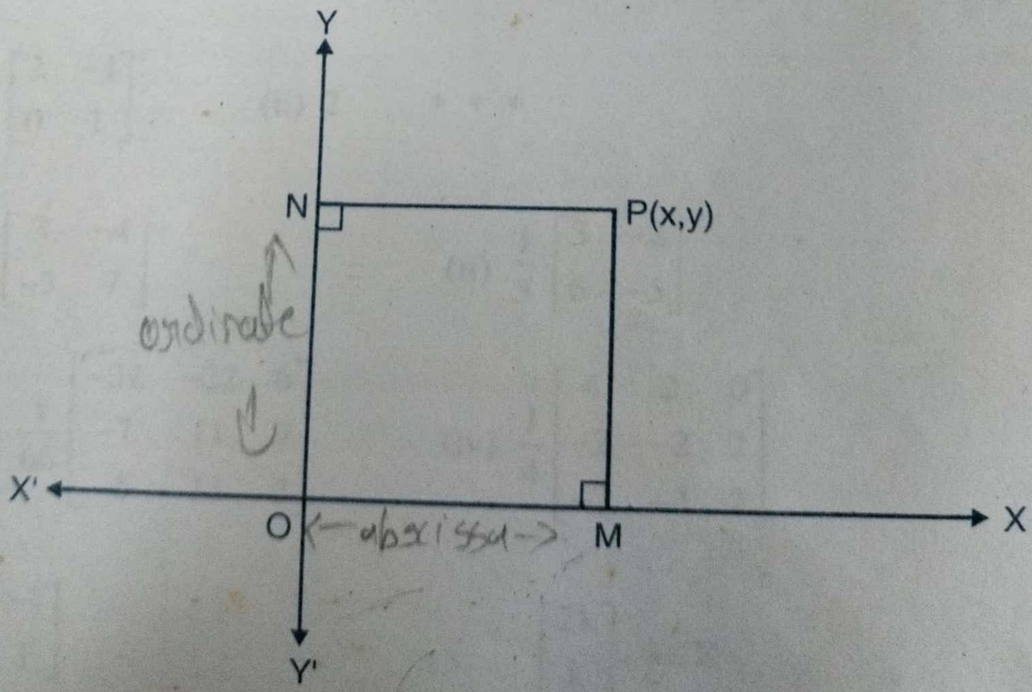


Fig. 4.1

into four parts, known as four quadrants. The four quadrants XOY, YOX', X'OY' and Y'OX which are respectively called the Ist, IInd, IIIrd and IVth quadrants. Let P be any point in the plane. Draw perpendiculars PM and PN from P on *X and Y–axis*. Then the length of directed line segment OM is called *abscissa* or *x–coordinate* of the point P and the length of directed line segment ON is called *ordinate* or *y–coordinate* of the point P. The ordered pair (x, y) is called the Cartesian co-ordinates or co-ordinates of point P.

Thus, for a given point in the plane, the abscissa and ordinate are the distances of the given point from y–axis and x–axis respectively.

The above system of co-ordinating an ordered pair (x, y) with every point in a plane is called the **Cartesian co-ordinate system** or **Rectangular co-ordinate system**.

The convention of sign for the four quadrants are as follows for the x–co-ordinate and y–co-ordinate. (Refer Fig. 4.2)

I^{st} quadrant $x > 0,\ y > 0$

II^{nd} quadrant $x < 0,\ y > 0$

III^{rd} quadrant $x < 0,\ y < 0$

IV^{th} quadrant $x > 0,\ y < 0$

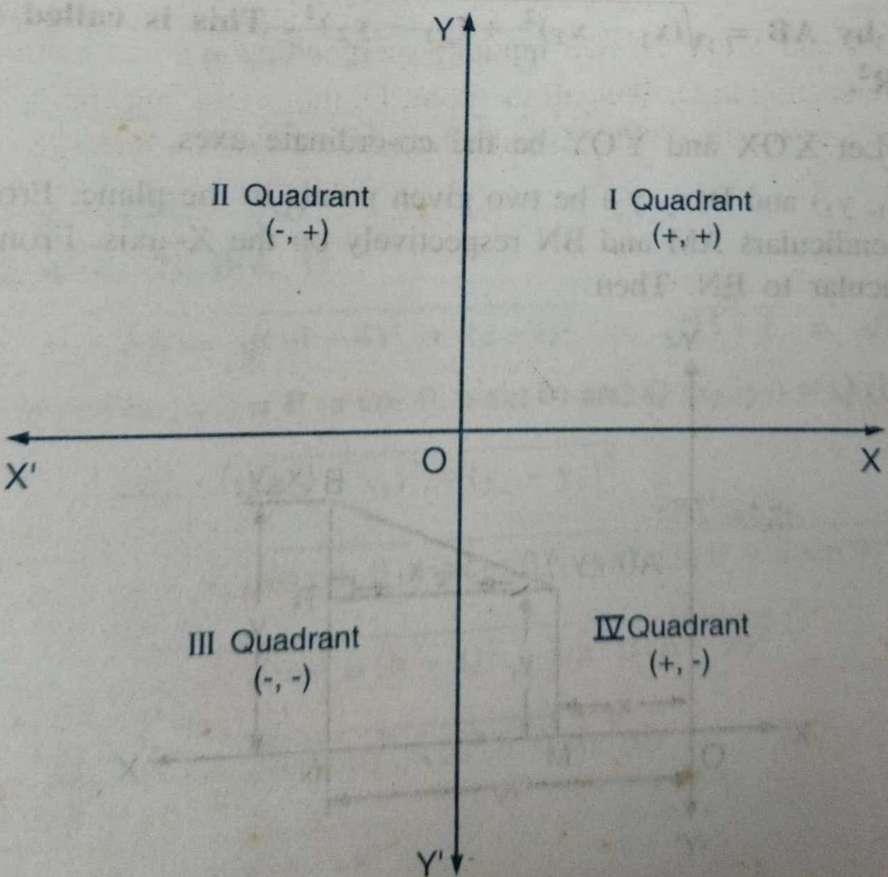


Fig. 4.2

Remarks

(i) The co-ordinates of the origin are taken as (0, 0).
(ii) The co-ordinates of any point on X–axis are of the form (x, 0).
(iii) The co-ordinates of any point on Y–axis are of the form (0, y).

EXAMPLE 1 In which quadrant do the following points lie ?
(i) (1, 2) (ii) (–3, –4) (iii) (4, –3) (iv) (–4, 5)

SOLUTION

(i) In the point (1, 2) both the co-ordinate are positive, so it lies in the first quadrant.
(ii) The point (–3, –4) lies in the third quadrant, because $x, y < 0$.
(iii) The point (4, –3) lies in the fourth quadrant, because $x > 0, y < 0$
(iv) The point (–4, 5) lies in the second quadrant, because $x < 0, y > 0$.

4.3 Distance Between Two Points (Distance Formula in R^2)

The distance between any two points in the plane is the length of the line segment joining them.

The distance between two distinct points $A(x_1, y_1)$ and $B(x_2, y_2)$ in R^2 is given by $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. This is called distance formula in R^2.

Proof Let X'OX and Y'OY be the co-ordinate axes.

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points in the plane. From A and B draw perpendiculars AM and BN respectively on the X–axis. From A draw AR perpendicular to BN. Then

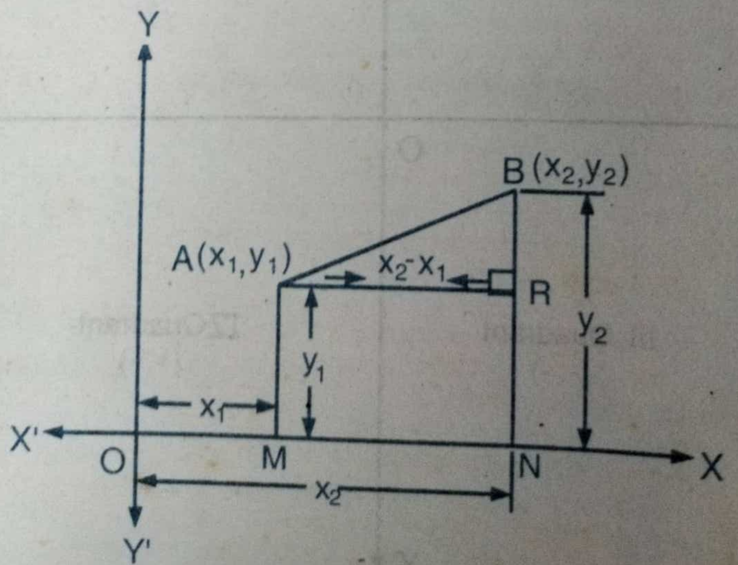


Fig. 4.3

$OM = x_1$, $ON = x_2$, $AM = y_1$ and $BN = y_2$

$AR = MN = ON - OM = x_2 - x_1$ and

$BR = ON - RN = BN - AM = y_2 - y_1$

From right angled ΔABR,

By Pythagoras theorem, we have

$$AB^2 = AR^2 + BR^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\therefore \quad AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\boxed{\therefore \quad AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}$$

EXAMPLE 1 Find the distance between the pair of points (i) (2, –1), (3, 2) (ii) (–1, 2), (4, 1) (iii) $(a\cos\theta, a\sin\theta)$, $(b\cos\theta, b\sin\theta)$

SOLUTION

(i) Let $A(x_1, y_1) = A(2, -1)$ and $B(x_2, y_2) = (3, 2)$

x_1 y_1 x_2 y_2

Using the distance formula,

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(3-2)^2 + (2+1)^2}$$
$$= \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\therefore \quad AB = \sqrt{10}$$

(ii) $A(-1, 2)$, $B(4, 1)$

$$\therefore \quad AB = \sqrt{(-1-4)^2 + (2-1)^2} = \sqrt{25+1} = \sqrt{26}.$$

(iii) Let $P(x_1, y_1) = P(a\cos\theta, a\sin\theta)$ and $Q(x_2, y_2) = Q(b\cos\theta, b\sin\theta)$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(b\cos\theta - a\cos\theta)^2 + (b\sin\theta - a\sin\theta)^2}$$
$$= \sqrt{\cos^2\theta\,(b-a)^2 + \sin^2\theta\,(b-a)^2}$$
$$= \sqrt{(b-a)^2\,(\cos^2\theta + \sin^2\theta)}$$
$$= \sqrt{(b-a)^2}$$

$$\therefore \quad PQ = |\,b - a\,|$$

EXAMPLE 2 Show that the points (1, 1), (–2, 7) and (3, –3) are collinear.

[**Note** : Points A, B and C are said to be collinear if they are on the same line.

In this case, either AB = AC + CB or BC = BA + AC or AC = AB + BC]

SOLUTION Let A(1, 1), B (–2, 7) and C (3, –2) be the given points, Then we have,

$$AB = \sqrt{(-2-1)^2 + (7-1)^2} = \sqrt{9+36} = 3\sqrt{5}$$

$$BC = \sqrt{(3+2)^2 + (-3-7)^2} = \sqrt{25+100} = 5\sqrt{5}$$

and $$AC = \sqrt{(3-1)^2 + (-3-1)^2} = \sqrt{4+16} = 2\sqrt{5}$$

$\therefore$ Clearly, BC = AB + AC;

Hence A, B, C are collinear.

EXAMPLE 3 If the distance between A(5, a) and B(2, 6) is $3\sqrt{2}$, find the value of a.

SOLUTION Here $AB = 3\sqrt{2}$

$\therefore$ $AB^2 = 18$

$\therefore$ $(2-5)^2 + (6-a)^2 = 18$

$\therefore$ $9 + 36 + a^2 - 12a = 18$

$\therefore$ $a^2 - 12a + 27 = 0$

$\therefore$ $(a-9)(a-3) = 0$

$\therefore$ $a - 9 = 0$ or $a - 3 = 0$

$\therefore$ $a = 9$ or $a = 3$

EXAMPLE 4 Show that the vertices of a triangle (7, 9), (3, –7) and (–3, 3) form a right angled isosceles triangle.

SOLUTION Let A(7, 9), B(3, –7) and C(–3, 3) are vertices of Δ ABC

$$AB^2 = (3-7)^2 + (-7-9)^2 = 16 + 256 = 272$$

$$BC^2 = (-3-3)^2 + (3+7)^2 = 36 + 100 = 136$$

$$AC^2 = (-3-7)^2 + (3-9)^2 = 100 + 36 = 136$$

Hence $AB^2 = BC^2 + AC^2$

$\therefore$ $\angle C = 90^\circ$

Also, $BC = AC = \sqrt{136}$ (Two sides are equal)

$\therefore$ The Δ ABC is right angled isosceles.

EXAMPLE 5 A(–1, 3), B(–1, x) and C(4, 3) are vertices of Δ ABC, $m\angle B = 90^\circ$. Find x.

SOLUTION Given that $m\angle B = 90^\circ$

By Pythagoras theorem,

$\therefore \quad AB^2 + BC^2 = AC^2$

$(-1 + 1)^2 + (x - 3)^2 + (4 + 1)^2 + (3 - x)^2 = (4 + 1)^2 + (3 - 3)^2$

$x^2 - 6x + 9 + 25 + 9 - 6x + x^2 = 25$

$\Rightarrow 2x^2 - 12x + 18 = 0$

$\Rightarrow x^2 - 6x + 9 = 0$

$\Rightarrow (x - 3)^2 = 0$

$\Rightarrow x - 3 = 0$

$\Rightarrow x = 3$

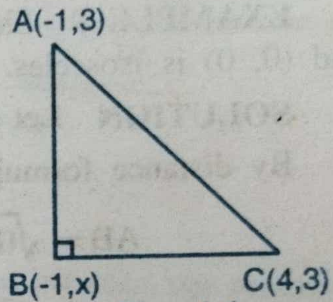


Fig. 4.4

EXAMPLE 6 Prove that the points A(4, 4), B(3, 5) and C(–1, –1) are the vertices of a right angled triangle.

SOLUTION Here, A(4, 4), B(3, 5), C(–1, –1)

By distance formula,

$AB^2 = (4 - 3)^2 + (4 - 5)^2$

$= 1 + 1$

$\therefore \quad AB^2 = 2$

$BC^2 = (3 + 1)^2 + (5 + 1)^2$

$= 16 + 36$

$\therefore \quad BC^2 = 52$

and, $AC^2 = (4 + 1)^2 + (4 + 1)^2$

$= 25 + 25$

$\therefore \quad AC^2 = 50$

$\therefore$ We observe that $AB^2 + AC^2 = BC^2$

$\therefore$ Δ ABC is right angled.

EXAMPLE 7 Find a point on the Y–axis which is equidistant from the points A(–5, –2) and B(3, 2).

SOLUTION Let the point be P(0, y)

$\Rightarrow AP^2 = BP^2$ (By squaring)

$\Rightarrow (-5-0)^2 + (-2-y)^2 = (3-0)^2 + (2-y)^2$

$\Rightarrow 25 + 4 + 4y + y^2 = 9 + 4 - 4y + y^2$

$\Rightarrow 8y = -16$

$\Rightarrow y = -2$

Thus, the required point is P(0, –2).

EXAMPLE 8 Show that a triangle whose vertices are (8, 2), (5, –3) and (0, 0) is isosceles.

SOLUTION Let A(8, 2), B(5, –3), C(0, 0).

By distance formula;

$$AB = \sqrt{(8-5)^2 + (2+3)^2}$$
$$= \sqrt{9 + 25}$$
$$= \sqrt{34}$$
$$BC = \sqrt{25 + 9}$$
$$= \sqrt{34}$$

and $$AC = \sqrt{64 + 4}$$
$$= \sqrt{68}$$

Colline

Thus, $AB = BC$

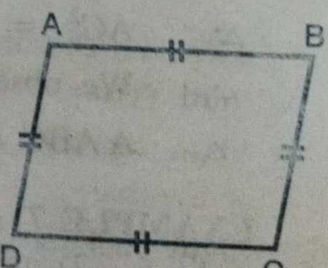
$\therefore$ ΔABC is an isosceles.

EXAMPLE 9 Show that the points A(1, 0), B(5, 3), C(2, 7) and D(–2, 4) are the vertices of a rhombus.

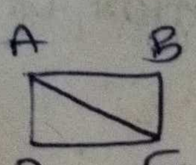
SOLUTION We note that rhombus is a quadrilateral in which measures of all four sides are equal.

Here, A(1, 0), B(5, 3), C(2, 7) and D(–2, 4).

By distance formula;



$$AB = \sqrt{(1-5)^2 + (0-3)^2} = \sqrt{16 + 9} = 5$$
$$BC = \sqrt{(5-2)^2 + (3-7)^2} = \sqrt{9 + 16} = 5$$
$$CD = \sqrt{(2+2)^2 + (7-4)^2} = \sqrt{16 + 9} = 5$$
$$DA = \sqrt{(-2-1)^2 + (4-0)^2} = \sqrt{9 + 16} = 5$$



ΔABC

ΔADC

Thus, AB = BC = CD = DA

∴ □ ABCD is a rhombus.

EXERCISES 4.1

1. Find the distance between each of following the pairs of points.
   (i) $(1, 2), (-1, 2)$
   (ii) $(x - y, y - x), (x + y, y + x)$
   (iii) $(1, 2), (1, 0)$
   (iv) $(2, -1), (1, -1)$
2. Find the value of k, if the distance between the points $(3, k)$ and $(4, 1)$ is $\sqrt{10}$.
3. Find a point on the X–axis which is equi distance from points $(7, 6)$ and $(-3, 4)$.
4. Find the point on X–axis which is equi distance from $(3, 2)$ and $(-5, -2)$.
5. Which point on the Y–axis is equi distance from $(-5 - 2)$ and $(3, 2)$?
6. If a point $P(x, y)$ is equidistant from $(6, -1)$ and $(2, 3)$ then prove that $x - y - 3 = 0$.
7. Prove that the points $(-2, 5)$, $(0, 1)$ and $(2, -3)$ are collinear.
8. Determine, by distance formula, whether the points.
   (i) $(0, 0), (2, 3), (6, 9)$
   (ii) $(0, 0), (1, 0), (0, -4)$ lie on a line ?
9. Show that the points $(1, 1)$, $(-1, 1)$ and $(0, 1 + \sqrt{3})$ are the vertices of an equilateral triangle.
10. The vertices of a triangle are $(1, 2\sqrt{3})$, $(3, 0)$ and $(-1, 0)$, the triangle right angled, equilateral or isosceles ?
11. An equilateral triangle has one vertex at the origin and another at $(3, \sqrt{3})$. Find the co-ordinates of the third vertex.
12. Two vertices of an equilateral triangle are $(0, 0)$ and $(0, 2\sqrt{3})$. Find the third vertex.
13. Show that the quadrilateral with vertices $(3, 2)$, $(0, 5)$, $(-3, 2)$, $(0, -1)$ is a square.
14. Show that the quadrilateral with vertices $(3, 2)$, $(0, 5)$, $(-3, 2)$, $(0, -1)$ is a square.
15. Show that $(-1, 0)$, $(2, 3)$, $(4, 1)$ and $(1, -2)$ are the vertices of a rectangle.

4.4 Area of a Triangle

Let us derive a formula to find the area of a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the three vertices of a Δ ABC as shown in Fig. 4.5.

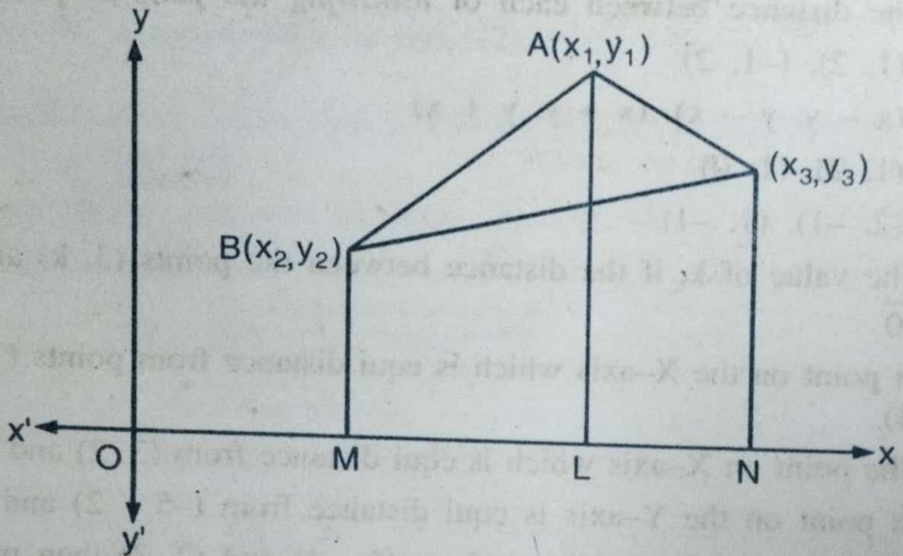


Fig. 4.5

Draw perpendiculars from points A, B and C to the X–axis meeting in L, M and N respectively.

$$\text{Area of } \Delta \text{ ABC} = \text{Area of trapezium BMLA} + \text{Area of trapezium ALNC} - \text{Area of trapezium BMNC} \quad ...(1)$$

Now, Area of a trepezium $= \frac{1}{2}$ (Sum of parallel sides) · (Distance between the parallel sides)

$\therefore$ Area of BMLA $= \frac{1}{2}(MB + LA) \times ML = \frac{1}{2}(y_2 + y_1)(x_1 - x_2)$

Area of ALNC $= \frac{1}{2}(LA + NC) \times LN = \frac{1}{2}(y_1 + y_3)(x_3 - x_1)$

and Areá of BMNC $= \frac{1}{2}(MB + NC) \times MN = \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$

Substituting the above in the right hand expression of (1), we get

$$\text{Area of } \Delta \text{ ABC} = \frac{1}{2} \mid (y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - (y_2 + y3)(x_3 - x_2) \mid$$

$$\therefore \text{ Area of } \Delta\ ABC = \frac{1}{2}\ |\ [x_1\ (y_2 - y_3) + x_2\ (y_3 - y_1) + x_3\ (y_1 - y_2)]\ |$$

$$\therefore \text{ Area of } \Delta\ ABC = \frac{1}{2}\ |\ x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3\ |$$

$$\boxed{\therefore \textbf{ Area of } \Delta\ \textbf{ABC} = \frac{1}{2}\ |\ (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\ |}$$

Remark

The above formula can be written as

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{array}$$

$$\therefore \text{ Area of } \Delta\ ABC = \frac{1}{2}\ |\ x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3\ |$$

Condition of Collinearity of three Points

Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear if and only if points are on the same line.

Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ lie on a line if and only if the area of the Δ ABC is zero.

$$\therefore \qquad x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2 = 0$$

EXAMPLE 1 Find the area of the triangle whose vertices are (4, 4), (3, –2) and (–3, 16).

SOLUTION

$$\begin{array}{cccc} 4 & 3 & -3 & 4 \\ 4 & -2 & 16 & 4 \end{array}$$

The area of the triangle formed by vertices A(4, 4), B(3, –2) and C(–3, 16) is given by

$$\Delta = \frac{1}{2}\ |\ 4(-2) + 3(16) + (-3)(4) - (4)(3) - (-2)(-3) - (16)(4)\ |$$

$$= \frac{1}{2}\ |\ -8 + 48 - 12 - 64 - 12 - 6\ |$$

$$= \frac{1}{2}\ |\ -54\ | = 27$$

EXAMPLE 2 Show that the points (–1, –1), (2, 3) and (8, 11) are collinear.

SOLUTION

$$\begin{array}{cccc} -1 & 2 & 8 & -1 \\ -1 & 3 & 11 & -1 \end{array}$$

The area of the triangle, formed by the given points as vertices, is given by

$$\Delta = \frac{1}{2}\left| -3 + 22 - 8 + 2 - 24 + 11 \right|$$

$$\Delta = \frac{1}{2}\left| 0 \right|$$

$\therefore \quad \Delta = 0$

$\therefore$ Points are collinear.

EXAMPLE 3 For what value of x the area of the triangle formed by the vertices (x, 4), (8, 2) and (6, 7) is 13 units ?

SOLUTION

$$\begin{array}{cccc} x & 8 & 6 & x \\ 4 & 2 & 7 & 4 \end{array}$$

$$\text{Area of the triangle} = \frac{1}{2}\left| 2x + 56 + 24 - 32 - 12 - 7x \right|$$

$$\therefore\ 13 = \frac{1}{2}\left| -5x + 36 \right|$$

$$\therefore\ 26 = \left| -5x + 36 \right|.$$

$$\therefore\ -5x + 36 = 26 \quad \text{or} \quad -5x + 36 = -26$$

$$\therefore\ x = 2 \ \text{or} \ x = \frac{62}{5}$$

EXAMPLE 4 If the points (k, 3), $\left(\frac{7}{2}, \frac{9}{2}\right)$ and (5, 6) are collinear, find the value of k.

SOLUTION

$$\begin{array}{cccc} k & \frac{7}{2} & 5 & k \\ 3 & \frac{9}{2} & 6 & 3 \end{array}$$

The points are collinear, if the area of a triangle is zero.

$\therefore \quad \Delta = 0$

$$\Rightarrow \left| \frac{9k}{2} + \frac{42}{2} + 15 - \frac{21}{2} - \frac{45}{2} - 6k \right| = 0$$

$$\Rightarrow \left| \frac{-3k}{2} - 12 + 15 \right| = 0$$

$$\Rightarrow | -3k + 6 | = 0$$

$$\Rightarrow | k - 2 | = 0$$

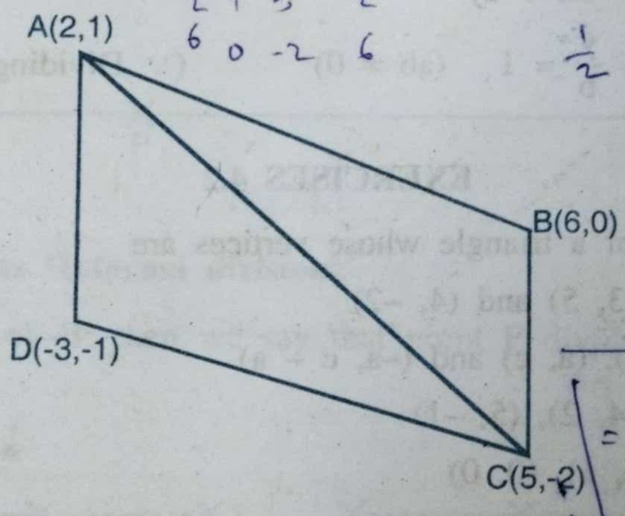
$$\Rightarrow k - 2 = 0$$

$$\therefore \quad k = 2$$

EXAMPLE 5 Find the area of the quadrilateral whose vertices are (2, 1), (6, 0), (5, –2) and (–3, –1).

SOLUTION

Let the points be A(2, 1), B(6, 0), C(5, –2) and D(–3, –1).



$= \frac{1}{2}[x_1(y_2 - y_3)] + x_2[y_3 - y_1] + x_3[y_1 - y_2]$

Fig. 4.6

Now, area of triangle ABC $= \frac{1}{2} | 2(0 + 2) + 6(-2 - 1) + 5(1 - 0) |$

$= 4.5$ sq. units

and area of triangle ACD $= \frac{1}{2} | 2\{-2 - (-1)\} + 5(-1 - 1) - 3(1 + 2) |$

$= 10.5$ sq. units

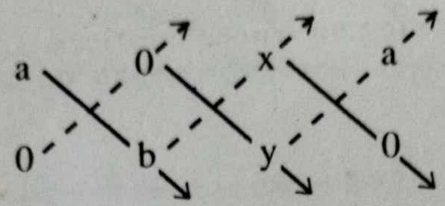
From the Fig. 7.5, it can be seen that

Area of quadrilateral ABCD = Area of Δ ABC + Area of Δ ACD

$= 4.5 + 10.5 = 15.0$ sq. units

EXAMPLE 6 The points A(a, 0), B(0, b) and C(x, y) are collinear, prove that $\frac{x}{a} + \frac{y}{b} = 1$.

SOLUTION As points A(a, 0), B(0, b) and C(x, y) are collinear.



$\therefore$ Area of Δ ABC = 0

$\Rightarrow \; | ab + 0 + 0 - 0 - bx - ay | = 0$

$\Rightarrow \; | ab - bx - ay | = 0$

$\Rightarrow \; ab - bx - ay = 0$

$\Rightarrow \; ab = bx + ay$

$\Rightarrow \; \frac{x}{a} + \frac{y}{b} = 1 \quad (ab \neq 0)$ $\quad$ ($\because$ Dividing by ab)

EXERCISES 4.2

1. Find the area of a triangle whose vertices are
   (i) (6, 3), (–3, 5) and (4, –2)
   (ii) (a, c + a), (a, c) and (–a, c – a).
   (iii) (3, 8), (–4, 2), (5, –1)
   (iv) (1, 0), (0, 1), (0, 0)
2. Prove that the points (a, 0), (0, b) and (1, 1) are collinear, if $\frac{1}{a} + \frac{1}{b} = 1$.
3. Show that the following sets of points are collinear.
   (a) (2, 5), (4, 6) and (8, 8)
   (b) (1, –1), (2, 1) and (4, 5).
   (c) (4, 7), (0, 1), (2, 4)
4. Find the value of x so that the points (–3, 12), (7, 6), (x, 9) are collinear.
5. Show that the points (a, b + c), (b, c + a) and (c, a + b) are collinear.
6. If (3, k), (9, 3) and (5, 2) are vertices of a triangle whose area is 7 sq. units, find the value of k.
7. The area of a triangle formed by the points (1, 1), (3, 4) and (x, –2) is 9 sq. units. What is the value of x ?

8. A(3, 4), B(5, −2). Find the co-ordinates of a point P such that PA = PB and the area of Δ PAB = 10.

9. Find the area of the quadrilateral with vertices (3, 2), (−3, 4), (−2, −3) and (2, −2).

10. Find the area of the Pentagon whose vertices are (1, 5), (−2, 4), (−3, −1), (2, −3), (5, 1).

4.5 Division of a Line Segment

Let A and B be two distinct points in R^2.

Let $P \neq A$, $P \neq B$.

If A–P–B; then we say that point P divides $\overline{AB}$ from A's side in the ratio $\frac{AP}{PB}$.

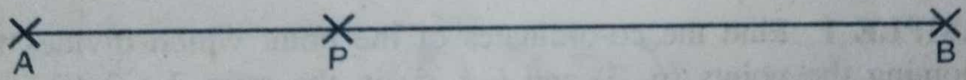


Fig. 4.7

This is known as **'Internal division'**.

If A–B–P or P–A–B, then we say that point P divides $\overline{AB}$ from A's side in ratio $\frac{AP}{PB}$.

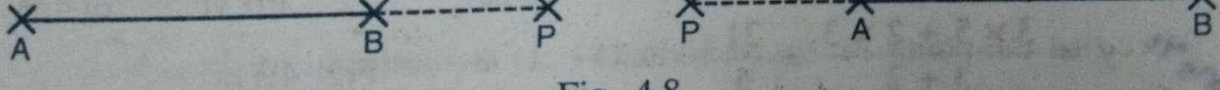


Fig. 4.8

This is known as **'External division'**.

Ratio corresponding to internal division is taken **to be positive** and for external division it is taken **to be negative.**

Section Formula (Division Formula)

If a point P divides $\overline{AB}$, where $A(x_1, y_1)$ and $B(x_2, y_2)$ from the A's side in the ratio m : n, then co-ordinates of P are $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$;

$m + n \neq 0$.

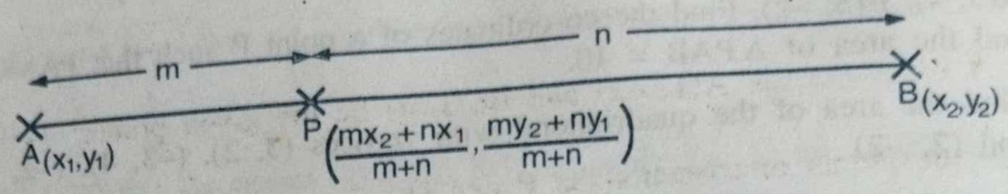


Fig. 4.9

Note If P is the mid-point of $\overline{AB}$ then m = n.

$\therefore$ Co-ordinates of $P = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

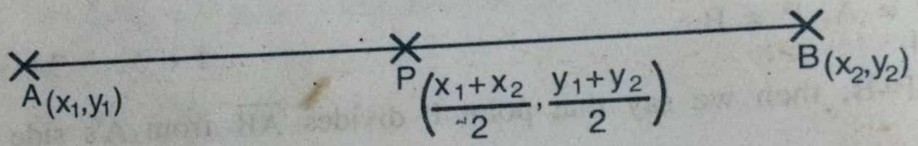


Fig. 4.10

EXAMPLE 1 Find the co-ordinates of the point which divides the line segment joining the points (6, 3) and (–4, 5) in the ratio 3 : 2 (i) internally and (ii) externally.

SOLUTION Let P(x, y) be the required point

(i) For internal division, we have

Internal. $x = \frac{m_1x_2 + m_2x_1}{m_1+m_2}$ $y = \frac{m_1y_2 + m_2y_1}{m_1+m_2}$

$y_1 \times x_1 +$

$$x = \frac{3 \times -4 + 2 \times 6}{3 + 2} = 0$$

$$y = \frac{3 \times 5 + 2 \times 3}{3 + 2} = \frac{21}{5}$$

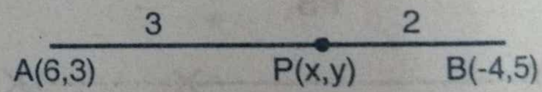


Fig. 4.11

$\therefore$ The co-ordinates of P are $\left(0, \frac{21}{5}\right)$

(ii) For external division, we have

External $x = \frac{m_1x_2 - m_2x_1}{m_1+m_2}$ $y = \frac{m_1y_2 + m_2y_1}{m_1+m_2}$.

$$x = \frac{3 \times -4 - 2 \times 6}{3 - 2} = -24$$

$$y = \frac{3 \times 5 - 2 \times 3}{3 - 2} = 9$$

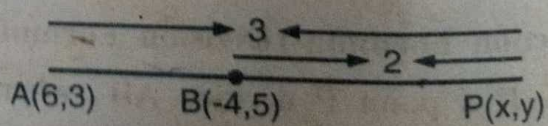


Fig. 4.12

$\therefore$ The co-ordinates of P are (–24, 9)

EXAMPLE 2 Find the co-ordinates of points which trisect the line segment joining (1, −2) and (−3, 4).

SOLUTION Let A(1, −2) and B(−3, 4) be the given points.

Let the points of trisection be P and Q.

Therefore P divides $\overline{AB}$ internally in the ratio 1 : 2.

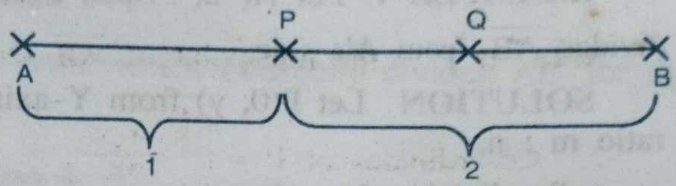


Fig. 4.13

Thus,

$$m_1x_2 + m_2x_1 \qquad m_1y_2 + m_2y_1$$

$$\text{the co-ordinates of } P = \left(\frac{1 \times -3 + 2 \times 1}{1 + 2}, \frac{1 \times 4 + 2 \times -2}{1 + 2}\right)$$

$$m_1 + m_2 \qquad m_1 + m_2$$

$$= \left(-\frac{1}{3}, 0\right)$$

Now, Q is the mid-point of $\overline{PB}$.

$$m_1 x_1 \quad m_2 . y_2$$

$$\therefore \quad \text{Co-ordinates of } Q\left(\frac{\frac{-1}{3} + (-3)}{2}, \frac{0 + 4}{2}\right) = Q\left(-\frac{5}{3}, 2\right)$$

$$m_2 \qquad m_2$$

$\therefore$ The points of trisection are $\left(-\frac{1}{3}, 0\right)$ and $\left(-\frac{5}{3}, 2\right)$.

EXAMPLE 3 Find the ratio in which P(−1, −1) divides $\overline{AB}$, where A(4, 4), B(7, 7).

SOLUTION Suppose P(−1, −1) divides $\overline{AB}$ in ratio m : n.

By division formula,

$$P(-1, -1) = \left(\frac{7m + 4n}{m + n}, \frac{7m + 4n}{m + n}\right)$$

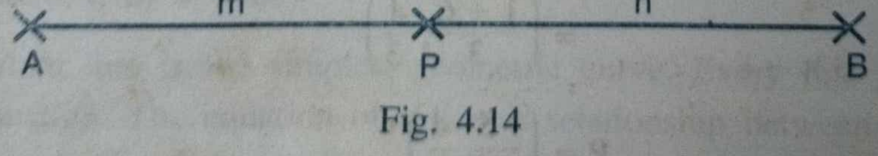


Fig. 4.14

$$\therefore \quad -1 = \frac{7m + 4n}{m + n}, \quad -1 = \frac{7m + 4n}{m + n}$$

$$\therefore \quad -m - n = 7m + 4n$$

$\therefore \quad -8m = 5n$

$\therefore \quad \dfrac{m}{n} = -\dfrac{5}{8}$

$\therefore$ P divides $\overline{AB}$ from A's side in ratio 5 : 8 externally.

EXAMPLE 4 For A(–2, 3) and B(3, 0) find the ratio in which Y–axis divides $\overline{AB}$ from A's side.

SOLUTION Let P(0, y) from Y–axis divides $\overline{AB}$ from A's side in the ratio m : n.

By division formula,

$$P(0,\ y) = \left(\frac{3(m) + n(-2)}{m + n}, \frac{m(0) + n(3)}{m + n}\right)$$

$$\Rightarrow \quad 0 = \frac{3m - 2n}{m + n}, \quad y = \frac{0 + n(3)}{m + n}$$

$$\Rightarrow \quad 3m - 2n = 0$$

$$\Rightarrow \quad 3m = 2n$$

$$\Rightarrow \quad \frac{m}{n} = \frac{2}{3}$$

$\therefore$ Ratio in which Y–axis divides $\overline{AB}$ from A's side is 2 : 3.

EXAMPLE 5 For A(1, 2), B(–3, 1), find the point dividing $\overline{AB}$ from B's side in the ratio 1 : 2.

SOLUTION Here $A(1,\ 2) = (x_2,\ y_2)$

$B(-3,\ 1) = (x_1,\ y_1)$, Ratio = 1 : 2

Let a point be P(x, y)

$$\therefore \quad \text{Co-ordinates of } P = \left(\frac{1(1) + 2(-3)}{1 + 2}, \frac{1(2) + 2(1)}{1 + 2}\right)$$

$$= \left(\frac{1 - 6}{3}, \frac{4}{3}\right)$$

$$P = \left(\frac{-5}{3}, \frac{4}{3}\right)$$

$\therefore$ Required point is $P\left(\dfrac{-5}{3}, \dfrac{4}{3}\right)$.

EXERCISES 4.3

1. Find the co-ordinates of the point which internally divides $\overline{AB}$ for the points and ratio given as

   (i) A(1, –3), B(–3, 9); Ratio 1 : 3.

   (ii) A(–2, –1), B(4, 3); Ratio 2 : 3.

2. Find the co-ordinates of the point which externally divides $\overline{AB}$ for the points and ratio given as

   (i) A(0, –4), B(8, 0); Ratio 4 : 3.

   (ii) A(2, –6), B(4, 3); Ratio 3 : 2.

3. Find the ratio in which the line joining (2, –3) and (5, 6) is divided by (i) the X–axis (ii) the Y–axis.

4. In what ratio does the point $\left(\frac{1}{2}, 6\right)$ divide the line segment joining the points (3, 5) and (–7, 9) ?

5. Find the co-ordinates of the points of trisection of the line–segment joining the poins A(1, 1), B(7, 4).

6. Three vertices of a triangle are A(1, 2), B(–1, 6) and C(5, 4). Find the co-ordinates of the mid–points of the sides of the triangle.

7. The mid–points of the sides of a triangle are at $\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{3}{2}\right)$ and (1, 1). Find the co-ordinates of the vertices of the triangle.

8. Three vertices of a triangle at (2, 2), (0, 6) and (8, 10). If D, E and F are the mid–points of the sides, show that the area of Δ ABC is four times the area of Δ DEF.

4.6 Introduction to a Line

A straight line is the simplest geometric curve. Every line is associated with an equation. The equation of a line is relationship between the abscissa and ordinate of a general point lying on the line under some suitable conditions that the line satisfies. In this chapter we will study various forms of equation of a line.

4.7 Slope of a Line

Definition

Let a $\overleftrightarrow{AB}$ passing through points $A(x_1, y_1)$ and $B(x_2, y_2)$, $x_1 \neq x_2$ then slope of the $\overleftrightarrow{AB}$ is defined as $\dfrac{y_2 - y_1}{x_2 - x_1}$, $x_1 \neq x_2$. It is denoted by m.

$$\therefore \quad m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of y-co-ordinates}}{\text{Difference of x-co-ordinates}}$$

We note that

(i) For any two different points of a line slope remains same.

(ii) Slope of X–axis (or any horizontal line) is zero.

(iii) Slope of Y–axis (or any vertical line) is not defined.

Geometrical meaning of slope of a line

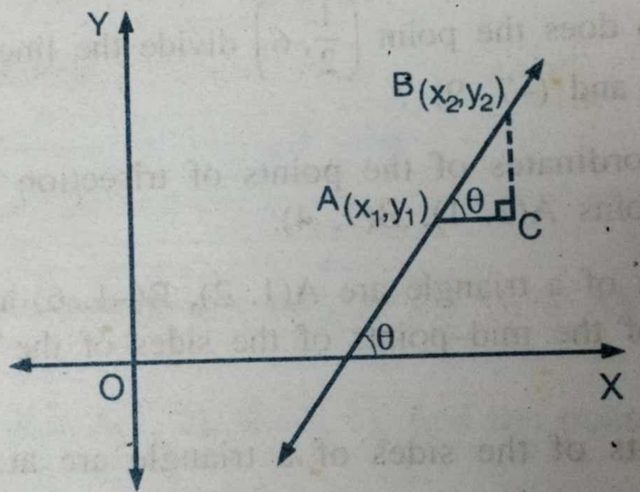


Fig. 4.15

If a $\overleftrightarrow{AB}$ makes angle θ with positive direction of X–axis.

Then in Δ ABC, as shown in the Fig. 4.15.

$$\tan\theta = \frac{BC}{AC} = \frac{y_2 - y_1}{x_2 - x_1} = \text{slope of the line}$$

$$\therefore \quad \boxed{m = \tan\theta}$$

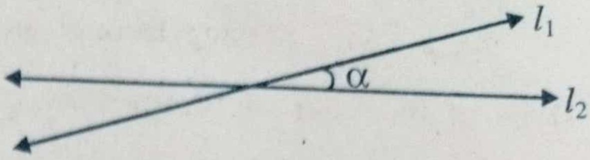
Thus, if a line makes angle θ with positive direction of X–axis, then m = tan θ is defined as slope of the line.

4.8 Angle between two Lines

We consider any two non perpendicular lines l_1 and l_2 neither of which is parallel to Y–axis and derive the formula for the angle between the lines l_1 and l_2 in terms of their slopes.

If α is the angle between lines l_1 and l_2 with slopes m_1 and m_2 respectively, then

$$\boxed{\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|}$$



Condition for parallel lines

If two lines having slopes m_1 and m_2 are parallel, then the angle α between them is of 0°.

$$\therefore \tan\alpha = \tan 0^\circ = 0 \Rightarrow \frac{m_2 - m_1}{1 + m_1 m_2} = 0$$

$$\Rightarrow m_1 = m_2$$

Thus, two lines are parallel $\Leftrightarrow$ their slopes are equal.

Condition for perpendicular lines

If two lines having slopes m_1 and m_2 are perpendicular, then the angle α between them is 90°.

$$\therefore \tan\alpha = \tan 90^\circ = \frac{m_2 - m_1}{1 + m_1 m_2} \quad \Rightarrow \cot 90^\circ = \frac{1 + m_1 m_2}{m_2 - m_1} = 0$$

$$\Rightarrow 1 + m_1 m_2 = 0$$

$$\Rightarrow m_1 m_2 = -1$$

Thus, two lines are perpendicular to each other $\Leftrightarrow$ the product of their slopes is –1.

$\therefore$ if m is the slope of a line, then the slope of a line perpendicular to it is $-\frac{1}{m}$.

EXAMPLE 1 Find the slope of the line through the points

(i) (1, 2), (4, 3)

(ii) (0, 4), (–6, –2)

(iii) (1, 4), (4, 4)

SOLUTION

(i) Slope of a line through (1, 2) and (4, 3) is $m = \frac{3-2}{4-1}$

$$= \frac{1}{3}$$

(ii) Slope of a line through (0, 4) and (–6, –2) is $m = \frac{-2-4}{-6-0}$

$$m = \frac{-6}{-6}$$

$$\therefore \quad m = 1$$

(iii) Slope of a line through (1, 4) and (4, 4) is $m = \frac{4-4}{4-1}$

$$\therefore \quad m = 0$$

EXAMPLE 2 Determine x such that 3 is the slope of a line through points (2, 5) and (x, 4).

SOLUTION Slope through the points (2, 5) and (x, 4) is

$$m = \frac{4-5}{x-2}$$

$$\Rightarrow 3 = \frac{-1}{x-2}$$

$$\Rightarrow 3x - 6 = -1$$

$$\Rightarrow 3x = 5$$

$$\Rightarrow x = 5$$

EXAMPLE 3 Find the slope of a line whose inclination to the positive direction of X-axis is (i) 30° (ii) 0° (iii) 90°.

SOLUTION

(i) Slope $= \tan 30^\circ = \frac{1}{\sqrt{3}}$

(ii) Slope $= \tan 0^\circ = 0$

(iii) Slope $= \tan 90^\circ =$ not defined

EXAMPLE 4 Show that points (1, 1), (2, 3) and (3, 5) are collinear.

SOLUTION

Remark

Three point A, B and C are collinear if and only if

slope of $\overleftrightarrow{AB}$ = slope of $\overleftrightarrow{BC}$ = slope of $\overleftrightarrow{AC}$

Let A(1, 1), B(2, 3) and C(3, 5)

Now,

Slope of $\overleftrightarrow{AB} = \frac{3-1}{2-1} = 2$

Slope of $\overleftrightarrow{BC} = \frac{5-3}{3-2} = 2$

Therefore, the slope of

$\overline{AB}$ = slope of $\overline{BC}$.

Thus, the points A, B and C are collinear.

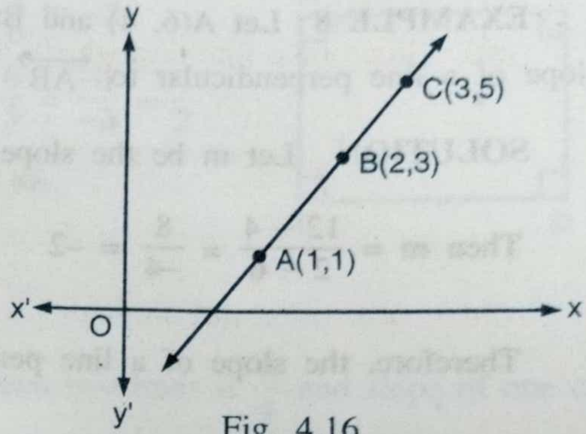


Fig. 4.16

EXAMPLE 5 The slope m of a line is given by $m = \sqrt{3}$. Find its inclination.

SOLUTION $\tan\theta = \sqrt{3}$, since $0^\circ \le \theta \le 180^\circ$,

we have,

$\therefore \quad \theta = 60^\circ$

EXAMPLE 6 Find the slope of a line which passes through points (3, 1) and (5, –2).

SOLUTION We know that the slope of a line passing through two points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Here, the line passing through (3, 1) and (5, –2).

$$m = \frac{-2-1}{5-3} = \frac{-3}{2}$$

EXAMPLE 7 If A(–2, 1), B(2, 3) and C(–2, –4) are three points, find the angle between $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$.

SOLUTION Let m_1 and m_2 be the slopes of $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ respectively. Then,

$$m_1 = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2} \text{ and } m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$$

Let α be the angle between $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$

We have,

$$\tan\alpha = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \times \frac{1}{2}}\right| = \left|\frac{\frac{10}{8}}{\frac{15}{8}}\right|$$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{2}{3}\right)$$

EXAMPLE 8 Let A(6, 4) and B(2, 12) be two given points. Find the slope of a line perpendicular to $\overleftrightarrow{AB}$.

SOLUTION Let m be the slope of $\overleftrightarrow{AB}$.

Then $m = \dfrac{12-4}{2-6} = \dfrac{8}{-4} = -2$

Therefore, the slope of a line perpendicular to $\overleftrightarrow{AB} = -\dfrac{1}{m} = \dfrac{1}{2}$

EXAMPLE 9 Without using Pythagoras theorem, show that A(4, 4), B(3, 5) and C(−1, −1) are the vertices of a right angled triangle.

SOLUTION In ΔABC

We have,

$$m_1 = \text{slope of } \overline{AB} = \frac{5-4}{3-4} = -1$$

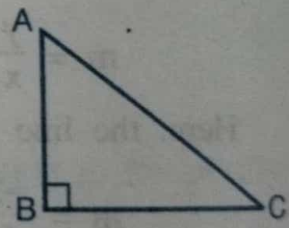
$$m_2 = \text{slope of } \overline{AC} = \frac{-1-4}{-1-4} = 1$$

Clearly, $m_1 \cdot m_2 = -1$

$\therefore \quad \overline{AB} \perp \overline{AC}$

$\therefore \quad m\angle A = 90^\circ$

$\therefore \quad \Delta\, ABC$ is right angled.



EXAMPLE 10 Prove that A(4, 3), B(6, 4), C(5, 6) and D(3, 5) are the vertices of a square.

SOLUTION We have by distance formula

$$AB = \sqrt{(6-4)^2 + (4-3)^2} = \sqrt{5}$$

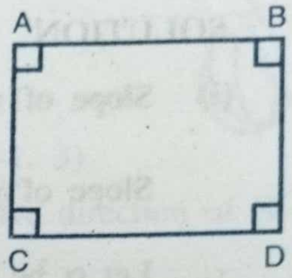
$$BC = \sqrt{(6-5)^2 + (4-6)^2} = \sqrt{5}$$

$$CD = \sqrt{(5-3)^2 + (5-6)^2} = \sqrt{5}$$

$$DA = \sqrt{(4-3)^2 + (3-5)^2} = \sqrt{5}$$

$\therefore \quad AB = BC = CD = DA$

Now, $m_1 = \text{slope of } \overline{AB} = \dfrac{4-3}{6-4} = \dfrac{1}{2}$



$$m_2 = \text{slope of } \overline{BC} = \frac{6-4}{5-6} = -2$$

and $$m_3 = \text{slope of } \overline{CD} = \frac{5-6}{3-5} = \frac{-1}{-2} = \frac{1}{2}$$

Clearly, $m_1 \cdot m_2 = -1$ and $m_1 = m_3$

$\therefore \quad \overline{AB} \perp \overline{BC}$

$\therefore \quad \square$ ABCD is a square.

EXAMPLE 11 If the angle between two lines is $\frac{\pi}{4}$ and slope of one of the lines is $\frac{1}{2}$ find the slope of the other line.

SOLUTION Here $\alpha = \frac{\pi}{4}$, $m_1 = \frac{1}{2}$

Let m_2 = slope of the other line

We use, $$\tan \alpha = \left| \frac{m_1 - m_2}{1 + m_1 \cdot m_2} \right|$$

$$\Rightarrow \tan \frac{\pi}{4} = \left| \frac{\frac{1}{2} - m_2}{1 + \frac{1}{2} m_2} \right|$$

$$\therefore \quad 1 = \left| \frac{1 - 2m_2}{2 + m_2} \right|$$

$$\therefore \quad \pm 1 = \frac{1 - 2m_2}{2 + m_2}$$

$$\therefore \quad 1 = \frac{1 - 2m_2}{2 + m_2} \quad \text{or} \quad -1 = \frac{1 - 2m_2}{2 + m_2}$$

$$\therefore \quad 2 + m_2 = 1 - 2m_2 \quad \text{or} \quad -2 - m_2 = 1 - 2m_2$$

$$\therefore \quad 3m_2 = 1 - 2 \quad \text{or} \quad m_2 = 1 + 2$$

$$\therefore \quad m_2 = -\frac{1}{3} \quad \text{or} \quad m_2 = 3$$

EXAMPLE 12 Obtain the measure of an angle between the following pairs of lines.

(i) $x - y + 3 = 0$, $y - 2 = 0$

(ii) $x - y + 4 = 0$, $5x - y + 3 = 0$

(iii) $x + y + 1 = 0$, $x - y = 0$

SOLUTION

(i) Slope of $x - y + 3 = 0$ is $m_1 = -\frac{1}{-1} = 1$

Slope of $y - 2 = 0$ is $m_2 = \frac{0}{1} = 0$

Let α be the angle between two lines

$$\therefore \quad \tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$$

$$\therefore \quad \tan\alpha = \left|\frac{1-0}{1+0}\right|$$

$$\therefore \quad \tan\alpha = 1$$

$$\therefore \quad \boxed{\alpha = \frac{\pi}{4}}$$

(ii) Slope of $x - y + 4 = 0$ is $m_1 = -\frac{1}{-1} = 1$

Slope of $5x - y + 3 = 0$ is $m_2 = \frac{5}{-1} = 5$

Let α be the measure of an angle between the lines

$$\therefore \quad \tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$$

$$\therefore \quad \tan\alpha = \left|\frac{1-5}{1+(1)(5)}\right|$$

$$\therefore \quad \tan\alpha = \left|\frac{-4}{6}\right|$$

$$\therefore \quad \tan\alpha = \frac{2}{3}$$

$$\therefore \quad \alpha = \tan^{-1}\left(\frac{2}{3}\right)$$

(iii) $x + y + 1 = 0 \Rightarrow$ slope $m_1 = -1$

$x - y = 0 \Rightarrow$ slope $m_2 = 1$

$\therefore \quad m_1 \cdot m_2 = -1$

$\therefore$ Lines are perpendicular to each other

$$\therefore \quad \alpha = \frac{\pi}{2}$$

EXERCISES 4.4

1. Find the slope of the line through the points.
(i) (1, 2), (4, 2) (ii) (4, –6), (–2, –5) (iii) (1, 3), (–1, 3)
2. Find the slope of a line whose inclination with positive direction of X–axis is (i) 60° (ii) 90°
3. Find the slope of the line bisecting the first quadrant angle.
4. Show that the line joining the points (–2, –3) and (–5, 1)
(i) is parallel to the line joining the points (3, –1) and (0, 3)
(ii) is perpendicular to the line joining (4, 1) and (0, –2).
5. State whether the two lines in each of the following are parallel, perpendicular or neither parallel nor perpendicular.
(i) Through (5, 6) and (2, 3); through (9, –2) and (6, –5).
(ii) Through (8, 2) and (–5, 3); through (16, 6) and (3, 15).
(iii) Through (2, –5) and (–2, 5); through (6, 3) and (1, 1).
(iv) Determine x so that 2 is the slope of the line through points (2, 5) and (x, 3).
6. What is the value of y so that the line through (3, y) and (2, 7) is parallel to the line through (–1, 4) and (0, 6) ?
7. Without using Pythagoras theorem, show that (4, 4), (3, 5) and (–1, –1) are the vertices of a right triangle.
8. By the concept of slope, show that (–2, –1), (4, 0), (3, 3) and (–3, 2) are vertices of a parallelogram.
9. Prove using slopes, that the points P(2, 5), Q(3, –2), R(–4, 1) and S(–5, 8) are the vertices of a parallelogram.
10. Prove that A(0, 0), B(1, 0), C(1, 1) and D(0, 1) are the vertices of a square.
11. Prove using slopes, that the points (3, 2), (0, 5), (–3, 2) and (0, –1) are the vertices of a square.
12. A(k, 1), B(3, –3), C(2, –4) and D(–3, –1) are the points in the plane.
Find k if (i) $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ (ii) $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.
13. Find the angle between the lines $x - y = 0$ and X–axis.
14. Find the measure of angle between $x + y = 0$ and $x - y = 0$.
15. Find the measure of angle between lines $y = 2$ and $x - y - 1 = 0$.

4.9 Various Forms of Equations of a Line

(1) Equation of a line parallel to Y–axis

For a line l parallel to Y–axis, P(x, y) on l is constant, say a.

However, the coordinate of the point varies continuously (Fig. 4.17)

Thus, the equation of this line is

$$\boxed{x = a}$$

that is x = constant

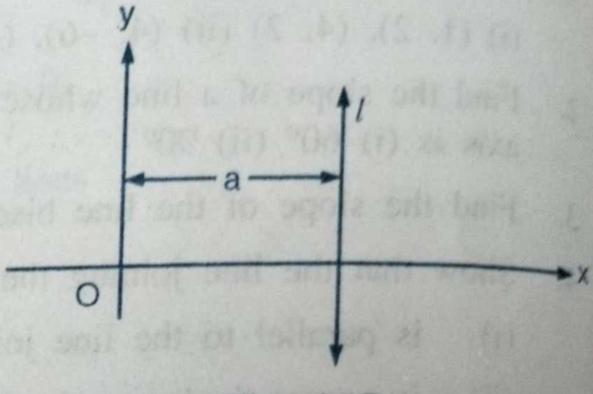


Fig. 4.17

(2) Equation of a line parallel to X–axis

For a line l parallel to X–axis (Fig. 4.18), the ordinate of any point P(x, y) on the line is constant, say, b which is equal to the directed distance of the line from X–axis. Thus, the equation of a line parallel to the X–axis is

$$\boxed{y = b}$$

that is y = constant

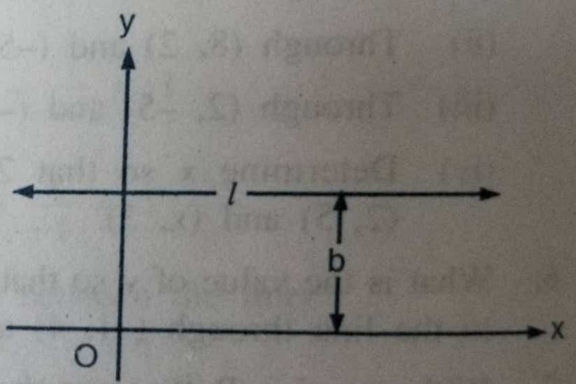


Fig. 4.18

EXAMPLE 1 Find the equation of the line which is parallel to X–axis and at a distance of 5 units below the X–axis.

SOLUTION The equation of a line parallel to X–axis is y = b. Now, the line is below the X–axis and is at a distance 5 units. Therefore b = –5.

Thus, the required equaion is y = –5.

EXAMPLE 2 Find the equation of the line which is parallel to Y–axis and passing through the point (3, –4).

SOLUTION The line is passing through the point (3, –4) and is parallel to Y–axis. Therefore, every point on the line must have same x–co-ordinate.

Thus, the equation of the line is x = 3.

(3) Point–slope form of the equation of a line

Let $P(x, y)$ be any point on the line l, and let $Q(x_1, y_1)$ be a fixed point on the line l (Fig. 4.19).

Let m be slope of line l. Then the slope of the line passing through Q and P is $\dfrac{y - y_1}{x - x_1}$.

Which is given to be m so that

$$m = \frac{y - y_1}{x - x_1}$$

or $\boxed{y - y_1 = m\,(x - x_1)}$

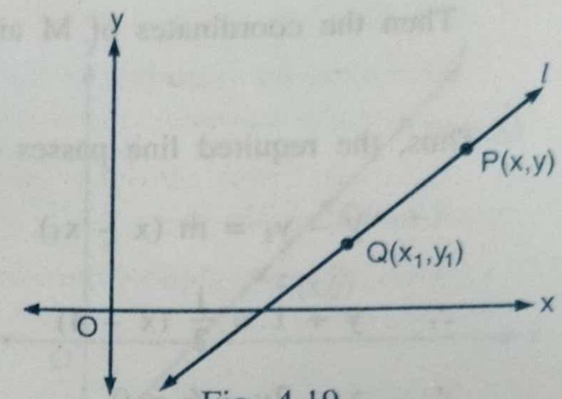


Fig. 4.19

This form of equation of a line is called *point–slope form.*

EXAMPLE 1 Find the equation of a line through the origin which makes an angle of 45° with the positive direction of X–axis.

SOLUTION The line passes through origin.

Therefore, a point on the line is (0, 0). (Fig. 4.20)

The slope of the line is given by

$m = \tan 45° = 1$

From the point–slope form the equation of a line, we have

$y - 0 = 1\,(x - 0)$

$\therefore\ y = x$

Which is the required equation of the line.

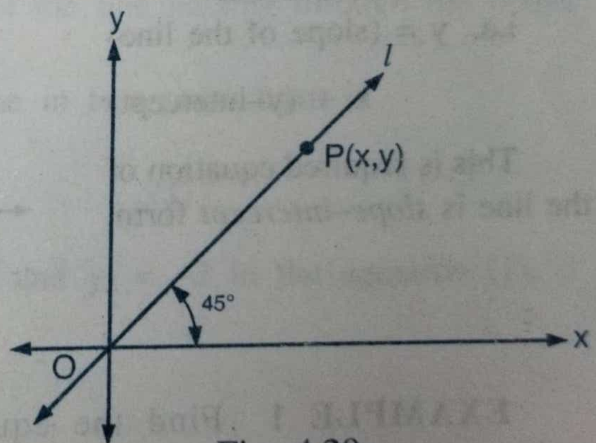


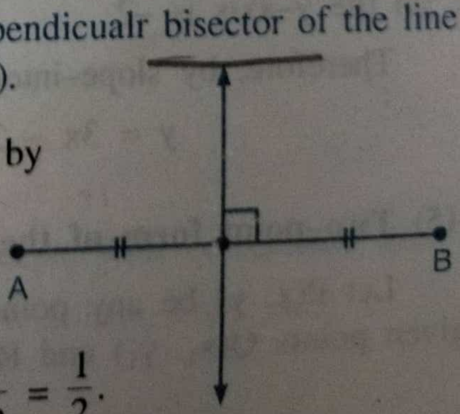
Fig. 4.20

EXAMPLE 2 Find the equation of the perpendicualr bisector of the line segment joining the points A(2, 3) and B(6, –5).

SOLUTION The slope of $\overleftrightarrow{AB}$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5 - 3}{6 - 2} = -2$$

$\therefore$ The slope of a line $\perp$ to $\overleftrightarrow{AB} = -\dfrac{1}{m} = \dfrac{1}{2}$.



Let M be the mid–point of $\overline{AB}$.

Then the coordinates of M are $\left(\frac{2+6}{2}, \frac{3-5}{2}\right) = (4, -1)$

Thus, the required line passes through M(4, –1) and has slope $\frac{1}{2}$ is

$$y - y_1 = m(x - x_1)$$

$$\therefore \quad y + 1 = \frac{1}{2}(x - 4)$$

$$\therefore \quad x - 2y - 6 = 0$$

(4) Slope-intercept form of the equation of a line

Let m be the slope and c be the intercept cut by the line *l* on the Y–axis is given by

$$\boxed{y = mx + c}$$

i.e. y = (slope of the line) x + (y–intercept)

This is required equation of the line is *slope–intercept* form.

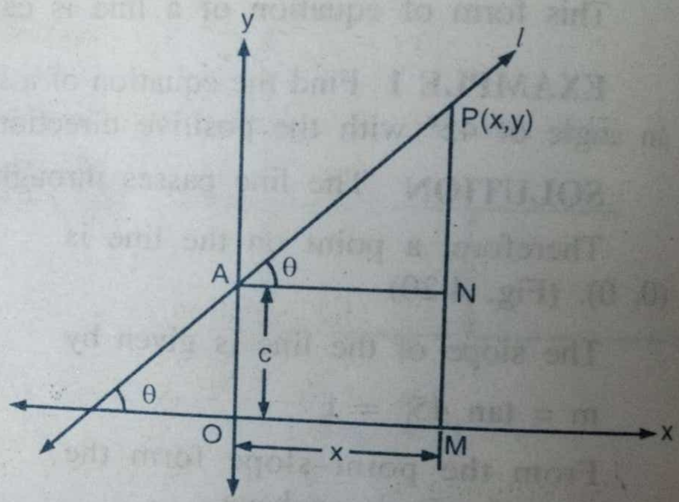


Fig. 4.21

EXAMPLE 1 Find the equation of the line with slope 3 and y–intercept –2.

SOLUTION It is given that the slope m of the line is 3 and intercept c on the y–axis is –2.

Therefore, by slope–intercept form, the required equation of the line is

$$y = 3x - 2.$$

(5) Two–point form of the equation of a line

Let P(x, y) be any point on the line *l*, and let line passes through two given points $Q(x_1, y_1)$ and $R(x_2, y_2)$.

Thus,

Slope of $\overleftrightarrow{PQ}$ = slope of $\overleftrightarrow{QR}$.

Therefore,

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \quad ...(1)$$

Which is the required equation of the line QR.

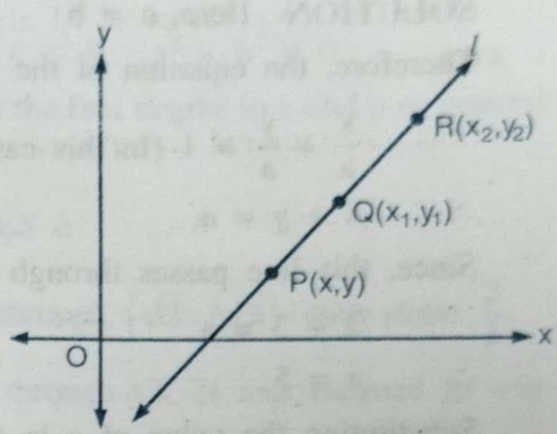


Fig. 4.22

Hence, the equation of the line through two points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

or $$\boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)} \quad ...(2)$$

This form of the equation of a line is called the *two–point form.*

EXAMPLE 1 Find the equation of the line passing through the points (2, 3) and (5, –2).

SOLUTION The equation of a line in two point form is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad ...(1)$$

Substituting $x_1 = 2$, $y_1 = 3$, $x_2 = 5$ and $y_2 = -2$ in the equation (1), we get,

$$y - 3 = \frac{-2 - 3}{5 - 2}(x - 2)$$

$$\therefore \quad y - 3 = -\frac{5}{2}(x - 2)$$

$$\therefore \quad 5x + 3y - 19 = 0$$

Which is the required equation of the line.

(6) Intercept form of the equation of a line

Equation of a line making 'a' and 'b' intercepts with co-ordinate axes is given by

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

EXAMPLE 1 Find the equation of a line which cuts off equal intercepts on the coordinate axes and passes through the point (2, 3).

SOLUTION Here, $a = b$

Therefore, the equation of the line in intercept form is

$$\frac{x}{a} + \frac{y}{a} = 1 \text{ (In this case } b = a)$$

$$\therefore \quad x + y = a \qquad \ldots(1)$$

Since, this line passes through the point (2, 3), we have

$$2 + 3 = a$$

$$\therefore \quad a = 5$$

Substituting the value of a in (1),

we get, the required equation is

$$x + y = 5.$$

EXAMPLE 2 Find the equation of the line which passes through the point (–5, 4) and is such that the portion intercepted between the axes is divided by the point in the ratio 1 : 2.

SOLUTION Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$.

This line meets the co-ordinate axes at A(a, 0) and B(0, b) respectively.

The co-ordinates of the point which divides the line joining A(a, 0) and B(0, b) in the ratio 1 : 2 are $\left(\frac{1 \times 0 + 2 \times a}{1 + 2}, \frac{1 \times b + 2 \times 0}{1 + 2}\right) = \left(\frac{2a}{3}, \frac{b}{3}\right)$.

It is given that the point (–5, 4) divides $\overline{AB}$ in the ratio 1 : 2.

Therefore,

$$\frac{2a}{3} = -5 \text{ and } \frac{b}{3} = 4$$

$$\therefore \quad a = -\frac{15}{2} \text{ and } b = 12$$

Hence, the required equation of the line is

$$\frac{x}{-\frac{15}{2}} + \frac{y}{12} = 1$$

$$\therefore \quad x - 5y + 60 = 0$$

$-\frac{2x}{15} \frac{y}{12}$

$-8x + 5y = 60$

$8x . 5y + 60 .. 0$

(7) General Equation of a line

An equation of the form

$$\boxed{\mathbf{ax + by + c = 0,}} \qquad a, b, c \in R; \quad a^2 + b^2 \neq 0$$

is called a general equation of a line in the first degree in x and y or general equation of a line linear in x and y.

EXERCISES 4

1. Get the equation of a line passing through $(\sqrt{2}, 2\sqrt{2})$ with slope $\frac{2}{3}$.
2. Get the equation of a line passing through (2, 2) and inclined to +ve direction of X–axis at 45°.
3. Get the equation of a line passing through (0, –3) and (5, 0).
4. Determine the equation of the line parallel to the X–axis and passing through the point (3, –4).
5. Find the equation of the line perpendicular to the X–axis and passing through the origin.
6. Find the equation of a line passing through points (1, 2) and (1, –1).
7. Find the equations of the sides of a triangle whose vertices are (2, 1), (–2, 3) and (4, 5).
8. Find the equation of the perpendicular bisector of the line segment joining the points A(1, 0) and B(2, 3).
9. Find the equation of the line passing through (–3, 5) and perpendicular to the line through the points (2, 5) and (–3, 6).
10. Find the equations of the lines passing through the point (2, 2), such that the sum of their intercepts on the axes is 9.
11. The mid-points of the sides of a triangle are (2, 1), (–5, 7) and (–5, –5). Find the equation of the sides.
12. Find the angle between the lines $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x + 6 = 0$.
13. The line through (4, 3) and (–6, 0) intersects the line $10x + 3y = 0$. Find the angle between them.
14. Find the equation of a lines passing through (1, 1) and parallel to $x - y + 2 = 0$.
15. Find the equation of the line through (1, 2) and perpendicular to $2x - y + 1 = 0$.
16. Find the equation of the line having intercepts 2 and –1 on the axes.

17. Find the equation of the line through the intersection of the lines $3x + y - 14 = 0$, $x - 2y = 0$ and
    (i) parallel to $3x + 2y - 1 = 0$
    (ii) perpendicular to $x - y + 4 = 0$
18. Find the equation of the perpendicular to the line segment joining (4, 3) and (–1, 1) and it cuts off an intercept –3 from Y–axis.
19. Find the equation of a straight line
    (i) with slope 2 and X–intercept 3.
    (ii) with slope –3 and Y–intercept 4.
20. Prove that the line $y - x + 2 = 0$ divides the join of points (3, –1) and (8, 9) in the ratio 2 : 3.
21. The vertices of a quadrilateral are A(–2, 6), B(1, 1), C(2, 3) and D(7, 8). Find the equations of its diagonals.
22. Find the equation of the line which passes through (1, 1) and cuts off equal intercepts on the axes.
23. Find the equation of a line which passes through the point (5, 6) and has intercepts on the axes
    (i) equal in magnitude and both positive
    (ii) equal in magnitude but opposite in sign
24. Find the equation of a line through the intersection $x - y - 1 = 0$ and $2x - 3y + 1 = 0$ also
    (i) having slope –2
    (ii) parallel to the line $x + y + 4 = 0$
    (iii) passes through (1, 2)
25. Find the equation of a line through the intersection of the lines $x + y = 0$, $x - y = 0$ and perpendicular to the line joining (1, 2) and (–1, 1).
26. Find the equation of the line passing through the origin and also through the point of intersection of the lines $x - 3y + 1 = 0$ and $y = 2$.
27. Find the equation of the line through the intersection of the lines $2x + 3y - 4 = 0$ and $x - 5y - 7 = 0$ that has its X–intercept equal to –4.
28. Find slope and intercepts (whenever defined) of the followings.
    (i) $x + y + 1 = 0$ (ii) $ax + by - ab = 0$
    (iii) $4x + 3y + 4 = 0$ (iv) $x - y = 0$
    (v) $x - 4 = 0$ (vi) $y - 5 = 0$

29. For each of the following, select the proper alternative from the given four alternatives.

(1) The distance of the point (a, b) from the origin is

(a) a (b) b (c) a + b (d) $\sqrt{a^2 + b^2}$

(2) A(3, –5) and B(2, 3) are given points. The point dividing $\overline{AB}$ from A in the ratio 2:3 is

(a) $\left(\frac{13}{5}, \frac{9}{5}\right)$ (b) (13, –9) (c) $\left(\frac{13}{5}, \frac{-9}{5}\right)$ (d) none

(3) A(3, –2), B(0, 7) are given points, a point that divides $\overline{AB}$ in – 4:3 ratio is

(a) (9, 34) (b) (–9, 34) (c) (9, –34) (d) none

(4) The points of trisection of $\overline{PQ}$, where P(3, 5), Q(12, 14) are

(a) (6, 8), (9, 11) (b) (6, 4) (9, 7)

(c) (–6, 8), (–9, 11) (d) none

(5) For A(7, 8), B(3, 5), the ratio in which X–axis divides $\overline{AB}$ from A is

(a) –8:5 (b) –5:8 (c) 5:8 (d) none

(6) Area of Δ ABC, where A(6, 3), B(–3, 5) and C(4, –2) is

(a) 49 (b) $\frac{49}{2}$ (c) $\frac{29}{2}$ (d) none

(7) Mid–point of the line–segment joining A(1, 2) and B(5, 8) is

(a) (3, 10) (b) (2, 3) (c) (3, 5) (d) none

(8) The points (a, b + c), (b, c + a), (c + a + b) are the vertices of a

(a) right triangle (b) equilateral

(c) acute angled triangle (d) none

(9) Slope of a line parallel to X–axis is

(a) 0 (b) undefined (c) 1 (d) none

(10) Slope of a $\overleftrightarrow{AB}$, where A(4, 1), B(3, –1) is

(a) 2 (b) –2 (c) 1 (d) none

(11) Slope of the line x – 1 = 0 is

(a) 0 (b) undefined (c) 1 (d) none

(12) The line 2x – 3y + 5 = 0 and 6x – 9y + 7 = 0 are

(a) parallel (b) perpendicular

(c) intersecting (d) none

(13) X–intercept of $4x - y + 2 = 0$ is

(a) 2 (b) –2 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

(14) Y–intercept of $x + 2y + 1 = 0$

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 2 (d) –2

(15) Slope of $lx + my + n = 0$ is

(a) $-\frac{m}{l}$ (b) $-\frac{l}{m}$ (c) $-\frac{n}{l}$ (d) none

(16) The point A, B and C are collinear if and only if area of triangle ABC is

(a) less than zero (b) zero

(c) greater than zero (d) none

(17) The triangle whose vertices are (0, 0), (1, 0) and (0, 2) is

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) none

(18) Slope of a line perpendicular to $4x - 3y + 1 = 0$ is

(a) $\frac{4}{3}$ (b) $-\frac{4}{3}$ (c) $-\frac{3}{4}$ (d) $\frac{3}{4}$

(19) Slope of a line parallel to the line $2x + 3y - 1 = 0$ is

(a) $\frac{2}{3}$ (b) $-\frac{3}{2}$ (c) $-\frac{2}{3}$ (d) none

(20) Angle between the lines $x = 2$ and $y = 3$ is

(a) $\frac{\pi}{2}$ (b) 0 (c) $\frac{\pi}{4}$ (d) none

(21) Angle between the lines $x + y + 1 = 0$ and $y = 2$ is

(a) $\frac{\pi}{2}$ (b) 0 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

: ANSWERS :

EXERCISES 4.1

1. (i) 2 (ii) $2\sqrt{x^2 + y^2}$ (iii) 2 (iv) 1
2. –2, 4
3. (3, 0)
4. (–1, 0)
5. (0, –2)
8. (i) collinear (ii) collinear
9. (3, –2)
10. Equilateral
11. $(0, 2\sqrt{3}), (3, -\sqrt{3})$
12. $(3, \sqrt{3})$ or $(-3, \sqrt{3})$

EXERCISES 4.2

1. (i) $\frac{49}{2}$ sq. units (ii) a^2 units (iii) 37.5 sq. units (iv) $\frac{1}{2}$ sq. units
4. 2
6. -2 or 5
7. $x = -7$ or 5
8. (1, 0) or (7, 2)
9. $\frac{55}{2}$ sq. units
10. 40

EXERCISES 4.3

1. (i) (0, 0) (ii) $\left(\frac{2}{5}, \frac{3}{5}\right)$
2. (i) (32, 12) (ii) (8, 21)
3. (i) 1 : 2 internally (ii) 2 : 5 externally
4. 1 : 3
5. (3, 2), (5, 2)
6. (0, 4), (2, 5) and (3, 3)
7. (0, 0), (1, 1) and (2, 1)

EXERCISES 4.4

1. (i) 0 (ii) $-\frac{1}{6}$ (iii) 0
2. (i) $\sqrt{3}$ (ii) not defined
3. 1
5. (i) parallel (ii) neither parallel nor perpendicular
   (iii) perpendicular (iv) $x = 1$
6. $y = 9$
12. (i) $\frac{-17}{3}$ (ii) 9
13. $\theta = 45^\circ$
14. $\frac{\pi}{2}$
15. $\frac{\pi}{4}$

EXERCISES 4

1. $2x - 3y + 4\sqrt{2} = 0$
2. $x - y = 0$
3. $3x - 5y - 15 = 0$
4. $y + 4 = 0$
5. $x = 0$ (Y–axis)
7. $x + 2y - 4 = 0,\ x - 3y + 11 = 0,\ 2x - y - 3 = 0$
8. $x + 3y - 6 = 0$
9. $5x - y + 20 = 0$

10. $x + 2y - 6 = 0,\ 2x + y - 6 = 0$

11. $x = 2,\ 6x - 7y + 79 = 0,\ 6x + 7y + 65 = 0$

12. 30°

13. 90°

14. $x - y = 0$

15. $x + 2y - 5 = 0$

16. $x - 2y - 2 = 0$

17. (i) $3x + 2y - 16 = 0$ (ii) $x + y - 6 = 0$

18. $5x + 2y + 6 = 0$

19. (i) $3x + y - 4 = 0$ (ii) $2x - y - 6 = 0$

21. $3x + 4y - 18 = 0,\ 7x - 6y - 1 = 0$

22. $x + y - 2 = 0,\ x - y = 0$

23. (i) $x + y - 11 = 0$ (ii) $x - y - 11 = 0$

24. (i) $2x + y - 11 = 0$ (ii) $x + y - 7 = 0$ (iii) $x - 3y + 5 = 0$

25. $2x + y = 0$

26. $2x - 5y = 0$

27. $10x + 93y + 40 = 0$

28. (i) $-1, -1, -1$ (ii) $-\frac{a}{b}$, b, a

(iii) $-\frac{4}{3}, -1, -\frac{4}{3}$ (iv) 1, 0, 0

(v) 0, Not defined, 4 (vi) Not defined, 5, Not defined

29. 1. (d), 2. (c), 3. (b), 4. (a), 5. (a), 6. (b), 7. (c), 8. (d), 9. (a), 10. (a), 11. (b), 12. (a), 13. (d), 14. (a), 15. (b), 16. (b), 17. (c), 18. (c), 19. (c), 20. (a), 21. (c).

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