Linear Equations

3/12/2021

Gauss –Jordan Elimination Process

Eg2.

$$3x_1 + 18x_2 + 9x_3 = 18$$
$$2x_1 + 3x_2 + 3x_3 = 117$$
$$4x_1 + x_2 + 2x_3 = 283$$

The augmented matrix can be written as

$$\begin{bmatrix} 3 & 18 & 9 & 18 \\ 2 & 3 & 3 & 117 \\ 4 & 1 & 2 & 283 \end{bmatrix}$$

Gauss –Jordan Elimination Process

$$\begin{bmatrix} 3 & 18 & 9 & 18 \\ 2 & 3 & 3 & 117 \\ 4 & 1 & 2 & 283 \end{bmatrix}$$

The first step is to divide the first row by 3 gives us

Replace R_1 by (1/3) R_1

$$\begin{bmatrix} 1 & 6 & 3 & 6 \\ 2 & 3 & 3 & 117 \\ 4 & 1 & 2 & 283 \end{bmatrix}$$

Gauss –Jordan Elimination Process

$$\begin{bmatrix} 1 & 6 & 3 & 6 \\ 2 & 3 & 3 & 117 \\ 4 & 1 & 2 & 283 \end{bmatrix}$$

- Subtract the two times reduced first row from the second row and also multiply the first row by 4 and then subtract from the third, gives us

Replace R_2 by $R_2 - 2R_1$ and R_3 by R_3 $-4R_1$ to get

- $\begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & -9 & -3 & 105 \\ 0 & -23 & -10 & 259 \end{bmatrix}$

Gauss –Jordan Elimination Process

- $\begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & -9 & -3 & 105 \\ 0 & -23 & -10 & 259 \end{bmatrix}$

The second step is to divide the second row by -9 gives us

Replace R_2 by (-1/9) R_2

- $\begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & 1 & {}^{1}/_{3} & {}^{-35}/_{3} \\ 0 & -23 & -10 & 259 \end{bmatrix}$

Gauss –Jordan Elimination Process

- $\begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & 1 & {}^{1}/_{3} & {}^{-35}/_{3} \\ 0 & -23 & -10 & 259 \end{bmatrix}$
- We reduce the second column to [0,1,0] by row operations

Replace R_1 by $R_1 - 6R_2$ and R_3 by $R_3 + 23R_2$ to get

- $\begin{bmatrix} 1 & 0 & 1 & 76 \\ 0 & 1 & {}^{1}/_{3} & {}^{-35}/_{3} \\ 0 & 0 & {}^{-7}/_{3} & {}^{28}/_{3} \end{bmatrix}$

Gauss –Jordan Elimination Process

- $\begin{bmatrix} 1 & 0 & 1 & 76 \\ 0 & 1 & {}^{1}/_{3} & {}^{-35}/_{3} \\ 0 & 0 & {}^{-7}/_{3} & {}^{28}/_{3} \end{bmatrix}$
- The third step is to divide the third row by -7/3 gives us

Replace R_3 by (-7/3) R_3

- $\begin{bmatrix} 1 & 0 & 1 & 76 \\ 0 & 1 & {}^{1}/_{3} & {}^{-35}/_{3} \\ 0 & 0 & 1 & 4 \end{bmatrix}$

Gauss –Jordan Elimination Process

- $\begin{bmatrix} 1 & 0 & 1 & 76 \\ 0 & 1 & {}^{1}/_{3} & {}^{-35}/_{3} \\ 0 & 0 & 1 & 4 \end{bmatrix}$

Replace R_1 by $R_1 -$ $R_{3,}$ R_3 by $R_2 -$ $(1/3)R_3$

- $\begin{bmatrix} 1 & 0 & 0 & 72 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & 4 \end{bmatrix}$
- The solution is $x_1 \; = 72,\; x_2 = -13, \;\; x_3 \quad = 4$

Gauss –Jordan Elimination Process

Eg3.

$$\begin{aligned} 2x_1 &+ x_2 + 2x_3 = 10 \\ x_1 &+ 2x_2 + x_3 = 8 \\ 3x_1 &+ x_2 - x_3 = 2 \end{aligned}$$

The augmented matrix can be written as

$$\begin{bmatrix} 2 & 1 & 2 & 10 \\ 1 & 2 & 1 & 8 \\ 3 & 1 & -1 & 2 \end{bmatrix}$$

Gauss –Jordan Elimination Process

$$\begin{bmatrix} 2 & 1 & 2 & 10 \\ 1 & 2 & 1 & 8 \\ 3 & 1 & -1 & 2 \end{bmatrix}$$

We want a 1 in row one, column one. This can be obtained by dividing the first row by 2 or interchanging the second row with the first. Interchanging the rows is a better choice because that way we avoid fractions.

$$\begin{bmatrix} 1 & 2 & 1 & 8 \\ 2 & 1 & 2 & 10 \\ 3 & 1 & -1 & 2 \end{bmatrix}$$

Gauss –Jordan Elimination Process

$$\begin{bmatrix} 1 & 2 & 1 & 8 \\ 2 & 1 & 2 & 10 \\ 3 & 1 & -1 & 2 \end{bmatrix}$$

Subtract the two times reduced first row from the second row and also multiply the first row by 3 and then subtract from the third, gives us

Replace R_2 by $R_2 - 2R_1$ and R_3 by R_3 $-3R_1$ to get

$$\begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & -5 & -4 & -22 \end{bmatrix}$$

Gauss –Jordan Elimination Process

- $\begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & -5 & -4 & -22 \end{bmatrix}$

The second step is to divide the second row by -3 gives us

Replace R_2 by (-1/3) R_2

- $\begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{bmatrix}$

Gauss –Jordan Elimination Process

- $\begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{bmatrix}$
- We reduce the second column to [0,1,0] by row operations

Replace R_1 by $R_1 - 2R_2$ and R_3 by $R_3 + 5R_2$ to get

- $\begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -12 \end{bmatrix}$

Gauss –Jordan Elimination Process

- $\begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -12 \end{bmatrix}$
- The third step is to divide the third row by -4 gives us

Replace R_3 by (-1/4) R_3

- $\begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Gauss –Jordan Elimination Process

- $\begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Replace R_1 by $R_1 - R_3$

- $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$
- The solution is $x_1 = 1,\ x_2 = 2,\ x_3 = 3$

Gauss –Jordan Elimination Process

- Three possible outcomes
- Unique Solution: If the reduced row echelon form has no free variables, then it looks like:

$$\begin{bmatrix} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{bmatrix}$$

and there is a unique solution, namely, $x_1 = b_1$, $x_2 = b_2$, $x_3 = b_3$.

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

A unique solution exists

$x_1 = 5$, $x_2 = 2$, $x_3 = 4$

Gauss –Jordan Elimination Process

- Infinite Solutions (dependent system): If the reduced row echelon form has free variables, then there are an infinite number of solutions. The parameter assigned to any one free variable can take on an infinite number of values.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equations are x_1 +2 x_3 =3 and x_2 -3 $x_3 = 4$

Solve for x_1 and x_2

x_1 = 3 - 2 x_3 and x_2 = 4+ x_3

Thus, the solution is

(3 - 2 x_3 , 4+ x_3 , x_3)

Gauss –Jordan Elimination Process

- No Solution (inconsistent system)
- If the reduced row echelon form has a row of the form [0,0,…,0,b] then the system of linear equations has no solution.
- Eg.

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Here we get 0 = 4 which is false. Hence, there is no solution for the system of equations

Rank

- The ***rank*** of a matrix is the number of leading ones in the reduced row echelon form.

Eg The following matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{bmatrix}$$

Has a reduced row echelon form of

$$\begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of matrix is 2

Inverse

Recall that square matrices A and B are inverses if $AB = BA = I$.

Or in other words

$B = IA^{-1}$

One way to find the inverse of matrix A is to employ the minors of its determinant but this is not efficient. The better way is to use an elimination method.

Inverse

If a matrix A is invertible, there are a set of steps to reduce it to the identity matrix, which also means that we have some set of elementary matrices such that

$E_n E_{n-1} \ldots E_2 E_1 A$ = I

However, by right-multiplying by A^{-1} (since A is invertible), we get $E_n E_{n-1} \ldots E_2 E_1$ I= A^{-1}

So by performing the steps to reduce A to the identity matrix, those same steps performed on the identity matrix create the inverse of A.

If we start with [A|I] and reduce the left side to the identity matrix, then we would end up with [I| A^{-1}]

Matrix Inversion

Eg1.

Find the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$$

Form the block matrix M = [A,I] and row reduce M to an echelon form

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Inversion

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{bmatrix}$$

Subtract the two times reduced first row from the second row and also multiply the first row by 4 and then subtract from the third, gives us

Replace R_2 by $R_2 - 2R_1$ and R_3 by R_3 $-4R_1$ to get

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{bmatrix}$$

Matrix Inversion

- $$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{bmatrix}$$
- The next step is to multiply the second row by -1 gives us

Replace R_2 by (-1) R_2

- $$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{bmatrix}$$

Matrix Inversion

- $\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{bmatrix}$
- We reduce the second column to [0,1,0] by row operations

Replace R_3 by R_3 - R_2 to get

- $\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{bmatrix}$

Matrix Inversion

- $\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{bmatrix}$
- The third step is to multiply the third row by -1 gives us

Replace R_3 by (-1) R_3

- $\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{bmatrix}$

Matrix Inversion

- $$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{bmatrix}$$
- Subtract the from first row two times the third row

Replace R_1 by $R_1 - 2R_3$

- $$\begin{bmatrix} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{bmatrix}$$
- The inverse of the matrix is $\begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$

Linear Equations

6/12/2021

Matrix Inversion

Eg2.

Find the inverse of the matrix

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 1 \end{bmatrix}$$

Form the block matrix M = [A,I] and row reduce M to an echelon form

$$\begin{bmatrix} 2 & 3 & -1 & 1 & 0 & 0 \\ 4 & 4 & -3 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Inversion

- $\begin{bmatrix} 2 & 3 & -1 & 1 & 0 & 0 \\ 4 & 4 & -3 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{bmatrix}$

- The next step is to multiply the first row by $1/2$ gives us

Replace R_1 by (1/2) R_1

- $\begin{bmatrix} 1 & {}^{3}/_{2} & {}^{-1}/_{2} & {}^{1}/_{2} & 0 & 0 \\ 4 & 4 & -3 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{bmatrix}$

Matrix Inversion

- $\begin{bmatrix} 1 & {}^{3}/_{2} & {}^{-1}/_{2} & {}^{1}/_{2} & 0 & 0 \\ 4 & 4 & -3 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{bmatrix}$

Subtract the two times reduced first row from the second row and also multiply the first row by 4 and then subtract from the third, gives us

Replace R_2 by $R_2 - 4R_1$ and R_3 by R_3 $-2R_1$ to get

$$\begin{bmatrix} 1 & {}^{3}/_{2} & {}^{-1}/_{2} & {}^{1}/_{2} & 0 & 0 \\ 0 & -2 & -1 & -2 & 1 & 0 \\ 0 & -6 & 2 & -1 & 0 & 1 \end{bmatrix}$$

Matrix Inversion

- $\begin{bmatrix} 1 & {}^{3}/_{2} & {}^{-1}/_{2} & {}^{1}/_{2} & 0 & 0 \\ 0 & -2 & -1 & -2 & 1 & 0 \\ 0 & -6 & 2 & -1 & 0 & 1 \end{bmatrix}$

- The next step is to multiply the second row by -1/2 gives us

Replace R_2 by (-1/2) R_2

- $\begin{bmatrix} 1 & {}^{3}/_{2} & {}^{-1}/_{2} & {}^{1}/_{2} & 0 & 0 \\ 0 & 1 & {}^{1}/_{2} & 1 & {}^{-1}/_{2} & 0 \\ 0 & -6 & 2 & -1 & 0 & 1 \end{bmatrix}$

Matrix Inversion

- $$\begin{bmatrix} 1 & {}^{3}/_{2} & {}^{-1}/_{2} & {}^{1}/_{2} & 0 & 0 \\ 0 & 1 & {}^{1}/_{2} & 1 & {}^{-1}/_{2} & 0 \\ 0 & -6 & 2 & -1 & 0 & 1 \end{bmatrix}$$

- We reduce the second column to [0,1,0] by row operations

Replace R_1 by $R_1 – (3/2)R_{2,}$ R_3 by $R_3 + 6R_2$ to get

- $$\begin{bmatrix} 1 & 0 & {}^{-5}/_{4} & -1 & {}^{3}/_{4} & 0 \\ 0 & 1 & {}^{1}/_{2} & 1 & {}^{-1}/_{2} & 0 \\ 0 & 0 & 5 & 5 & -3 & 1 \end{bmatrix}$$

Matrix Inversion

- $\begin{bmatrix} 1 & 0 & -5/4 & -1 & 3/4 & 0 \\ 0 & 1 & 1/2 & 1 & -1/2 & 0 \\ 0 & 0 & 5 & 5 & -3 & 1 \end{bmatrix}$
- The third step is to multiply the third row by 1/5 gives us

Replace R_3 by (1/5) R_3

- $\begin{bmatrix} 1 & 0 & -5/4 & -1 & 3/4 & 0 \\ 0 & 1 & 1/2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 1 & -3/5 & 1/5 \end{bmatrix}$

Matrix Inversion

- $\begin{bmatrix} 1 & 0 & -5/4 & -1 & 3/4 & 0 \\ 0 & 1 & 1/2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 1 & -3/5 & 1/5 \end{bmatrix}$

Replace R_1 by $R_1 + (5/4)R_3$, R_2 by $R_2 - (1/2)R_3$

- $\begin{bmatrix} 1 & 0 & 0 & 1/4 & 0 & 1/4 \\ 0 & 1 & 0 & 1/2 & -1/5 & -1/10 \\ 0 & 0 & 1 & 1 & -3/5 & 1/5 \end{bmatrix}$

- The inverse of the matrix is $\begin{bmatrix} 1/4 & 0 & 1/4 \\ 1/2 & -1/5 & -1/10 \\ 1 & -3/5 & 1/5 \end{bmatrix}$

Inverse

Let A be a square n×n matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

- A is an invertible matrix.
- A is row equivalent to the n×n identity matrix.
- A has n pivot positions.
- The columns of A form a linearly independent set.
- The equation A**x**=**b** has atleast one solution for each **b**.
- A^T is an invertible matrix.

Inverse

Singular or non-invertible matrix	**Non-singular or invertible matrix**
It has no inverse, A^{-1} does not exists	It has an inverse, A^{-1} exists
Its determinant is zero	The determinant is nonzero
There is no unique solution to the system ***Ax*** = b	There is a unique solution to the system ***Ax*** = b
Gaussian elimination cannot avoid a zero on the diagonal	Gaussian elimination does not encounter a zero on the diagonal
The rank is less than *n*	The rank equals *n*
Rows are linearly dependent	Rows are linearly independent
Columns are linearly dependent	Columns are linearly independent

Comparing Iterative and reduction methods

- When are iterative methods (Gauss-Seidel) useful? A major advantage of Gauss-Seidel is that roundoff errors are not given a chance to “accumulate,” as they are in Gaussian elimination and the GaussJordan Method, because each iteration essentially creates a new approximation to the solution. The only roundoff error that we need to consider with Gauss-Seidel method is the error involved in the most recent step.

Comparing Iterative and reduction methods

- Also, in many applications, the coefficient matrix for a given system contains a large number of zeroes (sparse matrix). When a linear system has a sparse matrix, each equation in the system may involve very few variables. If so, each step of the Gauss-Seidel process is relatively easy. However, neither the Gauss-Jordan Method nor Gaussian elimination would be very attractive in such a case because the cumulative effect of many row operations would tend to replace the zero coefficients with nonzero numbers. But even if the coefficient matrix is not sparse, Gauss-Seidel methods often give more accurate answers when large matrices are involved because fewer arithmetic operations are performed overall.

Comparing Iterative and reduction methods

- On the other hand, when Gauss-Seidel method take an extremely large number of steps to stabilize or do not stabilize at all (absence of diagonal dominance), it is much better to use the Gauss-Jordan Method or Gaussian elimination.

Ill-conditioned matrix

- System of equations, in which a very small change in a coefficient leads to a very large change in the solution set, are called

  ill-conditioned systems.

- Ill- conditioned or Nearly singular matrix—an invertible matrix that can become singular if some of its entries are changed ever so slightly. In this case, row reduction may produce fewer than n pivot positions, as a result of roundoff error. Also, roundoff error can sometimes make a singular matrix appear to be invertible.

Ill-conditioned matrix

Consider the following equations(the extra digits are in the ninth significant place)

$$
\begin{aligned}
x_1 + \quad 2x_2 &= \; 3 \\
1.000\ \ 000\ \ 01x_1 + \quad 2x_2 &= \quad 3.000\ \ 000\ \ 01
\end{aligned}
$$

The solution is x_1 = 1, x_2 = 1. A computer has more trouble. If it represents real numbers to eight significant places, called single precision, then it will represent the second equation internally as

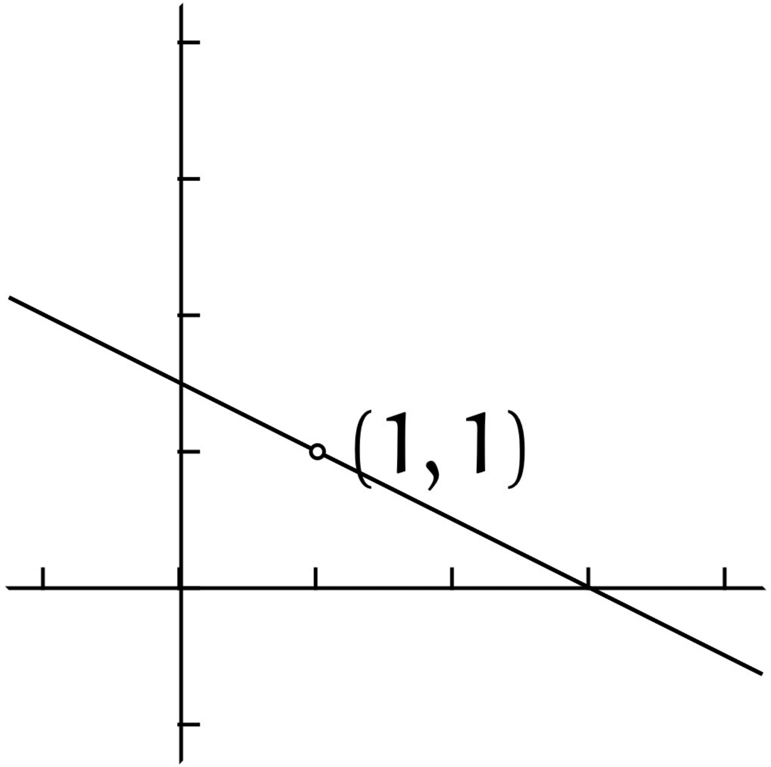
1.000 000 0 x_1 + 2 x_2 = 3.000 000 0,

losing the digits in the ninth place. Instead of reporting the correct solution, this computer will think that the two equations are equal and it will report that the system is singular.

Ill-conditioned matrix

Consider this graph of the system.

We cannot tell the two lines apart; this system is nearly singular in the sense that the two lines are nearly the same line. This gives the system the property that a small change in an equation can cause a large change in the solution.



Ill-conditioned matrix

- For instance, changing the 3.000 000 01 to 3.000 000 03 changes the intersection point from (1, 1) to (3, 0). The solution changes radically depending on the ninth digit, which explains why an eight-place computer has trouble. A problem that is very sensitive to inaccuracy or uncertainties in the input values is ill-conditioned
- The above example gives one way in which a system can be difficult to solve on a computer. It has the advantage that the picture of nearly-equal lines gives a memorable insight into one way for numerical difficulties to happen. Unfortunately this insight isn’t useful when we wish to solve some large system. We typically will not understand the geometry of an arbitrary large system

Ill-conditioned matrix

Consider the following equations

$$\begin{aligned} 0.001x_1 + \quad x_2 &= 1 \\ x_1 - \quad x_2 &= 0 \end{aligned}$$

The second equation gives $x_1 = x_2$, so $x_1 = x_2 = 1/1.001$ and thus both variables have values that are just less than 1. A computer using two digits represents the system internally in this way (we will do this example in two-digit floating point arithmetic for clarity but inventing a similar one with eight or more digits is easy).

Ill-conditioned matrix

- $(1.0\times10^{-3})x_1 + (1.0\times10^{0})x_2 = (1.0\times10^{0})$
- $(1.0\times10^{0})x_1 - (1.0\times10^{0})x_2 = (1.0\times10^{0})$

The row reduction step −1000 R_1 + R_2 produces a second equation −1001 x_2 = −1000, which this computer rounds to two places as

$$(-1.0\times10^{3})x_2 = (-1.0\times10^{3})$$

The computer decides from the second equation that x_2 = 1 and with that it concludes from the first equation that x_1 = 0.

The x_2 value is close but the x_1 is incorrect

Ill-conditioned matrix

- Another cause of unreliable output is the computer's reliance on floating point arithmetic when the system-solving code leads to using leading entries that are small.
- An experienced programmer may respond by using double precision, which retains sixteen significant digits, or perhaps using some even larger size. This will indeed solve many problems. However, double precision has greater memory requirements and besides we can obviously tweak the above to give the same trouble in the seventeenth digit, so double precision isn't a panacea. We need a strategy to minimize numerical trouble as well as some guidance about how far we can trust the reported solutions.

Ill-conditioned matrix

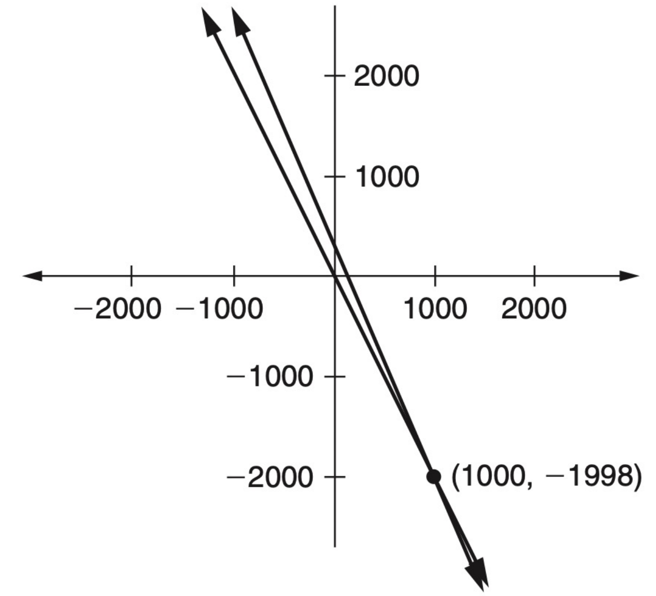
- Partial Pivoting: A basic improvement on the naive code above is to not determine the factor to use for row combinations by simply taking the entry in the row, row position, but rather to look at all of the entries in the row column below the row, row entry and take one that is likely to give reliable results because it is not too small. This is partial pivoting.

Ill-conditioned matrix

Consider the following similar systems

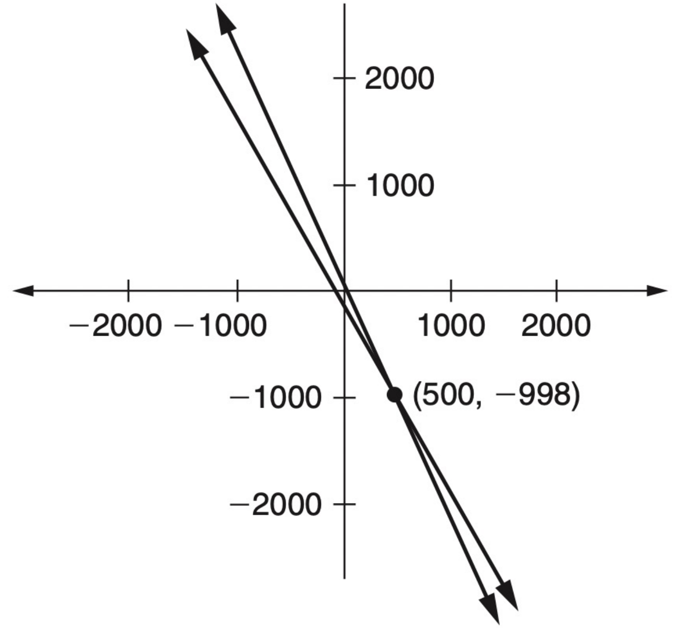
A:

$$\begin{aligned} 2x_1 + \quad x_2 &= 2 \\ 2.005x_1 + \quad x_2 &= 7 \end{aligned}$$



B:

$$\begin{aligned} 2x_1 + \quad x_2 &= 2 \\ 2.01x_1 + \quad x_2 &= 7 \end{aligned}$$



Ill-conditioned matrix

- Even though the coefficients of systems (A) and (B) are almost identical, the solutions to the systems are very different.
- Solution to (A) = (1000, -1998) and solution to (B) = (500, -998).
- In this case, there is a geometric way to see that these systems are ill-conditioned; the pair of lines in each system are almost parallel. Therefore, a small change in one line can move the point of intersection very far along the other line.

Ill-conditioned matrix

- Suppose the coefficients in system (A) had been obtained after a series of long calculations. A slight difference in the roundoff error of those calculations could have led to a very different final solution set. Thus, we need to be very careful when working with ill-conditioned systems. Special methods have been developed for recognizing ill-conditioned systems, and a technique known as iterative refinement is used when the coefficients are known only to a certain degree of accuracy.

Ill-conditioned matrix

- Condition number that describes the factor by which uncertainties in the input numbers could be magnified to become inaccuracies in the results returned
- Some matrix programs will compute a condition number for a square matrix. The larger the condition number, the closer the matrix is to being singular. The condition number of the identity matrix is 1.
- A singular matrix has an infinite condition number. In extreme cases, a matrix program may not be able to distinguish between a singular matrix and an ill-conditioned matrix.

Application in Machine Learning

- Why do we need these methods in machine learning?
  - Machine learning deals a lot with data
  - This data is at times represented in the form of equations
  - We need to solve these equations to arrive at a solution of our problem
  - Many applications require the inverse of matrices