The Dimension of a Subspace

- The **dimension** of a nonzero subspace H, denoted by dimH, is the number of vectors in any basis for H . The dimension of the zero subspace {**0**} is defined to be zero.
- The space R^n has dimension n. Every basis for R^n consists of n vectors. A plane through **0** in R^2 is two-dimensional, and a line through **0** is one-dimensional.

The Dimension of a Subspace

Example Find the dimension of the matrix

- $A = \begin{bmatrix} 3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

$$x_2 \underset{\substack{\uparrow \\ \mathbf{u}}}{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + x_4 \underset{\substack{\uparrow \\ \mathbf{v}}}{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}} + x_5 \underset{\substack{\uparrow \\ \mathbf{w}}}{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$$

- The null space of the matrix A has a basis of 3 vectors. So the dimension of Nul A in this case is 3. Observe how each basis vector corresponds to a free variable in the equation A**x** = **0**. Our construction always produces a basis in this way. So, to find the dimension of Nul A, simply identify and count the number of free variables in A**x** = **0**.

Rank of a matrix

- The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A.
- Since the pivot columns of A form a basis for Col A, the rank of A is just the number of pivot columns in A.

Rank of a matrix

EXAMPLE 3 Determine the rank of the matrix

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

Solution Reduce A to echelon form:

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

The matrix A has 3 pivot columns, so rank $A = 3$.

The Invertible Matrix Theorem (continued)

Let A be an n×n matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of R^n.

n. Col A = R^n
o. Dim Col A = n
p. Rank A=n

q. Nul A = {**0**}
r. Dim Nul A = 0

Length or Norm of a vector

- The length or norm of v is the nonnegative scalar $||v||$ defined by

$||v|| = \sqrt{v.v} = \sqrt{v_1^2 + v_2^2 + \ ... + v_n^2}$

and $||v||^2 = v.v$

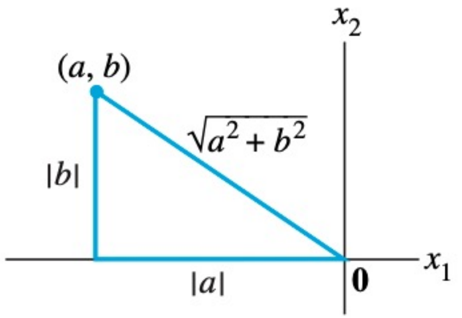


FIGURE 1
Interpretation of $\|\mathbf{v}\|$ as length.

Norm of a matrix

- For the rectangular n × d matrix A with (i,j)th entry denoted by a_{ij}, its Frobenius norm is defined as follows:
- $\|A\|_F = \|A^T\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{d} a_{ij}^2}$
- Note the use of $\| \cdot \|F$ to denote the Frobenius norm. The squared Frobenius norm is the sum of squares of the norms of the row-vectors (or, alternatively, column vectors) in the matrix.

Norm of a matrix

- The energy of a matrix A is an alternative term used in machine learning community for the squared Frobenius norm.
- The energy of a rectangular matrix A is equal to the trace of either AA^T or $A^T\mathrm{A}$
- $\|A\|_F^2 = Energy(A) = tr(AA^T) = tr(A^TA)$

Orthonormal Vectors

- Let q_1, q_2, ……. , q_n be vectors, they are said to be orthonormal

if $q_i^T q_j = \begin{cases} 0 \ if \ i \neq j \\ 1 \ if \ i = j \end{cases}$

In other words, the length of each vector is 1

Orthogonal Matrix

- An orthogonal matrix is a square matrix with orthonormal columns

$$\begin{bmatrix} - & q_1^T & - \\ - & q_i^T & - \\ - & q_n^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ q_1 & q_j & q_n \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}$$

When row i of Q^T multiplies column j of Q, the result is

$$q_j^T q_j = 0$$

On the diagonals where i =j, we have $q_j^T q_j = 1$, ie. The normalization to unit vector of length i

Orthogonal Matrix

An orthonormal matrix is a type of square matrix whose columns and rows are orthonormal unit vector, eg. Perpendicular and have a length or magnitude of 1.

Then Q^T is Q^{-1}

ie. $Q^T Q = Q Q^T = I$

Computing of Q^T is more time efficient as compared to computing Q^{-1}

- Eg
- $Q = \begin{bmatrix} \cos\theta & -sin\theta \\ sin\theta & \cos\theta \end{bmatrix}$
- $Q^T = Q^{-1} = \begin{bmatrix} \cos\theta & sin\theta \\ -sin\theta & \cos\theta \end{bmatrix}$

Linear Transformations

Transformation

- A transformation (or function or mapping) T from R^n to R^m is a rule that assigns to each vector x in R^n a vector T(x) in R^m. The set R^n is called the domain of T, and R^m is called the codomain of T .
- The notation T : $R^n \to R^m$ indicates that the domain of T is R^n and the codomain is R^m . For x in R^n , the vector T(x) in R^m is called the image of x (under the action of T). The set of all images T (x) is called the range of T.

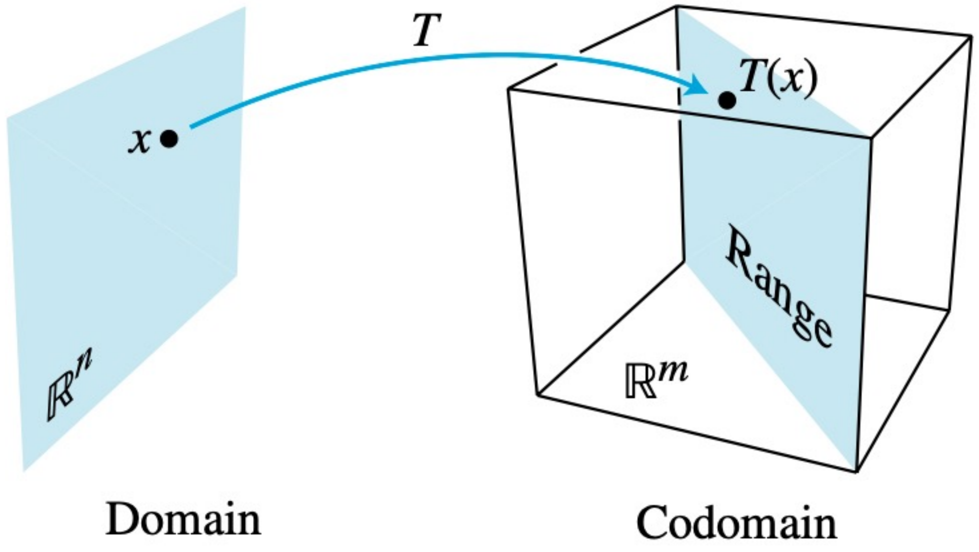
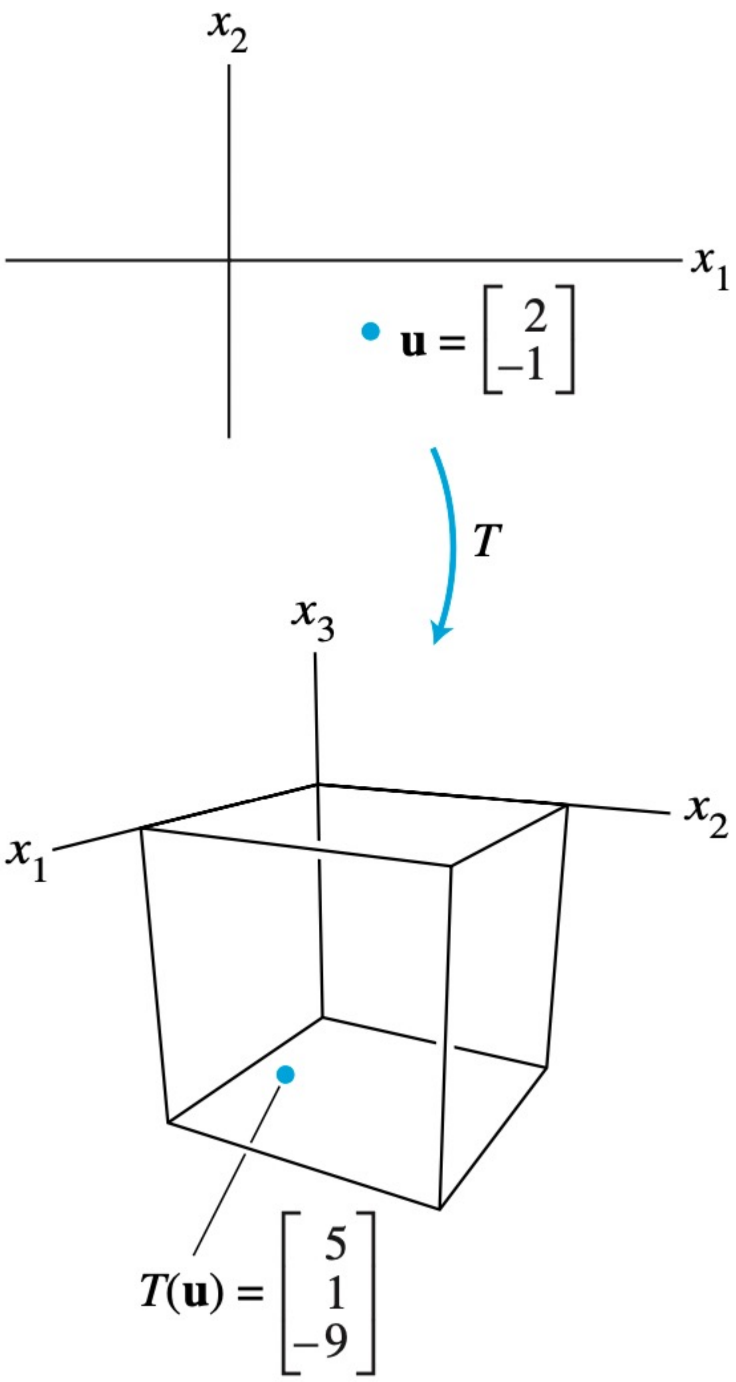


FIGURE 2 Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

- Example Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}, c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$,and define a transformation T : $R^2 \rightarrow R^3$ by T (**x**) = A**x**, so that
- $T(x) = Ax = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$. - (1)
- a. Find T(**u**), the image of **u** under the transformation T.
- b. Find an **x** in R^2 whose image under T is **b**.
- c. Is there more than one **x** whose image under T is **b**?
- d. Determine if **c** is in the range of the transformation T.

- a. Compute $T(u) = Au$
- $= \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$



- b. Solve T(**x**) = **b** for **x**. That is, solve A**x** = **b**,

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

Row reduced augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence x1 = 1.5, x2 = −0.5,and **x** = $\begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$. The image of this **x** under T is the given vector **b**.

c. Any **x** whose image under T is **b** must satisfy (1). From b. it is clear that equation (1) has a unique solution. So there is exactly one **x** whose image is **b**.

d. The vector **c** is in the range of T if **c** is the image of some **x** in R^2, that is, if **c** = T(**x**) for some **x**. This is just another way of asking if the system A**x** = **c** is consistent. To find the answer, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

The third equation, 0 = −35, shows that the system is inconsistent. So **c** is *not* in the range of T .

Example 2

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.

Find the images under T of $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

Example 2

$$T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

$$T(\mathbf{v}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}$$

Example 3

Let $A = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Define a transformation T : $R^2 \rightarrow R^3$ by T (**x**) = A**x**

Find T(**u**), T(**v**)

Example 3

$$T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} .5 \\ 0 \\ -2 \end{bmatrix}$$

$$T(\mathbf{v}) = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} .5a \\ .5b \\ .5c \end{bmatrix}$$

Example 4a

With T defined by T $(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$$

Example 4a

$$. \ \begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \text{ unique solution}$$

Example 4b

With T defined by T ($\mathbf{x}$) = A$\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique.

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$$

Example 4b

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 3 & -5 & -9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}, \text{ unique solution}$$

Example 5

With T defined by T ($\mathbf{x}$) = A$\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique.

$$A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$