Elementary Row Operations

Elementary Row Operations on a matrix

The following operations can be performed on the rows of a matrix:

1. (Replacement) Replace one row by the sum of itself and a multiple of another row
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a non-zero constant.

Elementary Matrices

- An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.
- Eg A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

Elementary Matrices Examples

A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ Addition of −4 times row 1 of I to row 3 produces A

B = $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ An interchange of rows 1 and 2 of I produces B

C = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ Multiplication of row 3 of I by 5 produces C

Elementary Row Operations on a matrix

- Row operations can be applied to any matrix. Two matrices are called row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other.
- It is important to note that row operations are *reversible*. If two rows are interchanged, they can be returned to their original positions by another interchange. If a row is scaled by a nonzero constant c, then multiplying the new row by 1/c produces the original row.
- A *nonzero* row or column in a matrix means a row or column that contains at least one nonzero entry;
- A leading entry of a row refers to the leftmost nonzero entry (in a nonzero row).

Row echelon form

A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

- All nonzero rows are above any rows of all zeros.
- Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- All entries in a column below a leading entry are zeros.

The following matrices are in echelon form.

eg. $\begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & -8 & 8 \\ 0 & 0 & 0 & 10 \end{bmatrix} \quad and \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 8 \\ 0 & 0 & -5 & 4 \end{bmatrix}$

Row echelon form

The following matrices are in echelon form. The leading entries ( ) may have any nonzero value; the starred entries ($*$) may have any values (including zero).

$$
\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad
\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}
$$

Reduced row echelon form

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form):

- The leading entry in each nonzero row is 1
- Each leading 1 is the only nonzero entry in its column.

- The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below *and above* each leading 1.

Eg. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad and \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 10 \end{bmatrix}$

Reduced row echelon form

- The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below and above each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Reduced row echelon form

- Any nonzero matrix may be row reduced (that is, transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique.

Pivot Positions

A pivot position in a matrix A is a location in A that corresponds to a leading entry in the reduced echelon form of A.

A pivot column is a column of A that contains a pivot position.

$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot Positions

Eg, $A = \begin{bmatrix} 0 & 3 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

Here, the entries a_{12} and a_{a23} are pivots and columns 2 and 3 are pivotal columns.

Row Reduction Algorithm

The algorithm that follows consists of four steps, and it produces a matrix in echelon form. A fifth step produces a matrix in reduced echelon form. We illustrate the algorithm by an example.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Row Reduction Algorithm

Step 1

Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Pivot column

Row Reduction Algorithm

Step 2

Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

$$\begin{array}{l} \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

Row Reduction Algorithm

Step 3

Use row replacement operations to create zeros in all positions below the pivot.

Replace R_2 with $R_2 - R_1$.

$$\begin{array}{l} \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

Row Reduction Algorithm

Step 4

Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, we'll select as a pivot the "top" entry in that column.

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \\ \text{New pivot column} \end{array}$$

Row Reduction Algorithm

Step 4

For step 3, we could insert an optional step of dividing the "top" row of the submatrix by the pivot, 2.

Instead, replace R_3 with $R_3 - (3/2)R_2$ This produces

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Row Reduction Algorithm

Step 4

When we cover the row containing the second pivot position for step 4, we are left with a new submatrix having only one row:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Pivot

Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. If we want the reduced echelon form, we perform one more step.

Row Reduction Algorithm

Step 5

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

Replace R_1 with $R_1 - 6R_3$ and replace R_2 with $R_2 - 2R_3$ This produces

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Row Reduction Algorithm

Step 5

The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Row Reduction Algorithm

Step 5

Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Row Reduction Algorithm

Step 5

Finally, scale row 1, dividing by the pivot, 3.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This is the reduced echelon form of the original matrix.

The combination of steps 1–4 is called the forward phase of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the backward phase.

Inverse of a Matrix

(contd.)

Inverse

Recall that square matrices A and B are inverses if $AB = BA = I$.

Or in other words

$B = IA^{-1}$

One way to find the inverse of matrix A is to employ the minors of its determinant but this is not efficient. The better way is to use an elimination method.

Matrix Inversion by Gauss Jordan method

If a matrix A is invertible, there are a set of steps to reduce it to the identity matrix, which also means that we have some set of elementary matrices such that

$E_n E_{n-1} \ldots E_2 E_1 A$ = I

However, by right-multiplying by A^{-1} (since A is invertible), we get
$E_n E_{n-1} \ldots E_2 E_1$ I= A^{-1}

So by performing the steps to reduce A to the identity matrix, those same steps performed on the identity matrix create the inverse of A.

Matrix Inversion by Gauss Jordan method

If we start with [A | I] and reduce the left side to the identity matrix, then we would end up with [I | A^{-1}]

[A | I]

[A^{-1}A | A^{-1} I]

[I | A^{-1}]

So A becomes I (because $A^{-1}A$ = I) and I becomes A-1 (because A^{-1} I = A^{-1})

Matrix Inversion by Gauss Jordan method

Eg1.

Find the inverse of the matrix

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

Matrix Inversion by Gauss Jordan method

Eg1.

Find the inverse of the matrix

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

Form the block matrix M = [A | I] and row reduce M to an echelon form

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

Solution 1

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

Subtract the 2 times first row from the second row

Replace R_2 by $R_2 - 2R_1$, gives us

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2-2(1) & 7-2(3) & 0-2(1) & 1-2(0) \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

Solution 1

- $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$
- We reduce the second column to [0,1] by row operations

Replace R_1 by $R_1 - 3R_2$ to get

- $\left[\begin{array}{cc|cc} 1-3(0) & 3-3(1) & 1-3(-2) & 0-3(1) \\ 0 & 1 & -2 & 1 \end{array}\right]$

Solution 1

- $\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$

- The inverse of the matrix is $\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$

Example 2

Find the inverse of the matrix

$$\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$$

Solution 2

Find the inverse of the matrix

$$\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$$

Form the block matrix M = [A | I] and row reduce M to an echelon form

$$\left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{array}\right]$$

Solution 2

$$\left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{array}\right]$$

Add the 2 times first row from the second row

Replace R_2 by $R_2 + 2\,R_1$

$$\left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ -2+2(1) & 6+2(-3) & 0+2(1) & 1+2(0) \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

As the last row is [0,0], the matrix in not invertible

Example 3

Find the inverse of the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Solution 3

Find the inverse of the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Form the block matrix M = [A | I] and row reduce M to an echelon form

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Solution 3

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Divide the first row by 2

Replace R_1 by $(1/2)R_1$ to get

$$\left[\begin{array}{ccc|ccc} 1 & {}^{1}\!/_{2} & {}^{1}\!/_{2} & {}^{1}\!/_{2} & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Solution 3

- $\left[\begin{array}{ccc|ccc} 1 & {}^{1}/_{2} & {}^{1}/_{2} & {}^{1}/_{2} & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$
- The next step is to multiply the first row by -3 and add to R_2 , this gives us

Replace R_2 by R_2 - 3 R_1

- $\left[\begin{array}{ccc|ccc} 1 & {}^{1}/_{2} & {}^{1}/_{2} & {}^{1}/_{2} & 0 & 0 \\ 0 & {}^{1}/_{2} & {}^{-1}/_{2} & {}^{-3}/_{2} & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$

Solution 3

- $\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & -1/2 & -3/2 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$
- Multiply the first row by -2 and add to R_3 , this gives us

Replace R_3 by R_3 - $2R_1$ to get

- $\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & -1/2 & -3/2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$

Solution 3

- $\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & -1/2 & -3/2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$
- The third step is to multiply the second row by 2 gives us

Replace R_2 by (2) R_2

- $\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$

Solution 3

- $\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$
- Subtract the from first row (1/2) times the second row

Replace R_1 by $R_1 - (\frac{1}{2})R_2$

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$

Solution 3

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$
- Subtract the third row from the first row

Replace R_1 by $R_1 - R_3$

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$

Solution 3

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$
- Add the third row to the second row

Replace R_2 by $R_2 + R_3$

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$
- The inverse of the matrix is $\begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

Matrix Inversion Examples

Find Inverses of the following matrices

1) $\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$

2) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Matrix Inversion Examples

Find Inverses of the following matrices

3) $\begin{bmatrix} 5 & 10 \\ 4 & 7 \end{bmatrix}$

4) $\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$

Matrix Inversion Examples

Find Inverses of the following matrices

5) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$

6) $\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 1 \end{bmatrix}$

Matrix Inversion Examples

Find Inverses of the following matrices

7) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ 8) $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$

9) $\begin{bmatrix} 1 & 5 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Matrix Inversion by Gauss Jordan method

Eg1.

Find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$$

Form the block matrix M = [A | I] and row reduce M to an echelon form

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right]$$

Solution 1

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right]$$

Subtract the four times first row from the second row

Replace R_2 by $R_2 - 4R_1$, gives us

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 4-4 & 7-8 & 0-4 & 1-0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -4 & 1 \end{array}\right]$$

Solution 1

- $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -4 & 1 \end{array}\right]$

The next step is to multiply the second row by -1 gives us

Replace R_2 by (-1) R_2

- $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 4 & -1 \end{array}\right]$

Solution 1

- $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 4 & -1 \end{array}\right]$

The next step is to multiply the second row by -2 and add to row one gives us

Replace R_1 by R_1 +(-2) R_2

- $\left[\begin{array}{cc|cc} 1+(-2)0 & 2+(-2)1 & 1+(-2)4 & 0+(-2)(-1) \\ 0 & 1 & 4 & -1 \end{array}\right]$

Solution 1

- $\left[\begin{array}{cc|cc} 1 & 0 & -7 & 2 \\ 0 & 1 & 4 & -1 \end{array}\right]$

The inverse of the matrix is $\begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}$

Solution 2

Find the inverse of the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Form the block matrix M = [A | I] and row reduce M to an echelon form

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

Solution 2

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

Interchange R_2 and R_1 gives us

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

The inverse of the matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example 3

Find the inverse of the matrix

$$\begin{bmatrix} 5 & 10 \\ 4 & 7 \end{bmatrix}$$

Form the block matrix M = [A | I] and row reduce M to an echelon form

$$\left[\begin{array}{cc|cc} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right]$$

Solution 3

$$\left[\begin{array}{cc|cc} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right]$$

Divide the first row by 5

Replace R_1 by $(1/5)R_1$, gives us

$$\left[\begin{array}{cc|cc} 5/5 & 10/5 & 1/5 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & 1/5 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right]$$

Solution 3

$$\left[\begin{array}{cc|cc} 1 & 2 & {}^{1}/_{5} & 0 \\ 4 & 7 & 0 & 1 \end{array}\right]$$

Subtract the 4 times first row from the second row

Replace R_2 by R_2 -4 R_1

$$\left[\begin{array}{cc|cc} 1 & 2 & {}^{1}/_{5} & 0 \\ 4-4 & 7-8 & 0-{}^{4}/_{5} & 1-0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & {}^{1}/_{5} & 0 \\ 0 & -1 & -{}^{4}/_{5} & 1 \end{array}\right]$$

Solution 3

- $\left[\begin{array}{cc|cc} 1 & 2 & {}^{1}/_{5} & 0 \\ 0 & -1 & -{}^{4}/_{5} & 1 \end{array}\right]$

Replace R_2 by - R_2 to get

- $\left[\begin{array}{cc|cc} 1 & 2 & {}^{1}/_{5} & 0 \\ 0 & 1 & {}^{4}/_{5} & -1 \end{array}\right]$

Solution 3

- $\left[\begin{array}{cc|cc} 1 & 2 & {}^{1}/_{5} & 0 \\ 0 & 1 & {}^{4}/_{5} & -1 \end{array}\right]$
- We reduce the second column to [0,1] by row operations

Replace R_1 by R_1 - $2R_2$ to get

- $\left[\begin{array}{cc|cc} 1 & 2-2 & {}^{1}/_{5} - {}^{8}/_{5} & 0+2 \\ 0 & 1 & {}^{4}/_{5} & -1 \end{array}\right]$

Solution 3

- $\left[\begin{array}{cc|cc} 1 & 0 & -{}^{7}/_{5} & 2 \\ 0 & 1 & {}^{4}/_{5} & -1 \end{array}\right]$

- The inverse of the matrix is $\begin{bmatrix} -{}^{7}/_{5} & 2 \\ {}^{4}/_{5} & -1 \end{bmatrix}$

Example 4

Find the inverse of the matrix

$$\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$$

Form the block matrix M = [A | I] and row reduce M to an echelon form

$$\left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 5 & -2 & 0 & 1 \end{array}\right]$$

Solution 4

$$\left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 5 & -2 & 0 & 1 \end{array}\right]$$

Divide the first row by 2

Replace R_1 by $(1/2)R_1$, gives us

$$\left[\begin{array}{cc|cc} {}^{2}/_{2} & {}^{-1}/_{2} & {}^{1}/_{2} & 0 \\ 5 & -2 & 0 & 1 \end{array}\right]$$

Solution 4

$$\left[\begin{array}{cc|cc} 1 & {}^{-1}/_{2} & {}^{1}/_{2} & 0 \\ 5 & -2 & 0 & 1 \end{array}\right]$$

Subtract the 5 times first row from the second row

Replace R_2 by $R_2 - 5\ R_1$

$$\left[\begin{array}{cc|cc} 1 & {}^{-1}/_{2} & {}^{1}/_{2} & 0 \\ 5-5(1) & -2-(5)({}^{-1}/_{2}) & 0-(5)({}^{1}/_{2}) & 1-(5)0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & {}^{-1}/_{2} & {}^{1}/_{2} & 0 \\ 0 & {}^{1}/_{2} & -{}^{5}/_{2} & 1 \end{array}\right]$$

Solution 4

- $\left[\begin{array}{cc|cc} 1 & -1/2 & 1/2 & 0 \\ 0 & 1/2 & -5/2 & 1 \end{array}\right]$

Replace R_2 by $2R_2$ to get

- $\left[\begin{array}{cc|cc} 1 & -1/2 & 1/2 & 0 \\ 0 & 1 & -5 & 2 \end{array}\right]$

Solution 4

- $\left[\begin{array}{cc|cc} 1 & -1/2 & 1/2 & 0 \\ 0 & 1 & -5 & 2 \end{array}\right]$
- We reduce the second column to [0,1] by row operations

Replace R_1 by $R_1 + (1/2)R_2$ to get

- $\left[\begin{array}{cc|cc} 1+(1/2)0 & -1/2+(1/2)1 & 1/2+(1/2)(-5) & 0+(1/2)2 \\ 0 & 1 & -5 & 2 \end{array}\right]$

Solution 4

- $\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right]$

- The inverse of the matrix is $\begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$

Solution 5

Find the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$$

Form the block matrix M = [A | I] and row reduce M to an echelon form

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right]$$

Solution 5

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right]$$

Subtract the two times reduced first row from the second row and also multiply the first row by 4 and then subtract from the third, gives us

Replace R_2 by $R_2 - 2R_1$ and R_3 by $R_3 - 4R_1$ to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$$

Solution 5

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$
- The next step is to multiply the second row by -1 gives us

Replace R_2 by (-1) R_2

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$

Solution 5

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$
- We reduce the second column to [0,1,0] by row operations

Replace R_3 by R_3 - R_2 to get

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right]$

Solution 5

- $$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right]$$
- The third step is to multiply the third row by -1 gives us

Replace R_3 by (-1) R_3

- $$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

Solution 5

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$
- Subtract the from first row two times the third row

Replace R_1 by $R_1 - 2R_3$ R_2 by $R_2 - R_3$

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$
- The inverse of the matrix is $\begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$

Solution 6

Find the inverse of the matrix

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 1 \end{bmatrix}$$

Form the block matrix M = [A | I] and row reduce M to an echelon form

$$\left[\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 4 & 4 & -3 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{array}\right]$$

Solution 6

- $\left[\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 4 & 4 & -3 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{array}\right]$

- The next step is to multiply the first row by 1/2 gives us

Replace R_1 by (1/2) R_1

- $\left[\begin{array}{ccc|ccc} 1 & {}^{3}/_{2} & {}^{-1}/_{2} & {}^{1}/_{2} & 0 & 0 \\ 4 & 4 & -3 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{array}\right]$

Solution 6

- $$\left[\begin{array}{ccc|ccc} 1 & 3/2 & -1/2 & 1/2 & 0 & 0 \\ 4 & 4 & -3 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{array}\right]$$

Subtract the two times reduced first row from the second row and also multiply the first row by 4 and then subtract from the third, gives us

Replace R_2 by $R_2 - 4R_1$ and R_3 by R_3 $-2R_1$ to get

$$\left[\begin{array}{ccc|ccc} 1 & 3/2 & -1/2 & 1/2 & 0 & 0 \\ 0 & -2 & -1 & -2 & 1 & 0 \\ 0 & -6 & 2 & -1 & 0 & 1 \end{array}\right]$$

Solution 6

- $\left[\begin{array}{ccc|ccc} 1 & {}^{3}/_{2} & {}^{-1}/_{2} & {}^{1}/_{2} & 0 & 0 \\ 0 & -2 & -1 & -2 & 1 & 0 \\ 0 & -6 & 2 & -1 & 0 & 1 \end{array}\right]$

- The next step is to multiply the second row by -1/2 gives us

Replace R_2 by (-1/2) R_2

- $\left[\begin{array}{ccc|ccc} 1 & {}^{3}/_{2} & {}^{-1}/_{2} & {}^{1}/_{2} & 0 & 0 \\ 0 & 1 & {}^{1}/_{2} & 1 & {}^{-1}/_{2} & 0 \\ 0 & -6 & 2 & -1 & 0 & 1 \end{array}\right]$

Solution 6

- $\left[\begin{array}{ccc|ccc} 1 & {}^{3}/_{2} & {}^{-1}/_{2} & {}^{1}/_{2} & 0 & 0 \\ 0 & 1 & {}^{1}/_{2} & 1 & {}^{-1}/_{2} & 0 \\ 0 & -6 & 2 & -1 & 0 & 1 \end{array}\right]$

- We reduce the second column to [0,1,0] by row operations

Replace R_1 by $R_1 - (3/2)R_{2,}$ R_3 by $R_3 + 6R_2$ to get

- $\left[\begin{array}{ccc|ccc} 1 & 0 & {}^{-5}/_{4} & -1 & {}^{3}/_{4} & 0 \\ 0 & 1 & {}^{1}/_{2} & 1 & {}^{-1}/_{2} & 0 \\ 0 & 0 & 5 & 5 & -3 & 1 \end{array}\right]$

Solution 6

- $\left[\begin{array}{ccc|ccc} 1 & 0 & -5/4 & -1 & 3/4 & 0 \\ 0 & 1 & 1/2 & 1 & -1/2 & 0 \\ 0 & 0 & 5 & 5 & -3 & 1 \end{array}\right]$
- The third step is to multiply the third row by 1/5 gives us

Replace R_3 by (1/5) R_3

- $\left[\begin{array}{ccc|ccc} 1 & 0 & -5/4 & -1 & 3/4 & 0 \\ 0 & 1 & 1/2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 1 & -3/5 & 1/5 \end{array}\right]$

Solution 6

- $\left[\begin{array}{ccc|ccc} 1 & 0 & -5/4 & -1 & 3/4 & 0 \\ 0 & 1 & 1/2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 1 & -3/5 & 1/5 \end{array}\right]$

Replace R_1 by $R_1 + (5/4)R_{3,}$ R_2 by $R_2 - (1/2)R_3$

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 0 & 1/4 \\ 0 & 1 & 0 & 1/2 & -1/5 & -1/10 \\ 0 & 0 & 1 & 1 & -3/5 & 1/5 \end{array}\right]$

- The inverse of the matrix is $\begin{bmatrix} 1/4 & 0 & 1/4 \\ 1/2 & -1/5 & -1/10 \\ 1 & -3/5 & 1/5 \end{bmatrix}$

Solution 7

Find the inverse of the matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

Form the block matrix M = [A | I] and row reduce M to an echelon form

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

Solution 7

- $\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$

- The next step is to interchange R_1 and R_2

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$

Solution 7

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$

Subtract the four times reduced first row from the third row

Replace R_3 by R_3 - $4R_1$ to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right]$$

Solution 7

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right]$

- We reduce the second column to [0,1,0] by row operations

Replace R_3 by $R_3 + 3R_2$ to get

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$

Solution 7

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$
- The third step is to multiply the third row by 1/2 gives us

Replace R_3 by (1/2) R_3

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & {}^{3}/_{2} & -2 & {}^{1}/_{2} \end{array}\right]$

Solution 7

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & {}^{3}/_{2} & -2 & {}^{1}/_{2} \end{array}\right]$

Replace R_1 by R_1 - $3R_{3,}$ R_2 by R_2 - $2R_3$

- $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & {}^{-9}/_{2} & 7 & {}^{-3}/_{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & {}^{3}/_{2} & -2 & {}^{1}/_{2} \end{array}\right]$

- The inverse of the matrix is $\begin{bmatrix} {}^{-9}/_{2} & 7 & {}^{-3}/_{2} \\ -2 & 4 & -1 \\ {}^{3}/_{2} & -2 & {}^{1}/_{2} \end{bmatrix}$

Solution 8

Find the inverse of the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$$

Form the block matrix M = [A | I] and row reduce M to an echelon form

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right]$$

Solution 8

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right]$$

Subtract the four times first row from the second row, add 2 times first row to the third row

Replace R_2 by $R_2 - 4R_1$, Replace R_3 by $R_3 + 2R_1$ to get

- $\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right]$

Solution 8

- $\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right]$

Subtract the two times second row from the third row

Replace R_3 by R_3 - $2R_2$ to get

- $\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{array}\right]$

This matrix is not invertible as the last row is [0 0 0]

Solution 9

- The inverse of the matrix is $\begin{bmatrix} 1 & 5 & -12 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$