Matrices

12/11/2021

Properties of Matrix Multiplication

- Associative : Matrix Multiplication is Associative

  $(AB)C = A(BC)$

- Distributive : Matrix Multiplication is Distributive

  $A(B + C) = AB + BC$

- Commutative : Matrix Multiplication is not commutative.

  $A\,B \neq BA$

- $k(AB) = (kA)B = A(Bk)$, where k is a scalar

Property of matrix multiplication

- Suppose A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} B = \begin{bmatrix} 5 & 6 \\ 0 & -2 \end{bmatrix}$

AB = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} . \begin{bmatrix} 5 & 6 \\ 0 & -2 \end{bmatrix}$ =

and

BA = $\begin{bmatrix} 5 & 6 \\ 0 & -2 \end{bmatrix} . \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ =

Property of matrix multiplication

- Suppose A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 0 & -2 \end{bmatrix}$

AB = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 5+0 & 6-4 \\ 15+0 & 18-8 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 15 & 10 \end{bmatrix}$

and

BA = $\begin{bmatrix} 5 & 6 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5+18 & 10+24 \\ 0-6 & 0-8 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ -6 & -8 \end{bmatrix}$

The above example shows that matrix multiplication is not commutative.

i.e. A B $\neq$ BA

Hadamard Product

- There are two important notions of multiplication of matrices, each of which has different properties.
- Hadamard product or element wise product is denoted $\odot$ or O
- The Hadamard product is only defined over matrices of equal size and returns a matrix of the same size
- Eg.

$$\begin{bmatrix} 5 & 10 \\ -2 & 0 \\ 1 & -1 \end{bmatrix} \odot \begin{bmatrix} 2 & -1 \\ -1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5\times 2 & 10\times -1 \\ -2\times -1 & 0\times 5 \\ 1\times 0 & -1\times 1 \end{bmatrix} = \begin{bmatrix} 10 & -10 \\ 2 & 0 \\ 0 & -1 \end{bmatrix}$$

Hadamard Product

- Eg.

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \odot \begin{bmatrix} 1 & 4 & 7 \\ 8 & 20 & 5 \\ 2 & 8 & 3 \end{bmatrix}$$

Hadamard Product

- Eg.

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \odot \begin{bmatrix} 1 & 4 & 7 \\ 8 & 20 & 5 \\ 2 & 8 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 21 \\ 32 & 100 & 30 \\ 14 & 64 & 27 \end{bmatrix}$$

Properties of Hadamard product

- Associative : Hadamard product is Associative

  $$(AB)C = A(BC)$$

- Distributive : Hadamard product is Distributive

  $$A(B + C) = AB + BC$$

- Commutative : Hadamard product is commutative.

  $$A\,B = BA$$

- Hadamard product is used in image compression techniques such as JPEG.
- It is also used in LSTM(Long Short-Term Memory) cells of Recurrent Neural Networks(RNNs)
- Used in Tensors

Scalar Multiplication

- The product of the matrix A by a scalar k, written k.a or simply kA, is the matrix obtained by multiplying each element of A by k.

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{1n} \\ ka_{21} & ka_{22} & ka_{2n} \\ ka_{m1} & ka_{m2} & ka_{mn} \end{bmatrix}$$

Eg $A = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad 3.A = \begin{bmatrix} 3.2 & 3.2 & 3.0 \\ 3.1 & 3.0 & 3.0 \\ 3.0 & 3.0 & 3.3 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 6 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Diagonal of a Matrix

- Let A= [a_{ij}] be a n-square matrix. The diagonal or main diagonal of A consists of the elements with the same subscripts i.e. a_{11},a_{22},.....a_{nn}. It is denoted by diag(A)

- Eg. $diag\left(\begin{bmatrix}1 & 4 & 5\\ 4 & 2 & 7\\ 5 & 2 & 6\end{bmatrix}\right) = [1, 2, 6]$

Trace of a Matrix

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- Trace of matrix A, denoted by trA, is the sum of the elements along the diagonal of A
- i.e. $trA = a_{11} + a_{22} + \ \dots . + a_{\mathrm{nn}}$
- eg. For $\left(A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 7 \\ 5 & 2 & 6 \end{bmatrix} \right)$

  trA= 1 + 2 + 6 = 9

Determinant of a Matrix

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- For a 2x2 Matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is denoted by |A|

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- Eg. A = $\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = 2(-2) - 1(-3) = -1$

Determinant of a Matrix

- For a mxn Matrix

$$A= \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}$$

$|A| = \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} a_{ij} C_{ij}$, where C_{ij} is the co-factor of a_{ij}

$C_{ij} = (-1)^{i+j} M_{ij,}$ where M_{ij} is Minor_{ij} which is the determinant of the reduced matrix obtained by removing the row i and column j

Determinant of a Matrix

- For a 3x3 Matrix

$$A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$$

$$|A| = a_{11}C_{11} + a_{12}\,C_{12} + a_{13}\,C_{13} = 7C_{11} + 2\,C_{12} + 1\,C_{13}$$

$$C_{11} = (\text{-}1)^{1+1}M_{11} = (-1)^2 \begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix} = -6+4 = -2$$

$$C_{12} = (\text{-}1)^{1+2}M_{12} = (-1)^3 \begin{vmatrix} 0 & -1 \\ -3 & -2 \end{vmatrix} = (-1)(0-3) = 3$$

$$C_{13} = (\text{-}1)^{1+3}M_{13} = (-1)^4 \begin{vmatrix} 0 & 3 \\ -3 & 4 \end{vmatrix} = 0+9 = 9$$

$|A| = 7(-2) + 2(3) + 9 = 1$

Determinant of a Matrix

- For a 3x3 Matrix

A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinant of a Matrix

- For a 3x3 Matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & 0 & -2 \end{bmatrix}$$

Determinant of a Matrix

- For a 3x3 Matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & 0 & -2 \end{bmatrix}$$

$$|A| = a_{11}C_{11} + a_{12}\,C_{12} + a_{13}\,C_{13} = 2C_{11} + 1\,C_{12} + 3\,C_{13}$$

$$C_{11} = (-1)^{1+1}M_{11} = (-1)^2 \begin{vmatrix} 0 & 2 \\ 0 & -2 \end{vmatrix} = 0$$

$$C_{12} = (-1)^{1+2}M_{12} = (-1)^3 \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = 6$$

$$C_{13} = (-1)^{1+3}M_{13} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = 0$$

$|A| = 2x0 + 1x6 + 3x0 = 6$

Properties of Determinant

- The determinant of identity matrix is 1.
- The determinant changes sign when two rows are exchanged.

Row exchange $\begin{vmatrix} c & d \\ a & b \end{vmatrix}$ = cb-ad = $-\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Properties of Determinant

- The determinant of matrix A and its transpose are equal

  $|A| = |A^T|$

  This property is useful in computation if there are many 0's in the matrix

- Let A be a square matrix.
  - If A has a row(column) of zeros, then $|A| = 0$
  - If A has two identical rows (columns) then $|A| = 0$
  - If A is a triangular (i.e. A has zeros above or below the diagonal),

    then $|A|$ = product of diagonal elements. Thus, in particular, $|I| = 1$, where I is the identity matrix.

Properties of Determinant

- The determinant of product of two matrices A and B is product of their determinants

$$\det(AB) = \det(A)\det(B)$$

- Let A be a square matrix.
  - A is invertible; that is A has an inverse A^{-1}
  - The determinant of A is not zero; that is $\det(A) \neq 0$

Programming Assignment 1

- Enter number of rows – Use input function to read the values
- Enter number of columns– Use input function to read the values
- Enter elements of rows – Use list to store the matrix elements
- Read elements and store(input) in a list
- Display options:

  a) Display matrix – Use print function to display the matrix on the screen

  b) Scalar multiplication – Multiply a scalar with every element of the list

  c) Transpose – Define a function to interchange the rows with the columns

  d) Diagonal of the matrix

  e) Trace of the matrix

  f) Determinant of a matrix

Programming Assignment 1

- Check the following exceptions- Define functions to make the following checks
  - Trace of non-square matrix
  - Determinant of a non-square matrix

Matrices

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Linear Independence of vectors

- Vectors $\bar{v}_1, \bar{v}_2, \ldots \bar{v}_n$ are said to be linearly independent.
If there exists scalars $x_1, x_2, \ldots x_n$
If $x_1\bar{v}_1 + x_2\bar{v}_2 + \ldots x_n\bar{v}_n=0$
$x_1 = x_2 = \ldots = x_n = 0$

OR

We can also say that there do not exist scalars
$x_1, x_2, \ldots x_n$ Not all Zeros
such that
$x_1\bar{v}_1 + x_2\bar{v}_2 + \ldots x_n\bar{v}_n=0$

Example

- Let $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$ be vectors, and x_1,x_2,x_3 be scalars

$$\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \quad \bar{v}_3 = \begin{bmatrix} 7 \\ -1 \\ -3 \end{bmatrix}$$

- $3\bar{v}_1 + 2\bar{v}_2 = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -3 \end{bmatrix} = \bar{v}3$

$3\bar{v}_1 + 2\bar{v}_2 - \bar{v}3 = 0$

We have scalars $x_1 = 3$, $x_2 = 2$, $x_3 = -1$ such that $x_1\bar{v}_1 + x_2\bar{v}_2 + x_3\bar{v}_3 = 0$

- $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$ are not linearly independent

Example

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- $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are linearly independent
- $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent
- $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are not linearly independent

  as $v_4 = v_1 + v_2 - v_3$

Rank of Matrix

- R(A) = maximum number of linearly independent columns(Column Rank)

  = maximum number of linearly independent rows(Row Rank)

- Column Rank = Row Rank = Rank of Matrix = R(A)
- Rank(A) $\leq$ min(m, n) #(As A only has n columns and m rows)

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- eg: $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ has rank = 2

  $X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has rank = 2 (Column 3 is sum of Column 1 & Column 2)

Minor of A

- Minor of matrix A is the determinant of some smaller square matrix cut down from A by removing one or more of its rows and columns.
- The minor M_{ij} is the determinant of the submatrix obtained by the deleting the i^{th} column.

Co-factor

- The cofactor C_{ij} is obtained by multiplying the minor by $(-1)^{i+j.}$
- To compute the minor $M_{2,\,3}$, and the cofactor $C_{2,\,3}$, we find the determinant of the above matrix with row 2 and column 3 removed.
- $M_{2,3} = det\begin{bmatrix} 1 & 4 & \\ & & \\ -1 & 9 & \end{bmatrix} = det\begin{bmatrix} 1 & 4 \\ -1 & 9 \end{bmatrix} = 9 - (-4) = 13$
- And $C_{2,\,3} = (-1)^{2+3} M_{2,\,3} = -13$

Adjoint of matrix

- Let A = [a_{ij}] be a nxn matrix and let C_{ij} denote the cofactor of a_{ij}. The classical adjoint or adjugate of A, denoted by the adjA, is the transpose of the matrix of cofactors of A
- $adjA = \left[C_{ij}\right]^T$

Inverse

- Let matrices A and B have the property

  $AB = BA = I$
- Then B is called the inverse of A and denoted by A^{-1}
- Not every matrix A possesses and inverse.
- If this inverse does exist, A is called regular/invertible/non-singular.
- If this inverse does not exist, A is called singular.
- If matrix inverse exists, it is unique.

Properties of inverse

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1} A^{-1}$
- $(A^{-1})^T = (A^T)^{-1}$

$Inverse\ of\ A$

$$A^{-1} = \frac{1}{|A|}\ (adjA)$$

$|A| \neq 0$

Eg. A = $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

$|A|\ 6 - 6 = 0$

Hence, A is not invertible

eg. $LetA = \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$

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$C_{11} = +\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} = -18, C_{12} = -\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = 2, C_{13} = +\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} = 4$

$C_{21} = -\begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix} = -11, C_{22} = +\begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} = 14, C_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = 5$

$C_{31} = +\begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix} = -10, C_{32} = -\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} = -4, C_{33} = +\begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix} = -8$

$C_{ij} = \begin{bmatrix} -18 & 2 & 4 \\ -11 & 14 & 5 \\ -10 & -4 & -8 \end{bmatrix}$

Continued…

The transpose of the above matrix of cofactors yields the adjoint of A

$$adjA = \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$$

$$det(A) = -40 + 6 + 0 - 16 + 4 + 0 = -46$$

$$det(A) \neq 0$$

$$A^{-1} = \frac{1}{|A|}(adjA) = \frac{-1}{46}\begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} {}^{9}/_{23} & {}^{11}/_{46} & {}^{5}/_{23} \\ {}^{-1}/_{23} & {}^{-7}/_{23} & {}^{2}/_{23} \\ {}^{-2}/_{23} & {}^{-5}/_{46} & {}^{4}/_{23} \end{bmatrix}$$

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$$\text{eg. } A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

eg. $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

- C_{11} = -1 C_{12} = 8 C_{13} = -5
- C_{21} = 1 C_{22} = -6 C_{23} = 3
- C_{31} = -1 C_{32} = 2 C_{33} = -1
- $adjA = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$
- $|A| = 0 - 1(11 - 9) + 2(1 - 6) = 8 - 10$ = -2
- $A^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$

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$$\text{eg. } A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

- $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$|A| = 9$

$$\text{Co-factor matrix} = \begin{bmatrix} 1 & 4 & -3 \\ -6 & 3 & 0 \\ 2 & -1 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/9 & -6/9 & 2/9 \\ 4/9 & 3/9 & -1/9 \\ -3/9 & 0 & 3/9 \end{bmatrix}$$