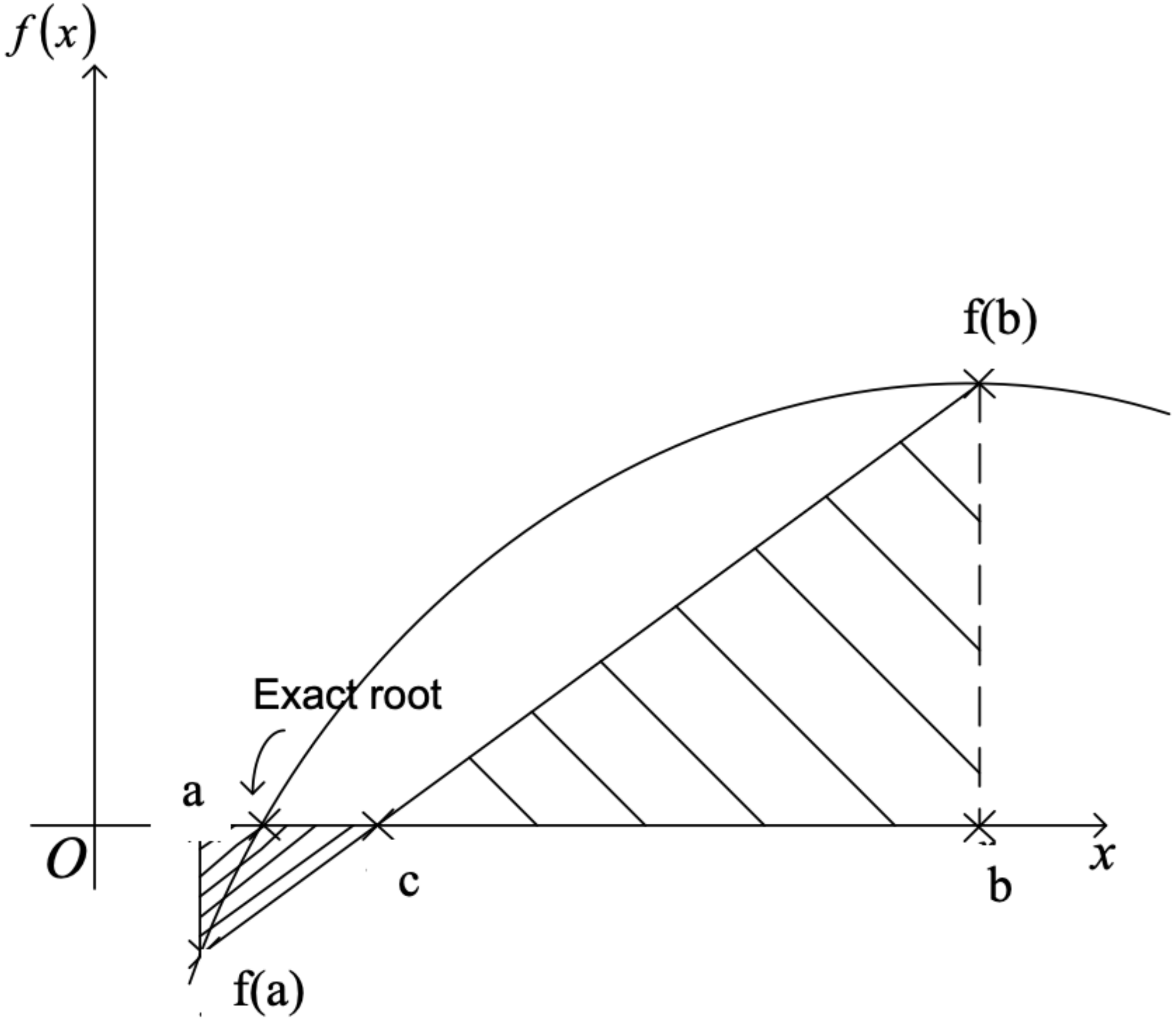
False Position

False position method



False position method

- In the bisection method, we identify proper values of a (lower bound value) and b (upper bound value) for the current bracket.
- However, in the example shown in Figure , the bisection method may not be efficient because it does not take into consideration that f(a) is much closer to the zero of the function f(x) as compared to f(b). In other words, the next predicted root c would be closer to a, than the mid-point between a and b. The false-position method takes advantage of this observation mathematically by drawing a straight line from the function value at a to the function value at b, and estimates the root as where it crosses the *x*-axis.

False position method

So, in regular falsi method, to take into consideration the function values at a and b, straight line is drawn, joining (a,f(a))and(b,f(b)).

The point, where it cuts X axis is, the new estimate of the root. Point of intersection of straight line with X – axis being on X-axis is (c , 0).

$$c = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Algorithm for the False Position method

1. Choose a and b as two guesses for the root such that f(a).f(b) < 0 Let k =1

2. Estimate the root, c, of the equation as a point between a and b as $c_k = \frac{af(b) - bf(a)}{f(b) - f(a)}$

3. Now check the following

   If f(a).f(c) < 0 , then the root lies between a and c; then b = c.

   If f(a).f(c) > 0 , then the root lies between c and b; then a = c

   If f(c) = 0; then the root is c. Stop the algorithm if this is true.

4. Find the new estimate of the root $c_k = \frac{af(b) - bf(a)}{f(b) - f(a)}$

5. Compare the absolute approximate error $|c_k - c_{k-1}|$ or absolute relative approximate error $\frac{|c_k - c_{k-1}|}{|c_k|}$ with the pre-specified relative error tolerance ϵ, where c_k = estimated root from present iteration, c_{k-1} = estimated root from previous iteration

If $|c_k - c_{k-1}| < \epsilon$ or $\frac{|c_k - c_{k-1}|}{|c_k|} < \epsilon$ then exit

Else

k = k + 1

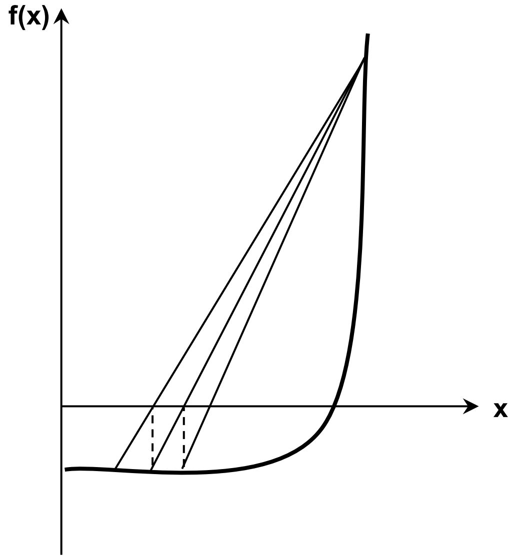
go to Step 3

Note one should also check whether the number of iterations is more than the maximum number of iterations allowed.

Advantages and Disadvantages

Disadvantages:

1. Not self starting. One needs two initial guesses a and b such that f(a)·f(b) < 0.
2. Though, faster than Bisection Method, still regarded as slow.
3. In rare cases, it may become slower than Bisection Method.



Slow convergence of the false-position method for $f(x) = x^{10} - 1 = 0$

Advantages and Disadvantages

Advantages:

1. It is faster than Bisection Method.
2. It is also simple.
3. It guarantees convergence.
4. Only one function evaluation per iteration is required.

Example

This polynomial obviously has roots at x = 2 and at x = 4; one is a double root, the other a triple root:

$$f(x) = (x-2)^3(x-4)^2$$
$$= x^5 - 14x^4 + 76x^3 - 200x^2 + 256x - 128$$

a. Which root can you get with bisection? Which root can't you get?

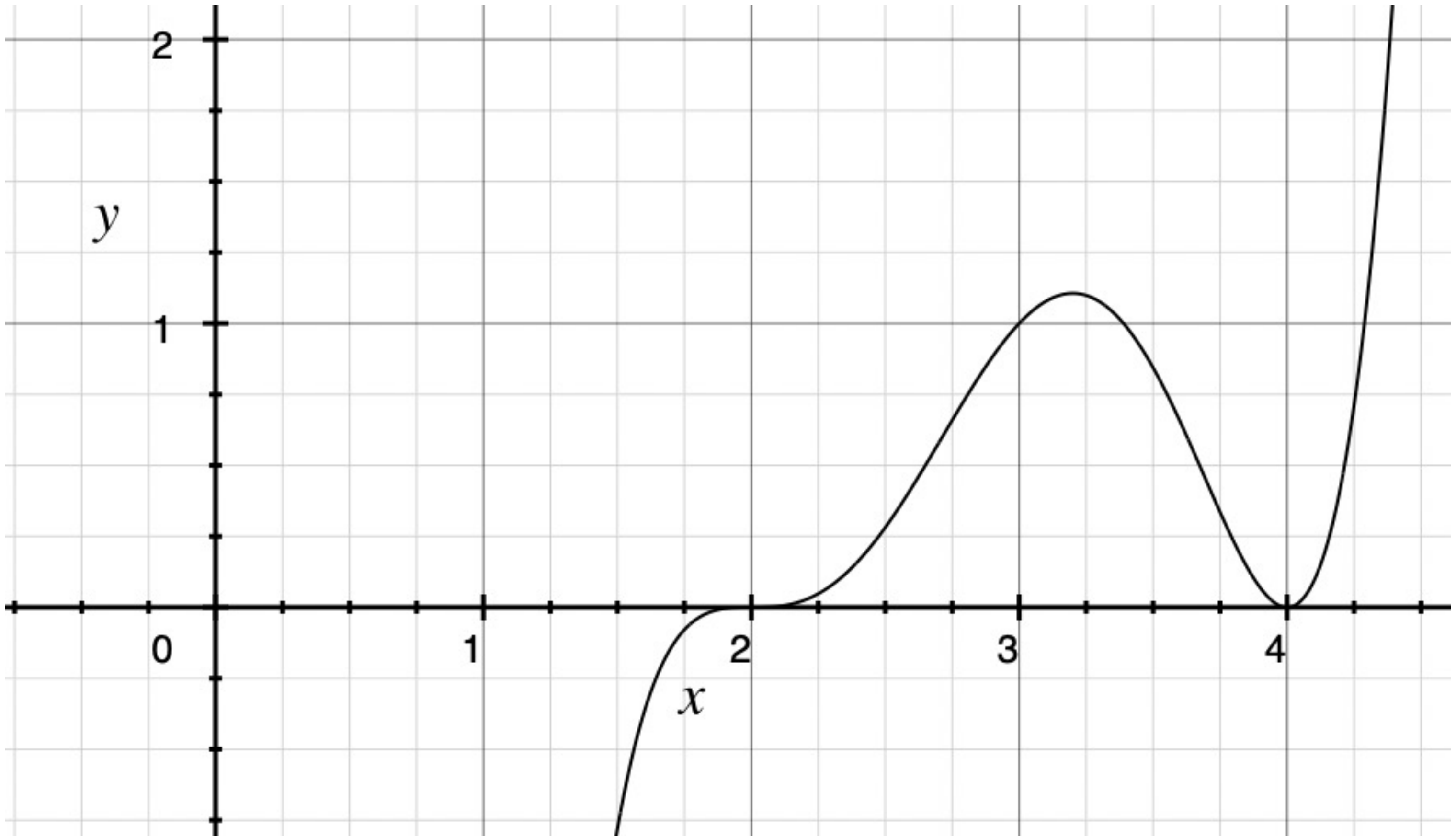
b. Repeat part (a) with the secant method.

c. If you begin with the interval [1,5], which root will you get with

(1) bisection, (2) the secant method, (3) false position?

d. Use Newton's method with x_0 = 3. Does it converge? To which root?

Solution



Solution

This polynomial obviously has roots at x = 2 and at x = 4; one is a double root, the other a triple root:

$f(x) = (x-2)^3(x-4)^2$

$= x^5 - 14x^4 + 76x^3 - 200x^2 + 256x - 128$

a.

b.

c. If you begin with the interval [1,5],

(3) False Position: Converge to 2

d.

Examples

Eg 1. *Find the real root of the equation* $x^3 - 4x - 9 = 0$ *by* False position method *correct.*

Take $a = 2$, $b = 3$, $\epsilon = 0.01$

Eg 2. *Find the real root of the equation* $x^3 - x - 1$ *by* False position method *correct to three decimal places.* $\epsilon = 0.001$

Take $a = 1$, $b = 1.5$

Solution 1

Eg 1. *Find the real root of the equation* $x^3 - 4x - 9 = 0$ *by* False position method *correct. Take* $a = 2$, $b = 3$, $\epsilon = 0.01$

$f(x) = x^3 - 4x - 9$

x	0	1	2	3
f(x)	-9	-12	-9	6

$f(2) = -9$ *i.e.*, (–)ve

and

$f(3) = 6$ *i.e.*, (+)ve

Hence, the root lies between 2 and 3.

a	f(a)	b	f(b)
2	-9	3	6

Solution 1

First approximation to the root is

$$c = \frac{(2)(6) - (3)(-9)}{6 - 9}$$

c = 2.6

a	f(a)	b	f(b)	c	f(c)	$\lvert c_k - c_{k-1} \rvert$
2	-9	3	6	2.6	– 1.824	-

Now $f(c)$ = – 1.824 *i.e.,* (–)ve and

f(b) = 6 *i.e.,* (+)ve

Hence, the root lies between 2. 6 and 3.

Solution 1

Second approximation to the root is

$$c = \frac{(2.6)(6) - (3)(-1.824)}{6 - (-1.824)}$$

= 2.6933

Now $f(c)$ = -0.2372 *i.e.,* (-)ve and $f(b)$ = 6 *i.e.,* (+)ve

Hence, the root lies between 2.6933 and 3

a	f(a)	b	f(b)	c	f(c)	$\lvert c_k - c_{k-1} \rvert$
2	-9	3	6	2.6	– 1.824	-
2.6	– 1.824	3	6	2.6933	-0.2372	0.0933

Solution 1

Third approximation to the root is

$$c = \frac{(2.6933)(6) - (3)(-0.2372)}{6 - (-0.2372)}$$

= 2.7049

Now $f(c)$ = -0.0289 *i.e.,* (-)ve and $f(b)$ = 6 *i.e.,* (+)ve

Hence, the root lies between 2.7049 and 3.

a	f(a)	b	f(b)	c	f(c)	$\lvert c_k - c_{k-1} \rvert$
2	-9	3	6	2.6	– 1.824	-
2.6	– 1.824	3	6	2.6933	-0.2372	0.0933
2.6933	-0.2372	3	6	2.7049	-0.0289	0.0116

Solution 1

Fourth approximation to the root is

$c = \frac{(2.7049)(6)-(3)(-0.0289)}{6-(-0.0289)}$ = 2.7063

ϵ = 0.01, $|c_k - c_{k-1}|$ = 0.0014

Hence, the root is 2.7063, correct to two decimal places.

a	f(a)	b	f(b)	c	f(c)	$\lvert c_k - c_{k-1} \rvert$
2	-9	3	6	2.6	– 1.824	-
2.6	– 1.824	3	6	2.6933	-0.2372	0.0933
2.6933	-0.2372	3	6	2.7049	-0.0289	0.0116
2.7049	-0.0289	3	6	2.7063	-0.0035	0.0014

Example 2

Eg 2. *Find the real root of the equation* $x^3 - x - 1$ *by* False position method *correct to three decimal places.* ϵ = 0.001

x	0	1	2
f(x)	-1	-1	5

Take a = 1, b = 1.5

Solution 2

Eg 2. *Find the real root of the equation* $x^3 - x - 1$ *by* False position method *correct to three decimal places. Take a = 1, b = 1.5,*

$\epsilon = 0.001$

x	0	1	2
f(x)	-1	-1	5

$f(x) = x^3 - x - 1$

$f(1) = -1$ (–)ve

$f(1.5) = 0.875$ (+)ve

Hence, the root lies between 1 and 1.5.

a	f(a)	b	f(b)
1	– 1	1.5	0.875

Solution 2

∴ First approximation to the root is

$$c = \frac{1(0.875) - (1.5)(-1)}{0.875 - (-1)}$$

c = 1.2667

a	f(a)	b	f(b)	c	f(c)
1	– 1	1.5	0.875	1.2667	-0.2344

Now f (c) = f(1.2667) = -0.2344 (-)ve

and f(b) = 0.875 (+)ve

Hence, the root lies between 1.2667 and 1.5

Solution 2

Second approximation to the root is

$$c = \frac{(1.2667\,)(0.875\,) - (1.5\,)(-0.2344\,)}{0.875 - (-0.2344\,)}$$

= 1.316

a	f(a)	b	f(b)	c	f(c)	$\lvert c_k - c_{k-1} \rvert$
1	– 1	1.5	0.875	1.2667	-0.2344	-
1.2667	-0.2344	1.5	0.875	1.316	-0.037	0.0493

Now $f(c)$ = f(1.316) = -0.037(-)ve and f(b) = 0.875 (+)ve

Hence, the root lies between 1.316 and 1.5.

Solution 2

Third approximation to the root is

$$c = \frac{(1.316)(0.875) - (1.5)(-0.037)}{0.875 - (-0.037)} = 1.3234$$

a	f(a)	b	f(b)	c	f(c)	$\lvert c_k - c_{k-1} \rvert$
1	– 1	1.5	0.875	1.2667	-0.2344	-
1.2667	-0.2344	1.5	0.875	1.316	-0.037	0.0493
1.316	-0.037	1.5	0.875	1.3234	-0.0055	0.0074

Now $f(c)$ = f(1.3234) = -0.0055 *i.e.,* (-)ve

and f(b) = 0.875 (+)ve

Hence, the root lies between 1.3234 and 1.5 .

Solution 2

Fourth approximation to the root is

$$c = \frac{(1.3234)(0.875)-(1.5)(-0.0055)}{0.875-(-0.0055)} = 1.3245$$

a	f(a)	b	f(b)	c	f(c)	$\lvert c_k - c_{k-1} \rvert$
1	– 1	1.5	0.875	1.2667	-0.2344	-
1.2667	-0.2344	1.5	0.875	1.316	-0.037	0.0493
1.316	-0.037	1.5	0.875	1.3234	-0.0055	0.0074
1.3234	-0.0055	1.5	0.875	1.3245	-0.0008	0.0011

Now $f(c) = f(1.3245) = -0.0008$ *i.e.,* (-)ve and

and f(b) = 0.875 (+)ve

Hence, the root lies between 1.3245 and 1.5

Solution 2

Fifth approximation to the root is

$$c = \frac{(1.3234)(0.875) - (1.5)(-0.0055)}{0.875 - (-0.0055)} = 1.3247$$

$\epsilon = 0.001$ $|c_k - c_{k-1}| = 0.0078 < \epsilon$

The approximate real root is 1.3247

a	f(a)	b	f(b)	c	f(c)	$\lvert c_k - c_{k-1} \rvert$
1	– 1	1.5	0.875	1.2667	-0.2344	-
1.2667	-0.2344	1.5	0.875	1.316	-0.037	0.0493
1.316	-0.037	1.5	0.875	1.3234	-0.0055	0.0074
1.3234	-0.0055	1.5	0.875	1.3245	-0.0008	0.0011
1.3245	-0.0008	1.5	0.875	1.3247	-0.0001	0.0002

Example

- In bisection and the method of false position, one tests to see that a function changes sign between x = a and x = b. If this is done by seeing if f(a) * f(b) < 0, underflow may occur. Is there an alternative way to make the test that avoids this problem?

Newton Raphson Method

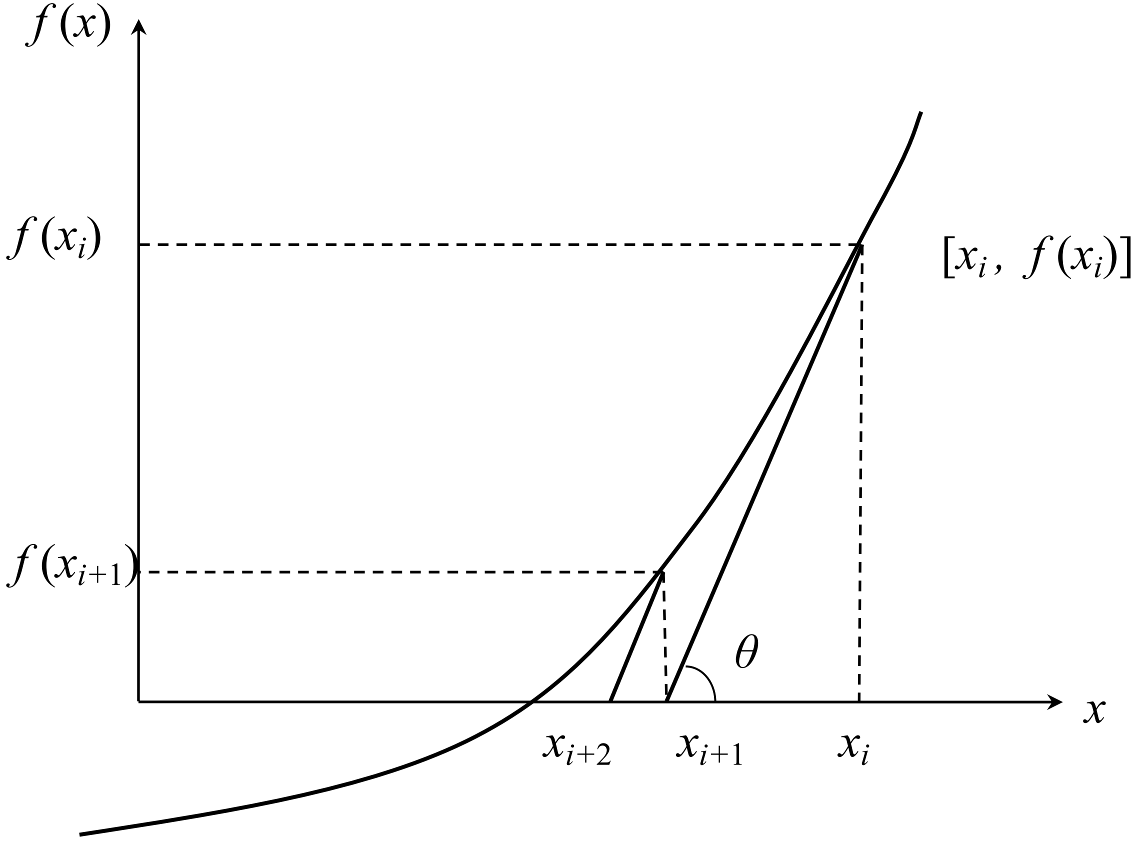
Newton Raphson Method

- Methods such as the bisection method and the false position method of finding roots of a nonlinear equation $f(x) = 0$ require bracketing of the root by two guesses. Such methods are called bracketing methods. These methods are always convergent since they are based on reducing the interval between the two guesses so as to zero in on the root of the equation.

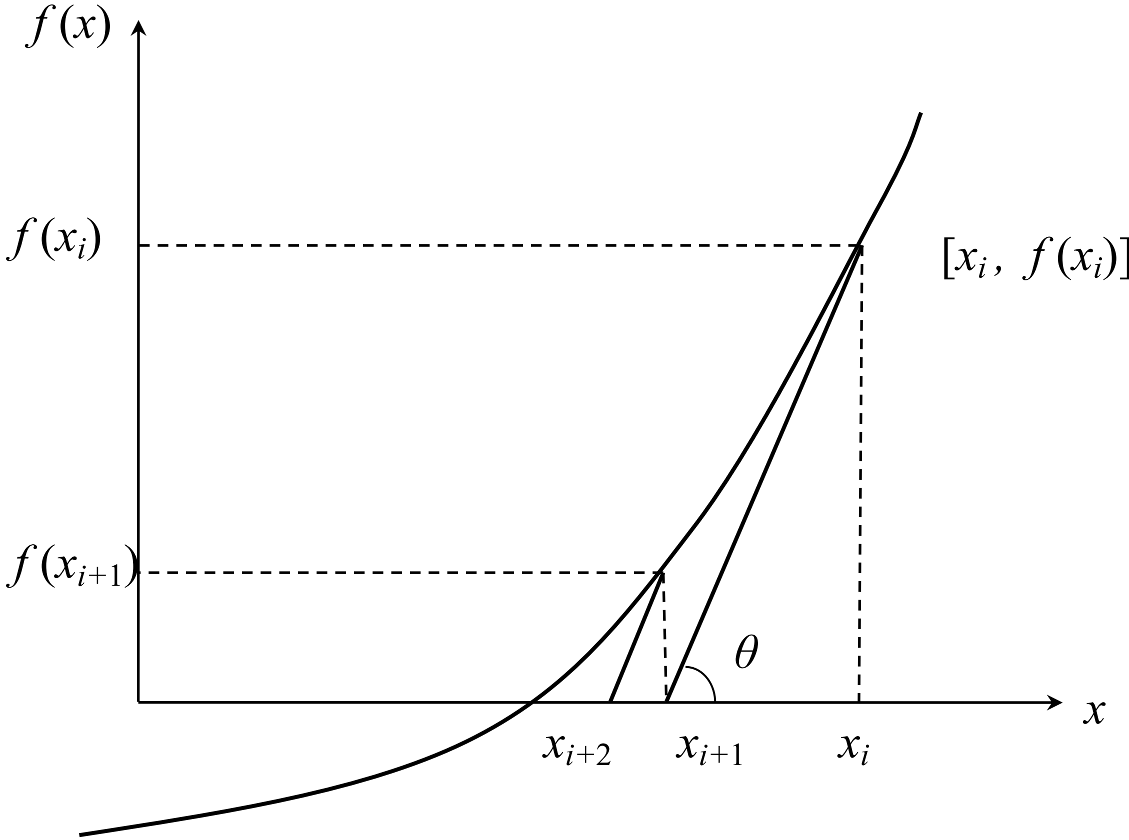
- In the Newton-Raphson method, the root is not bracketed. In fact, only one initial guess of the root is needed to get the iterative process started to find the root of an equation. The method hence falls in the category of open methods. Convergence in open methods is not guaranteed but if the method does converge, it does so much faster than the bracketing methods

Newton Raphson Method

- The Newton-Raphson method is based on the principle that if the initial guess of the root of f(x) =0 is at x_i , then if one draws the tangent to the curve at $f(x_i)$, the point x_{i+1} where the tangent crosses the x-axis is an improved estimate of the root



Newton Raphson Method



- Using the definition of the slope of

a function, at $x = x_i$

$$f'(x_i) = tan\theta$$

$$= \frac{f(x_i)-0}{x_i - x_{i+1}}$$

which gives

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- This equation is called the Newton-Raphson formula for solving nonlinear equations of the form f(x) = 0. So starting with an initial guess, x_i , one can find the next guess, x_{i+1} , by using the above equation. One can repeat this process until one finds the root within a desirable tolerance.

Algorithm

The steps of the Newton-Raphson method to find the root of an equation f(x) = 0 are

1. Let k =1, Evaluate $f'(x_i)$
2. Use an initial guess of the root, x_i , to estimate the new value of the root, x_{i+1} , as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

3. Compare the absolute approximate error $|c_k - c_{k-1}|$ or absolute relative approximate error $\frac{|c_k - c_{k-1}|}{|c_k|}$

with the pre-specified relative error tolerance ϵ.

where

c_k = estimated root from present iteration

c_{k-1}= estimated root from previous iteration

$If\ |c_k - c_{k-1}| < \epsilon\ or\ \frac{|c_k - c_{k-1}|}{|c_k|} < \epsilon$ then exit

Else

k = k + 1

go to Step 2

Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Advantages and Disadvantages

Advantages:

1. Only one initial guess is needed.
2. Very rapid convergence. It has quadratic convergence which implies, number of correct figures in the estimate is nearly doubled at each successive step.

Disadvantages:

1. If initial guess is not sufficiently near the root, it may diverge.
2. Function must be differentiable.
3. In case of multiple roots, pace of convergence slows down,
4. There can be many instances, when Newton Raphson may fail to converge,

Example 1

Eg 1. *Find the real root of the equationx*^{3} *– 4x – 9 = 0 by* Newton Raphson method *correct to two decimal places. Take* x_0 *= 2.5,*

ϵ = 0.001

$f'(x) = 3x^2 - 4$

x	0	1	2	3
f(x)	-9	-12	-9	6

$$f(2.5) = (2.5)^3 - 4(2.5) - 9 = -3.375$$

- $f'(2.5) = 3(2.5)^2 - 4 = 14.75$
- $x_{i+1} = 2.5 - \frac{-3.375}{14.75} = 2.7288$

Solution 1

x_i	f(x_i)	f ‘(x_i)	x_{i+1}	
2.5	-3.375	14.75	2.7288	-
2.7288	0.4046	18.3393	2.7067	0.0221

$$f(2.7288) = 0.4046$$

$$f'(2.7288) = 18.3393$$

$$x_{i+1} = 2.7288 - \frac{0.4046}{18.3393} = 2.7067$$

Solution 1

x_i	f(x_i)	f '(x_i)	x_{i+1}	
2.5	-3.375	14.75	2.7288	-
2.7288	0.4046	18.3393	2.7067	0.0221
2.7067	0.004	17.9795	2.7065	0.0002

$f(2.7067) = 0.004$

$f'(2.7067) = 17.9795$

$x_{i+1} = 2.7067 - \frac{0.004}{17.9795}$ = 2.7065

ϵ = 0.001 $|c_k - c_{k-1}|$ = 0.0002 < ϵ

The approximate real root is 2.7065

Solution 2

Eg 2. *Find the real root of the equation* $x^3 - x - 1$ *by* Newton Raphson method *correct to three decimal places. Take* x_0 *= 1.5,*

ϵ = 0.001

$f'(x) = 3x^2 - 1$

$f(1.5) = (1.5)^3 - 1.5 - 1 = 0.875$

- $f'(1.5) = 3(1.5)^2 - 1 = 5.75$
- $x_{i+1} = 1.5 - \frac{0.875}{5.75} = 1.3478$

Solution 2

x_i	f(x_i)	f '(x_i)	x_{i+1}	$\|c_k - c_{k-1}\|$
1.5	0.875	5.75	1.3478	-
1.3478	0.1007	4.4499	1.3252	0.0226

$$f(1.3478) = 0.1007$$

$$f'(1.3478) = 4.4499$$

$$x_{i+1} = 1.3478 - \frac{0.1007}{4.4499} = 1.3252$$

Solution 2

x_i	f(x_i)	f ‘(x_i)	x_{i+1}	$\|c_k - c_{k-1}\|$
1.5	0.875	5.75	1.3478	-
1.3478	0.1007	4.4499	1.3252	0.0226
1.3252	0.0021	4.2646	1.3247	0.0005

$f(1.3252) = 0.0021$

$f'(1.3252) = 4.2646$

$x_{i+1} = 1.3252 - \frac{0.0021}{4.2646}$ = 1.3247

ϵ = 0.001 $|c_k - c_{k-1}|$ = 0.0005 < ϵ

The approximate real root is 1.3247

Example 3

This quadratic has two nearly equal roots:

P (x) is $x^2 - 4x + 3.9999$

a. Which root do you get with Newton's method starting at x = 2.1?

b. Repeat part (a) but starting with x = 1.9.

c. What happens with Newton's method starting from

x = 2.0?

Solution 3

a. Starting from x_0 = 2.1, convergence is to x = 2.0108

b. Starting from x_0 = 1.9, convergence is to x = 1.9892.

c. Starting from x_0 = 2.0 fails, f'(2.0) = zero.

Example 4

This polynomial obviously has roots at x = 2 and at x = 4; one is a double root, the other a triple root:

$f(x) = (x-2)^3(x-4)^2$

$= x^5 - 14x^4 + 76x^3 - 200x^2 + 256x - 128$

a. Which root can you get with bisection? Which root can't you get?

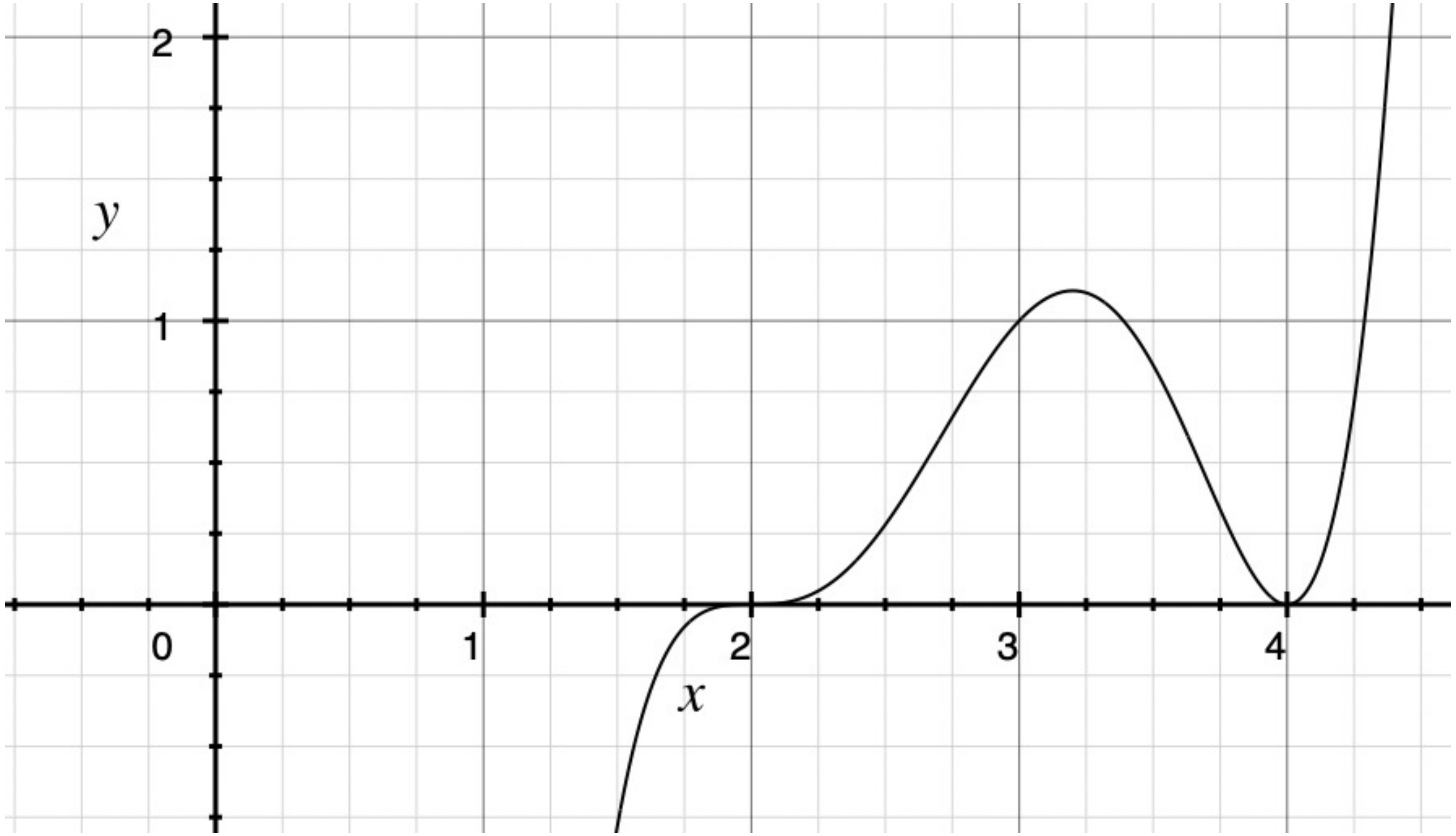
b. Repeat part (a) with the secant method.

c. If you begin with the interval [1,5], which root will you get with

(1) bisection, (2) the secant method, (3) false position?

d. Use Newton's method with x_0 = 3.2 Does it converge? Can Newton method converge to root x = 4?

Solution 4



Solution 4

This polynomial obviously has roots at x = 2 and at x = 4; one is a double root, the other a triple root:

$$f(x) = (x-2)^3(x-4)^2$$
$$= x^5 - 14x^4 + 76x^3 - 200x^2 + 256x - 128$$

a.

b.

c.

d. Derivate is zero, so the method does not converge.

Solution 4

d.

Yes x_0 = 4.1, can converge to x = 4

x_i	f(x_i)	f ‘(x_i)	x_{i+1}	$\lvert c_k - c_{k-1} \rvert$
4.1	0.0926	1.9845	4.0533	
4.0533	0.0246	0.9594	4.0277	0.0256
4.0277	0.0064	0.4707	4.0141	0.0136
4.0141	0.0016	0.233	4.0071	0.007
4.0071	0.0004	0.1159	4.0036	0.0035