10

MATRICES

In NDA exam, generally 1-2 questions are asked from this chapter which are based on equality of matrices, product of matrices, transpose of a matrix etc.

A rectangular array of mn numbers (real or complex) in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called an $m \times n$ matrix (to be read as m by n matrix) or matrix of order $m \times n$.

A $m \times n$ matrix is usually written as, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$

A matrix may be represented by the symbols $[a_{ij}], (a_{ij})$ or by a capital letter A, i.e.

$$A = [a_{ij}]_{m \times n} \text{ or } (a_{ij})_{m \times n}$$

The numbers $a_{11}, a_{12}, \dots$ etc. are known as elements of the matrix A, a_{ij} belongs to the ith row and jth column and is called the (i, j)th element of matrix $A = [a_{ij}]$

Note A matrix is not a number and it has no numerical value.

EXAMPLE 1. Construct a 2×2 matrix $A = (a_{ij})$, whose general element is given by $a_{ij} = (i + 2j)^2 / 2$.

a. $\begin{bmatrix} 9 & 25 \\ 4 & 9 \end{bmatrix}$ b. $\begin{bmatrix} 9/2 & 25/2 \\ 8 & 18 \end{bmatrix}$ c. $\begin{bmatrix} 9 & 5 \\ 4 & 18 \end{bmatrix}$ d. $\begin{bmatrix} 9/2 & 25 \\ 8 & 9 \end{bmatrix}$

Sol. *b.* $a_{11} = (1+2)^2/2 = 9/2$, $a_{12} = (1+4)^2/2 = 25/2$, $a_{21} = (2+2)^2/2 = 8$, $a_{22} = (2+4)^2/2 = 18$

$$\therefore \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 9/2 & 25/2 \\ 8 & 18 \end{bmatrix}$$

Types of Matrices

1. **Row matrix** A matrix having one row and n columns is called as row matrix. It is of the form
$$A = [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}]_{1 \times n} \quad \text{or} \quad A = [a_1 \quad a_2 \quad \cdots \quad a_n]_{1 \times n}$$
2. **Column matrix** A matrix having m rows and one column is called column matrix. It is of the form
$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}_{m \times 1} \quad \text{or} \quad A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}_{m \times 1}$$

3. **Zero matrix** A matrix in which all the elements are equal to zero is called zero matrix. It is also called a null matrix and is denoted by O.
4. **Singleton matrix** A matrix in which there is only one element is called singleton matrix.
   Thus, $A=[a_{ij}]_{m\times n}$ is singleton matrix, if $m=n=1$.
5. **Square matrix** A matrix in which the number of rows is equal to the number of columns, say n, is called a square matrix of order n, i.e. $m=n$. e.g. $\begin{bmatrix}2 & 3\\ 9 & 7\end{bmatrix}$ is a square matrix of order 2.
6. **Diagonal matrix** A square matrix in which all its elements are zero except those in the leading diagonal is called a diagonal matrix, i.e. $a_{ij}=0$ for $i\neq j$.
   e.g. The matrix, $C=\begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix}$ is diagonal matrix of order 2 and it can be rewritten as $C=\text{diag}(d_1, d_2)$
7. **Identity or unit matrix** A diagonal matrix in which all the diagonal elements are equal to 1 is called an identity matrix. It is also called a unit matrix. An identity matrix of order n is denoted by I or I_n.
8. **Scalar matrix** A diagonal matrix in which all the diagonal elements are equal is called a scalar matrix.
9. **Submatrix** Any matrix obtained by omitting some rows or columns or both from a given $m\times n$ matrix A is called a submatrix of A. As a convention, the given matrix A is also taken as a submatrix of A.
10. **Upper triangular matrix** A square matrix $A=[a_{ij}]$ is called upper triangular matrix, if $a_{ij}=0$ for all $i>j$.
11. **Lower triangular matrix** A square matrix $A=[a_{ij}]$ is called a lower triangular matrix, if $a_{ij}=0$ for all $i<j$.

Equal Matrices

Two matrices $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{p\times q}$ are said to be equal, if

(i) $m=p$ and $n=q$ (ii) $a_{ij}=b_{ij}, \forall\, i$ and j

Thus, two matrices are said to be equal, if they are of the same order and the corresponding elements are same.

EXAMPLE 2. Find x, y, z and w, if

$$\begin{bmatrix}3x & y\\ 2z & 3w\end{bmatrix}=\begin{bmatrix}x+4 & x-y\\ z+w & z-w+3x\end{bmatrix}$$

a. 2, 1, 2, 2 b. 2, 2, 2, 2 c. 2, 1, 1, 2 d. 1, 2, 1, 2

Sol. **a.** The matrix on each side is of order 2×2.
Equating the corresponding elements, we get

$$3x=x+4 \Rightarrow 2x=4\Rightarrow x=2$$
$$y=x-y \Rightarrow 2y=x=2\Rightarrow y=1$$
$$2z=z+w\Rightarrow z=w$$
$$3w=z-w+3x\Rightarrow z=4w-6$$

On substituting w in terms of z, we get

$$z=4z-6\Rightarrow 3z=6\Rightarrow z=2$$

Hence, we obtain $x=2, y=1, z=2$ and $w=2$

Algebra of Matrices

Addition of Matrices

If A and B be any two matrices of the same order $(m\times n)$, then their sum $A+B=[a_{ij}+b_{ij}]_{(m\times n)}$
where, $A=[a_{ij}]_{(m\times n)}$ and $B=[b_{ij}]_{(m\times n)}$

Properties of Addition

(i) Matrix addition is commutative, i.e. $A+B=B+A$.
(ii) Matrix addition is associative.
i.e. $(A+B)+C=A+(B+C)$
(iii) $A+O=A=O+A$
Here, the null matrix O is called additive identity.
(iv) $A+(-A)=O=(-A)+A$
Here, $(-A)$ is called the additive inverse of A.
(v) Matrix addition follows cancellation law,
i.e. $A+B=A+C\Rightarrow B=C$ [left cancellation law]
and $B+A=C+A\Rightarrow B=C$ [right cancellation law]

Subtraction of Matrices

If A and B are any two matrices of same order $(m\times n)$, then their difference $A-B=[a_{ij}-b_{ij}]_{m\times n}$
where, $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$

EXAMPLE 3. If $2A+3B=\begin{bmatrix}2 & -1 & 4\\ 3 & 2 & 5\end{bmatrix}$ and $A+2B=\begin{bmatrix}5 & 0 & 3\\ 1 & 6 & 2\end{bmatrix}$, then matrix B is

a. $\begin{bmatrix}8 & -1 & 2\\ -1 & 10 & -1\end{bmatrix}$ b. $\begin{bmatrix}8 & 1 & 2\\ -1 & 10 & -1\end{bmatrix}$

c. $\begin{bmatrix}8 & 1 & -2\\ -1 & 10 & -1\end{bmatrix}$ d. $\begin{bmatrix}8 & 1 & 2\\ 1 & 10 & 1\end{bmatrix}$

Sol. **b.** Given, $2A+3B=\begin{bmatrix}2 & -1 & 4\\ 3 & 2 & 5\end{bmatrix}$ ...(i)

and $A+2B=\begin{bmatrix}5 & 0 & 3\\ 1 & 6 & 2\end{bmatrix}\Rightarrow 2A+4B=\begin{bmatrix}10 & 0 & 6\\ 2 & 12 & 4\end{bmatrix}$ ...(ii)

On subtracting Eq. (i) from Eq. (ii), we get

$$B=\begin{bmatrix}8 & 1 & 2\\ -1 & 10 & -1\end{bmatrix}$$

Scalar Multiplication of Matrices

If a matrix is multiplied by the scalar k, then each element is multiplied by k. Thus, if $A = [a_{ij}]_{m \times n}$ then $kA = [ka_{ij}]_{m \times n}$.

Properties of Scalar Multiplication of Matrices

Let the matrices A and B be of the same order and λ, μ be scalars. Then,

(i) $\lambda(A + B) = \lambda A + \lambda B$ (ii) $(\lambda + \mu)A = \lambda A + \mu A$

(iii) $\lambda(\mu A) = \mu(\lambda A) = (\lambda)\mu A$ (iv) $(-\lambda)A = -(\lambda A) = \lambda(-A)$

EXAMPLE 4. Find the value or values of x such that

$$x^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

a. 0 b. 1 c. 1/2 d. Both (b) and (c)

***Sol.* b.** From the left hand side, we have

$$x^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x^2 \\ x^2 \end{bmatrix} + \begin{bmatrix} -3x \\ x \end{bmatrix} = \begin{bmatrix} 2x^2 - 3x \\ x^2 + x \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2x^2 - 3x \\ x^2 + x \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow 2x^2 - 3x = -1 \quad \text{...(i)}$$

and $x^2 + x = 2$ ...(ii)

On solving the first equation $2x^2 - 3x + 1 = 0$, we get $x = 1, 1/2$

The second equation $x^2 + x = 2$ is satisfied only when $x = 1$.

Hence, the solution is $x = 1$.

Multiplication of Matrices

If A and B are two matrices such that the number of columns of A is equal to the number of rows in B, i.e. if $A = [a_{ik}]$ is a $m \times n$ matrix and $B = [b_{kj}]$ be a $n \times p$ matrix, then the product AB of these matrices is $m \times p$ matrix and is defined as

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}$$

$$= [a_{i1} \; a_{i2} \cdots a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

= Sum of the product of elements of ith row of A with the corresponding elements of jth column of B.

Note In the matrix product AB, the matrix A is called pre multiplier or pre factor and B is called post multiplier or post factor.

Properties of Multiplication of Matrices

(i) Matrix multiplication is not commutative in general i.e. $AB \neq BA$

(ii) Multiplication is distributive, i.e. if A, B and C are the matrices of order $m \times n$, $n \times p$ and $n \times p$ respectively, then $A(B + C) = AB + AC$

(iii) Multiplication is associative, i.e. if A, B and C are the matrices of order $m \times n$, $n \times p$ and $p \times r$ respectively, then $(AB)C = A(BC)$.

(iv) If A is a $m \times n$ matrix and I_n, I_m are the identity matrices of order $m \times m$ and $n \times n$, then

$$I_m A = A = A I_n$$

Here, I is the multiplicative identity.

(v) If $AB = O$, then it is not necessary that either A or B is O or both are O.

EXAMPLE 5. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$.

If $AB = BA$, then what is the value of x?

a. −1 b. 0

c. 1 d. Any real number

***Sol.* b.** $AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & -1 \end{bmatrix}$

and $BA = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -x \\ 0 & -1 \end{bmatrix}$

$\because AB = BA$

$$\therefore \begin{bmatrix} 1 & x \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -x \\ 0 & -1 \end{bmatrix}$$

$\Rightarrow x = -x$

$\Rightarrow 2x = 0 \Rightarrow x = 0$

Transpose of a Matrix

Let A be a $m \times n$ matrix. Then, the $n \times m$ matrix obtained by interchanging the rows and columns of A is called the transpose of A and is denoted by A' or A^T.

e.g. If $A = \begin{bmatrix} 2 & -3 & -1 \\ 4 & 2 & 3 \end{bmatrix}_{2 \times 3}$, then $A' = \begin{bmatrix} 2 & 4 \\ -3 & 2 \\ -1 & 3 \end{bmatrix}_{3 \times 2}$

Properties of the Transpose Matrix

(i) $(A')' = A$ (ii) $(A \pm B)' = A' \pm B'$

(iii) $(kA)' = kA'$ (iv) $(AB)' = B'A'$

Symmetric and Skew-symmetric Matrices

A real square matrix $A = (a_{ij})$ is said to be symmetric, if $a_{ij} = a_{ji}$, $\forall\, i$ and j or $A = A^T$.

A real square matrix $A = (a_{ij})$ is said to be skew-symmetric (anti-symmetric), if $a_{ij} = -a_{ji}$, $\forall\, i$ and j or $A = -A^T$.

Properties of Symmetric and Skew-symmetric Matrices

(i) In a skew-symmetric matrix A, all its diagonal elements are zero, i.e. $a_{ii} = 0, \forall i$.

(ii) The matrix which is both symmetric and skew-symmetric is a null matrix.

(iii) For any real square matrix A; $A + A^T$ and AA^T or $A^T A$ is a symmetric matrix and $A - A^T$ is a skew-symmetric matrix.

(iv) A real square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

i.e. $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

EXAMPLE 6. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, then find the value of $A^T A$.

a. O b. I c. A d. A^T

Sol. **b.** We have, $A^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

Therefore, $A^T A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 \\ 0 & \sin^2\alpha + \cos^2\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Special Type of Matrix

1. **Orthogonal matrix** A square matrix A is called an orthogonal matrix if the product of matrix A and its transpose A' (or A^T) is an identity matrix, i.e. $AA' = I$

Note If A and B are orthogonal, then AB is also orthogonal.

2. **Conjugate of a matrix** The matrix obtained from any given matrix A containing complex numbers as its elements, on replacing its elements by the corresponding conjugate complex numbers is called conjugate of A and is denoted by $\overline{A}$.

e.g. If $A = \begin{bmatrix} 1+2i & 2-3i \\ 4-5i & 5+6i \end{bmatrix}$,

then $\overline{A} = \begin{bmatrix} 1-2i & 2+3i \\ 4+5i & 5-6i \end{bmatrix}$

3. **Hermitian matrix** A square matrix such that $(\overline{A}') = A$, then A is known as hermitian matrix.

4. **Skew-hermitian matrix** A square matrix such that $(\overline{A}') = -A$, then A is known as skew-hermitian matrix.

5. **Elementary matrix** A square matrix is called an elementary matrix if it can be obtained from identity matrix I by performing single elementary row or column operation.

Properties of Determinants

(i) If each entry in any row or column of a determinant is 0, then the value of the determinant is zero.

(ii) If rows be changed into columns and columns into rows, then the value of the determinant remains unchanged.

(iii) If any two adjacent rows (columns) of a determinant are interchanged, then the determinant remains its absolute value but changed in sign.

(iv) If a determinant have any two rows or columns identical, then its value is zero.

(v) If all the constituents (elements) of one row or of one column, multiplied by the same quantity, then the value of new determinant is k times the value of original determinant.

i.e. $$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(vi) If each constituent of any row or of any column be the sum of the two quantities, then the determinant can be expressed as the sum of the two determinants of the same order.

i.e. $$\begin{vmatrix} a_{11}+a & a_{12}+b & a_{13}+c \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a & b & c \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(vii) If each element of a row or column of a determinant is multiplied by a constant k and then added to the corresponding elements of some other row or column, then value of determinant remains same.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+ka_{21} & a_{32}+ka_{22} & a_{33}+ka_{23} \end{vmatrix}$$

(viii) Number of elements in nth order determinant $= n^2$

(ix) If A and B are two determinants of order n, then $|AB| = |A||B|$

(x) $\det(kA) = k^n \det(A)$, if A is of order $n \times n$.

(xi) If $AB = AC$, then $B = C$ is true only, when $|A| \neq 0$.

EXAMPLE 1. If x, y and z are all positive, then what is the value of $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$?

a. 1 b. 3 c. 0 d. −2

Sol. c. $$\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} = \begin{vmatrix} \frac{\log x}{\log x} & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & \frac{\log y}{\log y} & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & \frac{\log z}{\log z} \end{vmatrix}$$

$$= \frac{1}{\log x \log y \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix}$$

$= 0$ [since, all rows are identical]

EXAMPLE 2. If ω is the cube root of unity, then what is one root of the equation $\begin{vmatrix} x^2 & -2x & -2\omega^2 \\ 2 & \omega & -\omega \\ 0 & \omega & 1 \end{vmatrix} = 0$?

a. 1 b. −2 c. 2 d. ω

Sol. c. $$\begin{vmatrix} x^2 & -2x & -2\omega^2 \\ 2 & \omega & -\omega \\ 0 & \omega & 1 \end{vmatrix} = 0$$

$$\Rightarrow x^2\begin{vmatrix} \omega & -\omega \\ \omega & 1 \end{vmatrix} + 2x\begin{vmatrix} 2 & -\omega \\ 0 & 1 \end{vmatrix} - 2\omega^2\begin{vmatrix} 2 & \omega \\ 0 & \omega \end{vmatrix} = 0$$

$\Rightarrow x^2(\omega + \omega^2) + 2x(2) - 2\omega^2(2\omega) = 0$ [$\because 1 + \omega + \omega^2 = 0$]

$\Rightarrow -x^2 + 4x - 4\omega^3 = 0$

$\Rightarrow x^2 - 4x + 4 = 0$

$\Rightarrow (x-2)^2 = 0 \Rightarrow x = 2$

MINORS AND COFACTORS

Minor of an Element of a Determinant

If we delete the row and column passing through the element a_{ij}, the determinant, thus obtained is called the minor of a_{ij} and is usually denoted by M_{ij}.

e.g. For the 3×3 determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

The minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$.

Cofactor of an Element of a Determinant

The cofactor of an element a_{ij} is $(-1)^{i+j} M_{ij}$ and it is denoted by C_{ij}. Thus, $C_{ij} = \begin{cases} M_{ij}, \text{ when } i+j \text{ is even} \\ -M_{ij}, \text{ when } i+j \text{ is odd} \end{cases}$

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\therefore \Delta = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

(i) If $\Delta \neq 0$ and Δ^c denoted the determinant of cofactors, then $\Delta^c = \Delta^{n-1}$, where $n(>0)$ is the order of Δ.

(ii) If $\Delta = 0$, then $\Delta^c = 0$.

(iii) The sum of the product of constituents of any row (column) of a determinant with the cofactors of the corresponding elements of any other row (column) is zero.

i.e. $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$

EXAMPLE 3. If Δ is the determinant of the matrix $\begin{bmatrix} a & b \\ -b & -a \end{bmatrix}$ and Δ^c the determinant of the cofactors of the elements of the matrix. Then, which one of the following is correct?

a. $\Delta^c = \Delta$ b. $\Delta^c = \Delta^2$ c. $\Delta^c = \Delta^3$ d. $\Delta^c = \frac{1}{\Delta}$

Sol. *a.* Let $A = \begin{bmatrix} a & b \\ -b & -a \end{bmatrix}$, $\Delta = \begin{vmatrix} a & b \\ -b & -a \end{vmatrix} = -a^2 + b^2$

and matrix of cofactors of $A = \begin{bmatrix} -a & b \\ -b & a \end{bmatrix}$

$$\Delta^c = \begin{vmatrix} -a & b \\ -b & a \end{vmatrix} = -a^2 + b^2 \Rightarrow \Delta = \Delta^c$$

ADJOINT OF A MATRIX

Let $A = [a_{ij}]$ be a square matrix of order n and $C = [c_{ij}]$ be its cofactor matrix. Then, matrix $C^T = [C_{ji}]$, is called the adjoint of matrix A and is written as

$$\text{adj}\,(A) = C^T = [C_{ji}], 1 \le i, j \le n.$$

e.g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 4 & 3 & 2 \end{bmatrix}$, then $C_{11} = -3$, $C_{12} = 6$, $C_{13} = -3$,

$C_{21} = 5$, $C_{22} = -10$, $C_{23} = 5$

$C_{31} = 2$, $C_{32} = -4$ and $C_{33} = 2$

$$\therefore C = \begin{bmatrix} -3 & 6 & -3 \\ 5 & -10 & 5 \\ 2 & 4 & 2 \end{bmatrix}$$

Thus, $\text{adj}\,(A) = C^T = \begin{bmatrix} -3 & 5 & 2 \\ 6 & -10 & -4 \\ -3 & 5 & 2 \end{bmatrix}$

$A^T = A$

Properties of Adjoint Matrix

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

(i) $A(\text{adj}A) = |A| I_n = (\text{adj}A)A$ (ii) $|\text{adj}A| = |A|^{n-1}$

(iii) $\text{adj}(\text{adj}A) = |A|^{n-2} A; |A| \neq 0$

(iv) $|\text{adj}(\text{adj}A)| = |A|^{(n-1)^2}$ (v) $\text{adj}(A^T) = (\text{adj}A)^T$

(vi) $\text{adj}(AB) = (\text{adj}B)(\text{adj}A)$

(vii) $\text{adj}(A^m) = (\text{adj}A)^m, m \in N$

(viii) $\text{adj}(kA) = k^{n-1}(\text{adj}A), k \in R$

(ix) $\text{adj}(I_n) = I_n$ (x) $\text{adj}(O) = O$

(xi) A is symmetric matrix $\Rightarrow \text{adj}(A)$ is also symmetric matrix.

(xii) A is diagonal matrix $\Rightarrow$ adj (A) is also diagonal matrix.

(xiii) A is triangular matrix $\Rightarrow$ adj (A) is also triangular matrix.

EXAMPLE 4. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{bmatrix}$ and $B = (\text{adj}\,A)$ and $C = 5A$, then $\frac{|\text{adj}\,B|}{|C|}$ is equal to

a. 5 b. 25 c. -1 d. 1

Sol. *d.* $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{vmatrix} = 1(3) + 1(6) + 1(-4) = 5$

$$B = \text{adj}\,A = \begin{bmatrix} 3 & 1 & 1 \\ -6 & -2 & 3 \\ -4 & -3 & 2 \end{bmatrix}, \text{adj}\,B = \begin{bmatrix} 5 & -5 & 5 \\ 0 & 10 & -15 \\ 10 & 5 & 0 \end{bmatrix} = 5A$$

and $C = 5A$

$$\therefore \frac{|\text{adj}\,B|}{|C|} = \frac{|5A|}{|5A|} = 1$$

INVERSE OF A MATRIX

A non-singular square matrix $A = [a_{ij}]$ of order n is said to be invertible or has an inverse, if there exists another non-singular square matrix B of order n, such that

$$AB = BA = I_n$$

where, I is an identity matrix of order n. Then, we write

$$B = A^{-1} \text{ or } A = B^{-1}$$

Hence, we say that A^{-1} is the inverse of A, if

$$AA^{-1} = A^{-1}A = I$$

The inverse of a matrix A is given by $A^{-1} = \frac{1}{|A|}\text{adj}\,(A)$.

Note **Non-singular and singular matrices** A matrix A is said to be non-singular, if its determinant is non-zero, i.e. $|A| \neq 0$. The matrix whose determinant is zero, i.e. $|A| = 0$, is called a singular matrix.

Properties of Inverse Matrices

If A and B are invertible matrices of the same order, then

(i) $(A^{-1})^{-1} = A$ (ii) $(AB)^{-1} = B^{-1}A^{-1}$

(iii) $(A^k)^{-1} = (A^{-1})^k, k \in N$

(iv) $\text{adj}(A^{-1}) = (\text{adj} A)^{-1}$ (v) $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

Note If A is an invertible matrix, then
- If A is symmetric matrix, then A^{-1} is also symmetric matrix.
- If A is skew-symmetric matrix, then A^{-1} is also skew symmetric matrix.

EXAMPLE 5. If the inverse of $\begin{bmatrix} 1 & p & q \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -p & -q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then what is the value of x?

a. 1 b. 0 c. -1 d. $\frac{1}{p} + \frac{1}{q}$

Sol. a. Let $A = \begin{bmatrix} 1 & p & q \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -p & -q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thus, $B^{-1} = \begin{bmatrix} 1 & p & q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

But B is inverse of A, therefore $A = B^{-1}$

$$\Rightarrow \begin{bmatrix} 1 & p & q \\ 0 & x & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & p & q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow x = 1$$

SOLUTION OF SYSTEM OF LINEAR EQUATIONS

(i) Using Matrices

Consider, the system of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

We can write these equations in matrix form as

$$AX = B \quad \text{...(i)}$$

where, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Then, $X = A^{-1}B$

This method is also known as the matrix method to solve a system of equations.

Criterion of Consistency

Let $Ax = B$ be a system of n linear equation with n variables

(i) If $|A| \neq 0$ (i.e. A is non-singular), then the system of equations is consistent and has a unique solution given by $X = A^{-1}B$.

(ii) If $|A| = 0$ (i.e. A is singular) and $[\text{adj}(A)]B = 0$, i.e. null matrix, then the system of equations is consistent and infinitely many solutions.

(iii) If $|A| = 0$ and $[\text{adj}(A)]B \neq 0$, then the system of equations is inconsistent and has no solution.

Homogeneous Equations

The system of equations $AX = B$ is said to homogeneous, if $B = 0$

(i) If $|A| \neq 0$, then its only solution $X = 0$, is called trivial solution.

(ii) If $|A| = 0$, then $AX = 0$ has a non-trivial solution. It will have infinitely many solutions.

EXAMPLE 6. The equations $x + 2y + 3z = 1, x - y + 4z = 0$ and $2x + y + 7z = 1$ have

a. only two solutions b. only one solution
c. no solution d. infinitely many solutions

Sol. d. We have, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 4 \\ 2 & 1 & 7 \end{vmatrix} = 1(-11) - 2(-1) + 3(3) = 0$

$$\text{adj}(A) = \begin{bmatrix} -11 & 1 & 3 \\ -11 & 1 & 3 \\ 11 & -1 & -3 \end{bmatrix}^T = \begin{bmatrix} -11 & -11 & 11 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{bmatrix}$$

$$\therefore (\text{adj} A)B = \begin{bmatrix} -11 & -11 & 11 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

So, the given system of equations is consistent and has infinitely many solutions.

(ii) Cramer's Rule

Case I Let us consider a system of equations in two variables

$$a_1x + b_1y = c_1, a_2x + b_2y = c_2$$

then, $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \Delta_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

By Cramer's rule the solution of system of equation is $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}$, provided $\Delta \neq 0$.

(i) If $\Delta \neq 0$, then the system is consistent and has a unique solution.

(ii) If $\Delta = 0$ and atleast one of the determinants Δ_1 and Δ_2 is non-zero, then the system is inconsistent.

Case II Let us consider a system of equations in three variables

$$a_1x + b_1y + c_1z = d_1,\ a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Then, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$,

$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

By Cramer's rule, the solution of system of equations is

$$x = \frac{\Delta_1}{\Delta},\ y = \frac{\Delta_2}{\Delta} \text{ and } z = \frac{\Delta_3}{\Delta}$$

(i) If $\Delta \neq 0$, then the system is consistent and unique solution exists.

(ii) If $\Delta = 0$ and atleast one of the determinants Δ_1, Δ_2 and Δ_3 is non-zero, then the given system is inconsistent.

(iii) If $\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$, then the system is consistent and dependent and has infinitely many solutions.

EXAMPLE 7. For what value of p, is the system of equations $p^3x + (p+1)^3y = (p+2)^3$, $px + (p+1)y = p+2$ and $x + y = 1$ consistent?

a. $p = 0$ b. $p = 1$ c. $p = -1$ d. for all $p > 1$

Sol. **c.** The given system of equations is

$$p^3x + (p+1)^3y = (p+2)^3 \quad \text{...(i)}$$
$$px + (p+1)y = (p+2) \quad \text{...(ii)}$$
$$x + y = 1 \quad \text{...(iii)}$$

This system is consistent, if $\begin{vmatrix} p^3 & (p+1)^3 & (p+2)^3 \\ p & (p+1) & (p+2) \\ 1 & 1 & 1 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} p^3 & (p+1)^3 - p^3 & (p+2)^3 - p^3 \\ p & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$\left[\begin{array}{l}\text{apply } C_2 \to C_2 - C_1 \\ \text{and } C_3 \to C_3 - C_1\end{array}\right]$$

$$\Rightarrow 2(p+1)^3 - 2p^3 - (p+2)^3 + p^3 = 0$$
$$\Rightarrow 2(p^3 + 1 + 3p^2 + 3p) - 2p^3 - (p^3 + 8 + 12p + 6p^2) + p^3 = 0$$
$$\Rightarrow -6 - 6p = 0$$
$$\therefore p = -1$$

DIFFERENTIATION OF DETERMINANTS

Let $\Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{vmatrix}$,

where $f_1(x)$, $f_2(x)$, $g_1(x)$ and $g_2(x)$ are functions of x.

Then, $\Delta'(x) = \begin{vmatrix} f_1'(x) & g_1'(x) \\ f_2(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) \\ f_2'(x) & g_2'(x) \end{vmatrix}$

Also, $\Delta'(x) = \begin{vmatrix} f_1'(x) & g_1(x) \\ f_2'(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1'(x) \\ f_2(x) & g_2'(x) \end{vmatrix}$

Thus, to differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

INTEGRATION OF DETERMINANTS

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If $\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ \lambda_1 & \lambda_2 \end{vmatrix}$,

then $\int_a^b \Delta(x)\,dx = \begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx \\ \lambda_1 & \lambda_2 \end{vmatrix}$

Here, $f(x)$ and $g(x)$ are functions of x and λ_1, λ_2 are constants.

If the elements of more than one column or row are functions of x, then the integration can be done only after evalution/expansion of the determinant.

EXAMPLE 8. If $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2 \\ \tan x & x & 1 \end{vmatrix}$, then the value of $f'(x)$ at $x = 0$ is

a. -2 b. 2
c. 0 d. 1

Sol. **a.** $f'(x) = \begin{vmatrix} -\sin x & 1 & 0 \\ 2\sin x & x^2 & 2 \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\cos x & 2x & 0 \\ \tan x & x & 1 \end{vmatrix}$
$+ \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2 \\ \sec^2 x & 1 & 0 \end{vmatrix}$

$$\Rightarrow f'(0) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 0 + 0 + -2 = -2$$

Applications of Determinant in Geometry

1. **Area of triangle** If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of a triangle, then

$$\text{Area of triangle} = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

$$= \left|\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\right|$$

2. **Condition of collinearity of three points** Let three points be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then these points will be collinear, if Area of $\Delta ABC = 0$

3. **Equation of straight line passing through two points** Let two points be $A(x_1, y_1)$ and $B(x_2, y_2)$ and $P(x, y)$ be a point on the line joining points A and B, then the equation of line is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

EXAMPLE 9. The equation of straight line passing through the points $A(3, 1)$ and $B(9, 3)$ is

a. $x - 2 = 0$

b. $x - 3y = 0$

c. $y + 2 = 0$

d. $y - 2 = 0$

Sol. b. Let $P(x, y)$ be any point on the line joining $A(3, 1)$ and $B(9, 3)$.

Then, the points A, B and P are collinear. Therefore, the area of ΔABP will be zero.

$$\therefore \quad \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{1}{2}|3(3 - y) - 1(9 - x) + 1(9y - 3x)| = 0$$

$$\Rightarrow \quad 9 - 3y - 9 + x + 9y - 3x = 0$$

$$\Rightarrow \quad 6y - 2x = 0$$

$$\Rightarrow \quad x - 3y = 0$$

Hence, the equation of the line joining the given points is $x - 3y = 0$.