Example 5

With T defined by T ($\mathbf{x}$) = A$\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique.

$$A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Example 5

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{rcrcl} x_1 & & +\ 3x_3 & = & 3 \\ & x_2 & +\ 2x_3 & = & 1 \end{array}$$ and solve for the basic variables: $$\begin{cases} x_1 = 3 - 3x_3 \\ x_2 = 1 - 2x_3 \\ x_3 \text{ is free} \end{cases}$$

Example 5

$$\text{General solution } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3-3x_3 \\ 1-2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

- For a particular solution, one might choose

$$x_3 = 0 \text{ and } \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Linear Transformation

- A transformation (or mapping) T is linear if:

  (i) $T(u + v)=T(u) + T(v)$ for all u, v in the domain of T

  (ii) $T(cu) = cT(u)$ for all u and all scalars c.

- If T is a linear transformation, then

$$T(0)=0$$

and $$T(cu + dv) = cT(u) + dT(v)$$

for all vectors u, v in the domain of T and all scalars c, d.

Shear transformation

- The transformation T : $R^2 \rightarrow R^2$ defined by T (**x**) = A**x** is called a **shear transformation**. It can be shown that if T acts on each point in the 2×2 square shown in Fig. 4, then the set of images forms the shaded parallelogram. The key idea is to show that T maps line segments onto line segments and then to check that the corners of the square map onto the vertices of the parallelogram.

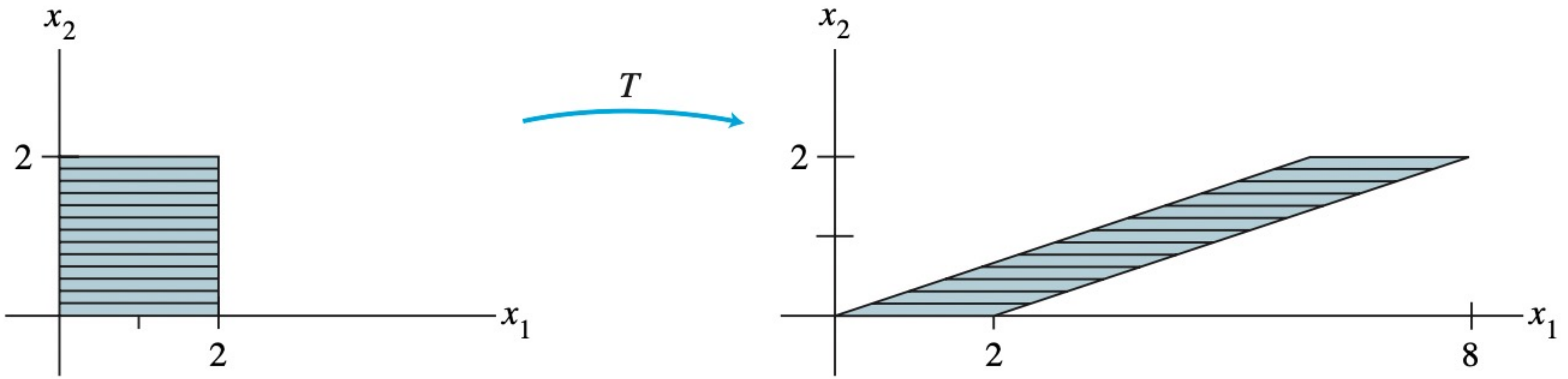


FIGURE 4 A shear transformation.

Shear transformation

- Example $Let\ A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, u = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \mathrm{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$
- Find $T(u)\ and\ T(v)$

Shear transformation

- Example $Let\ A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, u = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \mathrm{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$
- $T(u) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ $and\ T(v) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$
- T deforms the square as if the top of the square were pushed to the right while the base is held fixed.

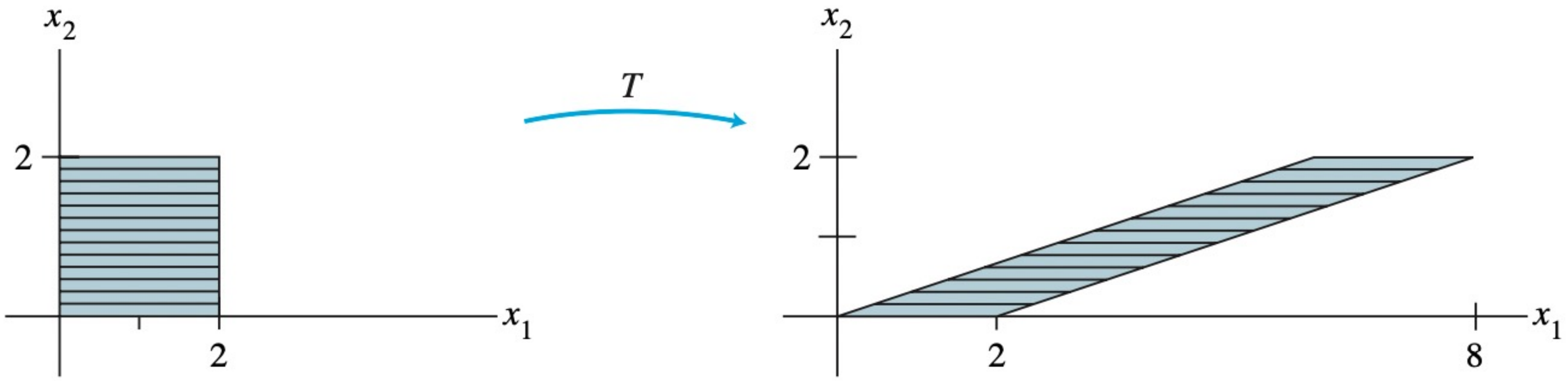
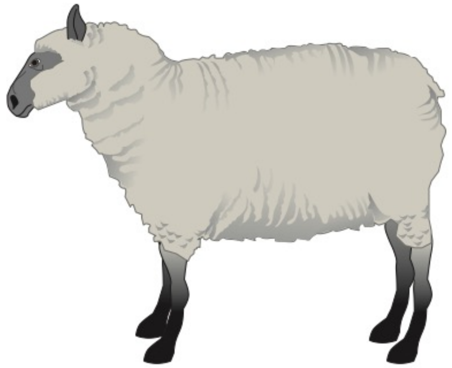


FIGURE 4 A shear transformation.

Shear transformation



sheep

sheared sheep

Contraction and Dilation

- Given a scalar r, define T : $R^2 \rightarrow R^2$ by T(**x**) = r**x**. T is called a **contraction** when 0 ≤r ≤ 1and a **dilation** when r > 1.

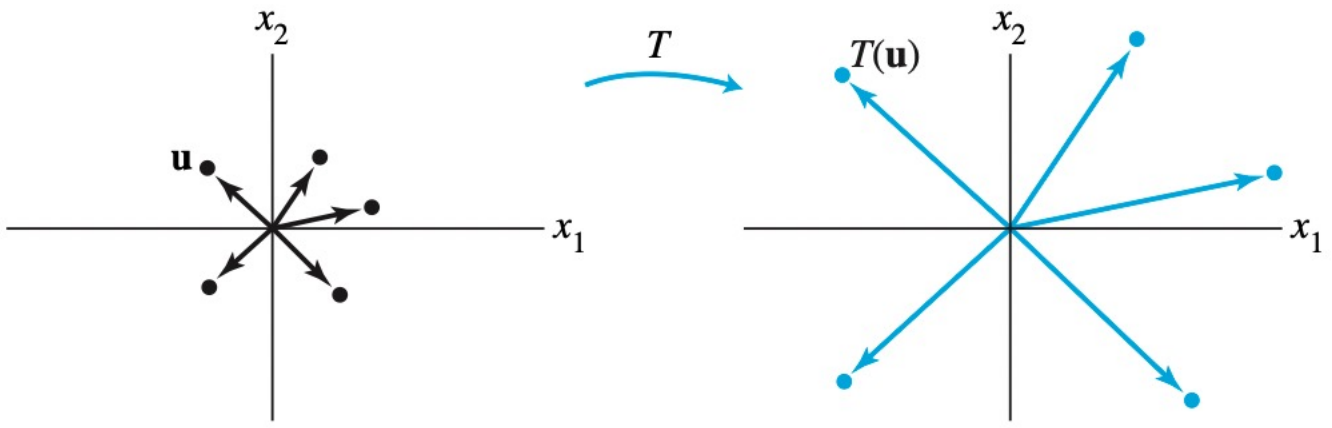


FIGURE 5 A dilation transformation.

Rotation

- The transformation T : $R^2 \rightarrow R^2$ defined by

$$T(u) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

is a rotation transformation counter clockwise about the origin through 90◦

Rotation

- The transformation T : $R^2 \to R^2$ defined by

$$T(u) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \mathrm{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Find, $T(u), T(v), T(u+v)$

Rotation

- The transformation T : $R^2 \rightarrow R^2$ defined by

$$T(u) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

$u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \mathrm{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Find, $T(u), T(v), T(u+v)$

$$T(u) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} and\ T(v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$T(u+v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

Rotation

- Note that T (u + v) is obviously equal to T (u) + T (v). It appears from Fig. 6 that T rotates u, v, and u + v counter clockwise about the origin through 90◦. In fact, T transforms the entire parallelogram determined by u and v into the one determined by T (u) and T (v).

$$T(u) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$T(u + v) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

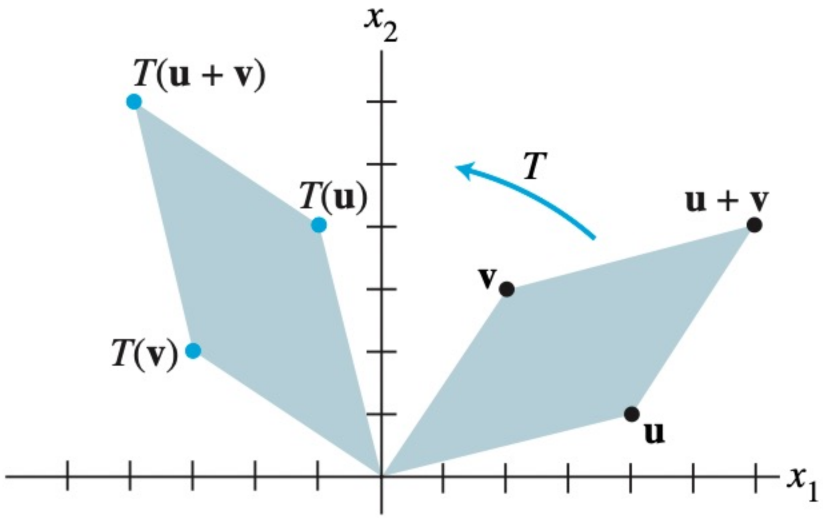


FIGURE 6 A rotation transformation.

Examples

1. For $u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, v = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, find their images under the given transformation T.

$$T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2. For $u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, v = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, find their images under the given transformation T.

$$T(\mathbf{x}) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

3. For $u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, v = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$,find their images under the given transformation T.

$$T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

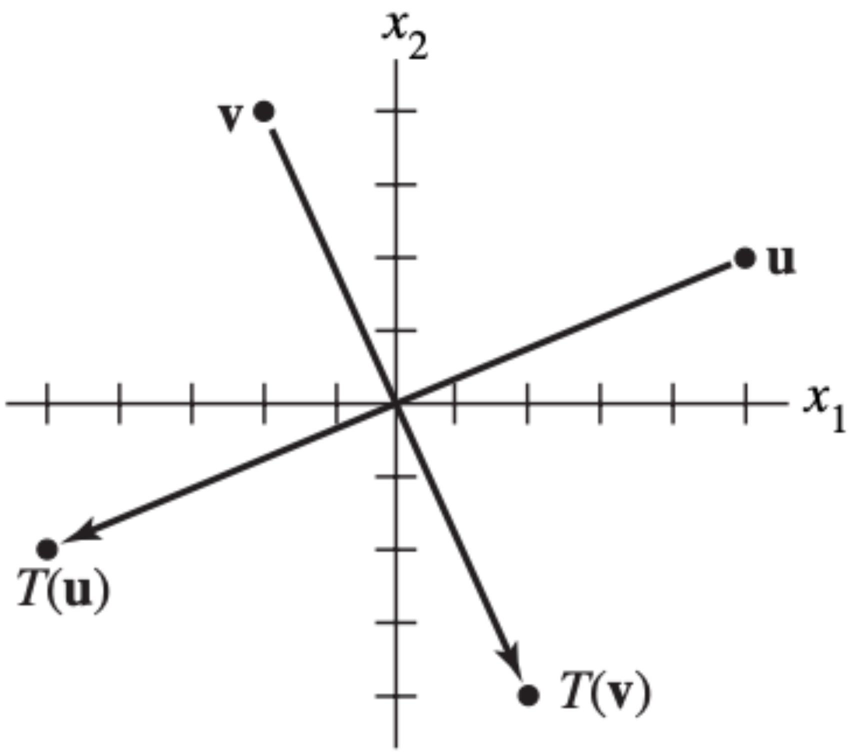
4. For $u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, v = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$,find their images under the given transformation T.

$$T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 1

$$T(u) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

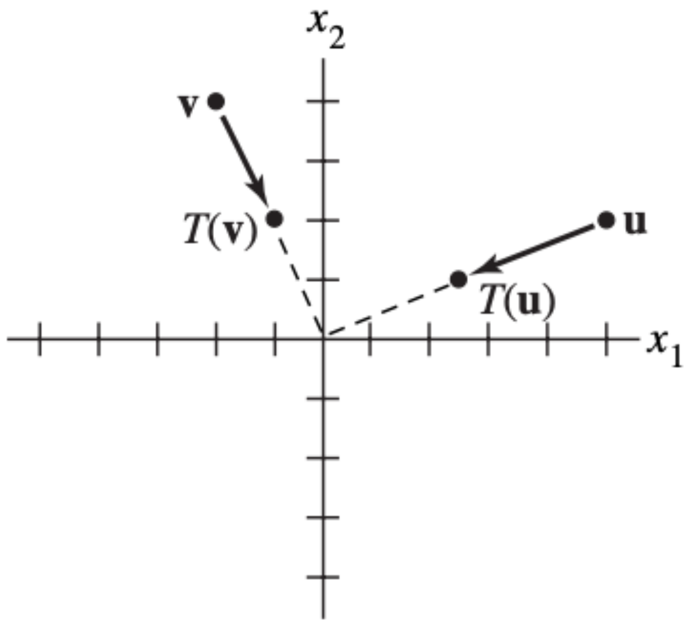


A reflection through the origin.

Example 2

$$T(u) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

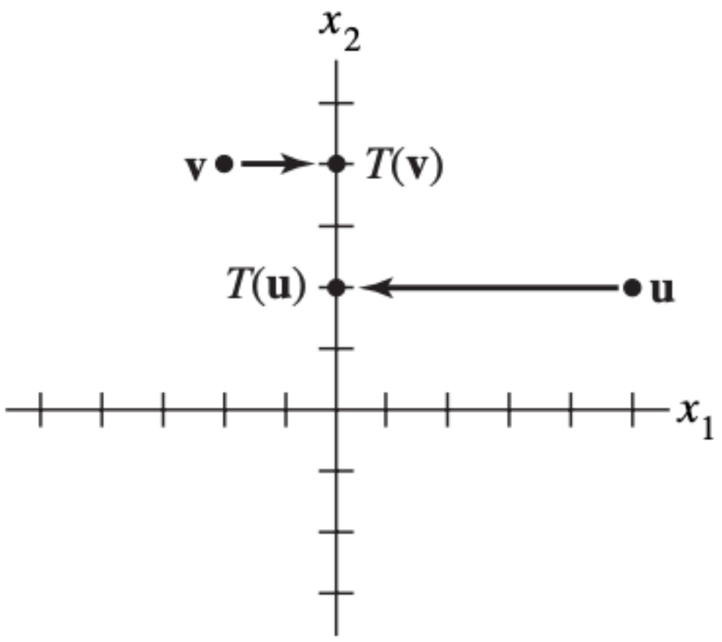


A contraction by the factor .5.

Example 3

$$T(u) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

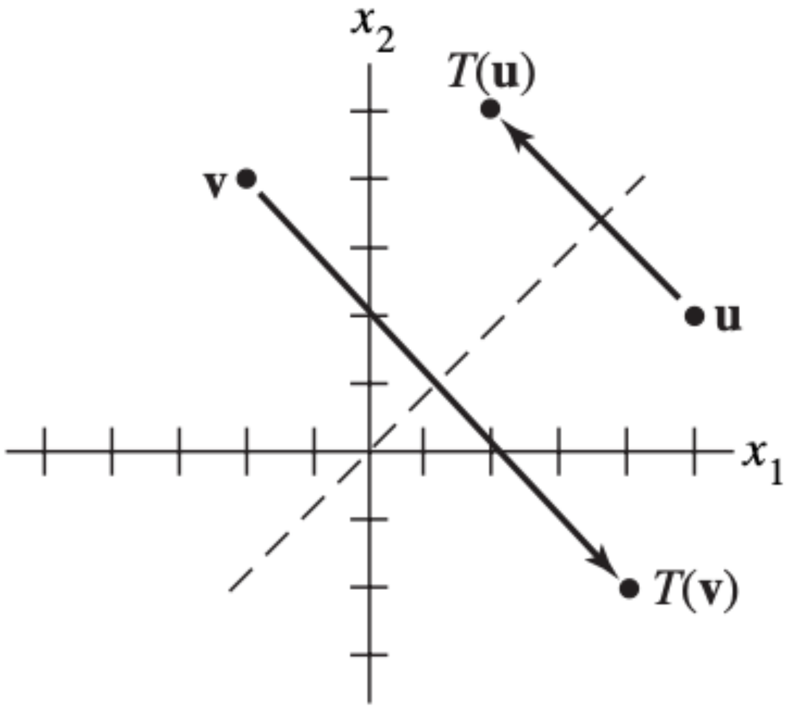


A projection onto the x_2-axis

Example 4

$$T(u) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$



A reflection through the line $x_2 = x_1$.

Matrix of a Linear Transformation

- Whenever a linear transformation T arises geometrically or is described in words, we usually want a "formula" for T (**x**). The discussion that follows shows that every linear transformation from R^n to R^m is actually a matrix transformation $\boldsymbol{x} \longmapsto \boldsymbol{Ax}$ and that important properties of T are intimately related to familiar properties of A. The key to finding A is to observe that T is completely determined by what it does to the columns of the n×n identity matrix I_n.

Standard Matrix of a Linear Transformation

- Let T : $R^n \to R^m$ be a linear transformation. Then there exists a unique matrix A such that T(**x**) = A**x** for all **x** in R^n
  In fact, A is the m×n matrix whose j^{th} column is the vector T (e_j), where e_j is the j^{th} column of the identity matrix in R^n :
  A=[T(e_1) ⋯ T(e_n)]
- The matrix A is called the standard matrix for the linear transformation T .
- We know now that every linear transformation from R^n to R^m is a matrix transformation, and vice versa.
- The term *linear transformation* focuses on a property of a mapping, while *matrix transformation* describes how such a mapping is implemented

Standard Rotation Matrix

- Let T : $R^2 \rightarrow R^2$ be the transformation that rotates each point in R^2 about the origin through an angle ϕ, with counter clockwise rotation for a positive angle. We could show geometrically that such a transformation is linear. Find the standard matrix A of this transformation.

$$\begin{bmatrix}1\\0\end{bmatrix} rotates\ into \begin{bmatrix}cos\phi\\sin\phi\end{bmatrix}\ and \begin{bmatrix}0\\1\end{bmatrix} rotates\ into \begin{bmatrix}-sin\phi\\cos\phi\end{bmatrix}$$

$$A = \begin{bmatrix}cos\phi & -sin\phi\\sin\phi & cos\phi\end{bmatrix}$$

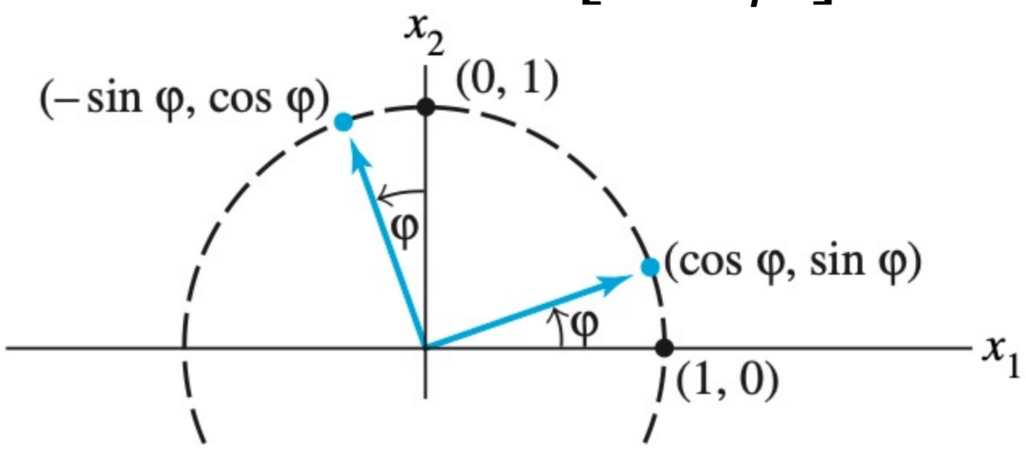
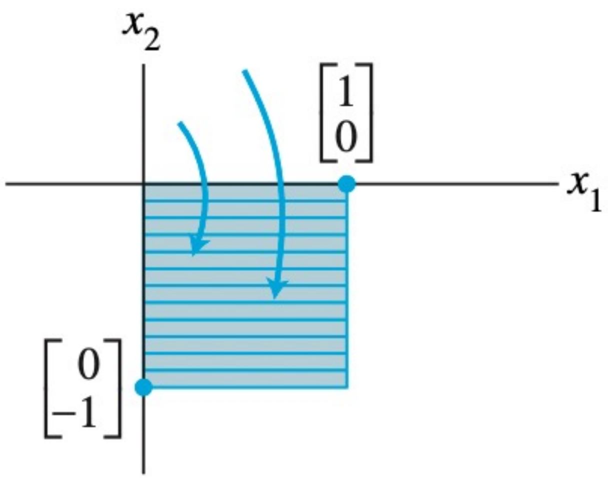
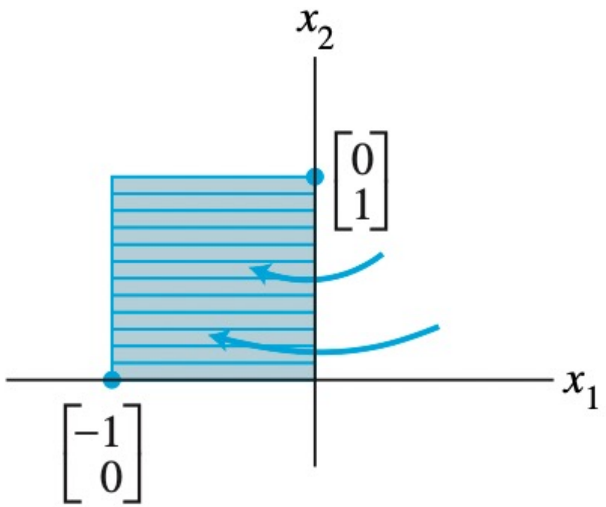
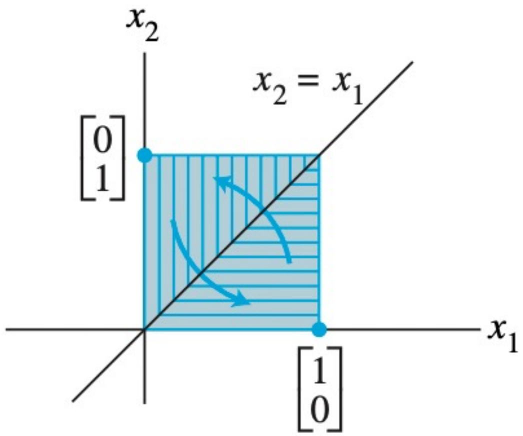
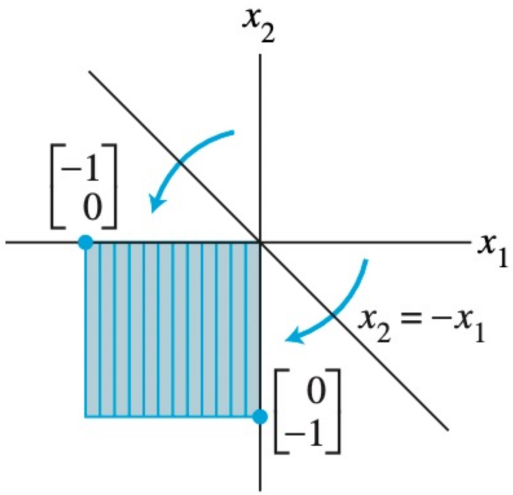


FIGURE 1 A rotation transformation.

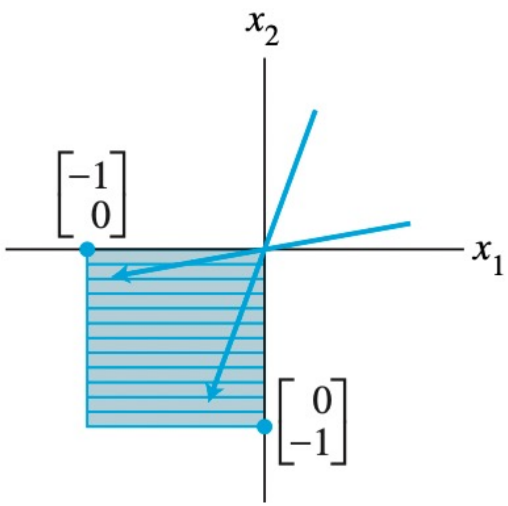
Reflections

Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$

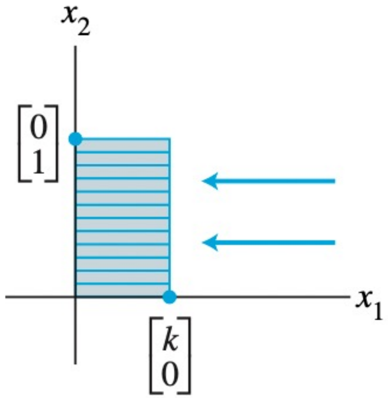
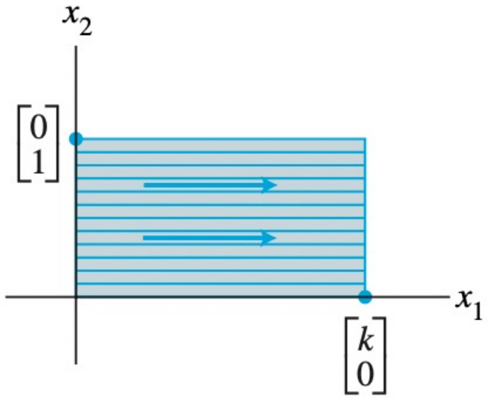
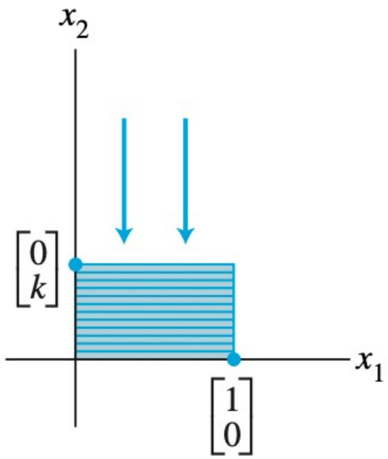
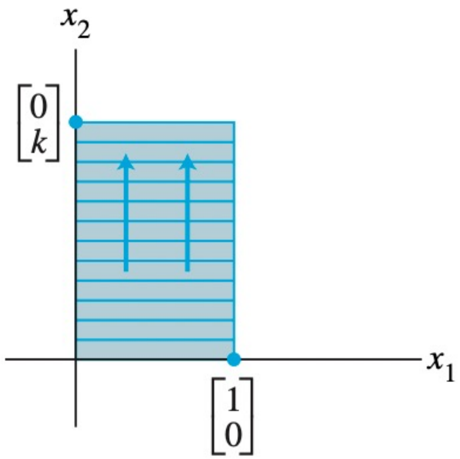
Reflections

<table>
<tr><th>Transformation</th><th>Image of the Unit Square</th><th>Standard Matrix</th></tr>
<tr><td>Reflection through the line $x_2 = x_1$</td><td>
</td><td>$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$</td></tr>
<tr><td>Reflection through the line $x_2 = -x_1$</td><td>
</td><td>$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$</td></tr>
</table>

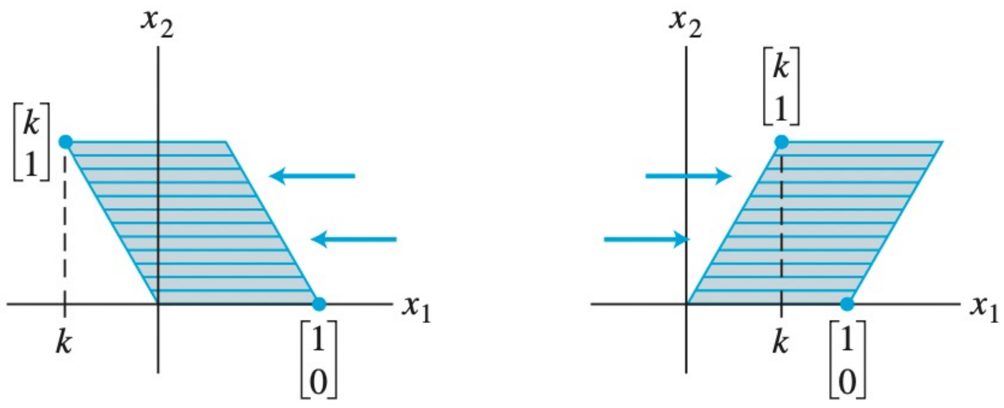
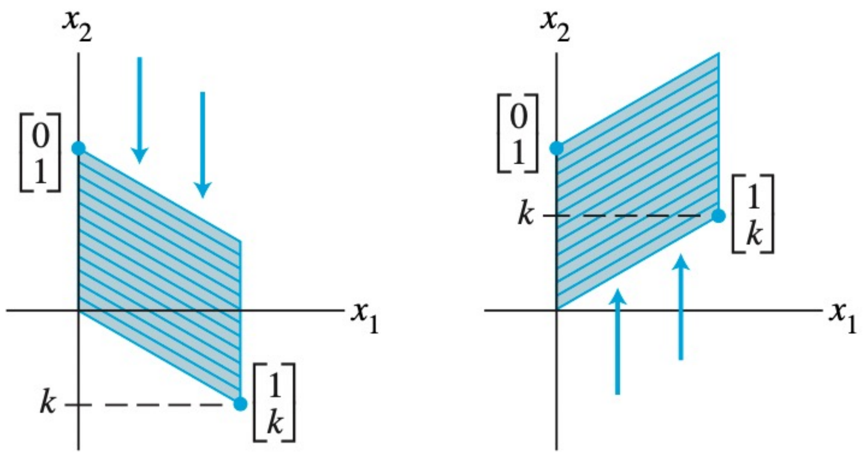
Reflections

Transformation	Image of the Unit Square	Standard Matrix
Reflection through the origin		$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

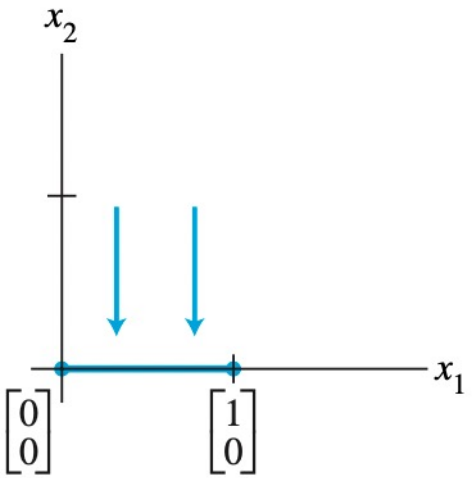
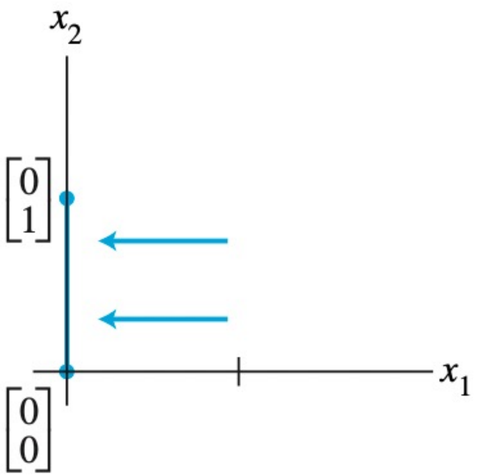
Contractions and Expansions

Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$	 $k > 1$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Shears

Transformation	Image of the Unit Square	Standard Matrix
Horizontal shear	 $k < 0$ $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$ $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Projections

Transformation	Image of the Unit Square	Standard Matrix
Projection onto the x_1-axis		$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$
Projection onto the x_2-axis		$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

Onto Mapping

- A mapping $T : R^n \rightarrow R^m$ is said to be onto R^m if each b in R^m is the image of at least one x in R^n.
- The mapping T is not onto when there is some b in Rm for which the equation T (x) = b has no solution.

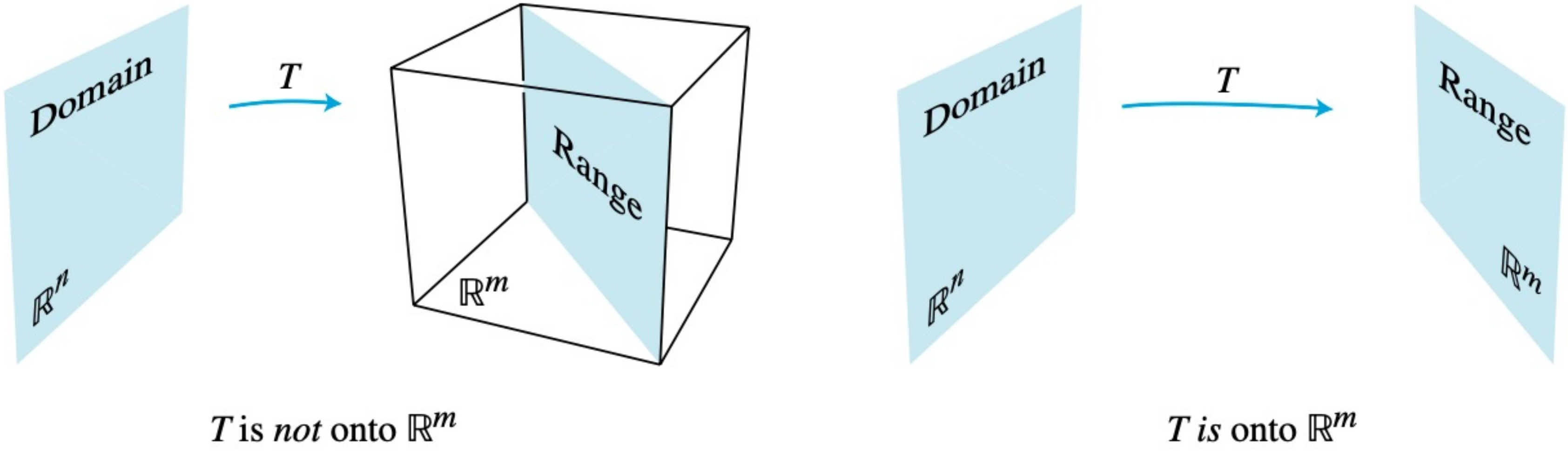


FIGURE 3 Is the range of T all of $\mathbb{R}^m$?

One-to-one Mapping

- A mapping $T : R^n \rightarrow R^m$ is said to be one-to-one R^m if each b in R^m is the image of at most one x in R^n.
- T is one-to-one if, for each b in R^m, the equation T (x) = b has either a unique solution or none at all.
- The mapping T is not one-to-one when some b in R^m is the image of more than one vector in R^n. If there is no such b, then T is one-to-one.

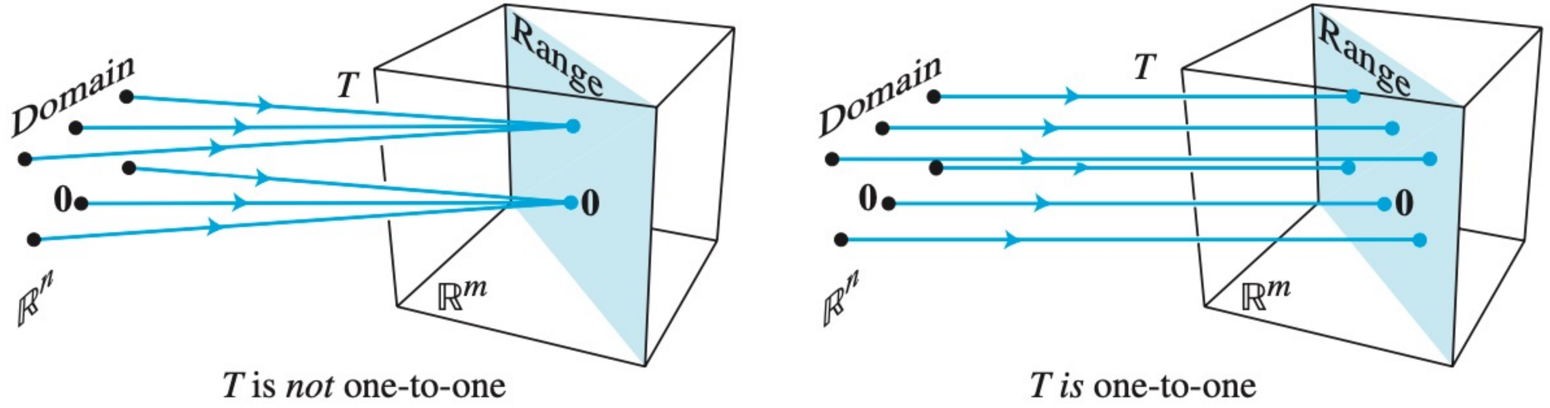


FIGURE 4 Is every **b** the image of at most one vector?

Mapping

- The projection transformations are not one-to-one and do not map R^2 onto R^2.
- The transformations in Reflection, Contraction, Expansion and Shear are one-to-one and do map R^2 onto R^2.

Invertible Linear Transformation

A linear transformation $T : R^n \rightarrow Rn$ is said to be **invertible** if there exists a function $S : R^n \rightarrow R^n$ such that

- $S(T(\mathbf{x}))=\mathbf{x}$ for all $\mathbf{x}$ in R^n
- $T(S(\mathbf{x}))=\mathbf{x}$ for all $\mathbf{x}$ in R^n

Application to Computer Graphics

- Computer graphics are images displayed or animated on a computer screen. Applications of computer graphics are widespread and growing rapidly. For instance, computer-aided design (CAD) is an integral part of many engineering processes, such as the aircraft design process
- The entertainment industry has made the most spectacular use of computer graphics—from the special effects in *The Matrix* to PlayStation 2 and the Xbox.
- Most interactive computer software for business and industry makes use of computer graphics in the screen displays and for other functions, such as graphical display of data, desktop publishing, and slide production for commercial and educational presentations. Consequently, anyone studying a computer language invariably spends time learning how to use at least two-dimensional (2D) graphics.

Application to Computer Graphics

- Basic mathematics used to manipulate and dis- play graphical images such as a wire-frame model of an airplane. Such an image (or picture) consists of a number of points, connecting lines or curves, and information about how to fill in closed regions bounded by the lines and curves. Often, curved lines are approximated by short straight-line segments, and a figure is defined mathematically by a list of points.
- Among the simplest 2D graphics symbols are letters used for labels on the screen. Some letters are stored as wire-frame objects; others that have curved portions are stored with additional mathematical formulas for the curves.

Example

- The capital letter N in Fig. 1 is determined by eight points, or *vertices*. The coordinates of the points can be stored in a data matrix, D. In addition to D, it is necessary to specify which vertices are connected by lines, but we omit this detail.

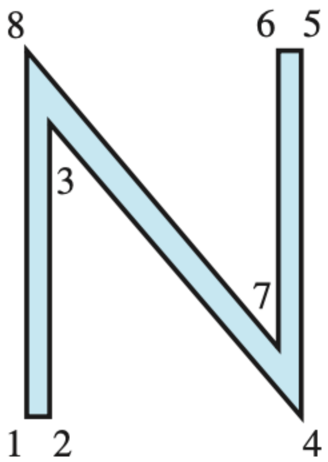


FIGURE 1
Regular N.

$$\begin{array}{r} \\ \\ x\text{-coordinate} \\ y\text{-coordinate} \end{array} \begin{array}{c} \text{Vertex:} \\ \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & .5 & .5 & 6 & 6 & 5.5 & 5.5 & 0 \\ 0 & 0 & 6.42 & 0 & 8 & 8 & 1.58 & 8 \end{bmatrix} \end{array} = D$$

Example

- Given A = $\begin{bmatrix} 1 & 0.25 \\ 0 & 1 \end{bmatrix}$, describe the effect of the shear transformation

x |→Ax on the letter N

Solution By definition of matrix multiplication, the columns of the product AD contain the images of the vertices of the letter N.

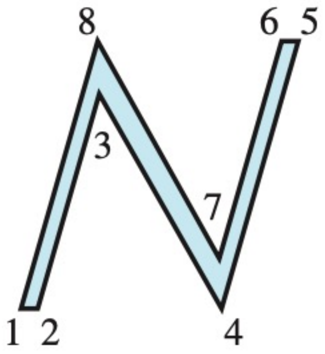


FIGURE 2
Slanted *N*.

$$\begin{array}{c} \quad\;\; \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ AD = \begin{bmatrix} 0 & .5 & 2.105 & 6 & 8 & 7.5 & 5.895 & 2 \\ 0 & 0 & 6.420 & 0 & 8 & 8 & 1.580 & 8 \end{bmatrix} \end{array}$$

Example

- The transformed vertices are plotted in Fig. 2, along with connecting line segments that correspond to those in the original figure.
- The italic N in Fig. 2 looks a bit too wide. To compensate, we can shrink the width by a scale transformation.

Example

Compute the matrix of the transformation that performs a shear transformation using A = $\begin{bmatrix} 1 & 0.25 \\ 0 & 1 \end{bmatrix}$, and then scales all x-coordinates by a factor of 0.75.

Solution

The matrix that multiplies the x-coordinate of a point by 0.75 is

$$S = \begin{bmatrix} .75 & 0 \\ 0 & 1 \end{bmatrix}$$

- So the matrix of the composite transformation is

$$SA = \begin{bmatrix} .75 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & .25 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} .75 & .1875 \\ 0 & 1 \end{bmatrix}$$

The result of this composite transformation is shown in Fig. 3.

FIGURE 3
Composite transformation of N.

Homogeneous Coordinates

- The mathematics of computer graphics is intimately connected with matrix multi- plication. Unfortunately, translating an object on a screen does not correspond directly to matrix multiplication because translation is not a linear transformation. The standard way to avoid this difficulty is to introduce what are called *homogeneous coordinates*.

Homogeneous Coordinates

- Each point (x, y) in R^2 can be identified with the point (x, y, 1) on the plane in R^3 that lies one unit above the xy-plane. We say that (x, y) has *homogeneous coordinates* (x, y, 1). For instance, the point (0, 0) has homogeneous coordinates (0, 0, 1). Homogeneous coordinates for points are not added or multiplied by scalars, but they can be transformed via multiplication by 3×3 matrices.

Translation

- A translation of the form (x, y) |→ (x + h, y + k) is written in homogeneous coordinates as

(x, y, 1) |→(x + h, y + k, 1). This transformation can be computed via matrix multiplication:

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + h \\ y + k \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- Any linear transformation on $\mathbb{R}^2$ is represented with respect to homogeneous coordinates by a partitioned matrix of the form $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$

where A is a 2×2 matrix. Typical examples are

$$\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Counterclockwise rotation about the origin, angle φ — Reflection through $y = x$ — Scale x by s and y by t

Composite Transformations

- Find the 3×3 matrix that corresponds to the composite transformation of a scaling by .3, a rotation of 90◦, and finally a translation that adds (−.5, 2) to each point of a figure.
- If $\phi = \pi/2$, then $\sin\phi = 1$ and $\cos\phi = 0$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \xrightarrow{\text{Scale}} \begin{bmatrix} .3 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\xrightarrow{\text{Rotate}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} .3 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\xrightarrow{\text{Translate}} \begin{bmatrix} 1 & 0 & -.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} .3 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Composite Transformations

The matrix for the composite transformation is

$$\begin{bmatrix} 1 & 0 & -.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} .3 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -.5 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} .3 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -.3 & -.5 \\ .3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous 3D Coordinates

- By analogy with the 2D case, we say that (x, y, z, 1) are homogeneous coordinates for the point (x, y, z) in R^3. In general, (X, Y, Z, H) are homogeneous coordinates for (x,y,z) if $H \neq 0$ and

$$x = \frac{X}{H}, \qquad y = \frac{Y}{H}, \quad \text{and} \quad z = \frac{Z}{H}$$

- Each nonzero scalar multiple of (x, y, z, 1) gives a set of homogeneous coordinates for (x,y,z). For instance, both (10,−6,14,2) and (−15,9,−21,−3) are homogeneous coordinates for (5, −3, 7).

Translation

- The translation operation is often used in machine learning for mean-centering the data, where a constant mean vector is subtracted from each row of the data set. As a result, the mean value of each column of the transformed data set becomes 0. An example of the effect of mean-centering on the scatter plot of a 2-dimensional data set is illustrated in Figure 2.1.

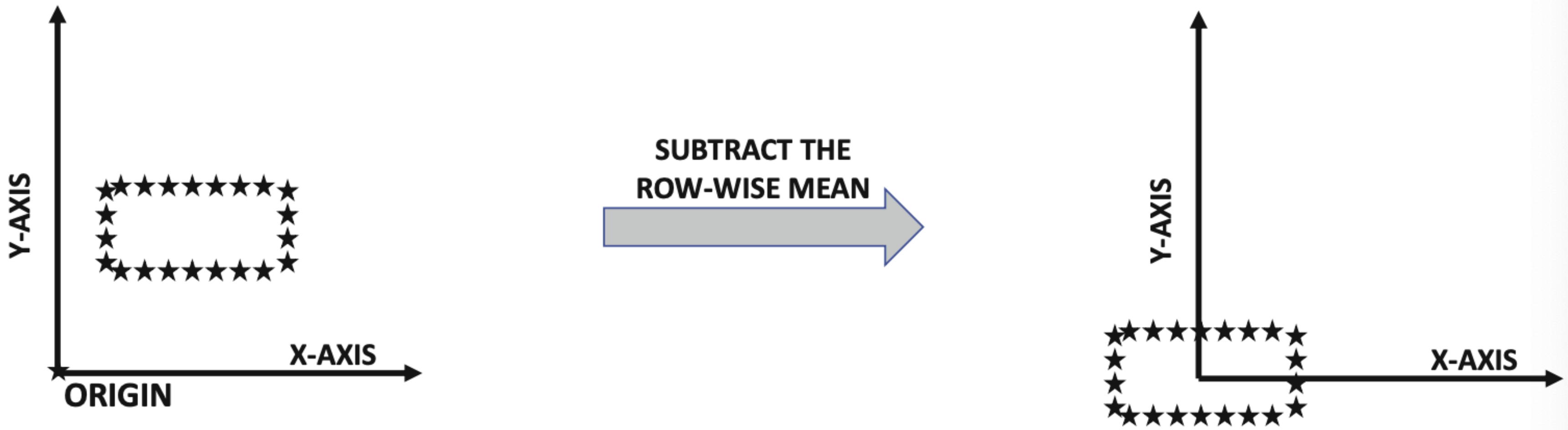


Figure 2.1: Mean-centering: a translation operation

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Orthogonal Transformation

- The orthogonal 2×2 matrices V_r and V_c that respectively rotate

2-dimensional row and column vectors by ϕ degrees in the counter-clockwise direction are as follows:

- $V_r = \begin{bmatrix} cos\phi & sin\phi \\ -sin\phi & cos\phi \end{bmatrix}, V_c = \begin{bmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{bmatrix}$
- If we have an n×2 data matrix D, then the product D V_r will rotate each row of D using V_r whereas the product $V_c D^T$ D^T will equivalently rotate each column of D^T . One can also view a data rotation D V_r in terms of projection of the original data on a rotated axis system. Counter-clockwise rotation of the data with a fixed axis system is the same as clockwise rotation of the axis system with fixed data.

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Orthogonal Transformation

- In essence, the two columns of the transformation matrix V_r represent the mutually orthogonal unit vectors of a new axis system that is rotated clockwise by ϕ. These two new columns are shown on the left of Figure 2.2 for a counter- clockwise rotation of 30◦.

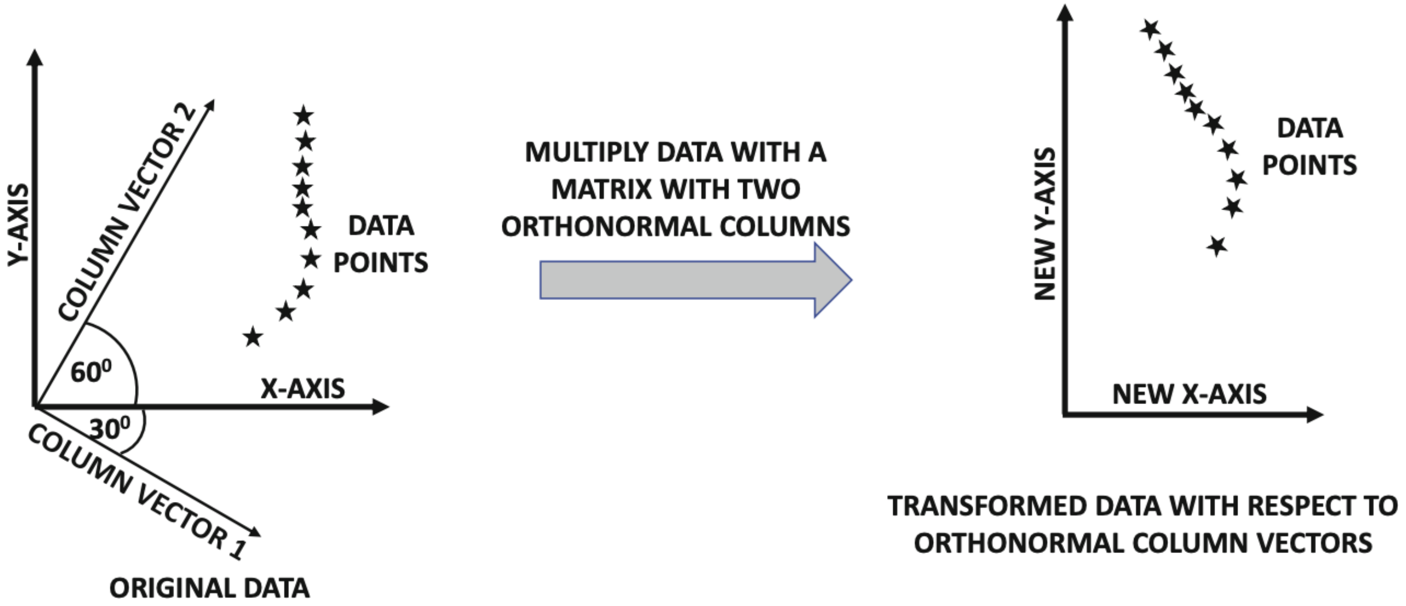

Figure 2.2: An example of counter-clockwise rotation with 30° with matrix multiplication. The two columns of the transformation matrix are shown in the figure on the left

Orthogonal Transformation

- The transformation returns the coordinates D V_r of the data points on these column vectors, because we are computing the dot product of each row of D with the (unit length) columns of V_r . In this case, the columns of V_r (orthonormal directions in new axis system) make counter-clockwise angles of −30◦ and 60◦ with the vector [1, 0].
- Therefore, the corresponding matrix V_r is obtained by populating the columns with vectors of the form [cos(ϕ), sin(ϕ)]T , where ϕ is the angle each new orthonormal axis direction makes with the vector [1,0]. This results in the following matrix V_r :
- $V_r = \begin{bmatrix} \cos(30) & \sin(30) \\ -\sin(30) & \cos(30) \end{bmatrix}$

Orthogonal Transformation

- After performing the projection of each data point on the new axes, we can reorient the figure so that the new axes are aligned with the original X- and Y -axes (as shown in the left- to-right transition of Figure 2.2). It is easy to see that the final result is a counter-clockwise rotation of the data points by 30◦ about the origin.

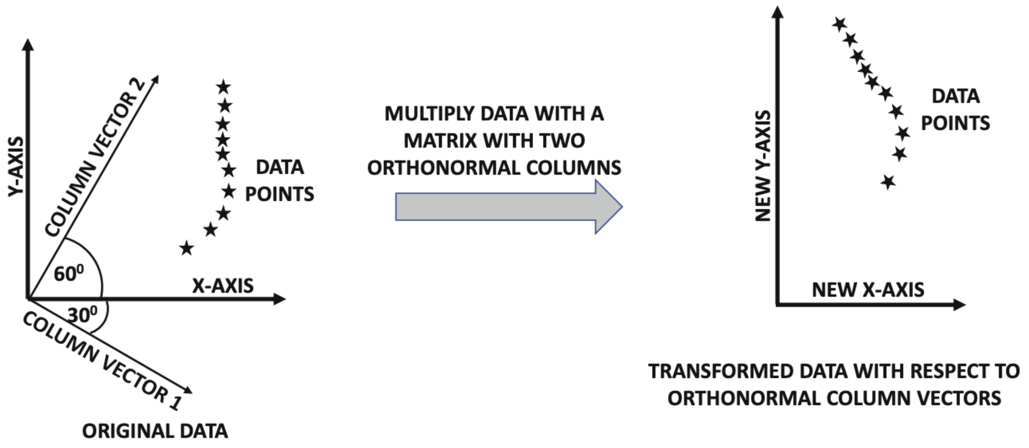

Figure 2.2: An example of counter-clockwise rotation with 30° with matrix multiplication. The two columns of the transformation matrix are shown in the figure on the left

Orthogonal Transformation

- Orthogonal matrices might include reflections. Consider the following matrix:

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- For any 2-dimensional data set contained in the $n \times 2$ matrix D, the transformation DV of the rows of D simply flips the two coordinates in each row of D. The resulting transformation cannot be expressed purely as a rotation. This is because this transformation changes the handedness of the data — for example, if the scatter plot of the n rows of the $n \times 2$ matrix D depicts a right hand, the scatter plot of the $n \times 2$ matrix DV will depict a left hand. Intuitively, when you look at your reflection in the mirror, your left hand appears to be your right hand. This implies that a reflection needs to performed somewhere.

Orthogonal Transformation

- The key point is that V can be expressed as the product of a counter-clockwise rotation of 90◦, followed by a reflection across the vector [0, 1]:

$$V = \begin{bmatrix} \cos(90) & \sin(90) \\ -\sin(90) & \cos(90) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$