Gauss Elimination Process

- Suppose, for example, that the augmented matrix of a linear system has been changed into the equivalent *reduced* echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 \qquad - 5x_3 &= 1 \\ x_2 + \ x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

- The variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables**. The other variable , x_3 ,is called a **free variable**.

Gauss Elimination Process

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

- By saying that , x_3 is "free," we mean that we are free to choose any value for , x_3 . Once that is done, we determine the values for x_1 and x_2 . For instance, when , x_3 = 0, the solution is (1, 4, 0); when , x_3 = 1, the solution is (6, 3, 1). *Each different choice of , x_3 determines a (different) solution of the system, and every solution of the system is determined by a choice of ,* x_3 .
- The solution is called a **general solution** of the system because it gives an explicit description of *all* solutions.

Gauss Elimination Process

$$\begin{aligned} x_1 + 6x_2 \qquad + 3x_4 \qquad &= 0 \\ x_3 - 4x_4 \qquad &= 5 \\ x_5 &= 7 \end{aligned}$$

- The pivot columns of the matrix are 1, 3, and 5, so the basic variables are x_1, x_3, and x_5 The remaining variables, x_2 and x_4, must be free.

we obtain the general solution:

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

Gauss Elimination Process

- A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form
- $[0 \cdots 0 \; b]$ with b nonzero
- If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Gauss Elimination Process

- $$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2 \end{aligned}$$

$$\left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

Gauss Elimination Process

- This echelon matrix shows that the system is *inconsistent*, because its rightmost column is a pivot column; the third row corresponds to the equation 0 = 5. There is no need to perform any more row operations. Note that the presence of the free variables in this problem is irrelevant because the system is inconsistent.

Gauss Elimination Process

Eg3.

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 3 \\
x_1 + 2x_2 + 2x_3 &= 5 \\
3x_1 + 4x_2 + 4x_3 &= 11
\end{aligned}
$$

Gauss Elimination Process

The augmented matrix can be written as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 3 & 4 & 4 & 11 \end{array}\right]$$

Replace R_2 by $R_2 - R_1$ and R_3 by $R_3 - 3R_1$ to get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1-1 & 2-1 & 2-1 & 5-3 \\ 3-3 & 4-3 & 4-3 & 11-3(3) \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

Gauss Elimination Process

Replace R_3 by R_3 - R_2 to get

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now back substitution gives us

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_2 + x_3 &= 2 \end{aligned}$$

Since there are 3 unknowns but only 2 constraints

The system has infinite number of solutions

Pivoting

We now come to the important case of the pivot being zero or very close to zero. If the pivot is zero, the entire process fails and if it is close to zero, round-off errors may occur. The process of swapping rows to avoid a zero element on the diagonal is called pivoting.

Complete Pivoting

We search both columns and rows for the largest element, the procedure is called complete pivoting. It is obvious that complete pivoting involves more complexity in computations since interchange of columns means change of order of unknowns which invariably requires more programming effort.

Partial Pivoting

- If a_{ii} is either zero or very small compared to the other coefficients of the equation, then we find the largest available coefficient in the columns below the pivot equation and then interchange the two rows. In this way, we obtain a new pivot equation with a nonzero pivot. Such a process is called partial pivoting, since in this case we search only the columns below for the largest element In comparison to complete pivoting, partial pivoting, i.e. row interchanges, is easily adopted in programming. Due to this reason, complete pivoting is rarely used.

Pivoting

- Partial Pivoting: A basic improvement on the naive code above is to not determine the factor to use for row combinations by simply taking the entry in the row, row position, but rather to look at all of the entries in the row column below the row, row entry and take one that is likely to give reliable results because it is not too small. This is partial pivoting.

Pivoting Example

0.0003120 x_1 +0.006032 x_2 = 0.003328

0.500000 x_1 + 0.89420 x_2 = 0.9471

The exact solution is x_1 = 1 and x_2 = 0.5

We first solve the system with pivoting. We write the given system as

$$\left[\begin{array}{cc|c} 0.500000 & 0.89420 & 0.9471 \\ 0.0003120 & 0.006032 & 0.003328 \end{array}\right]$$

Pivoting Example

Replace R_2 by $R_2 - \left(\frac{0.0003120}{0.0000050}\right) R_1$ to get

$$\left[\begin{array}{cc|c} 0.500000 & 0.89420 & 0.9471 \\ 0 & 0.005474 & 0.002737 \end{array}\right]$$

Back substitution gives us x_1 = 1 and x_2 = 0.5

Without pivoting, Gauss elimination gives

$$\left[\begin{array}{cc|c} 0.0003120 & 0.006032 & 0.003328 \\ 0 & -8.77725 & -5.3300 \end{array}\right]$$

Back substitution gives us x_1 = -1.0803 and x_2 = 0.6076

Gauss Seidel

- We shall now describe the iterative or indirect methods, which start from an approximation to the true solution and, if convergent, derive a sequence of closer approximations- the cycle of computations being repeated till the required accuracy is obtained. This means that in a direct method the amount of computation is fixed, while in an iterative method the amount of computation depends on the accuracy required.
- In general, one should prefer a direct method for the solution of a linear system, but in the case of matrices with a large number of zero elements, it will be advantageous to use iterative methods which preserve these elements.

Gauss Seidel

$$R_1 : a_{11}x_1 + a_{12}x_2 + \ \dots\dots.+ a_{1n}x_n = b_1$$
$$R_2 : a_{21}x_1 + a_{22}x_2 + \ \dots\dots.+ a_{2n}x_n = b_2$$
$$R_n : a_{n1}x_1 + a_{n2}x_2 + \ \dots\dots.+ a_{nn}x_n = b_n$$

in which the diagonal elements aii do not vanish. If this is not the case, then the equations should be rearranged so that this condition is satisfied.

Gauss Seidel

- Suppose $x_1^{(1)}, x_2^{(1)}, \dots\dots\ x_n^{(1)}$ are any first approximations to the unknowns $x_1, x_2, \dots\dots., x_n$
- We rewrite the equations as

$$x_1^{(1)} = \left(\frac{b_1}{a_{11}}\right) - \left(\frac{a_{12}}{a_{11}}\right) x_2 - \left(\frac{a_{13}}{a_{11}}\right) x_3 \dots\dots. - \left(\frac{a_{1n}}{a_{11}}\right) x_n$$

$$x_2^{(1)} = \left(\frac{b_2}{a_{22}}\right) - \left(\frac{a_{21}}{a_{22}}\right) x_1 - \left(\frac{a_{23}}{a_{22}}\right) x_3 \dots\dots. - \left(\frac{a_{2n}}{a_{22}}\right) x_n$$

$$x_3^{(1)} = \left(\frac{b_3}{a_{33}}\right) - \left(\frac{a_{31}}{a_{33}}\right) x_1 - \left(\frac{a_{32}}{a_{33}}\right) x_2 \dots\dots. - \left(\frac{a_{2n}}{a_{33}}\right) x_n$$

$$x_n^{(1)} = \left(\frac{b_n}{a_{nn}}\right) - \left(\frac{a_{n1}}{a_{nn}}\right) x_1 - \left(\frac{a_{n2}}{a_{nn}}\right) x_2 \dots\dots. - \left(\frac{a_{n,n-1}}{a_{nn}}\right) x_{n-1}$$

Gauss Seidel

- We get the second approximations as
- $x_1^{(2)} = \left(\frac{b_1}{a_{11}}\right) - \left(\frac{a_{12}}{a_{11}}\right) x_2^{(1)} - \left(\frac{a_{13}}{a_{11}}\right) x_3^{(1)} \dots \dots . - \left(\frac{a_{1n}}{a_{11}}\right) x_n^{(1)}$
- Since, we already have $x_1^{(2)}$, we can write second estimate of x_2 as
- $x_2^{(2)} = \left(\frac{b_2}{a_{22}}\right) - \left(\frac{a_{21}}{a_{22}}\right) x_1^{(2)} - \left(\frac{a_{23}}{a_{22}}\right) x_3^{(1)} \dots \dots . - \left(\frac{a_{2n}}{a_{22}}\right) x_n^{(1)}$
- Since, we already have$x_1^{(2)}, x_2^{(2)}$, we can write second estimate of x_3 as
- $x_3^{(2)} = \left(\frac{b_3}{a_{33}}\right) - \left(\frac{a_{31}}{a_{33}}\right) x_1^{(2)} - \left(\frac{a_{32}}{a_{33}}\right) x_2^{(2)} \dots \dots . - \left(\frac{a_{2n}}{a_{33}}\right) x_n^{(1)}$
- Since, we already have$x_1^{(2)}, x_2^{(2)}, x_{n-1}^{(2)}$ we can write second estimate of $x_n$as
- $x_n^{(2)} = \left(\frac{b_n}{a_{nn}}\right) - \left(\frac{a_{n1}}{a_{nn}}\right) x_1^{(2)} - \left(\frac{a_{n2}}{a_{nn}}\right) x_2^{(2)} \dots \dots . - \left(\frac{a_{n,n-1}}{a_{nn}}\right) x_{n-1}^{(n-1)}$

Gauss Seidel

- In this manner, we complete the first stage of iteration and the entire process is repeated till the values of $x_1, x_2, \dots \dots ., x_n$ are obtained to the accuracy required.
- It is clear, therefore, that this method uses an improved component as soon as it is available and it is called the method of successive displacements, or the Gauss-Seidel method.

Diagonal Dominance

- The Gauss-Seidel methods converge, for any choice of the first approximation $x_j^{(1)}$) (j = 1, 2, ..., n), if every equation of the system satisfies the condition that the sum of the absolute values of the coefficients $\left(\frac{a_{ij}}{a_{ii}}\right)$ is almost equal to, or in at least one equation less than unity, i.e. provided that

$$\sum_{j=1, j\neq 1}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| \leq 1, (i = 1,2, \ldots \ldots, n)$$

where the < sign should be valid in the case of ‘at least’ one equation.

Diagonal Dominance Example 1

Is the following matrices strictly diagonally dominant?

$$\begin{bmatrix} -6 & 0 & 3 \\ 7 & 8 & -2 \\ 1 & -1 & 2 \end{bmatrix}$$

Diagonal Dominance Solution 1

Is the following matrices strictly diagonally dominant?

$$\begin{bmatrix} -6 & 0 & 3 \\ 7 & 8 & -2 \\ 1 & -1 & 2 \end{bmatrix}$$

$|a_{11}| = 6$

$|a_{12}| + |a_{13}| = 0 + 3 = 3$

$|a_{22}| = 8$

$|a_{21}| + |a_{23}| = 7 + 2 = 9$

$|a_{33}| = 2$

$|a_{31}| + |a_{32}| = 1 + 1 = 2$

$|a_{22}| < |a_{21}| + |a_{23}|$ Matrix is not strictly diagonal dominant

Diagonal Dominance Examples

Which of the following matrices is strictly diagonally dominant?

$$a.\ \begin{bmatrix} -6 & 0 & 3 \\ 5 & 8 & -2 \\ 1 & -1 & 2 \end{bmatrix}$$

$$b.\ \begin{bmatrix} -6 & 0 & 3 \\ 1 & 2 & -2 \\ 1 & -1 & 8 \end{bmatrix}$$

Diagonal Dominance Solution a

Which of the following matrices is strictly diagonally dominant?

$$a.\ \begin{bmatrix} -6 & 0 & 3 \\ 5 & 8 & -2 \\ 1 & -1 & 2 \end{bmatrix}$$

$$|a_{11}| = 6$$
$$|a_{12}| + \ |a_{13}| = 0 + 3 = 3$$
$$|a_{22}| = 8$$
$$|a_{21}| + \ |a_{23}| = 5 + 2 = 7$$
$$|a_{33}| = 2$$
$$|a_{31}| + \ |a_{32}| = 1 + 1 = 2$$

Matrix is strictly diagonal dominant

Diagonal Dominance Solution b

Which of the following matrices is strictly diagonally dominant?

b. $\begin{bmatrix} -6 & 0 & 3 \\ 1 & 2 & -2 \\ 1 & -1 & 8 \end{bmatrix}$

$|a_{11}| = 6$

$|a_{12}| + |a_{13}| = 0 + 3 = 3$

$|a_{22}| = 2$

$|a_{21}| + |a_{23}| = 1 + 2 = 3$

$|a_{33}| = 8$

$|a_{31}| + |a_{32}| = 1 + 1 = 2$

$|a_{22}| < |a_{21}| + |a_{23}|$ Matrix is not strictly diagonal dominant

Stopping Criteria

Stopping criteria for iterations

1. Number of iterations
2. Calculate the Absolute Relative Approximate Error

$$|\epsilon_a| = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| * 100$$

The iterations are stopped when the absolute relative approximate error is less than a prespecified tolerance for all unknowns

Gauss Seidel

Eg1.

$$20x_1 + 2x_2 + x_3 = 30$$
$$x_1 - 40x_2 + 3x_3 = -75$$
$$2x_1 - x_2 + 10x_3 = 30$$

Take initial approx. as $x_1 = 1.5$, $x_2 = 2$, $x_3 = 3$

Check for diagonal dominance

$$|a_{11}| = 20 \geq |a_{12}| + |a_{13}| = 2 + 1 = 3$$
$$|a_{22}| = 40 \geq |a_{21}| + |a_{23}| = 1 + 3 = 4$$
$$|a_{33}| = 10 \geq |a_{31}| + |a_{32}| = 2 + 1 = 3$$

Gauss Seidel

We rewrite the equations as

$$x_1 = \frac{1}{20}(30 - 2x_2 - x_3)$$

$$x_2 = \frac{1}{40}(75 + x_1 + 3x_3)$$

$$x_3 = \frac{1}{10}(30 - 2x_1 + x_2)$$

Gauss Seidel

First iteration gives us

$$x_1^{(1)} = \frac{1}{20}(30 \ - 2(2) - 3) = 1.15$$

$$x_2^{(1)} = \frac{1}{40}(75 + 1.15 + 3(3)) = 2.14$$

$$x_3^{(1)} = \frac{1}{10}(30 \ - 2(1.15) + 2.14) = 2.98$$

Gauss Seidel

Second iteration gives us

$$x_1^{(2)} = \frac{1}{20}(30 \ - 2(2.14) - 2.98) = 1.137$$

$$x_2^{(2)} = \frac{1}{40}(75 + 1.137 + 3(2.98)) = 2.127$$

$$x_3^{(2)} = \frac{1}{10}(30 \ - 2(1.137) + 2.127) = 2.986$$

Gauss Seidel

Third iteration gives us

$$x_1^{(3)} = \frac{1}{20}(30 - 2(2.127) - 2.986) = 1.138$$

$$x_2^{(3)} = \frac{1}{40}(75 + 1.138 + 3(2.986)) = 2.127$$

$$x_3^{(3)} = \frac{1}{10}(30 - 2(1.138) + 2.127) = 2.985$$

The solution can be written as $x_1 = 1.14$, $x_2 = 2.13$, $x_3 = 2.98$

Gauss Seidel

Eg2.

$$
\begin{aligned}
10x_1 - 2x_2 - x_3 - x_4 &= 3 \\
-2x_1 + 10x_2 - x_3 - x_4 &= 15 \\
-x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\
-x_1 - x_2 - 2x_3 + 10x_4 &= -9
\end{aligned}
$$

Take initial approx. as $x_1 = 0.3$, $x_2 = 1.5$, $x_3 = 2.7$, $x_4 = -0.9$

Check for diagonal dominance

$$
\begin{aligned}
|a_{11}| = 10 &\geq |a_{12}| + |a_{13}| + |a_{14}| = 2 + 1 + 1 = 4 \\
|a_{22}| = 10 &\geq |a_{21}| + |a_{23}| + |a_{24}| = 2 + 1 + 1 = 4 \\
|a_{33}| = 10 &\geq |a_{31}| + |a_{32}| + |a_{34}| = 1 + 1 + 2 = 4 \\
|a_{44}| = 10 &\geq |a_{41}| + |a_{42}| + |a_{43}| = 1 + 1 + 2 = 4
\end{aligned}
$$

Gauss Seidel

We rewrite the equations as $x_2 = 1.5, x_3 = 2.7, x_4 = -0.9$

$$x_1 = (0.30 + 0.2x_2 + 0.1x_3 + 0.1x_4)$$

$$x_2 = (1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4)$$

$$x_3 = (2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4)$$

$$x_4 = (-0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3)$$

Gauss Seidel

$$|\epsilon_a| = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| * 100$$

	x_1	$\|\epsilon_a\|$	x_2	$\|\epsilon_a\|$	x_3	$\|\epsilon_a\|$	x_4	$\|\epsilon_a\|$
1	0.72		1.824		2.774		-0.0196	
2	0.9403	23%	1.9635	7.15%	2.9864	7.1%	-0.0125	56.8%
3	0.9901	5%	1.9954	1.6%	2.9960	0.32%	-0.0023	443%
4	0.9984	0.83%	1.9990	0.18%	2.9993	0.11%	-0.0004	475%
5	0.9997	0.13%	1.9998	0.04%	2.9998	0.016%	-0.0003	33%
6	0.9998	0.01%	1.9998	0	2.9998	0	-0.0003	0
7	1.0000	0	2.0000	0.01%	3.0000	0.006%	0.0000	-