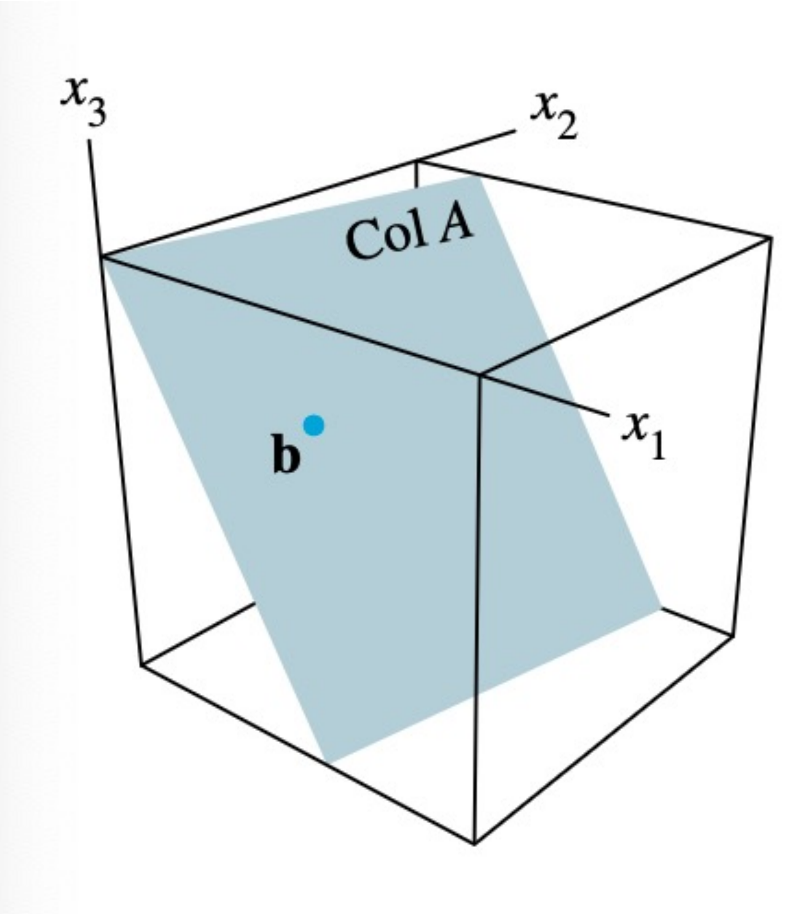
Column Space of a Matrix

- The column space of a matrix A is the set Col A of all linear combinations of the columns of A.
- If A = $[a_1 \quad \dots . \quad a_n]$ with the columns in R^m, then Col A is the same as Span $\{a_1 \quad \dots . \quad a_n\}$
- In other words, for an n × d matrix A, its column space is defined as the vector space spanned by its columns, and it is a subspace of R^n.

- The column space of an m×n matrix is a subspace of R^m .
- A plane through the origin is the standard way to visualize the subspace



Column Space Example 1

Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$, and b = $\begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$. Determine if b is in the the column space of A.

The vector b is a linear combination of the columns of A if and only if b can be written as Ax for some x, that is, if and only if the equation Ax = b has a solution.

Row reducing the augmented matrix [A b],

Column Space Example 1

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix}$$

Replace R_2 by $R_2 + 4\,R_1$ and R_3 by $R_3 + 3\,R_1$

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix}$$

Column Space Example 1

Replace R_3 by R_3 - (1/3) R_2

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we conclude that Ax = b is consistent and b is in Col A.

Column Space Example 2

Let $v_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix}$ and p = $\begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}$.

A = $[\, v_1 \quad v_2 \quad v_3]$

1. How many vectors are in $\{ v_1, \quad v_2, \quad v_3 \}$?
2. How many vectors are in Col A?
3. Is p in Col A? Why or why not?

Column Space Example 2

a. There are three vectors: v1, v2, and v3 in the set { v_1, v_2, v_3}.

b. There are infinitely many vectors in Span { v_1, v_2, v_3} = Col A.

c. The vector p is a linear combination of the columns of A if and only if p can be written as Ax for some x, that is, if and only if the equation Ax = p has a solution.

Row reducing the augmented matrix [A p],

Column Space Example 2

$$\begin{bmatrix} 2 & -3 & -4 & 6 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{bmatrix}$$

Replace R_2 by R_2 + 4 R_1 and R_3 by R_3 - 3 R_1

$$\begin{bmatrix} 2 & -3 & -4 & 6 \\ 0 & -4 & -10 & 14 \\ 0 & 2 & 5 & -7 \end{bmatrix}$$

Column Space Example 2

Replace R_3 by R_3 + (1/2) R_2

$$\begin{bmatrix} 2 & -3 & -4 & 6 \\ 0 & -4 & -10 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we conclude that Ax = p is consistent and p is in Col A.

Row Space

- For an n × d matrix A, its row space is defined as the vector space spanned by the columns of A^T (which are simply the transposed rows of A). The row space of A is a subspace of R^d.

Row Space Example

Consider the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$

a. Determine if b = $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the the column space of A.

b. Determine if b = $\begin{bmatrix} 4 & 5 \end{bmatrix}$ is in the the row space of A.

Row Space Example

a. The vector b is a linear combination of the columns of A if and only if b can be written as Ax for some x, that is, if and only if the equation Ax = b has a solution.

Row reducing the augmented matrix [A b],

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & 3 \end{bmatrix}$$

Row Space Example

Replace R_1 by $R_1 + R_2, R_3$ by R_3 - 3 R_1

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

we conclude that Ax = b is consistent and b is in Col A.

Row Space Example

b. If the vector w is in row(A), then w is a linear combination of the rows of A

Row reducing the augmented matrix $\begin{bmatrix} A \\ w \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \\ 4 & 5 \end{bmatrix}$$

Row Space Example

Replace R_3 by R_3 - 3 R_1 , R_4 by $R_4 - 4\,R_1$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 9 \end{bmatrix}$$

Replace R_4 by $R_4 - 9\,R_2$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Therefore, w is a linear combination of the rows of A

Null Space of a matrix

- The **null space** of a matrix A is the set Nul A of all solutions to the homogeneous equation A**x** = **0**.
- The null space of an m×n matrix A is a subspace of R^n . Equivalently, the set of all solutions to a system A**x** = **0** of m homogeneous linear equations in n unknowns is a subspace of R^n .

Null Space example 1

Let $v_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix}$ and p = $\begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}$.

A = $[\, v_1 \quad v_2 \quad v_3 \,]$

Determine if p is in Nul A.

p is in nul A if Ap = 0

Null Space example 1

Ap = $\begin{bmatrix} 2 & -3 & -4 \\ -8 & 8 & 6 \\ 6 & -7 & -7 \end{bmatrix} \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}$ = $\begin{bmatrix} -2 \\ -62 \\ 29 \end{bmatrix}$

Since Ap $\neq$ 0, p is not in Nul A.

Null Space example 2

Let u = $\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$ and A = $\begin{bmatrix} -3 & -2 & 0 \\ 0 & 2 & -6 \\ 6 & 3 & 3 \end{bmatrix}$

Determine if u is in Nul A.

u is in nul A if Au = 0

Null Space example 2

Au = $\begin{bmatrix} -3 & -2 & 0 \\ 0 & 2 & -6 \\ 6 & 3 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$ = $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Since $\mathrm{Ap} = 0$, u is in Nul A.

Null Space example 3

Let A = $\begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix}$

a. Find a nonzero vector in Nul A

b. A nonzero vector in Col A.

b. To produce a vector in Col A, select any column of A.

a. u is in nul A if Au = 0

Null Space example 3

Au = $\begin{bmatrix} 3 & 2 & 1 & -5 & 0 \\ -9 & -4 & 1 & 7 & 0 \\ 9 & 2 & -5 & 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is x_1 = x_3 –x_4 , and x_2 = –2 x_3 + 4 x_4, with x_3 and x_4 free. The general solution in parametric vector form is not needed. All that is required here is one nonzero vector. So choose any values for x_3 and x_4 (not both zero). For instance, set x_3 = 1 and x_4 = 0 to obtain the vector (1, –2, 1, 0) in Nul A.

Null Space

- The notion of a null space refers to a right null space by default. This is because the vector x occurs on the right side of matrix A in the product Ax, which must evaluate to the zero vector. Similar to the definition of a right null space, one can define the left null space of a matrix, which is the orthogonal complement of the vector space spanned by the columns of the matrix.

Left Null Space

- The left null space of an $n \times d$ matrix A is the sub- space of R^n containing all column vectors $x \in Rn$, such that $A^T x = 0$. The left null space of A is the orthogonal complementary subspace of the column space of A. (Let V be a vector space and W be a subspace of V. Then the orthogonal complement of W in V is the set of vectors u such that u is orthogonal to all vectors in W.)
- Alternatively, the left null space of a matrix A contains all vectors x satisfying $x^T A = 0^T$.
- The row space, column space, the right null space, and the left null space are referred to as the four fundamental subspaces of linear algebra.

The Invertible Matrix Theorem

Let A be a square n×n matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the n×n identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

The Invertible Matrix Theorem Contd..

Let A be a square n×n matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

f. The linear transformation **x** |→ A**x** is one-to-one.

g. The equation A**x** = **b** has atleast one solution for each **b** in R^n .

h. The columns of A span R^n .

i. The linear transformation **x** |→ A**x** maps R^n onto R^n.

j. There is an n×n matrix C such that CA = I.

k. There is an n×n matrix D such that AD = I.

l. A^T is an invertible matrix.

The Invertible Matrix Theorem Contd..

Let A be a square n×n matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

m. The columns of A form a basis of R^n .

n. Col A = R^n

o. Dim Col A = n

p. Rank A=n

q. Nul A={0}

r. Dim Nul A=0

Basis for a Subspace

- A **basis** for a subspace H of R^n is a linearly independent set in H that spans H .
- The columns of an invertible n×n matrix form a basis for all of R^n because they are linearly independent and span R^n , by the Invertible Matrix Theorem. One such matrix is the n×n identity matrix. Its columns are denoted by $e_1 \quad \dots. \quad e_n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ .. \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ .. \\ 0 \end{bmatrix}, \ldots\ldots\, e_n = \begin{bmatrix} 0 \\ .. \\ 0 \\ 1 \end{bmatrix}$$

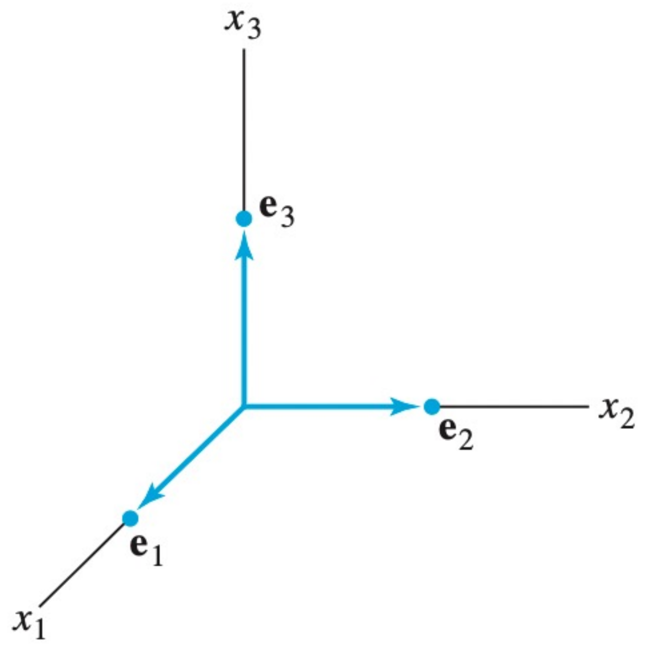


FIGURE 3
The standard basis for $\mathbb{R}^3$.

The set $\{e_1 \quad \dots. \quad e_n\}$ is called the **standard basis** for R^n .

Bases Example 1

- Determine if the set is bases for R^2 or R^3.

$\begin{bmatrix} 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 10 \\ -3 \end{bmatrix}$

Bases Example 1

- Determine if the set is bases for R^2 or R^3.

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 10 \\ -3 \end{bmatrix}$$

Yes. Let A be the matrix whose columns are the vectors given. Then A is invertible because its determinant is nonzero, and so its columns form a basis for R^2 , by the Invertible Matrix Theorem