Determinants

Determinant

A determinant is a number that is assigned to a square matrix in a certain way.

Determinant of a 2x2 Matrix

- For a 2x2 Matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is denoted by |A| or detA

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Determinant Example

- Eg. Find the determinant of A = $\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = 2(-2) - 1(-3) = -1$

Minor of A

- Minor of matrix A is the determinant of some smaller square matrix cut down from A by removing one or more of its rows and columns.
- The minor M_{ij} is the determinant of the submatrix obtained by the deleting the i^{th} row and j^{th} column.

Minor of A

- The minor $M_{i,j}$, is the determinant of the matrix with row i and column j removed

- Example, To compute the minor $M_{2,\,3}$, for the matrix

A = $\begin{bmatrix} 1 & 4 & 2 \\ 4 & 6 & 1 \\ -1 & 9 & 0 \end{bmatrix}$

- $M_{2,3} = det\begin{bmatrix} 1 & 4 & \\ & & \\ -1 & 9 & \end{bmatrix} = det\begin{bmatrix} 1 & 4 \\ -1 & 9 \end{bmatrix} = 9 - (-4) = 13$

Cofactor of A

- The cofactor C_{ij} is obtained by multiplying the minor $M_{i,j}$ by $(-1)^{i+j.}$

- For example, To compute the cofactor $C_{2,\,3}$, for the matrix

A = $\begin{bmatrix} 1 & 4 & 2 \\ 4 & 6 & 1 \\ -1 & 9 & 0 \end{bmatrix}$

- $C_{2,\,3} = (-1)^{2+3} M_{2,\,3} = (-1)\, det \begin{bmatrix} 1 & 4 & \\ & & \\ -1 & 9 & \end{bmatrix} = (-\ 1) det \begin{bmatrix} 1 & 4 \\ -1 & 9 \end{bmatrix}$

$= (-\ 1)[9 - (-4)] = -13$

Adjoint of matrix

- Let A = [a_{ij}] be a nxn matrix and let C_{ij} denote the cofactor of a_{ij} . The classical adjoint or adjugate of A, denoted by the adjA, is the transpose of the matrix of cofactors of A
- $adjA = \left[C_{ij}\right]^T$

Determinant of a Matrix

- For a 3x3 Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|\mathrm{A}| = \mathrm{a}_{11}\mathrm{C}_{11} + \mathrm{a}_{12}\,\mathrm{C}_{12} + \mathrm{a}_{13}\,\mathrm{C}_{13}$$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinant of a Matrix

- For a mxn Matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ .. & .. & .. & .. \\ a_{m1} & a_{m2} & .. & a_{mn} \end{bmatrix}$$

$|A| = \sum_{\substack{1\le i\le m \\ 1\le j\le n}} a_{ij}C_{ij}$, where C_{ij} is the co-factor of a_{ij}

C_{ij} = (-1)$^{i+j}$ M_{ij}, where M_{ij} is $Minor_{ij}$ which is the determinant of the reduced matrix obtained by removing the row i and column j

Determinant of a Matrix

- We can now give a recursive definition of a determinant.
- When $n = 3$, $\det A$ is defined using determinants of the 2×2 submatrices
- When $n = 4$, $\det A$ uses determinants of the 3×3 submatrices.
- In general, an $n\times n$ determinant is defined by determinants of $(n-1)\times(n-1)$ submatrices.

Determinant of a Matrix

- The determinant of an n×n matrix A can be computed by a cofactor expansion across any row or down any column.
- The expansion across the i^{th} row using the cofactors is

$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$, where C_{ij} is the co-factor of a_{ij}

- The cofactor expansion down the j^{th} column is

$|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$, where C_{ij} is the co-factor of a_{ij}

Properties of Determinant

- If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.
- The determinant of identity matrix is 1.
- A square matrix A is invertible if and only if det A $\neq 0$.

Properties of Determinant

- Let A be a square matrix.
  - If A has a row(or column) of zeros, then $|A| = 0$
  - If A has two identical rows (or columns) then $|A| = 0$
  - If a multiple of one row of A is added to another row to produce a matrix B,
  - then det B = det A.
  - If two rows of A are interchanged to produce B, then det B = −det A.
  - If one row of A is multiplied by k to produce B, then det B = k · det A.
  - The determinant of matrix A and its transpose are equal

    $|A| = |A^T|$
  - If A and B are n×n matrices, then det AB = (det A)(det B)

Determinant

- If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is |det A|. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is |det A|.

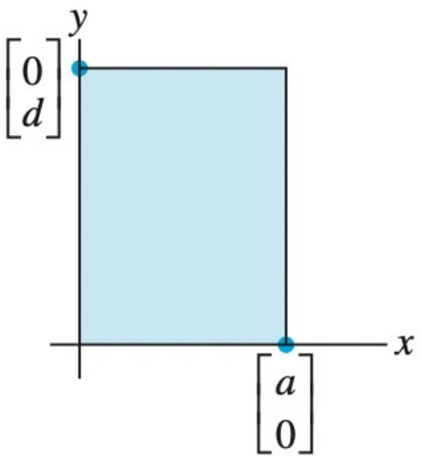

FIGURE 1
Area = $|ad|$.

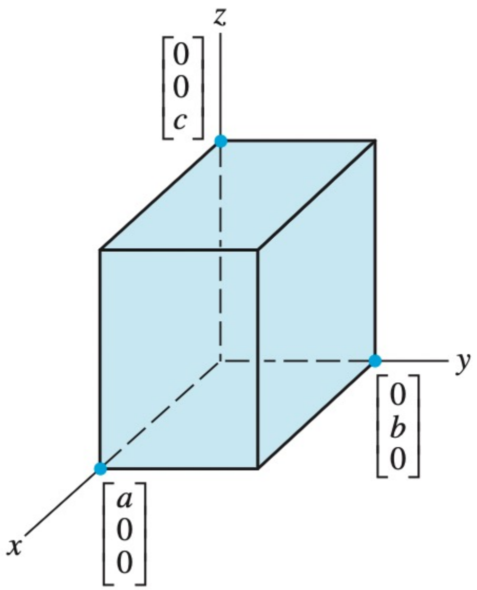

FIGURE 3
Volume = $|abc|$.

Determinant Examples

Find the determinant of the matrices

1) $\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$

2) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & 0 & -2 \end{bmatrix}$

3) $\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

4) $\begin{bmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{bmatrix}$

5) $\begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$

Determinant Examples

6) Let A and B be 3×3 matrices, with det A =4 and det B =−3. Use properties of to compute:

a) det AB b) det B^T c) det A^3

7) Let A and B be 4×4 matrices, with det A = −1 and det B = 2. Compute:

a) det AB b) det B^5 c) det $A^T A$

b) 8) Verify that det AB = (det A)(det B) for the matrices given below

a) $A = \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}$ and B $= \begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$ and B $= \begin{bmatrix} 4 & 2 \\ -1 & -1 \end{bmatrix}$

Solution 1

For a 3x3 Matrix

A = $\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$

$$|A| = a_{11}C_{11} + a_{12}\,C_{12} + a_{13}\,C_{13} = 7C_{11} + 2\,C_{12} + 1\,C_{13}$$

Solution 1

For a 3x3 Matrix

A = $\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$

$$|\mathrm{A}| = \mathrm{a}_{11}\mathrm{C}_{11} + \mathrm{a}_{12}\,\mathrm{C}_{12} + \mathrm{a}_{13}\,\mathrm{C}_{13} = 7\mathrm{C}_{11} + 2\,\mathrm{C}_{12} + 1\,\mathrm{C}_{13}$$

$\mathrm{C}_{11} = (\text{-}1)^{1+1}\mathrm{M}_{11} = (-1)^{2}\begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix}$ = -6+4 = -2

$\mathrm{C}_{12} = (\text{-}1)^{1+2}\mathrm{M}_{12} = (-1)^{3}\begin{vmatrix} 0 & -1 \\ -3 & -2 \end{vmatrix}$ = (-1)(0-3) = 3

$\mathrm{C}_{13} = (\text{-}1)^{1+3}\mathrm{M}_{13} = (-1)^{4}\begin{vmatrix} 0 & 3 \\ -3 & 4 \end{vmatrix}$ = 0+9= 9

|A| = 7(-2) + 2(3) +9 =1

Solution 2

- For a 3x3 Matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & 0 & -2 \end{bmatrix}$$

$$|A| = a_{11}C_{11} + a_{12}\,C_{12} + a_{13}\,C_{13} = 2C_{11} + 1\,C_{12} + 3\,C_{13}$$

$$C_{11} = (\text{-}1)^{1+1}M_{11} = (-1)^2 \begin{vmatrix} 0 & 2 \\ 0 & -2 \end{vmatrix} = 0$$

$$C_{12} = (\text{-}1)^{1+2}M_{12} = (-1)^3 \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = 6$$

$$C_{13} = (\text{-}1)^{1+3}M_{13} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = 0$$

$|A|$ = 2x0 + 1x6 + 3x0 = 6

Solution 3

- For a 3x3 Matrix

A = $\begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

$|A| = a_{31}C_{31} + a_{32}\, C_{32} + a_{33}\, C_{33} = 0C_{31}$ - 2 $C_{32} + 0C_{33}$

$C_{31} =$ (-1)$^{3+1}M_{31} = (-1)^4 \begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix}$ = 0

$C_{32} =$ (-1)$^{3+2}M_{32} = (-1)^5 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}$ = 1

$C_{33} =$ (-1)$^{3+3}M_{33} = (-1)^6 \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$ = 0

|A| = 0 - 2(1) + 0 = -2

Solution 4

A = $\begin{bmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{bmatrix}$

First expand along the third row

$|\mathrm{A}| = 2\mathrm{C}_{31} + 0\ \mathrm{C}_{32} + 0\ \mathrm{C}_{33} + 0\ \mathrm{C}_{34}$

Solution 4

$$|A| = (\text{-}1)^{1+3}\,.2\begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 8 \end{vmatrix}$$

Expand along the first row of the remaining matrix

$$|A| = 2.(\text{-}1)^{1+3}\,.\,5\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} = 10(1) = 10$$

Solution 5

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

The cofactor expansion down the first column of A has all terms equal to zero except the first.

$$|A| = 3C_{11} + 0\ C_{21} + 0\ C_{31} - 0\ C_{41} + 0\ C_{51}$$

Solution 5

$$|A| = 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} C_{11} + 0\ \mathrm{C}_{21} + 0\ \mathrm{C}_{31} \text{ - } 0\ \mathrm{C}_{41} + 0\ \mathrm{C}_{51}$$

Henceforth we will omit the zero terms in the cofactor expansion. Next, expand this 4×4 determinant down the first column, in order to take advantage of the zeros there.

$$|A| = 3.\,2.\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 3.2.(\text{-}2) = \text{-}12$$

Solution 6

Given $\det A = 4$ and $\det B = -3$

a) $\det AB = (\det A)(\det B) = (4)(-3) = -12$.

b) $\det B^{T} = \det B = -3$.

c) $\det A^3 = (det\ A)^3 = 4^3 = 64$.

Solution 7

Given det A = -1 and det B = 2

a) det AB = (det A)(det B) = (–1)(2) = –2.

b) det B^5 = $(det\ B)^5$ $= 2^5 = 32$.

c) det $A^T A$ = (det A^T)(det A)

= (det A)(det A)

$= (-1)(-1) = 1$.

Solution 8

a) $\det AB = 24 = (3)(8) = (\det A)(\det B)$.

b) $\det AB = 0 = (0)(-2) = (\det A)(\det B)$.

Inverse

- Let square matrices A and B have the property

  $$AB = BA = I$$

  then B is called the inverse of A and denoted by A^{-1}
- Not every matrix A possesses an inverse.
- If this inverse does exist, A is called regular/invertible/non-singular.
- If this inverse does not exist, A is called singular.
- If matrix inverse exists, it is unique.

Inverse of A

$$A^{-1} = \frac{1}{|A|}\,(adjA)$$

$$|A| \neq 0$$

Properties of inverse

- $A^{-1}A = I \; and \; AA^{-1} = I$
- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}\,A^{-1}$
- $(A^{-1})^{T} = (A^{T})^{-1}$ Transpose of inverse is inverse of the transpose

Inverse of A

Eg. Find the inverse of A = $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

Solution

$|A|$ = $6 - 6 = 0$

Hence, A is not invertible

Inverse Example 1

If A = $\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and C = $\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$

then show that A and C are inverses of each other.

$$AC =$$

$$CA =$$

Inverse Solution 1

If $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, then

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and}$$

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Thus, $C = A^{-1}$

Inverse Example 2

- Find the inverse of the following matrix using adjoint

$$\begin{bmatrix} 3 & 2 \\ 1 & -4 \end{bmatrix}$$

Inverse Solution 2

- Find the inverse of the following matrix using adjoint

A = $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

|A| = 3(4) – 2(1) = 10

Since |A| $\neq 0$, inverse exists

$C_{11} = (-1)^{1+1}\,4 \;\; = 4, \quad C_{12} = (-1)^{1+2} 1 \;\; = -1,$

$C_{21} = (-1)^{2+1}\,2 \;\; = -2 \;\; , C_{22} = (-1)^{2+2}\,3 \;\; = 3$

Co-factor matrix = $\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$

Inverse Solution 2

Adjoint of A = $\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}\,(adjA) = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} {}^{4}/_{10} & {}^{-2}/_{10} \\ {}^{-1}/_{10} & {}^{3}/_{10} \end{bmatrix}$$

= $\begin{bmatrix} {}^{2}/_{5} & {}^{-1}/_{5} \\ {}^{-1}/_{10} & {}^{3}/_{10} \end{bmatrix}$

Examples

Find inverse of the following matrices

1) $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ 2) $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

3) $\begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$

4) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

5) $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

Solution 1

Eg. Find inverse of A = $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

$|A|$ = $6 - 6 = 0$

Hence, A is not invertible

Solution 2

A = $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$|A|$ = $3(6) - 4(5) = -2$

Hence, A is invertible

$C_{11} = (-1)^{1+1}6 \ = 6, \quad C_{12} = (-1)^{1+2}5 \ = -5,$

$C_{21} = (-1)^{2+1}4 \ = -4 \ , C_{22} = (-1)^{2+2}3 \ = 3$

Co-factor matrix = $\begin{bmatrix} 6 & -5 \\ -4 & 3 \end{bmatrix}$

Solution 2

Adjoint of A = $\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}\,(adjA) = \frac{-1}{2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} {}^{6}/_{(-2)} & {}^{-4}/_{(-2)} \\ {}^{-5}/_{(-2)} & {}^{3}/_{(-2)} \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ {}^{5}/_{2} & {}^{-3}/_{2} \end{bmatrix}$$

Solution 3

$$Let A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$$

$$C_{11} = +\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} = -18, C_{12} = -\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = 2, C_{13} = +\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix} = -11, C_{22} = +\begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} = 14, C_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = 5$$

$$C_{31} = +\begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix} = -10, C_{32} = -\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} = -4, C_{33} = +\begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix} = -8$$

$$Co\ factor\ matrix\ C_{ij} = \begin{bmatrix} -18 & 2 & 4 \\ -11 & 14 & 5 \\ -10 & -4 & -8 \end{bmatrix}$$

Solution 3

The transpose of the above matrix of cofactors yields the adjoint of A

$$adjA = \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$$

$Now,$

$$|\mathrm{A}| = \mathrm{a}_{11}\mathrm{C}_{11} + \mathrm{a}_{12}\,\mathrm{C}_{12} + \mathrm{a}_{13}\,\mathrm{C}_{13}$$

$$= 2\mathrm{C}_{11} + 3\mathrm{C}_{12} - 4\,\mathrm{C}_{13}$$

$$= 2(-\,18) + 3(2) - 4(4)$$

$$= \ -36 + 6 - 16 = -46$$

$$det(A) \neq 0$$

Solution 3

$$A^{-1} = \frac{1}{|A|}\,(adjA) = \frac{-1}{46}\begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 9/_{23} & 11/_{46} & 5/_{23} \\ -1/_{23} & -7/_{23} & 2/_{23} \\ -2/_{23} & -5/_{46} & 4/_{23} \end{bmatrix}$$

Solution 4

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$C_{11} = -1$ $C_{12} = 8$ $C_{13} = -5$

$C_{21} = 1$ $C_{22} = -6$ $C_{23} = 3$

$C_{31} = -1$ $C_{32} = 2$ $C_{33} = -1$

$$\text{Cofactor of } A = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

Solution 4

- $adjA = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$
- $|A| = 0 - 1(11 - 9) + 2(1 - 6) = 8 - 10$ = -2
- $A^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} {}^{1}/_{2} & {}^{-1}/_{2} & {}^{1}/_{2} \\ -4 & 3 & -1 \\ {}^{5}/_{2} & {}^{-3}/_{2} & {}^{1}/_{2} \end{bmatrix}$

Solution 5

A = $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

|A| = 9

Co-factor matrix = $\begin{bmatrix} 1 & 4 & -3 \\ -6 & 3 & 0 \\ 2 & -1 & 3 \end{bmatrix}$

A^{-1} = $\begin{bmatrix} {}^{1}/_{9} & {}^{-6}/_{9} & {}^{2}/_{9} \\ {}^{4}/_{9} & {}^{3}/_{9} & {}^{-1}/_{9} \\ {}^{-3}/_{9} & 0 & {}^{3}/_{9} \end{bmatrix}$