Bases Example 3

- Determine if the set is bases for R^2 or R^3.

$$\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix}$$

Let A be the matrix whose columns are the vectors given.

$$A = \begin{bmatrix} 0 & 5 & 6 \\ 1 & -7 & 3 \\ -2 & 4 & 5 \end{bmatrix}$$

Row reduce the matrix A to get

Bases Example 3

Interchange R_2 and R_1

$$\begin{bmatrix} 1 & -7 & 3 \\ 0 & 5 & 6 \\ -2 & 4 & 5 \end{bmatrix}$$

Replace R_3 by $R_3 + 2\,R_1$

$$\begin{bmatrix} 1 & -7 & 3 \\ 0 & 5 & 6 \\ 0 & -10 & 11 \end{bmatrix}$$

Bases Example 3

Replace R_3 by $R_3 + 2\,R_2$

$$\begin{bmatrix} 1 & -7 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 23 \end{bmatrix}$$

The matrix A has three pivots, so A is invertible by the Invertible Matrix Theorem and its columns form a basis for R^3

Bases Example 4

Given matrix A and an echelon form of A.

a. Find a basis for Col A and

b. a basis for Nul A.

$$A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Bases Example 4

a. Basis for Col A

$$\begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The echelon form identifies columns 1 and 2 as the pivot columns.

A basis for Col A uses columns 1 and 2 of A: $\begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$

Bases Example 4

b. Basis for Nul A

For Nul A, obtain the reduced (and augmented) echelon form for Ax = 0

$$\begin{bmatrix} 1 & 2 & 6 & -5 & 0 \\ 0 & 1 & 5 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Replace R_1 by $R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -4 & 7 & 0 \\ 0 & 1 & 5 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Bases Example 4

b. Basis for Nul A

$$\begin{bmatrix} 1 & 0 & -4 & 7 & 0 \\ 0 & 1 & 5 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \text{ This corresponds to: } \quad \begin{aligned} x_1 \quad - 4x_3 + 7x_4 &= 0 \\ x_2 + 5x_3 - 6x_4 &= 0 \\ 0 &= 0 \end{aligned}.$$

Solve for the basic variables and write the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4x_3 - 7x_4 \\ -5x_3 + 6x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 6 \\ 0 \\ 1 \end{bmatrix}. \text{ Basis for Nul } A\text{: } \begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

Bases Example 5

Given matrix A and an echelon form of A.

a. Find a basis for Col A and

b. a basis for Nul A.

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Bases Example 5

a. Basis for Col A

$$\begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The echelon form identifies columns 1 and 3 as the pivot columns.

A basis for Col A uses columns 1 and 3 of A: $\begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}.$

Bases Example 5

b. Basis for Nul A

For Nul A, obtain the reduced (and augmented) echelon form for Ax = 0

$$\begin{bmatrix} 1 & -3 & 6 & 9 & 0 \\ 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Replace R_1 by R_1 – $(\frac{6}{4})R_2$,

$$\begin{bmatrix} 1 & -3 & 0 & 1.5 & 0 \\ 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Bases Example 5

b. Basis for Nul A

For Nul A, obtain the reduced (and augmented) echelon form for Ax = 0

Replace R_2 by $(\frac{1}{4})R_2$,

$$\begin{bmatrix} 1 & -3 & 0 & 1.5 & 0 \\ 0 & 0 & 1 & 1.25 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Bases Example 5

b. Basis for Nul A

For Nul A, obtain the reduced (and augmented) echelon form for $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & -3 & 0 & 1.50 & 0 \\ 0 & 0 & 1 & 1.25 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \text{ This corresponds to: } \begin{aligned} x_1 - 3x_2 \quad + 1.50x_4 &= 0 \\ x_3 + 1.25x_4 &= 0 \\ 0 &= 0 \end{aligned}.$$

Bases Example 5

b. Basis for Nul A

Solve for the basic variables and write the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - 1.5x_4 \\ x_2 \\ -1.25x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1.5 \\ 0 \\ -1.25 \\ 1 \end{bmatrix}. \text{ Basis for Nul } A\text{: } \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1.5 \\ 0 \\ -1.25 \\ 1 \end{bmatrix}.$$

Bases Example 6

Given matrix A and an echelon form of A.

a. Find a basis for Col A and

b. a basis for Nul A.

$$A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Bases Example 6

a. Find a basis for Col A

$$\begin{bmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \text{ Basis for Col } A: \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 5 \\ -5 \end{bmatrix}.$$

Bases Example 6

b. Find a basis for Nul A

For Nul A, obtain the reduced (and augmented) echelon form for $A\mathbf{x} = \mathbf{0}$:

$$[A \quad \mathbf{0}] \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 7 & 0 \\ 0 & 1 & 2.5 & 0 & -.5 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \begin{aligned} x_1 \quad - \quad 2x_3 \quad + \; 7x_5 &= 0 \\ x_2 + 2.5x_3 \quad - \; .5x_5 &= 0 \\ x_4 + \; 4x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

Bases Example 6

b. Find a basis for Nul A

The solution of $A\mathbf{x} = 0$ in parametric vector form : $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - 7x_5 \\ -2.5x_3 + .5x_5 \\ x_3 \\ -4x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -2.5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ .5 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$

Basis for Nul A: $\begin{bmatrix} 2 \\ -2.5 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -7 \\ .5 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$

Basis for a Subspace 7

Example Find the basis for the null space of the matrix

- $A = \begin{bmatrix} 3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$
- First, write the solution of A**x** = **0** in parametric vector form:
- $A \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{aligned} x_1 - 2x_2 \qquad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

- The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, x_3= -2x_4 + 2x_5, with x_2, x_4, and x_5 free

Basis for a Subspace

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underset{\substack{\uparrow \\ \mathbf{u}}}{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + x_4 \underset{\substack{\uparrow \\ \mathbf{v}}}{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}} + x_5 \underset{\substack{\uparrow \\ \mathbf{w}}}{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$$

$$= x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$$

Nul A coincides with the set of all linear combinations of **u**, **v**, and **w**. That is, {**u**, **v**, **w**} generates Nul A. In fact, this construction of **u**, **v**, and **w** automatically makes them linearly independent,

0 = x_2 **u** + x_4 **v** + x_5 **w** only if the weights x_2, x_4, and x_5 are all zero. So {**u**, **v**, **w**} is a *basis* for Nul A.