Set Theory.

Set : The set is well-defined collection of unique elements.

e.g.

$A = \{1, 2, ---, N\}$ → Rooster /Tabular form

$A = \{x : x$ is a Natural Number$\}$ → Builder form

A = set of all natural number → Statement form

Equal set: When two sets have the same elements, they are equal sets.

Equivalent set: If two sets have same no. of elements they are known as equivalent set.

equal set can be equivalent set
but
equivalent set cannot be equal set

Subset : Given two sets A and B, if each element of A is also an element of B, A is called subset of B and is denoted as $A \subseteq B$.

→ $Z = \{\{1\}, \{2\}, \{3\}\}$ → collection of set is also known as set.

→ $N \subset Z \subset R \subset C$.

example:
$Q = \{x : x \text{ is real no. } \notin x^2+1=0\}$
$Q = \{\ \} = \phi$

example:
$Q = \{x : x \text{ is a imaginary } \notin x^2+1=0\}$
$x^2 = -1$
$x = \sqrt{-1} = i$
$\therefore x^2+1=0$
$Q = \{x = i\}$

Union of sets: The union of sets A and B is a set of all the elements that are either in Set A or in Set B.

Intersection of sets: The intersection of sets A and B is set containing all the elements that are common to set A and set B.

Disjoint sets: Which have no elements in common are known as disjoint sets.

Overlapping sets: The set which have common element, are known as overlapping, joint or intersecting sets.

→ $A - B \neq B - A$

Complement of set: All elements in the universal set that are not elements of Set A, is known as complement of set.

Cartesian Product of sets: It set of all ordered pairs
$A \times B \neq B \times A$.
e.g.
$A = \{2, 5, 15, 20\}$ $B = \{3, 7, 9\}$

$A \times B = \{(2,3), (2,7), (2,9), (5,3), (5,7), (5,9), ------\}$

Cardinal number: The cardinal number of a set is the number of element present.
e.g.
$A = \{x : x \text{ is even from } 1 \text{ to } 10\}$. Find cardinal number?

$A = \{2, 4, 6, 8, 10\} = 5$

Properties of Set Operation:

1) Union:
a) $A \cup B = B \cup A$ [Commutative]
b) $(A \cup B) \cup C = A \cup (B \cup C)$ [Associative]
c) $A \cup \phi = A$ [Law of identity]
d) $A \cup A = A$ [Idempotent Law]
e) $A \cup (B \cup C) = (A \cup B) \cap (A \cap C)$

2) Intersection:
a) $A \cap B = B \cap A$
b) $(A \cap B) \cap C = A \cap (B \cap C)$
c) $A \cap \phi = \cancel{A}$ ϕ
d) $A \cap A = A$
e) $A \cap (A \cap C) = (A \cap B) \cap (A \cap C)$

Imp. Prop.
$(A')' = A$
$U' = \phi$
$\phi' = U$

De-Morgan's Law:

ⓐ $(A \cup B)' = A' \cap B'$

ⓑ $(A \cap B)' = A' \cup B'$

example:

$U = \{1, 3, 5, 7, 9\}$

$A = \{3, 5\}$ $B = \{5, 7, 9\}$

Prove: $(A \cup B)' = A' \cap B'$

$A \cup B = \{3, 5, 7, 9\}$ $A' = \{1, 7, 9\}$ $B' = \{1, 3, 9\}$

L.H.S $= (A \cup B)' = \{1\}$ R.H.S $= A' \cap B' = \{1\}$

$\therefore$ L.H.S = R.H.S.

Cardinal Property:

ⓐ $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

ⓑ $n(A - B) = n(A) - n(A \cap B)$

ⓒ $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$
$- n(A \cap C) + n(A \cap B \cap C)$

ⓓ $n(B - A) = n(B) - n(A \cap B)$

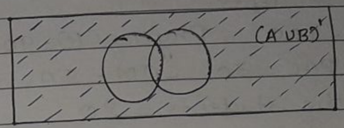
ⓔ $n(A') = n(U) - n(A)$

ⓕ $n(A \cup B)' = n(U) - n(A \cup B)$

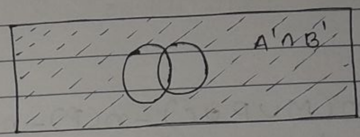
ⓖ $n(A \cap B)' = n(U) - n(A \cap B)$

ⓗ $n(A - B)' = n(U) - n(A - B)$

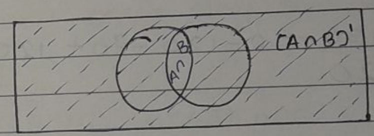
ⓐ $(A \cup B)'$



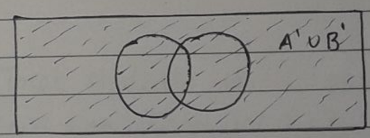
ⓑ $A' \cap B'$



ⓒ $(A \cap B)'$



ⓓ $A' \cup B'$



example:

$U = 200$ students

Maths(M) = 120 Physics(P) = 90 Chemistry(C) = 70

$M \cap P = 40$ $P \cap C = 30$ $C \cap M = 50$

None of the students = 20

subjects study by

Find $n(M \cap P \cap C)$?

Solution:

$$n(M \cap P \cap C) = n(M \cup P \cup C) - n(P) - n(M) - n(C) + n(P \cap M) + n(P \cap C) + n(C \cap M)$$

$$= (200 - 20) - 90 - 120 - 70 + 40 + 30 + 50$$

$$= 180 - 90 - 120 - 70 + 120$$

$$= 20$$

example:

Here x and y are finite set.

$x = 30$ $y = 41$

$x \cap y = 15$. Find $x \cup y$?

Solution:

$$n(x \cup y) = n(x) + n(y) - n(x \cap y)$$

$$= 30 + 41 - 15$$

$$= 56$$

example:

Basketball (B) = 70, Football (F) = 20

10 students play both game i.e. $B \cap F = 10$.

How many people play one of two game?

Solution:

$$n(B \cup F) = n(B) + n(F) - n(B \cap F)$$

$$= 70 + 20 - 10$$

$$= 80$$

example:

Total students $U = 300$, Hindi(H) = 150,

Gujarati(G) = 200, Find $H \cap G$ = ?

Solution: $300 = 150 + 200 - x$

$$x = 150 + 200 - 300$$

$$= 350 - 300$$

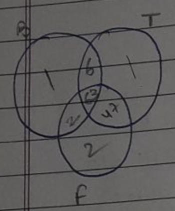
$$x = 50$$

$$\therefore H \cap G = 50$$

example:

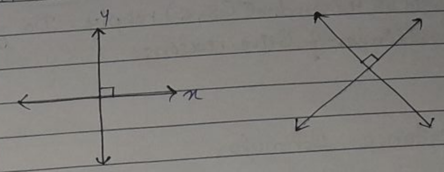
$U = 122$, $n(B) = 72$, $n(T) = 117$, $n(F) = 114$,
$n(B \cap T) = 69$, $n(F \cap T) = 110$, $n(F \cap B) = 65$
$n(B \cap T \cap F) = 63$

How many player play only one sport? 4



Coordinate Geometry

- It is used to represent object in plane.



This both representation are correct.

$x \to$ abscissa $\quad y \to$ ordinate

Distance Formula:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- In particular, the distance of a point $P(x, y)$ from the origin $O(0,0)$ is given by,

$$OP = \sqrt{x^2 + y^2}$$

- The sum of any two of these distane is greater than the third distance, therefore the points P, Q & R form a triangle.

- If $AB = BC = CD = DA$ and $AC = BD$, all four sides of the quadrilateral ABCD are equal & its diagonal AC & BD are also equal. Therefore, ABCD is a square.

- If $AB + BC = AC$, then we say that A, B and C are collinear. Therefore, they are seated in a line.

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example:

A(3,1), B(6,4) and C(8,6) resply. Do you think they are collinear? Give reasons.

solution:

Using distance formula,

$AB = \sqrt{(6-3)^2 + (4-1)^2} = \sqrt{(3)^2 + (3)^2}$
$= \sqrt{9+9} = \sqrt{18} = \sqrt{2\times 9} = 3\sqrt{2}$

$BC = \sqrt{(8-6)^2 + (6-4)^2} = \sqrt{(2)^2 + (2)^2} = \sqrt{8} = \sqrt{2\times 4}$
$= 2\sqrt{2}$

$CA = \sqrt{(8-3)^2 + (6-1)^2} = \sqrt{(5)^2 + (5)^2} = \sqrt{50}$
$= \sqrt{25\times 2}$
$= 5\sqrt{2}$

$\therefore AB + BC = 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2} = CA$

$\therefore$ A, B & C are collinear.

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example:

Find a relation b/w x & y such that the point (x,y) is equidistant from the points (7,1) and (3,5).

solution:

Let $P(x,y)$ be equidistant from the points $A(7,1)$ and $B(3,5)$.

Given that,

$AP = BP$

So, $AP^2 = BP^2$

$(x-7)^2 + (y-1)^2 = (x-3)^2 + (y-5)^2$

$\cancel{x^2} - 14x + 49 + \cancel{y^2} - 2y + 1 = \cancel{x^2} - 6x + 9 + \cancel{y^2} - 10y + 25$

$-14x + 6x - 2y + 10y = 9 + 25 - 1 - 49$

$-8x + 8y = -16$

$-8(x-y) = -16$

$\therefore x - y = 2$

$\therefore x - y = 2$ is a line and graph of $x - y = 2$ is the perpendicular bisector of AB.

example: Find a point on the y-axis which is equidistant from the points $A(6,5)$ and $B(-4,3)$.

solution: The point on y-axis is of the form $(0,y)$.

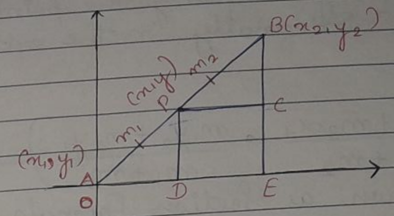
So,
Let the point $P(0,y)$ be equidistant from A and B.

$$(6-0)^2+(5-y)^2 = (-4-0)^2+(3-y)^2$$
$$36+25+\cancel{y^2}-10y = 16+9+\cancel{y^2}-6y$$
$$4y = 36$$
$$\therefore y = 9$$

So, required point is $(0,9)$

we can check by putting the value $y=9$ in above equ^n.

Section Formula:



$$\frac{AP}{PB} = \frac{AD}{PC} = \frac{PD}{BC}$$

$$\frac{m_1}{m_2} = \frac{x-x_1}{x_2-x} = \frac{y-y_1}{y_2-y}$$

Taking
$$\frac{m_1}{m_2} = \frac{x-x_1}{x_2-x}$$
we get,
$$x = \frac{m_1x_2+m_2x_1}{m_1+m_2}$$

Taking
$$\frac{m_1}{m_2} = \frac{y-y_1}{y_2-y}$$
we get,
$$y = \frac{m_1y_2+m_2y_1}{m_1+m_2}$$

- The coordinates of the point $P(x, y)$ which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally, in the ratio $m_1 : m_2$ are

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} \right)$$

This is known as Section Formula.

- If the ratio in which P divides AB is $k:1$, then the coordinate of the point P will be,

$$\left(\frac{k x_2 + x_1}{k+1}, \frac{k y_2 + y_1}{k+1} \right)$$

Special case:

If midpoint of a line-segment divides the line-segement in the ratio $1:1$.

$\therefore$ The coordinates of the midpoint P of the join of the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is,

$$\left(\frac{1 \cdot x_1 + 1 \cdot x_2}{1+1}, \frac{1 \cdot y_1 + 1 \cdot y_2}{1+1} \right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

example:

In what ratio does the point $(-4, 6)$ divide the line segment joining the points $A(-6, 10)$ & $B(3, -8)$?

Solution:

$P(-4, 6) = (x, y)$ $A(-6, 10) = (x_1, y_1)$

$B(3, -8) = (x_2, y_2)$

$$\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x}$$

$$= \frac{-4 + 6}{3 + 4}$$

$$\therefore \frac{m_1}{m_2} = \frac{2}{7} \qquad \therefore m_1 : m_2 = 2 : 7.$$

example:

Find the ratio in which y-axis divides the line-segment joining $(5, -6)$ & $(-1, -4)$.

solution:

Given that,

y-axis divides the line-segment $= (0, y) = (x, y)$

$A(5, -6) = (x_1, y_1)$ $B(-1, -4) = (x_2, y_2)$

$$\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x} = \frac{0 - 5}{-1 - 0} = \frac{-5}{-1} = 5$$

$\therefore \frac{m_1}{m_2} = 5$ i.e. $5:1$

Another method:

Let the ratio be $k:1$.

Then,

By section formula,

$$\left(\frac{-k+5}{k+1}, \frac{-4k-6}{k+1}\right)$$

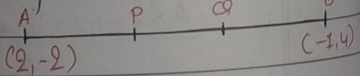
This point lies on y-axis, we know that on the y-axis the abscissa (x) is 0.

$$\therefore \frac{-k+5}{k+1} = 0$$

$\therefore k = 5$. i.e. $5:1$

example: Find the coordinates of the points of trisection (i.e. points dividing in three equal parts) of the line segment joining the points $A(2,-2)$ & $B(-7,4)$.

Solution: Let P & Q be the points of trisection of AB i.e. $AP = PQ = QB$



$\therefore$ P divides AB in ratios $1:2$

$$\therefore \left(\frac{1(-7)+2(2)}{1+2}, \frac{1(4)+2(-2)}{1+2}\right)$$

$$\therefore \left(\frac{-7+4}{3}, \frac{4-4}{3}\right) \text{ i.e. } (-1, 0)$$

Then,

Q divides AB in ratio $2:1$,

$$\therefore \left(\frac{2(-7)+1(2)}{2+1}, \frac{2(4)+1(-2)}{2+1}\right)$$

$$\therefore \left(\frac{-14+2}{3}, \frac{8-2}{3}\right) \text{ i.e. } (-4, 2)$$

$\therefore$ The coordinate of point are $(-1,0)$ & $(-4,2)$

example: If the points $A(6,1)$, $B(8,2)$, $C(9,4)$ and $D(p,3)$ are the vertices of a parallelogram, taken in order, find the value of p.

Solution: We know diagonal of parallelogram bisect each other.

coordinate of midpoint of AC = coordinate of the midpoint of BD.

$$\left(\frac{6+9}{2}, \frac{1+4}{2}\right) = \left(\frac{8+p}{2}, \frac{2+3}{2}\right)$$

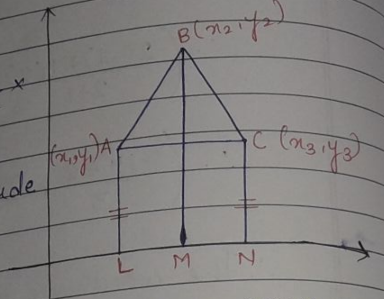
$$\left(\frac{15}{2}, \frac{5}{2}\right) = \left(\frac{8+p}{2}, \frac{5}{2}\right)$$

$$\frac{15}{2} = \frac{8+p}{2}$$

$p = 15 - 8 \quad \therefore p = 7$

Area of triangle:

Area of triangle $= \frac{1}{2} \times$ base $\times$ altitude



Area of ΔABC = Area of trapezium ABML + Area of trapezium BCNM − Area of trapezium ACNL

Area of trapezium $= \frac{1}{2}$ (sum of parallel sides) (distance between them)

$$\text{Area of } \Delta ABC = \frac{1}{2}(AL+BM)LM + \frac{1}{2}(CN+BM)MN - \frac{1}{2}(AL+CN)LN$$

$$= \frac{1}{2}(y_1+y_2)(x_2+x_1) + \frac{1}{2}(y_2+y_3)(x_2+x_3) - \frac{1}{2}(y_1+y_3)(x_3-x_1)$$

$$\therefore \text{Area of } \Delta ABC = \frac{1}{2}\left[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)\right]$$

example: Find the area of triangle A(5,2) B(4,7) and C(7,−4).

Solution: $A(5,2) = (x_1, y_1)$ $B(4,7) = (x_2, y_2)$ $C(7,-4) = (x_3, y_3)$.

$$\text{Area of } \Delta ABC = \frac{1}{2}\left[5(7+4) + 4(-4-2) + 7(2-7)\right]$$

$$= \frac{1}{2}\left[55 - 24 - 35\right] = \frac{-4}{2} = -2$$

Area cannot be negative i.e. 2 square units.

example: Find the area of triangle by point P(−1.5, 3), Q(6, −2) & R(−3, 4)

Solution:

$$\text{Area of triangle} = \frac{1}{2}\left[-1.5(-2-4) + 6(4-3) + (-3)(3+2)\right]$$

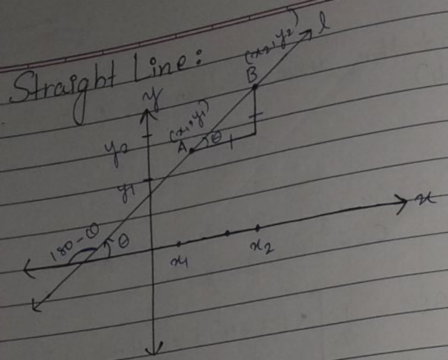
$$= \frac{1}{2}\left[-1.5(-6) + 6(1) - 3(5)\right]$$

$$= \frac{1}{2}\left[9 + 6 - 15\right] = \frac{1}{2}\left[15 - 15\right] = 0$$

i.e. zero square units

→ Area of triangle is zero square units, then its vertices will be collinear.

Straight Line:



- The angle θ made by the line l with positive direction of x-axis and measured anti-clockwise is called the inclination of the line.

$$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

- $0^\circ \leq \theta \leq 180^\circ$

- Inclination of line:
  $\tan\theta = \text{slope}$

- $\tan 90^\circ \Rightarrow$ slope = not defined
  $\tan 0^\circ \Rightarrow$ slope = 0

- Slope represent rate of change of function, when two line are parallel lines then slope is equal.

- Slope represent rate of change of function, when two line are perpendicular then slope is $m_1 m_2 = -1$

Example:

Find the slope of the lines:

(a) (3,-2) & (-1,4) (b) (3,-2) & (7,-2)

(c) (3,-2) & (3,4) (d) $\theta = 60^\circ$ i.e. making inclination of 60° with the positive direction of x-axis.

Solution:

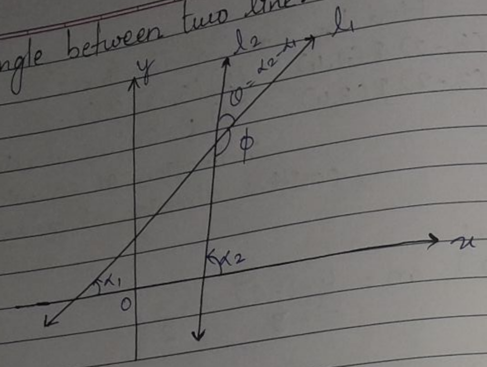
(a) $m = \frac{4-(-2)}{-1-3} = \frac{-3}{2}$

(b) $m = \frac{-2-(-2)}{7-3} = \frac{0}{4} = 0$

(c) $m = \frac{4-(-2)}{3-3} = \frac{6}{0}$, which is not defined.

(d) $\theta = 60^\circ$
$m = \tan\theta = \tan 60^\circ = \sqrt{3}$

Angle between two line:



$\theta = \alpha_2 - \alpha_1$

$$\tan\theta = \tan(\alpha_2 - \alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2}$$

Tangent of angle

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

Angle

$$\theta = \tan^{-1}\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)$$

example:

If the angle between two lines is $\pi/4$ and slope of one of the lines is $\frac{1}{2}$. Find the slope of the other line.

Solution:

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

$m_1 = 1/2 \quad m_2 = m \quad \theta = \pi/4$

$$\tan\frac{\pi}{4} = \left|\frac{m - 1/2}{1 + 1/2\, m}\right| \quad \text{or} \quad 1 = \left|\frac{m - 1/2}{1 + 1/2\, m}\right|$$

which gives

$$\frac{m - 1/2}{1 + 1/2\, m} = 1 \quad \text{or} \quad \frac{m - 1/2}{1 + 1/2\, m} = -1$$

$\therefore m = 3$ or $m = -1/3$

$$\frac{2m - 1/\cancel{2}}{2 + m/\cancel{2}} = 1$$

$$\frac{2m - 1}{2 + m} = 1$$

$2m - 1 = 2 + m$

$2m - m = 2 + 1$

$m = 3$

example:
Line through the points (-2, 6) and (4, 8) is perpendicular to the line through the points (8, 12) and (x, 24). Find the value of x.

solution:
Slope of the line through the points (-2, 6) and (4, 8) is

$$m_1 = \frac{8-6}{4-(-2)} = \frac{2}{6} = \frac{1}{3}$$

Slope of the line through the points (8, 12) and (x, 24) is

$$m_2 = \frac{24-12}{x-8} = \frac{12}{x-8}$$

Since,
two lines are perpendicular
$m_1 m_2 = -1$

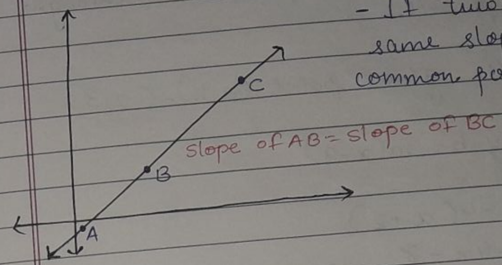
$$\frac{1}{\cancel{3}} \times \frac{\overset{4}{\cancel{12}}}{x-8} = -1$$

$$4 = -x + 8$$

$$4 - 8 = -x$$

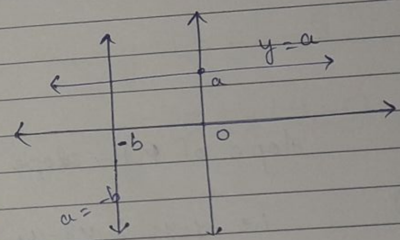
$$x = 4$$

Collinearity of three points:



- If two lines having the same slope pass through a common point, then two lines will coincide.

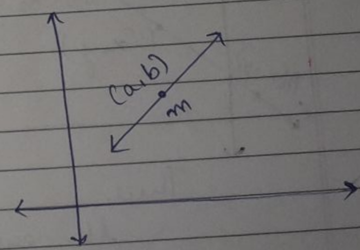
Horizontal and vertical lines:



point - slope form:

$$m = \frac{y-b}{x-a}$$

$$y - b = m(x-a)$$



example: Find the equation of the line through $(-2,3)$ with slope -4.

solution: $m = -4$ given point $(a,b) = (-2,3)$

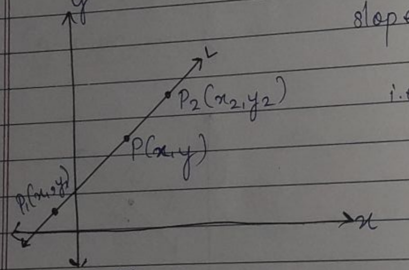
formula : $y - b = m(x-a)$

$y - 3 = -4(x+2)$

$y - 3 = -4x - 8$

$4x + y + 5 = 0$

Two-point form:



slope of P_1P = slope of P_1P_2

i.e. $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

Thus,

the equation of the line passing through the points (x_1, y_1) and (x_2, y_2)

$\therefore y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1)$

example: Write the equation of the line through the points $(1,-1)$ and $(3,5)$.

Solution: $x_1 = 1, y_1 = -1, x_2 = 3$ and $y_2 = 5$

$y - (-1) = \frac{5-(-1)}{3-1}(x-1)$

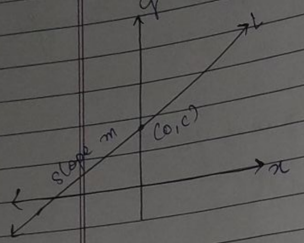
$y + 1 = \frac{5+1}{2}(x-1)$

$y + 1 = \frac{\cancel{6}^{3}}{\cancel{2}}(x-1)$

$y + 1 = 3x - 3$

$\therefore -3x + y + 4 = 0$

Slope-intercept form:



$$\frac{y-c}{x} = m$$

$$y - c = mx$$

$$y = mx + c$$

example: Write the equation of the line for which $\tan\theta = 1/2$, where θ is the inclination of the line and (i) y-intercept is $-3/2$

Solution:

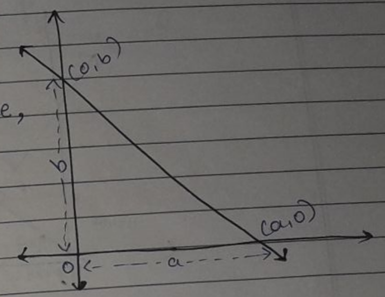
$m = \tan\theta = \frac{1}{2}$ y-intercept $c = -3/2$

$$y = mx + c$$

$$= \frac{1}{2}x - \frac{3}{2}$$

$\therefore 2y - x + 3 = 0$

Intercept-form:



By two-point form of the equ^n of the line,

$$y - b = \frac{b-0}{0-a}(x-a)$$

$$y - b = \frac{-b}{a}(x)$$

$$ay = ba - bx$$

$$ay = b(a-x)$$

$$\boxed{\frac{y}{b} + \frac{x}{a} = 1}$$

example: Find equ^n of line: -3 and 2 on x and y axes resply.

solution:

$a = -3 \quad b = 2$

$$\frac{x}{a} + \frac{y}{b} = 1$$

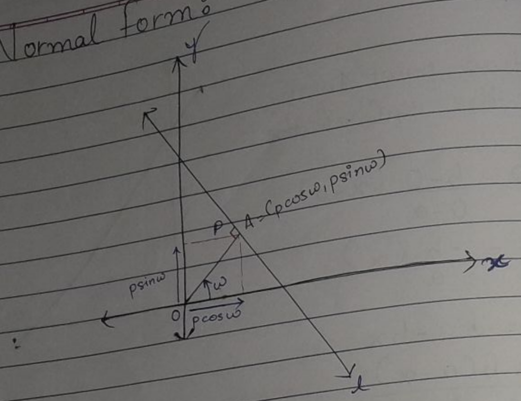
$$\frac{x}{-3} + \frac{y}{2} = 1$$

$$\frac{2x \cancel{+} -3y}{-6} = 1$$

$$2x - 3y = -6$$

$\therefore 2x - 3y + 6 = 0$

Normal form:



$$\text{slope of } OA = \frac{p\sin\omega}{p\cos\omega} = \tan\omega$$

$$\text{slope of line } L = \frac{-1}{\text{slope of } OA}$$

$$= \frac{-1}{\tan\omega}$$

$$= \frac{-\cos\omega}{\sin\omega}$$

$$\boxed{y - p\sin\omega = \frac{-\cos\omega}{\sin\omega}(x - p\cos\omega)}$$

$\therefore$ equation of line:

$$x\cos\omega + y\sin\omega = p$$

example: equ^n of a line is $3x-4y+10=0$. Find its (i) slope (ii) x- and y-intercept.

Solution:

(i) $3x - 4y + 10 = 0$

$$-4y = -3x - 10$$

$$y = \frac{-3x}{-4} - \frac{10}{-4}$$

$$y = \frac{3x}{4} + \frac{5}{2} \quad \text{i.e. } y = mx + c$$

$$m = \frac{3}{4} \quad \text{i.e. slope}$$

(ii) $3x - 4y + 10 = 0$

$$3x - 4y = -10$$

$$\therefore \frac{3x}{-10} + \frac{4y}{10} = 1 \qquad [\,x/a + y/b = 1\,]$$

$$\therefore \frac{x}{-10/3} + \frac{y}{10/4} = 1$$

$$\therefore \frac{x}{-10/3} + \frac{y}{5/2} = 1$$

x-intercept : $-10/3$

y-intercept : $5/2$

example: Find the angle between the lines $y-\sqrt{3}x-5=0$ and $\sqrt{3}y-x+6=0$.

solution:

$y-\sqrt{3}x-5=0$

$\therefore y=\sqrt{3}x+5$

$m_1=\sqrt{3}$

$\sqrt{3}y-x+6=0$

$\therefore \sqrt{3}y=x-6$

$y=\frac{x}{\sqrt{3}}-\frac{6}{\sqrt{3}}$

$y=\frac{x}{\sqrt{3}}-2\sqrt{3}$

$m_2=1/\sqrt{3}$

$$\tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right|$$

$$=\left|\frac{1/\sqrt{3}-\sqrt{3}}{1+\sqrt{3}(1/\sqrt{3})}\right|=\left|\frac{1-3}{2\sqrt{3}}\right|=\left|\frac{-2}{2\sqrt{3}}\right|$$

$\tan\theta=1/\sqrt{3}$

$\tan\frac{1}{\sqrt{3}}=\tan 30^\circ$

Hence, angle the two line between $\theta=30^\circ$

or $180^\circ-30^\circ=150^\circ$

example: Find the equation of a line perpendicular to the line $x-2y+3=0$ and passing through the point $(1,-2)$.

solution: Given line $x-2y+3=0$

$y=\frac{1}{2}x+\frac{3}{2}$

$m_1=\frac{1}{2}$ $\therefore m_2=-1/m_1=-2$

equ^n of line with slope -2 and passing through the point $(1,-2)$ is —

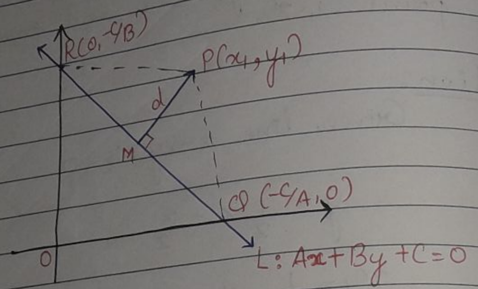
$y-(-2)=-2(x-1)$

or

$y+\cancel{2}=-2x+\cancel{2}$

$y=-2x$

Distance of a Point from a Line:



$$\text{area}(\Delta PQR) = \frac{1}{2} PM \cdot QR$$

$$PM = \frac{2\,\text{area}(\Delta PQR)}{QR} \quad \text{—— (1)}$$

$$\text{Area}(\Delta PQR) = \frac{1}{2}\left| x_1\left(0 + \frac{C}{B}\right) + \left(\frac{-C}{A}\right)\left(\frac{-C}{B} - y_1\right) + 0\,(y_1 - 0)\right|$$

$$= \frac{1}{2}\left| x_1 \frac{C}{B} + y_1 \frac{C}{A} + \frac{C^2}{AB} \right|$$

or

$$2\,\text{area}(\Delta PQR) = \left|\frac{C}{AB}\right| \cdot |Ax_1 + By_1 + C|$$

and

$$QR = \sqrt{\left(0 + \frac{C}{A}\right)^2 + \left(\frac{C}{B} - 0\right)^2} = \left|\frac{C}{AB}\right| \sqrt{A^2 + B^2}$$

Substituting the values of area (ΔPQR) and QR in (1),

$$PM = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

or

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Intercept of $Ax + By + C = 0$

for x intercept keep $y = 0$

$$Ax + 0 + C = 0$$

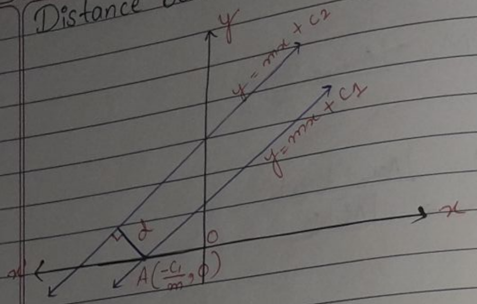
$$x = \frac{-C}{A}$$

for y intercept keep $x = 0$

$$0 + By + C = 0$$

$$y = \frac{-C}{B}$$

Distance between two parallel lines:



$y = mx + c$
$0 = mx + c$
$x = -\frac{c}{m}$

Two parallel lines can be,

$y = mx + c_1$ —— (1)
$y = mx + c_2$ —— (2)

Line (1) will intersect x-axis at the point

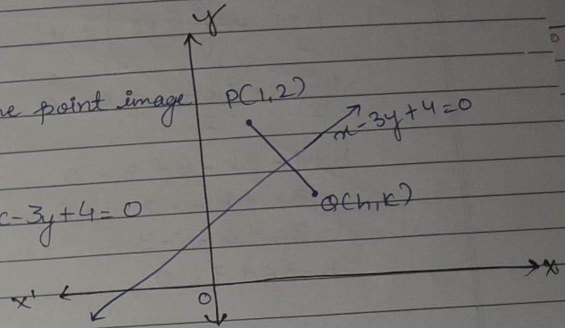
$A\left(\frac{-c_1}{m}, 0\right)$

$$d = \frac{\left|\left(\frac{-c_1}{\cancel{m}}\right)(-\cancel{m}) + (-c_2) + 1(0)\right|}{\sqrt{1+m^2}}$$

$$d = \frac{|c_1 - c_2|}{\sqrt{1+m^2}}$$

example: Assuming that straight lines work as the plane mirror for a point, find the image of the point (1,2) in the line $x-3y+4=0$.

Solution:



Let Q(h,k) be the point image P(1,2).

Slope of line $x-3y+4=0$

$y = mx + c$

$-3y = m(-x) - 4$

$-3y = -1x - 4$

$y = \frac{1}{3}x + \frac{4}{3}$ $\quad \therefore m = 1/3$

$(m_1 \cdot m_2 = -1)$ Slope of PQ $= \frac{-1}{\text{Slope of line } x-3y+4=0}$

$= \frac{-1}{1/3} = -3$

Slope of ⊥er line,

$\frac{k-2}{h-1} = \frac{-1}{1/3}$

$\frac{k-2}{h-1} = -3$

$k-2 = -3h+3$

$3h+k-5=0$

midpoint of PQ,

$\left(\frac{h+1}{2}, \frac{k+2}{2}\right)$ put in $x-3y+4=0$

$\frac{h+1}{2} - 3\left(\frac{k+2}{2}\right) + 4 = 0$

$\frac{h+1}{2} - \frac{3k-6}{2} + 4 = 0$

$h+1-3k-6+8=0$

$-3k+h=-8-1+6$

$-3k+h=-3$

$h-3k+3=0$

$\therefore$

$3h+k-5=0$ —— $\times 1$

$h-3k+3=0$ —— $\times 3$

$\cancel{3h}-9k+9=0$

$\cancel{3h}+k-5=0$

$-\quad-\quad+$

$-10k+14=0$

$-10k=-14$

$k = \frac{-\cancel{14}\,2\times7}{-\cancel{10}\,2\times5}$

$\therefore k = 7/5$

$3h+\frac{7}{5}-5=0$

$15h+7-25=0$

$15h-18=0$

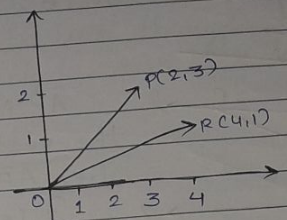
$h=\frac{18}{15}$

$\therefore h = 6/5$

$Q(6/5, 7/5)$

Vector

Vector:



$\vec{OP} = 2\hat{i}+3\hat{j}$

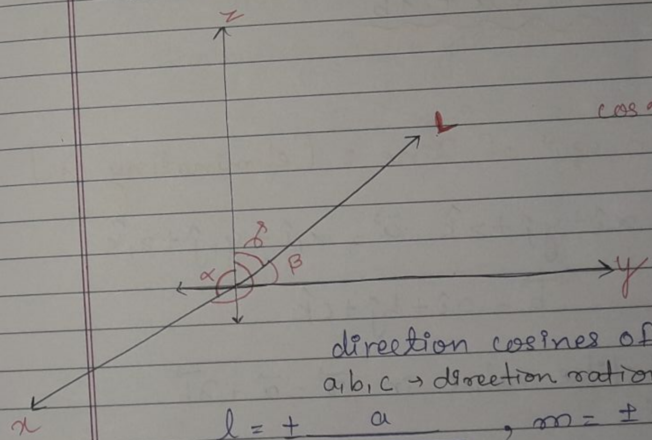
$\vec{OR} = 4\hat{i}+1\hat{j}$

$\vec{PR} = OR - OP$

$= (4\hat{i}+1\hat{j})-(2\hat{i}+3\hat{j})$

$= (2\hat{i}-2\hat{j})$

Direction Cosines and Direction ratio of a line:



$\cos\alpha, \cos\beta, \cos\gamma$

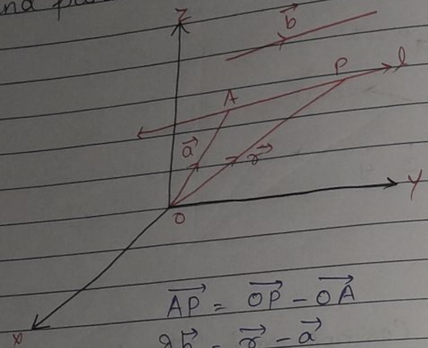
direction cosines of line:

$a,b,c \rightarrow$ direction ratios $\quad l,m,n \rightarrow$ direction cosines

$$l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}$$

$$n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Equation of a line through a given point and parallel to a given vector $\vec{b}$:



$\vec{AP}$ is ∥ to the vector $\vec{b}$

i.e. $\vec{AP} = \lambda \vec{b}$, λ is some real number.

$\vec{AP} = \vec{OP} - \vec{OA}$

$\lambda \vec{b} = \vec{r} - \vec{a}$

$\therefore \vec{r} = \vec{a} + \lambda \vec{b}$

Cartesian equ^n of line : [eliminating λ]

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$

Substitute value in $\vec{r} = \vec{a} + \lambda \vec{b}$

we get

$x = x_1 + \lambda a$, $y = y_1 + \lambda b$, $z = z_1 + \lambda c$

eliminating λ from above equ^n

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

example:

Find the vector and the cartesian equations of the line through the point (5, 2, -4) and which is parallel to the vector $3\hat{i} + 2\hat{j} - 8\hat{k}$.

solution:

$\vec{a} = 5\hat{i} + 2\hat{j} - 4\hat{k}$ $\vec{b} = 3\hat{i} + 2\hat{j} - 8\hat{k}$

vector equ^n of the line is,

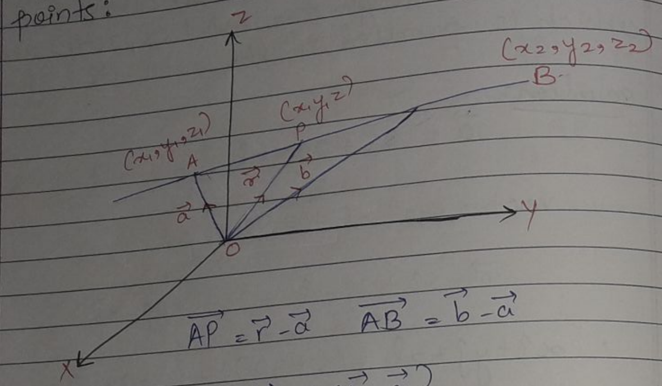
$\vec{r} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$

$x\hat{i} + y\hat{j} + z\hat{k} = (5 + 3\lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (-4 - 8\lambda)\hat{k}$

eliminating λ we get,

$$\frac{x - 5}{3} = \frac{y - 2}{2} = \frac{z + 4}{-8}$$

equation of line passing through two given points:



$\overrightarrow{AP} = \vec{r} - \vec{a} \quad \overrightarrow{AB} = \vec{b} - \vec{a}$

$\therefore \vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$

$\therefore \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}), \lambda \in R$

Derivation of cartesian form from vector form:

$x\hat{i} + y\hat{j} + z\hat{k} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} + \lambda[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}]$

$x = x_1 + \lambda(x_2 - x_1) \quad y = y_1 + \lambda(y_2 - y_1)$
$z = z_1 + \lambda(z_2 - z_1)$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Unit Vector:

- Vector with magnitude 1 is known as unit vector.

Dot product:

$$\vec{a} = \begin{bmatrix} \vec{a_1} \\ \vdots \\ \vec{a_n} \end{bmatrix} \quad \vec{b} = \begin{bmatrix} \vec{b_1} \\ \vdots \\ \vec{b_n} \end{bmatrix}$$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

magnitude of vector $||\vec{a}|| = \sqrt{a_1 \cdot a_1 + a_2 \cdot a_2 + \cdots + a_n \cdot a_n} = \sqrt{\vec{a} \cdot \vec{a}}$

- magnitude of vector is always positive.

Property:

1. Commutative:
$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

2. Distributive:
$(\vec{a} + \vec{b})\vec{c} = \vec{a}\vec{c} + \vec{b}\vec{c}$

3. Associative:
$(\lambda\vec{a})\vec{b} = \lambda(\vec{a} \cdot \vec{b})$

Schwarz Inequality:

$$|\vec{x}\cdot\vec{y}| \leq ||\vec{x}||\cdot||\vec{y}||$$

Proof:

$$f(t) = ||\vec{y}t - \vec{x}||^2 \geq 0$$

$$(\vec{y}t-\vec{x})\cdot(\vec{y}t-\vec{x}) \geq 0$$

$$\underbrace{\vec{y}\cdot\vec{y}}_{a}t^2 - \underbrace{2\vec{x}\cdot\vec{y}}_{b}t + \underbrace{\vec{x}\cdot\vec{x}}_{c} \geq 0$$

$$f\left(b/2a\right) = at^2 - bt + c \geq 0$$

$$a\left(\frac{b}{2a}\right)^2 - b\left(\frac{b}{2a}\right) + c \geq 0$$

$$\cancel{a}\frac{b^2}{4a^{\cancel{2}}} - \frac{2b^2}{4a} + c \geq 0$$

multiply & divide 2

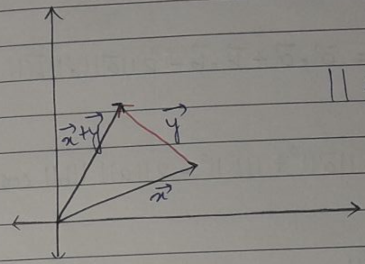
$$\therefore \frac{-b^2}{4a} + c \geq 0$$

$$\therefore b^2 \leq 4ac$$

$$(2(\vec{x}\cdot\vec{y}))^2 \leq 4(||\vec{y}||\cdot||\vec{x}||)^2$$

$$\cancel{4}(\vec{x}\cdot\vec{y})^2 \leq \cancel{4}(||\vec{y}||\cdot||\vec{x}||)^{\cancel{2}}$$

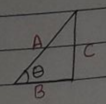
$$|\vec{x}\cdot\vec{y}| \leq ||\vec{y}||\cdot||\vec{x}||$$



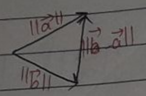
$$||\vec{x}+\vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$$

$$f(t) = ||\hat{y}t - \vec{x}||^2 \geq 0$$

Angle between two vectors:



$$C^2 = A^2 + B^2 + 2AB\cos\theta$$



$$(\vec{b}-\vec{a})^2 = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2\|\vec{a}\|\cdot\|\vec{b}\|\cos\theta$$

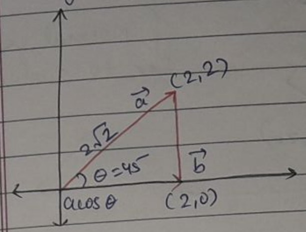
$$\therefore \|\vec{b}\|^2 + \|\vec{a}\|^2 - 2\vec{a}\cdot\vec{b} = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|a\|\,\|b\|\cos\theta$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}$$

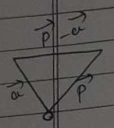
$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

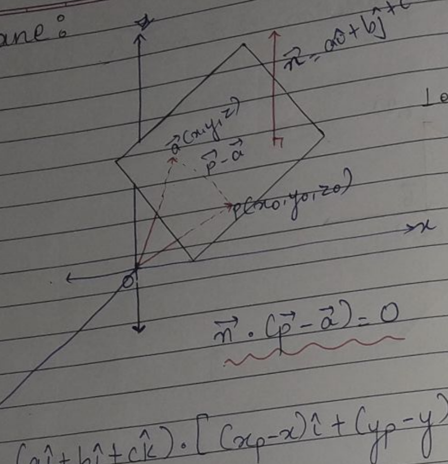
- If the line is ⊥er then dot product will be zero
- If the line are ⇈ then dot product will be 1.

(Length of projection of $\vec{a}$ on $\vec{b}$) x (Length of vector $\vec{b}$)



Plane:





Normal vector is $\perp$er to surface of plane

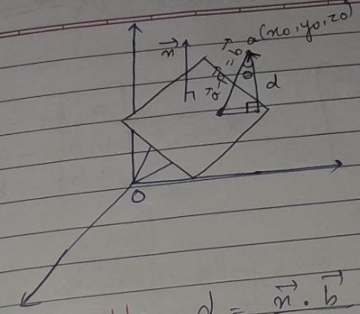
$$\vec{n}\cdot(\vec{p}-\vec{a})=0$$

$$(a\hat{i}+b\hat{j}+c\hat{k})\cdot[(x_p-x)\hat{i}+(y_p-y)\hat{j}+(z_p-z)\hat{k}]=0$$

$$ax_p\hat{i}+by_p\hat{j}+cz_p\hat{k}=ax+by+cz$$

$$ax_p\hat{i}+by_p\hat{j}+cz_p\hat{k}=D$$

$$\therefore Ax+By+Cz=D$$



$$\cos\theta=\frac{d}{|\vec{b}|}$$

$$d=|\vec{b}|\cos\theta$$

$$d=\frac{|\vec{n}||\vec{b}|\cos\theta}{|\vec{n}|}$$

distance b/w point on plane to point outside plane.

$$d=\frac{\vec{n}\cdot\vec{b}}{|n|}$$

$$\text{proj}(a,b)=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$ [normal and point are parallel]

Cosine angle between two plane $\vec{n_1}$ and $\vec{n_2}$

$$\cos\theta=\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} \qquad \vec{a_1}\cdot\vec{a_2}=|a_1||a_2|\cos\theta$$

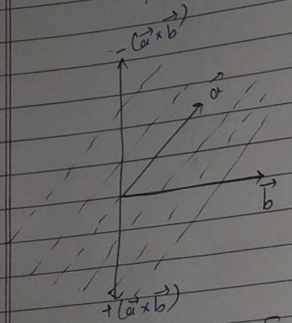
Angle between two plane:

$$\theta=\cos^{-1}\left(\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right)$$

Dot product : R^n

Cross product : R^3

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \vec{a} \times \vec{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ b_1 a_3 - a_1 b_3 \\ a_1 b_2 - b_1 a_2 \end{bmatrix}$$



cross product of two vector is $\perp$er to two vector.

$$\vec{a}(\vec{a} \times \vec{b}) = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ b_1 a_3 - a_1 b_3 \\ a_1 b_2 - b_1 a_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 (a_2 b_3 - a_3 b_2) \\ a_2 (b_1 a_3 - a_1 b_3) \\ a_3 (a_1 b_2 - b_1 a_2) \end{bmatrix}$$

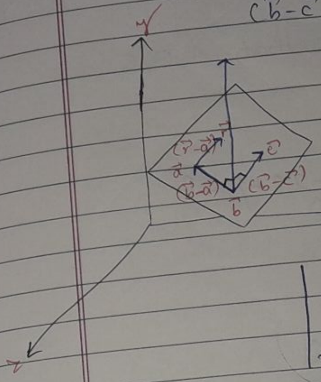
$$= \cancel{a_1 a_2 b_3} - \cancel{a_1 a_3 b_2} + \cancel{a_2 a_3 b_1} - \cancel{a_1 a_2 b_3} + \cancel{a_1 a_3 b_2} - \cancel{a_2 a_3 b_1}$$

$$\vec{a}(\vec{a} \times \vec{b}) = 0$$

Similarily,

$$\vec{b}(\vec{a} \times \vec{b}) = 0$$

Equation of plane with three distinct points:



$$(\vec{b} - \vec{c}) \times (\vec{b} - \vec{a})$$

$$\therefore (\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{b} - \vec{c})] = 0$$

$r = (x, y, z)$ $\quad \vec{a} = (x_1, y_1, z_1)$

$\vec{b} = (x_2, y_2, z_2)$ $\quad c = (x_3, y_3, z_3)$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_1 - x_2 & y_2 - y_1 & z_2 - z_1 \\ x_2 - x_3 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

example:

$A = 2\hat{i} - \hat{j} + \hat{k}$

$B = \hat{i} - 3\hat{j} - 5\hat{k}$

$C = 3\hat{i} - 4\hat{j} - 4\hat{k}$. Prove that A, B, C are right angled triangle.

solution:

$AB^2 = B - A$

$= \hat{i} - 3\hat{j} - 5\hat{k} - 2\hat{i} + \hat{j} - \hat{k}$

$= -\hat{i} - 2\hat{j} - 6\hat{k}$

$|AB| = \sqrt{41}$

$|BC| = C - B$

$= 3\hat{i} - 4\hat{j} - 4\hat{k} - \hat{i} + 3\hat{j} + 5\hat{k}$

$= 2\hat{i} - \hat{j} + \hat{k}$ $\quad |BC| = \sqrt{6}$

$|CA| = A - C$

$= 2\hat{i} - \hat{j} + \hat{k} - 3\hat{i} + 4\hat{j} + 4\hat{k}$

$= -\hat{i} + 3\hat{j} + 5\hat{k}$

$|CA| = \sqrt{35}$

$AB^2 = BC^2 + CA^2$

$= 6 + 35$

$AB^2 = 41$

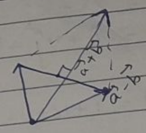
collinear or linear dependent

example:

$\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$

$\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$. State that $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular.

solution:



$\vec{a} + \vec{b} = (5\hat{i} - \hat{j} - 3\hat{k}) + (\hat{i} + 3\hat{j} - 5\hat{k})$

$= (6\hat{i} + 2\hat{j} - 8\hat{k})$

$\vec{a} - \vec{b} = (5\hat{i} - \hat{j} - 3\hat{k}) - (\hat{i} + 3\hat{j} - 5\hat{k})$

$= 4\hat{i} - 4\hat{j} + 2\hat{k}$

$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 24 - 8 - 16 = 0$

example:

Find $|\vec{a} - \vec{b}|$ if 2 vectors $\vec{a}$ & $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 3$ & $\vec{a} \cdot \vec{b} = 4$

solution:

$|\vec{a} - \vec{b}|^2 = a^2 + b^2 - 2\vec{a}\vec{b}$

$= 2^2 + 3^2 - 2(4)$

$= 4 + 9 - 8$

$= 5$

$|\vec{a} - \vec{b}| = \sqrt{5}$

0° → vector are close to each other.

90° → vector are opposing each other.

example:

$A = \hat{i} + \hat{j} + \hat{k}$ $B = 2\hat{i} + 5\hat{j}$

$C = 3\hat{i} + 2\hat{j} - 3\hat{k}$ $D = \hat{i} - 6\hat{j} - \hat{k}$

Find angle b/w $\vec{AB}$ & $\vec{CD}$.

$$\cos\theta = \frac{\vec{AB}\cdot\vec{CD}}{\|AB\|\,\|CD\|}$$

$$= \frac{-2-32-2}{\sqrt{18}\,\sqrt{72}}$$

$$= \frac{-36}{36} = -1$$

$AB = \vec{B} - \vec{A}$
$= 2\hat{i} + 5\hat{j} - \hat{i} - \hat{j} - \hat{k}$
$= \hat{i} + 4\hat{j} - \hat{k}$

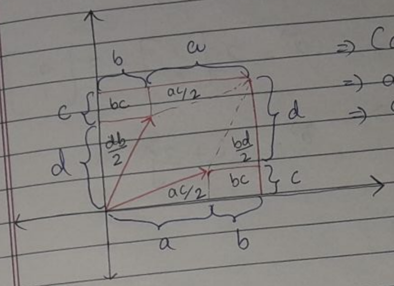
$\|AB\| = 1+16+1 = \sqrt{18}$

$CD = \vec{D} - \vec{C}$
$= \hat{i} - 6\hat{j} - \hat{k} - (3\hat{i} + 2\hat{j} - 3\hat{k})$
$= -2\hat{i} - 8\hat{j} + 2\hat{k}$

$\|CD\| = 4+64+4 = \sqrt{72}$

It is collinear
or linear dependent

$\vec{AB}\cdot\vec{CD} = (\hat{i} + 4\hat{j} - \hat{k})\cdot(-2\hat{i} - 8\hat{j} + 2\hat{k})$
$= -2 - 32 - 2 = -36$



$\Rightarrow (a+b)(d+c) - (2bc + ac + bd)$

$\Rightarrow ad + \cancel{ac} + \cancel{bd} + bc - 2bc - \cancel{ac} - \cancel{bd}$

$\Rightarrow ad - bc$

$\therefore \begin{vmatrix} a & b \\ d & c \end{vmatrix}$ i.e $ad - bc$

example: $A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$

$|A| = 2(-2) - (-1)(4) = -4 + 4 = 0$

$\therefore$ It is linearly dependent

Span: The span of a subset S of vector space V is the set of all finite linear combination of S.

$Span(S_1) = \{\alpha_1 v_1, \alpha_2 v_2, ----, \alpha_n v_n\} \mid \alpha_i \in R\}$

example: $S = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$
$\qquad v_1 \qquad v_2$

solution: $v_3 = \alpha_1 v_1 + \alpha_2 v_2$
$= \alpha_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\therefore \mathbb{R}^2$

if $|A|^{3\times 3} = 0$ is LD
if $|A| \neq 0$ is LI

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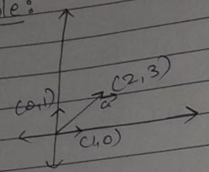
example:



$$\vec{\alpha} = 2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}$$

$$\vec{\alpha} = 2\hat{i} + 3\hat{j}$$

Linearly Independent:

If $u_1, u_2, \dots, u_n$ are n vectors of vector space V, then linear combination $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$ is LI iff $u_1\alpha_1 + u_2\alpha_n + \dots + u_n\alpha_n = 0 \ \forall\ \alpha_i = 0$

Linearly Dependent:

If at least one $\alpha_i \neq 0$ then $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = 0$ this linear combination is LD.

example:

$$S = \left\{ \begin{bmatrix}2\\3\end{bmatrix} \begin{bmatrix}4\\1\end{bmatrix} \begin{bmatrix}4\\0\end{bmatrix} \right\}$$

Solution:

$$\alpha_1\begin{bmatrix}2\\3\end{bmatrix} + \alpha_2\begin{bmatrix}4\\1\end{bmatrix} + \alpha_3\begin{bmatrix}4\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$2\alpha_1 + 4\alpha_2 + 4\alpha_3 = 0$

$3\alpha_1 + \alpha_2 = 0$

$3\alpha_1 = -\alpha_2$

$\alpha_1 = \frac{-1}{3}\alpha_2$

$\cancel{3\left[\frac{-1}{3}\alpha_2\right] + \alpha_2 = 0}$

$\cancel{-\alpha_2 + \alpha_2 = 0}$

By taking out det. we can also check wheather it is LI or LD.

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$2\left[\frac{-1}{3}\alpha_2\right] + 4\alpha_2 + 4\alpha_3 = 0$

$-\frac{2}{3}\alpha_2 + 4\alpha_2 + 4\alpha_3 = 0$

$-2\alpha_2 + 12\alpha_2 + 12\alpha_3 = 0$

$10\alpha_2 + 12\alpha_3 = 0$

$10\alpha_2 = -12\alpha_3$

$\alpha_2 = \frac{-6}{5}\alpha_3$

$\alpha_1 = \frac{-1}{3}\left(\frac{-6}{5}\right)\alpha_3$

$\alpha_1 = \frac{2}{5}\alpha_3$

example:

$$S = \{(1,2,1), (-1,1,0), (5,-1,2)\}$$

solution:

$$\alpha_1\begin{bmatrix}1\\2\\1\end{bmatrix} + \alpha_2\begin{bmatrix}-1\\1\\0\end{bmatrix} + \alpha_3\begin{bmatrix}5\\-1\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{vmatrix}1 & -1 & 5\\ 2 & 1 & -1\\ 1 & 0 & 2\end{vmatrix}$$

if $|A| = 0$ then LD

if $|A| \neq 0$ then LI

$\alpha_1 - \alpha_2 + 5\alpha_3 = 0$

$2\alpha_1 + \alpha_2 - \alpha_3 = 0$

$\alpha_1 + 2\alpha_3 = 0$

$2\alpha_1 - 2\alpha_2 + 10\alpha_3 = 0$

$2\alpha_1 + \alpha_2 - \alpha_3 = 0$

$-\quad -\quad +$

$-3\alpha_2 + 11\alpha_3 = 0$

$-3\alpha_2 = -11\alpha_3$

$\alpha_2 = \frac{11}{3}\alpha_3$

$\therefore \alpha_2 = \frac{11}{3}(0)$

$\therefore \alpha_2 = 0$

$\alpha_1 - \frac{11}{3}\alpha_3 + 5\alpha_3 = 0$

$3\alpha_1 - 11\alpha_3 + 15\alpha_3 = 0$

$3\alpha_1 + 4\alpha_3 = 0$

$3\alpha_1 = -4\alpha_3$

$\alpha_1 = \frac{-4}{3}\alpha_3$

$\frac{-4}{3}\alpha_3 + 2\alpha_3 = 0$

$-4\alpha_3 + 6\alpha_3 = 0$

$2\alpha_3 = 0$

$\alpha_3 = 0$

$\therefore \alpha_1 = \frac{-4}{3}(0)$

$\therefore \alpha_1 = 0$

linearly independent

example:
$S = \{(1,1,1),(1,2,3),(-1,-4,-7)\}$

solution: $\alpha_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}1\\2\\3\end{bmatrix} + \alpha_3 \begin{bmatrix}-1\\-4\\-7\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\alpha_1 + \alpha_2 - \alpha_3 = 0$
$\alpha_1 + 2\alpha_2 - 4\alpha_3 = 0$
$\alpha_1 + 3\alpha_2 - 7\alpha_3 = 0$

① - ②

$-\alpha_2 + 3\alpha_3 = 0$
$-\alpha_2 = -3\alpha_3$
$\alpha_2 = 3\alpha_3$

$\alpha_1 + 3\alpha_3 - \alpha_3 = 0$
$\alpha_1 + 2\alpha_3 = 0$
$\alpha_1 = -2\alpha_3$

$-2\alpha_3 + 3\alpha_3 - \alpha_3 = 0$
$-3\alpha_3 + 3\alpha_3 = 0$
$0 = 0$

linearly dependent.
α_3 is dependent on α_1 and α_2

→ If $S = \{v_1, v_2 \mid v_1, v_2 \in R^2\}$ such that S is LI then $span(S) = R^2$

→ Basis: A set B is said to be basis if
(i) B is LI and
(ii) B generates all vectors in V.

example:

① $B = \left\{\begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}-1\\3\end{bmatrix}\right\}$

$2\alpha_1 - \alpha_2 = 0 \quad -\times 4$
$4\alpha_1 + 3\alpha_2 = 0 \quad -\times 2$

$\cancel{8\alpha_1} - 4\alpha_2 = 0$
$\cancel{8\alpha_1} + 6\alpha_2 = 0$
$-10\alpha_2 = 0$
$\alpha_2 = 0$

$2\alpha_1 - 0 = 0$
$2\alpha_1 = 0$
$\alpha_1 = 0$

LI.

② $B = \left\{\begin{bmatrix}3\\-7\end{bmatrix}, \begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}5\\4\end{bmatrix}\right\}$

$3\alpha_1 - 2\alpha_2 + 5\alpha_3 = 0 \quad -\times 1$
$-7\alpha_1 + \alpha_2 + 4\alpha_3 = 0 \quad -\times 2$

$3\alpha_1 - \cancel{2\alpha_2} + 5\alpha_3 = 0$
$-14\alpha_1 + \cancel{2\alpha_2} + 8\alpha_3 = 0$
$-11\alpha_1 + 13\alpha_3 = 0$
$-11\alpha_1 = -13\alpha_3$
$\alpha_1 = \frac{13}{11}\alpha_3$

$3\left(\frac{13}{11}\right)\alpha_3 - 2\alpha_2 + 5\alpha_3 = 0$

$\frac{39}{11}\alpha_3 - 2\alpha_2 + 5\alpha_3 = 0$

$39\alpha_3 - 22\alpha_2 + 55\alpha_3 = 0$

$-22\alpha_2 + 94\alpha_3 = 0$

$-22\alpha_2 = -94\alpha_3$

$\alpha_2 = \frac{94}{22}\alpha_3$

$\alpha_2 = \frac{47}{11}\alpha_3$

$\therefore$ It is LD.

(3) $B = \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \end{bmatrix} \right\}$

$-\alpha_1 + 6\alpha_3 = 0$

$-\alpha_1 + 7\alpha_2 = 0$

$6\alpha_3 - 7\alpha_2 = 0$

$6\alpha_3 = 7\alpha_2$

$\alpha_3 = \frac{7}{6}\alpha_2$

$-\alpha_1 + 6\left(\frac{7}{6}\right)\alpha_2 = 0$

$-\alpha_1 + 7\alpha_2 = 0$

$\alpha_1 = +7\alpha_2$

α_3 and α_1 is dept. on α_2

$\therefore$ It is LD.

(4) $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

standard basis

$\therefore$ It is LI

$\alpha_1 = 0 \quad \alpha_2 = 0 \quad \alpha_3 = 0$

If we have 4 vector then we will ~~check~~ take 3 vectors and check whether it is LI or not.

(5) $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\alpha_1 + 3\alpha_3 = 0$

$\alpha_2 = 0$

$\alpha_3 = 0$

$\alpha_1 + 3(0) = 0$ LI.

$\alpha_1 = 0$

(6) $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}$

Take only 3 vectors and check

$\alpha_1 - 2\alpha_2 + \alpha_3 = 0$

$\alpha_1 + 3\alpha_2 - 2\alpha_3 = 0$

$\alpha_1 + 5\alpha_2 + 4\alpha_3 = 0$

$\alpha_1 - 2\alpha_2 + \alpha_3 = 0$

$\alpha_1 + 3\alpha_2 - 2\alpha_3 = 0$

$-5\alpha_2 + 3\alpha_3 = 0$

$-5\alpha_2 = -3\alpha_3$

$\alpha_2 = \frac{3}{5}\alpha_3$

$\alpha_1 - 2\left(\frac{3}{5}\right)\alpha_3 + \alpha_3 = 0$

$\alpha_1 - \frac{6}{5}\alpha_3 + \alpha_3 = 0$

$5\alpha_1 - 6\alpha_3 + 5\alpha_3 = 0$

$5\alpha_1 - \alpha_3 = 0$

$5\alpha_1 = \alpha_3$

$\alpha_1 = \frac{1}{5}\alpha_3$

Take $\alpha_3 = 1$

$\alpha_2 = \frac{3}{5} \quad \alpha_1 = \frac{1}{5}$ L.

If we get identity matrix in ~~eq~~ equ^n then if ↓ LI. we get identity matrix then it is and if we get one row zero then it is LD.

example:

$$B_1 = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} \right\}$$

solution:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 5 \end{bmatrix}$$

$R_1 = \frac{1}{2} R_1$

$$\begin{bmatrix} 1 & 1/2 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 5 \end{bmatrix}$$

$R_2 = R_2 - R_1 \quad R_3 = R_3 - R_1$

$$\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 3/2 & 0 \\ 0 & 3/2 & 5 \end{bmatrix}$$

$R_2 = \frac{2}{3} R_2$

$$\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 0 & 3/2 & 5 \end{bmatrix}$$

$R_3 = R_3 - \frac{3}{2} R_2 \quad R_1 = R_1 - \frac{1}{2} R_2$

$\dim V = n$

If $|A|$ is zero then LD.
If $|A| \neq 0$ then LI.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$R_3 = \frac{1}{5} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is identity matrix
∴ It is LI.

Q. What can be the length of largest set of basis of R^3?

A. 3. e.g $\{(1,0,1), (0,0,1), (1,1,0)\}$

$R^n = n$

Change of basis.

$$A^{-1}A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{nb} = A^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{Sb}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{nb} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{Sb}$$

examples:

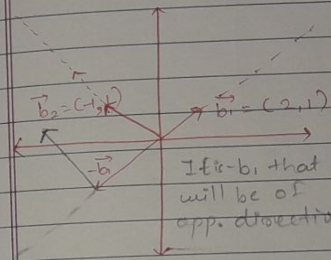
new basis	find standard matrix
(1, -5)	(5, 1)
(-2, 2)	(-2, -2)
(-3, 4)	(-4, -3)
(0, 7)	(-7, 0)

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

and if standard matrix is given and we have to find new basis.

① $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$



$P = -b_1 + 2b_2$

$\therefore$ new basis $P(-1, 2)$

find standard basis:

$\therefore \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$\therefore$ standard basis $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

→ Given standard basis $P = (3, 2)$ and $P = 3\hat{i} + 2\hat{j}$ find b_1 and b_2.

$P = (3, 2)$

$B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow |B| = 2(1) + 1(1) = 2 + 1 = 3$

$B^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$

$x_n = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}$

$\boxed{\begin{array}{l} Ax_n = x_s \\ x_n = A^{-1}x_s \end{array}}$

$A \cdot P' = P$

A = matrix of new basis.
P' = point under new basis.
P = point under standard basis

Orthogonal basis. A se.

example:

$$B_1 = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} \right\}$$

solution:

$(2,1,1) \cdot (1,2,1) = 2+2+1 = 5$

$(1,2,1) \cdot (0,0,5) = 0+0+5 = 5$

$(0,0,5) \cdot (2,1,1) = (0+0+5) = 5$

B_1 is not orthogonal basis.

example:

$$B_2 = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} \right\}$$

solution:

$(3,1,1) \cdot (-1,2,1) = -3+2+1 = 0$

$(-1,2,1) \cdot (-1/2,-2,7/2) = 1/2 - 4 + 7/2$

$= 1 - 8/2 + 7/2 = -7/2 + 7/2$

$= 0$

$(-1/2,-2,7/2) \cdot (3,1,1) = (-3/2 - 2 + 7/2)$

$= 0$

$\therefore B_2$ is orthogonal basis

example:

$$B_3 = \left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 5/2 \end{bmatrix} \right\}$$

solution:

$(-2,1,2) \cdot (1,2,0) = (-2+2+0) = 0$

$(1,2,0) \cdot (2,-1,5/2) = (2-2+0) = 0$

$(2,-1,5/2) \cdot (-2,1,2) = (-4-1+5) = 0$

* If we have to check orthogonal basis of more than 2×2 matrix

$$A^T \cdot A = diag(I) \longrightarrow \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & -1 & 5/2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & 0 & 5/2 \end{bmatrix} =$$

$$\begin{bmatrix} 9 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 45/4 \end{bmatrix}$$

Orthonormal basis:

→ A set B is said to be an orthonormal basis if the set B is linearly independent and each pair of vector in set B is ⊥er to each other having magnitude 1.

example:

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

solution:

$$A^T \cdot A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

∴ It is orthogonal basis.

Now orthonormal

$$\sqrt{(1)^2 + (1)^2 + (0)^2} = \sqrt{2}$$

$$\sqrt{(-1)^2 + (1)^2 + (0)^2} = \sqrt{2}$$

$$\sqrt{(0)^2 + (0)^2 + (1)^2} = 1$$

So, the orthonormal basis

$$B_4 = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

If $B_4^T B_4 = I$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Gram - Schmidt Process

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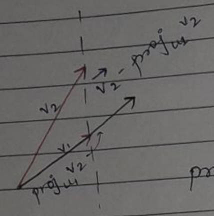
$B = \{v_1, v_2\}$ B is basis for R^2 and we have to find $B_1 = \{u_1, u_2\}$.

$$u_1 = \frac{\vec{v_1}}{||v_1||}$$



$$\vec{y_2} = v_2 - proj_{u_1} \vec{v_2}$$

$$proj_{u_1} \vec{v_2} = v_2 - (v_2 - u_1) \cdot \vec{u_1}$$

$$= \vec{v_2} - \left[\frac{\vec{v_2} \cdot \vec{v_1}}{||v_1||}\right] \cdot \frac{\vec{v_1}}{||\vec{v_1}||}$$

$$\vec{y_2} = proj_{u_1} \vec{v_2} = \vec{v_2} - \left[\frac{(v_2 \cdot v_1)}{||v_1||^2}\right] \vec{v_1}$$

$$\therefore \vec{u_2} = \frac{\vec{y_2}}{||y_2||}$$

example: $B = \left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$ find the orthonormal basis.

solution: $v_1 = \begin{bmatrix}3\\1\end{bmatrix}$ $v_2 = \begin{bmatrix}2\\2\end{bmatrix}$

$$u_1 = \frac{[3, 1]}{\sqrt{9+1}} = \frac{\begin{bmatrix}3\\1\end{bmatrix}}{\sqrt{10}} = \frac{1}{\sqrt{10}}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}3/\sqrt{10}\\1/\sqrt{10}\end{bmatrix}$$

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$$proj_{u_1} \vec{v_2} = \begin{bmatrix}2\\2\end{bmatrix} - \left[\frac{8}{(\sqrt{10})^2}\right]\begin{bmatrix}3\\1\end{bmatrix}$$

$$= \begin{bmatrix}2\\2\end{bmatrix} - \left[\frac{8}{10}\right]\begin{bmatrix}3\\1\end{bmatrix}$$

$$= \begin{bmatrix}2\\2\end{bmatrix} - 0.8\begin{bmatrix}3\\1\end{bmatrix}$$

$$= \begin{bmatrix}2\\2\end{bmatrix} - \begin{bmatrix}2.4\\0.8\end{bmatrix} = \begin{bmatrix}-0.4\\1.2\end{bmatrix}$$

$$\therefore u_2 = \begin{bmatrix}-0.4/\sqrt{1.6}\\1.2/\sqrt{1.6}\end{bmatrix}$$

for check we can see wheather dot product is 0 and magnitude 1

$$\therefore B' = \{u_1, u_2\}$$

$$= \left\{\begin{bmatrix}3/\sqrt{10}\\1/\sqrt{10}\end{bmatrix}, \begin{bmatrix}-0.4/\sqrt{1.6}\\1.2/\sqrt{1.6}\end{bmatrix}\right\}$$

Dr.

For R^3, $B = (v_1, v_2, v_3)$ $\quad B' = (u_1, u_2, u_3)$

$u_1 = \frac{v_1}{\|v_1\|}$

$y_2 = proj_{u_1} v_2 = v_2 - \frac{v_2 \cdot v_1}{(\|v_1\|)^2} \cdot v_1$ $\qquad u_2 = \frac{y_2}{\|y_2\|}$

$y_3 = v_3 - \frac{v_3 \cdot v_1}{\|v_1\|^2} \cdot v_1 - \frac{v_3 \cdot y_2}{\|y_2\|^2} \cdot y_2$ $\qquad u_3 = \frac{y_3}{\|y_3\|}$

$\therefore y_k = v_k - \sum_{j=1}^{k-1} proj_{y_j} (v_k)$

example:

$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Solution:

first check orthogonal

$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$A^T \cdot A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 9 & 3 & 2 \\ 3 & 5 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

~~It is not orthogonal~~

$\sqrt{\frac{121}{81} + \frac{16}{81} + \frac{196}{81}} = \sqrt{\frac{111}{27}}$

$(0,0,1) \cdot (1,2,2)$
$(0+0+2)$

$v_1 = (1,2,2)$, $v_2 = (-1,0,2)$ $\quad v_3 = (0,0,1)$

$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,2,2)}{\sqrt{1+4+4}} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

$y_2 = (-1,0,2) - \frac{2}{(3)^2} \cdot (1,2,2)$

$(0,0,1) \cdot (1,2,2)$
$(0,0,2)$
$(0,0,1) \cdot (-11/9, -4/9, 14/9)$
$(0,0,14/9)$

$= (-1,0,2) - \frac{2}{9} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \frac{2}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 2/9 \\ 4/9 \\ 4/9 \end{bmatrix}$

$= \begin{bmatrix} -11/9 \\ -4/9 \\ 14/9 \end{bmatrix}$ $\qquad u_2 = \frac{y_2}{\|y_2\|} = \sqrt{\frac{111}{27}} \begin{bmatrix} -11/9 \\ -4/9 \\ 14/9 \end{bmatrix}$

$y_3 = (0,0,1) - \frac{(0,0,2)}{(3)^2} \cdot (1,2,2) - \frac{14/9}{\sqrt{111/27}} \cdot \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

$= (0,0,1) - \frac{2}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{14/9}{\sqrt{111/27}} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

example: $B = \{(1,1,0), (-1,1,0), (0,0,3)\}$
$\quad\quad v_1 \quad\quad v_2 \quad\quad v_3$

$v = (1,-2,3)$

$v_1 \cdot v_2 = (1,1,0)\cdot(-1,1,0) = -1+1+0 = 0$

$v_2 \cdot v_3 = (-1,1,0)\cdot(0,0,3) = 0+0+0 = 0$

$v_3 \cdot v_1 = (0,0,3)\cdot(1,1,0) = 0+0+0 = 0$

$(1,-2,3) = \alpha_1(1,1,0) + \alpha_2(-1,1,0) + \alpha_3(0,0,3)$

$\alpha_1 - \alpha_2 = 1 \quad \alpha_1 + \alpha_2 = -2 \quad 3\alpha_3 = 3$

$\alpha_3 = 1$

$\alpha_1 - \cancel{\alpha_2} = 1$
$\alpha_1 + \alpha_2 = -2$
$2\alpha_1 = -1$
$\alpha_1 = \frac{-1}{2}$

$\alpha_1 + \alpha_2 = -2$

$\frac{-1}{2} + \alpha_2 = -2$

$\alpha_2 = -2 + 1/2$

$= \frac{-4+1}{2}$

$\alpha_2 = -3/2$

Linear Transformation

e.g. Define $T: V_3 \to V_2$ such that $T(x_1, x_2, x_3) = (x_1, x_2)$

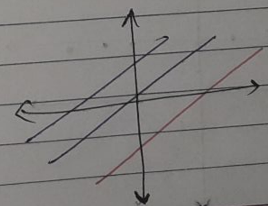
$\Rightarrow$ e.g. $T(1,2,3) = (1,2)$
$f(x_1, x_2, x_3) = (x_1, x_2)$
$R^3 \to R^2$

$\Rightarrow y = ax_1 + bx_2 + cx_3$
$R^3 \to R^1$

$\Rightarrow T: V_2 \to V_2$
$T(x_1, x_2) = (x_1 + x_2, x_1)$

I/P	O/P
(2,3)	(5,2)
(-5,7)	(2,-5)
(0,-4)	(-4,0)
(8,-16)	(-8,8)

Theorem:
Sum of 2 linear mapping. Let $T: U \to V$ and $S: U \to V$ be two linear transformations. Then the mapping $M: U \to V$ defined by $M(u) = S(u) + T(u)$ will be also linear.



two line linear then third line is also linear

$T: V_3 \to V_2$

$T(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3)$

$S: V_3 \to V_2$

$S(x_1, x_2, x_3) = (2x_1, x_2 - x_3)$

Then

$\alpha_1 S + \alpha_2 T$

$\alpha_1 = \alpha_2 = 1$

$S+T = S(x_1, x_2, x_3) + T(x_1, x_2, x_3)$

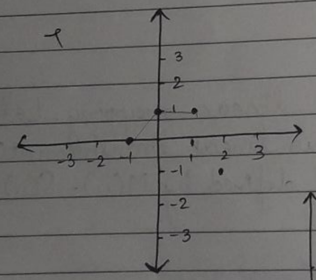
$= (2x_1, x_2 - x_3) + (x_1 - x_2, x_2 + x_3)$

$(S+T) = 3x_1 - x_2, 2x_2$

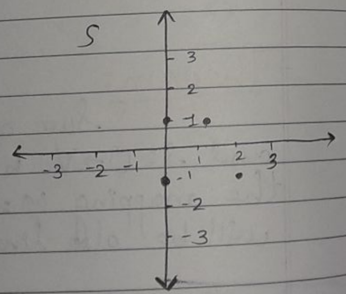
Take five point and draw graph of T, S and S+T

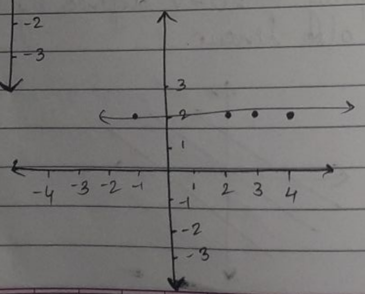
	T	S	S+T
(1, -1, 0)	(2, -1)	(2, -1)	(4, 2)
(1, 0, 1)	(1, 1)	(0, -1)	(3, 2)
(0, 0, 1)	(0, 1)	(0, -1)	(0, 0)
(1, 1, 0)	(0, 1)	(1, 1)	(2, 2)
(0, 1, 0)	(-1, 0)	(0, 1)	(-1, 2)

T



S





S+T

→ Suppose u and v are vector space (real/complex) then the map transformation U to V is said to be a linear map if

(a) $T(u_1 + u_2) = T(u_1) + T(u_2)$

(b) $T(\alpha u) = \alpha T(u), \alpha \in R$

Question:

$T, S: R^3 \to R^2$

$T(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3)$ — (a)

$S(x_1, x_2, x_3) = (2x_1, x_2 - x_3)$

Answer: $T(u_1 + u_2) = T(u_1) + T(u_2)$

(1) For (a)

$u_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad u_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$T(u_1 + u_2) = T \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix} = \begin{bmatrix} (a_1 + b_1) - (a_2 + b_2) \\ (a_2 + b_2) + (a_3 + b_3) \end{bmatrix}$

$T(u_1) + T(u_2) = \begin{bmatrix} a_1 - a_2 \\ a_2 + a_3 \end{bmatrix} + \begin{bmatrix} b_1 - b_2 \\ b_2 + b_3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a_1 + b_1 - (a_2 + b_2) \\ (a_2 + a_3) + (a_3 + b_3) \end{bmatrix}$

$\alpha T(u) = \alpha \begin{bmatrix} a_1 - a_2 \\ a_2 + a_3 \end{bmatrix} = \begin{bmatrix} \alpha(a_1 - a_2) \\ \alpha(a_2 + a_3) \end{bmatrix}$

$S(u_1) + S(u_2)$

$$\begin{bmatrix} 2a_1 \\ a_2 + a_3 \end{bmatrix} + \begin{bmatrix} 2b_1 \\ b_2 + b_3 \end{bmatrix}$$

$$\begin{bmatrix} 2(a_1 + b_1) \\ (a_2 + b_2) + (a_3 + b_3) \end{bmatrix}$$

example: $T: R^2 \to R^2$ and $S: R^2 \to R^2$.

$T(x_1, x_2) = (x_1 + x_2, 0)$

$S(x_1, x_2) = (2x_1, 3x_1 + 4x_2)$

Determine linear map.

(a) $2S + 3T$ (b) $3S - 7T$

solution: (a) $2S + 3T$

$2S = 2(2x_1, 3x_1 + 4x_2)$

$= (4x_1, 6x_1 + 8x_2)$

$3T = 3(x_1 + x_2, 0)$

$= (3x_1 + 3x_2, 0)$

$2S + 3T = (4x_1, 6x_1 + 8x_2) + (3x_1 + 3x_2, 0)$

$= (7x_1 + 3x_2, 6x_1 + 8x_2)$

(b) $3S = 3(2x_1, 3x_1 + 4x_2)$

$= 6x_1, 9x_1 + 12x_2$

$7T = 7(x_1 + x_2, 0)$

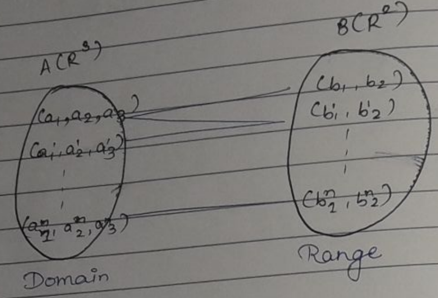
$= (7x_1 + 7x_2, 0)$

$3S - 7T = (6x_1, 9x_1 + 12x_2) - (7x_1 + 7x_2, 0)$

$= (-x_1 - 7x_2, 9x_1 + 12x_2)$

Functions

- Let A and B be any two sets and Let f denote a rule which associates to each member of A, a member of B we say that f is a function from A into B. Also A is said to be the domain of the function.

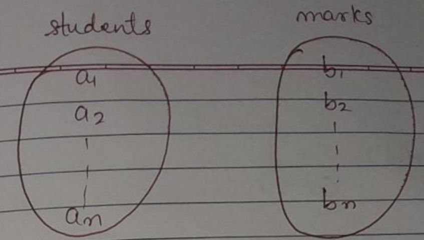


- If x denotes a member of the set A then the member of set B which associates to is denoted by $f(x)$ called the value of the function x.

$$x \to f(x)$$
$$y \to f(x)$$

Range: Let f be a function from the set A into a set B so the domain of f is the set A then the set of function values is called the range of f.

$$\text{Range of } f = \{f(x) : x \in A\}$$



- A symbol which denotes the member of the domain of a function is called independent variable.

- Similarity, which denote the member of range of a function is called dependent variable.

Operations of function:

Let f and g be two functions with domain of f (D_f) and Domain of g (D_g).

$\to f+g \quad \to f-g \quad \to f \cdot g \quad \to f \div g$

extensive use in ML.

For $f+g = z$
$\therefore D_z = D_f \cap D_g$

For $f \div g \to z = f/g$
$\to D_z = D_f \cap \{D_g - g(x) = 0\}$

example:

$f(x) = x^2 - 3x + 1$

$g(x) = \frac{1}{(x-2)}$

Domain of function.

(a) $f \cdot g$ (b) $f \div g$

(a) $D_f = R$ $D_g = R - \{2\}$

$D_f \cap D_g = R - \{2\}$

(b)

Relations

→ The relation shows the relationship between INPUT and OUTPUT.

→ Relation is a subset of Cartesian product.

→ In other words, the relation between the two sets is defined as the collection of the ordered pair, in which the ordered pair is formed by object from each set.

e.g.

$\{(2,3), (5,4), (0,3), (4,1)\}$

Function : yes

Domain : $\{0, 2, 4, 5\}$

Range : $\{1, 3, 4\}$