Stable Identification of Cliques with Restricted Sensing

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Abstract

In this paper we study the identification of common interest groups from low order interactive observations. We present a new algebraic approach based on the Radon basis pursuit on homogeneous spaces. We prove that if the common interest groups satisfy the condition that overlaps between different common interest groups are small, then such common interest groups can be recovered in a robust way by solving a linear programming problem. We demonstrate the applicability of our approach with examples in identifying social communities in the social network of *Les Miserables* and in inferring the most popular top 5-jokes in the Jester dataset.

1 Introduction

In this paper, we consider the problem of identifying common interest groups or cliques based on partial information. This problem arises from a variety of sources, from identity management [1], statistical rankings [2, 4], and in particular, social networks. The following three examples provide us a glimpse on the typical problems which could be addressed with the techniques discussed in this paper.

Motivating example 1 (Tracking and Identifying Teams) We consider the scenario of multiple targets moving in the environment monitored by sensors. We assume each moving target has an identity and they each belong to some teams. However, we only get partially observed interaction information due to sensing abilities. For example, consider watching a grey-scale video of a basketball game (we lack abilities to sense colors accurately in a grey scale video), we observe passing balls or collaborative offensive/defensive interactions between teammates. The observation being partial is due to the fact that people typically have pairwise interactions in basketball games. It is seldom for us to observe a single event which involves all team members. Our objective is to infer membership information (which team the players belong to) from partially observed interactions.

In Figure 1-(a), we show a weighted graph (shown on the top) illustrating pairwise interactions among 10 basketball players. Nodes in this graph represent players and weights on edges represent frequencies of pairwise interactions. Note that we may get noisy data due to observation errors where people from different teams have interactions. However, given that the noise is not too much, we hope to be capable to identify the two teams (shown at the bottom) from such partially observed pairwise interaction information.

Motivating example 2 (Detecting Communities in Social Networks) Detecting social communities in social networks is of extraordinary importance. It can be used to understand the organization or collaboration structures within the social network. However, we do not have direct mechanisms to sense what the social communities are. Instead, we have partial, low order interaction information. For example, we observe pairwise or triple-wise co-appearance among people who hang out for some leisure activities together. We hope to detect those social communities in the social network from such partially observed data.

In Figure 2-(a), we show an example as the social network of Victor Hugo's novel *Les Miserables* which was studied in [3]. In the weighted graph, the nodes represent 33 key characters and weights on edges represent frequencies of co-appearance. Several social communities arises in the network, formed by either friendships, street gangs, kinships, student society, or drama conflicts. We wish to detect those social communities from pairwise co-appearance frequencies data. Note that in this example, different social communities may have different sizes and one people may belong to several social communities.

Motivating example 3 (Inferring Partial Rankings of High Order) The problem of clique identification also arises in the ranking problems. Consider a collection of items is to be ranked by a collection of users. Each user can propose his/her top j, say 3, items in favor but without relative preference within. We wish to infer what are the first tier competitors as the top $k > j$, say 5. This problem is the inference of high order partial rankings from low order observations.

Among these examples, we are typically given a network with some nodes representing players, characters, or items, and with edges summarizing the pairwise interaction observations. Triple-wise and other low order information can be further considered if we consider small cliques in the networks (i.e. complete sub-graphs). *The basic problem here is to determine common interest groups or cliques within the network from observed low order interaction frequencies.* In reality such low order interactions are often governed by a considerably smaller number of high order teams or the first tier competitors. Thus, the clique identification problem can be formulated as a compressed sensing problem whose goal is to determine a sparse signal on those cliques.

Our problem can be regarded as an extension of the recent work in [4] which studies sparse recovery of *functions on permutation groups*, while we reconstruct *functions on k-subsets* (cliques), often called homogeneous space in literature [2]. In our studies the discrete Radon basis becomes the natural choice in stead of the Fourier basis considered in [4]. This leaves us a new challenge on addressing the noiseless exact recovery and stable recovery with noise. Unfortunately the greedy algorithm for exact recovery in [4] can not be applied to noisy settings, and in general the Radon basis does not satisfy the Restricted Isometry Property (RIP) [9] which is crucial for the universal recovery. In this paper, we develop new theories which guarantee the exact sparse recovery and stable recovery under the choice of Radon basis, which has a deep root in Basis Pursuit [5] and its extension with uniformly bounded noise.

The main content of this paper can be summarized as follows. Section 2 presents the formulation of our problem with a gentle introduction on Radon basis; Section 3 discusses exact recovery conditions without noise; Section 4 addresses stable recovery under uniformly bounded noise, which leads to a stage-wise algorithm detecting cliques of mixed sizes; The last section demonstrates three successful applications to the motivating examples discussed above.

2 Problem Formulation

We introduce a graph $G = (V, E)$ to facilitate our discussion. The set of vertices V represents individual identities such as people in the social network, basketball players, or items to be ranked. Each edge in E is associated with some weights which represent interactive frequency information.

We assume there are several common interest groups or communities within the network, represented by cliques or complete sub-graphs in graph G, which are perhaps of different sizes and may have overlaps. We assume every community has certain interaction frequency which can be viewed as a function on cliques. However, we can only receive partial measurements consisting of low order interaction frequency on subsets in a clique. For example, in the smallest case we only observe pairwise interactions represented by edge weights. Our problem is to reconstruct the function on cliques from partially observed data.

However, to resolve this problem, one has to answer two questions: *what is the suitable representation basis, and what is the reconstruction algorithm?* Below we shall provide an answer that Radon basis will be the appropriate representation for our purpose which allows the sparse recovery by a simple linear programming reconstruction algorithm.

2.1 Basis Construction

We first consider constructing basis so that we can use such basis to connect functions on j-subsets to functions on k-subsets ($j \leq k$). Our construction of basis is directly related to *Radon basis* for discrete combinatorial analysis.

2.1.1 Common Interest Groups of Equal Size

For simplicity, we restrict ourselves here to the case that all the common interest groups are all of the same size k ($k > j$). As we shall see below, the case with mixed sizes can be reduced to this simple setting in a stage-wise way. There are even some natural scenarios where such a simple case meets, for example the inference of two teams each of size $k = 5$ from pairwise ($j = 2$) interaction frequencies.

Let V_j denote the set of all $j-$subsets of $V = \{1, 2, \cdots, n\}$ and M^j be the set of functions on V_j. The observed partial interaction information, i.e., interaction frequencies on all $j-$subsets, can be viewed as a function on V_j, denoted by $b \in M^j$.

We build a matrix $\tilde{R}^{j,k} : M^k \to M^j$ ($j < k$) as a mapping from functions on all $k-$subsets of V to functions on all j-subsets of V. For example, $\tilde{R}^{2,5}$ is a $\binom{10}{2}$-by-$\binom{10}{5}$ matrix with rows representing all 2-subsets and columns representing all $5-$subsets. We let entries of $\tilde{R}^{j,k}$ are either 0 or 1 indicating whether the j-subset is a subset of the $k-$subset. Note that every column of $\tilde{R}^{j,k}$ has $\binom{k}{j}$ ones. Lacking *a priori* information, we assume that every $j-$subset has equal probability in interactions, whence choose the same constant 1 for each column. We further normalize $\tilde{R}^{j,k}$ to $R^{j,k}$ so that l_2 norm of each column of $R^{j,k}$ is one. In a summary, we have

$$R^{j,k}_{(\sigma,\tau)} = \begin{cases} \frac{1}{\sqrt{\binom{k}{j}}}, & \text{if } \sigma \subset \tau, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where σ is a $j-$subset and τ is a $k-$subset. As we shall see soon, this construction meets a canonical basis associated with discrete Radon transform. The size of of matrix $R^{j,k}$ clearly depends on the total number of items $n = |V|$, however, we omit n as its meaning will be clear from context.

In the example of basketball games, given information $b \in M^2$ as a function on 2-subsets, we wish to obtain a function $x \in M^5$ on 5-subsets such that $b = Ax$ where $A = R^{2,5}$. Note that $|V| = 10$. Ideally x has a sparse solution concentrating on two 5-subsets, representing the two disjoint teams. This is where compressive sensing techniques shall be applied.

2.1.2 Relation to Radon Basis

The matrix $R^{j,k}$ constructed above is related to discrete Radon transforms on homogeneous space M^k. In fact, up to a constant, the adjoint or transpose operator $(R^{j,k})^* : M^j \to M^k$ defined by $(R^{j,k})^* u(\tau) = c \sum_{\sigma \subset \tau} u(\sigma)$, is called in literature [2] the discrete Radon transform from homogeneous spaces M^j to M^k. The collection of all row vectors of $R^{j,k}$ is called as the j-th *Radon basis* for M^k. Our usage here is to exploit the transpose matrix of Radon transform to construct an over-complete dictionary for M^j, such that the observation $b \in M^j$ is represented by a possibly sparse $x \in M^k$ ($k \geq j$).

Radon basis was proposed to be an efficient way to study partially ranked data in [2], where it was shown that by looking at low order Radon coefficients of function on M^k, we usually get useful and interpretable information. Our approach here adds a reversal of this perspective, *i.e.* the reconstruction of sparse high order functions from low order Radon coefficients. We will discuss this in the following with a connection to the compressive sensing [5, 7].

2.2 Reconstruction Algorithms

Now we give some reconstruction algorithms for detecting high order cliques based on low order information exploiting the basis matrix we talked about in the last section.

Suppose x_0 is a sparse function on common interest groups or cliques. To reconstruct this sparse function based on low order observation data, we consider the following linear programming first known as Basis Pursuit [5], etc.

$$\begin{array}{rcl} \mathcal{P}_1: & \min & \|x\|_1, \\ & \text{subject to} & Ax = b, \end{array}$$

where the matrix A is $R^{j,k}$. For a robust construction against noise, we also consider the following algorithm

$$\begin{array}{rcl} \mathcal{P}_{1,\delta}: & \min & \|x\|_1, \\ & \text{subject to} & \|Ax - b\|_\infty \leq \delta. \end{array}$$

It differs to Lasso [6] or BPDN [5] in that a l_∞ norm is used to control the noise instead of the l_2 norm, and also differs to the Dantzig selector [8] which uses $\|A^*(Ax-b)\|_\infty \leq \delta$ in the constraint. The reason for our choice lies in the fact that the typical examples we discussed above are often with bounded noise rather than Gaussian-like noise. Our choice will be suitable to incorporate this kind of prior knowledge on noise.

2.3 Failure of Restricted Isometry Property and Universal Recovery

Recently it was shown by [7, 9] that $\mathcal{P}_1$ has a unique sparse solution x_0, if the matrix A satisfies the so called *Restricted Isometry Property* (RIP), i.e. for every set of columns T with $|T| \leq s$, there exist certain universal constant $\delta_s \in [0,1)$ (e.g. $\delta_{2s} < \sqrt{2}-1$ in [9]) such that

$$(1-\delta_s)\|x\|_{l_2}^2 \leq \|A_T x\|_{l_2}^2 \leq (1+\delta_s)\|x\|_{l_2}^2, \quad \forall x \in R^s.$$

This exact recovery holds for all $s-$sparse signal x_0, whence called the *universal recovery*.

Unfortunately, in our basis construction of matrix $A = R^{j,k}$, RIP is not satisfied unless $s < \binom{k+j+1}{k}$ which can not scale up with n. To see this, we extract a set of columns $T = \{\tau : \tau \subset \{1, 2, \cdots, k + j + 1\}\}$ (τ is interpreted as a $k-$subset) and form a submatrix $R_T^{j,k}$. By discarding zero rows, we know the rank of $R_T^{j,k}$ is determined by a small submatrix of $R_T^{j,k}$ of size $\binom{k+j+1}{j}$ by $\binom{k+j+1}{k}$. This matrix has more columns than rows. This means the extracted columns must be linear dependent. In other words, there exist a h where $\text{supp}(h) \subset T$ such that $R^{j,k}h = 0$. So in general, we can not expect that the sparse recovery holds universally for all $s-$sparse signals when $s \geq \binom{k+j+1}{k}$.

Therefore, in our case, the correct strategy is to look for the sparsity patterns corresponding to cliques which can be recovered by $\mathcal{P}_1$ or $\mathcal{P}_{1,\delta}$. In general, *we hope to be able to recover a collection of sparse signals x_0, whose sparsity pattern satisfies certain conditions instead of meeting a universal sparse recovery*. Such conditions might naturally occur in reality, which will be shown in the sequel as simply requiring small overlaps between cliques.

3 Exact Recovery Conditions

In this section we present our main results on noiseless exact recovery conditions of x_0 from the given information $b \in M^j$ by solving the linear program $\mathcal{P}_1$.

3.1 Irrepresentable Condition

Suppose A is a M-by-N matrix and x_0 is a sparse signal. Let $T = \text{supp}(x_0)$, T^c be the complement of T, and A_T (or A_{T^c}) be the submatrix of A where we only extract column set T (or T^c, respectively). A regularization path of $\mathcal{P}_{1,\delta}$ refers to the map $\delta \mapsto x_\delta$ where x_δ is a solution of $\mathcal{P}_{1,\delta}$.

Theorem 1 *Assume that $A_T^* A_T$ is invertible and there exists a vector $w \in R^N$ such that*

(1) $A_T w = \iota^* sgn(x_0)$*,*

(2) $\|A^*_{T^c} w\|_\infty < 1$*,*

where ι *is an imbedding operator* $\iota : l_2(T) \to l_2(N)$ *extending a vector on* T *to a vector in* R^N *by placing zeros outside of* T*, and* ι^* *is the dual restriction* $\iota^* sgn(x_0) = sgn(x_0)|_T$*. Then* x_0 *is the unique solution for* $\mathcal{P}_1$*, and it is also a necessary condition that* x_0 *lies on a unique regularization path of* $\mathcal{P}_{1,\delta}$

The sufficiency for the unique solution x_0 of $\mathcal{P}_1$ is shown by [7]. The necessity comes from KKT conditions of $\mathcal{P}_{1,\delta}$. Detailed proofs will be given in Appendix.

However this condition is difficult to check due to the presence of w. However if we further assume that $w \in im(A_T)$, then the condition in Theorem 1 reduces to the following condition.

Irrepresentable condition $A^*_T A_T$ *is invertible and*

$$\|A^*_{T^c} A_T (A^*_T A_T)^{-1}\|_\infty < 1, \tag{2}$$

where $*$ *denote matrix transpose and* $\|\cdot\|_\infty$ *stands for the matrix* ∞*-norm, i.e. the maximum absolute row sum of the matrix such that* $\|A\|_\infty := \max_j \sum_i |A_{ij}|$*.*

Note that this condition only depends on A and the true sparsity pattern of x_0, which is easy to check. The restriction $w \in im(A_T)$ does not put a too strong constraint, which is actually the necessary condition that x_0 can be reconstructed by Lasso [6] or Dantzig selector [8], even under some Gaussian-like noise assumptions [10, 11].

Corollary 1 *If the Irrepresentable condition holds, then* x_0 *is the unique solution of* $\mathcal{P}_1$ *and lies on a unique regularization path of* $\mathcal{P}_{1,\delta}$*.*

In the following we will present some further conditions which are easily checkable to satisfy the Irrepresentable condition in (2).

3.2 Common Interest Groups of Equal Size

We consider the case where A is $R^{j,k}$. Given data b defined on all $j-$subsets, we wish to infer common interest groups on all $k-$subsets so that low order interaction data b can be viewed as induced from high order common interest groups. Suppose x_0 is a sparse signal on all $k-$subsets. We have the following theorem:

Theorem 2 *Let* $T =$ *supp*(x_0)*, if we allow overlaps among common interest groups to be no larger than* r*, then the maximum* r *that can guarantee irrepresentable condition is* $j - 2$*.*

This is a direct conclusion of the following three results.

Lemma 1 Let $T = \text{supp}(x_0)$, and $j \geq 2$. Suppose that for any $\sigma_1, \sigma_2 \in T$, there holds $|\sigma_1 \cap \sigma_2| \leq r$.

1. If $r = j - 2$, then $\|A^*_{T^c} A_T (A^*_T A_T)^{-1}\|_\infty < 1$;
2. If $r = j - 1$, then $\|A^*_{T^c} A_T (A^*_T A_T)^{-1}\|_\infty \leq 1$ where equality holds with certain examples;
3. If $r = j$, there are examples such that $\|A^*_{T^c} A_T (A^*_T A_T)^{-1}\|_\infty > 1$.

Its proof are based on combinatorial arguments and will be given in Appendix. Theorem 2 thus provides us with a theoretical sufficient and necessary condition on how many overlaps we should allow to guarantee the Irrepresentable Condition. Clique overlaps no more than $j - 2$ will be suffice to guarantee the exact sparse recovery by $\mathcal{P}_1$, while larger overlaps may violate the Irrepresentable Condition. Note that this theorem is an analysis in the worse case, so in application, one may encounter examples which has larger overlaps than $j - 2$ where $\mathcal{P}_1$ still works.

4 Stable Recovery with Bounded Noise

In real applications, one almost always encounters examples with noise such that exact sparse recovery is impossible. In this case, $\mathcal{P}_{1,\delta}$ will be a good replacement of $\mathcal{P}_1$ as a robust reconstruction algorithm. In this section a stable recovery theorem will be given for $\mathcal{P}_{1,\delta}$ in the simple case that the clique sizes are of equal. This result is fundamental to deal with general cases with mixed clique sizes, where a stage-wise algorithm will be given by sequentially solving $\mathcal{P}_{1,\delta}$.

4.1 Stable Recovery Theorems

In the previous sections, we have given various sufficient conditions to recover sparse signal x_0 from the convex program $\mathcal{P}_1$, where b exactly equals Ax_0. In reality, one often meets with noisy observations with $b = Ax_0 + z$, where z accounts for noise. Extended algorithms from $\mathcal{P}_1$ to denoising has been studied extensively in the literature, under the names of BPDN [5], LASSO [6], and Dantzig selector [8], etc. These methods differ in the assumptions on the noise. In this paper, we choose $\mathcal{P}_{1,\delta}$ as we found it heuristically useful to assume bounded noise $|z| \leq \epsilon$ in our applications.

The following theorem is about the stable recovery of $\mathcal{P}_{1,\delta}$ under bounded noise assumptions, whose proof is given in the Appendix.

Theorem 3 *Assume that $\|z\|_\infty \leq \epsilon$, $|T| = s$, and the Irrepresentable condition*

$$\|A^*_{T^c} A_T (A^*_T A_T)^{-1}\|_\infty \leq \alpha < 1.$$

Then under the same condition of Theorem 2, the following error bound holds for any solution $\hat{x}_\delta$ of $\mathcal{P}_{1,\delta}$,

$$\|\hat{x}_\delta - x_0\|_1 \leq \frac{2s(\epsilon + \delta)}{1 - \alpha s} \sqrt{\binom{k}{j}}, \qquad \alpha s < 1.$$

In the particular case where $k = j + 1$, we have the following corollary.

Corollary 2 *Assume that $k = j + 1, |T| = s$, and overlap $|\sigma_1 \cap \sigma_2| \leq j - 2$ for any $\sigma_1, \sigma_2 \in T$. Then there holds $\|A^*_{T^c} A_T (A^*_T A_T)^{-1}\|_\infty \leq 1/(j + 1)$ and the following error bound for solution $\hat{x}_\delta$ of $\mathcal{P}_{1,\delta}$,*

$$\|\hat{x}_\delta - x_0\|_1 \leq \frac{2s(\epsilon + \delta)}{1 - \frac{s}{j+1}} \sqrt{j + 1}, \qquad s < j + 1.$$

4.2 Stage-wise Algorithm for Identifying Cliques of Mixed Sizes

In general, we need to deal with identifying cliques of mixed sizes. Equipped with the stability theory, we propose the following stage-wise algorithm which sequentially solve $\mathcal{P}_{1,\delta}$ under different sizes. This algorithm is found effective in the application to the Les Miserable social network.

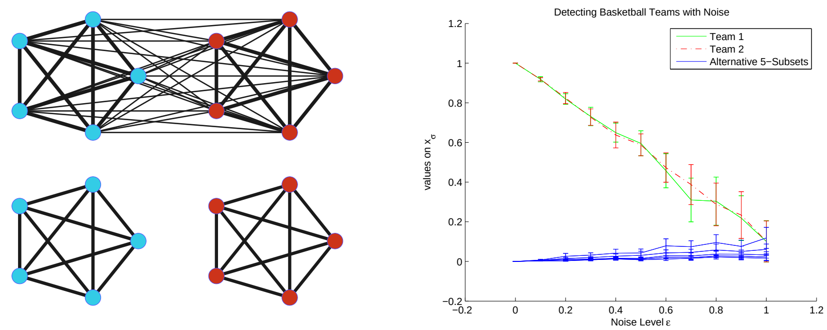
Suppose we wish to detect high order cliques of sizes $k_1, k_2, \cdots, k_l$ from low order information b on j-subsets. We built different linear programming problem $\mathcal{P}^{(i)}_{1,\delta_i}$'s with different A's and b's. We can detect cliques of sizes k_i from solving those linear programming problems to yield solutions $\hat{x}_i$. Once a solution $\hat{x}_i$ is obtained, we need to remove its effect by feeding the residue $b_i - A_i\hat{x}_i$ into the next stage as measurements. Algorithm 1 gives the pseudo code of this procedure.

Algorithm 1 Reconstructing Cliques of Different Sizes

Let $b_1 \leftarrow b$
for $i = 1$ to l **do**
  Build Matrix $A_i \leftarrow R^{j,k_i}$
  Form $\mathcal{P}^{(i)}_{1,\delta_i}$ with A_i and b_i
  Solve $\hat{x}_i$ as solution to $\mathcal{P}^{(i)}_{1,\delta_i}$
  $b_{i+1} \leftarrow b_i - A_i\hat{x}_i$
end for

5 Application Examples

We demonstrate three applications to the motivating examples given in the introduction. These applications show the effectiveness of the scheme proposed in this paper.



(a) (b)

Figure 1: Detecting Basketball Teams with Noise. (a) Two teams in a virtual Basketball Game, with intra-team interaction 1 and cross-team interaction noise no more than ϵ; (b) Under a large noise level $\epsilon < 0.9$, the two teams are identifiable.

5.1 Basket-ball team detection

Detecing two basketball teams from pairwise interactions among plays is an ideal scenario. Suppose we have x_0 which is a signal on all 5-subsets of the 10-player set. We assume it is sparsely concentrated on two 5-subsets which correspond to the two teams with magnitudes both equal to one. Assume we have observations b of pairwise interactions which are $b = Ax_0 + z$, where z is uniform random noise distributed in $[-\epsilon, \epsilon]$. We solve $\mathcal{P}_{1,\delta}$, with $\delta = \epsilon$, which is a linear programming searching over $x \in R^{\binom{10}{5}} = R^{252}$ with parameters $A \in R^{\binom{10}{2}\times\binom{10}{5}} = R^{45\times 252}$ and $b \in R^{45}$.

The two $5-$subsets correspond to the two teams have no overlap, hence satisfying the Irrepresentable Condition. In Figure 1, we try to detect the two teams under different noise levels $\epsilon \in [0, 1]$. The two basketball teams can be detected under fairly large noise level.

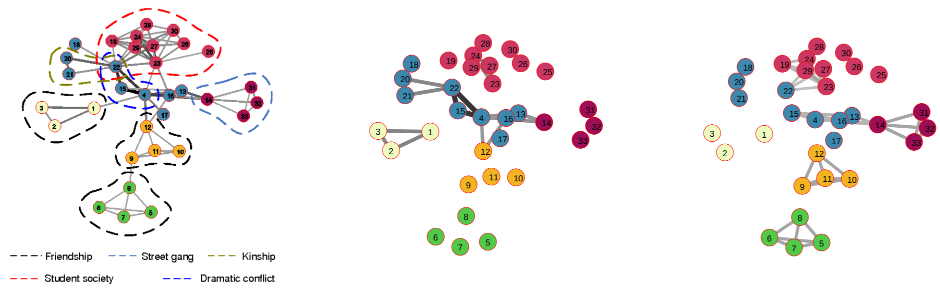
5.2 Communities in social networks

We consider the social network [3] of Victor Hugo's novel *Les Miserables*, where we extract 33 characters, and represent the social network of those characters in a weighted graph manner (Figure 2-(a)). The weights on edges represent frequencies of co-appearences. We try to detect triangles and tetrahedra from pairwise interactions by Algorithm 1, which solves $\mathcal{P}_{1,\delta}$ in a stage-wise way. Our

Table 1: The Social Network of Key Characters in *Les Miserables*

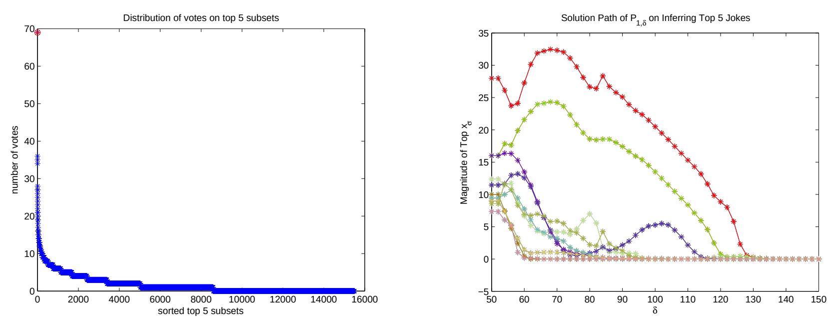
Cliques	Names of Characters	Relationships
$\{1, 2, 3\}$	{Myriel, Mlle Baptistine, Mme Magloire}	Friendship
$\{4, 12, 16\}$	{Valjean, Fantine, Javert}	Dramatic Conflicts
$\{4, 13, 14\}$	{Valjean, Mme Thenardier, Thenardier}	Dramatic Conflicts
$\{4, 15, 22\}$	{Valjean, Cosette, Marius}	Dramatic Conflicts
$\{20, 21, 22\}$	{Gillenormand, Mlle Gillenormand, Marius}	Kinship
$\{23, 24, 27\}$	{Enjolras, Combeferre, Courfeyrac}	Student Society
$\{4, 13, 14, 15\}$	{Valjean, Mme Thenardier, Thenardier, Cosette}	Dramatic Conflicts
$\{5, 6, 7, 8\}$	{Tholomyes, Listolier, Fameuil, Blacheville}	Friendship
$\{9, 10, 11, 12\}$	{Favourite, Dahlia, Zephine, Fantine}	Friendship
$\{14, 31, 32, 33\}$	{Thenardier, Gueulemer, Babet, Claquesous}	Street Gang
$\{19, 23, 24, 29\}$	{Gavroche, Enjolras, Combeferre, Bossuet}	Student Society
$\{22, 23, 27, 29\}$	{Marius, Enjolras, Courfeyrac, Bossuet}	Student Society

implementation first detects 3-cliques from pairwise interactions. Among $\binom{33}{3} = 5456$ triangles, the top 6 cliques contain those triples in Table 1. After those important 3-cliques are detected, we remove effects of pairwise interactions caused by them and go on to detect 4-cliques from the



(a) (b) (c)

Figure 2: Decomposition of *Les Miserables* social network. (a) Social Network of Characters in *Les Miserables*; (b) The identified 3-cliques; (c) The identified 4-cliques.



(a) (b)

Figure 3: (a) There is a significant top-5 jokes (in red) whose ID is $\{27, 29, 35, 36, 50\}$; (b) Regularization path where the top curve (red) selects this top group over $\delta \in [50, 130]$. Note that the top 2^{nd} curve (green) also identifies the fourth 5-subset in a persistent way.

remaining pairwise interactions. Similarly the top 6 cliques from $\binom{33}{4} = 40929$ tetrahedra are shown in Table 1. Figure 2-(b) and (c) depict such cliques respectively which contain important social community information about characters in the novel. The sparsity patterns of those cliques satisfy irrepresentable condition. However, they do not necessarily satisfy condition in Lemma 1.1 which is based on worst-case considerations. Running on a desktop with 2.4GHz CPU and 3G RAM, it takes 8.73 seconds in Matlab to compute the sparse signals on triangles and tetrahdra.

5.3 Inferring high order ranking

Jester dataset [12] contains about $24,000$ users who give ratings on 100 jokes. Those ratings are of real value ranging from -10.00 to $+10.00$. We extract top 20 jokes from the entire dataset according to mean scores. Among those 20 jokes, we count the voting on top 5-jokes by each user and view them as the ground truth. Figure 3-(a) shows that there is a top 5-subset, $\{27, 29, 35, 36, 50\}$, with an overwhelming voting than the others.

Now suppose we only know information as top 3 counts of the jokes and wonder if we can tell this most popular 5-joke group. By solving $\mathcal{P}_{1,\delta}$ with the whole regularization path by varying δ, we are capable to detect this subset (Figure 3-(b)) in a robust way.

6 Conclusion

In this paper, we propose a novel algebraic approach to study the identification of cliques based on low order interaction information. This approach exploits the Radon basis with sparse recovery algorithms rooted in Basis Pursuit. We prove that noiseless exact recovery and stable recovery with uniformly bounded noise hold under some natural conditions. We have successful applications in a simulated model of the basketball team identification, as well as two real examples, the detection of cliques in the *Les Miserables* social network and of the most popular 5-jokes in the Jester dataset. These results show the potential of broad applications of Radon basis pursuit in the studies of identity management, social networks, and statistical ranking, etc.

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A. Appendix

A.1 Notations

Given an M by N matrix A, denote by $(v_\tau) \in R^M$ the columns of the matrix A. Let $\bar{x} \in R^N$, $T = \text{supp}(\bar{x})$ and T^c be the complement of T. Denote by A_T the submatrix formed by all columns v_τ where $\tau \in T$ and A_{T^c} the submatrix formed by all columns when $\tau \in T^c$. A^* denote the conjugate of A, which is simply the matrix transpose in this paper.

A.2 Proof of Theorem 1 and Corollary 1

Proof of the following lemma can be found in [7].

Lemma A-1 *The linear programming $\mathcal{P}_1$ has a unique solution $\bar{x}$ if the matrix A_T has full rank and if one can find a vector $w \in R^N$ with the following two properties*

1. $\langle w, v_\tau \rangle = sgn((\bar{x})_\tau)$ *for all* $\tau \in T$,
2. $|\langle w, v_\tau \rangle| < 1$ *for all* $\tau \in T^c$,

where $sgn((\bar{x})_\tau)$ is the sign of $(\bar{x})_\tau$($sgn((\bar{x})_\tau) = 0$ for $(\bar{x})_\tau = 0$).

The following lemma is a result by the Karush-Kuhn-Tucker (KKT) condition of $\mathcal{P}_{1,\delta}$.

Lemma A-2 *The two conditions in Lemma A-1 are necessary and sufficient such that the linear programming $\mathcal{P}_{1,\delta}$ has a unique solution.*

Proof. Consider an alternative form of $\mathcal{P}_{1,\delta}$,

$$\begin{array}{rl} \min & 1^T \xi \\ \text{subject to} & -\delta \leq Ax - b \leq \delta \\ & -\xi \leq x \leq \xi \end{array}$$

whose Lagrangian is

$$L(x, \xi; \gamma, \lambda, \mu) = 1^T \xi - \gamma_+^T(\delta - Ax + b) - \gamma_-^T(Ax - b + \delta) - \lambda_+^T(\xi - x) - \lambda_-^T(\xi + x) - \mu^T \delta$$

Then the KKT condition gives

1. $A^*(\gamma_+ - \gamma_-) + (\lambda_+ - \lambda_-) = 0$,
2. $1 - (\lambda_+ + \lambda_-) - \mu = 0$,

with $\gamma, \lambda, \mu \geq 0$ and $\gamma_+(\tau)\gamma_-(\tau) = \lambda_+(\tau)\lambda_-(\tau) = 0$ for all τ.

Clearly $T = \{\tau : \delta_\tau > 0\}$. Define $w = \gamma_+ - \gamma_-$. Then the first equation leads to

$$\langle w, v_\tau \rangle = -(\lambda_+(\tau) - \lambda_-(\tau)) = -sgn(\bar{x}_\tau), \quad \tau \in T.$$

On the other hand, by the Strictly Complementary Theorem for linear programming [13], there are $1 > \mu_\tau > 0$ for $\tau \in T^c$ with $\delta_\tau = 0$ such that the second equation leads to

$$|\langle w, v_\tau \rangle| = |\lambda_+(\tau) - \lambda_-(\tau)| = 1 - \mu_\tau < 1,$$

which is the necessary and sufficient condition for the unique solution of $\mathcal{P}_{1,\delta}$. ◇

Theorem 1 is a direct result yielded from the two lemmas above. To see Corollary 1, note that with $M > |T|$ and the injectivity of A_T, if $w \in im(A_T)$, then the first condition in Lemma A-1 leads to

$$w = A_T(A_T^* A_T)^{-1} \iota^* sgn(\bar{x}),$$

where the imbedding operator $\iota : l_2(T) \to l_2(N)$ extends a vector on T to a vector in R^N by placing zeros outside of T and ι^* is the dual restriction $\iota^* \bar{x} = \bar{x}|_T$. With this the second condition in Lemma A-1 can be rewritten as

$$\|A_{T^c}^* A_T(A_T^* A_T)^{-1} \iota^* sgn(\bar{x})\|_\infty < 1,$$

which is exactly the Irrepresentable condition.

A.3 Proof of Lemma 1

To prove Lemma 1, given any $\tau \in T^c$, we define

$$\mu_\tau := \sum_{\sigma \in T} \frac{\binom{|\tau \cap \sigma|}{j}}{\binom{k}{j}},$$

then $\sup_{\tau \in T^c} \mu_\tau = \|A^*_{T^c} A_T\|_\infty$. As we will see in the following proofs, we essentially try to bound μ_τ for $\tau \in T^c$.

A.3.1 Proof of Lemma 1-1

Under condition 1, since any $\sigma_1, \sigma_2 \in T$ satisfy $|\sigma_1 \cap \sigma_2| \leq j - 2$, hence any two columns in T are orthogonal. This implies $A^*_T A_T$ is an identity matrix.

Now given $\tau \in T^c$, we will prove $\mu_\tau < 1$ under condition 1. If this is true, then

$$\sup_{\tau \in T^c} \mu_\tau = \|A^*_{T^c} A_T\|_\infty = \|A^*_{T^c} A_T (A^*_T A_T)^{-1}\|_\infty < 1$$

Let $T = \{\sigma_1, \sigma_2, \cdots, \sigma_{|T|}\}$ where $\sigma_i (1 \leq i \leq |T|)$ are $k-$subsets. We need to prove

$$\mu_\tau = \sum_{i=1}^{|T|} \frac{\binom{|\tau \cap \sigma_i|}{j}}{\binom{k}{j}} < 1$$

Let $A_i = \{\rho : |\rho| = j, \rho \subset \tau \cap \sigma_i\}$, so A_i is a collection of $j-$subsets of $\tau \cap \sigma_i$ (Here if $|\tau \cap \sigma_i| < j$, then A_i is simply an empty set). Obviously, we have $|A_i| = \binom{|\tau \cap \sigma_i|}{j}$. So

$$\sum_{i=1}^{|T|} \binom{|\tau \cap \sigma_i|}{j} = \sum_{i=1}^{|T|} |A_i|.$$

Now we note the fact that for any $1 \leq i, l \leq |T|$, we have $A_i \cap A_l = \emptyset$. This is true because otherwise suppose $\rho \in A_1 \cap A_2$, then this mean ρ is a $j-$subset of A_1 and A_2. Hence $\rho \subset \tau \cap \sigma_1, \rho \subset \tau \cap \sigma_2$, which implies that

$$|\sigma_1 \cap \sigma_2| \geq |(\tau \cap \sigma_1) \cap (\tau \cap \sigma_2)| \geq |\rho| \geq j$$

This contradicts with the condition that σ_i's$(1 \leq i \leq T)$ have overlaps at most $j - 2$. So A_i must be pairwise disjoint. Hence

$$\sum_{i=1}^{|T|} \binom{|\tau \cap \sigma_i|}{j} = \sum_{i=1}^{|T|} |A_i| = |\cup_{i=1}^{|T|} A_i|$$

For any $1 \leq i \leq |T|$, every $\rho \in A_i$ is a $j-$subset of $\tau \cap \sigma_i$. Hence ρ is of course a $j-$subset of τ. The set τ is of size k. So if we let $A_0 = \{\rho : |\rho| = j, \rho \subset \tau\}$ which is the collection of all $j-$subsets of τ, then we have $\cup_{i=1}^{|T|} A_i \subset A_0$. So $|\cup_{i=1}^{|T|} A_i| \leq |A_0| \leq \binom{k}{j}$.

Till now, we actually proved $\mu_\tau \leq 1$. All the above proof about $\mu_\tau \leq 1$ for any $\tau \in T^c$ will remain valid for condition 2. In the next, we prove if any $\sigma_i, \sigma_l \in T$ satisfy $|\sigma_i \cap \sigma_l| \leq j - 2$, then equality can not hold.

Without loss of generality, we assume $|\sigma_1 \cap \tau| \geq j$, otherwise if none of σ_i's satisfies $|\sigma_i \cap \tau| \geq j$, then $\mu_\tau = 0$ which actually finishes the proof. In this case, we can let $\tau = \{1, 2, \cdots, k\}$, $\sigma_1 = \{1, 2, \cdots, s, k+1, k+2, 2k-s\}$ where $j \leq s \leq k-1$($s \leq k-1$ because otherwise $\sigma_1 = \tau$ which contradicts with the fact that $\sigma_1 \in T, \tau \in T^c$). Now we show that $\rho_0 = \{1, 2, \cdots, j-1, s+1\}$ is not a member of $\cup_{i=1}^{|T|} A_i$. Clearly ρ_0 is not a member of A_1 because $s + 1 \notin \sigma_1$. Now it remains to show that ρ_0 is not a member of any $A_i (2 \leq i \leq |T|)$. If this was not true, say $\rho_0 \in A_2$, then $\rho_0 \subset (\tau \cap \sigma_2) \subset \sigma_2$, then $\{1, 2, \cdots, j-1\} \subset \sigma_1 \cap \sigma_2$, which contradicts with the condition that $|\sigma_1 \cap \sigma_2| \leq j - 2$.

While it is clear that $\rho_0 \in A_0$, so this means $\cup_{i=1}^{|T|} A_i$ is a proper subset of A_0. So $|\cup_{i=1}^{|T|} A_i| < \binom{k}{j}$ which means $\mu_\tau < 1$. $\diamond$

A.3.2 Proof of Lemma 1-2

Under condition 2, then almost the same as proof for lemma 1. We have $A_T^*A_T$ is an identity matrix and $\mu_\tau \leq 1$. However, one can not show $\mu_\tau < 1$ in this case. We have the following example where if n is large enough, then μ_τ can happens to be equal to one exactly.

Let $\tau = \{1, 2, \cdots, k\} \in T^c$. Denote all the $j-$subsets of τ to be $\rho_1, \rho_2, \cdots, \rho_{\binom{k}{j}}$. For n is large enough, we choose $\binom{k}{j}$ disjoint $(k-j)$-subsets of $\{k+1, k+2, \cdots, n\}$, denoted by $\omega_1, \omega_2, \cdots, \omega_{\binom{k}{j}}$.

Let $T = \{\sigma_1, \sigma_2, \cdots, \sigma_{|T|}\}$, where $\sigma_i = \rho_i \cup \omega_i$. Hence $|T| = \binom{k}{j}$ and σ_i's satisfy $|\sigma_i \cap \sigma_j| \leq j-1$. But

$$\sum_{i=1}^{|T|} \frac{\binom{|\tau \cap \sigma_i|}{j}}{\binom{k}{j}} = \sum_{i=1}^{|T|} \frac{1}{\binom{k}{j}} = 1$$

$\diamond$

A.3.3 Proof of Lemma 1-3

Under condition 3, we can construct examples where $\|A_{T^c}^*A_T(A_T^*A_T)^{-1}\|_\infty > 1$. Let $\rho_1, \rho_2, \cdots, \rho_{\binom{k}{j}}$ be all j-subsets of $\{1, 2, \cdots, k\}$. For large enough n, it is possible to choose $\binom{k}{j}+1$ disjoint $(k-j)$-subsets of $\{k+1, k+2, \cdots, n\}$, say $\omega_0, \omega_1, \omega_2, \cdots, \omega_{\binom{k}{j}}$. Let $\sigma_i = \rho_i \cup \omega_i$ for $1 \leq i \leq \binom{k}{j}$ and $\sigma_0 = \rho_1 \cup \omega_0$. Define $T = \{\sigma_0, \sigma_1, \sigma_2, \cdots, \sigma_{\binom{k}{j}}\}$ which is of size $|T| = \binom{k}{j} + 1$.

In this case, $|\sigma_i \cap \sigma_l| = j-1$ for any $1 \leq i, l \leq \binom{k}{j}$ and $|\sigma_0 \cap \sigma_1| = j$, $|\sigma_0 \cap \sigma_i| \leq j-1$ for any $2 \leq i \leq \binom{k}{j}$. Then $A_T^*A_T$ is a $\binom{k}{j}+1$ by $\binom{k}{j}+1$ matrix shown belog with rows and columns corresponds to $\{\sigma_0, \sigma_1, \cdots, \sigma_{\binom{k}{j}}\}$

$$A_T^*A_T = \left[\begin{array}{cc|cccc} 1 & \epsilon & 0 & 0 & \cdots & 0 \\ \epsilon & 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{array}\right]$$

Here $\epsilon = \frac{1}{\binom{k}{j}}$. The inverse of the matrix is

$$(A_T^*A_T)^{-1} = \left[\begin{array}{cc|cccc} \frac{1}{1-\epsilon^2} & -\frac{\epsilon}{1-\epsilon^2} & 0 & 0 & \cdots & 0 \\ -\frac{\epsilon}{1-\epsilon^2} & \frac{1}{1-\epsilon^2} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{array}\right]$$

Consider $\tau = \{1, 2, \cdots, k\} \in T^c$, then the row corresponds to τ for $A_{T^c}^*A_T$ is a vector of length $|T| = \binom{k}{j} + 1$ with each entry being $\epsilon = \frac{1}{\binom{k}{j}}$. So the row vector corresponds to τ in $A_{T^c}^*A_T(A_T^*A_T)^{-1}$ is a vector of length $\binom{k}{j}+1$, $[\frac{\epsilon}{1+\epsilon}, \frac{\epsilon}{1+\epsilon}, \epsilon, \epsilon, \cdots, \epsilon]$. This vector has row sum

$$\frac{2\epsilon}{1+\epsilon} + (\binom{k}{j} - 1)\epsilon = \frac{2\epsilon}{1+\epsilon} + (\frac{1}{\epsilon} - 1)\epsilon = \frac{1+2\epsilon-\epsilon^2}{1+\epsilon} > \frac{1+2\epsilon-\epsilon}{1+\epsilon} = 1$$

Hence in this example $\|A_{T^c}^*A_T(A_T^*A_T)^{-1}\|_\infty > 1$. $\diamond$

A.4 Proof of Theorem 3 and Corollary 2

Lemma A-3 *Assume that $\|z\|_\infty \leq \epsilon$, $|T| = s$, and the Irrepresentable condition*

$$\|A_{T^c}^*A_T(A_T^*A_T)^{-1}\|_\infty \leq \alpha < 1.$$

Then the following error bound holds for any solution $\hat{x}_\delta$ of $\mathcal{P}_{1,\delta}$,

$$\|\hat{x}_\delta - \bar{x}\|_1 \leq \frac{2s(\epsilon+\delta)}{1-\alpha s}\|A_T(A_T^*A_T)^{-1}\|_1.$$

Theorem 3 follows from the Lemma above. Note that when the conditions in Theorem 2 hold, $A_T^*A_T = I$ and $\|A_T\|_1 \leq \sqrt{\binom{k}{j}}$, which yields Theorem 3.

Proof of Lemma A-3. Note that $\|A\hat{x}_\delta - b\|_\infty \leq \delta$ and $z = A\bar{x} - b$ with $\|z\|_\infty \leq \epsilon$. Then

$$\|Ah\|_\infty = \|A\hat{x}_\delta - A\bar{x}\|_\infty = \|A\hat{x}_\delta - b + b - A\bar{x}\|_\infty \leq \|A\bar{x}_\delta - b\|_\infty + \|z\|_\infty \leq \delta + \epsilon. \tag{3}$$

Let $h = \hat{x}_\delta - \bar{x}$. By $\|\bar{x}\|_1 \geq \|\hat{x}\|_1$,

$$\|h_T\|_1 = \|\bar{x} - \hat{x}_\delta|_T\|_1 \geq \|\bar{x}\|_1 - \|\hat{x}_\delta|_T\|_1 \geq \|\hat{x}_\delta\|_1 - \|\hat{x}_\delta|_T\|_1 = \|\hat{x}_\delta|_{T^c}\|_1 = \|h_{T^c}\|_1. \tag{4}$$

Therefore,

$$\begin{aligned}
|\langle Ah, A_T(A_T^*A_T)^{-1}h_T\rangle| &= |\langle A_T h_T, A_T(A_T^*A_T)^{-1}h_T\rangle + \langle A_{T^c}h_{T^c}, A_T(A_T^*A_T)^{-1}h_T\rangle| \\
&\geq \|h_T\|_2^2 - |\langle h_{T^c}, A_{T^c}^*A_T(A_T^*A_T)^{-1}h_T\rangle| \\
&\geq \|h_T\|_2^2 - \|h_{T^c}\|_1\|A_{T^c}^*A_T(A_T^*A_T)^{-1}h_T\|_\infty \\
&\geq \frac{1}{s}\|h_T\|_1^2 - \alpha\|h_{T^c}\|_1\|h_T\|_\infty \\
&\geq \frac{1}{s}\|h_T\|_1^2 - \alpha\|h_{T^c}\|_1\|h_T\|_1 \\
&\geq \left(\frac{1}{s} - \alpha\right)\|h_T\|_1^2
\end{aligned}$$

where the last step is due to $\|h_T\|_1 \geq \|h_{T^c}\|_1$ in the inequality (4). On the other hand,

$$\begin{aligned}
|\langle Ah, A_T(A_T^*A_T)^{-1}h_T\rangle| &\leq \|Ah\|_\infty\|A_T(A_T^*A_T)^{-1}h_T\|_1 \\
&\leq (\delta+\epsilon)\|A_T(A_T^*A_T)^{-1}\|_1\|h_T\|_1
\end{aligned}$$

using (3). Combining these two inequalities yields

$$\|h_T\|_1 \leq \frac{s(\delta+\epsilon)}{1-\alpha s}\|A_T(A_T^*A_T)^{-1}\|_1,$$

as desired. $\diamond$

Proof of Corollary 2 It suffice to eatablish the fact that in this special case, we have

$$\|A_{T^c}^*A_T(A_T^*A_T)^{-1}\|_\infty \leq \frac{1}{j+1} < 1$$

Note that since any $\sigma_1, \sigma_2 \in T$ satisfy $|\sigma_1 \cap \sigma_2| \leq j-2$, we have $A_T^*A_T$ is an identity matrix. So $\|A_{T^c}^*A_T(A_T^*A_T)^{-1}\|_\infty = \|A_{T^c}^*A_T\|_\infty$. Now assume $\tau \in T^c$, let $S_\tau = \{\sigma : |\sigma\cap\tau| \geq j, \sigma \in T\}$, then $|S_\tau| \leq 1$. This is because otherwise, suppose $\{\sigma_1, \sigma_2\} \subset S_\tau$ such that $|S_\tau| \geq 2$, then we have

$$\begin{aligned}
|\tau| &\geq |\tau \cap (\sigma_1 \cup \sigma_2)| = |\tau\cap\sigma_1| + |t \cap \sigma_2| - |t \cap \sigma_1 \cap \sigma_2| \\
&\geq j + j - (j-2) = j+2
\end{aligned}$$

which contradicts with the fact that τ is a $j+1$-subset. So there exist at most one $\sigma_0 \in T$ such that $|\tau \cap \sigma| \geq j$. Let v_τ be the row vector of $A_{T^c}^*A_T$ with row index correspond to τ. Then $\|v_\tau\|_\infty \leq \frac{\binom{j}{j}}{\binom{j+1}{j}} = \frac{1}{j+1} < 1$. $\diamond$