Q5

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a) Consider two parallel lines with slope M = $\begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}$ and passing through the points:

$X_1 = \begin{pmatrix} x_{1x} \\ x_{1y} \\ x_{1z} \end{pmatrix}$ and $X_2 = \begin{pmatrix} x_{2x} \\ x_{2y} \\ x_{2z} \end{pmatrix}$

So in homogeneous coordinates arbitrary points on these lines will be represented in parametric form as:

$L_1 = \begin{pmatrix} x_{1x} \\ x_{1y} \\ x_{1z} \\ 1 \end{pmatrix} + t_1 \begin{pmatrix} m_x \\ m_y \\ m_z \\ 0 \end{pmatrix}$ and $L_2 = \begin{pmatrix} x_{2x} \\ x_{2y} \\ x_{2z} \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} m_x \\ m_y \\ m_z \\ 0 \end{pmatrix}$

Ideal perspective projection on image plane is given by the transformation matrix $P = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

On applying the tranformation matrix, equations of the lines become:

$L_1^1 = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{1x} + t_1 m_x \\ x_{1y} + t_1 m_y \\ x_{1z} + t_1 m_z \\ 1 \end{pmatrix} = \begin{pmatrix} cx_{1x} + ct_1 m_x \\ cx_{1y} + ct_1 m_y \\ x_{1z} + t_1 m_z \end{pmatrix}$

$L_2^1 = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{2x} + t_2 m_x \\ x_{2y} + t_2 m_y \\ x_{2z} + t_2 m_z \\ 1 \end{pmatrix} = \begin{pmatrix} cx_{2x} + ct_2 m_x \\ cx_{2y} + ct_2 m_y \\ x_{2z} + t_2 m_z \end{pmatrix}$

For intersection:

$$\frac{cx_{1x} + ct_1 m_x}{x_{1z} + t_1 m_z} = \frac{cx_{2x} + ct_2 m_x}{x_{2z} + t_2 m_z} \tag{1}$$

and

$$\frac{cx_{1y} + ct_1 m_y}{x_{1z} + t_1 m_z} = \frac{cx_{2y} + ct_2 m_y}{x_{2z} + t_2 m_z} \tag{2}$$

Solution exists when $t_1 \to \infty$ and $t_2 \to \infty$

So they will intersect each other on the image plane at the point:

$X_{12} = \begin{pmatrix} \frac{cm_x}{m_z} \\ \frac{cm_y}{m_z} \\ 1 \end{pmatrix}$

Hence Proved that the projections (in image plane) of any two parallel lines have an intersecting point, vanishing point.

b) We need 3 different set of parallel lines on a 3D plane. Let the slopes of these set of parallel lines be:

$M_1 = \begin{pmatrix} m_{1x} \\ m_{1y} \\ m_{1z} \end{pmatrix}$, $M_2 = \begin{pmatrix} m_{2x} \\ m_{2y} \\ m_{2z} \end{pmatrix}$ and $M_3 = \begin{pmatrix} m_{3x} \\ m_{3y} \\ m_{3z} \end{pmatrix}$

Since they are on same plane, dot product of direction of lines with normal to the plant is 0.

Let the normal to the plane be $N = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$

We have $M_1^T N = 0$, $M_2^T N = 0$ and $M_3^T N = 0$

$$\begin{pmatrix} m_{1x} & m_{1y} & m_{1z} \\ m_{2x} & m_{2y} & m_{2z} \\ m_{3x} & m_{3y} & m_{3z} \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Non zero solution exist for N, hence matrix is not invertible and its determinant is zero.

Therefore:

$$\begin{vmatrix} m_{1x} & m_{1y} & m_{1z} \\ m_{2x} & m_{2y} & m_{2z} \\ m_{3x} & m_{3y} & m_{3z} \end{vmatrix} = 0$$

$$\frac{1}{c^2} \begin{vmatrix} cm_{1x} & cm_{1y} & m_{1z} \\ cm_{2x} & cm_{2y} & m_{2z} \\ cm_{3x} & cm_{3y} & m_{3z} \end{vmatrix} = 0$$

$$\frac{m_{1z}m_{2z}m_{3z}}{c^2} \begin{vmatrix} \frac{cm_{1x}}{m_{1z}} & \frac{cm_{1y}}{m_{1z}} & 1 \\ \frac{cm_{2x}}{m_{2z}} & \frac{cm_{2y}}{m_{2z}} & 1 \\ \frac{cm_{3x}}{m_{3z}} & \frac{cm_{3y}}{m_{3z}} & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} \frac{cm_{1x}}{m_{1z}} & \frac{cm_{1y}}{m_{1z}} & 1 \\ \frac{cm_{2x}}{m_{2z}} & \frac{cm_{2y}}{m_{2z}} & 1 \\ \frac{cm_{3x}}{m_{3z}} & \frac{cm_{3y}}{m_{3z}} & 1 \end{vmatrix} = 0 \tag{3}$$

From solution of part a), we can conclude that the vanishing points for these 3 set of lines on projection will be:

$V_1 = \begin{pmatrix} \frac{cm_{1x}}{m_{1z}} \\ \frac{cm_{1y}}{m_{1z}} \\ 1 \end{pmatrix}$, $V_2 = \begin{pmatrix} \frac{cm_{2x}}{m_{2z}} \\ \frac{cm_{2y}}{m_{2z}} \\ 1 \end{pmatrix}$ and $V_3 = \begin{pmatrix} \frac{cm_{3x}}{m_{3z}} \\ \frac{cm_{3y}}{m_{3z}} \\ 1 \end{pmatrix}$

Area of triangle formed by these points is given by:

$$\frac{1}{2} \begin{vmatrix} \frac{cm_{1x}}{m_{1z}} & \frac{cm_{1y}}{m_{1z}} & 1 \\ \frac{cm_{2x}}{m_{2z}} & \frac{cm_{2y}}{m_{2z}} & 1 \\ \frac{cm_{3x}}{m_{3z}} & \frac{cm_{3y}}{m_{3z}} & 1 \end{vmatrix}$$

Using equation (3), we get area of the triangle formed by these points is 0

Hence proved that the vanishing points corresponding to three (different) sets of parallel lines on a 3D plane are collinear in the image plane.