南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Signals and Systems (Lab)

Lab 3 : Fourier Series Representation of Periodic Signals

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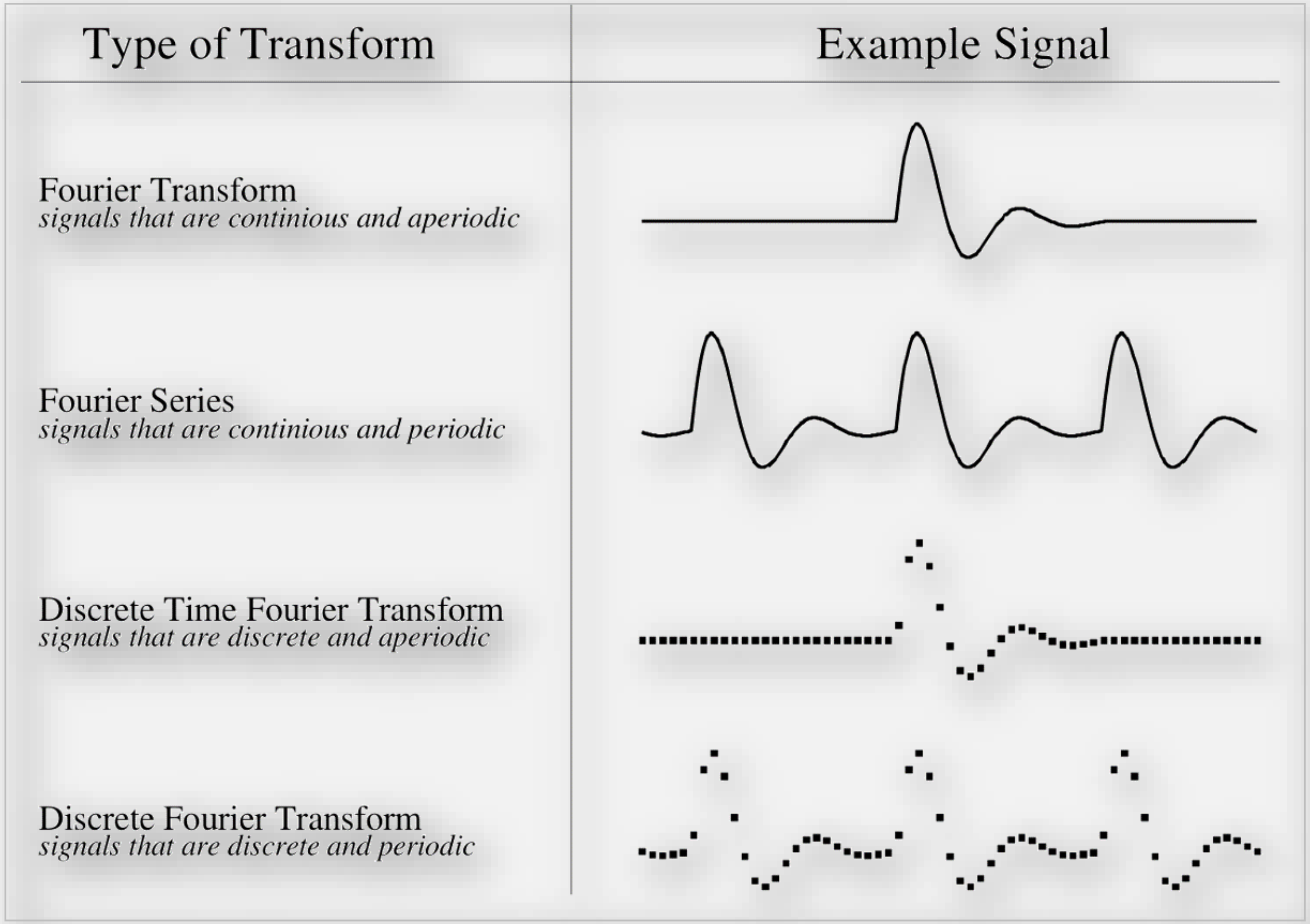
Feedback

Baron Jean Baptiste Joseph Fourier



Baron Jean Baptiste Joseph Fourier
法国数学家、物理学家，1768-1830

Overview

- In Lab 3, you will
  - Verify the **frequency** property of convolution.
  - Verify the **frequency** property of LTI systems.

- In this tutorial, you will learn
  - How to calculate the output of DT LTI system in frequency domain.
  - How to calculate the output of CT LTI system.
  - How to calculate the DTFS of signal.

Eigenvalue and eigenvector

$$\boldsymbol{A}\boldsymbol{X} = \underbrace{\lambda}_{\text{eigenvalue}}\underbrace{\boldsymbol{X}}_{\text{eigenvector}}$$

The eigenvector X has only scalar multiplication under the mapping A.

Example: Complex Exponentials

$$x(t) = e^{st} \longrightarrow \boxed{\;h(t)\;} \xrightarrow{h(t) * e^{st}}$$

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau \\
&= \left[\int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau \right] e^{st} \\
&= \underbrace{H(s)}_{\text{eigenvalue}} \underbrace{e^{st}}_{\text{eigenfunction}}
\end{aligned}
$$

$s = j\omega$ - purely imaginary,

i.e. signals of the form $e^{j\omega t}$

DT LTI System

$e^{j\omega n}$ $\rightarrow$ LTI System: H(.) $\rightarrow$ $H(e^{j\omega})e^{j\omega n}$

$$a_0 e^{j0\left(\frac{2\pi}{N}\right)n}$$
$$a_1 e^{j1\left(\frac{2\pi}{N}\right)n}$$
$$a_2 e^{j2\left(\frac{2\pi}{N}\right)n}$$
$$\dots$$
$$a_{N-1} e^{j(N-1)\left(\frac{2\pi}{N}\right)n}$$

$\Rightarrow$ LTI System: H(.) $\Rightarrow$

$$a_0 H\left(e^{j0\left(\frac{2\pi}{N}\right)}\right) e^{j0\left(\frac{2\pi}{N}\right)n}$$
$$a_1 H\left(e^{j1\left(\frac{2\pi}{N}\right)}\right) e^{j1\left(\frac{2\pi}{N}\right)n}$$
$$a_2 H\left(e^{j2\left(\frac{2\pi}{N}\right)}\right) e^{j2\left(\frac{2\pi}{N}\right)n}$$
$$\dots$$
$$a_{N-1} H\left(e^{j(N-1)\left(\frac{2\pi}{N}\right)}\right) e^{j(N-1)\left(\frac{2\pi}{N}\right)n}$$

Can we calculate response of an LTI system in another way ?

In Lab 2, difference equation is like …

- Causal DT LTI system can be specified by a linear constant-coefficient difference equation:

$$\boxed{\sum_{k=0}^{K} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]}$$

- Causal DT LTI system is uniquely specified by two vectors: A=[a_0 a_1 a_2 … a_K] and B=[b_0 b_1 … b_M]

We can use filter() to calculate h[n]

- ✓ For example:
  - $y[n]=0.5x[n]+x[n-1]+2x[n-2]$;
  - **h[n]=? Finite Impulse Response (FIR)**
  - $y[n]-0.8y[n-1]=2x[n]$;
  - **h[n]=? Infinite Impulse Response (IIR)**

- ✓ Causal DT LTI system is uniquely specified by two vectors: $A=[a_0\ a_1\ a_2 \ldots a_K]$ and $B=[b_0\ b_1 \ldots b_M]$
  - A=[1] B=[0.5 1 2]
  - A=[1 -0.8] B=[2]

Calculate Frequency Response

- In this Lab, we will use freqz() to calculate Frequency Response :

```
[H omega] = freqz(b, a, N);
```

$$H(e^{j\omega_k}) \quad \omega_k = \left(\frac{\pi}{N}\right)k, 0 \leq k \leq N-1$$

```
[H omega] = freqz(b, a, N, 'whole');
```

$$H(e^{j\omega_k}) \quad \omega_k = \left(\frac{2\pi}{N}\right)k, 0 \leq k \leq N-1$$

Exercise 1: Frequency response

- Consider LTI System:

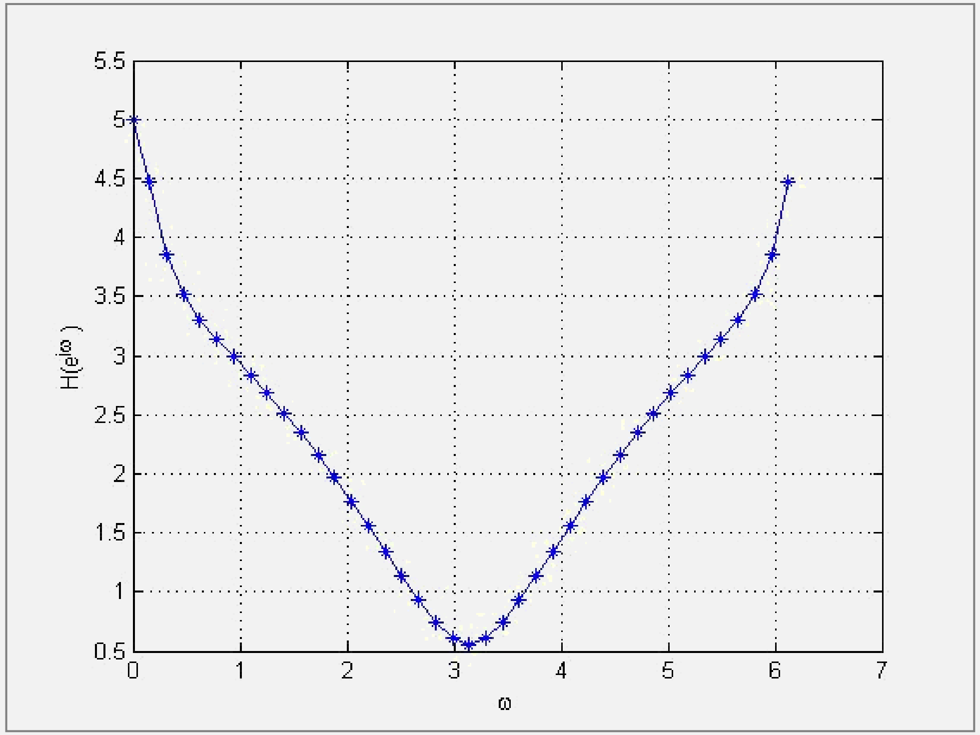
$$y[n]-0.8y[n-1]=2x[n]-x[n-2]$$

Define the vector of coefficients:

```
A=[1 -0.8];
B=[2 0 -1];
```

Plot the frequency response:

```
[H Omega] = freqz(B, A, 40, 'whole');
plot(Omega, abs(H), '*-');
xlabel('\omega');
ylabel('H(e^{j\omega} )');
grid;
```



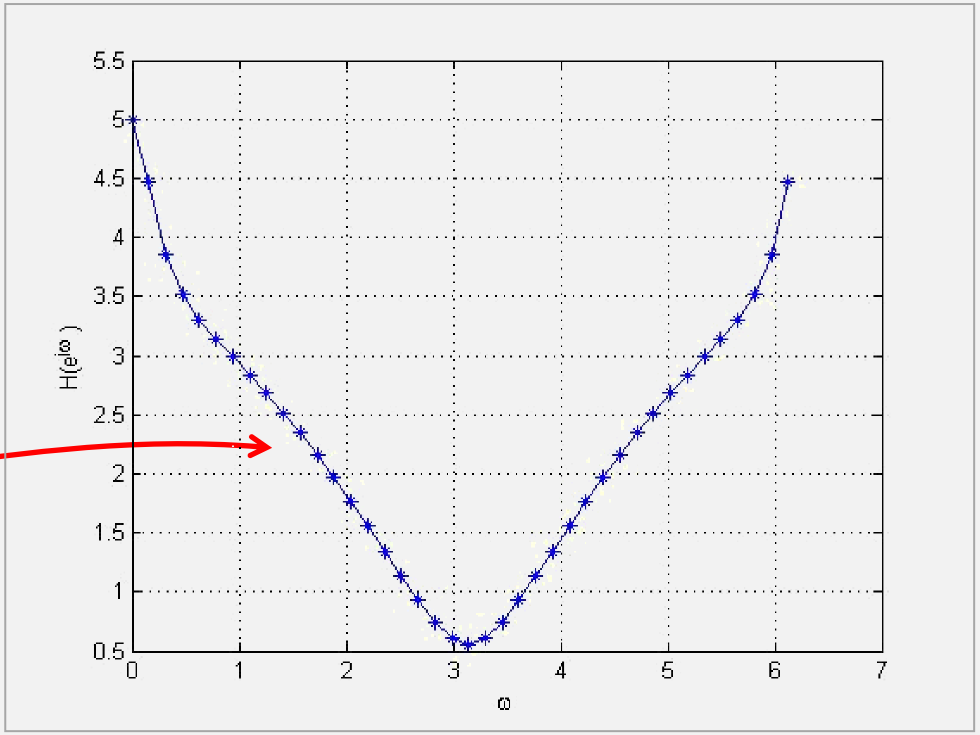
$$\boxed{a_0 H(e^{j0\left(\frac{2\pi}{N}\right)})e^{j0\left(\frac{2\pi}{N}\right)n}}$$

$$\boxed{a_1 H(e^{j1\left(\frac{2\pi}{N}\right)})e^{j1\left(\frac{2\pi}{N}\right)n}}$$

$$\boxed{a_2 H(e^{j2\left(\frac{2\pi}{N}\right)})e^{j2\left(\frac{2\pi}{N}\right)n}}$$

…

$$\boxed{a_{N-1} H(e^{j(N-1)\left(\frac{2\pi}{N}\right)})e^{j(N-1)\left(\frac{2\pi}{N}\right)n}}$$



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  - Verify the **frequency** property of LTI systems.

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  - **How to calculate the output of CT LTI system.**
  - How to calculate the DTFS of signal.

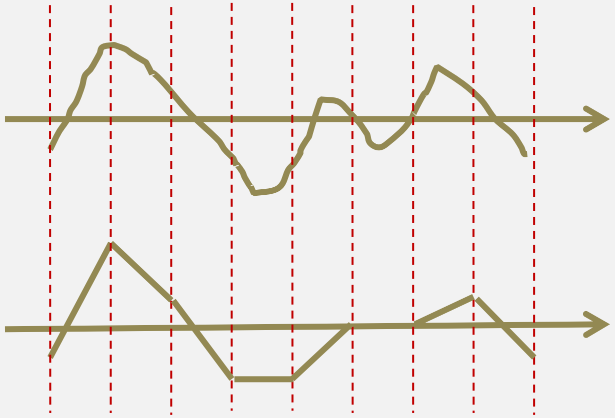
CT LTI System by Differential Equation

- Reading assignment: textbook 2.4.1.
- Causal CT LTI system can be specified by a linear constant-coefficient differential equation:

$$\boxed{\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}}$$

- Coefficient vectors:

$$\boxed{\begin{aligned} \boldsymbol{A} &= [\boldsymbol{a}_K\ \boldsymbol{a}_{K-1} \cdots\ \boldsymbol{a}_0] \\ \boldsymbol{B} &= [\boldsymbol{b}_K\ \boldsymbol{b}_{K-1} \cdots\ \boldsymbol{b}_0] \end{aligned}}$$



Attention !

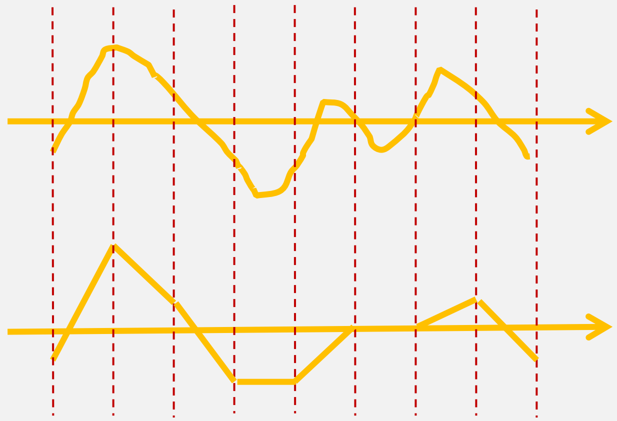
- CT LTI system by differential equation
  - $\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$
  - $A = [a_K, a_{K-1}, \dots a_0]$
  - $B = [b_M, b_{M-1}, \dots b_0]$
- DT LTI system by difference equation
  - $\sum_{k=0}^{K} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$
  - $A = [a_0, a_1, \dots a_K]$
  - $B = [b_0, b_1, \dots b_M]$

How to simulate CT systems ?

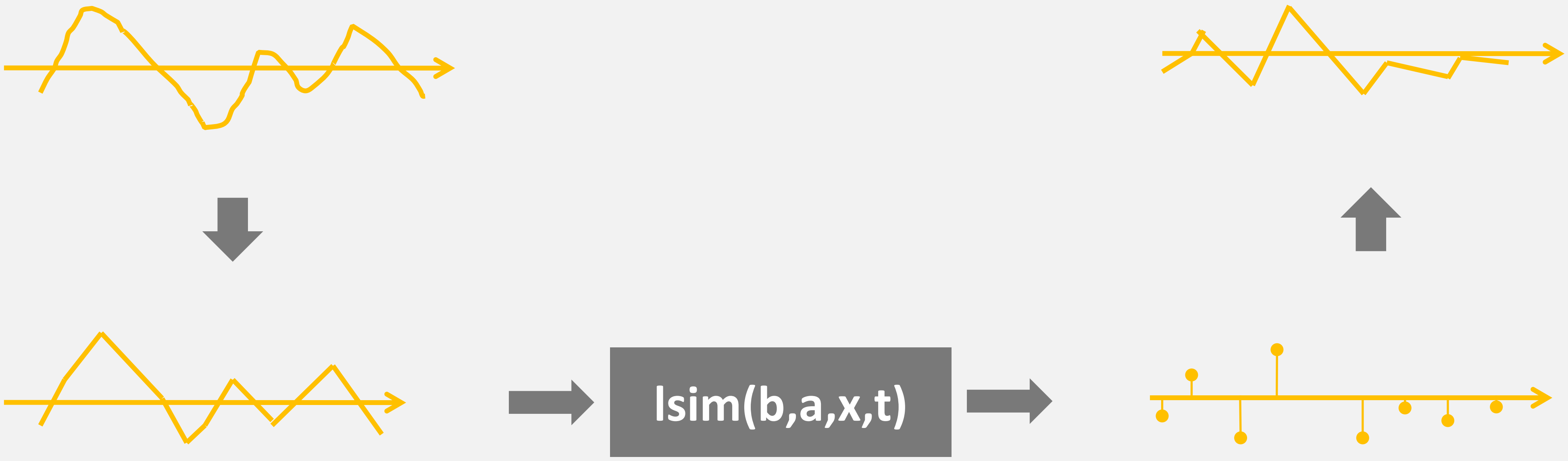
- How to simulate CT systems via Matlab?
- **lsim():** generate sampled output according to sampled input signal and CT system function
- **Syntax:** lsim(b,a,x,t)
- Sampled input signal

  Vector of sampling time: t

  Vector of sampled value: x



Simulation process



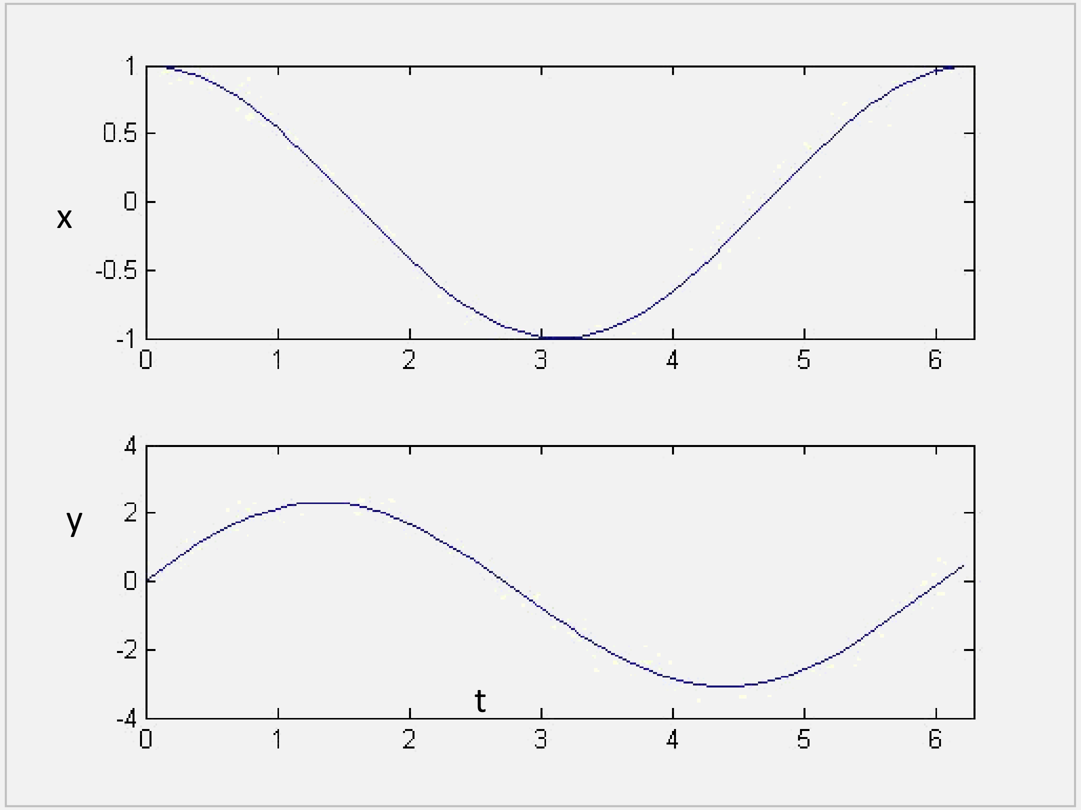
Exercise 2: CT System

- Consider LTI System: $0.3y(t)+dy(t)/dt = 3x(t)$

```
A=[1 0.3];
B=3;
```

Sample the input signal x=cos(t):

```
t=0:0.1:2*pi;
x=cos(t);
y=lsim(B,A,x,t)';
subplot(2,1,1), plot(t,x);
xlim([0 2*pi]);
subplot(2,1,2), plot(t,y);
xlim([0 2*pi]);
```



Tips

- Differential equation

$$\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

- System function

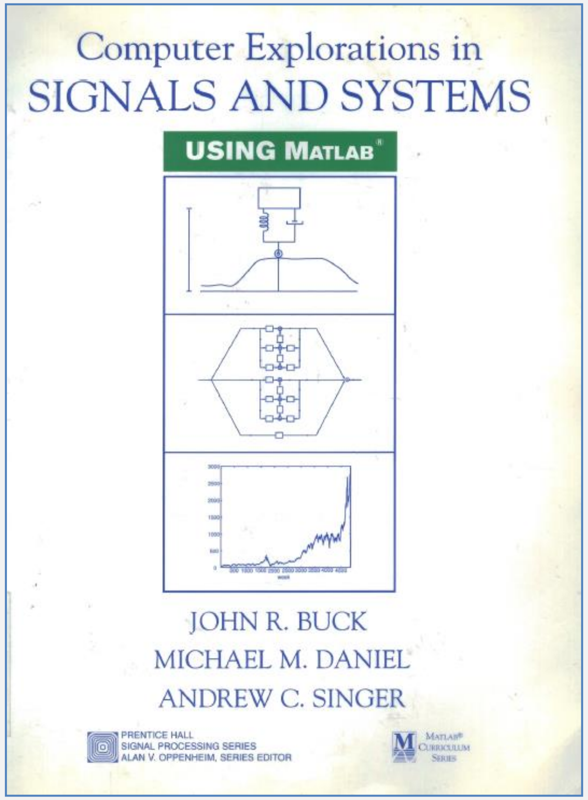
$$\Updownarrow$$

Tutorial 2.3 & 3.3

$$H(s) = \frac{\sum_{m=0}^{M} b_m s^m}{\sum_{k=0}^{K} a_k s^k}$$

Lab Assignment 3 (a)

- Read tutorial 3.2 & 3.3 by yourself
- 3.8 & 3.9 (Optional)
- Submit your report.



- 3.9(c)

Advanced Problems

(c). Analytically calculate the CTFS for the square wave x2. You may find it helpful to first find a relationship between the signal $x_2(t)$ and the signal $s(t)$ defined in Eq. (3.9). Use the ten lowest frequency nonzero CTFS coefficients of x2 to create the first 5 harmonic components individually. For example if you have the positive and

$$s(t) = \begin{cases} 1\,, & |t| < T/4\,, \\ 0\,, & T/4 \le |t| \le T/2 \end{cases} \tag{3.9}$$

CTFS coefficients a_k given by

$$a_k = \frac{\sin(\pi k/2)}{\pi k}.$$

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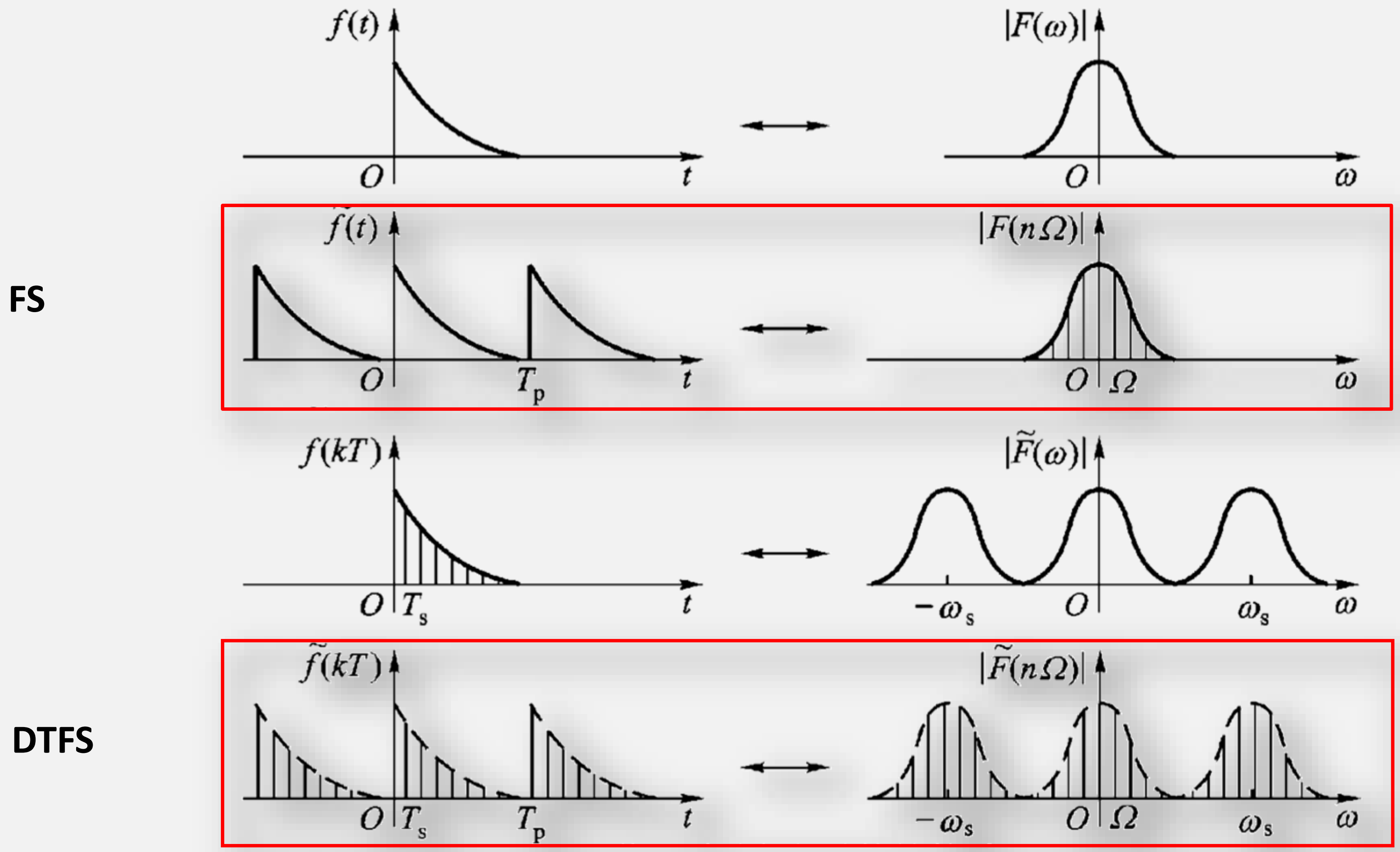


$f(t)$
O
t
$|F(\omega)|$
O
ω
FS
$\tilde{f}(t)$
O
T_p
t
$|F(n\Omega)|$
O
Ω
ω
$f(kT)$
O
T_s
t
$|\tilde{F}(\omega)|$
$-\omega_s$
O
ω_s
ω
DTFS
$\tilde{f}(kT)$
O
T_s
T_p
t
$|\tilde{F}(n\Omega)|$
$-\omega_s$
O
Ω
ω_s
ω

Fourier Series

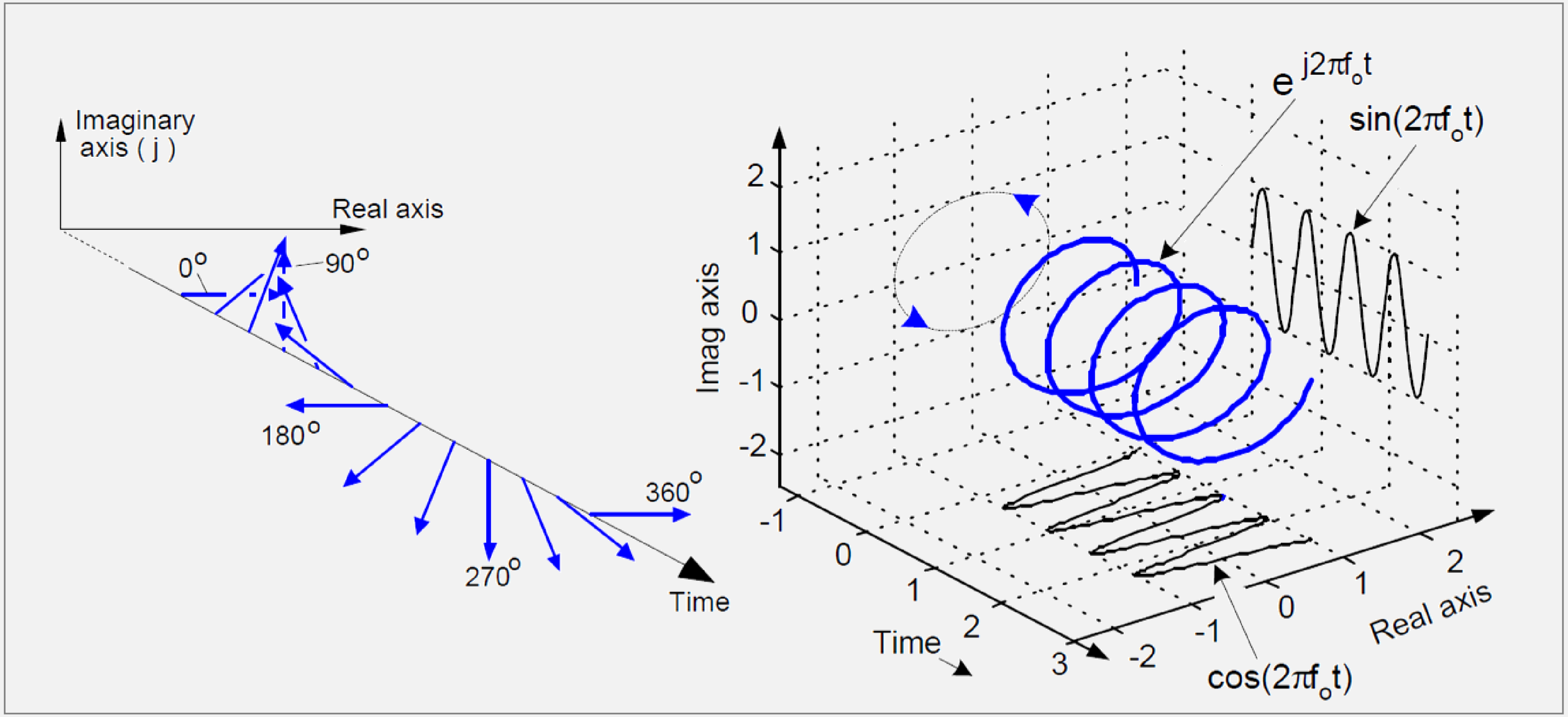
- Periodic signal with period T or N
- Synchesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \quad \text{v.s.} \quad x[n] = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

  – Summation of ***N harmonic components***
- Analysis equation:

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\left(\frac{2\pi}{T}\right)t} dt \quad \text{v.s.} \quad a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Understanding $e^{j2\pi f_0 t}$



Quadrature Signals: Complex, But Not Complicated--Richard Lyons

Matlab Function: fft()

- fft(): compute DTFS coefficients from signals

```
>> help fft
                    N
    X(k) =        sum  x(n)*exp(-j*2*pi*(k-1)*(n-1)/N), 1 <= k <= N.
                   n=1
```

Compare with our definition:

$$a_k = \sum_{n=1}^{N} \mathrm{x[n]} e^{-j(k-1)\left(\frac{2\pi}{N}\right)(n-1)} \quad \textbf{v.s.} \quad a_k = \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{x[n]} e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Calculate the DTFS of vector x:

```
a = (1/N) * fft(x)
```

Matlab Function: ifft()

- ifft(): reconstruct signals from DTFS coefficients

```
>> help fft
                    N
    x(n) = (1/N) sum  X(k)*exp( j*2*pi*(k-1)*(n-1)/N), 1 <= n <= N.
                   k=1
```

Compare with our definition:

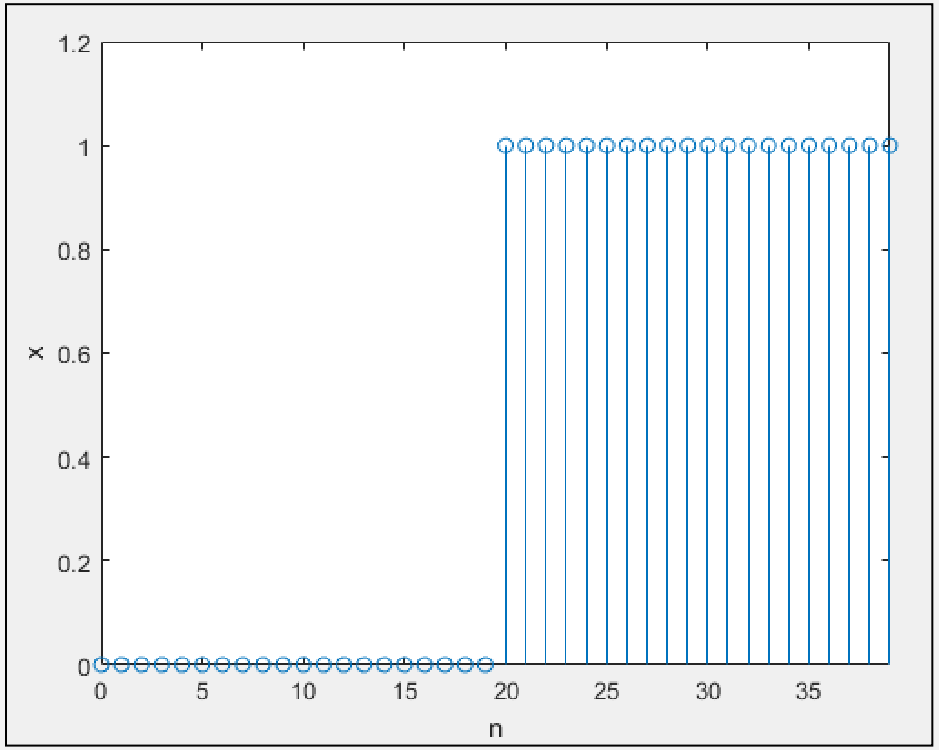
$$\mathbf{x[n]} = \frac{1}{N}\sum_{k=1}^{N} a_k e^{j(k-1)\left(\frac{2\pi}{N}\right)(n-1)} \quad \text{v.s.} \quad \mathbf{x[n]} = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

Calculate the DTFS of vector x: x = N * ifft(a)

Example

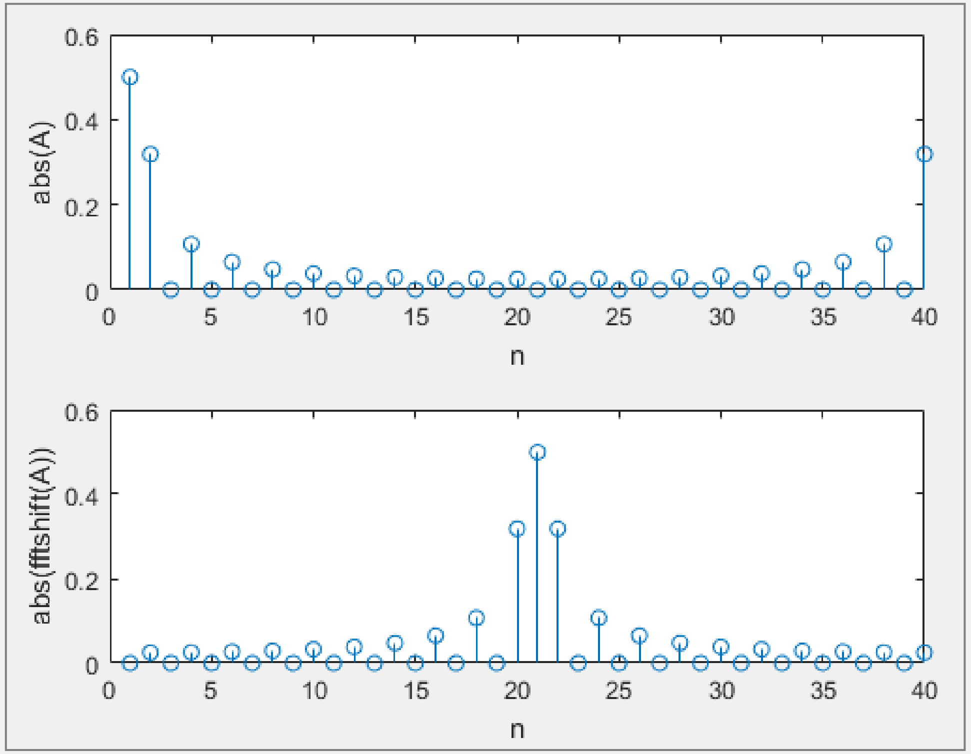
- Periodic DT rectangular wave with ***period = 40***

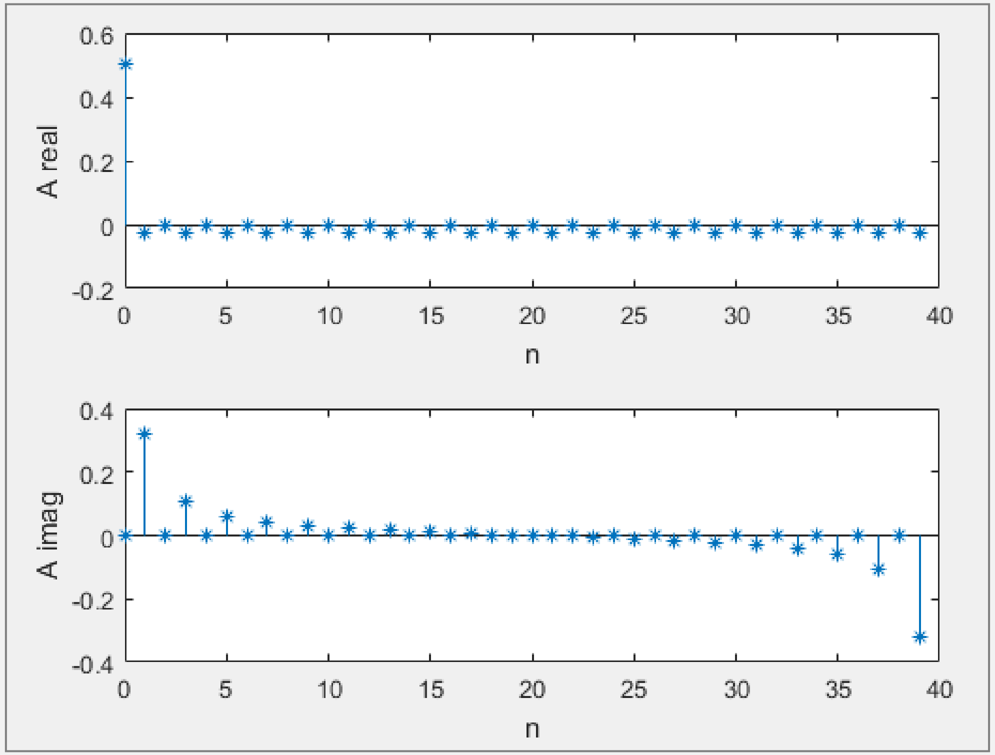
```
x=[zeros(1,20)
ones(1,20)];
stem(0:39, x);
xlim([0 39]);
ylim([0 1.2]);
```



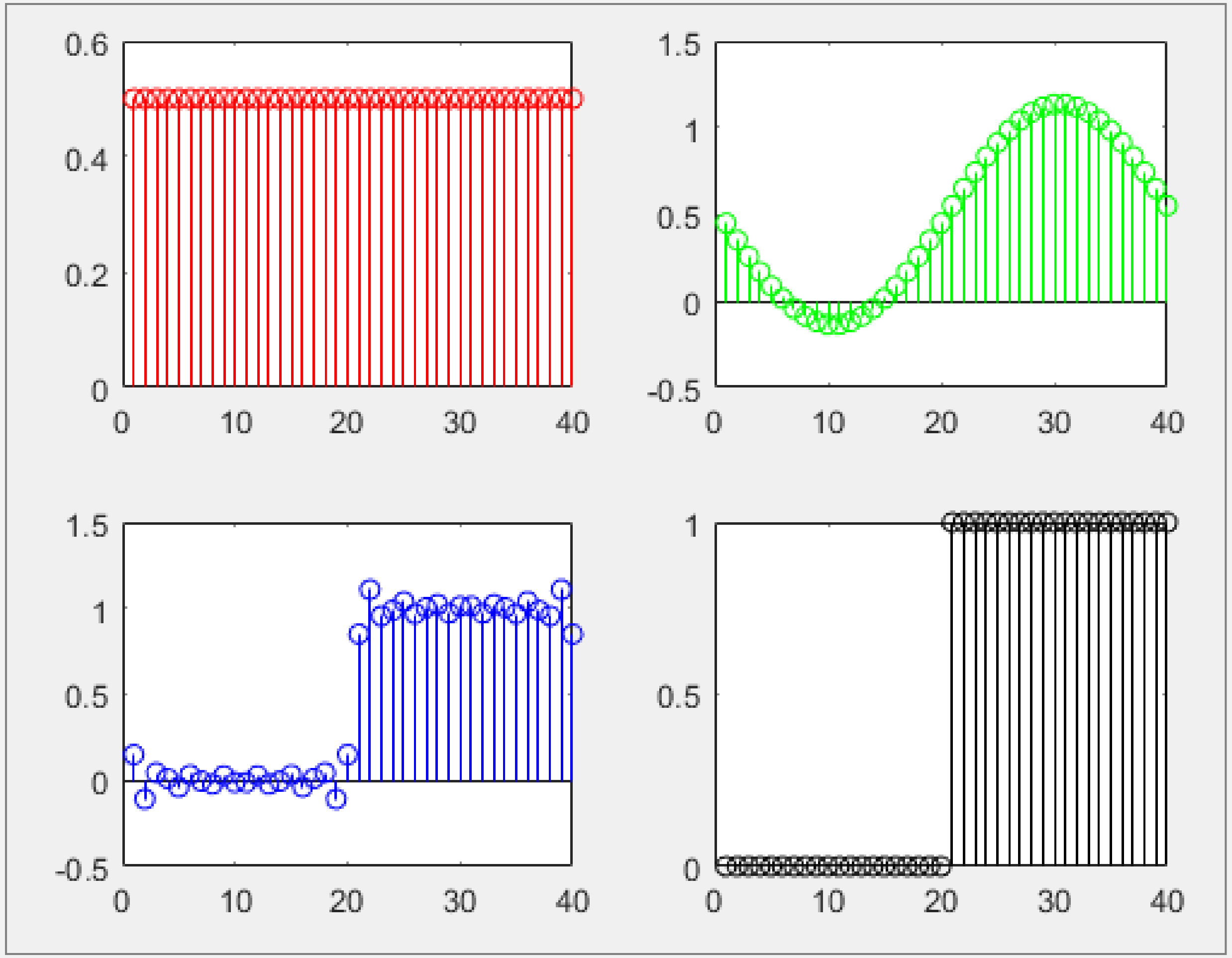
```
A = fft(x) / length(x);
figure(1)
subplot(2,1,1),stem(abs(A));
subplot(2,1,2),stem(abs(fftshift(A)));
```

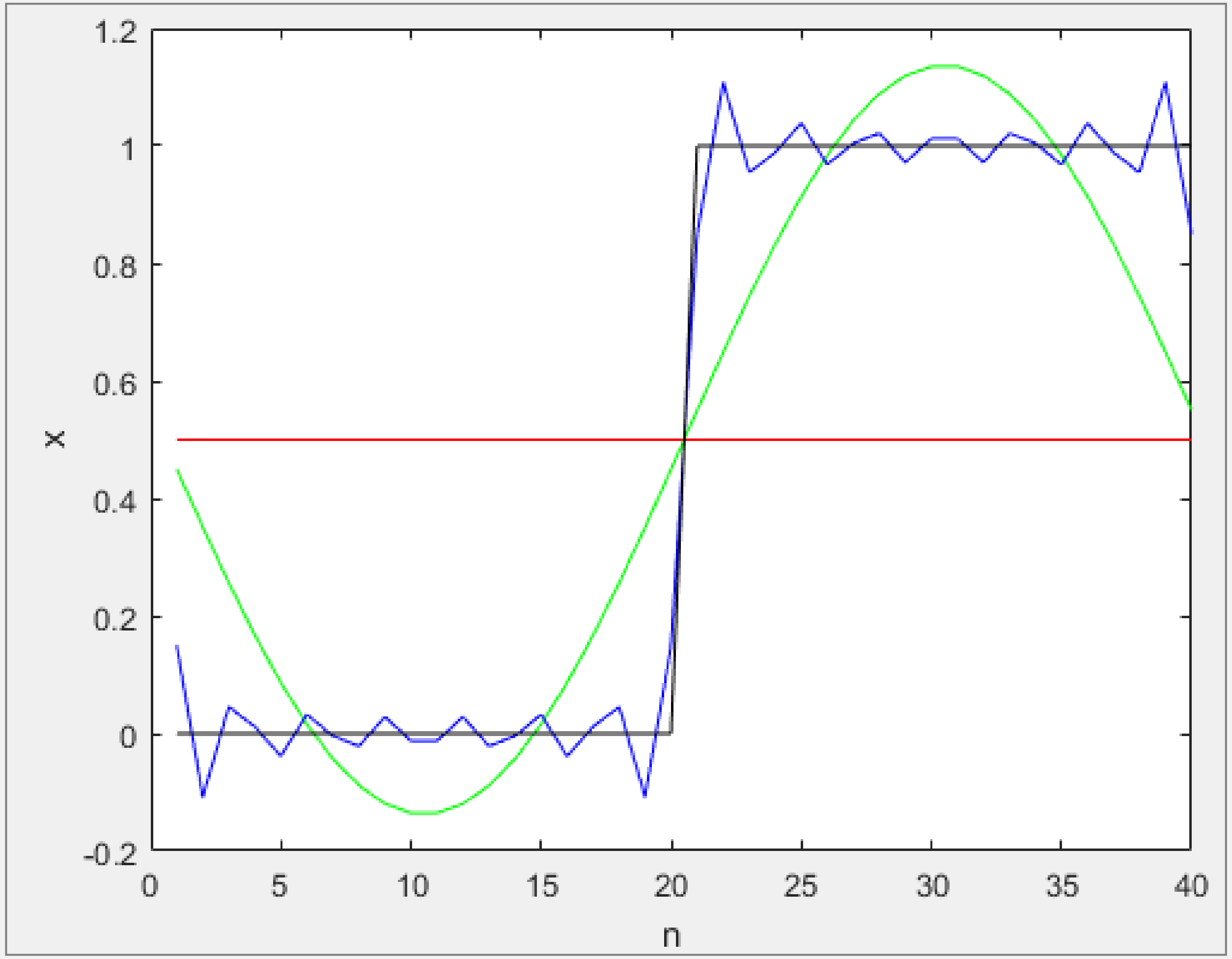
```
A = fft(x) / length(x);
figure(2)
subplot(2,1,1), stem(0: length(x)-1,real(A),'*-');
subplot(2,1,2), stem(0: length(x)-1,imag(A),'*-');
```





```
A1 = [A(1) zeros(1,39)];
A2 = [A(1) A(2) zeros(1,37) A(40)];
A3 = [A(1) A(2:15) zeros(1,11) A(27:40)];
subplot(2,2,1), stem(1:40,ifft(A1)*40, 'r');
subplot(2,2,2), stem(1:40,ifft(A2)*40, 'g');
subplot(2,2,3), stem(1:40,ifft(A3)*40, 'b');
subplot(2,2,4), stem(1:40,x, 'k');
```






```
plot(1:40,ifft(A1)*40, 'r', 1:40,ifft(A2)*40, 'g', 1:40,ifft(A3)*40, 'b', 1:40,x, 'k');
```

Complexity Analysis

- Suppose we know the matrix $E(n,k) = e^{-jk\left(\frac{2\pi}{N}\right)n}$

- How many multiplications & additions are needed to calculate Fourier series $[a_0, a_1, \ldots, a_{N-1}]$

$$a_0 = \frac{1}{N}\sum_{n=0}^{N-1} \mathrm{x[n]}E(n,0)$$

$$\ldots$$

$$a_{N-1} = \frac{1}{N}\sum_{n=0}^{N-1} \mathrm{x[n]}E(n,N-1)$$

Each: N+1 $\times$; N-1 $+$
Total: (N+1)N $\times$; N(N-1) $+$

Complexity Analysis

- Fast Fourier Transform:

  Calculation of Fourier series (transform) can be speeded up

  – Complexity reduces to O(NlogN)

N=4

$$a_0=(x[0]E(0,0)+x[1]E(1,0)+x[2]E(2,0)+x[3]E(3,0))/N$$

$$a_2=(x[0]E(0,2)+x[1]E(1,2)+x[2]E(2,2)+x[3]E(3,2))/N$$

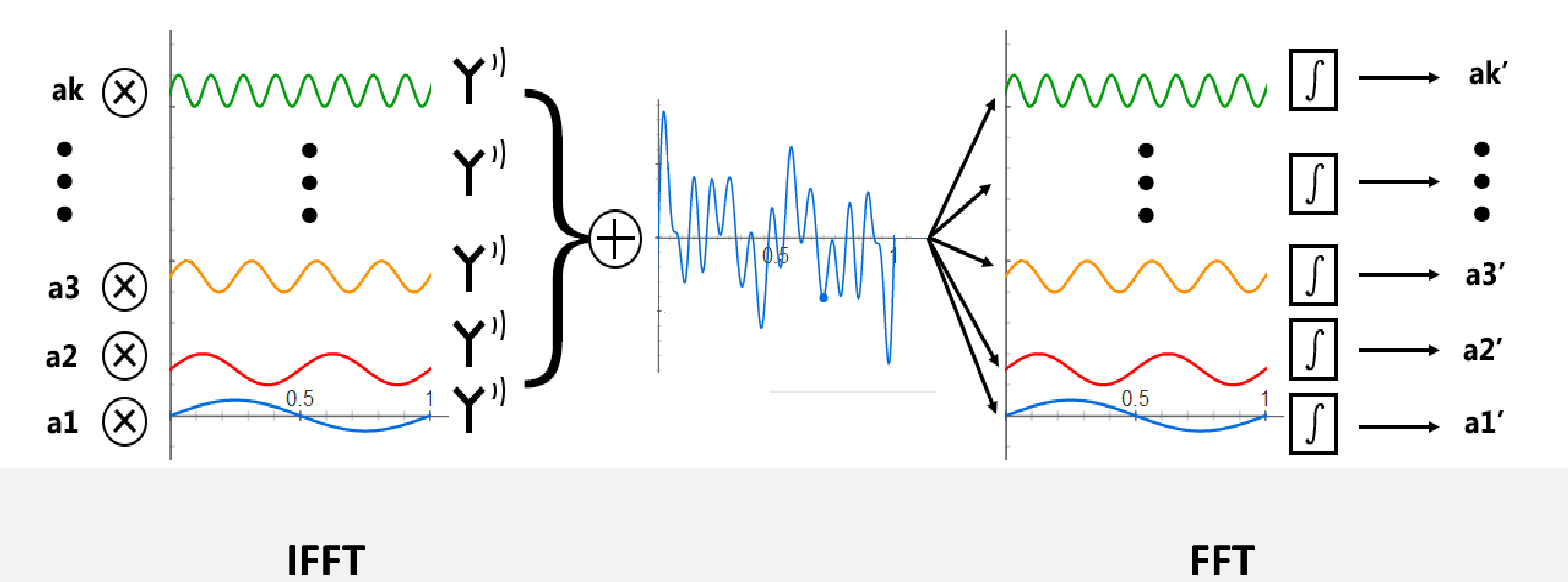
$$a_1=(x[0]E(0,1)+x[1]E(1,1)+x[2]E(2,1)+x[3]E(3,1))/N$$

$$a_3=(x[0]E(0,3)+x[1]E(1,3)+x[2]E(2,3)+x[3]E(3,3))/N$$

To calculate the DFT and FFT of a 1024*1024 image:

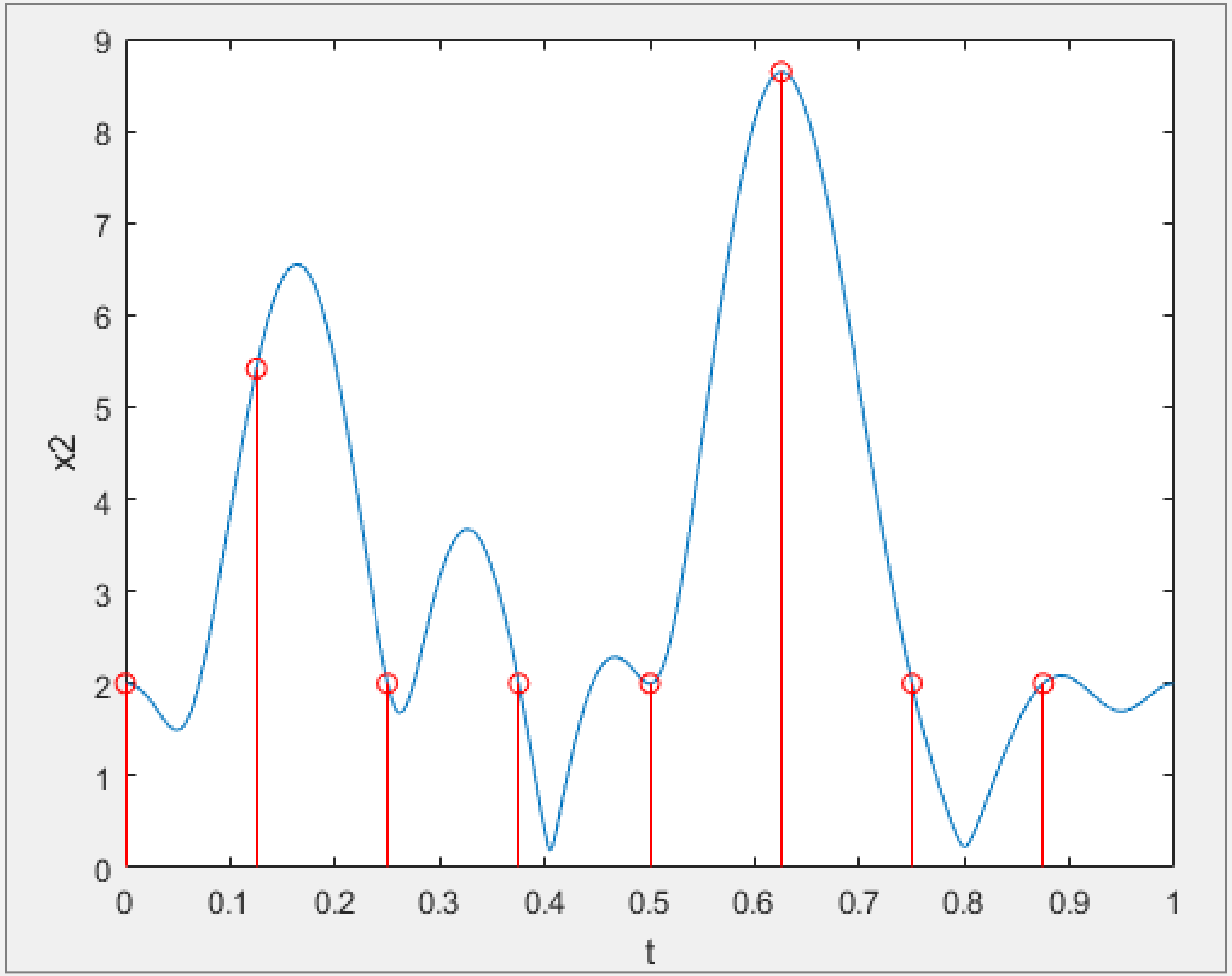
CPU	Clock Frequency	DFT	FFT
1941	60 Hz	152.3 y	271.4 d
1971（4004）	108KHz	30.8 d	3.6 h
1978（8086）	10MHz	8.0 h	2.3 min
1982（80286）	20MHz	4.0 h	1.2min
1985（80386）	33MHz	2.4h	42.6s
1989（80486）	100MHz	48.0min	14.1s
1995（Pentium）	200MHz	24.0min	7.0s
1999（Pentium Ⅲ）	450MHz	10.7min	3.1s
2000（Pentium 4）	1.4GHz	3.4min	1.0s
2001（Pentium 4）	2GHz	2.4min	0.7s

OFDM-Project



```
N=8;
x=randi([0 3],1,N);
x1=qammod(x,4);
f=1:N;
t=0:0.001:1-0.001;
w=2*pi*f'*t;
y1=x1*exp(j*w);
x2=ifft(x1,N);
plot(t,abs(y1));
hold on
stem(0:1/N:1-1/N,abs(x2)*N,'-r')
xlabel('t')
ylabel('x2')
x3=fft(x2)
```

help



Tips

- ***Periodic convolution*** in time domain is equivalent to multiplication in frequency domain

$$\boxed{x[n]\otimes\hat{h}[n] = \sum_{r=0}^{N-1} x[r]\hat{h}[n-r] \Leftrightarrow \; Na_k h_k}$$

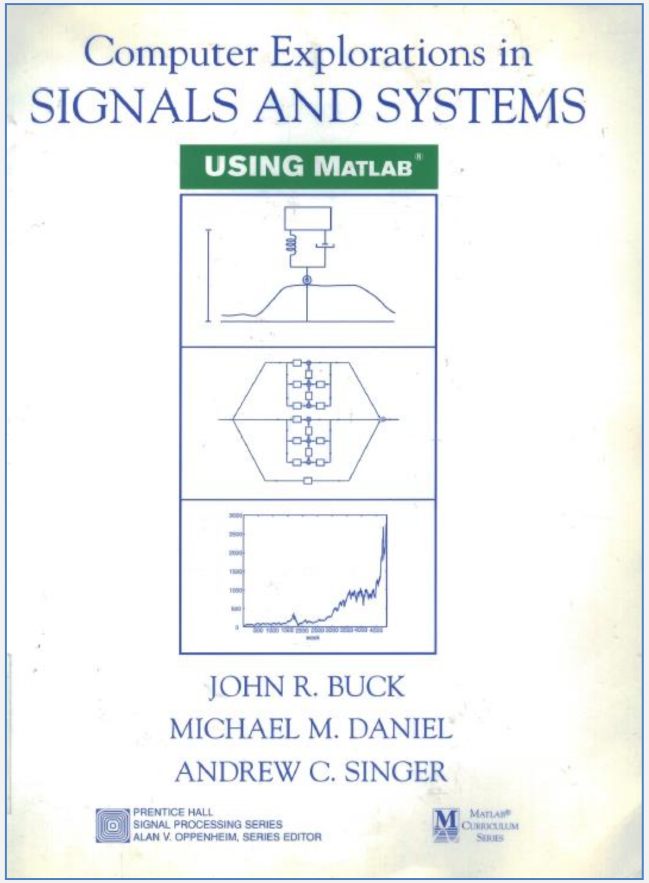
Table 3.2 of Textbook

$$\boxed{y[n] = x[n] * h[n] \; = \; x[n]\otimes\hat{h}[n]}$$

$\hat{h}[n]$ is a periodic version of h[n]

Lab Assignment 3 (b)

- Read tutorial 3.2 & 3.3 by yourself
- 3.5 & 3.10
- Submit your report



- **Question ?**