CHAPTER 24

GRAPHS AND APPLICATIONS

Objectives

- To model real-world problems using graphs and explain the Seven Bridges of Königsberg problem (§24.1).
- To describe graph terminologies: vertices, edges, simple graphs, weighted/unweighted graphs, and directed/undirected graphs (§24.2).
- To represent vertices and edges using lists, adjacent matrices, and adjacent lists (§24.3).
- To model graphs using the `Graph` class (§24.4).
- To represent the traversal of a graph using the `Tree` class (§24.5).
- To design and implement depth-first search (§24.6).
- To design and implement breadth-first search (§24.7).
- To solve the nine-tail problem using breadth-first search (§24.8).

24.1 Introduction

Shortest distance

Graphs play an important role in modeling real-world problems. For example, the problem to find a shortest path between two cities can be modeled using a graph, where the vertices represent cities and the edges represent the roads and distances between two adjacent cities, as shown in Figure 24.1. The problem of finding a shortest path between two cities is reduced to finding a shortest path between two vertices in a graph.

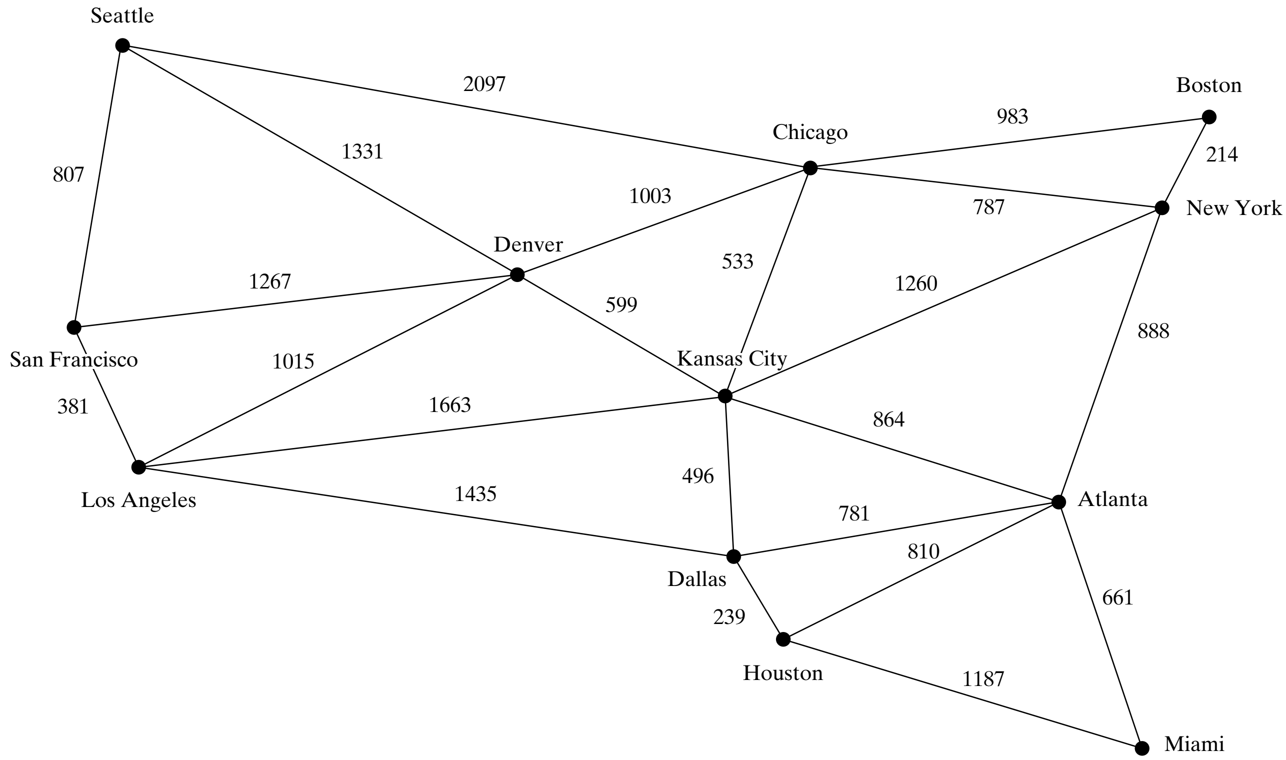


FIGURE 24.1 A graph can be used to model the distance between the cities.

graph theory
Seven Bridges of Königsberg

The study of graph problems is known as *graph theory*. Graph theory was founded by Leonard Euler in 1736, when he introduced graph terminology to solve the famous *Seven Bridges of Königsberg* problem. The city of Königsberg, Prussia, (now Kaliningrad, Russia) was divided by the Pregel River. There were two islands on the river. The city and islands were connected by seven bridges, as shown in Figure 24.2(a). The question is, can one take a walk, cross each bridge exactly once, and return to the starting point? Euler proved that it was not possible.

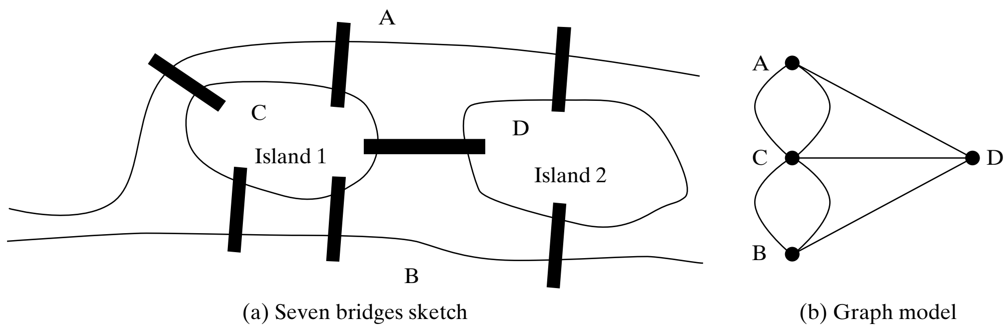


FIGURE 24.2 Seven bridges connecting islands and land.

To establish a proof, Euler first abstracted the Königsberg city map into the sketch shown in Figure 24.2(a), by eliminating all streets. Second, he replaced each land mass with a dot, called a vertex or a node, and each bridge with a line, called an edge, as shown in Figure 24.2(b). This structure with vertices and edges is called a graph.

Looking at the graph, we ask whether there is a path starting from any vertex, traversing all edges exactly once, and returning to the starting vertex. Euler proved that for such path to exist, each vertex must have an even number of edges. Therefore, the *Seven Bridges of Königsberg* problem has no solution.

Graph problems are often solved using algorithms. Graph algorithms have many applications in various areas, such as in computer science, mathematics, biology, engineering, economics, genetics, and social sciences. This chapter presents the algorithms for *depth-first search* and *breadth-first search*, and their applications. The next chapter presents the algorithms for finding a minimum *spanning tree* and shortest paths in weighted graphs, and their applications.

24.2 Basic Graph Terminologies

This chapter does not assume that the reader has prior knowledge of graph theory or discrete mathematics. We use plain and simple terms to define graphs.

what is a graph?

What is a graph? A *graph* is a mathematical structure that represents relationships among entities in the real world. For example, the graph is Figure 24.1 represents the roads and their distances among cities, and the graph in Figure 24.2(b) represents the bridges among land masses.

define a graph

A graph consists of a nonempty set of vertices, nodes, or points, and a set of edges that connect the vertices. For convenience, we define a graph as $G = (V, E)$, where V represents a set of vertices and E a set of edges. For example, V and E for the graph in Figure 24.1 are as follows:

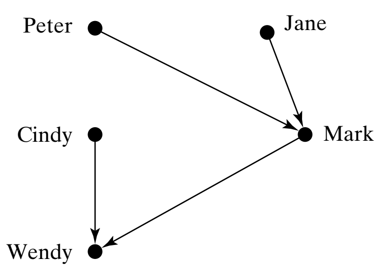
```
V = {"Seattle", "San Francisco", "Los Angeles",
  "Denver", "Kansas City", "Chicago", "Boston", "New York",
  "Atlanta", "Miami", "Dallas", "Houston"};

E = {{"Seattle", "San Francisco"}, {"Seattle", "Chicago"},
     {"Seattle", "Denver"}, {"San Francisco", "Denver"},
       ...
    };
```

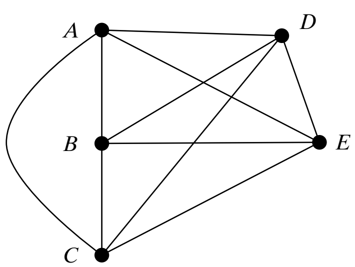
directed vs. undirected

A graph may be directed or *undirected*. In a *directed graph*, each edge has a direction, which indicates that you can move from one vertex to the other through the edge. You may model parent/child relationships using a directed graph, where an edge from vertex A to B indicates that A is a parent of B.

Figure 24.3(a) shows a directed graph. In an undirected graph, you can move in both directions between vertices. The graph in Figure 24.1 is undirected.



(a) A directed graph



(b) A complete graph

FIGURE 24.3 Graphs may appear in many forms.

weighted vs. unweighted

Edges may be *weighted* or *unweighted*. For example, each edge in the graph in Figure 24.1 has a weight that represents the distance between two cities.

adjacent

Two vertices in a graph are said to be *adjacent* if they are connected by the same edge. Similarly two edges are said to be *adjacent* if they are connected to the same vertex. An edge in a graph that joins two vertices is said to be *incident* to both vertices. The *degree* of a vertex is the number of edges incident to it.

incident
degree
neighbor

Two vertices are called *neighbors* if they are adjacent. Similarly two edges are called *neighbors* if they are incident.

loop
parallel edge
simple graph
complete graph
spanning tree

A *loop* is an edge that links a vertex to itself. If two vertices are connected by two or more edges, these edges are called *parallel edges*. A *simple graph* is one that has no loops and parallel edges. A *complete graph* is one in which every two pairs of vertices are connected, as shown in Figure 24.3(b).

Assume that the graph is connected and undirected. A *spanning tree* of a graph is a subgraph that is a tree and connects all vertices in the graph.

24.3 Representing Graphs

To write a program that processes and manipulates graphs, you have to store or represent graphs in the computer.

24.3.1 Representing Vertices

The vertices can be stored in an array. For example, you can store all the city names in the graph in Figure 24.1 using the following array:

```
string vertices[] = {"Seattle", "San Francisco", "Los Angeles",
  "Denver", "Kansas City", "Chicago", "Boston", "New York",
  "Atlanta", "Miami", "Dallas", "Houston"};
```

vertex type

Note

The vertices can be objects of any type. For example, you may consider cities as objects that contain the information such as name, population, and mayor. So, you may define vertices as follows:

```
City city0("Seattle", 563374, "Greg Nickels");
...
City city11("Houston", 1000203, "Bill White");
City vertices[] = {city0, city1, ..., city11};

class City
{
public:
  City(string &cityName, int population, string &mayor)
  {
    this->cityName = cityName;
    this->population = population;
    this->mayor = mayor;
  }

  string getCityName() const
  {
    return cityName;
  }

  int getPopulation() const
  {
    return population;
  }
```

```
    string getMayor() const
    {
      return mayor;
    }

    void setMayor(string &mayor)
    {
      this->mayor = mayor;
    }

    void setPopulation(int population)
    {
      this->population = population;
    }

  private:
    string cityName;
    int population;
    string mayor;
  };
```

The vertices can be conveniently labeled using natural numbers `0`, `1`, `2`, `...`, `n - 1`, for a graphs of n vertices. So, `vertices[0]` represents `"Seattle"`, `vertices[1]` represents `"San Francisco"`, and so on, as shown in Figure 24.4.

vertices[0]	Seattle
vertices[1]	San Francisco
vertices[2]	Los Angeles
vertices[3]	Denver
vertices[4]	Kansas City
vertices[5]	Chicago
vertices[6]	Boston
vertices[7]	New York
vertices[8]	Atlanta
vertices[9]	Miami
vertices[10]	Dallas
vertices[11]	Houston

FIGURE 24.4 An array stores the vertex names.

Note

You can reference a vertex by its name or its index, whichever is convenient. Obviously, it is easy to access a vertex via its index in a program.

reference vertex

24.3.2 Representing Edges (for input): Edge Array

The edges can be represented using a two-dimensional array. For example, you can store all the edges in the graph in Figure 24.1 using the following array:

```
int edges[][2] = {
  {0, 1}, {0, 3}, {0, 5},
  {1, 0}, {1, 2}, {1, 3},
  {2, 1}, {2, 3}, {2, 4}, {2, 10},
```

```
    {3, 0}, {3, 1}, {3, 2}, {3, 4}, {3, 5},
    {4, 2}, {4, 3}, {4, 5}, {4, 7}, {4, 8}, {4, 10},
    {5, 0}, {5, 3}, {5, 4}, {5, 6}, {5, 7},
    {6, 5}, {6, 7},
    {7, 4}, {7, 5}, {7, 6}, {7, 8},
    {8, 4}, {8, 7}, {8, 9}, {8, 10}, {8, 11},
    {9, 8}, {9, 11},
    {10, 2}, {10, 4}, {10, 8}, {10, 11},
    {11, 8}, {11, 9}, {11, 10}
  };
```

array edge

This is known as the *edge representation using arrays*.

24.3.3 Representing Edges (for input): **Edge** Objects

Another way to represent the edges is to define edges as objects and store them in a vector. The **Edge** class is defined in Listing 24.1:

LISTING 24.1 Edge.h

```
#ifndef EDGE_H
#define EDGE_H

class Edge
{
public:
  int u;
  int v;

  Edge(int u, int v)
  {
    this->u = u;
    this->v = v;
  }
};
#endif
```

For example, you can store all the edges in the graph in Figure 24.1 using the following vector:

```
vector<Edge> edgeVector;
edgeVector.push_back(Edge(0, 1));
edgeVector.push_back(Edge(0, 3));
edgeVector.push_back(Edge(0, 5));
...
```

Storing **Edge** objects in a vector is useful if you don't know the edges in advance.

Representing edges using edge array or **Edge** objects in §24.3.2 and §24.3.3 is intuitive for input, but not efficient for internal processing. The next two sections introduce the representation of graphs using adjacency matrices and *adjacency lists*. These two data structures are efficient for processing graphs.

24.3.4 Representing Edges: Adjacency Matrices

Assume that the graph has n vertices. You can use a two-dimensional $n \times n$ matrix, say **adjacencyMatrix**, to represent edges. Each element in the array is **0** or **1**. **adjacencyMatrix[i][j]** is **1** if there is an edge from vertex i to vertex j; otherwise, **adjacencyMatrix[i][j]** is **0**. If the graph is undirected, the matrix is symmetric, because `adjacencyMatrix[i][j]` is the same as `adjacencyMatrix[j][i]`. For example, the edges in the graph in Figure 24.1 can be represented using an *adjacency matrix* as follows:

```
int adjacencyMatrix[12][12] = {
  {0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0}, // Seattle
  {1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0}, // San Francisco
  {0, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0}, // Los Angeles
  {1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0}, // Denver
  {0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0}, // Kansas City
  {1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0}, // Chicago
  {0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0}, // Boston
  {0, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0}, // New York
  {0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 1}, // Atlanta
  {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1}, // Miami
  {0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 1}, // Dallas
  {0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0}  // Houston
};
```

The adjacency matrix for the directed graph in Figure 24.3(a) can be represented as follows:

```
int a[5][5] = {{0, 0, 1, 0, 0}, // Peter
               {0, 0, 1, 0, 0}, // Jane
               {0, 0, 0, 0, 1}, // Mark
               {0, 0, 0, 0, 1}, // Cindy
               {0, 0, 0, 0, 0}  // Wendy
              };
```

As discussed in §12.7, it is more flexible to represent arrays using vectors. When you pass an array to a function, you also have to pass its size, but when you pass a vector to a function, you don't have to pass its size, because a `vector` object contains the size information. The preceding adjacency matrix can be represented using a vector as follows:

representing arrays using vectors

```
vector< vector<int> > a(5);
a[0] = vector<int>(5); a[1] = vector<int>(5); a[2] = vector<int>(5);
a[3] = vector<int>(5); a[4] = vector<int>(5);

a[0][0] = 0; a[0][1] = 0; a[0][2] = 1; a[0][3] = 0; a[0][4] = 0;
a[1][0] = 0; a[1][1] = 0; a[1][2] = 1; a[1][3] = 0; a[1][4] = 0;
a[2][0] = 0; a[2][1] = 0; a[2][2] = 0; a[2][3] = 0; a[2][4] = 1;
a[3][0] = 0; a[3][1] = 0; a[3][2] = 0; a[3][3] = 0; a[3][4] = 1;
a[4][0] = 0; a[4][1] = 0; a[4][2] = 0; a[4][3] = 0; a[4][4] = 0;
```

24.3.5 Representing Edges: Adjacency Lists

To represent edges using adjacency lists, define an array of linked lists. The array has n entries. Each entry represents a vertex. The linked list for vertex i contains all the vertices j such that there is an edge from vertex i to vertex j. For example, to represent the edges in the graph in Figure 24.1, you may create an array of linked lists as follows:

```
list<int> neighbors[12];
```

`neighbors[0]` contains all vertices adjacent to vertex `0` (i.e., Seattle), `neighbors[1]` contains all vertices adjacent to vertex `1` (i.e., San Francisco), and so on, as shown in Figure 24.5.

To represent the edges in the graph in Figure 24.3(a), you may create an array of linked lists as follows:

```
list<int> neighbors[5];
```

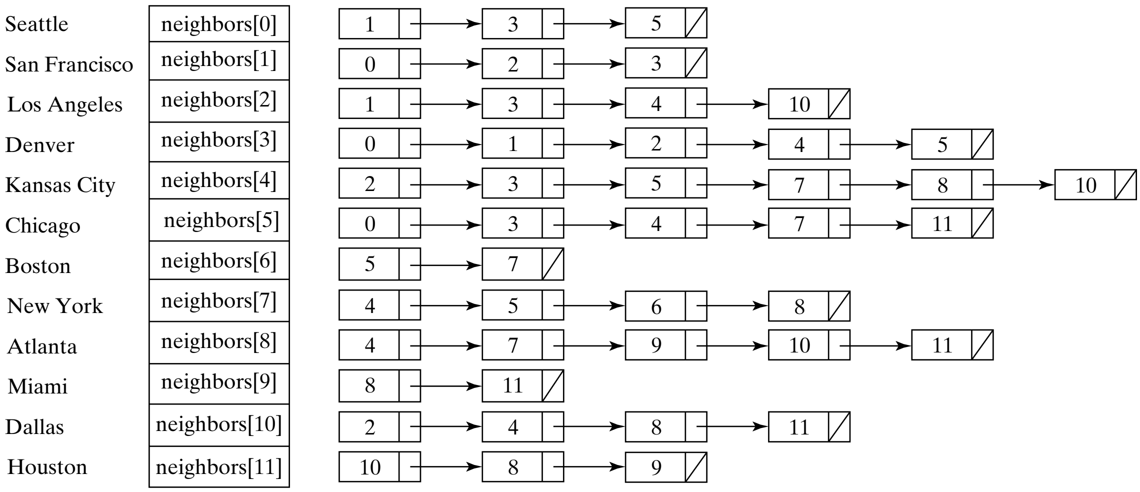


FIGURE 24.5 Edges in the graph in Figure 24.1 are represented using linked lists.

`neighbors[0]` contains all vertices pointed from vertex `0` via directed edges, `neighbors[1]` contains all vertices pointed from vertex `1` via directed edges, and so on, as shown in Figure 24.6.

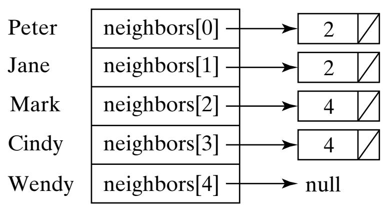


FIGURE 24.6 Edges in the graph in Figure 24.3(a) are represented using linked lists.

adjacency matrices vs. adjacency lists

Note

You can represent a graph using an adjacency matrix or adjacency lists. Which one is better? If the graph is dense (i.e., there are a lot of edges), using an adjacency matrix is preferred. If the graph is very sparse (i.e., very few edges), using adjacency lists is better, because using an adjacency matrix would waste a lot of space.

Both adjacency matrices and adjacency lists may be used in a program to make algorithms more efficient. For example, it takes $O(1)$ constant time to check whether two vertices are connected using an adjacency matrix and it takes linear time $O(m)$ to print all edges in a graph using adjacency lists, where m is the number of edges.

using vectors

For flexibility and simplicity, we will use vectors to represent arrays. Also we will use vectors instead of lists, because our algorithms only requires search for *adjacent vertices* in the list. Using vectors can simplify coding. Using vectors, the adjacency list in Figure 24.5 can be built as follows:

```
vector< vector<int> > neighbors(12);
neighbors[0] = vector<int>();
neighbors[0].push_back(1); neighbors[0].push_back(3);
neighbors[0].push_back(5);
neighbors[1] = vector<int>();
neighbors[1].push_back(0); neighbors[1].push_back(2);
neighbors[1].push_back(3);
...
...
```

24.4 The **Graph** Class

We now design a class to model graphs. What are the common operations for a graph? In general, you need to get the number of vertices in a graph, get all vertices in a graph, get the vertex object with a specified index, get the index of the vertex with a specified name, get the neighbors for a vertex, get the adjacency matrix, get the *degree* for a vertex, perform a depth-first search, and perform a breadth-first search. Depth-first search and breadth-first search will be introduced in the next section. Figure 24.7 illustrates these functions in the UML diagram.

Graph<T>	
#vertices: vector<T>	Vertices in the graph.
#neighbors: vector<vector<int>>	neighbors[i] stores all vertices adjacent to vertex with index i.
+Graph()	Constructs an empty graph.
+Graph(vertices: vector<T>, edges[][2]: int, numberOfEdges: int)	Constructs a graph with the specified vertices in a vector and edges in a 2-D array.
+Graph(numberOfVertices: int, edges[][2]: int, numberOfEdges: int)	Constructs a graph whose vertices are 0, 1, …, *n*-1 and edges are specified in a 2-D array.
+Graph(vertices: vector<T>, edges: vector<Edge>)	Constructs a graph with the vertices in a vector and edges in a vector of Edge objects.
+Graph(numberOfVertices: int, edges: vector<Edge>)	Constructs a graph whose vertices are 0, 1, …, *n*-1 and edges in a vector of Edge objects.
+getSize(): int	Returns the number of vertices in the graph.
+getDegree(v: int): int	Returns the degree for a specified vertex index.
+getVertex(index: int): T	Returns the vertex for the specified vertex index.
+getIndex(v: T): int	Returns the index for the specified vertex.
+getVertices(): vector<T>	Returns the vertices in the graph in a vector.
+getNeighbors(v: int): vector<int>	Returns the neighbors of vertex with index v.
+printEdges(): void	Prints the edges of the graph to the console.
+printAdjacencyMatrix(): void	Prints the adjacency matrix of the graph to the console.
+dfs(v: int): Tree	Obtains a depth-first search tree.
+bfs(v: int): Tree	Obtains a breadth-first search tree.

FIGURE 24.7 The **Graph** class defines the common operations for graphs.

vertices, a vector, is defined in the **Graph** class to represent vertices. The vertices may be of any type: **int**, string, and so on. So, we use a generic type **T** to define it. **neighbors**, a vector of vectors, is defined to represent edges. With these two data fields, it is sufficient to implement all the functions defined in the **Graph** class.

A no-arg constructor is provided for convenience. With a no-arg constructor, it is easy to use the class as a base class and as a data type for data fields in a class. You can create a **Graph** object using one of the four other constructors, whichever is convenient. If you have an edge array, use the first two constructors in the UML class diagram. If you have a vector of **Edge** objects, use the last two constructors in the UML class diagram.

The generic type **T** indicates the type of vertices—integer, string, and so on. You can create a graph with vertices of any type. If you create a graph without specifying the vertices, the vertices are integers 0, 1, . . . , n - 1, where n is the number of vertices. Each vertex is associated with an index, which is the same as the index of the vertex in the array for vertices.

Assume the class is available in Graph.h. Listing 24.2 gives a test program that creates a graph for the one in Figure 24.1 and another graph for the one in Figure 24.3(a).

Listing 24.2 TestGraph.cpp

```
#include <iostream>
#include <string>
#include <vector>
#include "Graph.h"
#include "Edge.h"
using namespace std;

int main()
{
  // Vertices for graph in Figure 24.1
  string vertices[] = {"Seattle", "San Francisco", "Los Angeles",
    "Denver", "Kansas City", "Chicago", "Boston", "New York",
    "Atlanta", "Miami", "Dallas", "Houston"};

  // Edge array for graph in Figure 24.1
  int edges[][2] = {
    {0, 1}, {0, 3}, {0, 5},
    {1, 0}, {1, 2}, {1, 3},
    {2, 1}, {2, 3}, {2, 4}, {2, 10},
    {3, 0}, {3, 1}, {3, 2}, {3, 4}, {3, 5},
    {4, 2}, {4, 3}, {4, 5}, {4, 7}, {4, 8}, {4, 10},
    {5, 0}, {5, 3}, {5, 4}, {5, 6}, {5, 7},
    {6, 5}, {6, 7},
    {7, 4}, {7, 5}, {7, 6}, {7, 8},
    {8, 4}, {8, 7}, {8, 9}, {8, 10}, {8, 11},
    {9, 8}, {9, 11},
    {10, 2}, {10, 4}, {10, 8}, {10, 11},
    {11, 8}, {11, 9}, {11, 10}
  };
  const int NUMBER_OF_EDGES = 46; // 46 edges in Figure 24.1

  // Create a vector for vertices
  vector<string> vectorOfVertices(12);
  copy(vertices, vertices + 12, vectorOfVertices.begin());

  Graph<string> graph1(vectorOfVertices, edges, NUMBER_OF_EDGES);
  cout << "The number of vertices in graph1: " << graph1.getSize();
  cout << "\nThe vertex with index 1 is " << graph1.getVertex(1);
  cout << "\nThe index for Miami is " << graph1.getIndex("Miami");

  cout << "\nedges for graph1: " << endl;
  graph1.printEdges();

  cout << "\nAdjacency matrix for graph1: " << endl;
  graph1.printAdjacencyMatrix();

  // vector of Edge objects for graph in Figure 24.3(a)
  vector<Edge> edgeVector;
  edgeVector.push_back(Edge(0, 2));
  edgeVector.push_back(Edge(1, 2));
  edgeVector.push_back(Edge(2, 4));
  edgeVector.push_back(Edge(3, 4));
  // Create a graph with 5 vertices
```

vertices

edges

create a graph

number of vertices

get vertex

get index

print edges

print adjacency matrix

vector of Edge objects

```
  Graph<int> graph2(5, edgeVector);

  cout << "The number of vertices in graph2: " << graph2.getSize();
  cout << "\nedges for graph2: " << endl;
  graph2.printEdges();

  cout << "\nAdjacency matrix for graph2: " << endl;
  graph2.printAdjacencyMatrix();

  return 0;
}
```

create a graph

print edges

print adjacency matrix

```
The number of vertices in graph1: 12
The vertex with index 1 is San Francisco
The index for Miami is 9

edges for graph1:
Vertex 0: (0, 1) (0, 3) (0, 5)
Vertex 1: (1, 0) (1, 2) (1, 3)
Vertex 2: (2, 1) (2, 3) (2, 4) (2, 10)
Vertex 3: (3, 0) (3, 1) (3, 2) (3, 4) (3, 5)
Vertex 4: (4, 2) (4, 3) (4, 5) (4, 7) (4, 8) (4, 10)
Vertex 5: (5, 0) (5, 3) (5, 4) (5, 6) (5, 7)
Vertex 6: (6, 5) (6, 7)
Vertex 7: (7, 4) (7, 5) (7, 6) (7, 8)
Vertex 8: (8, 4) (8, 7) (8, 9) (8, 10) (8, 11)
Vertex 9: (9, 8) (9, 11)
Vertex 10: (10, 2) (10, 4) (10, 8) (10, 11)
Vertex 11: (11, 8) (11, 9) (11, 10)

Adjacency matrix for graph1:
0 1 0 1 0 1 0 0 0 0 0 0
1 0 1 1 0 0 0 0 0 0 0 0
0 1 0 1 1 0 0 0 0 0 1 0
1 1 1 0 1 1 0 0 0 0 0 0
0 0 1 1 0 1 0 1 1 0 1 0
1 0 0 1 1 0 1 1 0 0 0 0
0 0 0 0 0 1 0 1 0 0 0 0
0 0 0 0 1 1 1 0 1 0 0 0
0 0 0 0 1 0 0 1 0 1 1 1
0 0 0 0 0 0 0 0 1 0 0 1
0 0 1 0 1 0 0 0 1 0 0 1
0 0 0 0 0 0 0 0 1 1 1 0

The number of vertices in graph2: 5
edges for graph2:
Vertex 0: (0, 2)
Vertex 1: (1, 2)
Vertex 2: (2, 4)
Vertex 3: (3, 4)
Vertex 4:

Adjacency matrix for graph2:
0 0 1 0 0
0 0 1 0 0
0 0 0 0 1
0 0 0 0 1
0 0 0 0 0
```

The program creates `graph1` for the graph in Figure 24.1 in lines 11–36. The vertices for `graph1` are defined in lines 11–13. The edges for `graph1` are defined in lines 16–29. The edges are represented using a two-dimensional array. For each row `i` in the array, `edges[i][0]` and `edges[i][1]` indicate that there is an edge from vertex `edges[i][0]` to vertex `edges[i][1]`. For example, the first row {`0`, `1`} represents the edge from vertex `0` (`edges[0][0]`) to vertex `1` (`edges[0][1]`). The row {`0`, `5`} represents the edge from vertex `0` (`edges[2][0]`) to vertex `5` (`edges[2][1]`). The graph is created in line 36. Line 42 invokes the `printEdges()` function on `graph1` to display all edges in `graph1`. Line 45 invokes the `printAdjacencyMatrix()` function on `graph1` to display the adjacency matrix for `graph1`.

The program creates `graph2` for the graph in Figure 24.3(a) in lines 48–54. The edges for `graph2` are defined in lines 48–52. `graph2` is created using a vector of `Edge` objects in line 54. Line 58 invokes the `printEdges()` function on `graph2` to display all edges in `graph2`. Line 61 invokes the `printAdjacencyMatrix()` function on `graph2` to display the adjacency matrix for `graph1`.

Note that `graph1` contains the vertices of strings and `graph2` contains the vertices with integers `0`, `1`, . . . , `n-1`, where `n` is the number of vertices. In `graph1`, the vertices are associated with indices `0`, `1`, . . . , `n-1`. The index is the location of the vertex in `vertices`. For example, the index of vertex `Miami` is `9`.

Now we turn our attention to implementing the `Graph` class, as shown in Listing 24.3.

Listing 24.3 Graph.h

```
#ifndef GRAPH_H
#define GRAPH_H

#include "Edge.h" // Defined in Listing 24.1
#include "Tree.h" // Defined in Listing 24.4
#include <vector>
#include <queue>

using namespace std;

template<typename T>
class Graph
{
public:
  /** Construct an empty graph */
  Graph();

  /** Construct a graph from vertices in a vector and
    * edges in 2-D array */
  Graph(vector<T> vertices, int edges[][2], int numberOfEdges);

  /** Construct a graph with vertices 0, 1, ..., n-1 and
    * edges in 2-D array */
  Graph(int numberOfVertices, int edges[][2], int numberOfEdges);

  /** Construct a graph from vertices and edges objects */
  Graph(vector<T> vertices, vector<Edge> edges);

  /** Construct a graph with vertices 0, 1, ..., n-1 and
    *  edges in a vector */
  Graph(int numberOfVertices, vector<Edge> edges);

  /** Return the number of vertices in the graph */
```

```
  int getSize() const;

  /** Return the degree for a specified vertex */
  int getDegree(int v) const;

  /** Return the vertex for the specified index */
  T getVertex(int index) const;

  /** Return the index for the specified vertex */
  int getIndex(T v) const;

  /** Return the vertices in the graph */
  vector<T> getVertices() const;

  /** Return the neighbors of vertex v */
  vector<int> getNeighbors(int v) const;

  /** Print the edges */
  void printEdges() const;

  /** Print the adjacency matrix */
  void printAdjacencyMatrix() const;

  /** Obtain a depth-first search tree */
  /** To be discussed in Section 24.6 */
  Tree dfs(int v) const;

  /** Starting bfs search from vertex v */
  /** To be discussed in Section 24.7 */
  Tree bfs(int v) const;

protected:
  vector<T> vertices; // Store vertices
  vector< vector<int> > neighbors; // Adjacency lists

private:
  /** Create adjacency lists for each vertex from an edge array */
  void createAdjacencyLists(int numberOfVertices, int edges[][2],
    int numberOfEdges);

  /** Create adjacency lists for each vertex from an Edge vector */
  void createAdjacencyLists(int numberOfVertices,
    vector<Edge> &edges);

  /** Recursive function for DFS search */
  void dfs(int v, vector<int> &parent,
    vector<int> &searchOrders, vector<bool> &isVisited) const;
};

template<typename T>
Graph<T>::Graph()
{
}

template<typename T>
Graph<T>::Graph(vector<T> vertices, int edges[][2],
  int numberOfEdges)
{
  this->vertices = vertices;
```

getSize
getDegree
getVertex
getIndex
getVertices
getNeighbors
printEdges
printAdjacencyMatrix
dfs
bfs
no-arg constructor
constructor

constructor

constructor

constructor

internal edge representation

internal edge representation

```
  createAdjacencyLists(vertices.size(), edges, numberOfEdges);
}

template<typename T>
Graph<T>::Graph( int numberOfVertices, int edges[][2],
  int numberOfEdges)
{
  for (int i = 0; i < numberOfVertices; i++)
    vertices.push_back(i); // vertices is {0, 1, 2, ..., n-1}

  createAdjacencyLists(numberOfVertices, edges, numberOfEdges);
}

template<typename T>
Graph<T>::Graph(vector<T> vertices, vector<Edge> edges)
{
  this->vertices = vertices;
  createAdjacencyLists(vertices.size(), edges);
}

template<typename T>
Graph<T>::Graph(int numberOfVertices, vector<Edge> edges)
{
  for (int i = 0; i < numberOfVertices; i++)
    vertices.push_back(i); // vertices is {0, 1, 2, ..., n-1}

  createAdjacencyLists(numberOfVertices, edges);
}

template<typename T>
void Graph<T>::createAdjacencyLists(int numberOfVertices,
  int edges[][2],  int numberOfEdges)
{
  for (int i = 0; i < numberOfVertices; i++)
  {
    neighbors.push_back(vector<int>(0));
  }

  for (int i = 0; i < numberOfEdges; i++)
  {
    int u = edges[i][0];
    int v = edges[i][1];
    neighbors[u].push_back(v);
  }
}

template<typename T>
void Graph<T>::createAdjacencyLists(int numberOfVertices,
  vector<Edge> &edges)
{
  for (int i = 0; i < numberOfVertices; i++)
  {
    neighbors.push_back(vector<int>(0));
  }

  for (int i = 0; i < edges.size(); i++)
  {
    int u = edges[i].u;
```

```
    int v = edges[i].v;
    neighbors[u].push_back(v);
  }
}

```

getSize()

```
template<typename T>
int Graph<T>::getSize() const
{
  return vertices.size();
}

```

getDegree()

```
template<typename T>
int Graph<T>::getDegree(int v) const
{
  return neighbors[v].size();
}

```

getVertex()

```
template<typename T>
T Graph<T>::getVertex(int index) const
{
  return vertices[index];
}

```

getIndex()

```
template<typename T>
int Graph<T>::getIndex(T v) const
{
  for (int i = 0; i < vertices.size(); i++)
  {
    if (vertices[i] == v)
      return i;
  }

  return -1; // If vertex is not in the graph
}

```

getVertices()

```
template<typename T>
vector<T> Graph<T>::getVertices() const
{
  return vertices;
}

```

getNeighbors()

```
template<typename T>
vector<int> Graph<T>::getNeighbors(int v) const
{
  return neighbors[v];
}

```

getEdges()

```
template<typename T>
void Graph<T>::printEdges() const
{
  for (int u = 0; u < neighbors.size(); u++)
  {
    cout << "Vertex " << u << ": ";
    for (int j = 0; j < neighbors[u].size(); j++)
    {
      cout << "(" << u << ", " << neighbors[u][j] << ") ";
    }
```

```
    cout << endl;
  }
}

template<typename T>
```

getAdjacencyMatrix()

```
void Graph<T>::printAdjacencyMatrix() const
{
  int size = vertices.size();
  // Use vector for 2-D array
  vector< vector<int> > adjacencyMatrix(size);

  // Initialize 2-D array for adjacency matrix
  for (int i = 0; i < size; i++)
  {
    adjacencyMatrix[i] = vector<int>(size);
  }

  for (int i = 0; i < neighbors.size(); i++)
  {
    for (int j = 0; j < neighbors[i].size(); j++)
    {
      int v = neighbors[i][j];
      adjacencyMatrix[i][v] = 1;
    }
  }

  for (int i = 0; i < adjacencyMatrix.size(); i++)
  {
    for (int j = 0; j < adjacencyMatrix[i].size(); j++)
    {
      cout << adjacencyMatrix[i][j] << " ";
    }

    cout << endl;
  }
}

template<typename T>
```

dfs(v)

```
Tree Graph<T>::dfs(int v) const
{
  vector<int> searchOrders;
  vector<int> parent(vertices.size());
  for (int i = 0; i < vertices.size(); i++)
    parent[i] = -1; // Initialize parent[i] to -1

  // Mark visited vertices
  vector<bool> isVisited(vertices.size());
  for (int i = 0; i < vertices.size(); i++)
  {
    isVisited[i] = false;
  }

  // Recursively search
  dfs(v, parent, searchOrders, isVisited);

  // Return a search tree
  return Tree(v, parent, searchOrders);
}
```

```
267 template<typename T>
268 void Graph<T>::dfs(int v, vector<int> &parent,
269   vector<int> &searchOrders, vector<bool> &isVisited) const
270 {
271   // Store the visited vertex
272   searchOrders.push_back(v);
273   isVisited[v] = true; // Vertex v visited
274
275   for (int j = 0; j < neighbors[v].size(); j++)
276   {
277     int i = neighbors[v][j];
278     if (!isVisited[i])
279     {
280       parent[i] = v; // The parent of vertex i is v
281       dfs(i, parent, searchOrders, isVisited); // Recursive search
282     }
283   }
284 }
285
286 template<typename T>
287 Tree Graph<T>::bfs(int v) const
288 {
289   vector<int> searchOrders;
290   vector<int> parent(vertices.size());
291   for (int i = 0; i < vertices.size(); i++)
292     parent[i] = -1; // Initialize parent[i] to -1
293
294   queue<int> queue; // Stores vertices to be visited
295   vector<bool> isVisited(vertices.size());
296   queue.push(v); // Enqueue v
297   isVisited[v] = true; // Mark it visited
298
299   while (!queue.empty())
300   {
301     int u = queue.front(); // Get from the front of the queue
302     queue.pop(); // remove the front element
303     searchOrders.push_back(u); // u searched
304     for (int j = 0; j < neighbors[u].size(); j++)
305     {
306       int w = neighbors[u][j];
307       if (!isVisited[w])
308       {
309         queue.push(w); // Enqueue w
310         parent[w] = u; // The parent of w is u
311         isVisited[w] = true; // Mark it visited
312       }
313     }
314   }
315
316   return Tree(v, parent, searchOrders);
317 }
318
319 #endif
```

bfs(v)

To construct a graph, you need to create vertices and edges. The vertices are stored in a **vector<T>** and the edges in a **vector< vector<int> >**, which is the adjacency list described in §24.3.4. The constructors (lines 88–120) create vertices and edges. The edges may be created from an edge array (discussed in §24.3.2) or a vector of **Edge** objects (discussed in §24.3.3). The private function **createAdjacencyList** (lines 122–137) is used to create the adjacency list from an edge array and the overloaded `createAdjacencyList` function (lines 139–154) is used to create the adjacency list from a vector of `Edge` objects. The `Edge` class is defined in Listing 24.1, which simply defines two vertices having an edge.

`vertices` and `neighbors` are declared protected so that they can be accessed from derived classes of `Graph`.

The function `getSize` returns the number of the vertices in the graph (lines 156–160). The function `getDegree(int v)` returns the degree of a vertex with index `v` (lines 162–166). The function `getVertex(int index)` returns the vertex with the specified index (lines 168–172). The function `getIndex(T v)` returns the index of the specified vertex (lines 174–184). The function `getVertices()` returns the vector for the vertex (lines 186–190). The function `printEdges()` (lines 198–210) displays all vertices and edges adjacent to each vertex. The function `getNeighbors()` returns the adjacent list (lines 192–196). The function `printAdjacencyMatrix()` (lines 212–243) displays the adjacency matrix.

The code in lines 235–304 gives the functions for finding a depth-first search tree and a breadth-first search tree, which will be introduced in subsequent sections.

24.5 Graph Traversals

depth-first
breadth-first

Graph traversal is the process of visiting each vertex in the graph exactly once. There are two popular ways to traverse a graph: *depth-first traversal* (or *depth-first search*) and *breadth-first traversal* (or *breadth-first search*). Both traversals result in a spanning tree, which can be modeled using a class, as shown in Figure 24.8. The `Tree` class describes the parent-child relationship of the nodes in the tree, as shown in Listing 24.4.

Tree	
-root: int	The root of the tree.
-parent: vector<int>	parent[i] stores the parent of vertex i.
-searchOrders: vector<int>	The orders for traversing the vertices.
+Tree()	Constructs an emtpy tree.
+Tree(root: int, parent: vector<int>, searchOrders: vector<int>)	Constructs a tree with the specified root, parent, and searchOrders.
+Tree(root: int, parent: vector<int>)	Constructs a tree with the specified root, parent.
+getRoot(): int	Returns the root of the tree.
+getSearchOrders(): vector<int>	Returns the order of vertices searched.
+getParent(v: int): int	Returns the parent of vertex v.
+getNumberOfVerticesFound(): int	Returns the number of vertices searched.
+getPath(v: int): vector<int>	Returns a path of all vertices leading to the root from v. The return values are in a vector.
+printTree(): void	Displays tree with the root and all edges.

FIGURE 24.8 The `Tree` class describes parent-child relationship of the nodes in a tree.

LISTING 24.4 Tree.h

```
#ifndef TREE_H
#define TREE_H

#include <vector>
using namespace std;

```

```
class Tree
{
public:
  /** Construct an empty tree */
  Tree()
  {
  }

  /** Construct a tree with root, parent, and searchOrder */
  Tree(int root, vector<int> &parent, vector<int> &searchOrders)
  {
    this->root = root;
    this->parent = parent;
    this->searchOrders = searchOrders;
  }

  /** Return the root of the tree */
  int getRoot() const
  {
    return root;
  }

  /** Return the parent of vertex v */
  int getParent(int v) const
  {
    return parent[v];
  }

  /** Return search order */
  vector<int> getSearchOrders() const
  {
    return searchOrders;
  }

  /** Return number of vertices found */
  int getNumberOfVerticesFound() const
  {
    return searchOrders.size();
  }

  /** Return the path of vertices from v to the root in a vector */
  vector<int> getPath(int v) const
  {
    vector<int> path;

    do
    {
      path.push_back(v);
      v = parent[v];
    }
    while (v != -1);

    return path;
  }

  /** Print the whole tree */
  void printTree() const
  {
    cout << "Root is: " << root << endl;
    cout << "Edges: ";
```

no-arg constructor

constructor

getRoot

getParent

getSearchOrders

getNumberOfVertices-Found

getPath

printTree

root
parent
searchOrders

```
67     for (int i = 0; i < searchOrders.size(); i++)
68     {
69       if (parent[searchOrders[i]] != -1)
70       {
71         // Display an edge
72         cout << "(" << parent[searchOrders[i]] << ", " <<
73           searchOrders[i] << ") ";
74       }
75     }
76     cout << endl;
77   }
78
79 private:
80   int root; // The root of the tree
81   vector<int> parent; // Store the parent of each vertex
82   vector<int> searchOrders; // Store the search order
83 };
84 #endif
```

The **Tree** class has two constructors. The no-arg constructor constructs an empty tree. The other constructor constructs a tree with a search order (lines 16–21).

The **Tree** class defines seven functions. The **getRoot()** function returns the root of the tree (lines 24–27). You can invoke **getParent(v)** to find the parent of vertex **v** in the search (lines 30–33). You can get the order of the vertices searched by invoking the **getSearchOrders()** function (lines 36–39). Invoking **getNumberOfVerticesFound()** returns the number of vertices searched (lines 42–45). The **getPath(v)** function returns a path from the **v** to root (lines 48–60). You can display all edges in the tree using the **printTree()** function (lines 63–75).

Sections 24.6 and 24.7 will introduce depth-first search and breadth-first search, respectively. Both searches will result in an instance of the **Tree** class.

24.6 Depth-First Search

The depth-first search of a graph is like the depth-first search of a tree discussed in §21.2.5, "Tree Traversal." In the case of a tree, the search starts from the root. In a graph, the search can start from any vertex.

A depth-first search of a tree first visits the root, then recursively visits the subtrees of the root. Similarly, the depth-first search of a graph first visits a vertex, then recursively visits all vertices adjacent to that vertex. The difference is that the graph may contain cycles, which may lead to an infinite recursion. To avoid this problem, you need to track the vertices that have already been visited and avoid visiting them again.

The search is called depth-first, because it searches "deeper" in the graph as much as possible. The search starts from some vertex v. After visiting v, visit an unvisited neighbor of v. If v has no unvisited neighbor, backtrack to the vertex from which we reached v.

24.6.1 Depth-First Search Algorithm

The algorithm for the depth-first search can be described in Listing 24.5.

LISTING 24.5 Depth-first Search Algorithm

visit v
check a neighbor
recursive search

```
dfs(vertex v)
{
  visit v;
  for each neighbor w of v
    if (w has not been visited)
      dfs(w);
}
```

You may use a vector named `isVisited` to denote whether a vertex has been visited. Initially, `isVisited[i]` is `false` for each vertex *i*. Once a vertex, say *v*, is visited, `isVisited[v]` is set to `true`.

Consider the graph in Figure 24.9(a). Suppose you start the depth-first search from vertex 0. First visit 0, then any of its neighbors, say 1. Now 1 is visited, as shown in Figure 24.9(b). Vertex 1 has three neighbors: 0, 2, and 4. Since 0 has already been visited, you will visit either 2 or 4. Let us pick 2. Now 2 is visited, as shown in Figure 24.9(c). 2 has three neighbors: 0, 1, and 3. Since 0 and 1 have already been visited, pick 3. 3 is now visited, as shown in Figure 24.9(d). At this point, the vertices have been visited in this order:

0, 1, 2, 3

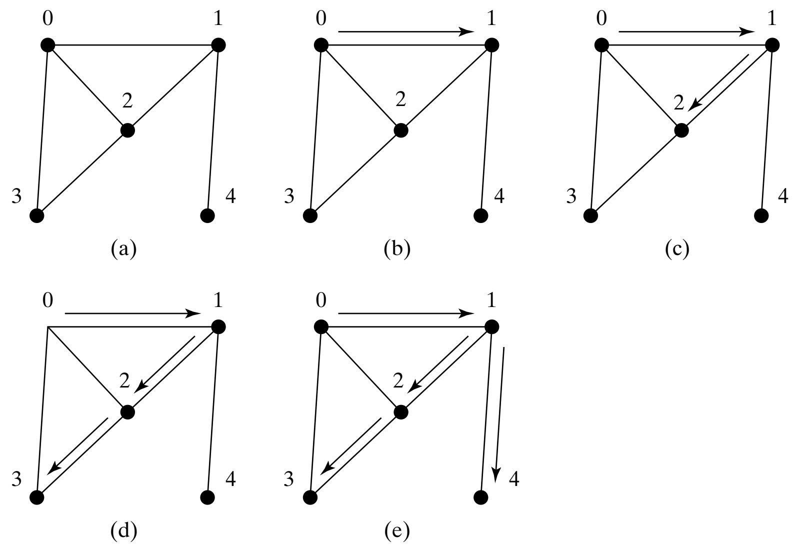


FIGURE 24.9 Depth-first search visits a node and its neighbors recursively.

Since all the neighbors of 3 have been visited, backtrack to 2. Since all the vertices of 2 have been visited, backtrack to 1. 4 is adjacent to 1, but 4 has not been visited. So, visit 4, as shown in Figure 24.9(e). Since all the neighbors of 4 have been visited, backtrack to 1. Since all the neighbors of 1 have been visited, backtrack to 0. Since all the neighbors of 0 have been visited, the search ends.

Since each edge and each vertex is visited only once, the time complexity of the `dfs` function is `O(|E| + |V|)`, where `|E|` denotes the number of edges and `|V|` the number of vertices. DFS time complexity

24.6.2 Implementation of Depth-First Search

The algorithm is described in Listing 24.5, using recursion. It is natural to use recursion to implement it. Alternatively, you can use a stack (see Exercise 24.4).

The `dfs(int v)` function is implemented in lines 245–265 in Listing 24.3. It returns an instance of the `Tree` class with vertex `v` as the root. The function stores the vertices searched in a list `searchOrders` (line 248), the parent of each vertex in an array `parent` (line 249), and uses the `isVisited` array to indicate whether a vertex has been visited (line 254). It invokes the helper function `dfs(v, parent, searchOrders, isVisited)` to perform a depth-first search (line 261).

In the recursive helper function, the search starts from vertex `v`. `v` is added to `searchOrders` (line 272) and is marked visited (line 273). For each unvisited neighbor of `v`, the function is recursively invoked to perform a depth-first search. When a vertex `i` is visited, the parent of `i` is stored in `parent[i]` (line 280). The function returns when all vertices are visited for a connected graph, or in a connected component.

Listing 24.6 gives a test program that displays a DFS for the graph in Figure 24.1, starting from `Chicago`.

LISTING 24.6 TestDFS.cpp

include Graph.h
include Edge.h
include Tree.h
vertices
edges
create a graph
get DFS
get search order

```
#include <iostream>
#include <string>
#include <vector>
#include "Graph.h" // Defined in Listing 24.3
#include "Edge.h" // Defined in Listing 24.1
#include "Tree.h"// Defined in Listing 24.4
using namespace std;

int main()
{
  // Vertices for graph in Figure 24.1
  string vertices[] = {"Seattle", "San Francisco", "Los Angeles",
    "Denver", "Kansas City", "Chicago", "Boston", "New York",
    "Atlanta", "Miami", "Dallas", "Houston"};

  // Edge array for graph in Figure 24.1
  int edges[][2] = {
    {0, 1}, {0, 3}, {0, 5},
    {1, 0}, {1, 2}, {1, 3},
    {2, 1}, {2, 3}, {2, 4}, {2, 10},
    {3, 0}, {3, 1}, {3, 2}, {3, 4}, {3, 5},
    {4, 2}, {4, 3}, {4, 5}, {4, 7}, {4, 8}, {4, 10},
    {5, 0}, {5, 3}, {5, 4}, {5, 6}, {5, 7},
    {6, 5}, {6, 7},
    {7, 4}, {7, 5}, {7, 6}, {7, 8},
    {8, 4}, {8, 7}, {8, 9}, {8, 10}, {8, 11},
    {9, 8}, {9, 11},
    {10, 2}, {10, 4}, {10, 8}, {10, 11},
    {11, 8}, {11, 9}, {11, 10}
  };
  const int NUMBER_OF_EDGES = 46; // 46 edges in Figure 24.1

  // Create a vector for vertices
  vector<string> vectorOfVertices(12);
  copy(vertices, vertices + 12, vectorOfVertices.begin());

  Graph<string> graph(vectorOfVertices, edges, NUMBER_OF_EDGES);
  Tree dfs = graph.dfs(5); // Vertex 5 is Chicago

  vector<int> searchOrders = dfs.getSearchOrders();
  cout << dfs.getNumberOfVerticesFound() <<
    " vertices are searched in this DFS order:" << endl;
  for (int i = 0; i < searchOrders.size(); i++)
    cout << graph.getVertex(searchOrders[i]) << " ";
  cout << endl << endl;

  for (int i = 0; i < searchOrders.size(); i++)
    if (dfs.getParent(i) != -1)
      cout << "parent of " << graph.getVertex(i) <<
        " is " << graph.getVertex(dfs.getParent(i)) << endl;

  return 0;
}
```

```
12 vertices are searched in this DFS order:
  Chicago Seattle San Francisco Los Angeles Denver Kansas City
  New York Boston Atlanta Miami Houston Dallas

parent of Seattle is Chicago
parent of San Francisco is Seattle
parent of Los Angeles is San Francisco
parent of Denver is Los Angeles
parent of Kansas City is Denver
parent of Boston is New York
parent of New York is Kansas City
parent of Atlanta is New York
parent of Miami is Atlanta
parent of Dallas is Houston
parent of Houston is Miami
```

The program creates a graph for Figure 24.1 in line 37 and obtains a DFS tree starting from vertex `Chicago` in line 38. The search order is obtained in line 40. The graphical illustration of the DFS starting from `Chicago` is shown in Figure 24.10.

Note it is not necessary to include Tree.h and Edge.h, because these two header files are already included in Graph.h.

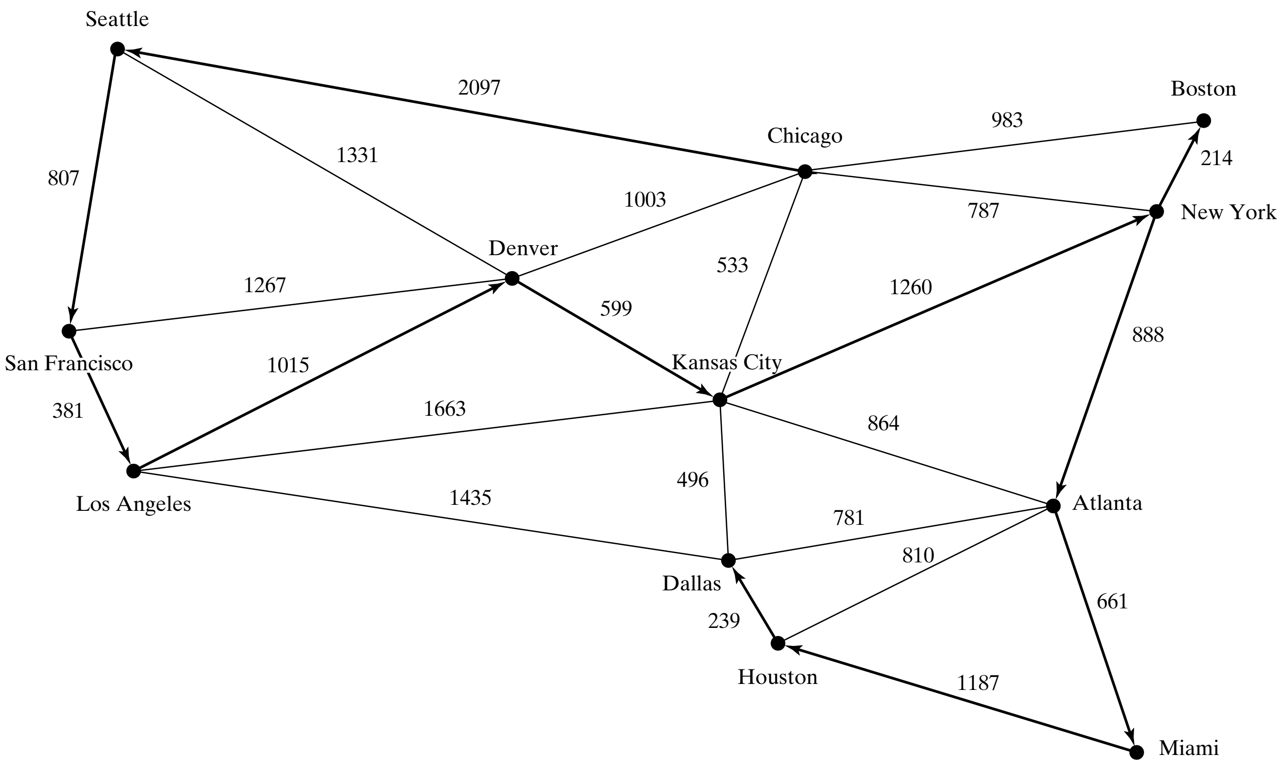


FIGURE 24.10 DFS search starts from Chicago.

24.6.3 Applications of the DFS

The depth-first search can be used to solve many problems, such as the following:

- Detecting whether a graph is connected. Search the graph starting from any vertex. If the number of vertices searched is the same as the number of vertices in the graph, the graph is connected. Otherwise, the graph is not connected.

- Detecting whether there is a path between two vertices.
- Finding a path between two vertices.
- Finding all connected components. A connected component is a maximal connected subgraph in which every pair of vertices are connected by a path.
- Detecting whether there is a cycle in the graph.
- Finding a cycle in the graph.

The first four problems can be easily solved using the **dfs** function in Listing 24.3. To detect or find a cycle in the graph, you have to slightly modify the **dfs** function.

24.7 Breadth-First Search

The breadth-first traversal of a graph is like the breadth-first traversal of a tree discussed in §21.2.5, "Tree Traversal." With breadth-first traversal of a tree, the nodes are visited level by level. First the root is visited, then all the children of the root, then the grandchildren of the root from left to right, and so on. Similarly, the breadth-first search of a graph first visits a vertex, then all its adjacent vertices, then all the vertices adjacent to those vertices, and so on. To ensure that each vertex is visited only once, skip a vertex if it has already been visited.

24.7.1 Breadth-First Search Algorithm

The algorithm for the breadth-first search starting from vertex v in a graph is described in Listing 24.7.

LISTING 24.7 Breadth-first Search Algorithm

create a queue
enqueue **v**
dequeue into **u**
visit **u**
check a neighbor **w**
is **w** visited?
enqueue **w**

```
bfs(vertex v)
{
  create an empty queue for storing vertices to be visited;
  add v into the queue;
  mark v visited;

  while the queue is not empty
  {
    dequeue a vertex, say u, from the queue
    visit u;
    for each neighbor w of u
      if w has not been visited
      {
        add w into the queue;
        mark w visited;
      }
  }
}
```

Consider the graph in Figure 24.11(a). Suppose you start the breadth-first search from vertex 0. First visit 0, then all its visited neighbors, 1, 2, and 3, as shown in 24.11(b). Vertex 1 has three neighbors, 0, 2, and 4. Since 0 and 2 have already been visited, you will now visit just 4, as shown in Figure 24.11(c). Vertex 2 has three neighbors, 0, 1, and 3, which have all been visited. Vertex 3 has three neighbors, 0, 2, and 4, which have all been visited. Vertex 4 has two neighbors, 1 and 3, which have all been visited. So, the search ends.

BFS time complexity

Since each edge and vertex is visited only once, the time complexity of the **bfs** function is **O(|E| + |V|)**, where **|E|** denotes the number of edges and **|V|** the number of vertices.

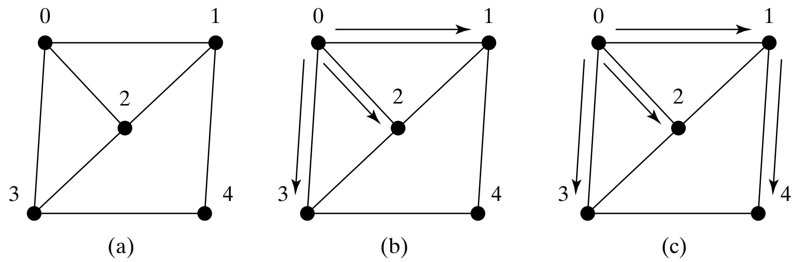


FIGURE 24.11 Breadth-first search visits a node, then its neighbors, and then its neighbors' neighbors, and so on.

24.7.2 Implementation of Breadth-First Search

The `bfs(int v)` function is defined in the `Graph` class in Listing 24.3 (lines 286–317). It returns an instance of the `Tree` class with vertex `v` as the root. The function stores the vertices searched in a list `searchOrders` (line 289), the parent of each vertex in an array `parent` (line 290), stores the vertices to be visited a queue (line 294), and uses the `isVisited` array to indicate whether a vertex has been visited (line 295). The search starts from vertex `v`. `v` is added to the queue in line 296 and is marked visited (line 297). The function now examines each vertex `u` in the queue (line 299) and adds it to `searchOrders` (line 303). The function adds each unvisited neighbor `w` of `u` to the queue (line 309), set its parent to `u` (line 310), and mark it visited (line 311).

Listing 24.8 gives a test program that displays a BFS for the graph in Figure 24.1, starting from `Chicago`.

LISTING 24.8 TestBFS.cpp

```
#include <iostream>
#include <string>
#include <vector>
#include "Graph.h"
using namespace std;

int main()
{
  // Vertices for graph in Figure 24.1
  string vertices[] = {"Seattle", "San Francisco", "Los Angeles",
    "Denver", "Kansas City", "Chicago", "Boston", "New York",
    "Atlanta", "Miami", "Dallas", "Houston"};

  // Edge array for graph in Figure 24.1
  int edges[][2] = {
    {0, 1}, {0, 3}, {0, 5},
    {1, 0}, {1, 2}, {1, 3},
    {2, 1}, {2, 3}, {2, 4}, {2, 10},
    {3, 0}, {3, 1}, {3, 2}, {3, 4}, {3, 5},
    {4, 2}, {4, 3}, {4, 5}, {4, 7}, {4, 8}, {4, 10},
    {5, 0}, {5, 3}, {5, 4}, {5, 6}, {5, 7},
    {6, 5}, {6, 7},
    {7, 4}, {7, 5}, {7, 6}, {7, 8},
    {8, 4}, {8, 7}, {8, 9}, {8, 10}, {8, 11},
    {9, 8}, {9, 11},
    {10, 2}, {10, 4}, {10, 8}, {10, 11},
    {11, 8}, {11, 9}, {11, 10}
  };
```

include Graph.h

vertices

edges

create a graph
get BFS
get search order

```
29   const int NUMBER_OF_EDGES = 46; // 46 edges in Figure 24.1
30
31   // Create a vector for vertices
32   vector<string> vectorOfVertices(12);
33   copy(vertices, vertices + 12, vectorOfVertices.begin());
34
35   Graph<string> graph(vectorOfVertices, edges, NUMBER_OF_EDGES);
36   Tree dfs = graph.bfs(5); // Vertex 5 is Chicago
37
38   vector<int> searchOrders = dfs.getSearchOrders();
39   cout << dfs.getNumberOfVerticesFound() <<
40     " vertices are searched in this BFS order:" << endl;
41   for (int i = 0; i < searchOrders.size(); i++)
42     cout << graph.getVertex(searchOrders[i]) << " ";
43   cout << endl << endl;
44
45   for (int i = 0; i < searchOrders.size(); i++)
46     if (dfs.getParent(i) != -1)
47       cout << "parent of " << graph.getVertex(i) <<
48         " is " << graph.getVertex(dfs.getParent(i)) << endl;
49
50   return 0;
51 }
```

```
12 vertices are searched in this BFS order:
  Chicago Seattle Denver Kansas City Boston New York
  San Francisco Los Angeles Atlanta Dallas Miami Houston

parent of Seattle is Chicago
parent of San Francisco is Seattle
parent of Los Angeles is Denver
parent of Denver is Chicago
parent of Kansas City is Chicago
parent of Boston is Chicago
parent of New York is Chicago
parent of Atlanta is Kansas City
parent of Miami is Atlanta
parent of Dallas is Kansas City
parent of Houston is Atlanta
```

The program creates a graph for Figure 24.1 in line 35 and obtains a DFS tree starting from vertex Chicago in line 36. The search order is obtained in line 38. The graphical illustration of the BFS starting from Chicago is shown in Figure 24.12.

24.7.3 Applications of the BFS

Many of the problems solved by the DFS can also be solved using the breadth-first search. Specifically, the BFS can be used to solve the following problems:

- Detecting whether a graph is connected. A graph is connected if there is a path between any two vertices in the graph.
- Detecting whether there is a path between two vertices.
- Finding a shortest path between two vertices. You can prove that the path between the root and any node in the BFS tree is the shortest path between the root and the node (see Review Question 24.10).

- Finding all connected components. A connected component is a maximal connected subgraph in which every pair of vertices are connected by a path.
- Detecting whether there is a cycle in the graph.
- Finding a cycle in the graph.
- Testing whether a graph is bipartite. A graph is bipartite if its vertices can be divided into two disjoint sets such that no edges exist between vertices in the same set.

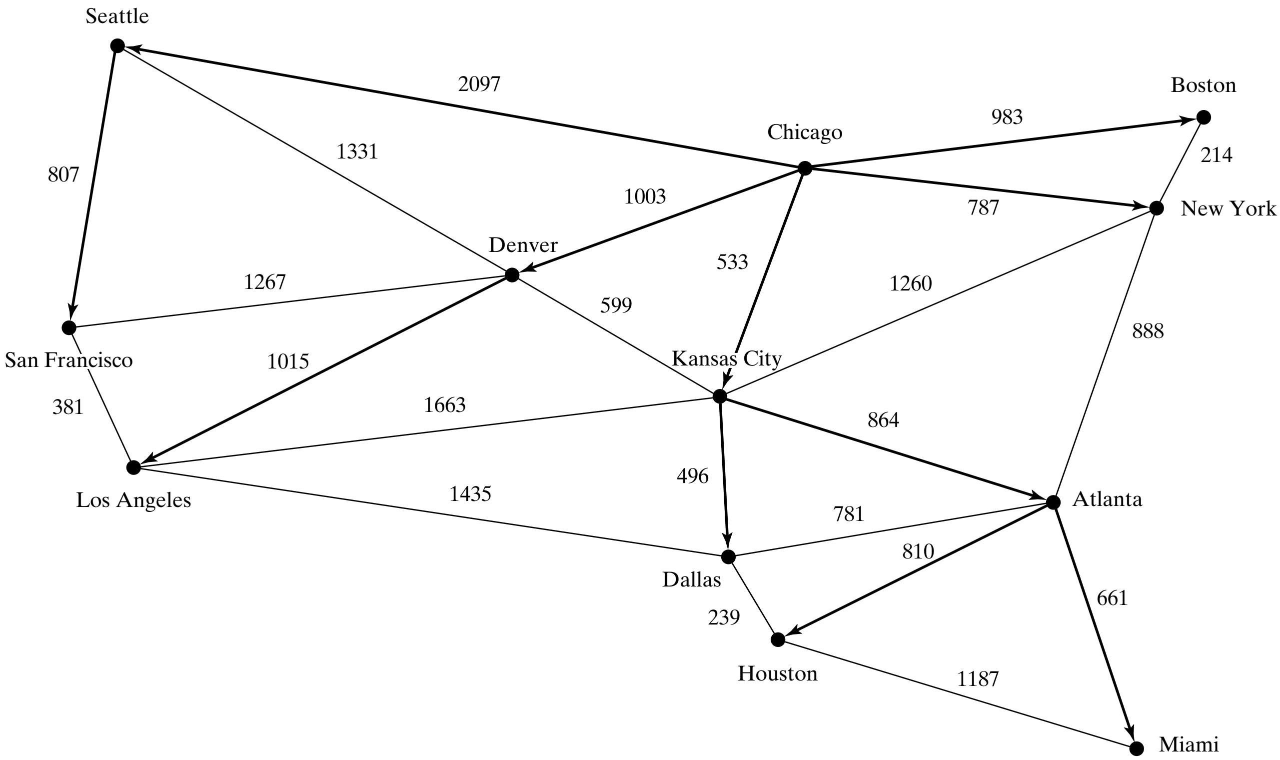


FIGURE 24.12 BFS search starts from Chicago.

24.8 Case Study: The Nine Tail Problem

The DFS and BFS algorithms have many applications. This section applies the BFS to solve the nine tail problem.

The problem is stated as follows. Nine coins are placed in a three-by-three matrix with some face up and some face down. A legal move is to take any coin that is face up and reverse it, together with the coins adjacent to it (this does not include coins that are diagonally adjacent). Your task is to find the minimum number of moves that lead to all coins being face down. For example, you start with the nine coins as shown in Figure 24.13(a). After flipping the second coin in the last row, the nine coins are now as shown in Figure 24.13(b). After flipping the second coin in the first row, the nine coins are all face down, as shown in Figure 24.13(c).

H	H	H
T	T	T
H	H	H

(a)

H	H	H
T	H	T
T	T	T

(b)

T	T	T
T	T	T
T	T	T

(c)

FIGURE 24.13 The problem is solved when all coins are face down.

We will write a C++ program that prompts the user to enter an initial state of the nine coins and displays the solution, as shown in the following sample run.

```
Enter an initial nine coin H and T's:
HHH ↵Enter
TTT ↵Enter
HHH ↵Enter

The steps to flip the coins are
HHH
TTT
HHH

HHH
THT
TTT

TTT
TTT
TTT
```

Each state of the nine coins represents a node in the graph. For example, the three states in Figure 24.13 correspond to three nodes in the graph. Intuitively, you can use a 3×3 matrix to represent all nodes and use **0** for head and **1** for tail. Since there are nine cells and each cell is either **0** or **1**, there are a total of 2^9 (512) nodes, labeled **0**, **1**, . . . , and **511**, as shown in Figure 24.14.

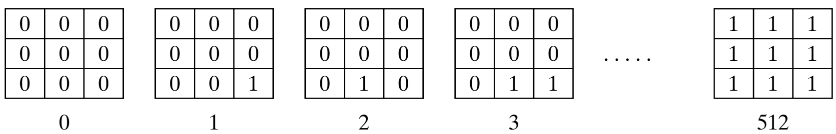


FIGURE 24.14 There are total of **512** nodes, labeled in this order as **0**, **1**, **2**, . . . , and **511**.

We assign an edge from node **u** to **v** if there is a legal move from **v** to **u**. Figure 24.15 shows the directed edges to node **56**.

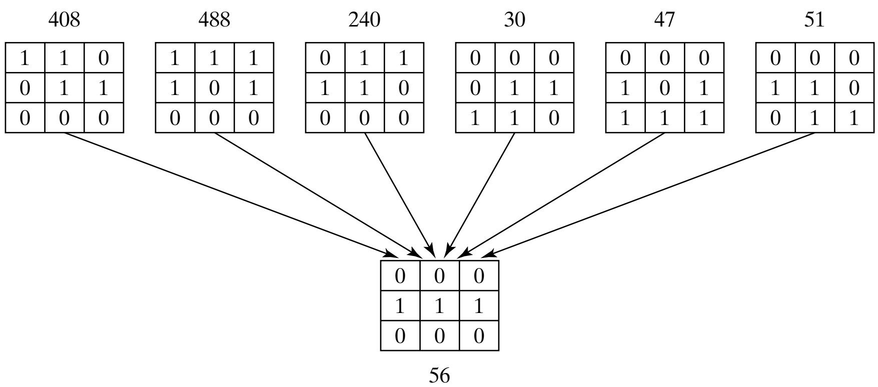


FIGURE 24.15 If node v becomes node u after flipping cells, assign an edge from **u** to **v**.

The last node in Figure 24.14 represents the state of nine face-down coins. For convenience, we call this last node the *target node*. So, the target node is labeled **511**. Suppose the initial state of the nine tail problem corresponds to the node **s**. The problem is reduced to finding a shortest path from the target node to **s** in a BFS tree rooted at the target node.

NineTailModel	
#tree: Tree	A tree rooted at node 511.
+NineTailModel()	Constructs a model for the Nine Tail problem and obtains the tree.
+getShortestPath(nodeIndex: int): vector<int>	Returns a path from the specified node to the root. The path returned consists of the node labels in a vector.
+getNode(index: int): vector<char>	Returns a node consisting of nine characters of H's and T's.
+getIndex(node: vector<char>): int	Returns the index of the specified node.
+printNode(node: vector<char>): void	Displays the node to the console.
#getEdges(): vector<Edge>	Returns a vector of Edge objects for the graph.
#getFlippedNode(node: vector<char>, position: int): int	Flips the node at the specified position and returns the index of the flipped node.
#flipACell(&node: vector<char>, row: int, column: int): void	Flips the node at the specified row and column.

FIGURE 24.16 The `NineTailModel` class models the Nine Tail problem using a graph.

Now the task is to build a graph that consists of `512` nodes labeled `0`, `1`, `2`, . . . , `511`, and edges among the nodes. Once the graph is created, obtain a BFS tree rooted at node `511`. From the BFS tree, you can find the shortest path from the root to any vertex. We will create a class named `NineTailModel`, which contains the function to get the shortest path from the target node to any other node. The class UML diagram is shown in Figure 24.16.

Visually, a node is represented in a 3×3 matrix with letters H and T. In a C++ program, you can use a vector of nine characters to represent nodes. The `getNode(index)` function returns the node with the specified index. For example, `getNode(0)` returns the node that contains nine H's. `getNode(511)` returns the node that contains nine T's. The `getIndex(&node)` function returns the index of the node. The `printNode(node)` function displays the node visually on the console.

Note that the data field `tree` and functions `getEdges()`, `getFlippedNode(node, position)`, and `flipACell(&node, row, column)` are defined protected so that they can be accessed from child classes in the next chapter.

The `getEdges()` function returns a vector of `Edge` objects.

The `getFlippedNode(node, position)` function flips the node at the specified position and returns the index of the new node. For example, for node `56` in Figure 24.16, flip it at position `0`, and you will get node `51`. If you flip node 56 at position `1`, you will get node `47`.

The `flipACell(&node, row, column)` function flips a node at the specified row and column. For example, if you flip node `56` at row `0` and column `0`, the new node is `51`. If you flip node `56` at row `2` and column `0`, the new node is `408`.

Listing 24.9 shows the source code for NineTailModel.h.

LISTING 24.9 NineTailModel.h

```
#ifndef NINETAILMODEL_H
#define NINETAILMODEL_H

#include "Graph.h" // Defined in Listing 24.3
#include "Edge.h" // Defined in Listing 24.1

using namespace std;

const int NUMBER_OF_NODES = 512;

```

include Graph.h

constant graph size

```
class NineTailModel
{
public:
  /** Construct a model for the Nine Tail problem */
  NineTailModel();

  /** Return the index of the node */
  int getIndex(vector<char> node) const;

  /** Return the node for the index */
  vector<char> getNode(int index) const;

  /** Return the shortest path of vertices from the specified
   *  node to the root */
  vector<int> getShortestPath(int nodeIndex) const;

  /** Print a node to the console */
  void printNode(vector<char> &node) const;

protected:
  Tree tree;

  /** Return a vector of Edge objects for the graph */
  /** Create edges among nodes */
  vector<Edge> getEdges() const;

  /** Return the index of the node that is the result of flipping
   *  the node at the specified position */
  int getFlippedNode(vector<char> node, int position) const;

  /** Flip a cell at the specified row and column */
  void flipACell(vector<char> &node, int row, int column);
};

NineTailModel::NineTailModel()
{
  // Create edges
  vector<Edge> edges = getEdges();

  // Build a graph
  Graph<int> graph(NUMBER_OF_NODES, edges);

  // Build a BFS tree rooted at the target node
  tree = graph.bfs(511);
}

vector<Edge> NineTailModel::getEdges() const
{
  vector<Edge> edges;

  for (int u = 0; u < NUMBER_OF_NODES; u++)
  {
    vector<char> node = getNode(u);
    for (int k = 0; k < 9; k++)
    {
      if (node[k] == 'H')
      {
        int v = getFlippedNode(node, k);
        // Add edge (v, u) for a legal move from node u to node v
        edges.push_back(Edge(v, u));
```

no-arg constructor
function **getNode**
function **getShortestPath**
function **printNode**
protected members
create a model
get edges
create a graph
a BFS tree
get edges
Edge vector
for each node
for H cell
get a flipped node
add an edge

```
      }
    }
  }

  return edges;
}

int NineTailModel::getFlippedNode(vector<char> node, int position)
  const
{
  int row = position / 3;
  int column = position % 3;

  flipACell(node, row, column);
  flipACell(node, row - 1, column);
  flipACell(node, row + 1, column);
  flipACell(node, row, column - 1);
  flipACell(node, row, column + 1);

  return getIndex(node);
}

void NineTailModel::flipACell(vector<char> &node,
  int row, int column)
{
  if (row >= 0 && row <= 2 && column >= 0 && column <= 2)
  { // Within boundary
    if (node[row * 3 + column] == 'H')
      node[row * 3 + column] = 'T'; // Flip from H to T
    else
      node[row * 3 + column] = 'H'; // Flip from T to H
  }
}

int NineTailModel::getIndex(vector<char> node) const
{
  int result = 0;

  for (int i = 0; i < 9; i++)
    if (node[i] == 'T')
      result = result * 2 + 1;
    else
      result = result * 2 + 0;

  return result;
}

vector<char> NineTailModel::getNode(int index) const
{
  vector<char> result(9);

  for (int i = 0; i < 9; i++)
  {
    int digit = index % 2;
    if (digit == 0)
      result[8 - i] = 'H';
    else
      result[8 - i] = 'T';
```

flip a node

flip the cell

flip neighbor cells

flip a cell

H to T

T to H

get index from node

get node from index

```
129     index = index / 2;
130   }
131
132   return result;
133 }
134
135 vector<int> NineTailModel::getShortestPath(int nodeIndex) const
136 {
137   return tree.getPath(nodeIndex);
138 }
139
140 void NineTailModel::printNode(vector<char> &node) const
141 {
142   for (int i = 0; i < 9; i++)
143     if (i % 3 != 2)
144       cout << node[i];
145     else
146       cout << node[i] << endl;
147
148   cout << endl;
149 }
150
151 #endif
```

getShortestPath

printNode

The constructor (lines 45–55) creates a graph with 512 nodes, and each edge corresponds to the move from one node to the other (line 48). From the graph, a BFS tree rooted at the target node `511` is obtained (line 54).

To create edges, the `getEdges` function (lines 57–76) checks each node `u` to see if it can be flipped to another node `v`. If so, add (`v`, `u`) to the `Edge` vector (line 70). The `getFlippedNode(node, position)` function finds a flipped node by flipping an `H` cell and its neighbors in a node (lines 78–91). The `flipACell(node, row, column)` function actually flips an `H` cell and its neighbors in a node (lines 93–103).

The `getIndex(node)` function is implemented in the same way as converting a binary number to a decimal (lines 105–116). The `getNode(index)` function returns a node consisting of letters H and T (lines 118–133).

The `getShortestpath(nodeIndex)` function invokes the `getPath(nodeIndex)` function the shortest from the specified node to the target node in a vector (lines 135–138).

The `printNode(node)` function displays the node visually on the console (lines 140–149).

Listing 24.10 gives a program that prompts the user to enter an initial node and displays the steps to reach the target node.

Listing 24.10 NineTail.cpp

```
#include <iostream>
#include <vector>
#include "NineTailModel.h"
using namespace std;

int main()
{
  // Prompt the user to enter nine coins H and T's
  cout << "Enter an initial nine coin H's and T's: " << endl;
  vector<char> initialNode(9);

  for (int i = 0; i < 9; i++)
    cin >> initialNode[i];

```

initial node

```
15   cout << "The steps to flip the coins are " << endl;
16   NineTailModel model;
17   vector<int> path =
18     model.getShortestPath(model.getIndex(initialNode));
19
20   for (int i = 0; i < path.size(); i++)
21     model.printNode(model.getNode(path[i]));
22
23   return 0;
24 }
```

create model

get shortest path

print node

The program prompts the user to enter an initial node with nine letters H and T in lines 9–13, creates a model to create a graph and get the BFS tree (line 16), obtains a shortest path from the initial node to the target node (lines 17–18), and displays the nodes in the path (lines 20–21).

Key Terms

adjacency list 24–6
adjacent vertices 24–8
adjacency matrix 24–7
breadth-first search 24–3
complete graph 24–4
degree 24–4
depth-first search 24–3
directed graph 24–3
graph 24–2
incident edges 24–4
parallel edge 24–4
Seven Bridges of Königsberg 24–2
simple graph 24–4
spanning tree 24–3
weighted graph 24–3
undirected graph 24–3
unweighted graph 24–3

Chapter Summary

1. A graph is a useful mathematical structure that represents relationships among entities in the real world.

2. A graph may be directed or undirected. In a directed graph, each edge has a direction, which indicates that you can move from one vertex to the other through the edge.

3. Edges may be weighted or unweighted. A weighted graph has weighted edges.

4. You can model graphs using classes and interfaces.

5. You can represent vertices and edges using arrays and linked lists.

6. Graph traversal is the process of visiting each vertex in the graph exactly once. Two popular ways of traversing a graph are depth-first search and breadth-first search.

7. The depth-first search of a graph first visits a vertex, then recursively visits all unvisited vertices adjacent to that vertex.

8. The breadth-first search of a graph first visits a vertex, then all its adjacent unvisited vertices, then all the unvisited vertices adjacent to those vertices, and so on.

9. DFS and BFS can be used to solve many problems, such as detecting whether a graph is connected, detecting whether there is a cycle in the graph, and finding a shortest path between two vertices.

Review Questions

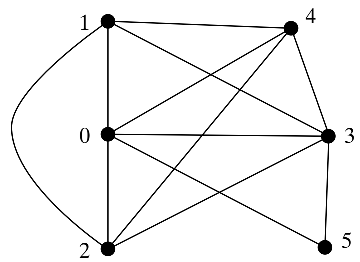
Sections 24.1–24.3

24.1 What is the famous *Seven Bridges of Königsberg* problem?

24.2 What is a graph? Explain terms: undirected graph, directed graph, weighted graph, degree of a vertex, parallel edge, simple graph, and complete graph.

24.3 How do you represent vertices in a graph? How do you represent edges using an edge array? How do you represent an edge using an edge object? How do you represent edges using an adjacency matrix? How do you represent edges using adjacency lists?

24.4 Represent the following graph using an edge array, a vector of edge objects, an adjacent matrix, and an adjacent list, respectively.



Sections 24.4–24.7

24.5 Describe the relationships among `Graph`, `Edge`, and `Tree`.

24.6 What is the return type from invoking `dfs(v)` and `bfs(v)`?

24.7 What are depth-first search and breadth-first search?

24.8 Show the DFS and BFS for the graph in Figure 24.1 starting from vertex `Atlanta`.

24.9 The depth-first search algorithm described in Listing 24.5 uses recursion. Alternatively you may use a stack to implement it, as shown below. Point out the error in this algorithm and give a correct algorithm.

```
// Wrong version
dfs(vertex v)
{
  push v into the stack;
  mark v visited;

  while (the stack is not empty)
  {
    pop a vertex, say u, from the stack
    visit u;
    for each neighbor w of u
      if (w has not been visited)
        push w into the stack;
  }
}
```

24.10 Prove that the path between the root and any node in the BFS tree is the shortest path between the root and the node.

Section 24.8

24.11 If the `flipACell` function in NineTailModel.h is redefined as follows:

```
void flipACell(vector<char> node, int row, int column);
```

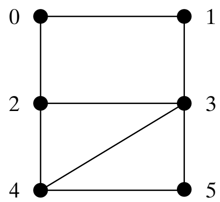
what is wrong?

Programming Exercises

Sections 24.6–24.7

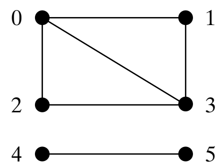
24.1* (*Testing whether a graph is connected*) Write a program that reads a graph from a file and determines whether the graph is connected. The first line in the file contains a number that indicates the number of vertices (`n`). The vertices are labeled as `0`, `1`, `...`, `n-1`. Each subsequent line, with the format `u: v1, v2, ...`, describes edges (`u`, `v1`), (`u`, `v2`), etc. Figure 24.17 gives the examples of two files for their corresponding graphs.

```
File
6
0: 1, 2
1: 0, 3
2: 0, 3, 4
3: 1, 2, 4, 5
4: 2, 3, 5
5: 3, 4
```



(a)

```
File
6
0: 1, 2, 3
1: 0, 3
2: 0, 3
3: 0, 1, 2
4: 5
5: 4
```



(b)

Figure 24.17 The vertices and edges of a graph can be stored in a file.

Your program should prompt the user to enter the name of the file, reads data from a file, creates an instance `g` of `Graph`, invokes `g.printEdges()` to display all edges, and invokes `dfs(0)` to obtain an instance `tree` of `Tree`. If `tree.getNumberOfVerticeFound()` is the same as the number of vertices in the graph, the graph is connected. Here is a sample run of the program:

```
Enter a file name: c:\exercise\Exercise24_1a.txt ↵Enter
The number of vertices is 6
Vertex 0: (0, 1) (0, 2)
Vertex 1: (1, 0) (1, 3)
Vertex 2: (2, 0) (2, 3) (2, 4)
Vertex 3: (3, 1) (3, 2) (3, 4) (3, 5)
Vertex 4: (4, 2) (4, 3) (4, 5)
Vertex 5: (5, 3) (5, 4)
The graph is connected
```

(*Hint*: Use `Graph(numberOfVertices, vectorOfEdges)` to create a graph, where `vectorOfEdges` contains a vector of `Edge` objects. Use `Edge(u, v)` to create an edge. Read the first line to get the number of vertices. Read each subsequent line to extract the vertices from the string and creates edges from the vertices.)

24.2* (*Creating a file for graph*) Modify Listing 24.2, TestGraph.cpp, to create a file for representing `graph1`. The file format is described in Exercise 24.1. Create the file from the array defined in lines 16–29 in Listing 24.2. The number of vertices for the graph is **12**, which will be stored in the first line of the file. The contents of the file should be as follows:

```
12
0: 1, 3, 5
1: 0, 2, 3
2: 1, 3, 4, 10
3: 0, 1, 2, 4, 5
4: 2, 3, 5, 7, 8, 10
5: 0, 3, 4, 6, 7
6: 5, 7
7: 4, 5, 6, 8
8: 4, 7, 9, 10, 11
9: 8, 11
10: 2, 4, 8, 11
11: 8, 9, 10
```

24.3* (*Finding a shortest path*) Write a program that reads a connected graph from a file. The graph is stored in a file using the same format specified in Exercise 24.1. Your program should prompt the user to enter the name of the file, then two vertices, and displays the shortest path between the two vertices. For example, for the graph in Figure 24.17(a), a shortest path between `0` and `5` may be displayed as `0 1 3 5`.

Here is a sample run of the program:

```
Enter a file name: c:\exercise\Exercise24_3a.txt ↵Enter
Enter two vertices (integer indexes): 0 5 ↵Enter
The number of vertices is 6
Vertex 0: (0, 1) (0, 2)
Vertex 1: (1, 0) (1, 3)
Vertex 2: (2, 0) (2, 3) (2, 4)
Vertex 3: (3, 1) (3, 2) (3, 4) (3, 5)
Vertex 4: (4, 2) (4, 3) (4, 5)
Vertex 5: (5, 3) (5, 4)
The path is 0 1 3 5
```

24.4* (*Implementing DFS using a stack*) The depth-first search algorithm described in Listing 24.5 uses recursion. Implement it without using recursion.

24.5* (*Finding connected components*) Add a new function in the `Graph` class to find all connected components in a graph with the following header:

```
vector< vector<int> > getConnectedComponents();
```

The function returns a vector. Each element in the vector is another vector that contains all the vertices in a connected component. For example, if the graph has three connected components, the function returns a vector with three elements, each of which contains the vertices in a connected component.

24.6* (*Finding paths*) Add a new function in `Graph` to find a path between two vertices with the following header:

```
vector<int> getPath(int u, int v);
```

The function returns a vector that contains all the vertices in a path from `u` to `v` in this order. Using the BFS approach, you can obtain a shortest path from `u` to `v`. If there is no path from `u` to `v`, the function returns an empty vector.

24.7* (*Detecting cycles*) Add a new function in `Graph` to determine whether there is a cycle in the graph with the following header:

```
bool containsCyclic();
```

24.8* (*Finding a cycle*) Add a new function in `Graph` to find a cycle in the graph with the following header:

```
vector<int> getACycle();
```

The function returns a vector that contains all the vertices in a cycle from `u` to `v` in this order. If the graph has no cycles, the function returns an empty vector.

24.9** (*Testing bipartite*) Recall that a graph is bipartite if its vertices can be divided into two disjoint sets such that no edges exist between vertices in the same set. Add a new function in `Graph` to detect whether the graph is bipartite:

```
bool isBipartite();
```

24.10** (*Getting bipartite sets*) Add a new function in `Graph` to return two bipartite sets if the graph is bipartite:

```
vector< vector<int> > getBipartiteSets();
```

The function returns a vector. Each element in the vector is another vector that contains a set of vertices.

24.11** (*Variation of the nine tail problem*) In the nine tail problem, when you flip a head, the horizontal and vertical neighboring cells are also flipped. Rewrite the program, assuming that all neighboring cells including the diagonal neighbors are also flipped.

24.12** (*4×4 16 tail model*) The nine tail problem in the text uses a 3×3 matrix. Assume that you have 16 coins placed in a 4×4 matrix. Create a new model class named `TailModel16`. Create an instance of the model and save the object into a file named Exercise24_12.dat.

24.13** (*Induced subgraph*) Given an undirected graph $G = (V, E)$ and an integer k, find an induced subgraph H of G of maximum size such that all vertices of H have degree $>= k$, or conclude that no such induced subgraph exists. Implement the function with the following header:

```
Graph maxInducedSubgraph(Graph edge, int k)
```

The function returns an empty graph if such subgraph does not exist. (*Hint*: An intuitive approach is to remove vertices whose degree is less than k. As vertices are removed with their adjacent edges, the degrees of other vertices may be reduced. Continue the process until no vertices can be removed, or all the vertices are removed.)