Lecture 01: Markov Chain.

STOCHASTIC PROCESS & MARKOV CHAIN – TRANSITION PROBABILITY MATRIX.

stochastic — random / changes with time.

Q1. In a certain market, there are three brands of lipstick – A, B and C.

- Given that a ~~day~~ lady purchased lipstick of brand A,
  - there is 70% chance that she would continue with brand A, 20% and 10% chances that ~~she~~ would shift to brands B and C, resp.
- Given that a lady last purchased lipstick of brand B,
  - there is 50% chances that she would shift to brand A and 10% chance to brand C.
- Given that a lady last purchased lipstick of brand C,
  - there is 60%, 20% chance that she would shift to brands A and B resp.

What are the market shares of the three brands at the end of the year?

→

$$P = \begin{array}{c|ccc} & A & B & C \\ A & 0.7 & 0.2 & 0.1 \\ B & 0.5 & 0.4 & 0.1 \\ C & 0.6 & 0.2 & 0.2 \end{array}$$

The study which evolves over time includes stochastic processes.
Markov model is a stochastic model.

Q2. A professor tried not to be late for class too often. If he is late one day, he is 90% sure to be on time the next day/time. If he is on time, then there is 30% chance of his being late. In the long run, how often is he late for class?

→

	Late	On-time
Late	0.1	0.9
On-time	0.3	0.7

Q3. Suppose that new razor blades were introduced in the market by three companies at the same time. When they were introduced each company has an equal share in the market, but during the year the following changes took place.

Company A retained 90% of its customers and lost 3% to company B and 7% to company C.

Company B retained 70% of its customers and lost ~~70~~ 10% to company A and 20% to company C.

Company C retained 80% of its customers and lost 10% to company A and 10% to company B.

The basic prop. of Markov chain is that X_{t+1} depends upon X_t and not $X_{t-1}, X_{t-2}, \cdots X_0$.

Assuming that no change takes place in the buying habits of the customers, what are the market shares of the three companies at the end of the 1st year and 2nd year?

→

Gain.

Loss	A	B	C
A	0.90	0.03	0.07
B	0.10	0.70	0.20
C	0.10	0.10	0.80

$P =$

extra:

$P(X_3 = B \mid X_2 = A) = P_{AB}^{(1)} = 0.03$

$P(X_3 = B \mid A_0 = A) = P_{AB}^{(3)} \neq 0.03$

- **Transition probabilities:**
  Probabilities from state i to state j after 1 step time period, denoted by p_{ij} is defined as.

$$p_{ij} = P\{X_{n+1} = j \mid X_n = i\}$$

time period state

eg. $P(X_2 = 3 \mid X_1 = 2) = P_{23}^{(1)}$

- **n-step probabilities:**
  Probabilities from state i to state j after n-step time period, denoted by $p_{ij}^{(n)}$ or $p_{ij}(n)$ is defined as,

$$P_{ij}^{(n)} = P\{X_{n+1} = j \mid X_1 = i\}$$

eg. $P(X_2 = 3 \mid X_0 = 2) = P_{23}^{(2)} \;/\; P_{23}(2)$

- **Transition Probability matrix (TPM):**
  Matrix describing Markov chain is called TPM.

$$P = \begin{array}{c|cccc} & 1 & 2 & \cdots & n \\ \hline 1 & p_{11} & p_{12} & \cdots & p_{1n} \\ 2 & p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ n & p_{n1} & p_{n2} & & p_{nn} \end{array} \begin{array}{l} \leftarrow \text{sum } 1 \\ \leftarrow \text{sum } 1 \\ \\ \\ \leftarrow \text{sum } 1 \end{array}$$

→ square matrix
→ $P_{ij} \geq 0$
→ $\Sigma P_{ij} = 1$ (row)

★ **Note:**
- Rows → NOW / FROM / PRESENT
- Columns → NEXT / TO / FUTURE.
- Entry (i, j) → probability of going FROM state i TO state j.

ie. $P_{ij} = P\{X_{n+1} = j \mid X_n = i\}$

eg1. A market survey is made on two brands of breakfast foods - A and B. The TPs are

FROM \ TO	BRAND A	BRAND B
BRAND A	0.8	0.2
BRAND B	0.6	0.4

eg2 Given that a person's last purchase was coke, there is 90% chance that his next cola purchase will also be coke. If a person's last cola purchase was PEPSI, there is 80% chance that his next cola purchase will also be PEPSI. construct the TPM.

→ There are two states – coke and pepsi.
So TPM will of 2X2.

TPM =		COKE	PEPSI
	COKE	0.9	
	PEPSI		0.8

row SUM = 1

∴ TPM =		COKE	PEPSI
	COKE	0.9	0.1
	PEPSI	0.2	0.8

eg3. Three boys – A, B and C are throwing a ball at each other. A always throws the ball to B & B always throws to C, but C is as likely to throw the ball to B as it is to A. Find TPM.

→

FROM \ TO	A	B	C
A	0	1	0
B	0	0	1
C	0.5	0.5	0

eg4. company A has 40% market share in the local markets for its cosmetics, while the other two companies, B and C, have equal share each on 1st Jan, 2018. A study by the market research company has disclosed the following data for every year.

company A retains 70% of its customers and gain 5% from company B and 10% from comp. C.

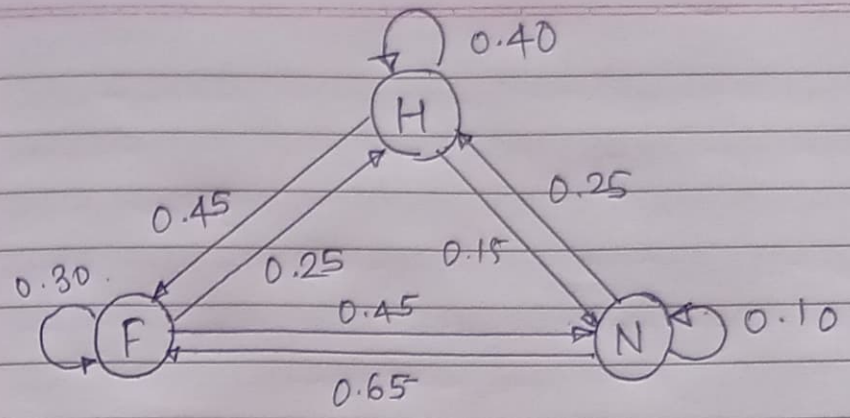
company B retains 90% of its customers and gains 14% from company A and 5% from C

company C retains 85% of its customers and gains 16% from company A and 5% from B.

construct TPM.

FROM \ TO	A	B	C
A	0.7	0.14	0.16
B	0.05	0.9	0.05
C	0.10	0.05	0.85

eg5. A prof. of statistics not wanting to be predictable, decides on an innovative way of assigning HW based on probabilities. The nodes of diagram represent full credit (F), half credit (H) and no credit (N). The TPs for 1 day are as shown in the fig.



→

	F	H	N
F	0.30	0.25	0.45
H	0.40	0.15	0.45
N	0.10	0.25	0.65