System of Non-Homogeneous linear equations

A system of m linear algebraic equations in n unknowns $x_1, x_2, x_3, \cdots\cdots, x_n$ is a set of equations of the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots\cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots\cdots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots\cdots + a_{3n}x_n = b_3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots\cdots + a_{mn}x_n = b_m$$

The above system of equation can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} \cdots\cdots (1)$$

The above system of equation can be written as

$$AX = D \cdots\cdots (2)$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, D = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$

$$[A/D] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Note:-

i) If $\rho(A) = \rho(A|D)$= n, the number of unknowns, then the system of equation (1) has unique solution

ii) If $\rho(A) = \rho(A|D)$< n, the number of unknowns, then the system of equation (1) has infinitely many solution

iii) If $\rho(A) \neq \rho(A|D)$,then the system of equation (1) is inconsistent i.e. it has no solution.

Type-1

Example-1

Is the following system of equations is consistent if consistent find it's solution

$$1x + 2y + 3z = 14, 3x + 1y + 2z = 11, 2x + 3y + 1z = 11$$

Solution: Given system of equations is

$1x + 2y + 3z = 14,$

$3x + 1y + 2z = 11,$

$2x + 3y + 1z = 11 \cdots$(1)

The above system of equations can be written as

$$[A/D] = \begin{bmatrix} 1 & 2 & 3 & 14 \\ 3 & 1 & 2 & 11 \\ 2 & 3 & 1 & 11 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$

$$[A/D] \sim \begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & -5 & -7 & -31 \\ 0 & -1 & -5 & -17 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$[A/D] \sim \begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & -1 & -5 & -17 \\ 0 & -5 & -7 & -31 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2, R_2 \to -1R_2$

$$[A/D] \sim \begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & 1 & 5 & 17 \\ 0 & 0 & 18 & 54 \end{bmatrix}$$

$R_3 \to \dfrac{1}{18} R_3$

$$[A/D] \sim \begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & 1 & 5 & 17 \\ 0 & 0 & 1 & 3 \end{bmatrix} \cdots (2)$$

Since $\rho(A) = \rho(A|D) = 3 = No. of\ unknowns$

The system of equation is consistent and it has unique solution

Equation (2) can be written as

$1x + 2y + 3z = 14, \cdots (3)$

$1y + 5z = 17, \cdots (4)$

1z=3⇒$z = 3$

Now put z=3 in equation (4) we get

$1y + (5 \times 3) = 17$

$y + 15 = 17 \ \therefore y = 2$

Now put $y = 2, and\ z = 3$ in equation (3) we get

$1x + (2 \times 2) + (3 \times 3) = 14$

$1x + 4 + 9 = 14$

$1x = 1 \Rightarrow x = 1$

$Thus\ x = 1, y = 2, and\ z = 3\ be\ the\ required\ solution$

Example-2

Is the following system of equations is consistent if consistent find it's solution

$2x + 1y + 1z = 4, 1x - 1y + 3z = 3,4x - 1y - 1z = 2$

Solution: Given system of equations is

$2x + 1y + 1z = 4,$

$1x - 1y + 3z = 3,$

$4x - 1y - 1z = 2 \cdots$(1)

The above system of equations can be written as

$$[A/D] = \begin{bmatrix} 2 & 1 & 1 & 4 \\ 1 & -1 & 3 & 3 \\ 4 & -1 & -1 & 2 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$[A/D] \sim \begin{bmatrix} 1 & -1 & 3 & 3 \\ 2 & 1 & 1 & 4 \\ 4 & -1 & -1 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 4R_1$

$$[A/D] \sim \begin{bmatrix} 1 & -1 & 3 & 3 \\ 0 & 3 & -5 & -2 \\ 0 & 3 & -13 & -10 \end{bmatrix}$$

$R_3 \to R_3 - 5R_2$

$$[A/D] \sim \begin{bmatrix} 1 & -1 & 3 & 3 \\ 0 & 3 & -5 & -2 \\ 0 & 0 & -8 & -8 \end{bmatrix}$$

$R_3 \to \frac{-1}{8} R_3$

$$[A/D] \sim \begin{bmatrix} 1 & -1 & 3 & 3 \\ 0 & 3 & -5 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdots (2)$$

Since $\rho(A) = \rho(A|D) = 3 = No. of\ unknowns$

The system of equation is consistent and it has unique solution

Equation (2) can be written as

$1x - 1y + 3z = 3, \cdots (3)$

$3y - 5z = -2, \cdots (4)$

1z=1⇒$z = 1$

Now put z=1 in equation (4) we get

$3y - (5 \times 1) = -2$

$3y - 5 = -2 \ \therefore 3y = 3 \Rightarrow y = 1$

Now put $y = 1, and\ z = 1$ in equation (3) we get

$1x - (1 \times 1) + (3 \times 1) = 3$

$1x - 1 + 3 = 3$

$1x = 1 \Rightarrow x = 1$

$Thus\ x = 1, y = 1, and\ z = 1\ be\ the\ required\ solution$

Example No. 03

Is the following system of equations is consistent if consistent find it's solution

$6x + 1y + 1z = -4, 2x - 3y - 1z = 0, 1x + 7y + 2z = -7$

Solution: Given system of equations is

$6x + 1y + 1z = -4$

$2x - 3y - 1z = 0$

$1x + 7y + 2z = -7 \cdots (1)$

The above system of equations can be written as

$$[A/D] = \begin{bmatrix} 6 & 1 & 1 & -4 \\ 2 & -3 & -1 & 0 \\ 1 & 7 & 2 & -7 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$[A/D] \sim \begin{bmatrix} 1 & 7 & 2 & -7 \\ 2 & -3 & -1 & 0 \\ 6 & 1 & 1 & -4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 6R_1$

$$[A/D] \sim \begin{bmatrix} 1 & 7 & 2 & -7 \\ 0 & -17 & -5 & 14 \\ 0 & -41 & -11 & 38 \end{bmatrix}$$

$R_2 \to \frac{-1}{17} R_2, R_3 \to \frac{-1}{41} R_3$

$$[A/D] \sim \begin{bmatrix} 1 & 7 & 2 & -7 \\ 0 & 1 & \frac{5}{17} & -\frac{14}{17} \\ 0 & 1 & \frac{11}{41} & -\frac{38}{41} \end{bmatrix}$$

$R_3 \to R_3$-$R_2$

$$[A/D] \sim \begin{bmatrix} 1 & 7 & 2 & -7 \\ 0 & 1 & \frac{5}{17} & -\frac{14}{17} \\ 0 & 0 & \frac{-18}{697} & \frac{-72}{697} \end{bmatrix}$$

$R_3 \to \frac{-697}{18} R_3$

$$[A/D] \sim \begin{bmatrix} 1 & 7 & 2 & -7 \\ 0 & 1 & \frac{5}{17} & -\frac{14}{17} \\ 0 & 0 & 1 & 4 \end{bmatrix} \cdots (2)$$

Since $\rho(A) = \rho(A|D) = 3 = No. of\ unknowns$

The system of equation is consistent and it has unique solution

Equation (2) can be written as

$1x + 7y + 2z = -7 \cdots (3)$

$1y + \frac{5}{17} z = -\frac{14}{17} \cdots (4)$

1z=4$\Rightarrow z = 4$

Now put z=4 in equation (4) we get

$1y + \left(\frac{5}{17} \times 4\right) = -\frac{14}{17}$

$1y = -\frac{20}{17} - \frac{14}{17} = -\frac{34}{17} = -2$

Thus $y = -2$

Now put $y = -2, and\ z = 4$ in equation (3) we get

$1x + \left(7 \times (-2)\right) + (2 \times 4) = -7$

$1x - 14 + 8 = -7$

$1x - 6 = -7$

$1x = -1 \Rightarrow x = -1$

$Thus\ x = -1, y = -2, and\ z = 4\ be\ the\ required\ solution$

Example-4

Is the following system of equations is consistent if consistent find it's solution

$1x + 2y - 1z = 1, 1x + 1y + 2z = 9, 2x + 1y - 1z = 2$

Solution: Given system of equations is

$1x + 2y - 1z = 1$

$1x + 1y + 2z = 9$

$2x + 1y - 1z = 2 \cdots (1)$

The above system of equations can be written as

$$[A/D] = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & 1 & 2 & 9 \\ 2 & 1 & -1 & 2 \end{bmatrix}$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1$

$$[A/D] \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 8 \\ 0 & -3 & 1 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$[A/D] \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 8 \\ 0 & 0 & -8 & -24 \end{bmatrix}$$

$R_3 \to \dfrac{-1}{8} R_3$

$$[A/D] \sim \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix} \cdots (2)$$

Since $\rho(A) = \rho(A|D) = 3 = No. of\ unknowns$

The system of equation is consistent and it has unique solution

Equation (2) can be written as

$1x + 2y - 1z = 1, \cdots (3)$

$-1y + 3z = 8, \cdots (4)$

1z=3$\Rightarrow z = 3$

Now put z=1 in equation (4) we get

$-1y + (3 \times 3) = 8$

$-1y + 9 = 8 \;\therefore -1y = -1 \Rightarrow y = 1$

Now put $y = 1, and\ z = 3$ in equation (3) we get

$1x + (2 \times 1) - (1 \times 3) = 1$

$1x + 2 - 3 = 1$

$1x = 2 \Rightarrow x = 2$

$Thus\ x = 2, y = 1, and\ z = 3\ be\ the\ required\ solution$

H.W.

1)$1x + 1y + 1z = 3, 1x + 2y + 3z = 4, 1x + 4y + 9z = 6\{x = 2, y = 1, z = 0\}$

2) $1x + 1y + 1z = 6, 1x - 1y + 2z = 5, 3x + 1y + 1z = 8, 2x - 2y + 3z = 7,$

$\{x = 1, y = 2, z = 3\}$

3) $2x + 1y - 1z + 3w = 11, 1x - 2y + 1z + 1w = 8, 4x + 7y + 2z - 1w = 0,$

$3x + 5y + 4z + 4w = 17, \{x = 2, y = -1, z = 1, w = 3\}$

4) $2x + 1y - 1z + 3w = 8, 1x + 1y + 1z - 1w = -2, 3x + 2y - 1z + 0w = 6,$

$0x + 4y + 3z + 2w = -8, \{x = 2, y = -1, z = -2, w = 1\}$

**Type-2(Infinite Solution)**

1) Is the following system of equations is consistent if consistent find it's solution

$1x + 4y - 6z = 1, 2x - 3y + 5z = 1, 3x + 1y - 1z = 2$

Solution: Given system of equations is

$1x + 4y - 6z = 1$

$2x - 3y + 5z = 1$

$3x + 1y - 1z = 2 \cdots (1)$

The above system of equations can be written as

$$[A/D] = \begin{bmatrix} 1 & 4 & -6 & 1 \\ 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \end{bmatrix}$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1$

$$[A/D] \sim \begin{bmatrix} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & -11 & 17 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$[A/D] \sim \begin{bmatrix} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdots (2)$$

Since $\rho(A) = \rho(A|D) = 2 < No. of\ unknowns$

The system of equation is consistent and it has infinite solutions

Equation (2) can be written as

$1x + 4y - 6z = 1 \cdots (3)$

$-11y + 17z = -1 \cdots (4)$

$11y = 17z + 1$

$$y = \frac{17}{11} z + \frac{1}{11} \cdots (5)$$

From (3), and (5)

$$1x + 4\left(\frac{17}{11}z + \frac{1}{11}\right) - 6z = 1$$

$$1x + \frac{68}{11}z + \frac{4}{11} - 6z = 1$$

$$1x = -\frac{68}{11}z - \frac{4}{11} + 6z + 1$$

$$1x = \frac{-68 + 66}{11}z + \frac{-4 + 11}{11}$$

$$1x = \frac{-2}{11}z + \frac{7}{11}$$

$$x = \frac{-2}{11}z + \frac{7}{11} \cdots (6)$$

$Let\ z = k,$

From (5), and (6)

$y = \frac{17}{11}k + \frac{1}{11}$, and $x = \frac{-2}{11}k + \frac{7}{11}$

Thus $x = \frac{-2}{11}k + \frac{7}{11}, y = \frac{17}{11}k + \frac{1}{11}, z = k$

For different value of k we get different values of x, y, and z

Thus we get infinite number of solutions

2) Is the following system of equations is consistent if consistent find it's solution

$2x - 1y - 1z = 2, 1x + 2y + 1z = 2, 4x - 7y - 5z = 2$

Solution: Given system of equations is

$2x - 1y - 1z = 2$

$1x + 2y + 1z = 2$

$4x - 7y - 5z = 2 \cdots (1)$

The above system of equations can be written as

$$[A/D] = \begin{bmatrix} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$[A/D] \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & -1 & -1 & 2 \\ 4 & -7 & -5 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 4R_1$

$$[A/D] \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & -15 & -9 & -6 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$[A/D] \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdots (2)$$

Since $\rho(A) = \rho(A|D) = 2 < No. of\ unknowns$

The system of equation is consistent and it has infinite solutions

Equation (2) can be written as

$1x + 2y + 1z = 2 \cdots (3)$

$-5y - 3z = -2 \cdots (4)$

$-5y = 3z - 2$

$$y = -\frac{3}{5}z + \frac{2}{5} \cdots (5)$$

Now from equations (3), and (5)

$$1x + 2\left(-\frac{3}{5}z + \frac{2}{5}\right) + 1z = 2$$

$$1x - \frac{6}{5}z + \frac{4}{5} + 1z = 2$$

$$1x = \frac{6}{5}z - \frac{4}{5} - 1z + 2$$

$$x = \frac{6-5}{5}z + \frac{-4+10}{5}$$

$$x = \frac{1}{5}z + \frac{6}{5} \cdots (6)$$

Thus From equations (5), and (6)
$x = \frac{1}{5}z + \frac{6}{5}, y = -\frac{3}{5}z + \frac{2}{5}$

Now Take $z = k, then\ x = \frac{1}{5}k + \frac{6}{5}, y = -\frac{3}{5}k + \frac{2}{5}$

Thus for different value of k we get different values of $x, y, and\ z$

In this way we get infinite number of solutions

3) Is the following system of equations is consistent if consistent find it's solution

$1x + 1y - 3z = -1, 4x - 2y + 6z = 8,5x - 1y + 3z = 7$

Solution: Given system of equations is

$1x + 1y - 3z = -1$

$4x - 2y + 6z = 8$

$5x - 1y + 3z = 7 \cdots (1)$

The above system of equations can be written as

$$[A/D] = \begin{bmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 5 & -1 & 3 & 7 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1, R_3 \to R_3 - 5R_1$

$$[A/D] \sim \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -6 & 18 & 12 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2, R_3 \to \frac{-1}{6} R_3$

$$[A/D] \sim \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdots (2)$$

Since $\rho(A) = \rho(A|D) = 2 < No.of\ unknowns$

The system of equation is consistent and it has infinite solutions

Equation (2) can be written as

$1x + 1y - 3z = -1 \cdots (3)$

$1y - 3z = -2 \cdots (4)$

$y = 3z - 2 \cdots (5)$

Now from equations (3), and (5)

$1x + 1(3z - 2) - 3z = -1$

$1x + 3z - 2 - 3z = -1$

$1x = -3z + 2 + 3z - 1$

$x = 1 \cdots (6)$

Now Take $z = k, then\ x = 1, y = 3k - 2$

Thus for different value of k we get different values of $x, y, and\ z$

In this way we get infinite number of solutions