Sem-I → complex Nos.

$z = a + ib$, $a, b \in \mathbb{R}$

(a → Real, b → Im.)

GATE

Sem-III complex variables : derivative of complex variable.

Sem-IV complex Integration :-

Complex variables

$z = x + iy$, $x, y \in \mathbb{R}$ $\qquad z = f(x, y)$

$|z| = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left[\frac{Im}{Re}\right] = \tan^{-1}\left[\frac{y}{x}\right]$

let $w = f(z) = z^2 = (x+iy)^2 = x^2 + 2ixy + i^2y^2 \quad \because i^2 = -1$

$= (x^2 - y^2) + i(2xy)$

$= u(x,y) + i\, v(x,y)$ → C.V.

$\boxed{w = f(z) = u + iv}$ → C.V $\qquad z = x + iy$

Modulus of $f(z) = |f(z)| = \sqrt{u^2 + v^2}$

Argument = Amplitude = $\theta = \tan^{-1}\left[\frac{Im}{Re}\right] = \tan^{-1}\left[\frac{v}{u}\right]$

Defⁿ: Analytic Function / Regular Function / Entire Fun / Holomorphic Function.

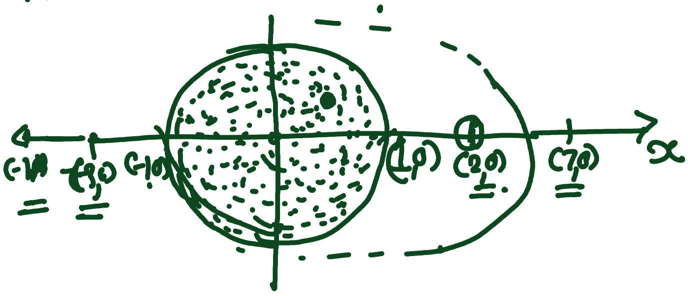
If $f(z) = u + iv$ is differentiable at each & every point in given Region of convergence (ROC). Then $f(z) = u + iv$ is called as an Analytic Function.

$z = 2 + 3i = (2, 3)$

Ex. $f(z) = \dfrac{z^3 - 4z^2 + 3z - 7}{(z-2)(z+3)(z-7)(z+10)}$

$f(z) = \infty = \dfrac{}{0}$

$\boxed{z = 2, -3, 7, -10}$



$z = 2 + i0 = (2, 0)$

Cauchy's Riemann equ^ns [C.R. equ^ns]

If $f(z)=u+iv$ is an Analytic Fun and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are exists, Then

(I) $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ or $\boxed{u_x = v_y}$

(II) $\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$ or $\boxed{u_y = -v_x}$

These conditions (equ^ns) are called Cauchy's Riemann equ^ns or C.R. equ^ns

Analytic $\longleftrightarrow$ C.R eqn exists

II] C.R equations in polar Form:-

If $f(z)=u+iv = u(r,\theta)+i\,v(r,\theta)$ is an Analytic Fun

and $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ are exists, Then

C.R. equ^ns in polar Form

(I) $\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$ or $\boxed{u_r = \frac{1}{r} v_\theta}$

(II) $\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$ or $\boxed{v_r = -\frac{1}{r} u_\theta}$

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Ex.(1) $f(z)=z^2$ is an Analytic Fun or Not?

$\rightarrow$ Step(I) $f(z)=z^2=(x+iy)^2=(x^2-y^2)+i(2xy)$

$f(z)=z^2=u+iv$

where $u=x^2-y^2$ and $v=2xy$

step(II) $\frac{\partial u}{\partial x}=2x-0=2x$; $\frac{\partial v}{\partial x}=2y$

$\frac{\partial u}{\partial y}=0-2y=-2y$; $\frac{\partial v}{\partial y}=2x$

$$\frac{\partial u}{\partial y} = 0 - 2y = -2y \quad ; \quad \frac{\partial v}{\partial y} = 2x$$

Step (III)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \boxed{2x} \Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \boxed{-2y} \Rightarrow \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

$\Rightarrow$ C.R. equ^{n}s are exist.

Then $f(z) = z^2$ is an Analytic Fu^n.

1 Find a, b, c, d, e If $f(z) = (ax^3 + bxy^2 + 3x^2 + cy^2 + x) + i.(dx^2y - 2y^3 + exy + y)$
Is an analytic function ?

$\rightarrow$ We have $f(z) = u + iv$ is an Analytic Fu^n (given)

$u = ax^3 + bxy^2 + 3x^2 + cy^2 + x$ —— (I)

$v = dx^2y - 2y^3 + exy + y$ —— (II)

given

C.R. equ^{n}s are exist. (given)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{—— (III)} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{—— (IV)}$$

$$\frac{\partial u}{\partial x} = 3ax^2 + by^2 + 6x + 0 + 1$$

$$\frac{\partial v}{\partial y} = dx^2 - 6y^2 + ex + 1$$

$$\frac{\partial u}{\partial y} = 0 + 2bxy + 0 + 2cy + 0$$

$$\frac{\partial v}{\partial x} = 2dxy - 0 + ey + 0$$

But By eq^n (III)

$$3ax^2 + by^2 + 6x + 1 = dx^2 - 6y^2 + ex + 1$$

$\Rightarrow \boxed{3a = d}$

$\Rightarrow \boxed{b = -6}$ $\quad \therefore \boxed{b = -6}$

$\Rightarrow \boxed{6 = e}$ $\quad \therefore \boxed{e = 6}$

From eq^n (I) & eq^n (II)

$$\frac{\partial u}{\partial y} = 2bxy + 2cy$$

$$\frac{\partial v}{\partial x} = 2dxy + ey$$

By eq^n (IV) $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$(2bxy + 2cy) = -(2dxy + ey)$$

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$\Rightarrow 2b = -2d \quad \Rightarrow \boxed{b=-d}$

$\Rightarrow -6 = -d \quad \Rightarrow \boxed{d=6}$

$\Rightarrow 2c = -e \quad \Rightarrow 2c = -6 \quad \Rightarrow \boxed{c=-3}$

We have $3a = d$

$3a = 6$

$\Rightarrow \boxed{a=2}$

$\therefore$ $\boxed{\text{Ans: } a=2,\ b=-6,\ c=-3,\ d=6,\ e=6}$

2. Find the constants k, if $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{kx}{y}$ is an Analytic function.

$\rightarrow$ We have $f(z) = u+iv \Rightarrow$ Analytic Fun (given)

Then C.R. equ's are exist.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{—(i)} \qquad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \quad \text{—(ii)}$$

$$u = \frac{1}{2}\log(x^2+y^2) \quad \text{—(iii)}$$

$$v = \tan^{-1}\left(\frac{kx}{y}\right) \quad \text{—(iv)}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{(x^2+y^2)} \times (2x+0) = \frac{x}{x^2+y^2} \quad \text{—(v)}$$

$$\frac{\partial v}{\partial y} = \frac{1}{\left[1+\left(\frac{kx}{y}\right)^2\right]} \times \left[\frac{kx(-1)}{y^2}\right] = \frac{-kx}{\left[\frac{y^2+k^2x^2}{\cancel{y^2}}\right]\cancel{y^2}} = \frac{-kx}{k^2x^2+y^2} \quad \text{—(vi)}$$

By eqn (i) $\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\frac{x}{x^2+y^2} = \frac{-kx}{k^2x^2+y^2}$$

$$\Rightarrow \boxed{k=-1}$$

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4. Show that $f(z) = e^{2z} - z$ is an analytic function ? [Biom-Dec-19,(5M)]

$\rightarrow$ We have $f(z) = e^{2z} - z = e^{2(x+iy)} - (x+iy)$

$2x \quad i(2y)$

$= e \cdot e \quad -x-iy$

$= e^{2x}[\cos(2y)+i\sin(2y)]-x-iy \qquad \therefore e^{i\theta}=\cos\theta+i\sin\theta$

$f(z)=[e^{2x}\cos(2y)-x]+i[e^{2x}\sin(2y)-y]=u+iv$

$\therefore u= e^{2x}\cos(2y)-x \quad \text{——— (I)}$

$v= e^{2x}\sin(2y)-y \quad \text{——— (II)}$

$\left.\begin{aligned}\frac{\partial u}{\partial x} &= 2e^{2x}\cos(2y)-1\\ \frac{\partial v}{\partial y} &= 2e^{2x}\cos(2y)-1\end{aligned}\right\} \Rightarrow \boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}}$

$\left.\begin{aligned}\frac{\partial u}{\partial y} &= -2e^{2x}\sin(2y)-0\\ \frac{\partial v}{\partial x} &= 2e^{2x}\sin(2y)-0\end{aligned}\right\} \Rightarrow \boxed{\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}}$

$\Rightarrow$ C.R. equ^n's are exist Hence f(z) is an Analytic fun.

4. Prove that an Analytic function with constant modulus is constant function. [Dec-08]

$\rightarrow$ We have $f(z)=u+iv$ is an Analytic Function. Then C.R equ^n's are always exist.

$$\boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}} \text{——— (I)} \qquad \boxed{\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}} \text{——— (II)}$$

$$|f(z)|=\sqrt{u^2+v^2}=\text{constant}=k \text{ (say)}$$

$$u^2+v^2=k^2 \quad \text{——— (III)}$$

diff eq^n (III) w.r.t x

$$2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=0 \quad \text{——— (IV)}$$

diff equ^n (III) w.r.t y

$$2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}=0$$

$$\Rightarrow u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}=0 \quad \text{——— (V)}$$

By C.R. equ^n's from eq^n (I) & (II)

$$-u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}=0 \quad \text{——— (VI)}$$

$$v\frac{\partial v}{\partial x}+u\frac{\partial u}{\partial x}=0 \quad \text{——— (IV) [copied]}$$

$u=$ const & derivative $u=0$

$$\frac{\partial u}{\partial x}=0,\ \frac{\partial u}{\partial y}=0$$

$$v\frac{\partial v}{\partial x}+u\frac{\partial u}{\partial x}=0 \quad \text{--- (IV) [copied]}$$

Multiple eq^n (VI) by v and multiple eq^n (IV) by u and Addition

$$-uv\frac{\partial v}{\partial x}+v^2\frac{\partial u}{\partial x}=0$$

$$+\quad + uv\frac{\partial v}{\partial x}+u^2\frac{\partial u}{\partial x}=0$$

$$0+(u^2+v^2)\frac{\partial u}{\partial x}=0$$

$$\Rightarrow (u^2+v^2)\frac{\partial u}{\partial x}=0$$

$$k^2\frac{\partial u}{\partial x}=0$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x}=0} \quad \text{--- (A)}$$

But By C-R eq^n $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=0$

$$\boxed{\frac{\partial v}{\partial y}=0} \quad \text{--- (B)}$$

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Multiple eq^n (VI) by u and eq^n (IV) by v and substraction

$$-u^2\frac{\partial v}{\partial x}+uv\frac{\partial u}{\partial x}=0$$

$$-\quad -v^2\frac{\partial v}{\partial x}\pm uv\frac{\partial u}{\partial x}=0$$

$$-(u^2+v^2)\frac{\partial v}{\partial x}+0=0$$

$$\Rightarrow -k^2\frac{\partial v}{\partial x}=0 \Rightarrow \boxed{\frac{\partial v}{\partial x}=0} \quad \text{--- (C)}$$

But C.R equ^n $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

$$\boxed{\frac{\partial u}{\partial y}=-0=0} \quad \text{--- (D)}$$

By eq^n (A), (B), (C), & (D)

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=0\\ \frac{\partial u}{\partial y}&=0\end{aligned}\right\} \boxed{u=\text{constant}}$$

$$\left.\begin{aligned}\frac{\partial v}{\partial x}&=0\\ \frac{\partial v}{\partial y}&=0\end{aligned}\right\} \boxed{v=\text{constant}}$$

$$f(z)=u+iv=\text{const.}$$

Orthogonal Trajectory:

Orthogonal Trajectory: In mathematics , if every member of one family of curves that intersect to each member of another family of curves at right angles (see figure) is called as Orthogonal trajectory in between both family of curves.

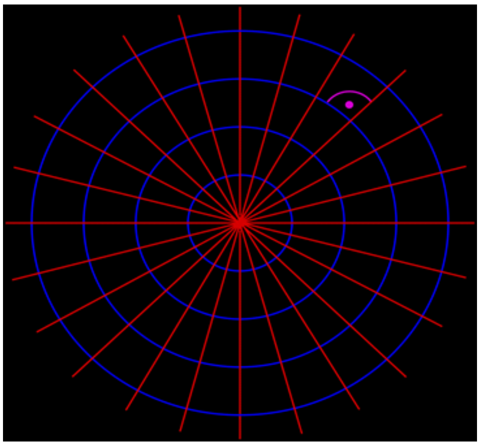

Fig.1. Family of concentric Circles and Straight lines passing though origin.

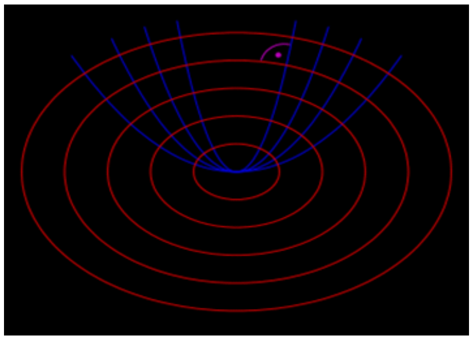

Fig.2. Family of ellipse and parabola

Theorem: : If $f(z) = (u + i\,v)$ is an Analytical function then the curves $u(x, y) = C_1$ and $v(x, y) = C_2$ represents family of orthogonal trajectories to each other.

Milne-Thompson Method :- (MTM)

case (I) IF we know $u = u(x,y)$ Then How to find an Analytic Fun $f(z) = u + iv$ → u = given

Step (I) $f(z) = u + iv$ → Analytic Fun and u = given $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$

Step (II) diff w.r.t. x; $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

Step (III) By C.R equns $\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

$$f'(z) = \left[\frac{\partial u}{\partial x}\right] - i\left[\frac{\partial u}{\partial y}\right]$$

$$f'(z) = \phi_1(x,y) - i\,\phi_2(x,y)$$

$u = x^2 + 2xy + y^2$

$\frac{\partial u}{\partial x} = 2x + 2y = \phi_1(x,y)$

$\frac{\partial u}{\partial y} = 0 + 2x + 2y = \phi_2(x,y)$

Step (IV) By MTM, put $x = z$ and $y = 0$

$$f'(x+iy) = \phi_1(x,y) - i\,\phi_2(x,y)$$

$$f'(z+i0) = \phi_1(z,0) - i\,\phi_2(z,0)$$

$$f'(z) = \phi_1(z) - i\,\phi_2(z)$$

Milne-Thompson Thm: If we put $x = z$ and $y = 0$ In derivative part Then, value of equn is Not change.

Thm

Step (V) Integrate w.r.t. z

Step (V) Integrate w.r.t. z

$$\int f'(z)\,dz = \int \phi_1(z)\,dz - i\int \phi_2(z)\,dz + C$$

$$\boxed{f(z) = \psi_1(z) - i\,\psi_2(z) + c} \rightarrow \text{Analytic}$$

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Case-II] IF $V = V(x,y) =$ given

Step-(I) $f(z) = u + iv \rightarrow$ Analytic Fun

(II) $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

Step (III) By C-R equn $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$f'(z) = f'(x+iy) = \phi_1(x,y) + i\,\phi_2(x,y)$$

Step (IV) By MTM, put $x = z$ and $y = 0$

$$f'(z+i0) = \phi_1(z,0) + i\,\phi_2(z,0)$$

Step (V) Integrating w.r.t z

$$\int f'(z)\,dz = \int \phi_1(z)\,dz + i\int \phi_2(z)\,dz + C$$

$$\boxed{f(z) = \psi_1(z) + i\,\phi_2(z) + C} \rightarrow \text{Analytic}$$

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Ex.1 Construct an Analytic function $f(z) = (u + iv)$ If $u = e^x \cos(y)$ [EXTC-May-19, (6M)]

$\rightarrow$ Step-(I) We have $f(z) = u + iv \rightarrow$ Analytic Fun

Step-(II) diff. w.r.t. 'x'; $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

Step-(III) By C.R equn $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

But We have $u = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y \quad \text{and} \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$f'(z) = (e^x \cos y) - i(-e^x \sin y)$$

Step-(IV) By Milne-Thompson Method, put $x=z$ and $y=0$

$$f'(x+iy) = (e^x \cos y)_{\substack{x=z\\y=0}} + i(e^x \sin y)_{\substack{x=z\\y=0}}$$

$$f'(z+i0) = e^z \cos(0) + i(e^z \sin(0))$$

$$f'(z) = e^z(1) + 0 = e^z$$

Step-(V) Integrate w.r.t z

$$\int f'(z)\,dz = \int e^z dz + C$$

$$\boxed{f(z) = e^z + c}$$

$$f(z) = e^{x+iy} + c = e^x \cdot e^{iy} + c = e^x[\cos y + i\sin y] + c$$

$$= e^x\cos y + i e^x \sin y + i\alpha \qquad \text{let } \boxed{c = i\alpha}$$

$$= (e^x\cos y) + i(e^x \sin y + \alpha)$$

$$= u + iv$$

$$\Rightarrow u = e^x \cos y$$

$$\boxed{v = e^x \sin y + \alpha}$$

→ {Harmonic conjugate of u

→ orthogonal trajectories of u

Ex.2 Construct an Analytic function $f(z)$ If $v = e^x[x\,Sin(y) + yCos(y)] = c$ [BIOM-Nov-18, (6M)]

→ step-① We have $f(z) = u + iv$ → Analytic fun

(ii) $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

(iii) By C.R equ^n $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

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$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

We have $v = e^x[x\sin y + y\cos y] - c = 0$

$$\frac{\partial v}{\partial x} = e^x[x\sin y + y\cos y] + e^x[\sin y + 0] - 0$$

$$= e^x[x\sin y + y\cos y + \sin y]$$

$$\frac{\partial v}{\partial y} = e^x[x\cos y + 1\cos y - y\sin y] - 0$$

$$f'(z) = e^x[x\cos y + \cos y - y\sin y] + i\, e^x[x\sin y + y\cos y + \sin y]$$

step (IV) By MTM, put $x=z$ & $y=0$

$$f'(z) = e^{z}[z(1)+(1)-0] + i\, e^{z}[z(0)+0+0]$$

$$f'(z) = e^{z}(z+1) + 0$$

step (V) Integrate w.r.t z

$$f(z) = \int (z+1)e^{z}\,dz + \alpha$$

$$= \{(z+1)[e^{z}] - (1)[e^{z}]\} + \alpha$$

$$\boxed{f(z) = e^{z}[z+1-1] + \alpha = ze^{z} + \alpha}$$

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Ex.3 Find orthogonal trajectories of family of curves $e^{x}\,[Cos(y)] - xy = c$ [EXTC-Nov-18, (6M)]

→ Assume $u = e^{x}\cos y - xy - c = 0$ → given

step (I) $f(z) = u + iv$ → Analytic

(II) $f'(z) = \frac{\partial u}{\partial x}$

(III) By C.R equ^n $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$; $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial x} = e^{x}\cos y - y - 0$$

$$\frac{\partial u}{\partial y} = -e^{x}\sin y - x - 0$$

$$f'(z) = (e^{x}\cos y - y) - i(-e^{x}\sin y - x)$$

(IV) By MTM, put $x=z$ & $y=0$

$$f'(z) = [e^{z}(1) - 0] + i(e^{z}(0) + z)$$

$$= e^{z} + iz$$

(V) Integrate w.r.t z

$$\int f'(z)\,dz = \int e^{z}\,dz + \int iz\,dz + \alpha$$

$$\boxed{f(z) = e^{z} + i\frac{z^{2}}{2} + \alpha}$$

$$f(z) = e^{x+iy} + \frac{i}{2}(x+iy)^{2} + \alpha$$

$$= e^{x}e^{iy} + \frac{i}{2}(x^{2} + 2ixy - y^{2}) + \alpha$$

$$= e^{x} e^{iy} + \frac{i}{2}(x^2 + 2ixy - y^2) + \alpha$$

$$= e^{x}[\cos y + i\sin y] + \frac{i}{2}(x^2 - y^2) - xy + \alpha$$

$$= e^{x}\cos y + i e^{x}\sin y + \frac{i}{2}(x^2 - y^2) - xy + \alpha$$

$$= [e^{x}\cos y - xy - c] + i\left[\frac{(x^2 - y^2)}{2} + e^{x}\sin y\right]$$

let $\boxed{\alpha = -c}$

$$= u + iv$$

$$u = e^{x}\cos y - xy - c$$

$$v = e^{x}\sin y + \frac{(x^2 - y^2)}{2}$$

$\Rightarrow$ orthogonal trajectories of u

$\Rightarrow$ Harmonic conjugate of u

$\boxed{\text{CASE-3}}$ IF $(u - v) =$ given

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step (1) We have $f(z) = u + iv \rightarrow$ Analytic fun — (1)

** step (2) Multiple 'i' to Both side of step (1)

$$if(z) = iu + i^2 v = iu - v \quad \text{— (2)}$$

*** step (3) Add step (1) & step (2)

$$f(z) + if(z) = (u + iv) + (iu - v)$$

$$(1 + i) f(z) = (u - v) + i(u + v)$$

step (4) $F(z) = U + iV \longrightarrow$ case-1

We have given $(u - v)$ i.e. $U = (u - v) =$ given

step (5) $F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$

step (6) By C-R equ^n $\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$

$$F'(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

step (7) By MTM, put $x = z$ & $y = 0$

$$F'(z) = \left(\frac{\partial U}{\partial x}\right)_{x=z} - i\left(\frac{\partial U}{\partial y}\right)_{x=z}$$

$$F'(z) = \left(\frac{\partial U}{\partial x}\right)_{\substack{x=z\\y=0}} - i\left(\frac{\partial U}{\partial y}\right)_{\substack{x=z\\y=0}}$$

step-(VIII) Integrate w.r.t. z

$$F(z) = \int \left(\frac{\partial U}{\partial x}\right)_{\substack{x=z\\y=0}} dz - i\int \left(\frac{\partial U}{\partial y}\right)_{\substack{x=z\\y=0}} dz + C_1$$

$$(1+i)f(z) = \phi_1(z) - i\,\phi_2(z) + C_1$$

divide by (1+i)

$$\boxed{f(z) = \frac{1}{(1+i)}\phi_1(z) - \frac{i}{(1+i)}\phi_2(z) + \frac{C_1}{(1+i)}}$$

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CASE-(4) IF $(u+v)$ = given

Step-(I) $f(z) = u+iv \to$ Analytic —(I)

(II) $if(z) = iu - v$ —(II)

(III) Add (I) & (II) $(1+i)f(z) = (u-v) + i(u+v)$

step-(IV) $F(z) = U + iV \longrightarrow$ case-(I)

We have given $V = (u+v)$ = given

Step (V) $F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$

Step (VI) By C-R equn $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$

$$F'(z) = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}$$

Step (VII) By MTM, put $x=z$ and $y=0$

$$F'(z) = \left(\frac{\partial V}{\partial y}\right)_{\substack{x=z\\y=0}} + i\left(\frac{\partial V}{\partial x}\right)_{\substack{x=z\\y=0}}$$

step-(VIII) Integrate w.r.t z

$$F(z) = \int\left(\frac{\partial V}{\partial y}\right)_{\substack{x=z\\y=0}} dz + i\int\left(\frac{\partial V}{\partial x}\right)_{\substack{x=z\\y=0}} dz + C_1$$

$$F(z) = \int \left(\frac{\partial}{\partial y}\right)_{\substack{x=z\\y=0}} \ldots \int (\quad)_{y=0}$$

$$(1+i)f(z) = \phi_1(z) + i\,\phi_2(z) + c_1$$

$$\boxed{f(z) = u+iv = \frac{\phi_1(z)}{(1+i)} + \frac{i}{(1+i)}\phi_2(z) + \frac{c_1}{(1+i)}}$$

Ex.4 Find an analytic function $f(z)$ *such that* $u - v = (x-y)(x^2+y^2+4xy)$

$\rightarrow$ step-(I) We have $f(z) = u+iv \rightarrow$ Analytic fun

step (II) $i f(z) = iu - v$

step (III) Add, $(1+i)f(z) = (u-v) + i(u+v)$

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step (IV) $F(z) = U + iV$

We have given $U = u - v = (x-y)(x^2+y^2+4xy)$ — given

step (V) $F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$

step (VI) By C.R. eqn's, $\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$

$$F'(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} \longrightarrow$$

$$U = (u-v) = (x-y)(x^2+y^2+4xy)$$

$$\frac{\partial U}{\partial x} = (1-0)(x^2+y^2+4xy) + (x-y)(2x+0+4y)$$

$$\frac{\partial U}{\partial y} = (0-1)(x^2+y^2+4xy) + (x-y)(0+2y+4x)$$

$$F'(z) = \left[1(x^2+y^2+4xy) + (x-y)(2x+4y)\right] - i\left[-(x^2+y^2+4xy) + (x-y)(2y+4x)\right]$$

step (VII) By M.T.M. put $x = z$ & $y = 0$

$$F'(z) = \left[(z^2+0+0) + (z-0)(2z+0)\right] - i\left[-(z^2+0+0) + (z-0)(0+4z)\right]$$

$$= (z^2 + 2z^2) - i(-z^2 + 4z^2)$$

$$= 3z^2 - i(3z^2)$$

$$F'(z) = (1-i)\,3z^2$$

step (VIII) Integrate w.r.t z

$$\int F'(z)\,dz = (1-i)3\int z^2\,dz + c_1$$

$$F(z) = (1-i)3\left(\frac{z^3}{3}\right) + c_1$$

$$F(z) = (1-i)\cancel{3}\left(\frac{z^3}{\cancel{3}}\right) + c_1$$

** $$(1+i)f(z) = (1-i)z^2 + c_1$$

$$f(z) = \frac{(1-i)}{(1+i)} z^2 + \frac{c_1}{(1+i)} = \frac{(1-i)}{(1+i)} \times \frac{(1-i)}{(1-i)} z^2 + C$$

$$= \frac{(1-i)^2}{1^2 - i^2} z^2 + C = \frac{(1-2i-1)}{2} z^2 + C$$

$$\boxed{f(z) = -iz^2 + C}$$

Ex.5. Find an analytic function $f(z)$ ***such that*** $\boldsymbol{u + v} = Cosx\ Coshy - Sinx\ Sinhy$

→ $f(z) = u + iv$ → Analytic Fun

$if(z) = iu - v$

$(1+i)f(z) = (u-v) + i(u+v)$.

$F(z) = U + iV$

But we have $V = (u+v) = \cos x \cosh y - \sin x \sinh y$

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$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$$

By C.R eq^n $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$

$$F'(z) = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x} \quad \text{--- (1)}$$

$$\frac{\partial V}{\partial x} = -\sin x \cosh y - \cos x \sinh y$$

$$\frac{\partial V}{\partial y} = \cos x \sinh y - \sin x \cosh y$$

By MTM
$$\left(\frac{\partial V}{\partial x}\right)_{\substack{x=z\\y=0}} = -\sin z(1) - \cos z(0) = -\sin z$$

$$\left(\frac{\partial V}{\partial y}\right)_{\substack{x=z\\y=0}} = \cos z(0) - \sin z(1) = -\sin z$$

$$F'(z) = \left(\frac{\partial V}{\partial y}\right)_{\substack{x=z\\y=0}} + i\left(\frac{\partial V}{\partial x}\right)_{\substack{x=z\\y=0}}$$

$$= -\sin z + i(-\sin z)$$

$$F'(z) = -\sin z(1+i)$$

Integrate w.r.t z

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$$F(z) = (1+i)\cos z + c_1$$

$$(1+i)f(z) = (1+i)\cos z + c_1$$

$$f(z) = \cos(z) + \boxed{\frac{c_1}{(1+i)}} = \boxed{\cos z + c}$$