<1> Laplace transform.

> Definition : $L[f(t)] = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

> LT of standard functions

1> $L\{1\} = \frac{1}{s}$

2> $L\{e^{at}\} = \frac{1}{s-a}$

3> $L[t^n] = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$

4> $L\{\sin at\} = \frac{a}{s^2+a^2}$

5> $L\{\cos at\} = \frac{s}{s^2+a^2}$

6> $L\{\sinh at\} = \frac{a}{s^2-a^2}$

7> $L\{\cosh at\} = \frac{s}{s^2-a^2}$

> Types: combined: Defactorization to be used

> Scaling property : $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

> First shifting property.

If $L\{f(t)\} = \bar{f}(s)$

then $L\{e^{at} f(t)\} = \bar{f}(s-a)$

Proof:

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\therefore \bar{f}(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= \int_0^\infty e^{-st+at} f(t)\, dt$$

$$= \int_0^\infty e^{-st} e^{at} f(t)\, dt = \int_0^\infty e^{at} e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} (e^{at} f(t))\, dt$$

$$= L\{e^{at} f(t)\}$$

> If cube is given then expand it.

> exponent $\times$ f^n

$\Rightarrow$ solve $L\{f^n\}$

then solve $L\{\exp. f^n\}$ using first shif.

> Multiplication by t theorem.

In general

$$L\{t^n f(t)\} = (-1)^n \bar{f}_n(s)$$

no. of derivatives to be done

> Second shift theorem.

$$L\{f(t-a)\} =$$

> $$L(\operatorname{erf}\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$$

> Division by t theorem.

$$L\left\{\frac{1}{t} f(t)\right\} = \int_s^{\infty} \bar{f}(s)\,ds.$$

> Laplace of derivative.

$$L\{f^n(t)\} = s^n \bar{F}(s) - s^{n-1} f(0) - s^{n-2} f'(0) \; \cdots\cdots - f^{n-1}(0)$$

Remember

indicates no. of " for this term.

$$L\{f'(t)\} = s\bar{f}(s) - f(0)$$

$$L\{f''(t)\} = s^2 \bar{f}(s) - s f(0) - f'(0)$$

Prolly....

$$L\{f'''(t)\} = s^3 \bar{f}(s) - s^2 f(0) - s f(0) - f''(0)$$

> Laplace transform of integral.

If $L\{f(t)\} = \bar{f}(s)$

then $L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}\bar{f}(s)$

Ex.

~~L.T of $\int_0^t te^{-3t}$~~

L.T of $\int_0^t te^{-3t}\sin^2 t\,dt.$

$\Rightarrow$

$$L\{\sin^2 t\} = L\left\{\frac{1-\cos t/2}{2}\right\}$$

$$= \frac{1}{2}L\{1-\cos t/2\} = \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+\frac{1}{4}}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s} - \frac{4s}{4s^2+1}\right\}$$

$$\therefore L\{e^{-3t}\sin^2 t\} = \frac{1}{2}\left\{\frac{1}{(s+3)} - \frac{4(s+3)}{4(s+3)^2+1}\right\}$$

$$\therefore L\{te^{-3t}\sin^2 t\} = \frac{(-1)}{2}\frac{d}{ds}\left[\frac{1}{s+3} - \frac{4s+12}{4(s+3)^2+1}\right]$$

$$= \frac{-1}{2}\left[\cancel{-(s+3)}\ \frac{-1}{(s+3)^2}\right]$$

Ex. L.T of $\int_0^t t e^{-3t} \sin^2 t \, dt$.

$\cos 2t = 1 - 2\sin^2 t$

$\Rightarrow \sin^2 t = (1 - \cos 2t)/2$

Now,

$$L\{\sin^2 t\} = \frac{L}{2}\{1 - \cos 2t\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Now

$$L\{e^{-3t} \sin^2 t\} = \frac{1}{2}\left[\frac{1}{(s+3)} - \frac{s+3}{(s+3)^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+3} - \frac{s+3}{s^2+6s+9+4}\right] = \frac{1}{2}\left[\frac{1}{s+3} - \frac{s+3}{s^2+6s+13}\right]$$

$$\therefore L\{t e^{-3t} \sin^2 t\} = (-1)^1 \frac{d}{ds} \frac{1}{2}\left[\frac{1}{s+3} - \frac{s+3}{s^2+6s+13}\right]$$

$$= \frac{-1}{2}\left[\frac{d}{ds} \cdot \frac{1}{s+3} - \frac{d}{ds} \frac{s+3}{s^2+6s+13}\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{(s+3)^2} - \frac{(s^2+6s+13)(1) - (s+3)(2s+6+0)}{(s^2+6s+13)^2}\right]$$

$$= \frac{1}{2(s+3)^2} + \frac{\cancel{s^2} + \cancel{6s} + 13 - 2\cancel{s^2} - \cancel{6s} - 6s - 18}{2(s^2+6s+13)^2}$$

$$\therefore L\left\{\int_0^t e^{-3t} t \sin^2 t \, dt\right\} = \frac{1}{2s}\left[\frac{1}{(s+3)^2} + \frac{(-s^2 - 6s - 5)}{(s^2+6s+13)^2}\right]$$

> Evaluation of integrals using L.T.

By definition $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$

> For a periodic function $f(t)$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^{T} e^{-st} f(t)\,dt.$$

wher T = period.

$$L\{H(t-a)\} = \int_0^\infty e^{-st} H(t-a)\,dt$$

$$= \int_0^a e^{-st} H(t-a)\,dt + \int_a^\infty e^{-st} H(t-a)\,dt$$

$$= \int_a^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{1}{s} e^{-as}$$

Laplace transform	Inverse laplace trans.
> $L\{1\} = \frac{1}{s}$	$L^{-1}\left\{\frac{1}{s}\right\} = 1$
> $L\{e^{at}\} = \frac{1}{s-a}$	$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
> $L\{\cos at\} = \frac{s}{s^2+a^2}$	$\therefore L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at.$
> $L\{\cosh at\} = \frac{s}{s^2-a^2}$	$\therefore L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$
> $L\{\sin at\} = \frac{a}{s^2+a^2}$	$\therefore L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$
> $L\{\sinh at\} = \frac{a}{s^2-a^2}$	$L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a}$
> $L\{t^{n-1}\} = \frac{\Gamma n}{s^n}$	$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{\Gamma n}$

> Convolution theorem:

$$L^{-1}\{\phi_1(s) * \phi_2(s)\} = \int_0^t f_1(u) f_2(t-u)\, du.$$

$f_1(t) = L^{-1}\{\phi_1(s)\}$ $\quad f_2(t) = L^{-1}\{\phi_2(s)\}$

classmate

Date

Page

Ex $\quad L^{-1}\left\{\frac{1}{s(s+a)}\right\}$

Let $\phi_1(s) = \frac{1}{s+a}$

$f_1(t) = L^{-1}\{\phi_1(s)\} = L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

$\therefore f_1(u) = e^{-au}$

Let $\phi_2(s) = \frac{1}{s}$

$f_2(t) = L^{-1}\{\phi_2(s)\} = L^{-1}\left\{\frac{1}{s}\right\} = 1$

$f_2(u) = 1$

$\therefore f_2(t-u) = 1$

$\therefore L^{-1}\{\phi_1(s) * \phi_2(s)\} = \int_0^t e^{-au} . 1 du$

$= \left[\frac{e^{-au}}{-a}\right]_0^t = \frac{1-e^{-at}}{a}$

> $L^{-1} \phi(s) = -\frac{1}{t} L^{-1}\{\phi'(s)\}$

Date
Page

fourier series.

> Parseval's Identity:

$$\frac{1}{l}\int_{-l}^{l}[f(x)]^2\,dx = 2\left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

> useful substitutions.
put $x=\pi$, $x=0$, add previous values, $\frac{\pi}{2}$,

> use parseval's identity when the series contains power of π more than or equal to 4

complex variables.

> If $f(z)$ is analytic ~~$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}$~~

~~$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial}$~~

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

> cartesian form: $f(z) = u(x,y) + iv(x,y)$
Polar form: $f(z) = u(r,\theta) + iv(r,\theta)$

Cartesian	Polar.
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$	$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$
$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$	$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$

> Harmonic function.

~~Polar~~ Cartesian:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{----} \quad \phi \text{ is harmonic } f^n$$

Polar:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Correlation and Regression.

> A correlation is a measure of association or relation.

> Covariance is a measure of joint variation between the two variables.

$$\text{cov}(x, y) = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

$$\bar{x} = \frac{\Sigma x_i}{n} \quad \& \quad \bar{y} = \frac{\Sigma y_i}{n}$$

$$r = \frac{\text{cov}(x, y)}{\sigma x \cdot \sigma y} \qquad r = \frac{\Sigma xy - N\bar{x}\bar{y}}{\sqrt{\Sigma x^2 - N\bar{x}^2}\sqrt{\Sigma y^2 - N\bar{y}^2}}$$

> $-1 \leq r \leq 1$

> Spearman's Rank correlation

$$R = 1 - \frac{6\,\Sigma d_i^2}{N^3 - N}.$$

for non-repeated ranks.

Date
Page

$$R = 1 - \frac{6\left[\Sigma d_i^2 + \frac{1}{12}[m_1^3 - m_1] + \frac{1}{12}[m_2^3 - m_2] + \frac{1}{12}[m_3^3 - m_3] \ldots\right]}{N^3 - N}$$

> line of regression of y on x is given by

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

i.e., $y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$, $b_{yx} = r\frac{\sigma_y}{\sigma_x}$

b_{yx} = Regression coefficient.
r = coefficient of correlation.

similarly of x i.e., line of regression of x on y

> $b_{xy} \cdot b_{yx} = r^2$

$$\Rightarrow r = \sqrt{b_{xy} \cdot b_{yx}}$$

> $\tan\theta = \frac{1 - r^2}{r}\left(\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\right)$

> pt. of intersection of regression line is $(\bar{x}, \bar{y})$

>

y on x   x on y

$\Sigma y = na + b \Sigma x$ $\Sigma x = na + b\Sigma y$

$\Sigma xy = a\Sigma x + b\Sigma x^2$ $\Sigma xy = a\Sigma y + b\Sigma y^2$

> general line of regression $y = a + bx$

> ~~plot~~

Probability.

> $P(A) = \dfrac{N_A}{N}$

> If odds in favour of A are $a:b$
then

$$P(A) = \frac{a}{a+b}.$$

If odds against A are $a:b$
then

$$P(A) = \frac{b}{a+b}$$

> Multiplication theorem: $P(A \cap B) = P(A)\, P\left(\frac{B}{A}\right)$

$= P(B)\, P\left(\frac{A}{B}\right)$

$P(B/A)$ is called as conditional probability of event B given that A has already happened

> If $P(B/A) = P(B)$

& $P(A/B) = P(A)$

then A & B are independent events.

> Average $= E[x] = \sum_{i=1}^{n} x_i P_i$

$E[x^2] = \sum_{i=1}^{n} x_i^2 P_i$

$V[x] = E[x^2] - (E[x])^2$

$s[x] = \sqrt{v[x]}$

Important formulae:

1) $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

2) $\cancel{2\sin a\cos b = \sin(a+b) + \sin(a-b)}$

$\cancel{2\cos a\sin b = \sin(a+b) - \sin(a+b)}$

$2\sin a\cos b = \sin(a+b) + \sin(a-b)$

$2\cos a\sin b = \sin(a+b) - \sin(a-b)$

$2\cos a\cos b = \cos(a+b) + \cos(a-b)$

$2\sin a\sin b = \cos(a-b) - \cos(a+b)$

3) $\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\sin a - \sin b = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$

$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\cos a - \cos b = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$

4) sine series: $\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \dots$

5) $\Gamma\frac{7}{2} = \cancel{\frac{7}{2}\cdot\frac{5}{2}}$

$= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

Main ~~pa~~class.

Main const.

main meth.

6) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$

$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \frac{x^n}{n!}$

7) $\sinh t = \left(\frac{e^t - e^{-t}}{2}\right)^5$

8)

0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1

9) $\sin^3 t = \frac{3\sin t - \sin 3t}{4}$

$\cos^3 t = (\cos t3 + 3\cos t)/4$

$\tan(3x) = (3\tan x - \tan x3) / 1 - 3\tan^2 x$

classmate

Date

Page

10) $\tan^{-1}\alpha - \tan^{-1}\alpha = \tan^{-1}\left(\frac{\alpha-\beta}{1+\alpha\beta}\right)$

11) $\Gamma n = (n-1)!$

$\Gamma n = (n-1)\Gamma n-1$

12) $\frac{p(x)+q}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$

$\frac{p(x)+q}{(x-a)^2} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2}$

$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$

$\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$

$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{(x-a)} + \frac{Bx+C}{x^2+bx+c}$.

$\frac{p(x)+r}{(x^2+bx+c)(x^2+dx+e)} = \frac{Ax+B}{(x^2+bx+c)} + \frac{Cx+D}{(x^2+dx+e)}$

13) $\tanh^{-1}\theta = \frac{1}{2}\log\left(\frac{1+\theta}{1-\theta}\right)$

14) $\int_{-a}^{a} f(x)dx = \begin{cases} = 2\int_0^a f(x)dx & \text{if } f(x) = \cancel{\text{odd}}\ \text{even } f^n \\ = 0 & \text{if } f(x) = \text{odd } f^n \end{cases}$

15) $\int e^{ax}\sin bx\ dx = \dfrac{e^{ax}}{a^2+b^2}\left(\overset{a\sin bx\ -\ b\cos bx}{\cancel{a\cos bx + b\sin bx}}\right)$

16) $\int e^{ax}\cos bx\ dx = \dfrac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$

17) $e^{x+iy} = e^{x}\cos y + ie^{x}\sin y$