⑥ Find Singular value decomposition of a matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$

Solution By given $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$A \cdot A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

~~To find Eigen~~

∵ A be a square matrix of order 3 ∴ it's characteristic equation is

$\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$ — ① where $s_1 = 5$, $s_2 = 1+4+1 = 6$, $|A| = 2\times1 - 1(2) = 2-2 = 0$

$\therefore \lambda^3 - 5\lambda^2 + 6\lambda - 0 = 0 \Rightarrow \lambda(\lambda^2 - 5\lambda + 6) = 0 \Rightarrow \lambda(\lambda-2)(\lambda-3) = 0$

$\therefore \lambda = \lambda_1 = 3$, $\lambda = \lambda_2 = 2$, and $\lambda = \lambda_3 = 0$ — ② be the Eigen values of a matrix

$\therefore \sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$, $\sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$, $\sigma_3 = \sqrt{\lambda_3} = \sqrt{0} = 0$ ③

To Find Eigen vector consider $(AA^T - \lambda I)X = 0$

$$\therefore \begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 1-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad — ④$$

Case-1: If $\lambda = \lambda_1 = 3$, $\begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\frac{x_1}{-1} = \frac{-x_2}{1} = \frac{x_3}{+1} = k = 1 \Rightarrow x_1 = 1, x_2 = 1, x_3 = -1$

$\therefore$ For $\lambda = \lambda_1 = 3$, $X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\|X_1\| = \sqrt{1+1+1} = \sqrt{3}$, $X_1' = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$

Case-2

If $\lambda = \lambda_2 = 2$, $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\frac{x_1}{-1} = \frac{-x_2}{0} = \frac{x_3}{-1} = k = 1 \Rightarrow x_1 = 1, x_2 = 0, x_3 = 1$,

$\therefore$ For $\lambda = \lambda_2 = 2$, $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\|X_2\| = \sqrt{1+0+1} = \sqrt{2}$, $\therefore X_2' = \frac{X_2}{\|X_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

Case-3

If $\lambda = \lambda_3 = 0$, $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\frac{x_1}{2} = \frac{-x_2}{4} = \frac{x_3}{-2} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{-1} = k = 1$. $x_1 = 1, x_2 = -2, x_3 = -1$

$\therefore$ For $\lambda = \lambda_3 = 0$, $X_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, $\|X_1\| = \sqrt{1+4+1} = \sqrt{6}$, $X_1' = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$

$$\therefore U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \quad — ⑤$$

Now $A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

∵ A^TA is diagonal matrix ∴ $\lambda = \lambda_1 = 3$ & $\lambda = \lambda_2 = 2$ be the Eigen value of matrix A^TA

$\therefore \sigma_1 = \sqrt{\lambda_1} = 3 \quad \sigma_2 = \sqrt{\lambda_2} = 2$

To find Eigen vector consider $(A^TA - \lambda I)X = 0$

$$\therefore \begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad - (6)$$

Case-1 If $\lambda = \lambda_1 = 3$, $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore -1x_1 + 0x_2 = 0 \Rightarrow x_1 = 0, \; x_2 = 1$ say

$\therefore$ For $\lambda = \lambda_1 = 3$ $X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\|X_1\| = \sqrt{1}$, $X_1' = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

If $\lambda = \lambda_2 = 2$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$0x_1 + 1x_2 = 0 \Rightarrow x_2 = 0, \; x_1 = 1$ say

$\therefore$ For $\lambda = \lambda_2 = 2$ $X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\|X_2\| = \sqrt{1}$, $X_2' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\therefore V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus $U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$, $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\because$ A is a matrix of order 3×2

$$\therefore D = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}_{3 \times 2}$$

$$\therefore A = UDV^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$