① Find Eigen value and Eigen vector of a Matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution: $\because$ A be a square matrix of order 3

$\therefore$ it's characteristic equation is

$$\therefore \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad -①$$

where $S_1 = 12$

$$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = 8 + 14 + 14 = 36, \quad |A| = 32$$

$$\therefore \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$\therefore \lambda = \lambda_1 = 8,\ \lambda = \lambda_2 = 2,\ \lambda = \lambda_2 = 2$ be the Eigen value of a matrix A

To Find Eigen vector consider $(A - \lambda I)X = 0$

$$\therefore \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -②$$

Case-1: If $\lambda = \lambda_1 = 8$
$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$$

$$\frac{x_1}{+12} = \frac{-x_2}{6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} = k = 1$$

$\therefore x_1 = 2,\ x_2 = -1,\ x_3 = 1$

Thus For Eigen value $\lambda = \lambda_1 = 8$, Eigen vector $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Case-2 If $\lambda = \lambda_2 = 2$
$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - \tfrac{1}{2}R_1,\quad R_2 \to R_2 + \tfrac{1}{2}R_1,\quad R_1 \to R_1 \times \tfrac{1}{2}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -③$$

$\therefore \rho(A - \lambda I) = 1,\ O(A - \lambda I) = 3$

$\therefore$ For $\lambda = \lambda_2 = 2$, G.M. $= O(A - \lambda I) - \rho(A - \lambda I) = 3 - 1 = 2$

$\therefore$ For $\lambda = \lambda_2 = 2$ two Eigen vector exist

Now From eqⁿ ③ $\quad 2x_1 - 1x_2 + 1x_3 = 0 \quad -④$

$\because$ G.M. = 2, we can select any value for two unknowns out of three unknown

$\therefore$ let $x_2 = 1,\ x_3 = 1$ $\therefore$ From ④ $x_1 = 0$

Thus For $\lambda = \lambda_2 = 2$, Eigen vector $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\because$ A be a symmetric matrix therefore for Eigen value $\lambda = \lambda_3 = 2$

we find Eigen vector $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$ such that $X_3^T X_1 = 0$ & $X_3^T X_2 = 0$

$$\begin{bmatrix} l & m & n \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 0 \ \& \ \begin{bmatrix} l & m & n \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$[2l - 1m + 1n] = [0] \ \& \ [0l + 1m + 1n] = [0]$$

$$\therefore \quad 2l - 1m + 1n = 0$$
$$0l + 1m + 1n = 0$$

$$\frac{l}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{-m}{\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}}$$

$$\therefore \frac{l}{-2} = \frac{-m}{2} = \frac{n}{2}$$

$$\therefore \frac{l}{-1} = \frac{m}{-1} = \frac{n}{1} = k = -1$$

$$\therefore l = +1, \ m = +1, \ n = -1$$

Thus For $\lambda = \lambda_3 = 2$, $X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Thus $\lambda = \lambda_1 = 8 \qquad \lambda = \lambda_2 = 2, \qquad \lambda = \lambda_3 = 2$

$$X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

② Find Eigen vector and Eigen value of a matrix $A = \begin{bmatrix} 2 & -4 & 2 \\ -4 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}$

Solution: $\because$ A be a square matrix of order 3

$\therefore$ it's characteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad ①$$

where $S_1 = 3$,

$S_2 = \begin{vmatrix} 2 & -2 \\ -2 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -4 \\ -4 & 2 \end{vmatrix} = -6 + (-6) + (-12) = -24$, $|A| = 28$

$\therefore \lambda^3 - 3\lambda^2 - 24\lambda - 28 = 0$

$\therefore \lambda = \lambda_1 = 7$, $\lambda = \lambda_2 = -2$, $\lambda = \lambda_3 = -2$ be the Eigen value of a matrix A

To find Eigen vector consider $(A - \lambda I)X = 0$

$$\therefore \begin{bmatrix} 2-\lambda & -4 & 2 \\ -4 & 2-\lambda & -2 \\ 2 & -2 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad - ②$$

<u>Case-1</u> If $\lambda = \lambda_1 = 7$

$$\begin{bmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -5 & -2 \\ -2 & -8 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -4 & -2 \\ 2 & -8 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -4 & -5 \\ 2 & -2 \end{vmatrix}}$$

$$\frac{x_1}{36} = \frac{-x_2}{36} = \frac{x_3}{18}$$

$$\therefore \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1} = k = 1$$

$\therefore x_1 = 2$, $x_2 = -2$, $x_3 = 1$

Thus For Eigen value $\lambda = \lambda_1 = 7$ Eigen vector $X_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

<u>Case-2</u> If $\lambda = \lambda_2 = -2$

$$\begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1 - 2R_3$, $R_2 \to R_2 + 2R_3$ & $R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad - ③$$

Now $\rho(A - \lambda I) = 1$, $O(A - \lambda I) = 3$

Now For $\lambda = \lambda_2 = -2$, G.M. $= O(A-\lambda I) - \rho(A - \lambda I) = 3 - 1 = 2$

For $\lambda = \lambda_2 = -2$ two Eigen vectors exist

Now from ③ $2x_1 - 2x_2 + 1x_3 = 0$

$\because$ For $\lambda = \lambda_2 = -2$, we can select any value for any two unknown out of any three unknown $\therefore$ let $x_1 = 1$, $x_2 = 1$ $\therefore x_3 = 0$

$\therefore$ For $\lambda = \lambda_2 = -2$, Eigen vector $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\because$ A be symmetric matrix $\therefore$ For Eigen value $\lambda = \lambda_3 = -2$

consider Eigen vector $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$, such that $X_3^T X_1 = 0$ & $X_3^T X_2 = 0$

$$[l \ m \ n]\begin{bmatrix}2\\-2\\1\end{bmatrix} = 0 \ \& \ [l \ m \ n]\begin{bmatrix}1\\1\\0\end{bmatrix} = 0$$

$$[2l - 2m + 1n] = [0] \ \& \ [1l + 1m + 0n] = [0]$$

$\therefore$ $2l - 2m + 1n = 0$ &

$1l + 1m + 0n = 0$

$$\frac{l}{\begin{vmatrix}-2 & 1\\1 & 0\end{vmatrix}} = \frac{-m}{\begin{vmatrix}2 & 1\\1 & 0\end{vmatrix}} = \frac{n}{\begin{vmatrix}2 & -2\\1 & 1\end{vmatrix}}$$

$$\frac{l}{-1} = \frac{-m}{-1} = \frac{n}{4} = k = 1$$

$\therefore l = -1, \ m = 1, \ n = 4$

Thus For Eigen value $\lambda = \lambda_3 = -2$, $X_3 = \begin{bmatrix}-1\\1\\4\end{bmatrix}$

Thus $\lambda = \lambda_1 = 7$ $\quad \lambda = \lambda_2 = -2$ $\quad \lambda = \lambda_3 = -2$

$$X_1 = \begin{bmatrix}2\\-2\\1\end{bmatrix} \quad X_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad X_3 = \begin{bmatrix}-1\\1\\4\end{bmatrix}$$

③ Find Eigen value and Eigen vector of a matrix $A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$

Solution: $\because$ A be a square matrix of order 3

$\therefore$ it's characteristics equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad -①$$

$S_1 = -9$

$S_2 = \begin{vmatrix} -8 & -1 \\ -1 & -8 \end{vmatrix} + \begin{vmatrix} 7 & -4 \\ -4 & -8 \end{vmatrix} + \begin{vmatrix} 7 & 4 \\ 4 & -8 \end{vmatrix} = 63 - 72 - 72 = -81, \quad |A| = 729$

$\therefore \lambda^3 + 9\lambda^2 - 81\lambda - 729 = 0$

$\lambda = \lambda_1 = 9, \lambda = \lambda_2 = -9, \lambda = \lambda_3 = -9$ be the Eigen value of a matrix A

To find Eigen vector, consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad - ②$$

Case-1 If $\lambda = \lambda_1 = 9$ $\begin{bmatrix} -2 & 4 & -4 \\ 4 & -17 & -1 \\ -4 & -1 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\frac{x_1}{\begin{vmatrix} 4 & -4 \\ -17 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & -4 \\ 4 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 4 \\ 4 & -17 \end{vmatrix}}$$

$$\frac{x_1}{-72} = \frac{-x_2}{18} = \frac{x_3}{18}$$

$$\therefore \frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{1} = k = -1$$

$\therefore x_1 = 4, \ x_2 = 1, \ x_3 = -1$

Thus For $\lambda = \lambda_1 = 9$, Eigen vector $X_1 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$

Case-2: If $\lambda = \lambda_2 = -9$ $\begin{bmatrix} 16 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_1 \to R_1 + 4R_3, \ R_2 \to R_2 + R_3, \ R_1 \leftrightarrow R_3$

$$\begin{bmatrix} -4 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -③$$

$\therefore O(A - \lambda I) = 3, \ \rho(A - \lambda I) = 1$

Thus For $\lambda = \lambda_2 = -9$, G.M. $= O(A-\lambda I) - \rho(A - \lambda I) = 3 - 1 = 2$

$\therefore$ For $\lambda = \lambda_2 = -9$ two Eigen vector exist

Now From equation ③ $-4x_1 - 1x_2 + 1x_3 = 0$

$\because$ For $\lambda = \lambda_2 = -9$, G.M $= 2$ $\therefore$ we select any value for any two unknow out of any three unknown $\therefore$ let $x_2 = 1, \ x_3 = 1 \ \therefore x_1 = 0$

$\therefore$ For Eigen value $\lambda = \lambda_2 = -9$ Eigen vector $X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\because$ A be a symmetric matrix

$\therefore$ For Eigen value $\lambda = \lambda_3 = -9$ consider Eigen vector $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

Such that $X_3^T X_1 = 0$ & $X_3^T X_2 = 0$

$$\begin{bmatrix} l & m & n \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} = 0 \ \& \ \begin{bmatrix} l & m & n \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$[4l + 1m - 1n] = [0] \ \& \ [0l + 1m + 1n] = [0]$$

$$4l + 1m - 1n = 0$$

$$\& \ 0l + 1m + 1n = 0$$

$$\frac{l}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{-m}{\begin{vmatrix} 4 & -1 \\ 0 & 1 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{l}{2} = \frac{-m}{4} = \frac{n}{4}$$

$$\frac{l}{1} = \frac{m}{-2} = \frac{n}{2} = k = 1$$

$$l = 1, \ m = -2, \ n = 2$$

Thus For $\lambda = \lambda_3 = -9$ $X_3 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

Thus $\lambda = \lambda_1 = 9$ $\quad \lambda = \lambda_2 = -9$ $\quad \lambda = \lambda_3 = -9$

$$X_1 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad X_3 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$