Recitation Guide - Week 1

Topics Covered: Proofs.

Problem 1: Let m and n be two integers. Prove that $mn+m$ is odd if and only if m is odd and n is even.

Solution:

Lemma 1: For two integers x and y, if xy is odd, then x and y are both odd.

We will prove the Lemma through a proof by contrapositive. In other words, we will prove "If x or y is even, then xy is even."

WLOG, let x be the even integer.

We can write $x = 2k$, for some $k \in \mathbb{Z}$. Then, we have,

$$\begin{aligned} xy &= (2k)(y) \\ &= 2(ky) \end{aligned}$$

which is even, as $ky \in \mathbb{Z}$.

($\Longrightarrow$) If $mn+m$ is odd, then m is odd and n is even.

We can write $mn+m = m(n+1)$. Then, according to Lemma 1, m and $n+1$ must both be odd.

Since we know that $n+1$ is odd and 1 is odd, then n must be an even integer (because only even + odd = odd).

Therefore, m is odd and n is even.

($\Longleftarrow$) If m is odd and n is even, then $mn+m$ is odd.

We can write $m = 2k+1$, for some $k \in \mathbb{Z}$ and $n = 2l$, for some $l \in \mathbb{Z}$. Then, we have,

$$\begin{aligned} mn+m &= (2k+1)(2l) + (2k+1) \\ &= 4kl + 2l + 2k + 1 \\ &= 2(2kl + l + k) + 1 \end{aligned}$$

which is odd, as $2kl + l + k \in \mathbb{Z}$.

Problem 2: Suppose $x, y \in \mathbb{R}$. Prove that if $y^3 + yx^2 \leq x^3 + xy^2$, then y $\leq$ x.

Solution:

We will prove the claim by proving its contrapositive. Recall from the lecture that

$$p \implies q \equiv \overline{q} \implies \overline{p}$$

Thus we will prove that if $y > x$ then $y^3 + yx^2 > x^3 + xy^2$.
Since $y > x$, we have

$$\begin{aligned} y - x &> 0 \\ (y-x)(x^2+y^2) &> 0 \qquad \text{(multiplying both sides by the positive value } x^2+y^2\text{)} \\ yx^2 + y^3 - x^3 - xy^2 &> 0 \\ y^3 + yx^2 &> x^3 + xy^2 \end{aligned}$$

Problem 3:

Prove that the product of a non-zero rational and irrational number is irrational.

Solution:

Let us first rewrite the claim as: if a is a non-zero rational number and b is an irrational number, then their product ab is irrational.

We prove the claim using contradiction. In order to do a proof by contradiction, we need to first assume the negation of our statement and then arrive at a false, or contradictory, statement. **Assume for contradiction that a is a non-zero rational number and b is an irrational number, and their product ab is rational.**

Rational numbers can be written as a fraction of two integers where the denominator is non-zero. Since a is a non-zero rational number, we can rewrite it as $a = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $p, q \neq 0$.

Let $c = ab$. Since c is a rational number, we can rewrite it as $c = \frac{y}{z}$ where $y, z \in \mathbb{Z}$ and $z \neq 0$. Plugging these in, we get that,

$$c = ab \longrightarrow \frac{y}{z} = \frac{pb}{q}$$

Rearranging the equation and solving for b gives us,

$$b = \frac{yq}{zp}$$

Since yq and zp are integers and $zp \neq 0$, then by definition, b is rational. This is a contradiction (since we assumed that b is an irrational number), and this proves our initial statement.