

Matrix: A matrix is a system of mn numbers arranged in m rows and n columns it is called an $m \times n$ matrix.

$$\text{e.g. } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \vdots & a_{ij} & \vdots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Types of matrices

1) Matrix which contains only one row is called a row matrix.

$$\text{e.g. } A = [2 \quad 3 \quad 21 \quad 63 \quad 21]$$

2) Matrix which contains only one column is called a column matrix.

$$\text{e.g. } A = \begin{bmatrix} 5 \\ 23 \\ 15 \\ 8 \\ 10 \end{bmatrix}$$

3) Matrix in which the number of rows and columns are the same is called a square matrix.

$$\text{e.g. } A = \begin{bmatrix} 4 & 2 & 6 & 8 \\ 8 & 5 & 5 & 9 \\ 4 & 2 & 7 & 8 \\ 5 & 3 & 9 & 8 \end{bmatrix}$$

4) Diagonal elements: In a square matrix the elements lying along the diagonal of a matrix are called diagonal elements.

$$\text{e.g. } A = \begin{bmatrix} 4 & 2 & 6 & 8 \\ 8 & 5 & 5 & 9 \\ 4 & 2 & 7 & 8 \\ 5 & 3 & 9 & 8 \end{bmatrix} \quad \text{Here } 4, 5, 7, 8 \text{ are the diagonal elements}$$

5) Diagonal Matrix: a square matrix in which all the non-diagonal elements are equal to zero is called a diagonal matrix.

$$\text{e.g. } A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \quad \text{Here } 4, 5, 7, 8 \text{ are the diagonal elements}$$

6) Scalar matrix: a diagonal matrix whose all diagonal elements are equal is called a scalar matrix. e.g.

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

7) Unit matrix: a diagonal matrix whose all diagonal elements are equal to one is called a unit matrix.

e.g. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

8) Trace of a matrix: The sum of all the diagonal elements of a matrix is called trace of a matrix

If $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ then Trace of matrix $A = \sum_{i=1}^n a_{ii} = a_{11} + a_{22} + a_{33} + \dots + a_{nn}$

In given matrix Trace of $A = 4 + 5 + 7 + 8 = 24$

9) Determinant of a square matrix A is denoted by $|A|$

10) Singular matrix: A square matrix A is said to be singular matrix if $|A| = 0$

e.g. , $C = [0]$, $|C| = 0$, $A = \begin{bmatrix} 3 & 2 \\ 18 & 12 \end{bmatrix}$ then $|A| = 0$, $B = \begin{bmatrix} 1 & 3 & 5 \\ 8 & 4 & 3 \\ 2 & 6 & 10 \end{bmatrix}$ then $|B| = 0$

So matrices C, A, and B are singular matrices.

11) Non-singular matrix: A square matrix A is said to be non-singular matrix if $|A| \neq 0$

e.g. $A = [5]$, $|A| = 5 \neq 0$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $|B| = -1 \neq 0$, $C = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, $|C| = 27 \neq 0$

So matrices C, A, and B are singular matrices.

12) Upper triangular matrix: A matrix $A = [a_{ij}]$ is upper triangular matrix if $a_{ij} = 0$ for $i > j$

e.g. . $A = \begin{bmatrix} 4 & 2 & 6 & 8 \\ 0 & 5 & 5 & 9 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ i.e. all the elements below the diagonal elements are zero.

13) Lower triangular matrix: A matrix $A = [a_{ij}]$ is lower triangular matrix if $a_{ij} = 0$ for $i < j$

. $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 8 & 5 & 0 & 0 \\ 4 & 2 & 7 & 0 \\ 5 & 3 & 9 & 8 \end{bmatrix}$ i.e. all the elements above the diagonal elements are zero.

14) Transpose of a matrix: Matrix obtained by interchange of rows and column of a matrix A is called Transpose of a matrix and is denoted by A^T

If . $A = \begin{bmatrix} 4 & 2 & 6 & 8 \\ 8 & 5 & 5 & 9 \\ 4 & 2 & 7 & 8 \\ 5 & 3 & 9 & 8 \end{bmatrix}$ then $A^T = \begin{bmatrix} 4 & 8 & 4 & 5 \\ 2 & 5 & 2 & 3 \\ 6 & 5 & 7 & 9 \\ 8 & 9 & 8 & 8 \end{bmatrix}$

15) Conjugate of a matrix: The matrix obtained from a given matrix by replacing each element by its complex conjugate is called the conjugate of the given matrix and is denoted by \bar{A}

$$\text{Thus if } A = \begin{bmatrix} -2+3i & 3-2i & -6i \\ 4i & 1-2i & -5+2i \\ 3 & -4-6i & 8+2i \end{bmatrix} \text{ then } \bar{A} = \begin{bmatrix} -2-3i & 3+2i & 6i \\ -4i & 1+2i & -5-2i \\ 3 & -4+6i & 8-2i \end{bmatrix}$$

16) Transposed conjugate of a matrix: The transpose of a complex conjugate of a given is called the transposed conjugate of a matrix A and is denoted by A^θ

$$\text{i.e. } A^\theta = \bar{A}' = \bar{A}'$$

$$\text{e.g. if } A = \begin{bmatrix} -2+3i & 3-2i & -6i \\ 4i & 1-2i & -5+2i \\ 3 & -4-6i & 8+2i \end{bmatrix} \text{ then } \bar{A} = \begin{bmatrix} -2-3i & 3+2i & 6i \\ -4i & 1+2i & -5-2i \\ 3 & -4+6i & 8-2i \end{bmatrix}$$

$$\text{and } A^\theta = \bar{A}' = \begin{bmatrix} -2-3i & -4i & 3 \\ 3+2i & 1+2i & -4+6i \\ 6i & -5-2i & 8-2i \end{bmatrix}$$

17) Symmetric matrix: A matrix $A = [a_{ij}]$ is said to be symmetric matrix if $a_{ij} = a_{ji} \forall i, j$

$$\text{Or } A^T = A$$

$$A = \begin{bmatrix} 4 & -2 & 6 & 8 \\ -2 & 5 & -1 & -3 \\ 6 & -1 & 7 & 9 \\ 8 & -3 & 9 & 8 \end{bmatrix}, A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 5 & -7 \\ 3 & -7 & 8 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

18) Skew-symmetric matrix: A matrix $A = [a_{ij}]$ is said to be skew-symmetric matrix if $a_{ij} = -a_{ji} \forall i, j$

$$\text{Or } A^T = -A$$

$$A = \begin{bmatrix} 0 & 2 & -6 & -8 \\ -2 & 0 & 1 & 3 \\ 6 & -1 & 0 & -9 \\ 8 & -3 & 9 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 7 \\ 3 & -7 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

Note: In skew-symmetric matrix all the diagonal elements equal to zero

19) Hermitian matrix: Matrix $A = [a_{ij}]$ is said to be Hermitian matrix if $\bar{a}_{ij} = a_{ji} \forall i, j$

$$\text{Or } A^\theta = A$$

$$A = \begin{bmatrix} 1 & 2-3i & 5+6i \\ 2-3i & 3 & 8-2i \\ 5-6i & 8+2i & 6 \end{bmatrix}, A = \begin{bmatrix} 2 & 3-5i \\ 3+5i & 7 \end{bmatrix}$$

Note: In Hermitian matrix all the diagonal elements are purely real.

20) Skew-Hermitian matrix: Matrix $A = [a_{ij}]$ is said to be Skew-Hermitian matrix if

$$\bar{a}_{ij} = -a_{ji} \forall i, j \text{ Or } A^\theta = -A$$

$$A = \begin{bmatrix} 0 & 2-3i & 5+6i \\ -2-3i & 3i & 8-2i \\ -5+6i & -8-2i & -6i \end{bmatrix}, A = \begin{bmatrix} 2i & 3-5i \\ -3-5i & 0 \end{bmatrix}$$

Note: In Skew-Hermitian matrix all the diagonal elements are either zero or purely imaginary

Note:

1) If a Matrix A is Hermitian then iA is skew – Hermitian Matrix

2) If a matrix A is Skew – Hermitian then iA is Hermitian Matrix

Theorem: Every matrix A can be expressed as sum of symmetric and skew-symmetric matrices. i.e. $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

Where $\frac{1}{2}(A + A^T)$ is symmetric matrix, and $\frac{1}{2}(A - A^T)$ is Skew – symmetric matrix

e.g. Let $A = \begin{bmatrix} 6 & 10 & 16 \\ 20 & 26 & 30 \\ 40 & 50 & 60 \end{bmatrix} \therefore A^T = \begin{bmatrix} 6 & 20 & 40 \\ 10 & 26 & 50 \\ 16 & 30 & 60 \end{bmatrix}$

Now $A + A^T = \begin{bmatrix} 12 & 30 & 56 \\ 30 & 52 & 80 \\ 56 & 80 & 120 \end{bmatrix}$, and $A - A^T = \begin{bmatrix} 0 & -10 & -24 \\ 10 & 0 & -20 \\ 24 & 20 & 0 \end{bmatrix}$

$\therefore \frac{1}{2}(A + A^T) = \begin{bmatrix} 6 & 15 & 28 \\ 15 & 26 & 40 \\ 28 & 40 & 60 \end{bmatrix}$ is symmetric, and

$\frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & -5 & -12 \\ 5 & 0 & -10 \\ 12 & 10 & 0 \end{bmatrix}$ is skew – symmetric

And $\begin{bmatrix} 6 & 10 & 16 \\ 20 & 26 & 30 \\ 40 & 50 & 60 \end{bmatrix} = \begin{bmatrix} 6 & 15 & 28 \\ 15 & 26 & 40 \\ 28 & 40 & 60 \end{bmatrix} + \begin{bmatrix} 0 & -5 & -12 \\ 5 & 0 & -10 \\ 12 & 10 & 0 \end{bmatrix}$ i.e. $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

Example: Express the following matrices as sum of symmetric and skew-symmetric matrices

1) $A = \begin{bmatrix} 5 & 7 \\ -3 & 4 \end{bmatrix}$, 2) $A = \begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 3 & 5 & 11 \end{bmatrix}$, 3) $A = \begin{bmatrix} 0 & 5 & -3 \\ 1 & 1 & 1 \\ 4 & 5 & 9 \end{bmatrix}$, 4) $A = \begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & 2 & 0 \end{bmatrix}$, 5) $A = \begin{bmatrix} 1 & 5 & 7 \\ -1 & -2 & -4 \\ 8 & 2 & 13 \end{bmatrix}$

Theorem: Every matrix A can be expressed as sum of Hermitian and skew-Hermitian matrices. i.e. $A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$

Where $\frac{1}{2}(A + A^\theta)$ is Hermitian matrix, and $\frac{1}{2}(A - A^\theta)$ is Skew - Hermitian matrix

e.g. Let $A = \begin{bmatrix} 2+4i & 4-6i & 6+8i \\ 8-10i & 10+12i & 12-14i \\ 14+16i & 16-18i & 18+20i \end{bmatrix} \therefore \bar{A} = \begin{bmatrix} 2-4i & 4+6i & 6-8i \\ 8+10i & 10-12i & 12+14i \\ 14-16i & 16+18i & 18-20i \end{bmatrix}$

$$A^\theta = \bar{A}^T = \begin{bmatrix} 2-4i & 8+10i & 14-16i \\ 4+6i & 10-12i & 16+18i \\ 6-8i & 12+14i & 18-20i \end{bmatrix}$$

Now

$$A + A^\theta = \begin{bmatrix} 4 & 12+4i & 20-8i \\ 12-4i & 20 & 28+4i \\ 20+8i & 28-4i & 36 \end{bmatrix}, \text{ and } A - A^\theta = \begin{bmatrix} 8i & -4-16i & -8+24i \\ 4-16i & 24i & -4-32i \\ 8+24i & 4-32i & 40i \end{bmatrix}$$

$$\therefore \frac{1}{2}(A + A^\theta) = \begin{bmatrix} 2 & 6+2i & 10-4i \\ 6-2i & 10 & 14+2i \\ 10+4i & 14-2i & 18 \end{bmatrix} \text{ is Hermitian matrix, and}$$

$$\frac{1}{2}(A - A^\theta) = \begin{bmatrix} 4i & -2-8i & -4+12i \\ 2-8i & 12i & -2-16i \\ 4+12i & 2-16i & 20i \end{bmatrix} \text{ is skew - Hermitian matrix}$$

$$\text{And } \begin{bmatrix} 2+4i & 4-6i & 6+8i \\ 8-10i & 10+12i & 12-14i \\ 14+16i & 16-18i & 18+20i \end{bmatrix} = \begin{bmatrix} 2 & 6+2i & 10-4i \\ 6-2i & 10 & 14+2i \\ 10+4i & 14-2i & 18 \end{bmatrix} + \begin{bmatrix} 4i & -2-8i & -4+12i \\ 2-8i & 12i & -2-16i \\ 4+12i & 2-16i & 20i \end{bmatrix}$$

$$\text{i.e. } A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$$

Example: Express the following matrices as sum of symmetric and skew-symmetric matrices

$$1) A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}, 2) A = \begin{bmatrix} 2 & 2+i & 3 \\ -2+i & 0 & 4i \\ -i & 3-i & 1-i \end{bmatrix}, 3) A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & i & 3i \end{bmatrix}, 4) A =$$

$$\begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 0 \end{bmatrix}, 5) A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}, 6) A = \begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}$$

$$7) A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3+2i & -1+2i & 0 \end{bmatrix}, 8) A = \begin{bmatrix} 2 & 4+i & 4i \\ 3i & 6-i & 2 \\ 6 & 4-2i & 1-i \end{bmatrix}, 9) A = \begin{bmatrix} 2 & 4+i & 6i \\ 6 & 5-i & 4 \\ 0 & 1-i & 8i \end{bmatrix}$$

Theorem: Every square matrix A can be expressed as $P + iQ$ where P , and Q are Hermitian matrices

Note: $(i)^\theta = -i$, $(-i)^\theta = i$

Proof: $A = \frac{1}{2}[A + A] = \frac{1}{2}[A + A^\theta + A - A^\theta] = \frac{1}{2}[(A + A^\theta) + i\frac{1}{i}(A - A^\theta)] = \left[\frac{1}{2}(A + A^\theta) + i\frac{1}{2i}(A - A^\theta)\right]$

$A = P + iQ$ where $P = \frac{1}{2}(A + A^\theta)$, and $Q = \frac{1}{2i}(A - A^\theta)$

Now $P^\theta = \left(\frac{1}{2}(A + A^\theta)\right)^\theta = \frac{1}{2}(A^\theta + (A^\theta)^\theta) = \frac{1}{2}(A^\theta + A) = P$ so P is hermitian Matrix

$Q = \left(\frac{1}{2i}(A - A^\theta)\right)^\theta = \frac{1}{2i^\theta}(A^\theta - (A^\theta)^\theta) = \frac{1}{2(-i)}(A^\theta - A) = \frac{-1}{2i}(A^\theta - A) = \frac{1}{2i}(A - A^\theta) = Q$

So Q is hermitian matrix

Thus

$A = P + iQ$ where $P = \frac{1}{2}(A + A^\theta)$, and $Q = \frac{1}{2i}(A - A^\theta)$ then P and Q are hermitian matrices

Example: Express the matrix $A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix}$ as $P + iQ$, where P and Q are hermitian matrices

Solution: By given $A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix}$, So $\bar{A} = \begin{bmatrix} 2 & 3+i & 2-i \\ -i & 0 & 1+i \\ 1-2i & 1 & -3i \end{bmatrix}$

$$A^\theta = (\bar{A})^T = \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & 1 \\ 2-i & 1+i & -3i \end{bmatrix}$$

Now $(A + A^\theta) = \begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & 2-i \\ 3+i & 2+i & 0 \end{bmatrix}$, and $(A - A^\theta) = \begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & -i \\ -1+3i & -i & 6i \end{bmatrix}$

$\therefore \frac{1}{2}(A + A^\theta) = \frac{1}{2} \begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & 2-i \\ 3+i & 2+i & 0 \end{bmatrix}$, and $\frac{1}{2i}(A - A^\theta) = \frac{1}{2i} \begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & -i \\ -1+3i & -i & 6i \end{bmatrix}$

Let $P = \frac{1}{2}(A + A^\theta) = \frac{1}{2} \begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & 2-i \\ 3+i & 2+i & 0 \end{bmatrix}$, $Q = \frac{1}{2i}(A - A^\theta) = \frac{1}{2i} \begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & -i \\ -1+3i & -i & 6i \end{bmatrix}$

Then P and Q are Hermitian Matrices

Thus $A = P + iQ$

i.e. $\begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & 2-i \\ 3+i & 2+i & 0 \end{bmatrix} + i \left(\frac{1}{2i} \begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & -i \\ -1+3i & -i & 6i \end{bmatrix} \right)$

Example: Express the matrix $A = \begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}$ as $A = P + iQ$

Theorem: Every Hermitian matrix A can be written as $A = B + iC$, where B is real symmetric matrix and C is real skew-symmetric matrix

Solution: By given A is hermitian matrix $\therefore A^H = A$ i.e. $(\bar{A})^T = A$, or $\overline{(A^T)} = A \dots (*)$

Now $A = \frac{1}{2}(A + A) = \frac{1}{2}(A + \bar{A} + A - \bar{A}) = \frac{1}{2}(A + \bar{A}) + i\frac{1}{2i}(A - \bar{A}) = P + iQ \dots (1)$

Where $P = \frac{1}{2}(A + \bar{A})$, $Q = \frac{1}{2i}(A - \bar{A}) \dots (2)$

We know if $z = x + iy$, $\bar{z} = x - iy$, where x and y are real numbers, then

$z + \bar{z} = 2x$, $z - \bar{z} = i2y \Rightarrow \frac{1}{2}(z + \bar{z}) = x$, and $\frac{1}{2i}(z - \bar{z}) = y \Rightarrow \frac{1}{2}(z + \bar{z})$ and $\frac{1}{2i}(z - \bar{z})$ are real numbers

From equation (2) $P = \frac{1}{2}(A + \bar{A})$, $Q = \frac{1}{2i}(A - \bar{A})$ are real numbers i.e. P and Q are real numbers

Now From equation (2) $P^T = \left(\frac{1}{2}(A + \bar{A})\right)^T = \frac{1}{2}(A^T + \bar{A}^T) = \frac{1}{2}(A^T + A^H) = \frac{1}{2}(A^T + A)$ {using (*)}

$P^T = \frac{1}{2}(A + A^T) = P$ {Using (2)} $\Rightarrow P$ is symmetric

Now From equation (2) $Q^T = \left(\frac{1}{2i}(A - \bar{A})\right)^T = \frac{1}{2i}(A^T - \bar{A}^T) = \frac{1}{2i}(A^T - A^H) = \frac{1}{2i}(A^T - A)$ {using (*)}

$Q^T = \frac{-1}{2i}(A - A^T) = -Q$ {Using (2)} $\Rightarrow Q$ is skew-symmetric

Matrix A can be expressed as $A = P + iQ$ where

$P = \frac{1}{2}(A + \bar{A})$ is real and symmetric and $Q = \frac{1}{2i}(A - \bar{A})$ is real and skew-symmetric

Example: Express the matrix $A = \begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix}$ as $A = P + iQ$, where

P is real and symmetric and Q is real and skew-symmetric

Solution: $A = \begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix}$, $\bar{A} = \begin{bmatrix} 2 & 1-i & i \\ 1+i & 0 & -3+i \\ -i & -3-i & -1 \end{bmatrix}$

$A + \bar{A} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 0 & -6 \\ 0 & -6 & -2 \end{bmatrix} \Rightarrow \frac{1}{2}(A + \bar{A}) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix}$ is symmetric

And $A - \bar{A} = \begin{bmatrix} 0 & 2i & -2i \\ -2i & 0 & -2i \\ 2i & 2i & 0 \end{bmatrix} \Rightarrow \frac{1}{2i}(A - \bar{A}) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ is skew-symmetric

Thus $A = P + iQ$, Where $P = \frac{1}{2}(A + \bar{A}) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix}$ is real and symmetric matrix

And $Q = \frac{1}{2i}(A - \bar{A}) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ is real and skew-symmetric matrix

Example: Express the matrix 1) $A = \begin{bmatrix} 1 & 2+i & -1+i \\ 2-i & 1 & 2i \\ -1-i & -2i & 0 \end{bmatrix}$, 2) $A = \begin{bmatrix} 2 & 2+i & -2i \\ 2-i & 3 & i \\ 2i & -i & 1 \end{bmatrix}$

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Theorem: Every Skew-Hermitian matrix A can be expressed as $A = P + iQ$ where P is real and skew-symmetric and Q is real and symmetric.

Solution: By given A is skew-hermitian matrix $\therefore A^\theta = -A$, or $(\bar{A})^T = -A$, or $\overline{(A^T)} = -A$

$$\text{Now } A = \frac{1}{2}(A + A) = \frac{1}{2}(A + \bar{A} + A - \bar{A}) = \frac{1}{2}(A + \bar{A}) + i\frac{1}{2i}(A - \bar{A}) = P + iQ \dots (1)$$

$$\text{Where } P = \frac{1}{2}(A + \bar{A}), Q = \frac{1}{2i}(A - \bar{A}) \dots (2)$$

We know if $z = x + iy$, $\bar{z} = x - iy$, where x and y are real numbers, then

$$z + \bar{z} = 2x, z - \bar{z} = i2y \Rightarrow \frac{1}{2}(z + \bar{z}) = x, \text{ and } \frac{1}{2i}(z - \bar{z}) = y \Rightarrow \frac{1}{2}(z + \bar{z}), \text{ and } \frac{1}{2i}(z - \bar{z}) \text{ are real numbers}$$

From equation (2) $P = \frac{1}{2}(A + \bar{A})$, $Q = \frac{1}{2i}(A - \bar{A})$ are real numbers i.e. P and Q are real numbers

$$\text{Now From equation (2) } P^T = \left(\frac{1}{2}(A + \bar{A})\right)^T = \frac{1}{2}(A^T + \bar{A}^T) = \frac{1}{2}(A^T + A^\theta) = \frac{1}{2}(A^T - A) \text{ \textit{\textcolor{violet}{using (*)}}}$$

$$P^T = \frac{-1}{2}(A - A^T) = -P \quad \text{\textit{\textcolor{violet}{Using (2)}}} \Rightarrow P \text{ is skew-symmetric}$$

$$\text{Now From equation (2) } Q^T = \left(\frac{1}{2i}(A - \bar{A})\right)^T = \frac{1}{2i}(A^T - \bar{A}^T) = \frac{1}{2i}(A^T - A^\theta) = \frac{1}{2i}(A^T + A) \text{ \textit{\textcolor{violet}{using (*)}}}$$

$$Q^T = \frac{1}{2i}(A + A^T) = Q \quad \text{\textit{\textcolor{violet}{Using (2)}}} \Rightarrow Q \text{ is symmetric}$$

Matrix A can be expressed as $A = P + iQ$ where

$$P = \frac{1}{2}(A + \bar{A}) \text{ is real and symmetric and } Q = \frac{1}{2i}(A - \bar{A}) \text{ is real and skew - symmetric}$$

- 1) Example: Express the matrix $A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$ as $A = P + iQ$, where
P is real and skew - symmetric and Q is real and symmetric

$$\text{Solution: } A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}, \bar{A} = \begin{bmatrix} -2i & 2-i & 1+i \\ -2-i & i & -3i \\ -1+i & -3i & 0 \end{bmatrix}$$

$$A + \bar{A} = \begin{bmatrix} 0 & 4 & 2 \\ -4 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \Rightarrow \frac{1}{2}(A + \bar{A}) = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \text{ is skew-symmetric}$$

$$\text{And } A - \bar{A} = \begin{bmatrix} 4i & 2i & -2i \\ 2i & -2i & 6i \\ -2i & 6i & 0 \end{bmatrix} \Rightarrow \frac{1}{2i}(A - \bar{A}) = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ -1 & 3 & 0 \end{bmatrix} \text{ is symmetric}$$

Thus $A = P + iQ$, Where $P = \frac{1}{2}(A + \bar{A}) = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ is real and skew-symmetric matrix

And $Q = \frac{1}{2i}(A - \bar{A}) = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ -1 & 3 & 0 \end{bmatrix}$ is real and symmetric matrix

Example: Express the matrix 1) $A = \begin{bmatrix} i & 1-i & 2+3i \\ -1-i & 2i & -3i \\ -2+3i & -3i & -i \end{bmatrix}$, 2) $A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$

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