Graph Theory

* **Definition of graph** $\Rightarrow$ A planar representation of non-empty set of vertices and set of edges is called as a graph

$G = (V, E)$

Formal defn of graph - $G = (V, E, \gamma)$

$V(G)$ = Non-empty set of vertices

$E(G)$ = set of edges

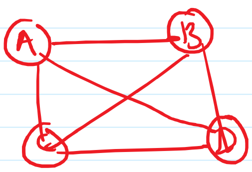
γ = function of an edge joining vertex (V, W)

* **Types of graph** $\rightarrow$

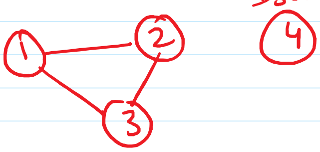
1. Directed graph
2. Undirected graph
3. Null graph — empty graph / no vertices, no edges
4. Finite / Infinite graph
5. Trivial graph - only one vertex and no edges
6. Connected / Disconned graph
7. Multigraph - A multigraph $G = (V, E)$ where G is called as multigraph iff any of the follo condn exists

   (a) More that one edge is used to join / connect same vertices

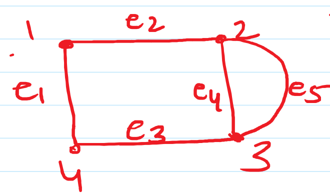
   (b) An edge joining vertex itself (self loop)



Isolated vertex



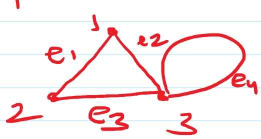
ex.



$\Rightarrow$ Multigraph

$\because$ e_4 & e_5 are joining same vertices

$e_4 = (2, 3)$ $e_5 = (2, 3)$

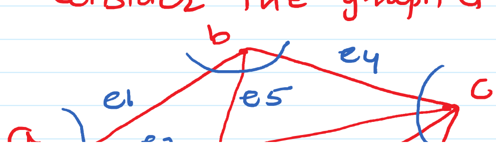


$\Rightarrow$ Multigraph

edge e_4 joining same vertex (loop)

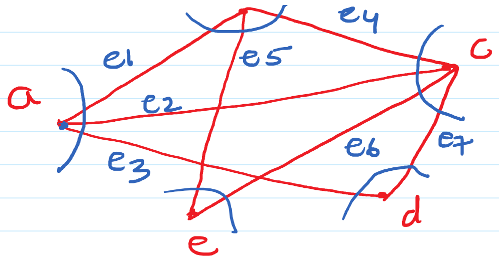
$e_4 = (3, 3)$

* Consider the graph $G = (V, E)$ in the fig below



(a) Describe G formally

(b) Find the deg & parity of each



(a) Describe G formally
(b) Find the deg & parity of each vertex of G
(c) Verify that the sum of deg. of the vertices of graph is equal to twice the number of edges.

Soln (1) Formal defn of graph G

$G = (V, E, \gamma)$

$V(G) = \{a, b, c, d, e\}$

$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

$\gamma(e_1) = (a, b)$ $\gamma(e_5) = (b, e)$

$\gamma(e_2) = (a, c)$ $\gamma(e_6) = (c, e)$

$\gamma(e_3) = (a, d)$ $\gamma(e_7) = (c, d)$

$\gamma(e_4) = (b, c)$

(2) Find deg & parity of each vertex of G

vertex	a	b	c	d	e
deg	3	3	4	2	2
parity	odd	odd	Even	Even	Even

(3) For any connected graph G, the sum of deg. of vertices of G is equal to twice the no. of edges of graph G.

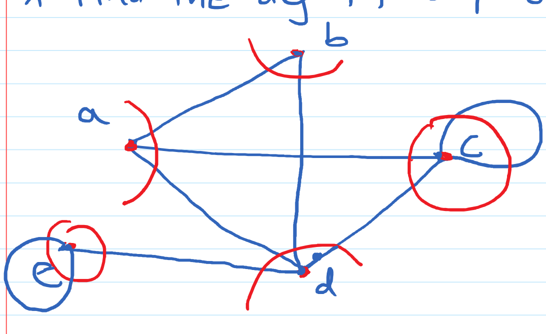
$$\sum \deg(v_i) = 2 \times E$$

$$\deg(a) + \deg(b) + \deg(c) + \deg(d) + \deg(e) = 2 \times E$$

$$3 + 3 + 4 + 2 + 2 = 2 \times 7$$

$$14 = 14$$

* Find the deg & parity of each vertex



Vertex	a	b	c	d	e
deg	3	2	4	4	3
Parity	odd	Even	Even	Even	odd

* A connected planar graph has 9 vertices having deg 2, 2, 2, 3, 3, 3, 4, 4 and 5. How many edges are there?

Soln – For any connected graph G

$$\sum \deg(v_i) = 2 \times E$$

$$2+2+2+3+3+3+4+4+5 = 2\times E$$
$$28 = 2\times E$$
$$\boxed{E=14}$$

* **Graph Isomorphism / Isomorphic Graph**

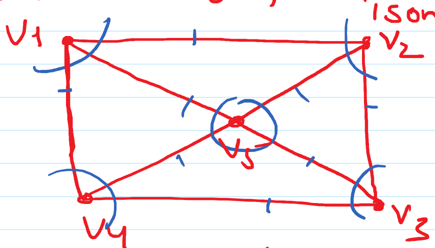
let G_1 and G_2 be the any two given graph then they are said to be isomorphic to each other iff

(a) No. of vertices of G_1 = No. of vertices of G_2
$$V(G_1) = V(G_2)$$

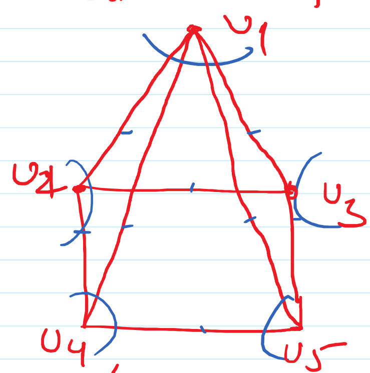
(b) No. of edges of G_1 = No. of edges of G_2
$$E(G_1) = E(G_2)$$

(c) Both the graph must have ~~same~~ equal no. of vertices ~~have~~ having same deg.

* ST the graph G_1 & G_2 are isomorphic



Graph G1



Graph G2

Soln (1) No. of vertices of G_1 = No. of vertices of G_2
$$V(G_1) = V(G_2)$$
$$5 = 5$$

(2) No. of edges of G_1 = No. of edges of G_2
$$E(G_1) = E(G_2)$$
$$8 = 8$$

(3) For graph G_1

$\deg(v_1) = 3$
$\deg(v_2) = 3$
$\deg(v_3) = 3$
$\deg(v_4) = 3$
$\deg(v_5) = 4$

For Graph G_2

$\deg(u_1) = 4$
$\deg(u_2) = 3$
$\deg(u_3) = 3$
$\deg(u_4) = 3$
$\deg(u_5) = 3$

∴ Graph G_1 & G_2 are isomorphic graph.

∴ Graph G_1 & G_2 are isomorphic graph.

*Eulerian and Hamiltonian Graph

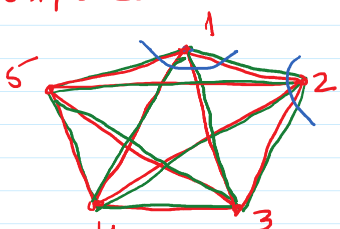
Eulerian Graph :-

Let G be the given graph
⇓
Find the deg. of each vertex of graph G

sabka deg even hoga

If the deg of each vertex of graph G is EVEN.
⇓
∴ Eulerian circuit is present in graph G
⇓
∴ Graph G is eulerian graph
⇓
∴ Eulerian path is present in graph G.

Eulerian path is a closed path where each edge travelled only ones.



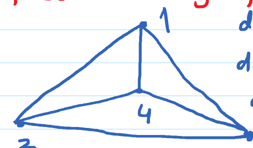
$deg(1) = 4$
$deg(2) = 4$
$deg(3) = 4$
$deg(4) = 4$
$deg(5) = 4$

∵ All the vertices of G is having EVEN deg
∴ Eulerian circuit is present in graph G
∴ Graph G is Eulerian graph
∴ Eulerian path is present in G

Eulerian path ⇒ 1 − 2 − 3 − 4 − 5 − 1 − 3 − 5 − 2 − 4 − 1

Agar atleast ek vertex ka bhi deg. odd nikala

If atleast one vertex is having odd deg
OR
If all the vertices of G is not having even deg
⇓
∴ Eulerian circuit is not present in graph G
⇓
∴ Graph G is not Eulerian graph
⇓
∴ Eulerian path is not present in graph G



$deg(1) = 3$
$deg(2) = 3$
$deg(3) = 3$
$deg(4) = 3$

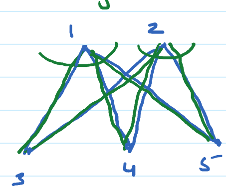
∵ The deg of all vertices of G is not Even.
∴ Eulerian ckt is not present
∴ Graph G is not eulerian graph.
∴ Eulerian path is not present in graph G

special case

Exactly two vertices of G is having ODD deg.

If exactly two vertices of graph G is having odd deg.
⇓
Eulerian circuit is not present in G
⇓
∴ Graph G is not Eulerian graph
⇓
But eulerian path is present in G

Eulerian path must start with a vertex having odd deg and must end with another vertex of G having odd deg.



Eulerian path ⇒ 1 − 3 − 2 − 4 − 1 − 5 − 2 (odd ... odd)

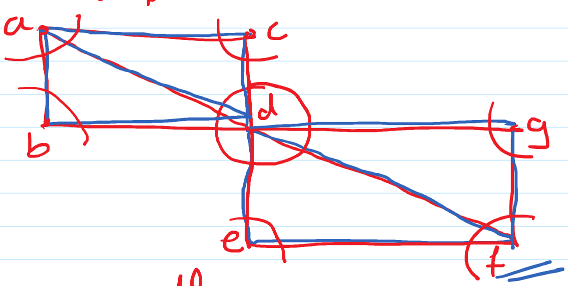
$deg(1) = 3$
$deg(2) = 3$
$deg(3) = 2$
$deg(4) = 2$
$deg(5) = 2$

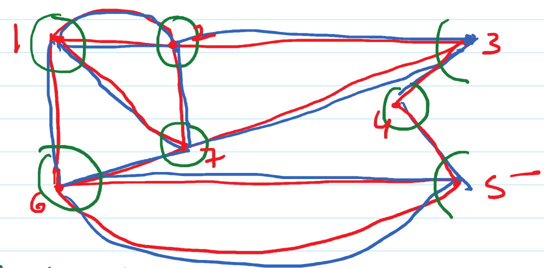
∵ The deg of all vertices of G is not even
∴ Eulerian ckt is not present in G
∴ Graph G is not

Eulerian graph
But Eulerian path is present in G because here exactly 2 vertices is having odd deg

path ⇒

* Which of the graphs shown below have eulerian ckt, an eulerian path but not eulerian ckt or neither?





deg(a) = 3 odd
deg(b) = 2
deg(c) = 2
deg(d) = 6

deg(e) = 2
deg(f) = 3 odd
deg(g) = 2

∵ All the vertices of G is not having even deg
∴ Eulerian ckt is not present in G
∴ Graph G is not Eulerian graph.
∵ In G, exactly two vertices (a & f) are having odd deg
∴ Eulerian path is present in graph G

path ⇒ a(odd) – b – d – c – a – d – g – f – e – d – f(odd)

deg(1) = 4
deg(2) = 4
deg(3) = 3

deg(4) = 2
deg(5) = 3

deg(6) = 4
deg(7) = 4

Eulerian path ⇒ 3 – 2 – 1 – 2 – 7 – 1 – 6 – 7 – 3 – 4 – 5 – 6 – 5

* Hamiltonian Graph –

Let G be the given graph
⇓
Find the deg. of each vertex of graph G

If $\deg(V_i) \geq \frac{N}{2}$

If deg of each vertex of graph G greater than or equal to $\frac{N}{2}$
where N ⇒ total no. of vertices of G
⇓
Hamiltonian circuit is present in G

If $\deg(V_i) \ngeq N/2$

If deg of each vertex of G is not greater than equal to $N/2$.
where N ⇒ total no. of vertices of G
⇓
Hamiltonian ckt is not present in G

$\Downarrow$

Hamiltonian circuit is present in G

$\Downarrow$

$\therefore$ Graph G is Hamiltonian graph

$\Downarrow$

$\therefore$ Hamiltonian path is present in G

<u>Ham path</u> $\Rightarrow$ Hamiltonian path is a closed path where each vertex visited/travelled only ones, but starting vertex is visited twice [because closed path]

$\Downarrow$ of G

Hamiltonian ckt is not present in G

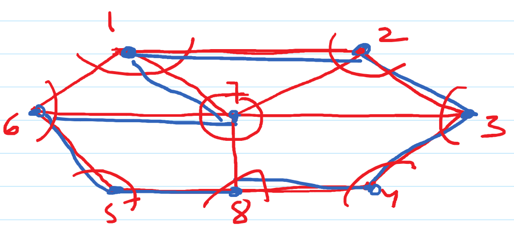
$\Downarrow$

$\therefore$ Graph G is not Hamiltonian graph

$\Downarrow$

But hamiltonian path may or may not be present

* Check that the follo. graph are Hamiltonian or not. Also find the path if possible



deg(1) = 3
deg(2) = 3
deg(3) = 3
deg(4) = 2
deg(5) = 2
deg(6) = 3
deg(7) = 5
deg(8) = 3

$deg(V_i) \geq \frac{N}{2}$ N = total no. of vertices of G $\Rightarrow$ 8

$\because deg(v_i) \ngeq \frac{N}{2}$

$deg(4) \geq \frac{8}{2}$ $2 \ngeq 4$

$deg(5) \geq N/2$ $2 \ngeq \frac{8}{2}$

$\therefore$ Hamiltonian ckt is not present

$\therefore$ Graph G is not hamiltonian graph

<u>Ham path</u> $\Rightarrow$ 1 − 2 − 3 − 4 − 8 − 5 − 6 − 7 − 1

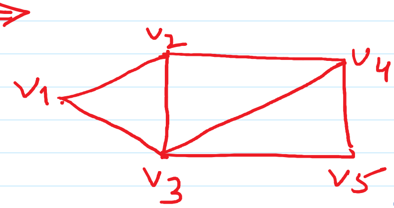
* <u>Representation of graph</u> :- How to represent a graph in memory using matrix.

$\longrightarrow$ Adjacency matrix (vertex-vertex adjacency matrix)

$\longrightarrow$ Incidence matrix [vertex-edge incidence matrix]

① <u>Adjacency matrix for graph</u> [or <u>vertex-vertex</u>]

ⓐ Undirected graph $\Rightarrow$

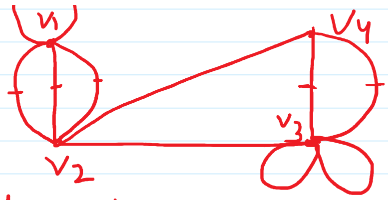


$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

consider only directly connected edge

ⓑ multigraph $\Rightarrow$

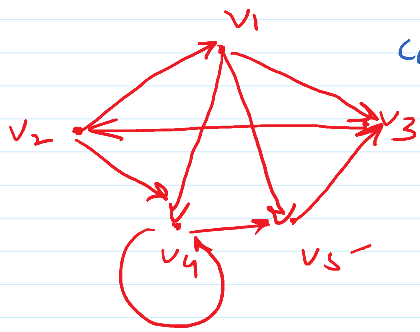
$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix} \end{array}$$



directly connected

$$A = \begin{array}{c} v_2 \\ v_3 \\ v_4 \end{array} \begin{bmatrix} 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

(c) Directed graph ⇒

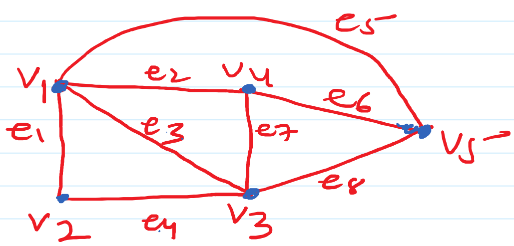


consider only outgoing edges

$$A = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 0 & 1 & 0 \\ v_3 & 0 & 1 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 1 & 1 \\ v_5 & 0 & 0 & 1 & 0 & 0 \end{array}$$

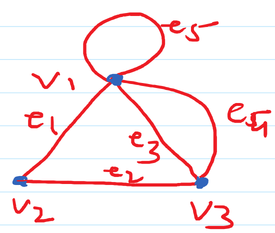
(2) Incidence matrix. (vertex-edge incidence matrix)

Undirected graph



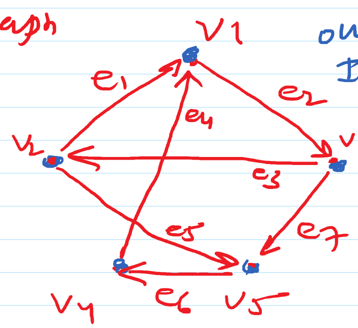
$$A = \begin{array}{c|cccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \hline v_1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ v_2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ v_4 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ v_5 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}$$

multigraph



$$A = \begin{array}{c|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline v_1 & 1 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 1 & 1 & 1 & 0 \end{array}$$

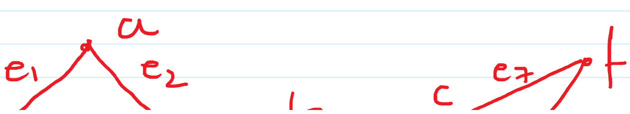
Directed graph



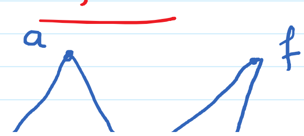
outgoing ⇒ +1
Incoming ⇒ −1

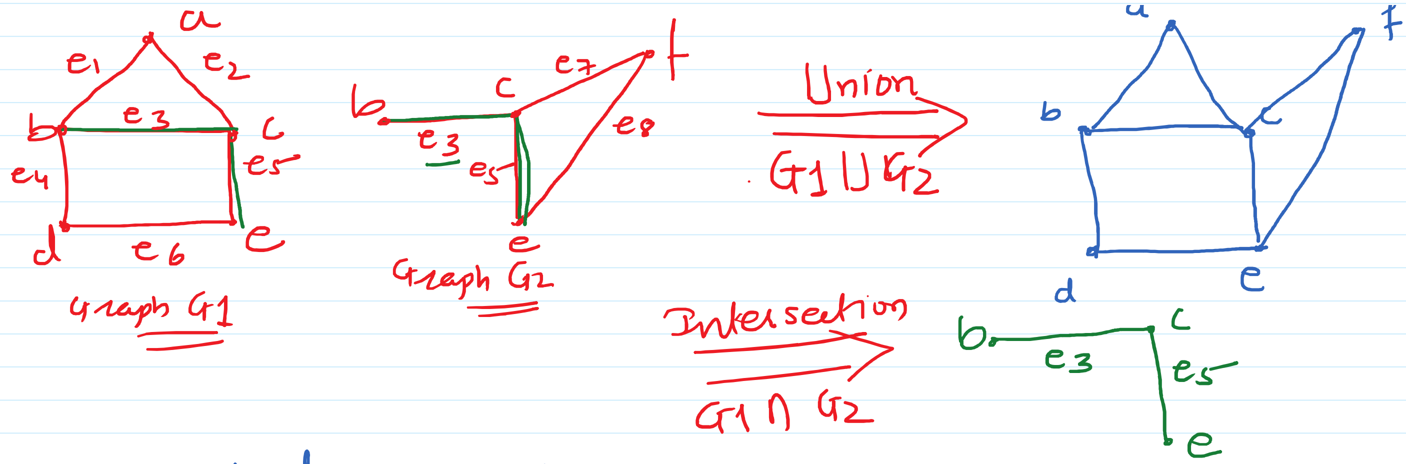
$$A = \begin{array}{c|ccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \hline v_1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 \\ v_2 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ v_3 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ v_4 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ v_5 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}$$

* Operations on graph ⇒ Union | Intersection | complement —

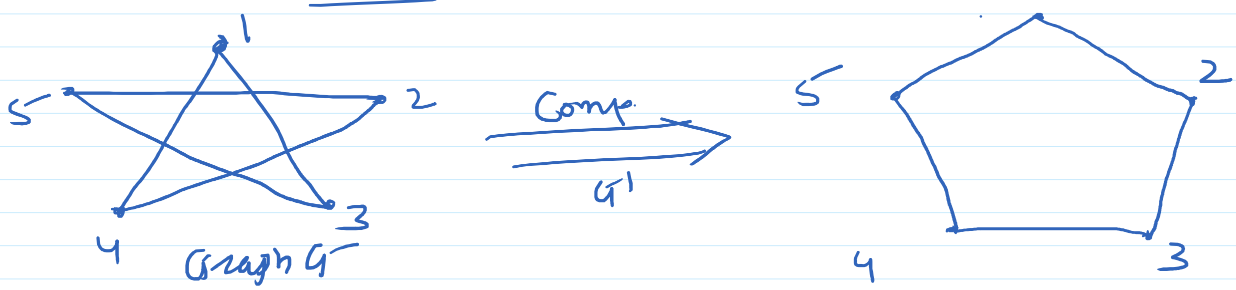


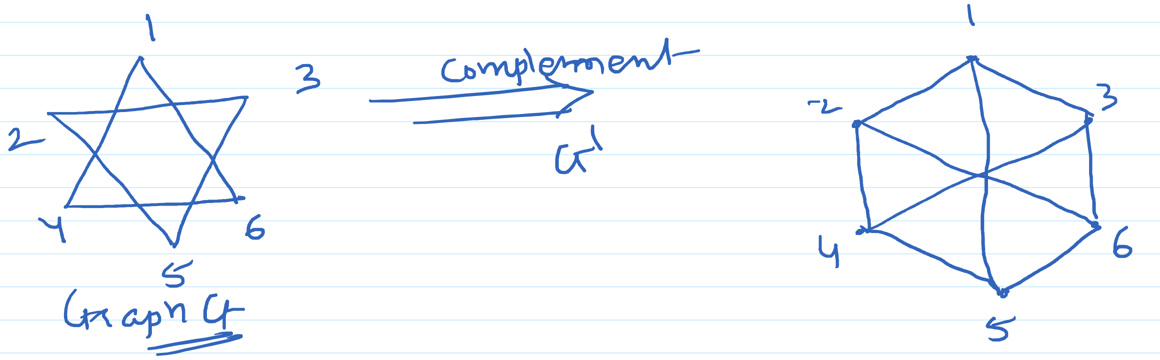
Union





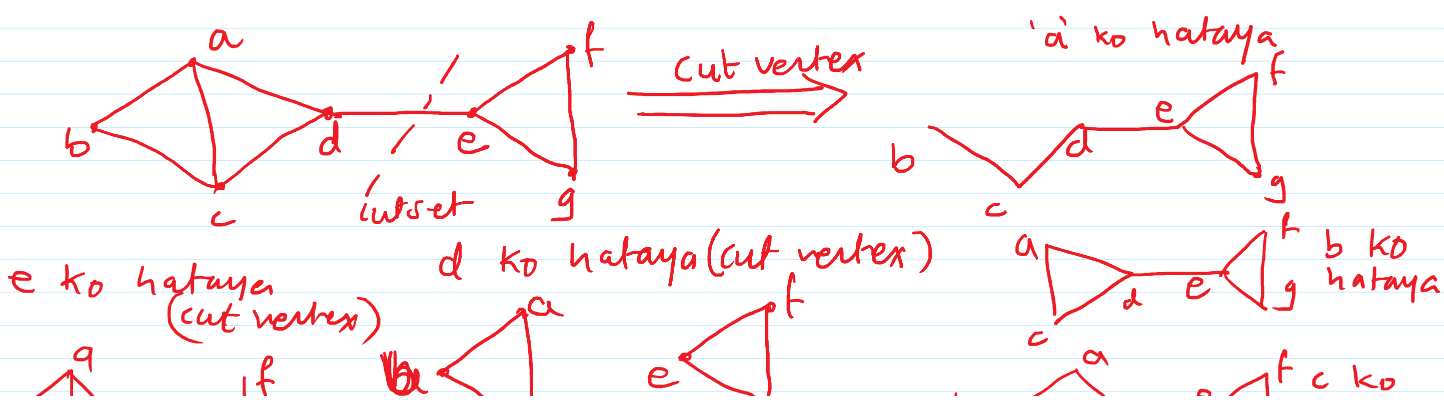
Complement of a graph



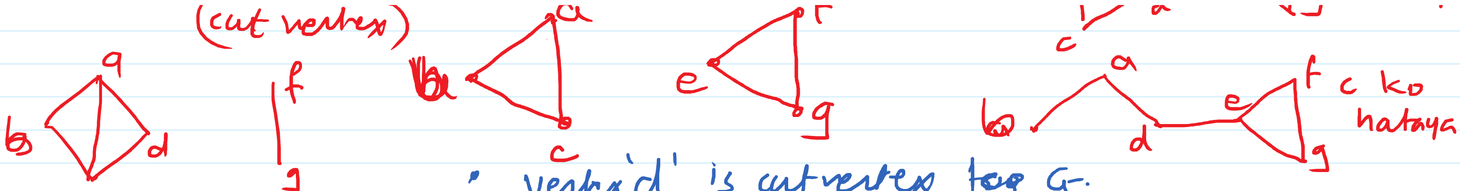


* Cut vertex / Cutset / Bridge

① Cut vertex → Agar koi ek vertex ko cut karne k badd agar graph 2 parts me divide ho jaata hai toh us vertex ko cut vertex bolte hai.



(cut vertex)

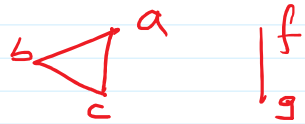


vertex e also is cut vertex

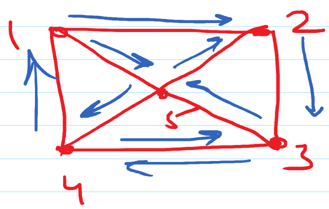
∴ vertex 'd' is cut vertex for G.

cutset ⇒ Ek se jyada vertex ko hatane k baad agar graph 2 parts me divide hota hai toh usko ~~cutve~~ cutset bolte hai.

In the above graph {d, e} is cutset



* **Walk** :- vertex & edges can be repeated.



* **Trails** ⇒ vertex can be repeated but edges should not be repeated

* **Circuit** ⇒ It is trails with condⁿ that trail should be closed

* **path** ⇒ vertex & edges should not repeat

cycle ⇒ Closed path