* Eigen Values & Eigen Vectors *

matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

vecto

No relation

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

same scaled

Relation

$AX = \lambda X$, $X \neq 0$, $X \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

eigen value

eigen vector

$AX = \lambda X$, $X \neq 0$

$AX - \lambda X = 0$ ⟵ Null vecto $\Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$(A - \lambda I) X = 0$

Point

$A - \lambda I = 0, \quad X \neq 0$

$\Rightarrow |A - \lambda I| = 0 \longleftarrow$.

2^{nd}

Eigen value $\quad |A - \lambda I| = 0$

$A - \lambda I \Rightarrow$ char. poly

$|A - \lambda I| = 0 \Rightarrow$ characterist equation

- Roots of char. eqn. is called eigen values

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Eigen Values and Eigen Vectors

Any non zero vector X is said to be characteristic vector or eigen vector of a matrix A, if there exist a number λ such that
$AX = \lambda X$

where $A = [a_{ij}]_{n\times n}$ is a n-rowed square matrix and $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a column vector.

Also, λ is said to be characteristic root or latent root or characteristic value or eigen value or proper value of the matrix A.

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① Find the eigen values of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$$

Sol: The characteristic equation is $|A-\lambda I| = 0$

$A - \lambda I$

$\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 5-\lambda \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 5-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(5-\lambda) - 0\times 0 = 0$

$(1-\lambda)(5-\lambda) = 0$

$1-\lambda = 0$ or $5-\lambda = 0$

$\lambda = 1$ or $\lambda = 5$

$\therefore$ Eigen Values are 1, 5.

② $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

Sol:

$\lambda^2 - (0+0)\lambda + (0-2) = 0$

$\lambda^2 - 0\lambda - 2 = 0$

$\lambda^2 - 2 = 0$

$\lambda^2 = 2$

$\lambda = \pm\sqrt{2} \longleftarrow$ Eigen values

$\mathbb{R}, \mathbb{C}$

③ Find the eigen Values of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Sol: The char. eqn. is $|A-\lambda I| = 0$

$$\begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{vmatrix} = 0$$

$$(\cos\theta-\lambda)(\cos\theta-\lambda) + \sin^2\theta = 0$$

$$\cos^2\theta - 2\cos\theta\lambda + \lambda^2 + \sin^2\theta = 0$$

$$1 - 2\cos\theta\lambda + \lambda^2 = 0$$

$$\lambda^2 - 2\cos\theta\lambda + 1 = 0$$

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4(1)(1)}}{2(1)}$$

$\cos^2\theta - 1 = -(1-\cos^2\theta)$
$= -\sin^2\theta$

$$= \frac{2\cos\theta \pm 2\sqrt{\cos^2\theta - 1}}{2}$$

$$= \cos\theta \pm \sqrt{-\sin^2\theta}$$

Eigen value is complex

$$= \cos\theta \pm i\sin\theta$$

Note:

① $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\lambda = \cos\theta \pm i\sin\theta$

② $\theta = 0°$
$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\lambda = 1 \pm i(0) = 1, 1$

③ $\theta = 45°$
$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $\lambda = \frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$

Fundamental Theorem Alge (R/C) Complex

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0$$

deg n $\implies$ n roots ✓

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0$

deg $n \Rightarrow n$ roots ✓

Shortcut method

① $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Sol: The char. eqn. is $|A - \lambda I| = 0$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

$\lambda^2 - (\text{sum of diag elts of } A)\lambda + \det(A) = 0$

$$\boxed{\lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0}$$

② $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Sol: The char. eqn. is $|A - \lambda I| = 0$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

$\lambda^3 - (\text{sum of diag elts of } A)\lambda^2 + (\text{sum of minors of diag elts})\lambda - \det A = 0$

$$\lambda^3 - (a_{11}+a_{22}+a_{33})\lambda^2 + \left(\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right)\lambda - \det A = 0$$

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① find the eigen Values of $A = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 7 & 3 \\ 0 & 0 & 8 \end{bmatrix}$ $\rightarrow 5 \times 7 \times 8$

Sol: The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 0 & 2 \\ 0 & 7-\lambda & 3 \\ 0 & 0 & 8-\lambda \end{vmatrix} = 0$$

$$\begin{array}{r} 35 \\ \times\ 8 \\ \hline 280 \end{array}$$

$\therefore \lambda^3 - (\text{sum of diag. elts})\lambda^2 + (\text{sum of minors of dia. elts of } A)\lambda - \det A$

$\therefore \lambda^3 - (5+7+8)\lambda^2 + \left(\begin{vmatrix} 7 & 3 \\ 0 & 8 \end{vmatrix} + \begin{vmatrix} 5 & 2 \\ 0 & 8 \end{vmatrix} + \begin{vmatrix} 5 & 0 \\ 7 & 8 \end{vmatrix} \right)\lambda - 280 = 0$

$\therefore \lambda^3 - 20\lambda^2 + (56 + 40 + 40)\lambda - 280 = 0$

$\therefore \lambda^3 - 20\lambda^2 + 136\lambda - 280 = 0$

$$\lambda = 5, 7, 8$$

* Properties of Eigen Values *

① Sum of eigen values of a matrix is equal to sum of principal diagonal elements

② Product of eigen values is equal to value of the determinant of that matrix.

(2) Find the sum and product of the eigen values of the following matrix without solving the characteristic equation.

(a) $\begin{bmatrix} -2 & -9 & 5 \\ -5 & -10 & 7 \\ -9 & -21 & 14 \end{bmatrix}$ **(M.U. 2000)** (b) $\begin{bmatrix} -17 & 18 & -6 \\ -18 & 19 & -6 \\ -9 & 9 & -2 \end{bmatrix}$

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2 ⓐ sum of eigen values = sum of diag. elts.

$= -2 - 10 + 14 = 2$

Product of eigen values = value of the determinant

$= -2(-140+147) + 9(-70+63) + 5(105-90)$

$= -14 - 63 + 75 = -2$

$= -2$

ⓑ $A = \begin{bmatrix} -17 & 18 & -6 \\ -18 & 19 & -6 \\ -9 & 9 & -2 \end{bmatrix}$

sum of e.values = sum of diag. elts of A

$= -17 + 19 - 2 = 0$

Product of e.values = Value of the determinant of A

$= -17\begin{vmatrix} 19 & -6 \\ 9 & -2 \end{vmatrix} - 18\begin{vmatrix} -18 & -6 \\ -9 & -2 \end{vmatrix} - 6\begin{vmatrix} -18 & 19 \\ -9 & 9 \end{vmatrix}$

$= -2$

3

(3) If the product of two eigen values of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16, find the third eigen value.

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Sol :

Let $\lambda_1, \lambda_2, \lambda_3$ be the e.values

Product of eigen values = value of det. of A

$$\lambda_1 \lambda_2 \lambda_3 = 32$$

$$16 \lambda_3 = 32 \Longrightarrow \boxed{\lambda_3 = 2}$$

4

(4) Two of the eigen values of a 3×3 matrix are -1 , 2, if the determinant of the matrix is 4 , find its third eigen value. **(M.U. 2000)**

Sol :

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$$\lambda_1 \lambda_2 \lambda_3 = 4$$

$$(-1)(2)\lambda_3 = 4$$

$$(-2)\lambda_3 = 4 \Longrightarrow \lambda_3 = \frac{4}{-2} = -2$$

5

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(5) If $A = \begin{bmatrix} x & 4x \\ 2 & y \end{bmatrix}$ has eigen values 5 and -1 then find the values of x and y.

Sol :

sum of eigen values = sum of diagonal element

$$5 + (-1) = x + y$$

$$4 = x + y \quad \text{——— (1)}$$

Product of eigen values = value of the determinant of A

$$(5)(-1) = xy - 8x$$

$$-5 = x(y-8) \quad \text{——— (2)}$$

$$(1) \Longrightarrow \boxed{x = 4 - y} \quad \text{——— (3)}$$

$$(3) \,\&\, (2) \Longrightarrow \quad -5 = (4-y)(y-8)$$

$$-5 = 4y - 32 - y^2 + 8y$$

$$-5 = 12y - 32 - y^2$$

$$y^2 - 12y + 27 = 0$$
$$y = 3, 9$$

When $y = 3$, $x = 4 - y$
$= 4 - 3 = 1$ $\therefore x = 1, y = 3 \Rightarrow \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Also, $y = 9$, $x = 4 - y$
$= 4 - 9$
$= -5$ $\therefore x = -5, y = 9 \Rightarrow \begin{bmatrix} -5 & -20 \\ 2 & 9 \end{bmatrix}$

Note: Two different matrices may have same eigen values

Properties of Eigen Values

(1) The sum of eigen values of a matrix is the sum of the elements of the principal diagonal.

(2) The product of the eigen values of a matrix is equal to the determinant of the matrix.

(3) If λ is the eigen value of matrix A, then eigen values of the following matrices are given as

Sr. No.	Matrix	Eigen Value
(i)	A^t	λ
(ii)	A^{-1}	$1/\lambda$
(iii)	A^n	λ^n
(iv)	kA	$k\lambda$
(v)	$A \pm kI$	$\lambda \pm k$
(vi)	$adj.A$	$\|A\|/\lambda$
(vii)	A^θ	$\bar{\lambda}$
(viii)	Singular	at least one zero
✗ (ix)	Hermitian	all real
✗ (x)	Skew Hermitain	either zero or purely imaginary
✗ (xi)	Real Symmetric	all real
✗ (xii)	Skew real symmetric	either zero or purely imaginary
✗ (xiii)	Unitary	± 1
✗ (xiv)	Orthogonal	± 1
(xv)	Triangular	diagonal elements

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(9) If $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, Find the eigen values and eigen vectors for the following matrices:

(i) A^t (ii) A^{-1} ~~(iii) A^θ~~ (iv) $4A^{-1}$ (v) A^2 (vi) $A^2 - 2A + I$ (vii) $A^3 + 2I$ (viii) Adj(A)

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Sol: The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (\text{sum of diag. ele of } A)\lambda^2 + (\text{sum of minors of diag. ele of } A)\lambda - \det A = 0$$

$$\lambda^3 - (3+5+3)\lambda^2 + \left(\begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix} \right)\lambda - \quad = 0$$

$$\lambda^3 - 11\lambda^2 + (15-1+9-1+15-1)\lambda - 36 = 0$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 6, 3, 2$$

Eigen values of A	6	3	2
Eigen values of A^t	6	3	2
Eigen values of A^{-1}	$6^{-1} = 1/6$	$3^{-1} = 1/3$	$2^{-1} = 1/2$
Eigen values of $4A^{-1}$	$4 \times 6^{-1} = 2/3$	$4 \times 3^{-1} = 4/3$	$4 \times 2^{-1} = 2$
Eigen values of A^2	$6^2 = 36$	$3^2 = 9$	$2^2 = 4$
Eigen value of $A^2 - 2A + I$	$6^2 - 2(6) + 1 = 25$	$3^2 - 2(3) + 1 = 4$	$2^2 - 2(2) + 1 = 1$
Eigen value of $A^3 + 2I$	$6^3 + 2 = 218$	$3^3 + 2 = 29$	$2^3 + 2 = 10$
Eigen value of $Adj(A) = \frac{\lvert A \rvert}{\lambda}$	$\frac{36}{6} = 6$	$\frac{36}{3} = 12$	$\frac{36}{2} = 18$

Case (I) When Eigen Values are distinct

(1) Find the eigen values & Eigen vectors of

$$A = \begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$$

Sol: The characteristic equation is $|A-\lambda I| = 0$

$$\therefore \begin{vmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - (\text{sum of diag. elts. of } A)\lambda^2 + (\text{sum of minors of diag. elts. of } A)\lambda - |A| = 0$$

$$\lambda^3 - (-2+7-2)\lambda^2 + \left(\begin{vmatrix} 7 & 5 \\ 5 & -2 \end{vmatrix} + \begin{vmatrix} -2 & 4 \\ 4 & -2 \end{vmatrix} + \begin{vmatrix} -2 & 5 \\ 5 & 7 \end{vmatrix} \right)\lambda - 216 = 0$$

$$\lambda^3 - 3\lambda^2 + (-14-25+4-16-14-25)\lambda - 216 = 0$$

$$\lambda^3 - 3\lambda^2 - 90\lambda - 216 = 0$$

$$\lambda = 12, -3, -6$$

(a) for $\lambda = 12$, $[A-\lambda I]X = 0$

$\therefore [A - 12I]X = 0$

$$\begin{bmatrix} -2-12 & 5 & 4 \\ 5 & 7-12 & 5 \\ 4 & 5 & -2-12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -14 & 5 & 4 \\ 5 & -5 & 5 \\ 4 & 5 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By crammer's Rule

$$\frac{x_1}{\begin{vmatrix} 5 & 4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -14 & 4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -14 & 5 \end{vmatrix}}$$

(b) For $\lambda = -3$, $[A-\lambda I]X = 0$

$\therefore [A+3I]X = 0$

$$\therefore \begin{bmatrix} -2+3 & 5 & 4 \\ 5 & 7+3 & 5 \\ 4 & 5 & -2+3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 5 & 4 \\ 5 & 10 & 5 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By crammer's Rule

$$\frac{x_1}{\begin{vmatrix} 10 & 5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & 10 \end{vmatrix}}$$

(c) for $\lambda = -6$, $[A-\lambda I]X = 0$

$\therefore [A+6I]X = 0$

$$\begin{bmatrix} -2+6 & 5 & 4 \\ 5 & 7+6 & 5 \\ 4 & 5 & -2+6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 & 4 \\ 5 & 13 & 5 \\ 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By crammer's Rule

$$\frac{x_1}{\begin{vmatrix} 5 & 4 \\ 13 & 5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 4 & 4 \\ 5 & 5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 4 & 5 \\ 5 & 13 \end{vmatrix}}$$

$$\frac{x_1}{\begin{vmatrix}5&4\\-5&5\end{vmatrix}} = \frac{-x_2}{\begin{vmatrix}-14&4\\5&5\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}-14&5\\5&-5\end{vmatrix}}$$

$$\frac{x_1}{45} = \frac{-x_2}{-90} = \frac{x_3}{45}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

for $\lambda = 12$, $X_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$

$$\frac{x_1}{\begin{vmatrix}10&5\\5&1\end{vmatrix}} = \frac{-x_2}{\begin{vmatrix}5&5\\4&1\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}5&10\\4&5\end{vmatrix}}$$

$$\frac{x_1}{-15} = \frac{-x_2}{-15} = \frac{x_3}{-15}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$\therefore \lambda = -3$, $X_2 = \begin{bmatrix}1\\-1\\1\end{bmatrix}$

$$\frac{x_1}{\begin{vmatrix}5&4\\13&5\end{vmatrix}} = \frac{-x_2}{\begin{vmatrix}4&4\\5&5\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}4&5\\5&13\end{vmatrix}}$$

$$\frac{x_1}{-27} = \frac{-x_2}{0} = \frac{x_3}{27}$$

$$\frac{x_1}{-1} = \frac{-x_2}{0} = \frac{x_3}{1}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

for $\lambda = -6$, $X_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$

(2) find the eigen values & Eigen Vector of $A = \begin{bmatrix}-2&2&-3\\2&1&-6\\-1&-2&0\end{bmatrix}$

Sol:

(3) find the eigen values & eigen vectors of $A = \begin{bmatrix}2&0&1\\0&2&0\\1&0&2\end{bmatrix}$

Sol:

$$\lambda^3 - (2+2+2)\lambda^2 + \left(\begin{vmatrix}2&0\\0&2\end{vmatrix} + \begin{vmatrix}2&1\\1&2\end{vmatrix} + \begin{vmatrix}2&0\\0&2\end{vmatrix}\right)\lambda - 6 = 0$$

$$\lambda^3 - 6\lambda^2 + (4+3+4)\lambda - 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1, 2, 3$$

(a) for $\lambda = 1$, $[A - \lambda I]X = 0$

$$\begin{bmatrix}2&0&1\\0&2&0\\1&0&2\end{bmatrix}$$

$\therefore [A - I]X = 0$

$$\begin{bmatrix}2-1&0&1\\0&2-1&0\\1&0&2-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1&0&1\\ \end{bmatrix}\begin{bmatrix}x_1\\x_2\\ \end{bmatrix} = \begin{bmatrix}0\\ \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By crammer's Rule

$$\frac{x_1}{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-1} = \frac{-x_2}{0} = \frac{x_3}{1}$$

for $\lambda = 1$, $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(b) For $\lambda = 2$, $[A - \lambda I]X = 0$

$\therefore [A - 2I]X = 0$

$$\therefore \begin{bmatrix} 2-2 & 0 & 1 \\ 0 & 2-2 & 0 \\ 1 & 0 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ By crammer's Rule

$$\frac{x_1}{\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}}$$

$$\frac{x_1}{0} = \frac{-x_2}{-1} = \frac{x_3}{0}$$

for $\lambda = 2$, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(c) for $\lambda = 3$, $[A - \lambda I]X = 0$

$\therefore [A - 3I]X = 0$

$\therefore [A-3I]X=0$

$$\begin{bmatrix} 2-3 & 0 & 1 \\ 0 & 2-3 & 0 \\ 1 & 0 & 2-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ By crammer's Rule

$$\frac{x_1}{\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{x_1}{1} = \frac{-x_2}{0} = \frac{x_3}{1}$$

for $\lambda = 3$, $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Q4 Verify that $X = [2, 3, -2, -3]^t$ is an eigen vector corresponding to the eigen value $\lambda = 2$ of the matrix
$$A = \begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}.$$

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Sol:

$$AX = \begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2-12+2+12 \\ 4+0-10+12 \\ -2+3+4-9 \\ -2+12+2-18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -4 \\ -6 \end{bmatrix}$$

$$\lambda X = 2\begin{bmatrix} 2 \\ 3 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -4 \\ -6 \end{bmatrix}$$

$\therefore AX = \lambda X$ $\therefore$ verified.

* To find Eigen Vectors of repeated Eigen Values

① find Eigen Values & Eigen Vectors of

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$$

Sol:

$$\lambda^3 - (4+3-2)\lambda^2 + \left(\begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix} \right)\lambda - 4 = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda = 1, 2, 2$$

ⓐ for $\lambda = 1$, $[A - \lambda I]X = 0$

$$[A - I]X = 0$$

$$\begin{bmatrix} 4-1 & 6 & 6 \\ 1 & 3-1 & 2 \\ -1 & -5 & -2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By crammers Rule

$$\frac{x_1}{\begin{vmatrix} 2 & 2 \\ -5 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2 \\ -1 & -5 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{-x_2}{-1} = \frac{x_3}{-3}$$

∴ for $\lambda = 1$, $X_1 = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$

ⓑ for $\lambda = 2$, $[A - \lambda I]X = 0$

$$[A - 2I]X = 0$$

$$\begin{bmatrix} 4-2 & 6 & 6 \\ 1 & 3-2 & 2 \\ -1 & -5 & -2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

∴ By crammer's Rule

$$\frac{x_1}{\begin{vmatrix} 1 & 2 \\ -5 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 1 \\ -1 & -5 \end{vmatrix}}$$

$$\frac{x_1}{6} = \frac{-x_2}{-2} = \frac{x_3}{-4}$$

$$\frac{x_1}{3} = \frac{x_2}{1} = \frac{x_3}{-2}$$

∴ for $\lambda = 2$, $X_2 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

② Find the Eigen values & Eigen vectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol:

$$\lambda^3 - (2+2+2)\lambda^2 + \left(\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} \right)\lambda - 8 = 0$$

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

$$\lambda = 2, 2, 2$$

ⓐ For $\lambda = 2$, $[A - \lambda I]X = 0$

$$[A - 2I]X = 0$$

$$\begin{bmatrix} 2-2 & 1 & 0 \\ 0 & 2-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

∴ By crammers Rule

$$\frac{x_1}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x_1}{1} = -\frac{x_2}{0} = \frac{x_3}{0}$$

∴ For $\lambda = 2$, $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

③ find the eigen values & Eigen Vectors of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol :

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\lambda = 8, 2, 2$$

(a) for $\lambda = 8$, $[A - \lambda I]X = 0$

$$[A - 8I]X = 0$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

∴ By crammer's Rule

$$\frac{x_1}{\begin{vmatrix} -5 & -1 \\ -1 & -5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & -1 \\ 2 & -5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -5 \\ 2 & -1 \end{vmatrix}}$$

$$\frac{x_1}{24} = \frac{-x_2}{12} = \frac{x_3}{12}$$

$$\frac{x_1}{2} = \frac{-x_2}{1} = \frac{x_3}{1}$$

∴ for $\lambda = 8$, $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

(b) for $\lambda = 2$ $[A - \lambda I]X = 0$

$$[A - 2I]X = 0$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

∵ Two distinct Rows are not available ∴ we cannot apply crammer's Rule

Rank $= r = 1$ (∵ All minors of order 2x2 are zero)

∴ No. of variable $= n = 3$

∴ No. of Linearly Independent Eigen Vectors $= n - r$
$= 3 - 1 = 2$

consider, $2x_1 - x_2 + x_3 = 0$ —— *

put $x_3 = 0$ in * $\Rightarrow$ $2x_1 - x_2 = 0$

$x_1 = 1, \ x_2 = 2$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

put $x_2 = 0$ in * $\Rightarrow$ $2x_1 + x_3 = 0$

$x_1 = -1, \ x_3 = 2$

$$\therefore X_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$\boxed{\lambda = 8, X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \lambda = 2, 2 \to \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}}$$

④ Find the Eigen Values & Eigen Vectors of

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Sol: $\lambda = 5, 2, 2$

ⓑ For $\lambda = 2$, $[A - \lambda I]X = 0$

$[A - 2I]X = 0$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank $= r = 1$ ($\because$ All minors of order 2×2 are zero)

No. of variable $= n = 3$

$\therefore$ No. of Linearly Independent Eigen vectors $= n - r$

$= 3 - 1 = 2$

consider $x_1 - x_2 + x_3 = 0$ ——— *

put $x_3 = 0$ in $* \implies x_1 - x_2 = 0$

$$x_1 = 1,\ x_2 = 1$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

put $x_2 = 0$ in $* \implies x_1 + x_3 = 0$

$$x_1 = 1,\ x_3 = -1$$

$$\therefore X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

correct $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Q Find the Eigen Values & Eigen Vectors of $A^3 + 2A + I$ if

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Sol: copy above answer.

Eigen values of A	5	2	2
Eigen vectors of A	$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$
Eigen values of $A^3 + 2A + I$	$5^3 + 2(5) + 1 = 136$	$2^3 + 2(2) + 1 = 13$	$2^3 + 2(2) + 1 = 13$
Eigen Vectors of $A^3 + 2A + I$	$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Homework

(7) Find the eigen values and eigen vectors for the following matrices.

(i) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ **(M.U. 2003)** (ii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Type:IV A is symmetric and eigen values are repeated

(8) Find the eigen values and eigen vectors for the following matrices.

(i) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **(M.U. 1996,2003)** (ii) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

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$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$\lambda = 2, 3, 6$

$[A - \lambda I]X = 0$

$[A - 2I]X = 0$

$$\begin{bmatrix} 3-2 & -1 & 1 \\ -1 & 5-2 & -1 \\ 1 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$[\ \ ]_{3\times 3}$

$\boxed{1, 2, 3}$

$A - \lambda I$

rank $\neq 3$

$\boxed{1, 2}$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \leftarrow \text{rank} \neq 3$$

rank 1, rank 2

$$\begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} = 3 - 1 = 2 \neq 0$$

$\therefore$ rank $= 2$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$\lambda = -1, -1, 5$

$$A - I \quad \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

3 rank

$$\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 0$$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda = 2, 2, 2$

$[A - \lambda I]X = 0$

$[A - 2I]X = 0$

$\cancel{|A - \lambda I| = 0}$

rank $\neq 3$

rank $= 2$

rank $= 1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \quad \therefore \text{rank} = 2$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \quad \text{rank} \neq 3$$

2, 1

1 Hora

$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix}$

1 Hoga

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\lambda = 1, 1, 1$

(a) for $\lambda = 1$, $[A - \lambda I] X = 0$

$[A - I] X = 0$

$\begin{bmatrix} 1-1 & 0 & 0 \\ 0 & 1-1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Null mat. $= 0$

No. of L.I. vec $= n - r$

$= 3 - 0$

$= \underline{\underline{3}}$

Consider $0x_1 + 0x_2 + 0x_3 = 0$

put $x_3 = 0 \Rightarrow 0x_1 + 0x_2 = 0$

let $x_1 = 1, x_2 = 0$

$X_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Put $x_2 = 0 \Rightarrow 0x_1 + 0x_3 = 0$

Let $x_1 = 0, x_3 = 1$

$X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Put $x_1 = 0 \Rightarrow 0x_2 + 0x_3 = 0$

$x_2 = 1, x_3 = 0$

$x_2 = 1,\ x_3 = 0$

$$X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = 1, 1, 1, \quad X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 1, 1, \Rightarrow X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ \& \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

det $= \underline{\underline{0}}$

✓

$a + ib$

upper Δ $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

$\lambda = 0, 0$

$$A = \begin{bmatrix} \ \end{bmatrix}_{3\times 3}$$

$$D = M^{-1} A M$$

$A =$ $\quad M^{-1} A M = D$

M $\quad$ A is diagonal

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}, M = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

verify $M^{-1} A M = D$

$$\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}\begin{bmatrix}4 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 5\end{bmatrix}$$

Diag

eix value

MAT^{-1} x MATA x MATB

$D = M^{-1}AM \leftarrow$ Diagonalise

$$D = \begin{bmatrix}2 & 0\\ 0 & 5\end{bmatrix} \Rightarrow D^{100} = \begin{bmatrix}2^{100} & 0\\ 0 & 5^{100}\end{bmatrix}$$

$D = M^{-1}AM$

$$\Rightarrow A = MDM^{-1}$$

$$A^{100} = (MDM^{-1})^{100}$$

$$= MD^{100}M^{-1}$$

Cayley-Hamilton Theorem (C.H.T.)

Statement:

Every matrix satisfies its own characteristic equation.

eg. $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

The char. eqn. is $|A-\lambda I| = 0$

$$\lambda^2 - (2+3)\lambda + 6 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$\therefore$ By Cayley Hamilton Theorem

$$A^2 - 5A + 6I = 0$$

Verification:

$$L.H.S = A^2 - 5A + 6I$$

$$= \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 10 & 5 \\ 0 & 15 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4-10+6 & 5-5+0 \\ 0-0+0 & 9-15+6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= RHS$$

① Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Hence find $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A.

Sol: The characteristic eqn. is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (\text{sum of diag. elts. of } A)\lambda + \det(A) = 0$$

$$\lambda^2 - (1+3)\lambda + (3-8) = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$\therefore$ By Cayley Hamilton Theorem

$$A^2 - 4A - 5I = 0$$

Verification

$$LHS = A^2 - 4A - 5I$$

$$= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 16-16-0 \\ 8-8-0 & 17-12-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 = RHS. \quad \therefore \text{CHT is verified}$$

To find $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

consider

Division (char. poly) $\rightarrow$ $\lambda^2 - 4\lambda - 5$

$$\begin{array}{r} \lambda^3 - 2\lambda + 3 \leftarrow \text{quotient} \\ \lambda^2 - 4\lambda - 5 \;\big)\; \lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 \;\big(\leftarrow \text{Dividend} \\ \lambda^5 - 4\lambda^4 - 5\lambda^3 \qquad\qquad\qquad \\ - \quad + \qquad + \qquad\qquad\qquad\qquad \\ \hline -2\lambda^3 + 11\lambda^2 - \lambda - 10 \\ -2\lambda^3 + 8\lambda^2 + 10\lambda \quad\; \\ + \qquad - \qquad - \qquad\quad \\ \hline 3\lambda^2 - 11\lambda - 10 \\ 3\lambda^2 - 12\lambda - 15 \\ - \qquad + \qquad + \quad\; \\ \hline \text{Remainder} \rightarrow \lambda + 5 \end{array}$$

Dividend = (division)(quotient) + Remainder

$$\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 = (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5$$

$$= (0)\,(\lambda^3 - 2\lambda + 3) + \lambda + 5$$

$$= \lambda + 5$$

∴ By Cayley Hamilton Theorem

$$A^5-4A^4-7A^3+11A^2-A-10I = A+5I$$

$$=\begin{bmatrix}1&4\\2&3\end{bmatrix}+5\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$=\begin{bmatrix}6&4\\2&8\end{bmatrix}$$

Q2

(18) Find the characteristic equation of the matrix A given below and hence find the matrix represented by

$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$ where $A=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}$

Sol: The char. eqn. is $|A-\lambda I|=0$

$$\begin{vmatrix}2-\lambda&1&1\\0&1-\lambda&0\\1&1&2-\lambda\end{vmatrix}=0$$

$$\lambda^3-(2+1+2)\lambda^2+\left(\begin{vmatrix}1&0\\1&2\end{vmatrix}+\begin{vmatrix}2&1\\1&2\end{vmatrix}+\begin{vmatrix}2&1\\0&1\end{vmatrix}\right)\lambda-3=0$$

$$\lambda^3-5\lambda^2+7\lambda-3=0$$

∴ By Cayley Hamilton Theorem

$$A^3-5A^2+7A-3I=0$$

To find $A^8-5A^7+7A^6-3A^5+A^4-5A^3-2A+I$

Consider

$$\underbrace{\lambda^8-5\lambda^7+7\lambda^6-3\lambda^5+\lambda^4-5\lambda^3-2\lambda+1}_{\text{Dividend}}$$

$$=\underbrace{(\lambda^3-5\lambda^2+7\lambda-3)}_{\text{Division}}\underbrace{(\lambda^2+\lambda)}_{\text{Quotient}}+\underbrace{(\lambda^2+\lambda+1)}_{\text{Remainder}}$$

0

∴ $\lambda^8-5\lambda^7+\cdots+1 = 0+\lambda^2+\lambda+1$

∴ By Cayley Hamilton Theorem

$$A^8-5A^7-\cdots+I = A^2+A+I$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} * \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

***** 06, 10, 15, 18, 19, 20

(4) Verify Cayley Hamilton Theorem for the matrix A and hence find A^{-1} and A^4 where

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Sol: The char. eqn. is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

$\therefore$ By Cayley Hamilton Theorem $A^3 - 5A^2 + 9A - I = 0$

Verification:

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = A^2 A$$

$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

$$A^3 - 5A^2 + 9A - I = \begin{bmatrix} -13 & +42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -13 \end{bmatrix} - 5\begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dots \end{bmatrix}$$

$\therefore$ C.H.T. is verified.

① To find A^{-1}

Consider

$$A^3-5A^2+9A-I=0$$

multiply both sides by A^{-1}

$$A^{-1}(A^3-5A^2+9A-I)=A^{-1}(0)$$

$$A^2-5A+9I-A^{-1}=0$$

$$\therefore A^{-1}=A^2-5A+9I$$

$$=\begin{bmatrix}-1&12&-4\\-4&7&2\\2&-8&1\end{bmatrix}-5\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}+9\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}3&2&6\\1&1&2\\2&2&5\end{bmatrix}$$

ⓑ To find A^4

Consider

$$A^3-5A^2+9A-I=0$$

$$A(A^3-5A^2+9A-I)=A(0)$$

$$A^4-5A^3+9A^2-A=0$$

$$A^4=5A^3-9A^2+A$$

$$=5\begin{bmatrix}-13&42&-2\\-11&9&10\\-10&-22&-3\end{bmatrix}-9\begin{bmatrix}-1&12&-4\\-4&7&2\\2&-8&1\end{bmatrix}+\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}$$

$$=\begin{bmatrix}-55&104&24\\-20&-15&32\\32&-42&13\end{bmatrix}$$

H.W

ⓒ Verify C.H.T. and hence find matrix represented by

(5) Verify C.H.T. and hence find matrix represented by

$$A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I$$ where A is

$$\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$

Diagonalisation of a matrix

Algebraic Multiplicity: The No. of times an eigen value is repeated (occurs) is called Algebraic multiplicity

Geometric Multiplicity: The No. of linearly independent eigen vector of corresponding eigen value is called geometric multiplicity.

Diagonalisation of a matrix:

A square matrix A is said to be diagonalisable or similar to a diagonal matrix if there exist a non singular matrix M such that

$$D = M^{-1}AM$$

where D = Diagonal Matrix
M = modal matrix (transforming form)

Note:

① $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, $M = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$

Theorem ② A square matrix is diagonalisable iff Algebraic multiplicity should be equal to Geometric multiplicity for each of its eigen Values.

Theorem ③ If A matrix has distinct eigen values then A.M = G.M. for each eigen values of that matrix. Hence that matrix is diagonalisable.

Type I: To Prove that Matrix is not diagonalisable

① P.T. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ is NOT diagonalisable.

Sol: The char. eqn. is $|A - \lambda I| = 0$

$$\lambda^2 - 5\lambda + 8\lambda - 4 = 0$$

Sol. The char. eqn. is $|A-\lambda I|=0$

$$\lambda^2-5\lambda+8\lambda-4=0$$

$$\lambda=1,2,2$$

(a) for $\lambda=2$, $[A-\lambda I]X=0$

$$[A-2I]X=0$$

$$\begin{bmatrix} 1-2 & 2 & 3 \\ 0 & 2-2 & 5 \\ 0 & 0 & 2-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank $= r = 2$

for $\lambda=2$, A.M. $=2$

G.M $=1$

$\therefore$ A.M $\neq$ G.M. for $\lambda=2$

$\therefore$ A is not diagonalisable

(2) MU 2017 Show that $A=\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$ is not diagonalisable

Sol: $\lambda^3-(2+2-1)\lambda^2+\left(\begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix}+\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}+\begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\right)\lambda-1=0$

$$\lambda^3-3\lambda^2+3\lambda-1=0$$

$$\lambda=1,1,1$$

(a) for $\lambda=1$, $[A-\lambda I]X=0$

$\therefore [A-I]X=0$

$$\therefore \begin{bmatrix} 2-1 & -1 & 1 \\ 1 & 2-1 & -1 \\ 1 & 2 & -1-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rank $= r = 2$

No. of variables $= n = 3$

No. of Linearly independent Eigen vector $= n-r$

$= 3-2$

$= 1$

$\therefore$ for $\lambda=1$, A.M $=3$

G.M $=1$

$\therefore$ A.M $\neq$ G.M. for $\lambda=1$

$\therefore$ A is not diagonalisable

③ Show that $A=\begin{bmatrix}1&2\\0&1\end{bmatrix}$ is not diagonalisable

Sol:
$$\lambda^2-(1+1)\lambda+1=0$$
$$\lambda^2-2\lambda+1=0$$
$$\lambda=1,1$$

ⓐ for $\lambda=1$, $[A-\lambda I]X=0$
$$[A-I]X=0$$
$$\begin{bmatrix}0&2\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Rank $= r = 1$

No. of variables $= n = 2$

No. of L.I. E. vect. $= n-r$
$= 2-1 = 1$

for $\lambda=1$, A.M. $=2$
G.M. $=1$

$\therefore$ A.M $\neq$ G.M for $\lambda=1$ $\therefore$ A is not diago.

H.W

④ If $A=\begin{bmatrix}1&2\\0&1\end{bmatrix}$, $B=\begin{bmatrix}2&0\\\frac{1}{2}&2\end{bmatrix}$ P.T. Both A & B are not diagonalisable but AB is diagonalisable.

Type-II : To show that A is diagonalisable.

① Show that $A=\begin{bmatrix}2&1&1\\1&2&1\\0&0&1\end{bmatrix}$ is diagonalisable.

Hence find the diagonal matrix & transforming matrix

$\boxed{\text{OR}}$

Show that $A=\begin{bmatrix}2&1&1\\1&2&1\\0&0&1\end{bmatrix}$ is similar to diagonal matrix

Hence find diagonal form D & diagonalising matrix M.

Sol:
$$\lambda^3-5\lambda^2+7\lambda-3=0$$
$$\lambda=1,1,3$$

ⓐ for $\lambda=3$, $[A-\lambda I]X=0$
$$[A-3I]X=0$$

ⓑ for $\lambda=1$, $[A-\lambda I]X=0$
$$[A-I]X=0$$

For $\lambda=3$, $[A-\lambda I]X=0$

$[A-3I]X=0$

$$\begin{bmatrix}2-3 & 1 & 1\\ 1 & 2-3 & 1\\ 0 & 0 & 1-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 0 & 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

rank $= r = 2$

No. of vari. $= n = 3$

No. of L.I. E.V $= n-r = 3-2 = 1$

by crammers Rule

$$\frac{x_1}{\begin{vmatrix}-1 & 1\\ 0 & -2\end{vmatrix}} = \frac{-x_2}{\begin{vmatrix}1 & 1\\ 0 & -2\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}1 & -1\\ 0 & 0\end{vmatrix}}$$

$$\frac{x_1}{2} = \frac{-x_2}{-2} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$

For $\lambda=3$, $X_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$

For $\lambda=1$, $[A-\lambda I]X=0$

$[A-I]X=0$

$$\begin{bmatrix}2-1 & 1 & 1\\ 1 & 2-1 & 1\\ 0 & 0 & 1-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Rank $= r = 1$

No. of variable $= n = 3$

No. of L.I. E.V $= n-r = 3-1 = 2$

consider, $x_1+x_2+x_3=0$

put $x_3=0 \Rightarrow x_1+x_2=0$

$x_1=1,\ x_2=-1$

$\therefore X_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$

put $x_2=0 \Longrightarrow x_1+x_3=0$

$x_1=1,\ x_3=-1$

$X_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$

$\therefore$ for $\lambda=3$, $X_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$

for $\lambda=1$, $X_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$, $X_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$

$\therefore$ A.M. = G.M. for each eigen value of A

$\therefore$ A is diagonalisable.

$$D = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix},\quad M = [X_1\ X_2\ X_3]$$

$$= \begin{bmatrix}3 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\quad M = \begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

Home work

(2) Show that the matrix $A = \begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}$ is diagonalisable

find the diagonal form & diagonalising matrix M

*** [M.U. 1992, 93, 94, 04, 20014, 15, 16, 19]

$\lambda = -1, -1, 3.$

(3) Show that $A = \begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$ is diagonalisable.

Find the transforming matrix and the diagonal form.

sol:

$$\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$$

$$\lambda = 0, 3, 15$$

∵ Each eigen value is distinct ∴ A.M = G.M. for each Eigen value ∴ A is diagonalisable

(a) for $\lambda = 0$, $[A-\lambda I]X = 0$

$[A-0I]X = 0$

$$\begin{bmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By crammer's Rule

$$\frac{x_1}{|\ \ |} = \frac{-x_2}{|\ \ |} = \frac{x_3}{|\ \ |}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

(b) for $\lambda = 3$, $[A-\lambda I]X = 0$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

(c) for $\lambda = 15$, $(A-\lambda I]X = 0$

$$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad M = [X_1 \ X_2 \ X_3]$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \qquad = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

∴ $D = M^{-1}AM$.

* Probability Distribution & Sampling Theory *

① n, the No. of trials is infinitely large ie. $n \to \infty$
② p, the prob. of success is infinitely small ie. $p \to 0$
③ np, average success is finite ie. $np = m > 0$

2.1 Poisson Distribution

A discrete random variable X is said to follow the **Poisson Distribution** if the probability of x is given by

$$\mathbf{P(X = x) = \frac{e^{-m} m^x}{x!}}, \mathbf{x = 0, 1, 2, 3, \ldots\ldots} \text{and } \mathbf{(m > 0)}$$

Properties
1)If two independent variates have Poisson distribution with means m_1 and m_2 then their sum also is Poisson distribution with mean $m_1 + m_2$.
2)Mean and variance of Poisson distribution is m

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② standard deviation $= \sigma$ (sigma)
Variance $= \sigma^2$

Std. deviation $= \sqrt{\text{variance}}$

Variance ≥ 0
$\therefore \sigma^2 \geq 0$

① If the mean of Poisson Variate is 2, find the probabilities of $x = 1, 2, 3, 4$.

Sol: Given $m = 2$,

To find $P(X=1)$, $P(X=2)$, $P(X=3)$, $P(X=4)$

$$P(X=x) = \frac{e^{-m} m^x}{x!}, \quad x = 0, 1, 2, 3, \ldots\ldots$$

$$P(X=x) = \frac{e^{-2}\, 2^{x}}{x!}, \quad x = 1, 2, 3, 4$$

$$P(X=1) = \frac{e^{-2}\, 2^{1}}{1!} = 0.2706$$

$$P(X=2) = \frac{e^{-2}\, 2^{2}}{2!} = 0.2706$$

$$P(X=3) = \frac{e^{-2}\, 2^{3}}{3!} = 0.1804$$

$$P(X=4) = \frac{e^{-2}\, 2^{4}}{4!} = 0.0902$$

(Q) If the mean of the Poisson distribution is 4 , find $P(m-2\sigma < X < m+2\sigma)$
(M.U.2005) **Ans.** [0.93]

Sol: $m = 4$

std. devi^n $= \sqrt{\text{variance}}$ for Poisson Dist. mean = Variance = m

$$\sigma = \sqrt{m} = \sqrt{4} = 2$$

$$\therefore \quad P(X=x) = \frac{e^{-m}\, m^{x}}{x!}, \quad x = 0, 1, 2, 3, \ldots$$

$$P(X=x) = \frac{e^{-4}\, 4^{x}}{x!}, \quad x = 0, 1, 2, 3, \ldots$$

$$P(m-2\sigma < X < m+2\sigma) = P(4-2(2) < X < 4+2(2))$$

$$= P(0 < X < 8)$$

$$= P(X = 1, 2, 3, 4, 5, 6, 7)$$

$$= P(X=1) + P(X=2) + \cdots + P(X=7)$$

$$= \sum^{7} e^{-4}\, 4^{x}$$

$$-\sum_{x=1} \quad x!$$

$$= 0.9305$$

(3) Find out the fallacy if any in the following statement. "If X is a Poisson variate such that $P(X=2) = 9P(X=4) + 90P(X=6)$ then mean of $X = 1$"

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Sol:

Let m be the mean of poisson distribution

$$P(X=x) = \frac{e^{-m} m^x}{x!}, \quad x = 0,1,2,3,\ldots\ldots$$

$$P(X=2) = 9\,P(X=4) + 90\,P(X=6)$$

$$\frac{e^{-m} m^2}{2!} = 9\left[\frac{e^{-m} m^4}{4!}\right] + 90\left[\frac{e^{-m} m^6}{6!}\right]$$

$$\frac{e^{-m} m^2}{2} = 9\left[\frac{e^{-m} m^4}{24}\right] + 90\left[\frac{e^{-m} m^6}{720}\right]$$

Dividing each term by $e^{-m} m^2$, we get

$$\frac{1}{2} = \frac{9}{24} m^2 + \frac{90}{720} m^4$$

$$\frac{1}{2} = \frac{3}{8} m^2 + \frac{m^4}{8}$$

$$4 = 3m^2 + m^4$$

$$m^4 + 3m^2 - 4 = 0$$

$$m = 1, -1, \pm 2i$$

$$\boxed{m = 1}$$

$\therefore$ Statement is True

(3) A variate X follows a Poisson distribution with variance 3, calculate $(i)\ P(X=2)$ $\quad (ii)\ P(X \geq 4)$ **(M.U.1996)** **Ans.** $[0.224, 0.353]$

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Sol: $m = \text{mean} = \text{Variance} = 3$

$$P(X=x) = \frac{e^{-m} m^x}{x!}, \quad x = 0,1,2,3,\ldots$$

$$P(X=x) = \frac{e^{-3} 3^x}{x!}, \quad x = 0,1,2,3,\ldots$$

(a) $P(X=2) = \dfrac{e^{-3} 3^2}{2!} = 0.224$ $\qquad P(E) = 1 - P(E^c)$

(b) $P(X \geq 4) = 1 - P(X < 4)$

$$= 1 - P(X = 0,1,2,3)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$$

$$= 1 - \left[\sum_{x=0}^{3} P(X=x)\right]$$

$$= 1 - \sum_{x=0}^{3} \frac{e^{-3} 3^x}{x!}$$

$$= 1 - 0.6472$$

$$= 0.3528$$

Q5

An insurance company found that only 0.01% of the population is involved in a certain type of accident each year. If it's 1000 policy were randomly selected from the population. What is the probability that no more than two of its clients are involved in such accident next year? **(M.U.2002)**
Ans. [0.9998]

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Sol: $\text{mean} = m = 1000 \times 0.01\%$

$$= 1000 \times \frac{0.01}{100} = \frac{1}{10} = 0.1$$

$$P(X=x) = \frac{e^{-m} m^x}{x!}, \quad x = 0,1,2,\ldots$$

$$P(X=x) = \frac{e^{-0.1}(0.1)^x}{x!}, \quad x=0,1,2,3,\ldots$$

Required Prob. $= P(X \leq 2)$

$$= P(X=0,1,2) = P(X=0)+P(X=1)+P(X=2)$$

$$= \sum_{x=0}^{2} \frac{e^{-0.1}(0.1)^x}{x!}$$

$$= 0.9998$$

Homework

(6) A hospital Switch board receives on an average of 4 emergency calls ina 10 minutes intervals. What is the probability that (a) There are at least two emergency calls (b) There are exactly 3 emergency call in an interval of 10 minutes.

(7) The number of accidents ina year attributed to taxi driver in a city follows Poisson distribution with mean 3. out of 1,000 taxi drivers, find approximately the number of driver with (i) no accident in a year (ii) more than 3 accidents in a year. **(M.U.2002)** **Ans.** $[50, 577]$

(8) A transmission channel has a per digit error probability $p = 0.01$.Calculate the probability of more than 1 error in 10 received digits using (i) ~~Binomial Distribution~~ (ii)Poisson Distribution. **(M.U.2004)** **Ans.** $[0.0045, 0.0047]$

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sol 7 X = No. of accidents attributed to taxi drivers

$m = 3$

$$P(X=x) = \frac{e^{-m} m^x}{x!}, \quad x=0,1,2,3,\ldots$$

$$P(X=x) = \frac{e^{-3} 3^x}{x!}, \quad x=0,1,2,3,\ldots$$

(a) $P(X=0) = \frac{e^{-3} 3^0}{0!} = e^{-3} = 0.04978$

$\therefore$ No. of drivers with 0 accidents $= 1000 \times 0.04978$

$$= 49.78 \approx 50$$

(b) $P(X \geq 3) = 1 - P(X \leq 2)$

$= 1 - P(X = 0,1,2) = 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - \sum_{x=0}^{2} \frac{e^{-3} 3^x}{x!} = 0.5768$

$\therefore$ No. of drivers with more than 3 accidents $= 1000 \times 0.5768$

$\simeq 576.8 \simeq 577$

(9) Fit a Poisson distribution to the following data.

No. of deaths	0	1	2	3	4
Frequency	123	59	14	3	1

(M.U.2000,01,04)

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Sol:

$$m = \text{mean} = \frac{\sum x_i f_i}{\sum f_i} = \frac{100}{200} = \frac{1}{2} = 0.5$$

$$P(X=x) = \frac{e^{-m} m^x}{x!}, \quad x = 0,1,2,3, \ldots\ldots$$

$$P(X=x) = \frac{e^{-0.5} (0.5)^x}{x!}, \quad x = 0,1,2,3,4$$

X	$P(X=x)$	$200 \times P(X=x)$
0	0.6065	121.3 $\simeq$ 121
1	0.3033	60.653 $\simeq$ 61
2	0.0758	15.163 $\simeq$ 15
3	0.0126	2.527 $\simeq$ 03
4	0.0015	0.3159 $\simeq$ 0
		Total 200

Normal distribution

A continuous discrete random variable X is said to follow the **Normal Distribution** with parameter m**(called mean)** and σ^2 **(called variance)**, if it's probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \quad -\infty < X < \infty,\ -\infty < m < \infty,\ \sigma^2 > 0$$

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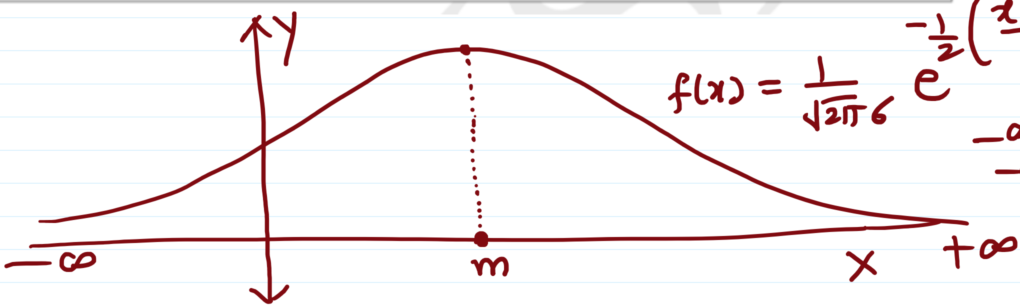
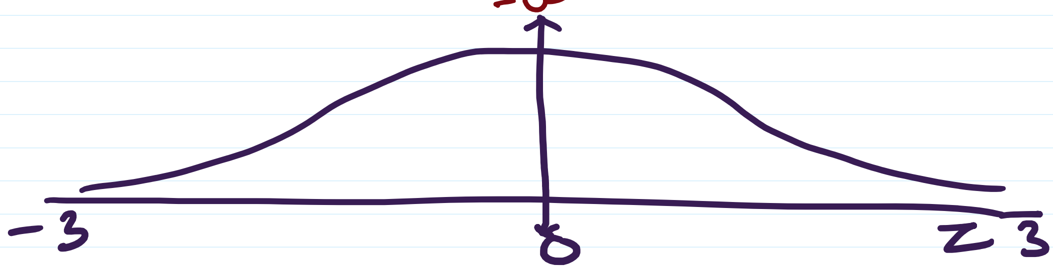


$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

$-\infty < x < \infty$
$-\infty < m < \infty$
$\sigma^2 > 0$

Standard Normal variate

$$f(z) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, dz, \quad -\infty \le z \le \infty$$



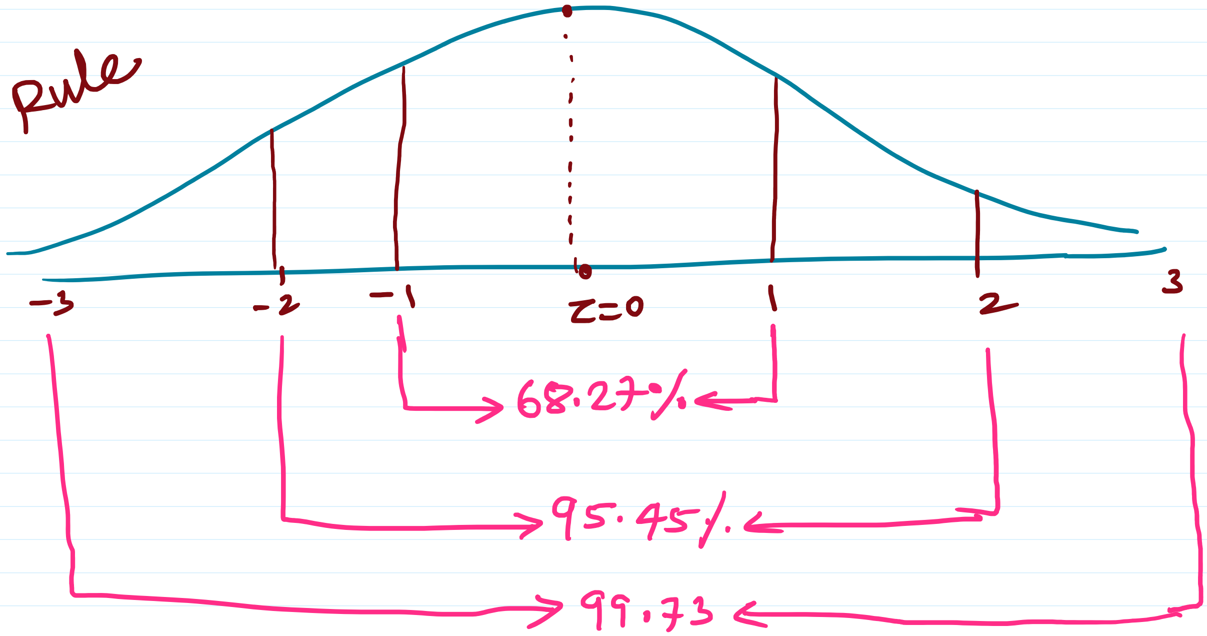
Area Property:

The Prob. of continuous random Normal variable X in the interval (a, b) is the area under the curve bounded by $x = a$ & $x = b$ and is given by

$$P(a < X < b) = \int_a^b f(x)\, dx$$

* Percentage of Area under Standard Normal distribution

68–95–99 Rule

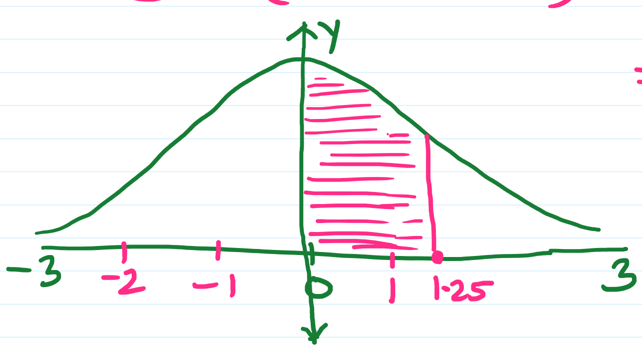


① If X is normally distributed variable with mean = 30 and std. deviation = 4. find

ⓐ $P(30 < X < 35)$ ⓑ $P(X < 40)$ ⓒ $P(X > 21)$

Sol: $m = 30,\ \sigma = 4$

ⓐ $P(30 < X < 35) = P\left(\frac{30-m}{\sigma} < \frac{X-m}{\sigma} < \frac{35-m}{\sigma}\right)$



$= P\left(\frac{30-30}{4} < z < \frac{35-30}{4}\right)$

$= P(0 < z < 1.25)$

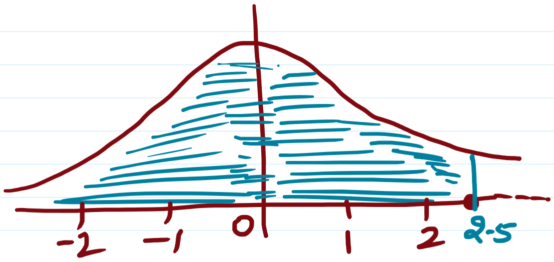
= Area from z=0 to z=1.25

$= 0.3944$

ⓑ $P(X < 40) = P\left(\frac{X-m}{\sigma} < \frac{40-m}{\sigma}\right)$

$= P\left(\quad < 40-30\right)$

$$= P\left(Z < \frac{40-30}{4}\right)$$



$$= P(Z < 2.5)$$

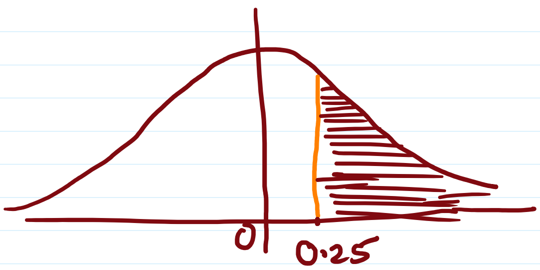
$$= 0.5 + (\text{Area from } z=0 \text{ to } z=2.5)$$

$$= 0.5 + 0.4938$$

$$= 0.9938$$

(c) $P(X > 31) = P\left(\frac{X-m}{\sigma} > \frac{31-m}{\sigma}\right)$

$$= P\left(Z > \frac{31-30}{4}\right)$$



$$= P(Z > 0.25)$$

$$= 0.5 - (\text{Area from } z=0 \text{ to } z=0.25)$$

$$= 0.5 - 0.0987$$

$$= 0.4013$$

Homework

(2) If X is Normal variate with mean 10 & std. dev = 4

find (a) $P(5 < X < 18)$ Ans 0.8716

(b) $P(X < 12)$ Ans 0.6915

(c) $P(|X-14| < 1)$ Ans 0.0655

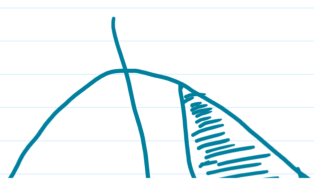
Sol:

(c) $P(|X-14|<1) = P(-1 < X-14 < 1)$

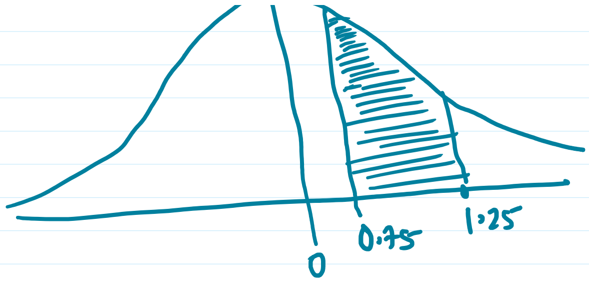
$$= P(-1+14 < X < 1+14)$$

$$= P(13 < X < 15)$$

$$= P\left(\frac{13-m}{\sigma} < \frac{X-m}{\sigma} < \frac{15-m}{\sigma}\right)$$



$$= P\left(\frac{13-10}{4} < Z < \frac{15-10}{4}\right)$$



$$= P\left(\frac{13-10}{4} < z < \frac{15-10}{4}\right)$$

$$= P(0.75 < z < 1.25)$$

= (Area from $z=0$ to $z=1.25$)

− (Area from $z=0$ to $z=0.75$)

$$= 0.3944 - 0.2734$$

$$= 0.1210$$

Z-table

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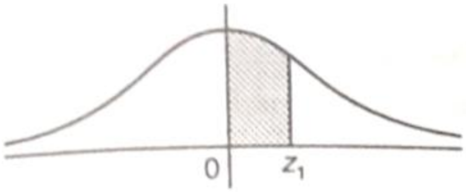
Area Under Standard Normal Curve

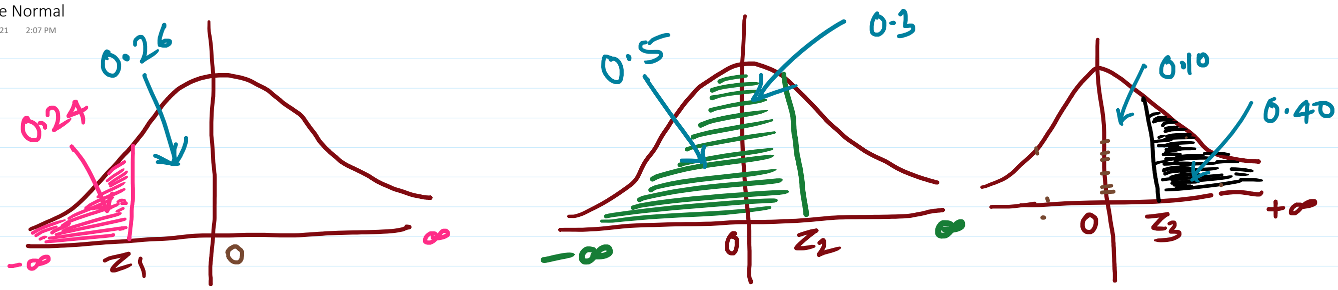


The table gives the area under the standard normal curve from $z = 0$ to $z = z_1$ which is the probability that z will lie between $z = 0$ and $z = z_1$.

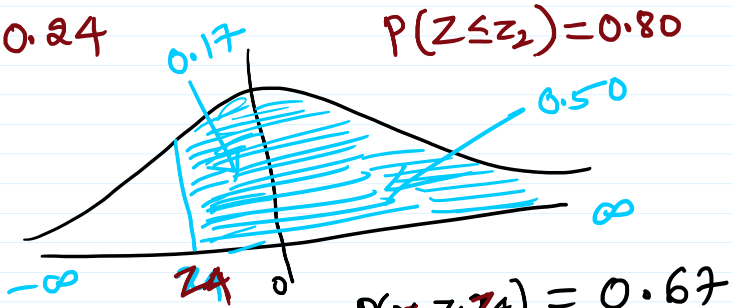
Z=0

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.8832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4415	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4841	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4560	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990



$P(Z \le z_1) = 0.24$ $\quad$ $P(Z \le z_2) = 0.80$ $\quad$ $P(Z \geq z_3) = 0.40$



$P(\cancel{x} \geq z_4) = 0.67$

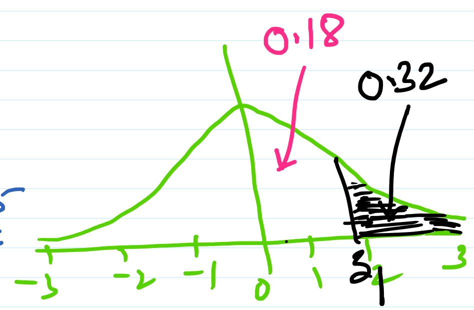
(6) If X is a normal variate with mean 25 and standard deviation 5, find
(i) $P(X \geq x_1) = 0.32$
(ii) $P(X \leq x_2) = 0.73$
(iii) $P(X \leq x_3) = 0.24$

(a) $P(X \geq x_1) = 0.32$

$P\left(\frac{X-m}{\sigma} \geq \frac{x_1 - m}{\sigma}\right) = 0.32$

$P\left(Z \geq \frac{x_1 - 25}{5}\right) = 0.32$

$P(Z \geq z_1) = 0.32$ where $z_1 = \frac{x_1 - 25}{5}$



$\therefore$ from Normal table

$z_1 = 0.46$

$\frac{x_1 - 25}{5} = 0.46$

$x_1 = 0.46 \times 5 + 25$

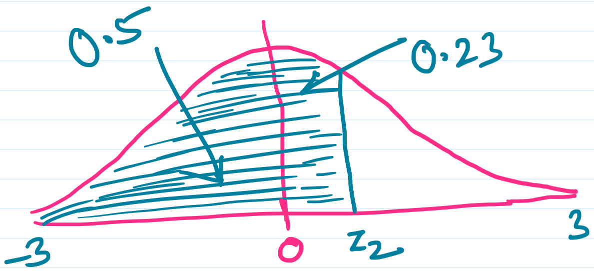
$x_1 = 27.3$

(b) $P(X \le x_2) = 0.73$

Sol: $P\left(\frac{X-m}{\sigma} \le \frac{x_2 - m}{\sigma}\right) = 0.73$

$P\left(Z \le \frac{x_2 - 25}{5}\right) = 0.73$

$$P(Z \le z_2) = 0.73 \quad \text{where } z_2 = \frac{x_2 - 25}{5}$$



from z-table

$$z_2 = 0.61$$

$$\frac{x_2 - 25}{5} = 0.61$$

$$x_2 = 5 \times 0.61 + 25$$

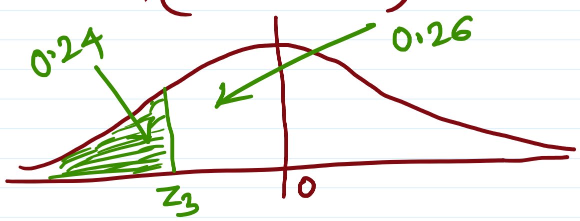
$$x_2 = 28.05$$

(c) $P(X \le x_3) = 0.24$

Sol: $$P\left(\frac{X - m}{\sigma} \le \frac{x_3 - m}{\sigma}\right) = 0.24$$

$$P\left(Z \le \frac{x_3 - 25}{5}\right) = 0.24$$

$$P(Z \le z_3) = 0.24 \quad \text{where } z_3 = \frac{x_3 - 25}{5}$$



$\therefore$ from z-table

$$z_3 = -0.7$$

$$\frac{x_3 - 25}{5} = -0.7$$

$$x_3 = 25 + (-0.7 \times 5)$$

$$= 21.5$$

Q1

The weights of 4000 students are found to be normally distributed with mean 50 kgs and standard deviation 5 kgs. Find the probability that a student selected at random will have weight (a) Less than 45 kg (b) between 45 and 60 kgs

Sol: Given

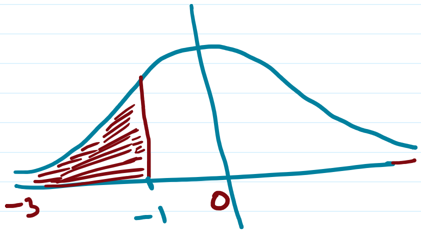
$$m = 50 \text{ kg}$$

$$\sigma = 5 \text{ kg}$$

Let X be the weight of the student

(a) $P(X<45) = P\left(\frac{X-m}{\sigma} < \frac{45-m}{\sigma}\right)$

$= P\left(Z < \frac{45-50}{5}\right)$



$= P(Z<-1)$

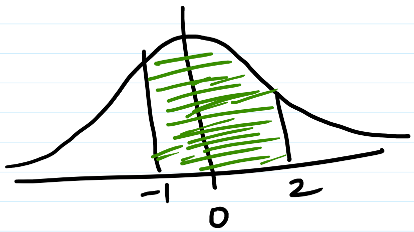
$= 0.5 - (\text{Area from } z=0 \text{ to } z=1)$

$= 0.5 - 0.3413$

$= 0.1587$

(b) $P(45<X<60) = P\left(\frac{45-m}{\sigma} < \frac{X-m}{\sigma} < \frac{60-m}{\sigma}\right)$

$= P\left(\frac{45-50}{5} < Z < \frac{60-50}{5}\right)$



$= P(-1<Z<2)$

$= (\text{Area from } z=0 \text{ to } z=1) + (\text{Area from } z=0 \text{ to } z=2)$

$= 0.3413 + 0.4772$

$= 0.8185$

Q2

The incomes of a group of 10000 persons were found to be normally distributed with mean Rs 520 and S.D. 60. Find
(a) The Number of Persons having incomes between s 400 and 550
(b) The lowest income of the richest 500

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Sol: Let X be the income

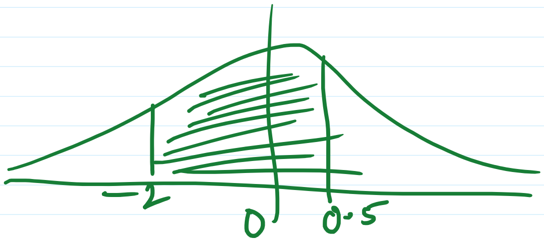
$m = 520, \ \sigma = 60$

(a) $P(400<X<550) = P\left(\frac{400-m}{\sigma} < \frac{X-m}{\sigma} < \frac{550-m}{\sigma}\right)$

$= P\left(\frac{400-520}{60} < Z < \frac{550-520}{60}\right)$

$$= P\left(\frac{-120}{60} < Z < \frac{30}{60}\right)$$

$$= P(-2 < Z < 0.5)$$



$$= (\text{Area from } z=0 \text{ to } z=2)$$

$$+ (\text{Area from } z=0 \text{ to } z=0.5)$$

$$= 0.4772 + 0.1915$$

$$= 0.6687$$

$\therefore$ No. of people having income between 400 & 550 $= 10000 \times 0.6687$

$$= 6687$$

(b) The prob. that a person belongs to top 500 richest person $= \frac{500}{10000}$

$$= 0.05$$

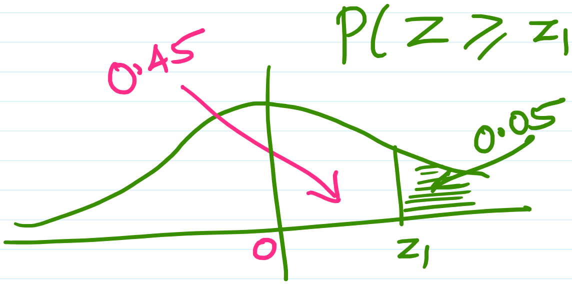
Let x_1 be the lowest amount of richest 500 people

$$\therefore \quad P(X \geq x_1) = 0.05$$

$$P\left(\frac{X-m}{\sigma} \geq \frac{x_1 - m}{\sigma}\right) = 0.05$$

$$P\left(Z \geq \frac{x_1 - 520}{60}\right) = 0.05$$

$$P(Z \geq z_1) = 0.05 \quad \text{where } z_1 = \frac{x_1 - 520}{60}$$



$\therefore$ from Z-Table

$$z_1 = 1.66$$

$$\frac{x_1 - 520}{60} = 1.66$$

$$x_1 = (60 \times 1.66) + 520$$

$$= 619.6$$

Homework

The IQ of army volunteers in a given year are normally distributed with mean 110 and standard deviation 10. The army wants to give advanced training to 20% of those recruits with the highest score. What is the lowest IQ score acceptable for advanced training?

Sol:

$m = 110$

$\sigma = 10$

X : be the IQ of the volunteers

$$P(X \geq x_1) = 0.20$$

$$P\left(\frac{X-m}{\sigma} \geq \frac{x_1-m}{\sigma}\right) = 0.20$$

$$P\left(Z \geq \frac{x_1-110}{10}\right) = 0.20$$

$$P(Z \geq z_1) = 0.20 \quad \text{where } z_1 = \frac{x_1-110}{10}$$

from Normal table,

$$z_1 = 0.84$$

$$\frac{x_1-110}{10} = 0.84$$

$$x_1 = 110 + 10 \times 0.84$$

$$x_1 = 118.4$$

Testing of Hypothesis

Large sample → ≥ 30 (Not in syllabus)

Small sample → < 30 ← syllabus

* Procedure of Testing Hypothesis *

Step I: set up Null & Alternate Hypothesis

Step II: set up level of significance (1%, 5%, 10%)

Step III: Test statistics

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}}$$ (When unbiased std. dev. of sample is given)

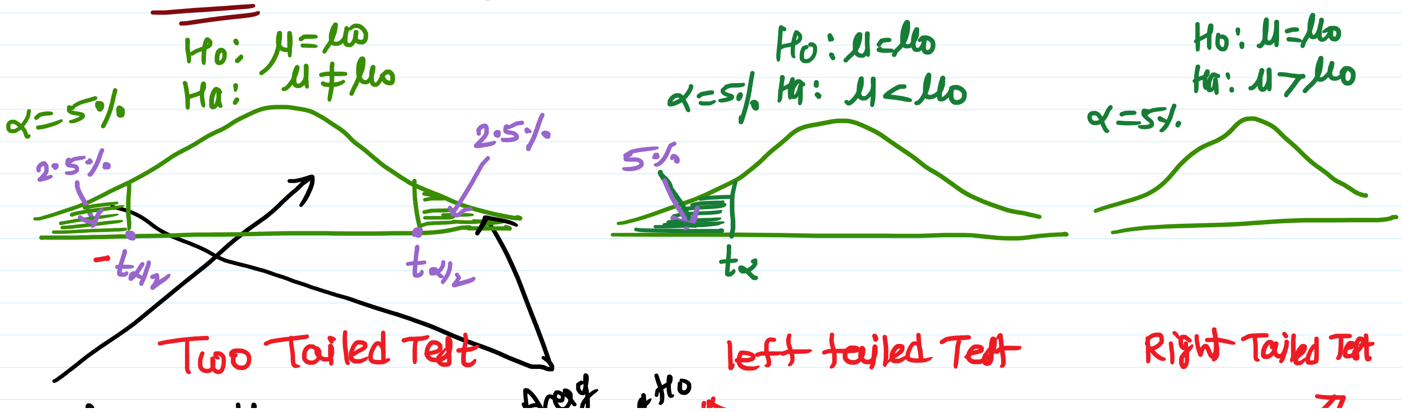
capital S →

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}}$$ (When std. dev. of sample is given)

small s →

where $\bar{x}$ = sample mean
S = Unbiased std. dev. of sample
s = std. dev. of sample
μ = Population mean
n = sample size.

Step 4: Critical Value



Area of Acceptance of Ho — Two Tailed Test — Area of Rejecting Ho

left tailed Test — One Tailed Test — Right Tailed Test

Step 5: Decision :

If t (computed) lies in Area of acceptance of Ho then Ho is accepted else Ho is rejected

* Testing the Hypothesis that the population mean is μ

1. A college professor wants to compare his students score with national average. He chooses a simple random sample of 20 students who score an average of 50.2 on a standardized test. Their score have a S.D. of 2.5. The national average on the test is 60. He wants to know if his students scored significantly lower than national average.

Sol:

$\bar{x} = 50.2$

$s = 2.5$

$\mu = 60$

$n = 20$

Step I: Null Hypothesis $H_0 : \mu = 60$

Alternate Hypothesis $H_a : \mu < 60$

Step II: Level of significance $= 5\%$ (If not given take it as 5%)

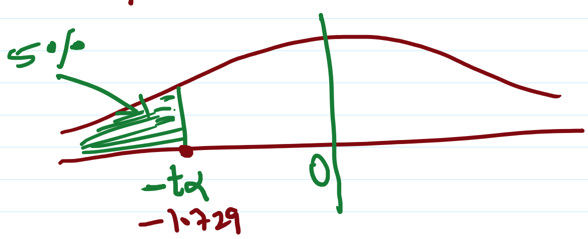
Step III: $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n-1}} = \dfrac{50.2 - 60}{2.5/\sqrt{20-1}} = -17.08$

Step IV: critical value

The critical value t_α at 5% level of significance with $n-1 = 20-1 = 19$ degrees of freedom is -1.729

$$\boxed{t_\alpha = -1.729}$$

Step V: Decision



$\because$ t(computed) $= -17.08$ lies in the region of rejection of Ho

$\therefore$ Ho is Rejected

$\therefore$ Ha is accepted

Q2

A random sample of 10 items gives mean 6.2 and standard deviation 10.24. Can it be reasonably regarded as a sample drawn from a large population having mean 5.4?

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Sol:

$\bar{x} = 6.2$

$S = 10.24$

$n = 10$

$\mu = 5.4$

Step I: Null Hypothesis $H_0: \mu = 5.4$

Alternate Hypo $H_a: \mu \neq 5.4$

Step II: Level of significance $= 5\%$

Step III:
$$t = \frac{\bar{x}-\mu}{S/\sqrt{n-1}} = \frac{6.2-5.4}{10.24/\sqrt{10-1}} = 0.234$$

Step IV: critical value:

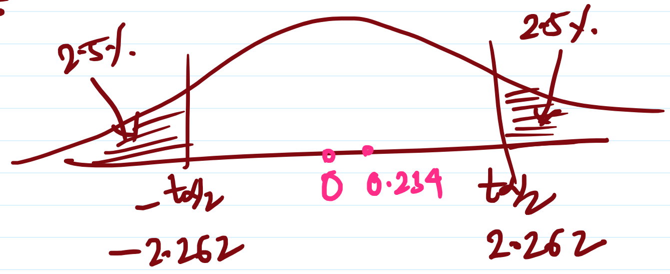
The critical value at 5% Level of significance with $n-1 = 10-1 = 9$ degrees of freedom is 2.262

$\therefore t_{\alpha/2} = 2.262$

Step 5 Decision

$\because$ t (computed) $= 0.0234$ lies in the region of

Step 5 Decision



$\therefore$ t (computed) = 0.0234 lies in the region of acceptance of Ho

$\therefore$ Ho is Accepted.

$\therefore$ Ha is Rejected

Q3

Ten individuals are chosen at random from a population and their heights are found to be 63,63,64,65,66,69,69,70,70,71 inches. Discuss the suggestion that the mean height of the universe is 65 inches. **(M.U.2003)**

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Sol: $n = 10$, $\bar{x} = 67$

$\mu = 65$, $s = 2.9664$

Step I: Null Hypothesis $H_0 : \mu = 65$

Alternate Hypo. $H_a : \mu \neq 65$

Step II: Level of significance = 5%

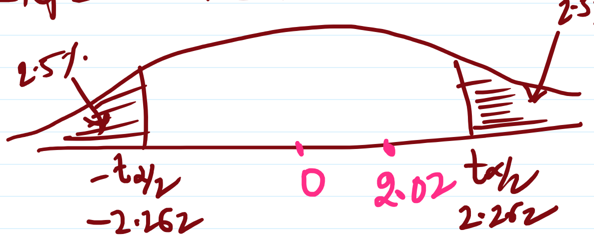
Step III: Test statistics

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{67 - 65}{2.9664/\sqrt{10-1}} = 2.02$$

Step IV: critical values

The critical value at 5% L.O.S. with $n-1 = 10-1 = 9$ d.o.f is $t_{\alpha/2} = 2.262$

Step V: Decision



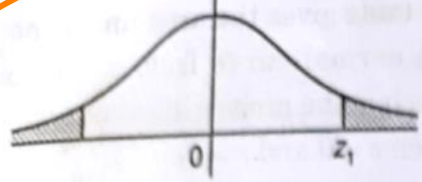
$\therefore$ t (computed) = 2.02 lies in the region of acceptance of Ho

$\therefore$ Ho is accepted.

T table

Tuesday, March 2, 2021 9:26 AM

Percentage Points of t - distribution



Example

For $\Phi = 10$ d. o. f.

$P(|t| > 1.812) = 0.1$

5% L.O.S.

one Tailed

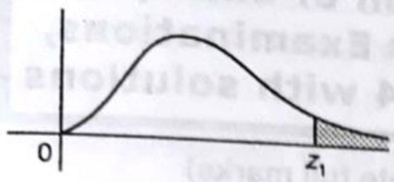
F \ P	0.20	0.10	0.05	0.02	0.01
1	3.078	6.314	12.706	31.812	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.287
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
[illegible]	[illegible]	[illegible]	[illegible]	2.467	2.763

26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
40	1.303	1.684	2.021	2.423	2.704
60	1.296	1.671	2.000	2.390	2.660
120	1.289	1.658	1.980	2.358	2.617
∞	1.282	1.645	1.960	2.325	2.576

Chi squate table

Tuesday, March 2, 2021 9:29 AM

Percentage Points of χ^2 - Distribution



Example

For Φ = 10 d. o. f.

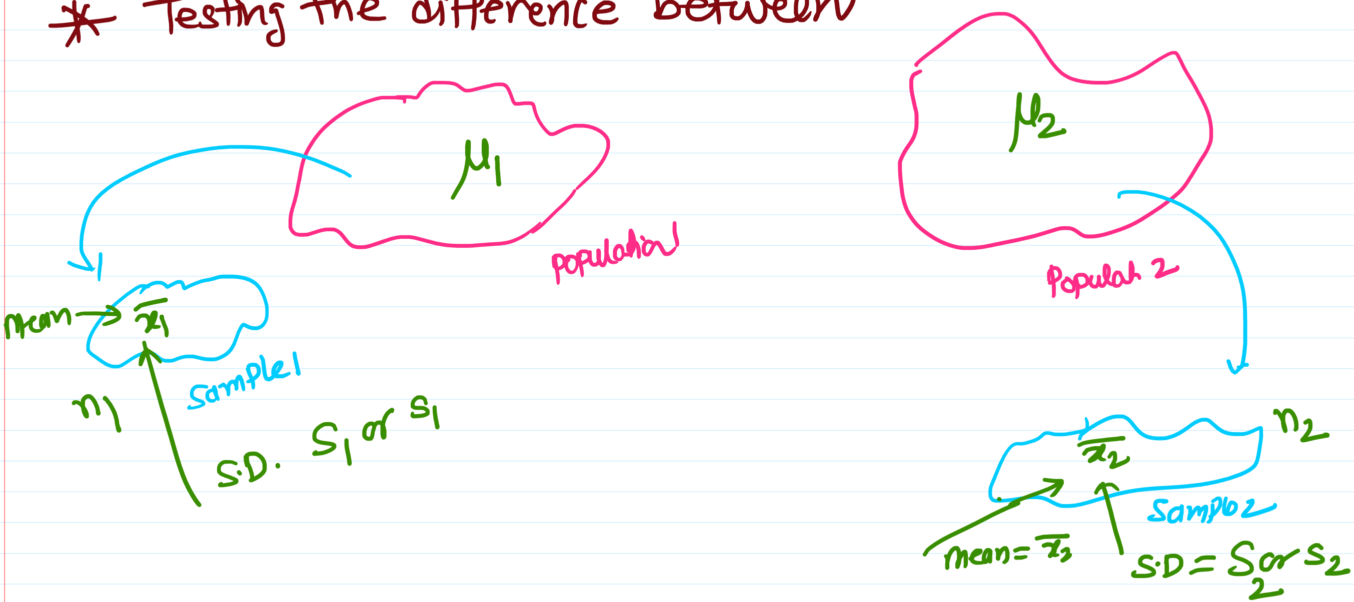
$P(\chi^2 > 15.99) = 0.10$

F \ P	0 = .99	0.95	0.50	0.10	0.05	0.02	0.01
1	.000157	.00393	.455	2.706	3.841	5.214	6.635
2	.0201	.103	1.386	4.605	5.991	7.824	9.210
3	.115	.352	2.366	6.251	7.815	9.837	11.341
4	.297	.711	3.357	7.779	9.488	11.668	13.277
5	.554	1.145	4.351	9.236	11.070	13.388	15.086
6	.872	1.635	5.348	10.645	12.592	15.033	16.812
7	1.339	2.167	6.346	12.017	14.067	16.622	18.475
8	1.646	2.733	7.344	13.362	15.507	18.168	20.090
9	2.088	3.325	8.343	14.684	16.919	19.679	21.666
10	2.558	3.940	9.340	15.987	18.307	21.161	23.209
11	3.053	4.575	10.341	17.275	19.675	22.618	24.725
12	3 571	5.226	11.340	18.549	21.026	24.054	26.217
13	4.107	5.892	12.340	19.812	22.362	25.472	27.688
14	4.660	6.571	13.339	21.064	23.685	26.873	29.141
15	4.229	7.261	14.339	22.307	24.996	28.259	30.578
16	5.812	7.962	15.338	23.542	26.296	29.633	32.000
17	6.408	8.672	16.338	24.769	27.587	30.995	33.409
18	7.015	9.390	17.338	25.989	28.869	32.346	34.805
19	7.633	10.117	18.338	27.204	30.144	33.687	36.191
20	8.260	10.851	19.337	28.412	31.410	35.020	37.566
21	8.897	11.591	20.337	29.615	32.671	36.349	38.932
22	9.542	12.338	21.337	30.813	33.924	37.659	40.289
23	10.196	13.091	22.337	32.007	35.172	38.968	41.638
24	10.856	13.848	23.337	32.196	36.415	40.270	42 980
25	11.524	14.611	24.337	34.382	37.652	41.566	44.314
26	12.198	15.379	25.336	35.363	38.885	41.856	45.642
27	12.879	16.151	26.336	36.741	40.113	44.140	46.963
28	13.565	16.928	27.336	37.916	41.337	45.419	48.278
29	14.256	17.708	28.336	39.087	42.557	46.693	49.588
30	14.953	18 493	29.336	40.256	43.773	47.962	50.892

29	14.256	17.708	28.336	39.087	[illegible]	[illegible]	[illegible]
30	14.953	18 493	29.336	40.256	43.773	47.962	50.892

Thursday, March 4, 2021 2:19 PM

* Testing the difference between



	Sample 1	Sample 2	
mean	$\bar{x}_1$	$\bar{x}_2$	
sample size	n_1	n_2	
Std. dev.	s_1	s_2	(small)
Unbiased std dev.	S_1	S_2	(capital)
sum of squares of deviation from mean	$\sum(x_{i1}-\bar{x}_1)^2$	$\sum(x_{i2}-\bar{x}_2)^2$	

* Null Hypothesis : $H_0 : \mu_1 = \mu_2$

H_a:

- $\mu_1 \neq \mu_2$ — Two Tail
- $\mu_1 < \mu_2$ — left tail
- $\mu_1 > \mu_2$ — Right Tail

* Test statistics

- sum of squares of deviation from mean is given
- Unbiased std. dev. is given OR (If you are using your
- only S.D. is given

deviation from mean is given

OR
(If you are using your calculator)

is given

$$Sp = \sqrt{\frac{\Sigma(x_{i1}-\bar{x}_1)^2 + \Sigma(x_{i2}-\bar{x}_2)^2}{n_1+n_2-2}}$$

$$Sp = \sqrt{\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}}$$

(capitals)

$$Sp = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}}$$

(smaller s)

$$\boxed{t = \frac{\bar{x}_1 - \bar{x}_2}{Sp \times \sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}}$$

1. The heights of six randomly chosen sailors are in inches 63, 65, 68, 69, 71 and 72. The heights of 10 randomly chosen soldiers are 61, 62, 65, 66, 69, 69, 70, 71, 72, 73.
Discuss in the light that these data throw on the suggestion that the soldiers on on average are taller than sailors

Sol:

Sailors	Soldiers
$n_1 = 6$	$n_2 = 10$
$\bar{x}_1 = 68$	$\bar{x}_2 = 67.8$
$S_1 = 3.46$	$S_2 = 4.13$

Step I: Null Hypothesis $H_0: \mu_1 = \mu_2$
Alternate Hypothesis $H_a: \mu_1 < \mu_2$

Step II: Level of significance $= 5\%$

Step III: Test statistics

$$Sp = \sqrt{(n_1-1)S_1^2 + (n_2-1)S_2^2}$$

$$S_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}$$

$$= \sqrt{\frac{(6-1)(3.46)^2 + (10-1)(4.13)^2}{6+10-2}}$$

$$= 3.903$$

$$t = \frac{\overline{x_1} - \overline{x_2}}{S_p \times \sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{68 - 67.8}{3.903 \times \sqrt{\frac{1}{6}+\frac{1}{10}}} = 0.099$$

Step 4: Critical Values

The critical value with 5% L.O.S. with $n_1+n_2-2 = 6+10-2 = 14$ d.o.f is

$$t_\alpha = -1.761$$

Step 5: Decision

∵ t (computed) = 0.099 lies in the region of acceptence of H_0

∴ H_0 is accepted.

② S

If two independent random samples of sizes 15 and 8 have respectively the following means and ~~population~~ Sample standard deviation,

$\overline{x_1} = 980$ $\quad \overline{x_2} = 1012$

$S_1 = 75$ $\quad S_2 = 80$ $\quad$ Test the hypothesis that $\mu_1 = \mu_2$ at 5% level of significance.

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→ Small

Sol: Step I: $H_0: \mu_1 = \mu_2$

$H_a: \mu_1 \neq \mu_2$

step II: Level of significance = 5%

step III: Test statistics

$$S_p = \sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}} = 80.34$$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S_p \times \sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = -0.90$$

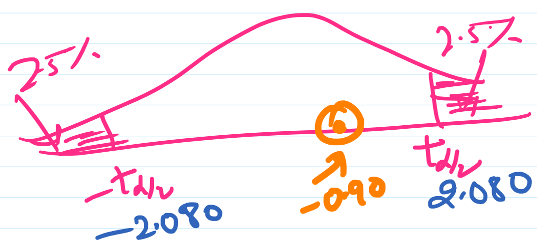
Step4: Critical Value:

$d.f = n_1+n_2-2 = 15+8-2 = 21$

L.O.S = 5%

$$\boxed{t_{\alpha/2} = 2.080}$$

Step5: decision



∴ $t_{(computed)} = -0.90$ lies in the region of acceptance of H_0

∴ H_0 is accepted.

(3) Six guinea pigs injected with 0.5 mg of a medication took on an average 15.4 secs to fall asleep with an unbiased standard deviation of 2.2 secs. While six other guinea pigs injected with 1.5 mg of the medication took on an average 11.2 secs. to fall asleep with an unbiased std. dev. 2.6 secs. Use 1% Level of significance to test the null hypothesis that the difference in the dosage has no effect.

Sol:

Step I: $H_0: \mu_1 = \mu_2$

$H_a: \mu_1 \neq \mu_2$

step2: L.O.S. = 1%

step3 test statistic

$$S_p = 2.408$$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\quad} = 3.02$$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p \times \sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = 3.02$$

Step 4: Critical value

$d.o.f = n_1 + n_2 - 2 = 6 + 6 - 2 = 10$

$L.O.S = 1\% \quad 5\%$

$t_{\alpha/2} = 3.169 \quad (2.228)$

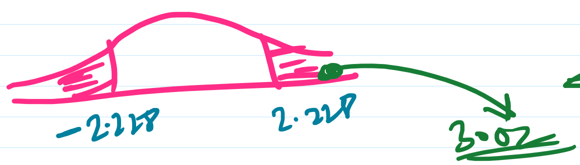
Step 5: Decision

$t(\text{computed}) = 3.02$

1% LOS

Ho is accepted

5%



Ho is Rejected

H.W

Q The means of two random samples of size 9 & 7 are 196.42 and 198.82 resp. The sum of squares of deviation from means re 26.94 & 18.73 resp. Can the samples be considered to have been drawn from the same population.

χ^2 - Test

χ^2 - chi square Test (χ^2 - kai)

- Non parametric Test
- Goodness of fit
- Association betn two or more variables
- Null Hypothesis : equal fit is good
  Alternate Hypo. not equal : fit is not good
- Test statistics

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

where O = observed freq.
E = Expected freq.

(2) 300 digits are chosen at random from a table of random numbers. The frequency of digits was as follows:

Digit	0	1	2	3	4	5	6	7	8	9	Total
Frequency	28	29	33	31	26	35	32	30	31	25	300

Using χ^2 Test examine the hypothesis that the digits were distributed in equal numbers in the table.

Sol:

Step I: Null Hypothesis H_0: digits are distributed in equal numbers

Alternate Hypo. H_a: digits are not distributed in equal numbers

Step II: Level of significance $(\alpha) = 5\%$

Step III: Test statistics.

O	E	$(O-E)$	$(O-E)^2$	$(O-E)^2/E$
28	30	-2	4	$4/30$

O	E	$(O-E)$	$(O-E)^2$	$(O-E)^2/E$
28	30	−2	4	4/30
29	30	−1	1	1/30
33	30	3	9	9/30
31	30	1	1	1/30
26	30	−4	16	16/30
35	30	5	25	25/30
32	30	2	4	4/30
30	30	0	0	0/30
31	30	1	1	1/30
25	30	−5	25	25/30

$$\chi^2 = \sum\left[\frac{(O-E)^2}{E}\right] = \frac{86}{30} = 2.87$$

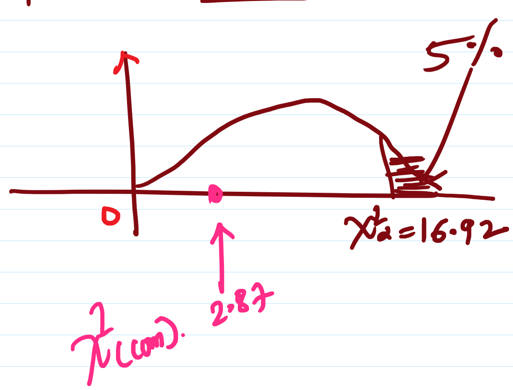
Step4: critical value

$d.o.f. = 10-1 = 9$

$L.O.S = 5\%$

The critical value with 9 d.o.f & with 5% L.O.S is 16.92

Step5: decision



χ^2(computed) $= 2.87$ lies in the region of acceptance of Ho

$\therefore$ Ho is accepted.

(1) The number of car accidents in a metropolitan city was found to be 20,17,12,6,7,15,8,5,16 and 14 per month respectively. Use χ^2 test to check whether these frequencies are in agreement with he belief that occurrence of accident was the same during 10 months period. Test at 5% level of significance.

Sol: Step1:

Ho : Accidents are equally distributed

Ha : Accidents are not equally distributed

Step II: $\alpha = 5\%$ (L.O.S.)

Step III: Test statistics

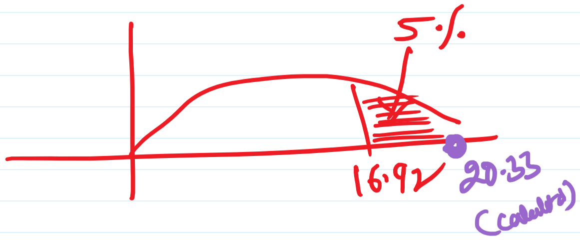
O	E	$O-E$	$(O-E)^2$	$(O-E)^2/E$

$\chi^2 = 20.33$

Step IV: critical value

The critical value with $10-1=9$ d.o.f with 5% L.O.S is 16.92

Step V: Decision



χ^2 (calculated) $= 20.33$ lies the region of Rejection of H_0

$\therefore$ H_0 is rejected

$\therefore$ H_a is accepted.

(3) A die was thrown 132 times and the following frequencies were observed.

No. Obtained	1	2	3	4	5	6	Total
Frequency	15	20	25	15	29	28	132

Test the Hypothesis that the die is unbiased.

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Sol:

Step 1: H_0: The die is unbiased

H_a: The die is not unbiased.

Step 2: Level of significance $= 5\%$

Step 3: Test statistics.

O	E	$O-E$	$(O-E)^2$	$(O-E)^2/E$
15	22	-7	49	49/22

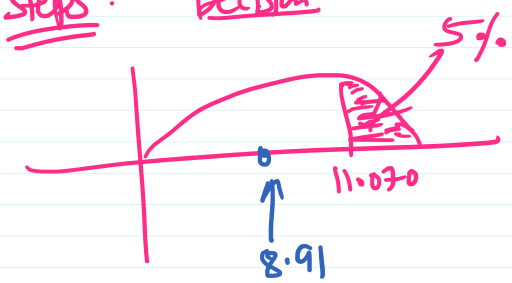
O	E	O−E	$(O-E)^2$	$(O-E)^2/E$
15	22	−7	49	49/22
20	22	−2	4	4/22
25	22	3	9	9/22
15	22	−7	49	49/22
29	22	7	49	49/22
28	22	6	36	36/22

$$\chi^2 = \frac{196}{22} = 8.91$$

Step4: Critical value

The critical value with $6-1=5$ d.of for 5% L.O.S is 11.070

Step5: Decision



χ^2(computed) = 8.91 lies in Region of acceptance of Ho

∴ Ho is accepted.

Type II: Association betn two variables

Ho: There is NO association

Ha: There is association

(6) In an experiment on immunisation of cattle from tuberculosis the folowing results were obtained.

	Affected	Not Affected	Total
Inoculated	267	27	294
Not Inoculated	757	155	912
Total	1024	182	1206

use χ^2 test to determine the efficacy of vaccine in preventing tuberculosis.

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Sol:

step 1: Ho: The vaccine is not efficient in preventing tuberculosis

Ha: The vaccine is efficient in preventing t.b.

Step2 L.O.S = 5%

Step3: Test statistics

O	E	(O−E)	$(O-E)^2$	$(O-E)^2/E$
267	$\frac{294\times1024}{1206} = 250$	17	289	289/250
27	$\frac{294\times182}{1206} = 44$	−17	289	289/44
757	$\frac{912\times1024}{1206} = 774$	−17	289	289/774
155	$\frac{182\times912}{1206} = 138$	17	289	289/138
				$\chi^2 = 10.19$

step4: critical value

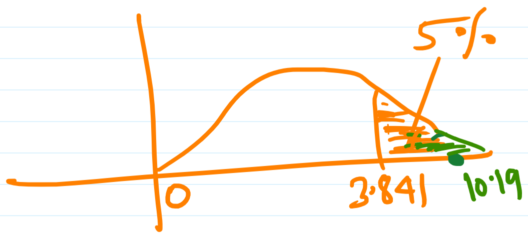
$$d.of = (r-1)(c-1) = (2-1)(2-1) = 1$$

$$L.O.S = 5\%$$

The Critical value with 1 d.of for 5% L.O.S is

3.841

Step5: decision



$\chi^2_{(computed)} = 10.19$

lies in the rejection of Ho

∴ Ho is rejected.

∴ Ha is accepted.

(5) Investigate the association between the darkness of eye colour in father and son from the following data

Colour of Father's eye

Colour of son's eye

	Dark	Not Dark	Total
Dark	48	90	138
Not Dark	80	782	862
Total	128	872	1000

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Sol: Step 1: Null Hypothesis H_0: There is No association betn eye color of father & son

Alternate Hypo H_a: There is association betn eye colour of father & son

Step 2: L.O.S = 5%

Step 3: Test statistics.

Z- Transform

* Some Useful series

① $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{r=0}^{\infty} \frac{x^r}{r!}$

② $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

③ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

④ $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \begin{cases} \frac{a(1-r^n)}{1-r}, & |r|<1 \\ \frac{a(r^n-1)}{r-1}, & |r|>1 \end{cases}$

Sum of finite terms of G.P.

⑤ $a + ar + ar^2 + \cdots = \frac{a}{1-r}, \quad |r|<1$

Sum of Infinite terms of G.P.

⑥ $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \cdots + {}^nC_n x^n$

Binomial Theorem

⑦ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots, \quad |x|<1$

⑧ $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots, \quad |x|<1$

$\boxed{\text{Series}}$

$\boxed{\text{Differentiation}}$ $\boxed{\text{Integration}}$

① $\frac{1}{1-x} = 1+x+x^2+x^3+\cdots, |x|<1$

diff. w.r.t. x, we get

② $\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\cdots, |x|<1$

diff w.r.t x we get

③ $\frac{1}{(1-x)^3} = 1+3x+6x^2+\cdots, |x|<1$

$\boxed{\text{Also}}$

ⓐ $\frac{1}{1+x} = 1-x+x^2-x^3+\cdots, |x|<1$

ⓑ $\frac{1}{(1+x)^2} = 1-2x+3x^2-4x^3 \cdots, |x|<1$

ⓒ $\frac{1}{(1+x)^3} = 1-3x+6x^2 \cdots, |x|<1$

① $\frac{1}{1-x} = 1+x+x^2+x^3+\cdots$

Integ. both sides, w.r.t. x, we get

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots, |x|<1$

$\boxed{\text{Also}}$

② $\frac{1}{1+x} = 1-x+x^2-x^3+\cdots$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$|x|<1$

Pascal's Triangle

$\frac{1}{1-x}$, $\frac{1}{(1-x)^2}$, $\frac{1}{(1-x)^3}$, $\frac{1}{(1-x)^4}$

1

1 2 1 $\rightarrow (1+x)^2$

1 3 3 1 $\rightarrow (1+x)^3$

1 4 6 4 1 $\rightarrow (1+x)^4$

1 5 10 10 5 1 $\rightarrow (1+x)^5$

1 6 15 20 15 6 1 $\rightarrow (1+x)^6$

Also

ⓐ $\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{}$

ⓒ $\sinh\alpha = \frac{e^{\alpha} - e^{-\alpha}}{2}$

(a) $\sin\alpha = \dfrac{e^{i\alpha} - e^{-i\alpha}}{2i}$

(c) $\sinh\alpha = \dfrac{e^{\alpha} - e^{-\alpha}}{2}$

(b) $\cos\alpha = \dfrac{e^{i\alpha} + e^{-i\alpha}}{2}$

(d) $\cosh\alpha = \dfrac{e^{\alpha} + e^{-\alpha}}{2}$

Z transform

The Z tranform of a sequence $\{f(k)\}$ is denoted as $Z\{f(k)\}$.

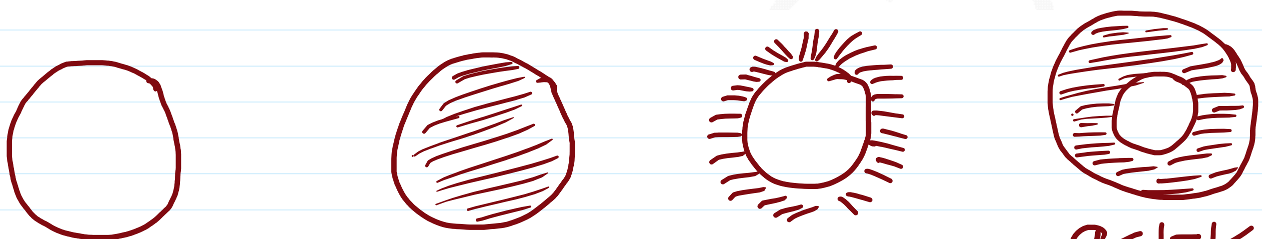
It is defined as $Z\{f(k)\} = F(z) = \sum\limits_{k=-\infty}^{k=\infty} f(k)z^{-k}$

where

1. Z is a complex number.
2. Z is an operator of Z transform .
3. $F(z)$ is the z transform of $\{f(k)\}$.

Region of Convergence

The series $\sum\limits_{k=-\infty}^{k=\infty} f(k)z^{-k}$ will be convergent only for certain value of z. The region in which the series is convergent is called the region of convergence of z- transform.



$|z| = a$ $\quad$ $|z| < a$ $\quad$ $|z| > a$ $\quad$ $a < |z| < b$

ROC: circle $\quad$ Interior of circle $\quad$ Exterior of circle $\quad$ Annular region

(1) Find the Z transform of the following sequence.

(a) $f(k) = \{1, 2, 5_{\downarrow}, 7, 0, 1\}$ where $\downarrow$ is used to denote the term in zero position

(b) $f(k) = \{1, 2_{\downarrow}, 5, 7, 0, 1\}$

(a) $f(k) = \{\underset{f(-2)}{1}, \underset{f(-1)}{2}, \underset{f(0)}{5_{\uparrow}}, \underset{f(1)}{7}, \underset{f(2)}{0}, \underset{f(3)}{1}\}$

$$\begin{array}{cccccc} -2 & -1 & \overset{\downarrow}{0} & 1 & 2 & 3 \end{array}$$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-2}^{3} f(k)\, z^{-k}$$

$$= f(-2)\, z^{2} + f(-1)\, z^{1} + f(0)\, z^{0} + f(1)\, z^{-1} + f(2)\, z^{-2} + f(3)\, z^{-3}$$

$$= 1z^{2} + 2z^{1} + 5z^{0} + 7z^{-1} + 0z^{-2} + 1z^{-3}$$

$$= z^{2} + 2z + 5 + \frac{7}{z} + \frac{1}{z^{3}}$$

Region of Convergence : Entire z-plane except at $z=0$

(b) $f(k) = \{1, \underset{\uparrow}{2}, 5, 7, 0, 1\}$

Sol:

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=-1}^{4} f(k)\, z^{-k}$$

$$= f(-1)\, z^{1} + f(0)\, z^{-0} + f(1)\, z^{-1} + f(2)\, z^{-2} + f(3)\, z^{-3} + f(4)\, z^{-4}$$

$$= 1z + 2z^{-0} + 5z^{-1} + 7z^{-2} + 0z^{-3} + 1z^{-4}$$

$$= z + 2 + \frac{5}{z} + \frac{7}{z^{2}} + \frac{1}{z^{4}}$$

Region of convergence: Entire z-plane except at $z=0$

(c) $f(k) = a^k, k \geq 0$

Sol:
$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$
$$= \sum_{k=0}^{\infty} a^k z^{-k}$$
$$= \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k$$
$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots\cdots$$

first term $= 1$, common ratio $= \frac{a}{z}$

$$= \frac{1}{1-\frac{a}{z}}, \quad \left|\frac{a}{z}\right| < 1$$
$$= \frac{z}{z-a}, \quad |a| < |z|$$

ROC: $|a| < |z|$ i.e. exterior of the circle

(d) $f(k) = b^k, k < 0$

(e) $f(k) = \begin{cases} a^k & k \geq 0 \\ b^k & k < 0 \end{cases}$

(f) $f(k) = k, k \geq 0$

(g) $f(k) = \frac{1}{k}, k \geq 1$

(h) $f(k) = \frac{a^k}{k!}, k \geq 0$

(i) $\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$

(j) $u(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$

Sol: (f) $$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} k z^{-k} = \sum_{k=0}^{\infty} \frac{k}{z^k}$$

$$= 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \cdots\cdots$$

$$= 1\left(\frac{1}{z}\right) + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + 4\left(\frac{1}{z}\right)^4 + \cdots\cdots$$

$$\frac{1}{(1-\theta)^2} = 1 + 2\theta + 3\theta^2 + 4\theta^3 + \cdots\cdots \quad , |\theta| < 1$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + 4\left(\frac{1}{z}\right)^3 + \cdots\cdots\right]$$

$$= \frac{1}{z}\left[\frac{1}{\left(1-\frac{1}{z}\right)^2}\right] \quad , \left|\frac{1}{z}\right| < 1$$

$$= \frac{z}{(z-1)^2} \quad , \left|\frac{1}{z}\right| < 1$$

$$|1| < |z|$$

ROC : Exterior of the circle $|z| > |1|$

(g) $f(k) = \frac{1}{k} \quad , k \geq 1$

Sol: $$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k} z^{-k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k z^k}$$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \frac{1}{5z^5} + \cdots\cdots$$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \frac{1}{5z^5} + \cdots$$

$$\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \frac{1}{4}\left(\frac{1}{z}\right)^4 + \cdots$$

$$\log(1-\theta) = -\theta - \frac{\theta^2}{2} - \frac{\theta^3}{3} - \frac{\theta^4}{4} - \cdots, \quad |\theta| < 1$$

$$-\log(1-\theta) = \theta + \frac{\theta^2}{2} + \frac{\theta^3}{3} + \frac{\theta^4}{4} + \cdots \quad |\theta| < 1$$

$$= -\log\left(1 - \frac{1}{z}\right), \quad \left|\frac{1}{z}\right| < 1$$

$$|1| < |z|$$

ROC : Exterior of the circle $|z| > |1|$

(h) $f(k) = \dfrac{a^k}{k!}, \quad k \geq 0$

Sol :

$$Z\{f(k)\} = \sum_{k=0}^{\infty} \frac{a^k}{k!} z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{a}{z}\right)^k$$

$$= 1 + \left(\frac{a}{z}\right)^1 + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \frac{1}{3!}\left(\frac{a}{z}\right)^3 + \cdots$$

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots$$

$$= e^{\frac{a}{z}}$$

ROC : Entire z-plane

(i) $\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$

$$\delta(k) = \{ \cdots 0, 0, 0, \underset{\uparrow}{1}, 0, 0, 0, \cdots \}$$

$$-3, -2, -1, 0, 1, 2, 3, \cdots$$

$$Z\{\delta(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= f(0) z^{-0} = 1 z^{-0} = 1$$

(j)

$$U(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$U(k) = \{ \cdots 0, 0, 0, 0, \underset{\uparrow}{1}, 1, 1, 1, 1, 1 \cdots \}$$

$$0 \; 1 \; 2 \; 3 \; 4 \; \cdots$$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$Z\{U(k)\} = \sum_{k=0}^{\infty} 1 z^{-k}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$= 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots$$

$$= \frac{1}{1 - \frac{1}{z}}, \quad \left|\frac{1}{z}\right| < 1$$

$$= \frac{z}{z-1}, \quad |1| < |z|$$

ROC: Exterior of the circle $|z| > |1|$

① $f(k) = k, \; k \geq 0$

Sol: $$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} k\, z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{k}{z^k}$$

$$= 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \cdots\cdots$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + 4\left(\frac{1}{z}\right)^4 + \cdots\right]$$

$$= \frac{1}{z}\, \frac{1}{\left(1-\frac{1}{z}\right)^2} = \frac{z}{(z-1)^2}$$

② $f(k) = \frac{1}{k}, \; k \geq 1$

Sol: $$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k} z^{-k} = \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{z}\right)^k$$

$$= \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \frac{1}{4}\left(\frac{1}{z}\right)^4 + \cdots$$

$$= -\log\left(1 - \frac{1}{z}\right), \quad \left|\frac{1}{z}\right| < 1$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots\cdots$$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

③ H.W $f(k) = \frac{1}{k+1}, \; k \geq 0$

④ H.W $f(k) = \frac{a^k}{k!}, \; k \geq 0$

⑤ H.W $f(k) = {}^{n}C_{k}, \; 0 \leq k \leq n$

Q Find Z transform of $\sin \alpha k, \; k \geq 0$

Sol: $$Z\{f(k)\} = \sum_{k=0}^{\infty} \sin \alpha k\, z^{-k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{e^{i\alpha k} - e^{-i\alpha k}}{2i}\right) z^{-k} \qquad \left(\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\right)$$

$$= \frac{1}{2i}\left[\sum_{k=0}^{\infty} e^{i\alpha k} z^{-k} - \sum_{k=0}^{\infty} e^{-i\alpha k} z^{-k}\right]$$

$$= \frac{1}{2i}\left[\sum_{k=0}^{\infty} \left(\frac{e^{i\alpha}}{z}\right)^k - \sum_{k=0}^{\infty} \left(\frac{e^{-i\alpha}}{z}\right)^k\right]$$

$$= \frac{1}{2i}\left\{\left[1 + \left(\frac{e^{i\alpha}}{z}\right) + \left(\frac{e^{i\alpha}}{z}\right)^2 + \cdots\right] - \left[1 + \left(\frac{e^{-i\alpha}}{z}\right) + \left(\frac{e^{-i\alpha}}{z}\right)^2 + \cdots\right]\right\}$$

$$\left\{\left[1+\left(\frac{e^{i\alpha}}{z}\right)+\left(\frac{e^{i\alpha}}{z}\right)^2+\cdots\right]-\left[1+\left(\frac{e^{-i\alpha}}{z}\right)+\left(\frac{e^{-i\alpha}}{z}\right)^2+\cdots\right]\right\}$$

$$= \frac{1}{2i}\left[\frac{1}{1-\frac{e^{i\alpha}}{z}} - \frac{1}{1-\frac{e^{-i\alpha}}{z}}\right] \qquad \left|\frac{e^{i\alpha}}{z}\right|<1,\ \left|\frac{e^{-i\alpha}}{z}\right|<1$$

$$= \frac{1}{2i}\left[\frac{z}{z-e^{i\alpha}} - \frac{z}{z-e^{-i\alpha}}\right], \quad |e^{i\alpha}|<|z|,\ |e^{-i\alpha}|<|z|$$

$$\frac{z}{2i}\left[\frac{1}{z-e^{i\alpha}}-\frac{1}{z-e^{-i\alpha}}\right]$$

$$= \frac{z}{2i}\left[\frac{(z-e^{-i\alpha})-(z-e^{i\alpha})}{z^2-(e^{i\alpha}+e^{-i\alpha})z+1}\right]$$

$$(z-e^{i\alpha})(z-e^{-i\alpha})$$

$$z^2-e^{-i\alpha}z-e^{i\alpha}z+e^{i\alpha}e^{-i\alpha}$$

$$z^2-(e^{-i\alpha}+e^{i\alpha})z+1$$

$$|e^{\pm i\alpha}|=1$$

$$= \frac{z}{2i}\left[\frac{e^{i\alpha}-e^{-i\alpha}}{z^2-(e^{i\alpha}+e^{-i\alpha})z+1}\right]$$

$$= \frac{z\sin\alpha}{z^2-2z\cos\alpha+1}, \quad 1<|z|$$

$$\boxed{\therefore \quad Z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2-2z\cos\alpha+1}}$$

Please Note:

1. $Z\{\sin\alpha k\} = \dfrac{z\sin\alpha}{z^2-2z\cos\alpha+1}$

2. $Z\{\cos\alpha k\} = \dfrac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}$

3. $Z\{\sinh\alpha k\} = \dfrac{z\sinh\alpha}{z^2-2z\cosh\alpha+1}$

4. $Z\{\cosh\alpha k\} = \dfrac{z(z-\cosh\alpha)}{z^2-2z\cosh\alpha+1}$

List of standard functions

1. $Z\{1\} = \frac{z}{z-1}$

2. $Z\{a^k\} = \frac{z}{z-a}, \; k \geq 0$

3. $Z\{a^k\} = \frac{-z}{z-a}, \; k < 0$

4. $Z\{\delta(k)\} = 1$ (dirac delta function)

5. $Z\{U(k)\} = \frac{z}{z-1}$ (discrete Unit step function)

6. $Z\{\sin \alpha k\} =$

7. $Z\{\cos \alpha k\} =$

8. $Z\{\sinh \alpha k\} =$

9. $Z\{\cosh \alpha k\} =$

* Properties of Z-Transform

1. Linearity

If $Z\{f_1(k)\} = F_1(z)$ with ROC : R_1 &

$Z\{f_2(k)\} = F_2(z)$ with ROC : R_2 then

$$Z\{a f_1(k) \pm b f_2(k)\} = a F_1(z) \pm b F_2(z) \text{ with ROC : } R_1 \cap R_2$$

eg 1. find $Z\{3(2^k) - 4(3^k)\}, \; k \geq 0$

sol:
$$Z\{3(2^k) - 4(3^k)\} = 3\, Z\{2^k\} - 4\, Z\{3^k\}$$
$$= 3\left[\frac{z}{z-2}\right] - 4\left[\frac{z}{z-3}\right]$$

$$= \frac{3z}{z-2} - \frac{4z}{z-3}$$

$$= z\left[\frac{3}{z-2} - \frac{4}{z-3}\right]$$

$$= z\left[\frac{3z-9-4z+8}{z^2-5z+6}\right]$$

$$= \frac{z(-z-1)}{z^2-5z+6}$$

$$= \frac{-z(z+1)}{z^2-5z+6}$$

② Find $Z\{\sin(3k+2)\}, k \geq 0$

Sol: $Z\{\sin(3k+2)\} = Z\{\sin 3k \cos 2 + \cos 3k \sin 2\}$

$$= \cos 2 \, Z\{\sin 3k\} + \sin 2 \, Z\{\cos 3k\}$$

$$= \cos 2\left[\frac{z \sin 3}{z^2-2z\cos 3+1}\right] + \sin 2\left[\frac{z(z-\cos 3)}{z^2-2z\cos 3+1}\right]$$

$$= \frac{z \sin 3 \cos 2 + z^2 \sin 2 - z \sin 2 \cos 3}{z^2-2z\cos 3+1}$$

$$= \frac{z(\sin 3 \cos 2 - \cos 3 \sin 2) + z^2 \sin 2}{z^2-2z\cos 3+1}$$

$$= \frac{z \sin 1 + z^2 \sin 2}{z^2-2z\cos 3+1}$$

② Change of scale.

If $Z\{f(k)\} = F(z)$ with ROC : R

then $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$ with ROC : $|a|R$

eg ① Find $Z\{\alpha^k \sin\alpha k\}$, $k \geq 0$

Sol :

$$Z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$$

$$Z\{\alpha^k \sin\alpha k\} = \frac{\frac{z}{\alpha}\sin\alpha}{\left(\frac{z}{\alpha}\right)^2 - 2\left(\frac{z}{\alpha}\right)\cos\alpha + 1}$$

$$= \frac{\frac{z\sin\alpha}{\alpha}}{\frac{z^2 - 2z\alpha\cos\alpha + \alpha^2}{\alpha^2}}$$

$$= \frac{z\sin\alpha}{\alpha} \times \frac{\alpha^2}{z^2 - 2z\alpha\cos\alpha + \alpha^2}$$

$$= \frac{\alpha z\sin\alpha}{z^2 - 2z\alpha\cos\alpha + \alpha^2}$$

② Find $Z\{2^k \cosh\alpha k\}$

Sol :

$$Z\{\cosh\alpha k\} = \frac{z(z - \cosh\alpha)}{z^2 - 2z\cosh\alpha + 1}$$

$$Z\{2^k \cosh\alpha k\} = \frac{\frac{z}{2}\left(\frac{z}{2} - \cosh\alpha\right)}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cosh\alpha + 1}$$

③ Multiplication By K

If $Z\{f(k)\} = F(z)$ with ROC : R then

$$Z\{k\, f(k)\} = -z \frac{d}{dz} F(z)$$

① Find $Z\{k \cdot a^k\}$, $k \geq 0$

Sol:

$$Z\{a^k\} = \frac{z}{z-a}$$

$$Z\{k\, a^k\} = -z \frac{d}{dz}\left(\frac{z}{z-a}\right)$$

$$= -z\left[\frac{(z-a)(1)-(z)(1)}{(z-a)^2}\right]$$

$$= -z\left[\frac{z-a-z}{(z-a)^2}\right] = \frac{az}{(z-a)^2}$$

② Find $Z\{k^2\}$, $k \geq 0$

Sol:

$$Z\{k\} = \sum_{k=0}^{\infty} k z^{-k} = 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^2} + \frac{4}{z^3} + \cdots$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + 4\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= \frac{1}{z}\left[\frac{1}{\left(1-\frac{1}{z}\right)^2}\right] = \frac{z}{(z-1)^2}$$

$k \cdot k$

$$Z\{k^2\} = -z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2(1) - z[2(z-1)]}{(z-1)^4}\right]$$

$$= -z\left[\frac{(z-1) - 2z}{(z-1)^3}\right]$$

$$= -z\left[\frac{-z-1}{\quad}\right] = \frac{z(z+1)}{\quad}$$

$$= -z\left[\frac{-z-1}{(z-1)^3}\right] = \frac{z(z+1)}{(z-1)^3}$$

Property No. 4

Time shifting

If $Z\{f(k)\} = F(z)$ with ROC : R_1

then $Z\{f(k \pm n)\} = z^{\pm n} F(z)$

① $Z\{\delta(k-n)\}$

Sol:

$$Z\{\delta(k)\} = 1$$

$$Z\{\delta(k-n)\} = z^{-n} \cdot 1 = z^{-n}$$

② $Z\{a^k \delta(k-n)\}$

Sol:

$$Z\{\delta(k)\} = 1$$

$$Z\{\delta(k-n)\} = z^{-n} \cdot 1 = z^{-n}$$

$$Z\{a^k \delta(k-n)\} = \left(\frac{z}{a}\right)^{-n} = \left(\frac{a}{z}\right)^{n}$$

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Property No. 5

Convolution Theorem

If $Z\{f_1(k)\} = F_1(z)$ with ROC : R_1 &

$Z\{f_2(k)\} = F_2(z)$ with ROC : R_2

then $Z\{f_1(k) * f_2(k)\} = F_1(z) \cdot F_2(z)$

where $f_1(k) * f_2(k) = \sum_{n=-\infty}^{\infty} f_1(n) f_2(n-k)$

① Using convolution Theorem find z transform of $f_1(k) * f_2(k)$ where $f_1(k) = \{2,3,4\}$, $f_2(k) = \{-1,2,3\}$

Sol:

$$F_1(z) = Z\{f_1(k)\} = \sum_{k=0}^{2} f_1(k) z^{-k} = 2z^0 + 3z^{-1} + 4z^{-2} = 2 + \frac{3}{z} + \frac{4}{z^2}$$

$$F_2(z) = Z\{f_2(k)\} = \sum_{k=0}^{2} f_2(k) z^{-k} = -1z^{-0} + 2z^{-1} + 3z^{-2} = -1 + \frac{2}{z} + \frac{3}{z^2}$$

∴ By convolution Theorem

$$Z\{f_1(k) * f_2(k)\} = F_1(z) \cdot F_2(z)$$
$$= \left(2 + \frac{3}{z} + \frac{4}{z^2}\right)\left(-1 + \frac{2}{z} + \frac{3}{z^2}\right)$$
$$= -2 + \frac{4}{z} + \frac{6}{z^2} - \frac{3}{z} + \frac{6}{z^2} + \frac{9}{z^3} - \frac{4}{z^2} + \frac{8}{z^3} + \frac{12}{z^4}$$
$$= -2 + \frac{1}{z} + \frac{8}{z^2} + \frac{17}{z^3} + \frac{12}{z^4}$$

② Using convol. Thm find $Z\{f_1(k) * f_2(k)\}$ where

$$f_1(k) = \frac{1}{3^k},\ k \geq 0, \quad f_2(k) = \frac{1}{4^k},\ k \geq 0$$

Sol:

$$F_1(z) = Z\{f_1(k)\} = Z\left\{\frac{1}{3^k}\right\} = Z\left\{\left(\frac{1}{3}\right)^k\right\}$$
$$= \frac{z}{z - \frac{1}{3}} = \frac{3z}{3z-1}$$

similarly $F_2(z) = \frac{4z}{4z-1}$

$\therefore$ By convolution Theorem

$$Z\{f_1(k) * f_2(k)\} = F_1(z) * F_2(z)$$

$$= \left(\frac{3z}{3z-1}\right)\left(\frac{4z}{4z-1}\right) = \frac{12z^2}{12z^2-7z+1}$$

③ $f_1(k) = \left(\frac{1}{2}\right)^k, \ k \geq 0,$ $\quad f_2(k) = \cos \pi k, \ k \geq 0$

Sol:

$$F_1(z) = Z\{f_1(k)\}$$
$$= Z\left\{\left(\tfrac{1}{2}\right)^k\right\}$$
$$= \frac{z}{z-\frac{1}{2}}$$
$$= \frac{2z}{2z-1}$$

$$F_2(z) = Z\{\cos \pi k\}$$
$$= \frac{z(z-\cos\pi)}{z^2-2z\cos\pi+1}$$
$$= \frac{z(z+1)}{z^2+2z+1} \quad (\cos\pi = -1)$$
$$= \frac{z}{(z+1)}$$

$\therefore$ By convol. thm.

$$Z\{f_1(k) * f_2(k)\} = F_1(z) F_2(z)$$
$$= \left(\frac{2z}{2z-1}\right)\left(\frac{z}{z+1}\right) = \frac{2z^2}{2z^2+2z-z-1} = \frac{2z^2}{2z^2+z-1}$$

H.W

④ $f_1(k) = u(k)$, $f_2(k) = \delta(k) + \left(\frac{1}{2}\right)^k u(k)$

* Inverse Z-Transform *

If $Z\{f(k)\} = F(z)$ then $f(k)$ is called an inverse z-transform of $F(z)$ and symbolically written as

$$f(k) = Z^{-1}\{F(z)\}$$

where Z^{-1} is called inverse z-operator

Inverse z-transform can be found by foll. method.

(a) Long division
(b) Binomial Expansion
(c) Partial Fractions

(1) Find the inverse z transform of following functions
(i) $\frac{z}{z-a}$ (a) $|z|>a$ (b) $|z|<a$

Sol:

(a) ROC : $|z|>a$

$$\frac{1}{|z|} < \frac{1}{a}$$
$$\frac{a}{|z|} < 1$$
$$\left|\frac{a}{z}\right| < 1$$

$$F(z) = \frac{z}{z-a}$$
$$= \frac{z}{z\left(1-\frac{a}{z}\right)}$$
$$= \frac{1}{1-\frac{a}{z}}$$
$$= 1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\left(\frac{a}{z}\right)^3+\cdots\cdots$$
$$= 1+az^{-1}+a^2z^{-2}+a^3z^{-3}+\cdots\cdots$$
$$= \sum_{k=0}^{\infty} a^k z^{-k} = Z\{a^k\}$$

$$\therefore f(k) = a^k \quad, k \geq 0$$

(b) ROC : $|z|<a$

$$\frac{|z|}{a} < 1$$
$$\left|\frac{z}{a}\right| < 1$$

$$F(z) = \frac{z}{z-a}$$
$$= \frac{z}{-(a-z)}$$
$$= \frac{-z}{a\left(1-\frac{z}{a}\right)}$$
$$= -\frac{z}{a}\cdot\frac{1}{1-\frac{z}{a}}$$
$$= -\frac{z}{a}\left[1+\frac{z}{a}+\left(\frac{z}{a}\right)^2+\left(\frac{z}{a}\right)^3+\cdots\right]$$
$$= -\left[\frac{z}{a}+\left(\frac{z}{a}\right)^2+\left(\frac{z}{a}\right)^3+\left(\frac{z}{a}\right)^4+\cdots\right]$$
$$= -\left[a^{-1}z+a^{-2}z^2+a^{-3}z^3+\cdots\right]$$
$$= -\sum_{k=-1}^{-\infty} a^k z^{-k} \quad, k<0$$
$$= Z\{-a^k\}, \quad k<0$$

$$\therefore f(k) = -a^k \quad, k<0$$

(2) Find the inverse z-transform of

$F(z) = \frac{1}{(z-a)^2}$ (a) $|z|>a$ (b) $|z|<a$

Sol: (a) $|z|>a$ (b) $|z|<a$

$f(z) = \frac{1}{(z-a)^2}$ (a) $|z| > a$ (b) ...

Sol: (a) $|z| > a$

$$\frac{1}{|z|} < \frac{1}{a}$$

$$\left|\frac{a}{z}\right| < 1 \qquad \frac{1}{1 \pm \frac{a}{z}}$$

$$F(z) = \frac{1}{(z-a)^2}$$

$$= \frac{1}{z^2\left(1-\frac{a}{z}\right)^2}$$

$$= \frac{1}{z^2}\left[\frac{1}{\left(1-\frac{a}{z}\right)^2}\right]$$

$$= \frac{1}{z^2}\left[1 + 2\left(\frac{a}{z}\right) + 3\left(\frac{a}{z}\right)^2 + 4\left(\frac{a}{z}\right)^3 + \cdots\right]$$

$$= \frac{1}{z^2} + \frac{2a}{z^3} + \frac{3a^2}{z^4} + \frac{4a^3}{z^5} + \cdots$$

k^{th} term $= \frac{(k-1)a^{k-2}}{z^k}, \; k \geq 2$

$$= \sum_{k=2}^{\infty} (k-1)a^{k-2} z^{-k}$$

$$= Z\{(k-1)a^{k-2}\}, \; k \geq 2$$

$$\therefore \; f(k) = (k-1)a^{k-2}, \; k \geq 2$$

(b) $|z| < a$

$$\frac{|z|}{a} < 1$$

$$\left|\frac{z}{a}\right| < 1 \rightarrow \frac{1}{1 \pm \frac{z}{a}}$$

$$F(z) = \frac{1}{(z-a)^2} = \frac{1}{(a-z)^2}$$

$$= \frac{1}{a^2\left(1-\frac{z}{a}\right)^2}$$

$$= \frac{1}{a^2}\left[\frac{1}{\left(1-\frac{z}{a}\right)^2}\right]$$

$$= \frac{1}{a^2}\left[1 + \frac{2z}{a} + 3\left(\frac{z}{a}\right)^2 + 4\left(\frac{z}{a}\right)^3 + \cdots\right]$$

$$= \frac{1}{a^2} + \frac{2z}{a^3} + \frac{3z^2}{a^4} + \frac{4z^3}{a^5} + \cdots$$

k^{th} term $= \frac{(k+1)z^k}{a^{k+2}}, \; k \geq 0$

Replace k with $-k$, we get

$$= \frac{(-k+1)z^{-k}}{a^{-k+2}}, \; k \leq 0$$

$$= \sum_{k=0}^{-\infty} \frac{-k+1}{a^{-k+2}} z^{-k}$$

$$\therefore \; Z\left\{\frac{-k+1}{a^{-k+2}}\right\}$$

$$\therefore \; f(k) = \frac{-k+1}{a^{-k+2}}, \; k \leq 0$$

* Problems Based on Partial Fractions *

1. Find the inverse Z transform of

$$\frac{z}{(z-2)(z-3)}, \; |z| > 3$$

Sol: $F(z) = \frac{z}{(z-2)(z-3)} = z\left[\frac{1}{(z-2)(z-3)}\right] = z\left[\frac{1}{z-3} - \frac{1}{z-2}\right]$

$$= \frac{z}{z-3} - \frac{z}{z-2}$$

$\because |z| > 3$ Also $|z| > 2$

$\frac{1}{|z|} < \frac{1}{3}$ $\frac{1}{|z|} < \frac{1}{2}$

$\left|\frac{3}{z}\right| < 1$ $\left|\frac{2}{z}\right| < 1$

$$= \frac{z}{z\left(1-\frac{3}{z}\right)} - \frac{z}{z\left(1-\frac{2}{z}\right)}$$

$$= \frac{1}{1-\frac{3}{z}} - \frac{1}{1-\frac{2}{z}}$$

$$= \left[1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \cdots\right] - \left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right]$$

$$= \sum_{k=0}^{\infty}\left(\frac{3}{z}\right)^k - \sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k$$

$$= \sum_{k=0}^{\infty} 3^k z^{-k} - \sum_{k=0}^{\infty} 2^k z^{-k}$$

$$= \sum_{k=0}^{\infty} 3^k z^{-k} - \sum_{k=0}^{\infty} 2^k z^{-k}$$

$$= Z\{3^k\} - Z\{2^k\}, \quad k \geq 0$$

$$= Z\{3^k - 2^k\}, \quad k \geq 0$$

$$\therefore f(k) = 3^k - 2^k, \quad k \geq 0$$

Q2 Find the inverse z transform of the function

$$F(z) = \frac{z^2}{(z-1)(z-\frac{1}{2})}$$

(a) $|z| > 1$

(b) $|z| < 1/2$

(c) $\frac{1}{2} < |z| < 1$

Sol:

$$F(z) = \frac{z^2}{(z-1)(z-\frac{1}{2})}$$

$$\frac{F(z)}{z} = \frac{z}{(z-1)(z-\frac{1}{2})}$$

Let $\frac{z}{(z-1)(z-\frac{1}{2})} = \frac{A}{z-1} + \frac{B}{z-\frac{1}{2}}$

$$A = \frac{1}{1-1/2} = 2$$

$$B = \frac{1/2}{\frac{1}{2}-1} = \frac{1/2}{-1/2} = -1$$

$$= \frac{2}{z-1} - \frac{1}{z-\frac{1}{2}}$$

$$\therefore F(z) = \frac{2z}{z-1} - \frac{z}{z-\frac{1}{2}}$$

(1) ROC: $|z| > 1$

$\because |z| > 1 \quad \therefore |z| > 1/2$

$\left|\frac{1}{z}\right| < 1 \qquad \left|\frac{1}{z}\right| < 2 \Rightarrow \left|\frac{1}{2z}\right| < 1$

$$F(z) = \frac{2z}{z-1} - \frac{z}{z-\frac{1}{2}}$$

$$= \frac{2z}{z(1-\frac{1}{z})} - \frac{z}{z(1-\frac{1}{2z})}$$

$$= \frac{2}{1-\frac{1}{z}} - \frac{1}{1-\frac{1}{2z}}$$

$$= 2\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots\right] - \left[1 + \left(\frac{1}{2z}\right) + \left(\frac{1}{2z}\right)^2 + \cdots\right]$$

$$= 2\left[\sum_{k=0}^{\infty} \left(\frac{1}{z}\right)^k\right] - \left[\sum_{k=0}^{\infty} \left(\frac{1}{2z}\right)^k\right]$$

$$= 2\left[\sum_{k=0}^{\infty} 1^k z^{-k}\right] - \left[\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^{-k}\right], \quad k \geq 0$$

$$= 2\left[Z\{1\}\right] - \left[Z\left\{\left(\frac{1}{2}\right)^k\right\}\right]$$

$$= 2-\left(\frac{1}{2}\right)^k, \quad k \geq 0$$

$$\therefore f(k) = 2-\left(\frac{1}{2}\right)^k, \quad k \geq 0$$

(b) ROC: $|z| < 1/2$

Sol: $|z| < \frac{1}{2} \quad \therefore |z| < 1$

$|2z| < 1 \quad \therefore |z| < 1$

$$F(z) = \frac{2z}{z-1} - \frac{z}{z-\frac{1}{2}}$$

$$= \frac{-2z}{1-z} - \frac{2z}{2z-1}$$

$$= \frac{-2z}{1-z} + \frac{2z}{1-2z}$$

$$= -2z\left[\frac{1}{1-z}\right] + 2z\left[\frac{1}{1-2z}\right]$$

$$= -2z\left[1+z+z^2+z^3+\cdots\right] + 2z\left[1+2z+(2z)^2+(2z)^3+\cdots\right]$$

$$= -2\left[z+z^2+z^3+z^4+\cdots\right] + 2\left[z+2z^2+2^2z^3+2^3z^4+\cdots\right]$$

$$= -2\left[\sum_{k=1}^{\infty} z^k\right] + 2\left[\sum_{k=1}^{\infty} 2^{k-1} z^k\right]$$

$$= \sum_{k=1}^{\infty} -2z^k + \sum_{k=1}^{\infty} 2^k z^k$$

Replace k with $-k$, we get

$$= \sum_{k=-1}^{-\infty} -2z^{-k} + \sum_{k=-1}^{-\infty} 2^{-k} z^{-k}$$

$$= Z\{-2\} + Z\{2^{-k}\}, \quad k < 0$$

$$= Z\{-2+2^{-k}\}, \quad k < 0$$

$$\therefore f(k) = -2+2^{-k}, \quad k < 0$$

Inverse - Z transform

① find the inverse z-transform of $\frac{z^3}{(z-3)(z-2)^2}$, $|z|>3$

sol:

$$F(z) = \frac{z^3}{(z-3)(z-2)^2}, \quad |z|>3$$

$$\frac{F(z)}{z} = \frac{z^2}{(z-3)(z-2)^2}, \quad |z|>3$$

$$\frac{z^2}{(z-3)(z-2)^2} = \frac{A}{z-3} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$$z^2 = A(z-2)^2 + B(z-2)(z-3) + C(z-3)$$

put $z=3 \implies 9 = A$

put $z=2 \implies 4 = 0+0+C(-1) \implies C=-4$

put $z=0 \implies 0 = 4A+6B-3C$

$$0 = 4\times9 + 6B - 3\times-4$$

$$0 = 36 + 6B + 12$$

$$-48 = 6B \quad \therefore B = -8$$

$\therefore$
$$\frac{z^2}{(z-3)(z-2)^2} = \frac{9}{z-3} + \frac{-8}{z-2} + \frac{-4}{(z-2)^2}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{9}{z-3} - \frac{8}{z-2} - \frac{4}{(z-2)^2}$$

$$\therefore \quad F(z) = 9\left(\frac{z}{z-3}\right) - 8\left(\frac{z}{z-2}\right) - 4\left(\frac{z}{(z-2)^2}\right), \quad |z|>3$$

$$\therefore \quad z^{-1}\{F(z)\} = 9\, z^{-1}\left\{\frac{z}{z-3}\right\} - 8\, z^{-1}\left\{\frac{z}{z-2}\right\} - 4\left\{\frac{z}{(z-2)^2}\right\}$$

$$= 9(3^k) - 8(2^k) - 4k\,2^k, \quad k \geq 0$$

$$f(k) = 3^{k+2} - 2^{k+3} - k\,2^{k+2}, \quad k \geq 0$$

② $F(z) = \dfrac{z^2}{(z+2)(z^2+4)}, \quad |z|>2$

Sol: $\dfrac{F(z)}{z} = \dfrac{z}{(z+2)(z^2+4)}, \quad |z|>2$

$$\therefore \quad \frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{Bz+C}{z^2+4}, \quad |z|>2$$

$$z = A(z^2+4) + (Bz+C)(z+2)$$

put $z=-2$ $\quad \therefore -2 = A(4+4) + 0 \quad \therefore A = \frac{-2}{8} = \frac{-1}{4}$

put $z=0$ $\quad \therefore 0 = 4A + 2C$

$$0 = 4\left(\frac{-1}{4}\right) + 2C \quad \therefore C = \frac{1}{2}$$

put $z=1$ $\quad \therefore 1 = 5A + (B+C)(3)$

$$= 5\left(\frac{-1}{4}\right) + \left(B+\frac{1}{2}\right)(3)$$

$$-\frac{5}{4} + \frac{3\times 2}{2\times 2}$$

$$= -\frac{5}{4} + 3B + \frac{3}{2}$$

$$= \frac{1}{4} + 3B \qquad \therefore 3B = 1 - 1/4$$

$$3B = 3/4$$

$$B = 1/4$$

$$\therefore \frac{f(z)}{z} = \frac{-1/4}{z+2} + \frac{\frac{1}{4}z + \frac{1}{2}}{z^2+4}$$

$$\therefore \frac{f(z)}{z} = -\frac{1}{4} \cdot \frac{1}{z+2} + \frac{1}{4} \cdot \frac{z}{z^2+4} + \frac{1}{2} \cdot \frac{1}{z^2+4}$$

$$\therefore f(z) = -\frac{1}{4}\left[\frac{z}{z+2}\right] + \frac{1}{4}\left[\frac{z^2}{z^2+4}\right] + \frac{1}{2}\left[\frac{z}{z^2+4}\right]$$

$$= -\frac{1}{4}\left[\frac{z}{z-(-2)}\right] + \frac{1}{4}\left[\frac{(z/2)^2}{\left(\frac{z}{2}\right)^2+1}\right] + \frac{1}{4}\left[\frac{z/2}{\left(\frac{z}{2}\right)^2+1}\right]$$

$$z\left\{\frac{z}{z+a}\right\} = -a^k, \quad k \geq 0$$

Also

$$z\{\sin \alpha k\} = \frac{z \sin\alpha}{z^2 - 2z\cos\alpha + 1}$$

$$z\{\cos \alpha k\} = \frac{z(z - \cos\alpha)}{z^2 - 2z\cos\alpha + 1}$$

$$= -\frac{1}{4}(-2)^k + \frac{1}{4}\left[2^k \cos k\frac{\pi}{2}\right] + \frac{1}{4}\left[2^k \sin k\frac{\pi}{2}\right]$$

Non Linear Programming Problems (NLPP)

2.2 Equality Constrainted Problems

Langarange's Multiplier Method

(a) Optimize $Z = f(x)$
subject to $g_i(x) = 0$
where $x = x_1, x_2, x_3 \ldots, x_n$ (n=no. of variables)
$g = g_1, g_2, g_3 \ldots, g_m$ (m=no. of constraints)

$L(x,\lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x)$ is called Langrange's function

(b) the necessary condition for Z to have extreme values are

(1) $\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0$ and so on.

(2) $\frac{\partial L}{\partial \lambda} = 0 \Rightarrow \frac{\partial L}{\partial \lambda_1} = 0, \quad \frac{\partial L}{\partial \lambda_2} = 0$ and so on.

(c) Solving these equations, we get different set of values of $x_1, x_2, x_3 \ldots, x_n$

(d) Let $P \equiv (x_1, x_2, x_3 \ldots, x_n)$be one of the points. At the point P, find the bordered Hessian matrix given by

$$H_B = \begin{bmatrix} O & P \\ P^t & Q \end{bmatrix}_{(m+n)\times(m+n)}$$

where
$P^t =$ transpose of P
$O =$ Null matrix of order $m \times m$
$Q =$ Hessian Matrix of $L(x,\lambda)$
Also find the sign of $(-1)^m$ and $(-1)^{m+1}$

$$P = \begin{bmatrix} \nabla g_1(x) \\ \nabla g_2(x) \\ \nabla g_3(x) \\ \vdots \\ \nabla g_m(x) \end{bmatrix}_{(n\times n)}$$

(e) (i) If starting with principle minor determinant of order $(2m+1)$, the last $(n-m)$ principle minor determinant of H_B have the same sign of $(-1)^m$ then there is minimum value.

(ii) If starting with principle minor determinant of order $(2m+1)$, the last $(n-m)$ principle minor determinant of H_B have an alternate pattern starting with a sign of $(-1)^{m+1}$ then there is maximum value.

* Hessian Matrix *

$$Z = f(x_1,x_2,x_3) = x_1^2 + x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 - 8x_2x_3$$

$$\frac{\partial z}{\partial x_1} = 2x_1 - 2x_2 + 4x_3 \quad \therefore \frac{\partial^2 z}{\partial x_1^2} = 2$$

$$\frac{\partial z}{\partial x_2} = 2x_2 - 2x_1 - 8x_3 \quad \therefore \frac{\partial^2 z}{\partial x_2^2} = 2$$

$$\frac{\partial z}{\partial x_3} = 6x_3 + 4x_1 - 8x_2 \quad \therefore \frac{\partial^2 z}{\partial x_3^2} = 6$$

$$\frac{\partial f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left[\frac{\partial f}{\partial x_2}\right] = \frac{\partial}{\partial x_1}\left[2x_2 - 2x_1 - 8x_3\right]$$

$$Q = \begin{array}{c|ccc} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial}{\partial x_3} & \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{array} = \begin{bmatrix} 2 & -2 & 4 \\ -2 & 2 & -8 \\ 4 & -8 & 6 \end{bmatrix}$$

(1) **Optimize** : $Z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$
Subject to : $x_1 + x_2 + x_3 = 20$
$x_1, x_2, x_3 \geq 0$ **(M.U. 1996,99,2004,06)** **Ans.** $[Z_{min} = 281$ at $x_1 = 5, x_2 = 11, x_3 = 4]$

Sol:

(a) Given: No. of variables $(x_1, x_2, x_3) = n = 3$
No. of constraint $(g_i) = m = 1$

The Langranges function is given by

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} g_i(x)$$

$$= (2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100) - \lambda(x_1 + x_2 + x_3 - 20)$$

(b) The necessary condition for z to have stationary values are

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 4x_1 + 10 - \lambda = 0 \quad \text{--- (1)}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 + 8 - \lambda = 0 \quad \text{--- (2)}$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 6x_3 + 6 - \lambda = 0 \quad \text{--- (3)}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 0 - (x_1 + x_2 + x_3 - 20) = 0$$

$$\Rightarrow x_1 + x_2 + x_3 = 20 \quad \text{--- (4)}$$

(c) Solving (1), (2), (3) & (4), we get

$$x_1 = 5, \; x_2 = 11, \; x_3 = 4, \; \lambda = 30$$

$$\therefore p = (x_1, x_2, x_3) = (5, 11, 4)$$

(d) $O_{m \times m} = O_{1 \times 1}$ = Null matrix of order 1

$$Q = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}_{3 \times 3} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}_{3 \times 3}$$

$g_1(x) = x_1 + x_2 + x_3 - 20$

$\frac{\partial g_1(x)}{\partial x_1} = 1$

$\frac{\partial g_1(x)}{\partial x_2} = 1$

$\frac{\partial g_1(x)}{\partial x_3} = 1$

$$P = [\nabla g_1(x)]$$

$$= \begin{bmatrix} \frac{\partial g_1(x)}{\partial x_1} & \frac{\partial g_1(x)}{\partial x_2} & \frac{\partial g_1(x)}{\partial x_3} \end{bmatrix}_{1 \times 3}$$

$$= [\,1 \quad 1 \quad 1\,]$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}_{1\times 3}$$

$\frac{\partial g_1(x)}{\partial x_3} = 1$

$$P^t = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{3\times 1}$$

$$H_B = \left[\begin{array}{c|c} O & P \\ \hline P^T & Q \end{array}\right] = \left[\begin{array}{c|ccc} 0 & 1 & 1 & 1 \\ \hline 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{array}\right]_{4\times 4} = (H_B)_p$$

$(-1)^m = (-1)^1 = -1$ $\therefore$ sign of $(-1)^m$ is Negative

$(-1)^{m+1} = (-1)^{1+1} = 1$ $\therefore$ sign of $(-1)^{m+1}$ is positive

(e) The principal minor determinants are

$$|0| = 0, \quad \begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} = -1, \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6, \quad \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} = -44$$

Starting order $= 2m+1 = 2\times 1 + 1 = 3$

No. of principal minor determi $= n - m = 3 - 1 = 2$

$\therefore$ starting with Principal minor determinants of order 3, the last 2 principal minor determinant matches with the sign of $(-1)^m$

$\therefore$ z has minima at $p = (5, 11, 4)$

$$\therefore \quad z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$$

$$= 2(5)^2 + 11^2 + 3(4)^2 + 10(5) + 8(11) + 6(4) - 100$$

$$= 281$$

$\therefore$ $Z_{min} = 281$ at $x_1 = 11$, $x_2 = 5$, $x_3 = 4$

(2) **Optimize** : $Z = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$
Subject to : $x_1 + x_2 + x_3 = 10$
$x_1, x_2, x_3 \geq 0$ **(M.U. 2007)** **Ans.** $[Z_{max} = 35 \text{ at } x_1 = 5, x_2 = 3, x_3 = 2]$

Sol: Given $n = 3$
$m = 1$

(a) The Langranges function is given by

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x)$$

$$L(x, \lambda) = \left(12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23\right) - \lambda(x_1 + x_2 + x_3 - 10)$$

(b) The necessary condition for z to have extreme values

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 12 - 2x_1 - \lambda = 0 \quad \text{——— (1)}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 8 - 2x_2 - \lambda = 0 \quad \text{——— (2)}$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 6 - 2x_3 - \lambda = 0 \quad \text{——— (3)}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 0 - (x_1 + x_2 + x_3 - 10) = 0$$

$$\Rightarrow x_1 + x_2 + x_3 = 10 \quad \text{——— (4)}$$

(c) solving (1), (2), (3) & (4), We get

$x_1 = 5, x_2 = 3, x_3 = 2, \lambda = 2$

$p \equiv (x_1, x_2, x_3) = (5, 3, 2)$

(d)

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

O = Null matrix of order 1×1 = $[0]$

$$P = [\nabla g_1(x)] = \left[\frac{\partial g_1(x)}{\partial x_1} \quad \frac{\partial g_1(x)}{\partial x_2} \quad \frac{\partial g_1(x)}{\partial x_3}\right]$$

$$= [1 \quad 1 \quad 1]$$

$$P^t = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$H_B = \left[\begin{array}{c|c} O & P \\ \hline P^t & Q \end{array}\right] = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}$$

$$[H^B]_p = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}$$

(e) $|0| = 0$, $\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1$, $\begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 4$, $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{vmatrix} = -12$

$(-1)^m = (-1)^1 = -1$

$(-1)^{m+1} = (-1)^{1+1} = 1$

starting order $= 2m+1 = 2\times1+1 = 3$

No. of principle minor determinants $= n-m = 3-1 = 2$

sign of $(-1)^{m+1}$ matches alternately with last $n-m$ principle minor determinants

$\therefore$ Z has maxima at $p = (5, 3, 2)$

$$Z = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$$

$$= 12(5) + 8(3) + 6(2) - 5^2 - 3^2 - 2^2 - 23$$

$$= 60 + 24 + 12 - 25 - 9 - 4 - 23$$

$$= 35$$

$\therefore$ $Z_{max} = 35$ at $x_1 = 5, x_2 = 3, x_3 = 2$

Q3 Optimise $Z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

subject to $x_1 + x_2 + x_3 = 15$

$2x_1 - x_2 + 2x_3 = 20$ & $x_1, x_2, x_3 \geq 0$

Sol: Given, $n = 3$

$m = 2$

(a) The Langrangel's function is given by

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x)$$

$$L(x, \lambda) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2 - \lambda_1(x_1 + x_2 + x_3 - 15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20)$$

(b) The necessary condition for extreme values are

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0$ ——— (1)

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0$ ——— (2)

$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2x_3 - \lambda_1 - 2\lambda_2 = 0$ ——— (3)

$\frac{\partial L}{\partial} = 0 \Rightarrow 0 - (x_1 + x_2 + x_3 - 15) = 0$ ——— (4)

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2x_3 - \lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow 0 - (x_1 + x_2 + x_3 - 15) = 0$$

$$\Rightarrow x_1 + x_2 + x_3 = 15 \quad \text{--- (4)}$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow 0 - 0 - (2x_1 - x_2 + 2x_3 - 20) = 0$$

$$\Rightarrow 2x_1 - x_2 + 2x_3 = 20 \quad \text{--- (5)}$$

(c) (1) + (2) $\Rightarrow 4x_1 - 2\lambda_1 - \lambda_2 = 0$

$$4x_1 = 2\lambda_1 + \lambda_2$$

$$x_1 = \frac{2\lambda_1 + \lambda_2}{4} \quad \text{--- (6)}$$

(2) $\Rightarrow 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0$

$$4x_2 - (2\lambda_1 + \lambda_2) - \lambda_1 + \lambda_2 = 0$$

$$4x_2 - 3\lambda_1 = 0$$

$$x_2 = \frac{3}{4}\lambda_1 \quad \text{--- (7)}$$

(3) $\Rightarrow 2x_3 = \lambda_1 + 2\lambda_2$

$$x_3 = \frac{\lambda_1}{2} + \lambda_2 \quad \text{--- (8)}$$

Now (4) $\Rightarrow x_1 + x_2 + x_3 = 15$

$$\frac{2\lambda_1 + \lambda_2}{4} + \frac{3\lambda_1}{4} + \frac{\lambda_1}{2} + \lambda_2 = 15$$

$$2\lambda_1 + \lambda_2 + 3\lambda_1 + 2\lambda_1 + 4\lambda_2 = 60$$

$$7\lambda_1 + 5\lambda_2 = 60 \quad \text{--- (9)}$$

(5) $\Rightarrow 2x_1 - x_2 + 2x_3 = 20$

$$2\left(\frac{2\lambda_1 + \lambda_2}{4}\right) - \left(\frac{3\lambda_1}{4}\right) + 2\left(\frac{\lambda_1}{2} + \lambda_2\right) = 20$$

$$\frac{2\lambda_1 + \lambda_2}{2} - \frac{3\lambda_1}{4} + \lambda_1 + 2\lambda_2 = 20$$

$$4\lambda_1 + 2\lambda_2 - 3\lambda_1 + 4\lambda_1 + 8\lambda_2 = 80$$

$$5\lambda_1 + 10\lambda_2 = 80 \quad \text{--- (10)}$$

Solving (9) & (10), we get

$$\lambda_1 = \frac{40}{9}, \quad \lambda_2 = \frac{52}{9}$$

(6) $\Rightarrow x_1 = \frac{2\lambda_1 + \lambda_2}{4}$

$$= \frac{\frac{80}{9} + \frac{52}{9}}{4} = \frac{\cancel{132}^{\,33}}{9 \times \cancel{4}} = \frac{11}{3}$$

(7) $\Rightarrow x_2 = +\frac{3}{4}\lambda_1 = +\frac{3}{4}\left(+\frac{40}{9}\right) = \frac{10}{3}$

(8) $\Rightarrow x_3 = \frac{\lambda_1}{2} + \lambda_2$

$$= \frac{\frac{40}{9}}{2} + \frac{52}{9} = \frac{20}{9} + \frac{52}{9} = \frac{72}{9} = 8$$

$\therefore\ p \equiv (x_1, x_2, x_3) \equiv \left(\frac{11}{3}, \frac{10}{3}, 8\right)$

$$\therefore\ p \equiv (x_1, x_2, x_3) \equiv \left(\frac{11}{3}, \frac{10}{3}, 8\right)$$

$x^1\ 1$

(d) $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$

$$P = \begin{bmatrix} \nabla g_1(x) \\ \nabla g_2(x) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} g_1(x) & \frac{\partial}{\partial x_2} g_1(x) & \frac{\partial}{\partial x_3} g_1(x) \\ \frac{\partial}{\partial x_1} g_2(x) & \frac{\partial}{\partial x_2} g_2(x) & \frac{\partial}{\partial x_3} g_2(x) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}$$

$$P^t = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$H^B = \begin{bmatrix} O & P \\ P^t & Q \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{bmatrix}$$

$$(-1)^m = (-1)^2 = 1$$

$$(-1)^{m+1} = (-1)^{2+1} = -1$$

(e)

Starting order $= 2m+1 = 2\times 2+1 = 5$

No. of P.M.D $= n-m = 3-2 = 1$

$$\begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{vmatrix} = \ ?$$

+ve value

H.Work

2nd method

1. Solve the foll. NLPP

maximise $Z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

subject to $2x_1 + x_2 \leq 5$

$x_1, x_2 \geq 0$

Sol: Given:

No. of variable $n = 2$

No. of constrains $m = 1$

The Langrangels function is given

$$L(x,\lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x)$$

$$\therefore L(x,\lambda) = 10x_1 + 4x_2 - 2x_1^2 - x_2^2 - \lambda(2x_1 + x_2 - 5)$$

The kuhn Tucker necessary conditions are

(a) $\frac{\partial L}{\partial x_j} = 0$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 10 - 4x_1 - 2\lambda = 0 \quad \text{——— (1)}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 4 - 2x_2 - \lambda = 0 \quad \text{——— (2)}$$

(b) $\lambda_i g_i(x) = 0$

$$\lambda(2x_1 + x_2 - 5) = 0 \quad \text{——— (3)}$$

(c) $g_m(x_n) \leq 0$

$$2x_1 + x_2 - 5 \leq 0 \quad \text{——— (4)}$$

(d) case (i) when $\lambda = 0$

$$(1) \Rightarrow x_1 = \frac{5}{2} \quad \text{——— (5)}$$

$$(2) \Rightarrow x_2 = 2 \quad \text{——— (6)}$$

$\lambda = 0$, $x_1 = \frac{5}{2}$, $x_2 = 2$ satisfies eqn. (3)

$(4) \Rightarrow$ $2x_1 + x_2 - 5 \leq 0$

$2\left(\frac{5}{2}\right) + 2 - 5 \leq 0$

$5 + 2 - 5 \leq 0$

$2 \le 0$ [false]

$\therefore$ These values do not satisfy eqn. (4)

$\therefore$ We discard this case.

Case (ii) When $\lambda \neq 0$

(1) $\Rightarrow \lambda = \dfrac{10-4x_1}{2}$

$\lambda = 5-2x_1$ ——— (7)

(2) $\Rightarrow \lambda = 4-2x_2$ ——— (8)

(7) & (8) $\Rightarrow 5-2x_1 = 4-2x_2$

$2x_1-2x_2=1$ ——— (9)

(3) $\Rightarrow 2x_1+x_2-5=0$

$2x_1+x_2=5$ ——— (10)

(9) & (10) $\Rightarrow x_1=\dfrac{11}{6},\ x_2=\dfrac{4}{3}$

Also $\lambda = 4-2x_2 = 4-2\left(\frac{4}{3}\right) = 4/3 > 0$

(4) $\Rightarrow 2x_1+x_2-5 \le 0$

$2\left(\frac{11}{6}\right)+\frac{4}{3}-5 \le 0$

$\frac{11}{3}+\frac{4}{3}-5 \le 0$

$0 \le 0$ [True]

$\therefore \lambda=\frac{4}{3},\ x_1=\frac{11}{6},\ x_2=\frac{4}{3}$ satisfied (1), (2), (3) & (4)

$\therefore$ Z has maxima at $x_1=\frac{11}{6},\ x_2=4/3$

$$Z = 10x_1+4x_2-2x_1^2-x_2^2$$

$$=10\left(\frac{11}{6}\right)+4\left(\frac{4}{3}\right)-2\left(\frac{11}{6}\right)^2-\left(\frac{4}{3}\right)^2$$

$$=\frac{91}{6}$$

$\therefore Z_{max}=\dfrac{91}{6}$ at $x_1=\dfrac{11}{6},\ x_2=\dfrac{4}{3}$

(2) Using Kuhn Tucker conditions to solve the foll. NLPP

Minimise $Z = x_1^3 - 4x_1 - 2x_2$

subject to $x_1 + x_2 \leq 1$

$x_1, x_2 \geq 0$

Sol: Given

No. of variables $= n = 2$

No. of constraints $= m = 1$

The Langrangele function is given by

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x)$$

$$L(x, \lambda) = x_1^3 - 4x_1 - 2x_2 - \lambda(x_1 + x_2 - 1)$$

(a) $\frac{\partial L}{\partial x_j} = 0$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 3x_1^2 - 4 - \lambda = 0$ ——(1)

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow -2 - \lambda = 0$ ——(2)

(b) $\lambda_i g_i(x) = 0$

$\lambda(x_1 + x_2 - 1) = 0$ ——(3)

(c) $g_m(x_n) \leq 0$

$x_1 + x_2 - 1 \leq 0$ ——(4)

(d) case (I) $\lambda = 0$

(1) $\Rightarrow 3x_1^2 = 4$

$x_1 = \sqrt{4/3}$

(2) $\Rightarrow \lambda = 2$

(3) $\Rightarrow 2\left(\sqrt{4/3} + x_2 - 1\right) = 0$

$x_2 = 1 - \sqrt{4/3}$

$x_2 < 0$

Which is not possible

$\therefore$ we reject this case

(d) case (II) $\lambda \neq 0$

(1) $\Rightarrow \lambda = 3x_1^2 - 4$ ——(5)

(2) $\Rightarrow \boxed{\lambda = -2}$ ——(6)

(5) & (6) $\Rightarrow$

$3x_1^2 - 4 = -2$

$3x_1^2 = 2$

$\boxed{x_1 = \sqrt{2/3}}$

(3) $\Rightarrow$

$-2\left(\sqrt{2/3} + x_2 - 1\right) = 0$

$\boxed{x_2 = 1 - \sqrt{2/3}}$

(4) $\Rightarrow$

$x_1 + x_2 - 1 \leq 0$

$\sqrt{\frac{2}{3}} + 1 - \sqrt{\frac{2}{3}} - 1 \leq 0$

$0 \leq 0$ [True]

$\therefore$ These values satisfies all the (4) conditions

$\therefore$ z has minima at $x_1 = \sqrt{2/3}$, $x_2 = 1 - \sqrt{2/3}$

$$\therefore \quad Z = x_1^3 - 4x_1 - 2x_2$$

$$= \left(\sqrt{\tfrac{2}{3}}\right)^2 - 4\left(\sqrt{2/3}\right) - 2\left(1 - \sqrt{2/3}\right)$$

$$= -3.093$$

$\therefore$ $Z_{min} = -3.093$ at $x_1 = \sqrt{2/3}$, $x_2 = 1 - \sqrt{2/3}$

③ Using Kuhn Tucker conditions solve the following NLPP

Maximise $Z = 10x_1 + 10x_2 - x_1^2 - x_2^2$

subject to $x_1 + x_2 \le 8$

$-x_1 + x_2 \le 5$, $x_1, x_2 \ge 0$

Sol: No. of variables $= n = 2$

No. of constraints $= m = 2$

The Langrange's function is given by

$$L(x,\lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x)$$

$$L(x,\lambda) = 10x_1 + 10x_2 - x_1^2 - x_2^2 - \lambda_1(x_1 + x_2 - 8) - \lambda_2(-x_1 + x_2 - 5)$$

The Kuhn Tucker necessary conditions are

(a) $\frac{\partial L}{\partial x_j} = 0$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 10 - 2x_1 - \lambda_1 + \lambda_2 = 0$ —— ①

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 10 - 2x_2 - \lambda_1 - \lambda_2 = 0$ —— ②

(b) $\lambda_i g_i(x) = 0$

$\lambda_1(x_1 + x_2 - 8) = 0$ —— ③

$\lambda_2(-x_1 + x_2 - 5) = 0$ —— ④

(c) $g_m(x_n) \le 0$

$x_1 + x_2 - 8 \le 0$ —— ⑤

$-x_1 + x_2 - 5 \le 0$ —— ⑥

(d) Case (I) $\lambda_1 = 0, \lambda_2 = 0$

① $\Rightarrow 10 - 2x_1 = 0$

$\boxed{\therefore x_1 = 5}$

② $\Rightarrow 10 - 2x_2 = 0$

$\boxed{\therefore x_2 = 5}$

③ $\Rightarrow$ True

④ $\Rightarrow$ True

⑤ $\Rightarrow x_1 + x_2 - 8 \le 0$

$5 + 5 - 8 \le 0$

$2 \le 0$ [false]

$\therefore$ x_1 & x_2 does not satisfy eqn. ⑤

$\therefore$ we discard this case.

(d) Case (II) $\lambda_1 = 0, \lambda_2 \ne 0$

① $\Rightarrow \lambda_2 = 2x_1 - 10$ —— ⑦

② $\Rightarrow \lambda_2 = 10 - 2x_2$ —— ⑧

⑦ & ⑧ $\Rightarrow$

$2x_1 - 10 = 10 - 2x_2$

$2x_1 + 2x_2 = 10 + 10$

$x_1 + x_2 = 10$ —— ⑨

③ $\Rightarrow$ True

④ $\Rightarrow -x_1 + x_2 - 5 = 0$ —— ⑩

⑨ & ⑩ $\Rightarrow x_2 = \frac{15}{2}, x_1 = \frac{5}{2}$

$\lambda_2 = 10 - 2x_2$

$= 10 - 2\left(\frac{15}{2}\right)$

$= -5$

⑤ $\Rightarrow x_1 + x_2 - 8 \le 0$

$\frac{15}{2} + \frac{5}{2} - 8 \le 0$

$10 - 8 \le 0$

$2 \le 0$ [F]

$\therefore$ We reject this case

ⓐ $\frac{\partial L}{\partial x_j} = 0$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 10 - 2x_1 - \lambda_1 + \lambda_2 = 0$ —— ①

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 10 - 2x_2 - \lambda_1 - \lambda_2 = 0$ —— ②

ⓑ $\lambda_i g_i(x) = 0$

$\lambda_1(x_1 + x_2 - 8) = 0$ —— ③

$\lambda_2(-x_1 + x_2 - 5) = 0$ —— ④

ⓒ $g_m(x_n) \le 0$

$x_1 + x_2 - 8 \le 0$ —— ⑤

$-x_1 + x_2 - 5 \le 0$ —— ⑥

ⓓ Case ③ $\lambda_1 \ne 0, \lambda_2 = 0$

① $\Rightarrow \lambda_1 = 10 - 2x_1$ —— ⑪

② $\Rightarrow \lambda_1 = 10 - 2x_2$ —— ⑫

⑪ & ⑫ $10 - 2x_1 = 10 - x_2$

$x_1 = x_2$

$x_1 - x_2 = 0$ —— ⑬

③ $\Rightarrow x_1 + x_2 - 8 = 0$ —— ⑭

solving ⑬ & ⑭, we get

$x_1 = 4, x_2 = 4$

④ $\Rightarrow$ True

⑤ $\Rightarrow x_1 + x_2 - 8 \le 0$

$4 + 4 - 8 \le 0$

$0 \le 0$ True

⑥ $\Rightarrow -x_1 + x_2 - 5 \le 0$

$-5 + 5 - 5 \le 0$

$-5 \le 0$ [True]

$\lambda_1 = 10 - 2x_1$

$= 10 - 2(4) = 2$

$\therefore \lambda_1 = 2, \lambda_2 = 0, x_1 = x_2 = 4$

satisfies all condition

$\therefore$ Z has maxima at

$x_1 = x_2 = 4$

$Z = 10x_1 + 10x_2 - x_1^2 - x_2^2$

$= 10(4) + 10(4) - 4^2 - 4^2$

$= 48$

$\therefore Z_{max} = 48$ at $x_1 = x_2 = 4$

Case ④ $\lambda_1 \ne 0, \lambda_2 \ne 0$

③ $\Rightarrow x_1 + x_2 - 8 = 0$ —— ⑮

④ $\Rightarrow -x_1 + x_2 - 5 = 0$ —— ⑯

⑮ & ⑯ $\Rightarrow$

$x_1 = 3/2, x_2 = 13/2$

⑤ $\Rightarrow x_1 + x_2 - 8 \le 0$

$\frac{3}{2} + \frac{13}{2} - 8 \le 0$

$0 \le 0$ [True]

⑥ $\Rightarrow -x_1 + x_2 - 5 \le 0$

$-\frac{3}{2} + \frac{13}{2} - 5 \le 0$

$\frac{-3 + 13 - 10}{2} \le 0$

$0 \le 0$ [True]

① $\Rightarrow 10 - 2x_1 - \lambda_1 + \lambda_2 = 0$

$10 - 2\left(\frac{3}{2}\right) - \lambda_1 + \lambda_2 = 0$

$-\lambda_1 + \lambda_2 = 7$ —— ⑰

② $\Rightarrow 10 - 2x_2 - \lambda_1 - \lambda_2 = 0$

$10 - 2\left(\frac{13}{2}\right) - \lambda_1 - \lambda_2 = 0$

$-3 - \lambda_1 - \lambda_2 = 0$

$-\lambda_1 - \lambda_2 = 3$ —— ⑱

⑰ & ⑱ $\Rightarrow$

$\lambda_1 = 2, \lambda_2 = -5$

We reject this pair

($\because$ for maxima $\lambda_1 \ge 0, \lambda_2 \ge 0$

& for minima $\lambda_1 \le 0, \lambda_2 \le 0$)

H.W

④ Using Kuhn Tucker Condition Solve the foll. NLPP

Maximize $Z = x_1^2 + x_2^2$

subject to

$$x_1 + x_2 - 4 \leq 0$$

$$2x_1 + x_2 - 5 \leq 0$$

$$x_1, x_2 \geq 0$$

Linear Programming Problems

Simplex Method

① Solve the following L.P.P. by simplex Method

Maximise $Z = 4x_1 + 10x_2$

subject to $2x_1 + x_2 \le 50$

$2x_1 + 5x_2 \le 100$

$2x_1 + 3x_2 \le 90$

$x_1, x_2 \geq 0$

Sol: The standard form of L.P.P. is

Maximise $Z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$

subject to $2x_1 + x_2 + s_1 = 50$

$2x_1 + 5x_2 + s_2 = 100$

$2x_1 + 3x_2 + s_3 = 90$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

θ = min. +ve Ratio

	$c_j \to$	4	10	0	0	0		
C_B	x_B	x_1	x_2	s_1	s_2	s_3	b	$\theta = \frac{b}{\text{key col.}}$
0	s_1	2	1	1	0	0	50	$\frac{50}{1} = 50$
0	s_2	2	5	0	1	0	100	$\frac{100}{5} = 20 \leftarrow$
0	s_3	2	3	0	0	1	90	$\frac{90}{3} = 30$
	z_j	0	0	0	0	0	0	
	$c_j - z_j$	4	10 ↑	0	0	0		
0	s_1	8/5	0	1	$-\frac{1}{5}$	0	30	$R_2(N) = \frac{R_2(O)}{5}$
10	x_2	2/5	1	0	1/5	0	20	$R_1(N) = R_1(O) - 1R_2(N)$
0	s_3	4/5	0	0	$-3/5$	1	30	$R_3(N) = R_3(O) - 3R_2(N)$
	z_j	4	10	0	2	0	200	
	$c_j - z_j$	0	0	0	-2	0		

$\because c_j - z_j \le 0 \ \forall j \ \therefore Z_{max} = 200$ at $x_1 = 0, x_2 = 20$

② Using simplex method solve the foll. L.P.P.

Std form of LPP is

② Using simplex method solve the foll. L.P.P.

$$\text{Max. } Z = 10x_1 + x_2 + x_3$$

subject to

$$x_1 + x_2 - 3x_3 \le 10$$
$$4x_1 + x_2 + x_3 \le 20$$
$$x_1, x_2, x_3 \geq 0$$

std form of LPP is

$$\max. Z = 10x_1 + x_2 + x_3 + 0s_1 + 0s_2 + 0s_3$$

s/t

$$x_1 + x_2 - 3x_3 + 0s_1 = 10$$
$$4x_1 + x_2 + x_3 + 0s_2 = 20$$
$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

Sol:

	$c_j \rightarrow$	10	1	1	0	0		
C_B	x_B	x_1	x_2	x_3	S_1	S_2	b	$\theta = \frac{b}{\text{key col.}}$
0	S_1	1	1	−3	1	0	10	$\frac{10}{1} = 10$
0	S_2	$\boxed{4}$	1	1	0	1	20	$\frac{20}{4} = 5 \leftarrow$
	z_j	0	0	0	0	0	0	
	$c_j - z_j$	10 ↑	1	1	0	0		
0	S_1	0	3/4	−13/4	1	−1/4	5	
10	x_1	1	1/4	1/4	0	1/4	5	
	z_j	10	5/2	5/2	0	5/2	50	
	$c_j - z_j$	0	−3/2	−3/2	0	−5/2		

$R_2(N) = R_2(0)/4$
$R_1(N) = R_1(0) - 1\, R_2(N)$

$\because$ $c_j - z_j \le 0 \ \forall j$ $Z_{max} = 50$ at $x_1 = 5, x_2 = 0$
$x_3 = 0$

③ Solve using simplex Method

$$\text{Max. } Z = 6x_1 - 2x_2 + 3x_3$$

subject to

$$2x_1 - x_2 + 2x_3 \le 2$$
$$x_1 + 4x_3 \le 4$$
$$x_1, x_2, x_3 \geq 0$$

Sol:

	$c_j \rightarrow$	6	−2	3	0	0		
C_B	x_B	x_1	x_2	x_3	S_1	S_2	b	$\theta = \frac{b}{\text{key col.}}$
0	S_1	$\boxed{2}$	−1	2	1	0	2	$2/2 = 1 \leftarrow$
0	S_2	1	0	4	0	1	4	$4/1 = 4$
	z_j	0	0	0	0	0	0	
	$c_j - z_j$	6 ↑	−2	3	0	0		
6	x_1	1	−1/2	1	1/2	0	1	$\frac{1}{-1/2} = -2$
0	S_2	0	1/2	3	−1/2	1	3	$\frac{3}{1/2} = 6 \leftarrow$

$R_1(N) = \frac{R_1(0)}{2}$
$R_2(N) = R_2(0) - 1\, R_1(N)$

C_B	Basis						b
6	x_1	1	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	1
0	s_2	0	$\frac{1}{2}$	3	$-\frac{1}{2}$	1	3
	z_j	6	-3	6	3	0	6
	$c_j - z_j$	0	1 ↑	-3	-3	0	
6	x_1	1	0	4	0	1	4
-2	x_2	0	1	6	-1	2	6
	z_j	6	-2	12	2	2	12
	$c_j - z_j$	0	0	-9	-2	-2	

$\frac{1}{-1/2} = -2$ $R_1(N) = \ldots$

$\frac{3}{1/2} = 6$ ← $R_2(N) = R_2(O) \ldots$

$R_1(N) = R_1(O) + \frac{1}{2} R_2(N)$

$\therefore\ c_j - z_j \le 0 \ \forall j$ $Z_{max} = 12$ at $x_1 = 4$, $x_2 = 6$

objective function

cost coefficients

$$\text{Max / min. } Z = c_{11}x_1 + c_{12}x_2 + c_{13}x_3$$

subject to

Constraints

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \le b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \le b_3$$

$$x_1, x_2, x_3 \geq 0$$

Right Hand sol.

x_1, x_2, x_3 are called decision variables

Big M Method / Penalty Method / Charne's Method.

(1)

Simplex

Max./min. $Z = c_1x_1 + c_2x_3$

subject to

$a_{11}x_1 + a_{12}x_2 \le b_1$

$a_{21}x_1 + a_{22}x_2 \le b_2$

$x_1, x_2, \geq 0$

$\boxed{b_1, b_2 \geq 0}$ & $\boxed{\text{All constraints must be } \le}$

Big Method

max/min. $Z = c_1x_1 + c_2x_3$

Subject to

$a_{11}x_1 + a_{12}x_2 \le b_1$

$a_{21}x_1 + a_{22}x_2 \geq b_2$

$a_{31}x_1 + a_{32}x_2 = b_3$

$x_1, x_2, \geq 0$

$\boxed{\text{All } b_1, b_2, b_3 \geq 0}$ & $\boxed{\text{Atleast one constraint must be } \geq \text{ or } = \text{ type}}$

(2) In Big M Method

(a) '$\le$' type use slack variable (i.e. S_1, S_2, S_3)

$2x_1 + 3x_2 \le 5$

$2x_1 + 3x_2 + S_1 = 5$

(b) '$\geq$' type use surplus variable (ie. $-S_1, -S_2$) & Also add artificial variable ($A_1, A_2, ..$)

$2x_1 + 3x_2 \geq 5$

$2x_1 + 3x_2 - S_1 + A_1 = 5$

(c) '$=$' type then use Artificial variable (ie. $A_1, A_2 ..$) only

$2x_1 + 3x_2 = 5$

$2x_1 + 3x_2 + A_2 = 5$

(d) In objective function add $0S_1 + 0S_2 + 0S_3$ & $-MA_1 - MA_2$ & so on

(1)

Using Penalty OR Big M Method Solve the followin

(1) **Maximize** : $Z = 3x_1 - x_2$
Subject to : $2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \le 3$

$x_2 \le 4$

The std. form of L.P.P. is

Max. $Z = 3x_1 - x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1$

Subject to $2x_1 + x_2 - S_1 + A_1 = 2$

Subject to : $2x_1 + x_2 \ge 2$

$x_1 + 3x_2 \le 3$

$x_2 \le 4$

$x_1, x_2 \ge 0$ **(M.U. M2001,D2009)**

Subject to
$$2x_1 + x_2 - S_1 + A_1 = 2$$
$$x_1 + 3x_2 + S_2 = 3$$
$$x_2 + S_3 = 4$$
$$x_1, x_2, S_1, S_2, S_3, A_1 \geq 0$$

	$C_j \rightarrow$	3	-1	0	0	0	-M		
C_B	x_B	x_1	x_2	S_1	S_2	S_3	A_1	b	$\theta = \frac{b}{\text{Key col.}}$
-M	A_1	2	1	-1	0	0	1	2	$2/2 = 1$ ←
0	S_2	1	3	0	1	0	0	3	$3/1 = 3$
0	S_3	0	1	0	0	1	0	4	$4/0 = \infty$
	Z_j	-2M	-M	M	0	0	-M	-2M	
	$C_j - Z_j$	3+2M ↑	-1+M	-M	0	0	0		
3	x_1	1	1/2	-1/2	0	0	1/2	1	$R_1(N) = \frac{R_1(O)}{2}$; $\frac{1}{-1/2} = -2$
0	S_2	0	5/2	1/2	1	0	-1/2	2	$R_2(N) = R_2(O) - 1R_1(N)$; $\frac{2}{1/2} = 4$ ←
0	S_3	0	1	0	0	1	0	4	$R_3(N) = R_3(O) - 0R_1(N)$; $4/0 = \infty$
	Z_j	3	3/2	-3/2	0	0	3/2	3	
	$C_j - Z_j$	0	-5/2	3/2 ↑	0	0	-M-3/2		
3	x_1	1	3	0	1	0	0	3	
0	S_1	0	5	1	2	0	-1	4	
0	S_3	0	1	0	0	1	0	4	
	Z_j	3	9	0	3	0	0	9	
	$C_j - Z_j$	0	-10	0	-3	0	-M		

$R_2(N) = \frac{R_2(O)}{1/2}$

$R_1(N) = R_1(O) - (-\frac{1}{2}) R_2(N)$

$R_3(N) = R_3(O) - 0 R_2(N)$

$\because C_j - Z_j \leq 0 \ \forall j \quad Z_{max} = 9$ at $x_1 = 3, x_2 = 0$

Q2 Solve by Big M Method

$$\text{Max. } Z = 6x_1 + 4x_2$$

Subject to
$$2x_1 + 3x_2 \leq 30$$
$$3x_1 + 2x_2 \leq 24$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

The standard form is

$$\text{Max. } Z = 6x_1 + 4x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1$$

subject to
$$2x_1 + 3x_2 + S_1 = 30$$
$$3x_1 + 2x_2 + S_2 = 24$$
$$x_1 + x_2 - S_3 + A_1 = 3$$
$$x_1, x_2, S_1, S_2, S_3, A_1 \geq 0$$

Sol:

	$C_j \rightarrow$	6	4	0	0	0	-M		
C_B	x_B	x_1	x_2	S_1	S_2	S_3	A_1	b	$\theta = \frac{b}{\ldots}$

	$C_j \rightarrow$	6	4	0	0	0	$-M$			
C_B	x_B	x_1	x_2	s_1	s_2	s_3	A_1	b	$\theta = \frac{b}{\text{key col.}}$	
0	s_1	2	3	1	0	0	0	30	$\frac{30}{2} = 15$	
0	s_2	3	2	0	1	0	0	24	$\frac{24}{3} = 8$	
$-M$	A_1	1	1	0	0	-1	1	3	$3/1 = 3 \leftarrow$	
	z_j	$-M$	$-M$	0	0	M	$-M$	$-3M$		
	$c_j - z_j$	$6+M \uparrow$	$4+M$	0	0	$-M$	0			
0	s_1	0	1	1	0	2	-2	24	$24/2 = 12$	$R_3(N) = \frac{R_3(0)}{1}$
0	s_2	0	-1	0	1	3	-3	16	$16/3 = 5.\dot{}\leftarrow$	$R_1(N) = R_1(0) - 2R_3(N)$
6	x_1	1	1	0	0	-1	1	3	$3/-1 = -3$	
	z_j	6	6	0	0	-6	6	18		$R_2(N) = R_2(0) - 3R_3(N)$
	$c_j - z_j$	0	-2	0	0	$6 \uparrow$	$-M-6$			
0	s_1	0	$5/3$	1	$-2/3$	0	0	$40/3$		
0	s_3	0	$-1/3$	0	$1/3$	1	-1	$16/3$		
6	x_1	1	$2/3$	0	$1/3$	0	0	$25/3$		
	z_j	6	4	0	2	0	0	50		$R_1(N) = R_1(0) - 2R_2(N)$
	$c_j - z_j$	0	0	0	-2	0	$-M$			$R_3(N) = R_3(0) + 1R_2(N)$

$\because c_j - z_j \le 0 \ \forall j \quad Z_{max.} = 50$ at $x_1 = \frac{25}{3}, \ x_2 = 0$

Q Use Penalty Method to solve the foll. L.P.P.

Minimise $Z = 2x_1 + 3x_2$

subject to
$$x_1 + x_2 \geq 5$$
$$x_1 + 2x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

Sol:

Maximise $Z' = -2x_1 - 3x_2$

subject to
$$x_1 + x_2 \geq 5$$
$$x_1 + 2x_2 \geq 6$$
$$x_1, x_2 \geq 0$$

The standard form is

$$\text{Max. } Z' = -2x_1 - 3x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$$

subject to
$$x_1 + x_2 - s_1 + A_1 = 5$$
$$x_1 + 2x_2 - s_2 + A_2 = 6$$
$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

		$C_j \rightarrow$	-2	-3	0	0	$-M$	$-M$		
C_B	x_B		x_1	x_2	S_1	S_2	A_1	A_2	b	$\theta = \frac{b}{\text{Key col.}}$
$-M$	A_1		1	1	-1	0	1	0	5	$5/1 = 5$
$-M$	A_2		1	$\boxed{2}$	0	-1	0	1	6	$6/2 = 3 \leftarrow$
	Z_j'		$-2M$	$-3M$	M	M	$-M$	$-M$	$-11M$	
	$C_j - Z_j'$		$-2+2M$	$-3+3M \uparrow$	$-M$	$-M$	0	0		
$-M$	A_1		$\boxed{1/2}$	0	-1	$1/2$	1	$-1/2$	2	$\frac{2}{1/2} = 4 \leftarrow$
-3	x_2		$1/2$	1	0	$-1/2$	0	$1/2$	3	$\frac{3}{1/2} = 6$
	Z_j'		$-\frac{M}{2} - 3/2$	-3	M	$-\frac{M}{2} + 3/2$	$-M$	$M/2 - 3/2$	$-2M-9$	
	$C_j - Z_j'$		$\frac{M}{2} - 1/2 \uparrow$	0	$-M$	$\frac{M}{2} - 3/2$	0	$-\frac{3M}{2} + 3/2$		
-2	x_1		1	0	-2	1	2	-1	4	
-3	x_2		0	1	1	-1	-1	1	1	
	Z_j'		-2	-3	1	1	-1	-1	-11	$R_2(N) = R_2(0) - \frac{1}{2} R_1(N)$
	$C_j - Z_j'$		0	0	-1	-1	$-M+1$	$-M+1$		

$\therefore C_j - Z_j' \le 0 \;\forall j \qquad Z'_{max} = -11$ at $x_1 = 4,\ x_2 = 1$

$Z_{min} = 11$ at $x_1 = 4,\ x_2 = 1$

Duality

Problem A

$\text{Max. } Z = 6x_1 + 10x_2$

subject to

$2x_1 + 4x_2 \le 18$

$2x_1 + x_2 \le 8$

$x_1 + 3x_2 \le 20$

$x_1, x_2 \ge 0$

Primal

Problem B

$\text{Min. } W = 18y_1 + 8y_2 + 20y_3$

subject to

$3y_1 + 2y_2 + y_3 \ge 6$

$4y_1 + y_2 + 3y_3 \ge 10$

$y_1, y_2, y_3 \ge 0$

Dual

Formulation of Dual

(1) $\max Z = c_1x_1 + c_2x_2 + c_3x_3$

subject to $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le b_1$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 \le b_m$

$x_1, x_2, x_3 \ge 0$

(All constraints must be '$\le$' type)

$\min. Z = c_1x_1 + c_2x_2 + c_3x_3$

s/t

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \ge b_1$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 \ge b_m$

$x_1, x_2, x_3 \ge 0$

(All constraints must be '$\ge$' type)

(2) $2x_1 + 3x_2 \le 5$

$-2x_1 - 3x_2 \ge -5$

(3) $2x_1 + 3x_2 = 5$

min.

$2x_1 + 3x_2 \le 5$

$2x_1 + 3x_2 \ge 5$

$-2x_1 - 3x_2 \ge -5$

$2x_1 + 3x_2 \ge 5$

max

$2x_1 + 3x_2 \le 5$

$2x_1 + 3x_2 \ge 5$

$2x_1 + 3x_2 \le 5$

$-2x_1 - 3x_2 \le -5$

(4) All decision variables must be ≥ 0 type. If any variable say x_2 is unrestricted then express it as differ of two positive ~~dec~~ variable

$x_2 = x_2' - x_2''$ where x_2' & $x_2'' \geq 0$

	Primal		Dual
(1)	max.	$\Leftrightarrow$	min
(2)	coef matrix	$\Leftrightarrow$	transpose of coefficient matrix
(3)	$c_1, c_2, \ldots c_n$	$\Leftrightarrow$	$b_1, b_2, b_3, \ldots b_m$
(4)	$\leq$	$\Leftrightarrow$	$\geq$
(5)	No. of constraint	$\Leftrightarrow$	No. of variable
(6)	'='	$\Leftrightarrow$	unrestricted variable

(1) Write the dual of foll. L.P.P.

Maximize $Z = 2x_1 - x_2 + 4x_3$

Subject to
$$x_1 + 2x_2 - x_3 \leq 5$$
$$2x_1 - x_2 + x_3 \leq 6$$
$$x_1 + x_2 + 3x_3 \leq 10$$
$$4x_1 + x_3 \leq 12$$
$$x_1, x_2, x_3 \geq 0$$

$$\begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 3 \\ 4 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 6 \\ 10 \\ 12 \end{bmatrix}$$

Sol:

Primal
$m = 3$ (variables)
$n = 4$ (constraints)

Dual
$m = 4$ (variable)
$n = 3$ (constraints)

Min. $W = 5y_1 + 6y_2 + 10y_3 + 12y_4$

Subject to
$$y_1 + 2y_2 + y_3 + 4y_4 \geq 2$$
$$2y_1 - y_2 + y_3 + 0y_4 \geq -1$$
$$-y_1 + y_2 + 3y_3 + y_4 \geq 4$$
$$y_1, y_2, y_3, y_4 \geq 0$$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -1 & 1 & 0 \\ -1 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

(2) Write the dual of foll. LPP

Min. $Z = x_2 + 3x_3$

S/t
$$2x_1 + x_2 \leq 3$$
$$x_1 + 2x_2 + 6x_3 \geq 5$$
$$-x_1 + x_2 + 2x_3 = 2$$
$$x_1, x_2, x_3 \geq 0$$

$-x_1 + x_2 + 2x_3 \leq 2 \Rightarrow x_1 - x_2 - 2x_3 \geq -2$

$-x_1 + x_2 + 2x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

Sol: $\text{Min. } Z = 0x_1 + x_2 + 3x_3$

subject to
$$-2x_1 - x_2 + 0x_3 \geq -3$$
$$x_1 + 2x_2 + 6x_3 \geq 5$$
$$x_1 - x_2 - 2x_3 \geq -2$$
$$-x_1 + x_2 + 2x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

$$[0 \quad 1 \quad 3]$$

$$\begin{bmatrix} -2 & -1 & 0 \\ 1 & 2 & 6 \\ 1 & -1 & -2 \\ -1 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} -3 \\ 5 \\ -2 \\ 2 \end{bmatrix}$$

Primal	Dual
$m=3$ (vari)	$m=3$ (variables)
$n=3$	$n=3$ (constraint)

$$\text{Max. } W = -3y_1 + 5y_2 - 2y_3 + 2y_4$$

subject to
$$-2y_1 + y_2 + y_3 - y_4 \leq 0$$
$$-y_1 + 2y_2 - y_3 + y_4 \leq 1$$
$$0y_1 + 6y_2 - 2y_3 + 2y_4 \leq 3$$
$$y_1, y_2, y_3, y_4 \geq 0$$

Let $y_5 = y_3 - y_4$

$$\text{Max } W = -3y_1 + 5y_2 - 2y_5$$

subject to
$$-2y_1 + y_2 + y_5 \leq 0$$
$$-y_1 + 2y_2 - y_5 \leq 1$$
$$0y_1 + 6y_2 - 2y_5 \leq 3$$
$y_1, y_2 \geq 0$, y_5 is unrestricted

Q3 find the dual of foll. L.P.P.

Sol: $\text{Max. } Z = 2x_1 - x_2 + 3x_3$

subject to
$$x_1 - 2x_2 + x_3 \geq 4$$
$$2x_1 + x_3 \leq 10$$
$$x_1 + x_2 + 3x_3 = 20$$
$x_1, x_3 \geq 0$, x_2 is unrestricted

$m = 3$ (var)
$n = 3$ (const)

Ans
$\text{min. } W = -4y_1 + 10y_2 + 20y_3$
subject to $-y_1 + 2y_2 + y_3 \geq 2$
$2y_1 + y_3 = -1$
$-y_1 + y_2 + 3y_3 \geq 3$
$y_1, y_2 \geq 0$, y_3 unrestricted

Sol: Let $x_2 = x_2' - x_2''$ where $x_2', x_2'' \geq 0$

$$\text{Max. } Z = 2x_1 - x_2' + x_2'' + 3x_3$$

subject to
$$x_1 - 2x_2' + 2x_2'' + x_3 \geq 4$$
$$2x_1 + x_3 \leq 10$$

$x_1 + x_2' - x_2'' + 3x_3 \leq 20$

$\ldots \geq 20$

$$x_1 + x_2' - x_2'' + 3x_3 = 20$$
$$x_1, x_2, x_3', x_3'' \geq 0$$

$$\text{Max. } Z = 2x_1 - x_2 + x_2'' + 3x_3$$

subject to

$$-x_1 + 2x_2 - 2x_2'' - x_3 \leq -4$$
$$2x_1 + 0x_2 + 0x_2'' + x_3 \leq 10$$
$$x_1 + x_2' - x_2'' + 3x_3 \leq 20$$
$$-x_1 - x_2' + x_2'' - 3x_3 \leq -20$$
$$x_1, x_2', x_2'', x_3 \geq 0$$

$[2\ -1\ \ 3]$

$$\begin{bmatrix} -1 & 2 & -2 & -1 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & -1 & 3 \\ -1 & -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} -4 \\ 10 \\ 20 \\ -20 \end{bmatrix}$$

$$\text{Min } W = -4y_1 + 10y_2 + 20y_3 - 20y_4$$

subject to

$$-y_1 + 2y_2 + y_3 - y_4 \geq 2$$
$$2y_1 + 0y_2 + y_3 - y_4 \geq -1$$
$$-2y_1 + 0y_2 - y_3 + y_4 \geq 1$$
$$-y_1 + y_2 + 3y_3 - 3y_3 \geq 3$$
$$y_1, y_2, y_3, y_4 \geq 0$$

$-2y_1 + 0y_2 - y_3 + y_4 \leq 1$

$-2y_1 + 0y_2 - y_3 + y_4 \geq 1$

$-2y_1 + 0y_2 - y_3 + y_4 = 1$

$\therefore -2y_1 + 0y_2 - y_5 = 1$

$-2y_1 + 0y_2 - y_5 = 1$

$-2y_1 + 0y_2 - y_3 + y_4 = 1$

$-2y_1 + y_5 = 1$

$A \geq B$

$A \leq B$

$\Rightarrow A = B$

Let $y_5 = y_3 - y_4$ where $y_3, y_4 \geq 0$

$$\text{Min } W = -4y_1 + 10y_2 + 20y_5$$

subject to

$$-y_1 + 2y_2 + y_5 \geq 2$$
$$-2y_1 + y_5 = 1$$
$$-y_1 + y_2 + 3y_5 \geq 3$$

$y_1, y_2 \geq 0$, y_5 is unrestricted.

* Solving LPP using Duality *

① Using Duality solve the following L.P.P.

$$\text{minimise } Z = 4x_1 + 3x_2 + 6x_3$$

subject to

$$x_1 + x_3 \geq 2$$
$$x_2 + x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Sol: The Dual is

$$\max.\ W = 2y_1 + 5y_2$$

subject to
$$y_1 + 0y_2 \le 4$$
$$0y_1 + y_2 \le 3$$
$$y_1 + y_2 \le 6$$
$$y_1, y_2 \ge 0$$

The std. form of LPP is

$$\max.\ W = 2y_1 + 5y_2 + 0s_1 + 0s_2 + 0s_3$$

subject to
$$y_1 + 0y_2 + s_1 = 4$$
$$0y_1 + y_2 + s_2 = 3$$
$$y_1 + y_2 + s_3 = 6$$
$$y_1, y_2, s_1, s_2, s_3 \ge 0$$

	$c_j \rightarrow$	2	5	0	0	0			
C_B	x_B	y_1	y_2	s_1	s_2	s_3	b	$\theta = \frac{b}{\text{key col.}}$	
0	s_1	1	0	1	0	0	4	$\frac{4}{0} = \infty$	
0	s_2	0	1	0	1	0	3	$\frac{3}{1} = 3 \leftarrow$	
0	s_3	1	1	0	0	1	6	$\frac{6}{1} = 6$	
	z_j	0	0	0	0	0	0		
	$c_j - z_j$	2	5 ↑	0	0	0			
0	s_1	1	0	1	0	0	4	$\frac{4}{1} = 4$	$R_2(N) = \frac{R_2(O)}{1}$
5	y_2	0	1	0	1	0	3	$\frac{3}{0} = \infty$	$R_1(N) = R_1(O)$
0	s_3	1	0	0	−1	1	3	$\frac{3}{1} = 3 \leftarrow$	$R_3(N) = R_3(O) - 1R_2(N)$
	z_j	0	5	0	5	0	15		
	$c_j - z_j$	2 ↑	0	0	−5	0			
0	s_1	0	0	1	1	−1	1		
5	y_2	0	1	0	1	0	3		
2	y_1	1	0	0	−1	1	3		
	z_j	2	5	0	3	2	21		$R_3(N) = \frac{R_3(O)}{1}$
	$c_j - z_j$	0	0	0	−3	−2			$R_2(N) = R_2(O)$
									$R_1(N) = R_1(O) - 1R_3(N)$

$\therefore\ c_j - z_j \le 0\ \forall j$ $\quad W_{max} = 21$ at $y_1 = 3, y_2 = 3$

$\therefore\ Z_{min} = 21$ at $x_1 = 0, x_2 = 3, x_3 = 2$

($x_1 \rightarrow s_1,\ x_2 \rightarrow s_2,\ x_3 \rightarrow s_3$
the value of x_1 is absolute value of s_1 in $c_j - z_j$)

$x_1 \rightarrow s_1$, $x_2 \rightarrow s_2$, $x_3 \rightarrow s_3$

the value of x_1 is absolute value of s_1 in $c_j - z_j$

x_2 — — — s_2 in $c_j - z_j$

x_3 - - - - s_3 in $c_j - z_j$

Dual simplex Method

Simplex	Big M	Dual simplex
Max.	max.	max.
① $b_i \geq 0$	① $b_i \geq 0$	① At least one $b_i < 0$
② & all consts ≤ 0	② At least one constraint should be $\geq$ or $=$	② All constraint must be '$\leq$' type
③ Slack is used	③ slack, surplus & artificial variable is used	③ slack variables are used only

① Use the dual simplex method to solve the foll. L.P.P.

Minimise $Z = 2x_1 + 2x_2 + 4x_3$

subject to

$$2x_1 + 3x_2 + 5x_3 \geq 2$$
$$3x_1 + x_2 + 7x_3 \leq 3$$
$$x_1 + 4x_2 + 6x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

Sol: Maximise $Z' = -2x_1 - 2x_2 - 4x_3$

subject to

$$-2x_1 - 3x_2 - 5x_3 \leq -2$$
$$3x_1 + x_2 + 7x_3 \leq 3$$
$$x_1 + 4x_2 + 6x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

The standard form of LPP is

Maximise $Z' = -2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3$

subject to

$$-2x_1 - 3x_2 - 5x_3 + s_1 = -2$$
$$3x_1 + x_2 + 7x_3 + s_2 = 3$$
$$x_1 + 4x_2 + 6x_3 + s_3 = 5$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

	$c_j \rightarrow$	-2	-2	-4	0	0	0	
C_B	x_B	x_1	x_2	x_3	s_1	s_2	s_3	b
0	s_1	-2	-3	-5	1	0	0	-2 ←
0	s_2	3	1	7	0	1	0	3
0	s_3	1	4	6	0	0	1	5
	z_j	0	0	0	0	0	0	
	$c_j - z_j$	-2	-2	-4	0	0	0	
$\theta = \frac{c_j - z_j}{[\text{illegible}]}$		$\frac{-2}{-2} = 1$	$\frac{-2}{-3} = \frac{2}{3}$	$\frac{-4}{-5} = 0.8$	$\frac{0}{1} = 0$	$\frac{0}{0}$	$\frac{0}{0}$	

	z_j	0	0	0	0	0	0	
	$c_j - z_j$	-2	-2	-4	0	0	0	
	$\theta = \frac{c_j - z_j}{\text{Key Row}}$	$\frac{-2}{-2}=1$	$\frac{-2}{-3}=\frac{2}{3}$	$\frac{-4}{-5}=0.8$	$\frac{0}{-1}=0$	$\frac{0}{0}$	$\frac{0}{0}$	
-2	x_2	$2/3$	1	$5/3$	$-\frac{1}{3}$	0	0	$\frac{2}{3}$
0	s_2	$7/3$	0	$16/3$	$1/3$	1	0	$7/3$
0	s_3	$-5/3$	0	$-2/3$	$4/3$	0	1	$7/3$
	z_j	$-4/3$	-2	$-10/3$	$2/3$	0	0	$-4/3$
	$c_j - z_j$	$-2/3$	0	$-2/3$	$-2/3$	0	0	

$R_1(N) = \frac{R_1(0)}{-3}$

$R_2(N) = R_2(0) - R_1(N)$

$R_3(N) = R_3(0) - 4R_1(N)$

$\because c_j - z_j \le 0 \ \forall j \ \& \ b_i \ge 0 \quad \therefore Z'_{max} = -4/3$ at $x_1 = 0, x_2 = 2/3, x_3 = 0$

$\therefore Z_{min} = 4/3$ at $x_1 = 0, x_2 = 2/3, x_3 = 0$

(2) Use the dual simplex method to solve the foll. LPP

Max. $Z = -3x_1 - 2x_2$

subject to

$x_1 + x_2 \ge 1 \longrightarrow -x_1 - x_2 \le -1$

$x_1 + x_2 \le 7$

$x_1 + 2x_2 \ge 10 \longrightarrow -x_1 - 2x_2 \le -10$

$x_2 \le 3$

$x_1, x_2 \ge 0$

Sol: The standard form of LPP is

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to

$-x_1 - x_2 + s_1 = -1$

$x_1 + x_2 + s_2 = 7$

$-x_1 - 2x_2 + s_3 = -10$

$x_2 + s_4 = 3$

$x_1, x_2, s_1, s_2, s_3, s_4 \ge 0$

	$c_j \rightarrow$	-3	-2	0	0	0	0	
C_B	x_B	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	-2	0	0	1	0	-10
0	s_4	0	1	0	0	0	1	3
	z_j	0	0	0	0	0	0	
	$c_j - z_j$	-3	-2	0	0	0	0	
	$\theta = \frac{c_j - z_j}{\text{Key Row}}$	$\frac{-3}{-1}=3$	$\frac{-2}{-2}=1$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{1}$	$\frac{0}{0}$	
0	s_1	$-1/2$	0	1	0	$-1/2$	0	4
0	s_2	$1/2$	0	0	1	$1/2$	0	2
-2	x_2	$1/2$	1	0	0	$-1/2$	0	5
0	s_4	$-1/2$	0	0	0	$1/2$	1	-2
	z_j	-1	-2	0	0	1	0	-10
	$c_j - z_j$	-2	0	0	0	-1	0	
	$\theta = \frac{c_j - z_j}{\text{Key Row}}$	4	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{0}{0}$	$\frac{-1}{1/2}$	$\frac{0}{1}$	
0	s_1	0	0	1	0	-1	-1	6
0	s_2	0	0	0	1	1	1	0
-2	x_2	0	1	0	0	-1	1	3
-3	x_1	1	0	0	0	-1	-2	4

$R_3(N) = \frac{R_3(0)}{-2}$

$R_1(N) = R_1(0) + 1R_3(N)$

$R_2(N) = R_2(0) - R_3(N)$

$R_4(N) = R_4(0) - R_3(N)$

$R_4(N) = \frac{R_4(0)}{-1/2}$

$R_1(N) = R_1(0) + \frac{1}{2}R_4(N)$

$R_2(N) = R_2(0) - \frac{1}{2}R_4(N)$

$R_3(N) = R_3(0) - \frac{1}{2}R_4(N)$

0	s_2	0	0	0	1	1	1	0
−2	x_2	0	1	0	0	−1	1	3
−3	x_1	1	0	0	0	−1	−2	4
	z_j	−3	−2	0	0	5	4	−18
	$c_j - z_j$	0	0	0	−2	−5	−4	

$R_1(N) = R_1(0) \ldots$
$R_2(N) = R_2(0) - \frac{1}{2}R_4(N)$
$R_3(N) = R_3(0) - \frac{1}{2}R_4(N)$

$\because c_j - z_j \le 0 \ \forall j \ \& \ b_i \ge 0$ $\quad Z_{max} = -18$ at $x_1 = 4, x_2 = 0$

Q Solve by Dual simplex method

minimize $Z = 6x_1 + x_2$

subject to $2x_1 + x_2 \ge 3$

$x_1 - x_2 \ge 0$

$x_1, x_2 \ge 0$

$Z_{min} = 7$ at $x_1 = x_2 = 1$

$\max. Z' = -6x_1 - x_2 + 0s_1 + 0s_2$

$-2x_1 - x_2 + s_1 = -3$

$-x_1 + x_2 + s_2 = 0$

$x_1, x_2, s_1, s_2 \ge 0$

	$c_j \rightarrow$	−6	−1	0	0	
C_B	x_B	x_1	x_2	s_1	s_2	b
0	s_1	−2	−1	1	0	−3
0	s_2	−1	1	0	1	0
	z_j	0	0	0	0	0
	$c_j - z_j$	−6	−1	0	0	
	$\theta = \frac{c_j - z_j}{\text{Key Row}}$	$\frac{-6}{-2} = 3$	$\frac{-1}{-1} = 1$	$\frac{0}{1} = 0$	$\frac{0}{0}$	
−1	x_2	2	1	−1	0	3
0	s_2	−3	0	1	1	−3
	z_j	−2	−1	1	0	−3
	$c_j - z_j$	−4	0	−1	0	
	$\theta = \frac{c_j - z_j}{\text{Key Row}}$	4/3 ↑	0/0	$\frac{-1}{1}$	$\frac{0}{0}$	
−1	x_2	0	1	−1/3	2/3	1
−6	x_1	1	0	−1/3	−1/3	1
	z_j	−6	0	7/3	4/3	−7
	$c_j - z_j$	0	−1	−7/3	−4/3	

$R_2(N) = \frac{R_2(0)}{-3}$

$R_1(N) = R_1(0) - 2R_2(N)$

$-\frac{2}{3} + \frac{6}{3}$

$\because c_j - z_j \le 0 \ \forall j \ \& \ b_i \ge 0 \quad \therefore Z'_{max} = -7$ at $x_1 = 1, x_2 = 1$

$Z_{min} = 7$ at $x_1 = 1, x_2 = 1$

* — Complex Integration — *

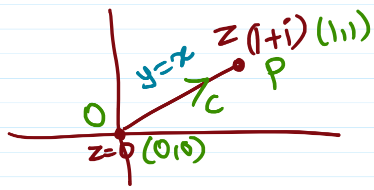
Type I: Problems Based on Line & Parabola

① Evaluate $\int_0^{1+i} (x - y + ix^2)\, dz$

ⓐ along the line from $z=0$ to $z=1+i$

ⓑ Along the parabola $y^2 = x$

Sol. ⓐ



Equation of line OP is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{1-0} = \frac{x-0}{1-0}$$

$$\boxed{y = x}$$

The equation of the path C is $\boxed{y=x}$

$$\frac{dy}{dx} = 1$$

$$\boxed{dy = dx}$$

$$\int_0^{1+i} (x - y + ix^2)\, dz = \int_0^{1+i} (x - y + ix^2)(dx + i\,dy)$$

$$= \int_0^1 (x - x + ix^2)(dx + i\,dx)$$

$$= \int_0^1 (ix^2)(1+i)\, dx$$

$$= i(1+i) \int_0^1 x^2\, dx$$

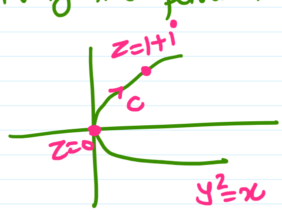
$$= (i-1)\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}(i-1)$$

ⓑ The path of Integration C is along the parabola

$$\boxed{y^2 = x}$$

$$2y\frac{dy}{dx} = 1$$

$$\boxed{2y\,dy = dx}$$



$$\int_0^{1+i} (x - y + ix^2)\, dz = \int_0^{1+i} (x - y + ix^2)(dx + i\,dy)$$

$$= \int_0^1 (y^2 - y + i(y^2)^2)(2y\,dy + i\,dy)$$

$$= \int_0^1 (y^2 - y + iy^4)(2y + i)\, dy$$

$$= \int_0^1 (y^2-y)2y - y^4 + i\left(y^2 y + 2y^5\right)dy$$

$$= \int_0^1 \left(2y^3 - 2y^2 - y^4\right) + i\left(y^2 y + 2y^5\right)dy$$

$$= \left[\frac{y^4}{2} - 2\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 + i\left[\frac{y^3}{3} - \frac{y^2}{2} + \frac{y^6}{3}\right]_0^1$$

$$= \left(\frac{1}{2} - \frac{2}{3} - \frac{1}{5}\right) + i\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{3}\right)$$

$$= \frac{15-20-6}{30} + i\left(\frac{1}{6}\right)$$

$$= \frac{-11}{30} + \frac{i}{6}$$

(2) Evaluate $\int_0^{1+i} z^2 dz$ along the (a) line $y = x$
(b) parabola $x = y^2$

Is the line Integral Independent of path?

Sol:

(a) The path of integration c is along line $y = x$

$\therefore dy = dx$

$$\int_0^{1+i} z^2 dz = \int_0^{1+i} (x+iy)^2 (dx + i dy)$$

$$= \int_0^{1+i} \left(x^2 - y^2 + i2xy\right)(dx + i dy)$$

$$= \int_0^1 \left(x^2 - x^2 + i2xx\right)(dx + i dx)$$

$$= \int_0^1 \left(i2x^2\right)(1+i)\,dx$$

$$= i(1+i)\int_0^1 2x^2 dx$$

$$= i(1+i)\left[\frac{2x^3}{3}\right]_0^1$$

$$= \frac{2}{3}(i-1)$$

(b) The path of Integration c is along the line $x = y^2$

$dx = 2y\,dy$

$$\int_0^{1+i} z^2 dz = \int_0^{1+i} \left(x^2 - y^2 + i2xy\right)(dx + i dy)$$

$$= \int_0^1 \left(y^4 - y^2 + i2y^2 y\right)(2y\,dy + i\,dy)$$

$$= \int_0^1 \left(y^4 - y^2 + i2y^3\right)(2y + i)\,dy$$

$$= \int_0^1 \left[\left(y^4 - y^2\right)(2y) - 2y^3 + i\left(4y^4 + y^4 - y^2\right)\right]dy$$

$$= \int_0^1 \left(2y^5 - 4y^3\right) + i\left(5y^4 - y^2\right)dy$$

$$= \left[\frac{y^6}{3} - y^4\right]_0^1 + i\left[y^5 - \frac{y^3}{3}\right]_0^1$$

$$= \left(\frac{1}{3} - 1\right) + i\left(1 - \frac{1}{3}\right)$$

$$= -\frac{2}{3} + i\frac{2}{3} = -\frac{2}{3}(1-i)$$

$$= \frac{2}{3}(i-1)$$

$\therefore$ line integral is independent of path as $f(z) = z^2$ is analytic.

Problems Based on Parameters

Problems Based on Parameters

① Evaluate $\int_C f(z)\,dz$ where $f(z) = x^2 + ixy$ from A (1,1) to B(2,4) along the curve $x = t, y = t^2$

OR

————————————————||———————— along the curve $z = t + it^2$

Sol: The path of integration C is along the curve

$$\boxed{x = t},\quad \boxed{y = t^2}$$

$$\frac{dx}{dt} = 1,\quad \frac{dy}{dt} = 2t$$

$$\boxed{dx = dt},\quad \boxed{dy = 2t\,dt}$$

$$\int_C f(z)\,dz = \int_C (x^2 + ixy)\,dz$$

$$= \int_C (x^2 + ixy)(dx + i\,dy)$$

$$= \int_1^2 (t^2 + i\,t\,t^2)(dt + i\,2t\,dt)$$

$$= \int_1^2 (t^2 + it^3)(1 + i2t)\,dt$$

As $x : 1 \longrightarrow 2$
But $x = t \Rightarrow$
$t : 1 \longrightarrow 2$

As $y : 1 \longrightarrow 4$
But $y = t^2 \Rightarrow$
$t^2 : 1 \longrightarrow 4$
$t : 1 \longrightarrow 2$

$$= \int_1^2 (t^2 - 2t^4) + i\,(t^3 + 2t^3)\,dt$$

$$= \int_1^2 (t^2 - 2t^4) + i(3t^3)\,dt$$

$$= \left[\frac{t^3}{3} - \frac{2t^5}{5}\right]_1^2 + i\left[\frac{3t^4}{4}\right]_1^2$$

$$= \left(\frac{8}{3} - \frac{64}{5}\right) - \left(\frac{1}{3} - \frac{2}{5}\right) + i\left(\frac{48}{4} - \frac{3}{4}\right)$$

$$= \frac{-151}{15} + i\,\frac{45}{4}$$

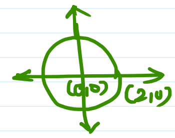
H.W

④ Evaluate $\int \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve $z = t^2 + it$

$\boxed{\text{Ans } \dfrac{30-8i}{3}}$

Type 2: Problems Based on circle

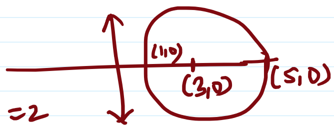
① $|z| = 2$ Complex eqn. | $x^2+y^2=2^2$ Cartesian eqn. | $r = 2$ polar equation



② $|z - z_0| = r$

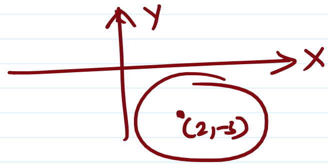
z_0 ↙ centre, r ↖ radius

ⓐ $|z-3| = 2 \Longrightarrow$ centre $= (3,0)$, radius $= 2$



ⓑ $|z-2+3i| = 1 \Longrightarrow$

$C \equiv (2,-3)$, radius $= 1$



③ Substitution

$$|z - z_0| = r$$

put
$$z - z_0 = re^{i\theta}$$
$$dz = ri\,e^{i\theta}d\theta$$

eg ① $|z| = 2$
$z = 2e^{i\theta}$
$dz = 2ie^{i\theta}d\theta$

② $|z-1| = 2$
$z-1 = 2e^{i\theta}$
$dz = 2ie^{i\theta}d\theta$

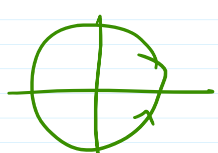
③ $|z-1-i| = 3$
$z-1-i = 3e^{i\theta}$
$dz = 3ie^{i\theta}d\theta$

④ $z = x+iy$
$\bar{z} = x - iy$
$|z| = \sqrt{x^2+y^2}$

$|z|^2 = x^2+y^2$
$|z|^2 = z\bar{z}$
$Im(z) = y$

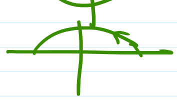
$Real(z) = x$

⑤ circle $\Longrightarrow$



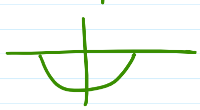
$\theta : 0$ to 2π

Upper Half $\Longrightarrow$



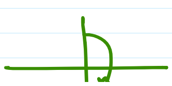
$\theta : 0$ to π

Lower Half $\Longrightarrow$



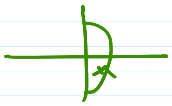
$\theta : \pi$ to 2π

Right Half $\Longrightarrow$



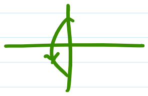
$\theta : -\frac{\pi}{2}$ to $\frac{\pi}{2}$

Right Half $\Rightarrow$



$\theta: -\frac{\pi}{2}$ to $\frac{\pi}{2}$

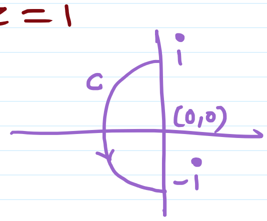
Left Half $\Rightarrow$



$\theta: \frac{\pi}{2}$ to $\frac{3\pi}{2}$

① Evaluate $\int_C |z|\,dz$ where C is the left Half of unit circle $|z|=1$ from $z=-i$ to $z=i$

Sol:



$|z|=1$

Let $I = \int_C |z|\,dz$

put $z = 1\,e^{i\theta}$

$dz = i e^{i\theta} d\theta$

$$I = \int_C |e^{i\theta}|\, i e^{i\theta} d\theta$$

$$\boxed{\begin{aligned}|e^{i\theta}| &= |\cos\theta + i\sin\theta| \\ &= \sqrt{\cos^2\theta + \sin^2\theta} = 1\end{aligned}}$$

$$= \int_C 1\, i e^{i\theta} d\theta$$

$$= i \int_C e^{i\theta} d\theta$$

$$= i \int_{\pi/2}^{3\pi/2} e^{i\theta} d\theta = i \left[\frac{e^{i\theta}}{i}\right]_{\pi/2}^{3\pi/2}$$

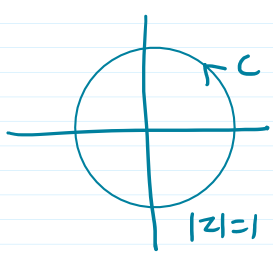
$$= \left[e^{i\theta}\right]_{\pi/2}^{3\pi/2} = e^{i3\pi/2} - e^{i\pi/2}$$

$$= \left(\underset{0}{\cos\tfrac{3\pi}{2}} + i\underset{-1}{\sin\tfrac{3\pi}{2}}\right) - \left(\underset{0}{\cos\tfrac{\pi}{2}} + i\underset{1}{\sin\tfrac{\pi}{2}}\right)$$

$$= (0 + i(-1)) - (0 + i(1)) = -2i$$

② Show that $\int_C \log z\,dz = 2\pi i$ where C is the unit circle in z-plane (xy plane)

Sol:



$I = \int_C \log z\,dz$

put $z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$

$$I = \int_0^{2\pi} \log e^{i\theta}\, i e^{i\theta} d\theta$$

$$\log e^{i\theta} = i\theta \log e = i\theta$$

$$= \int_0^{2\pi} i\theta\, i e^{i\theta} d\theta$$

$= \int_0^{2\pi} i\theta \, i e^{i\theta} d\theta$ $\qquad \log e^{i\theta} = i\theta$

$= i^2 \int_0^{2\pi} \theta e^{i\theta} d\theta$ $\qquad \int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$

$= i^2 \left[\theta \left(\frac{e^{i\theta}}{i}\right) - (1)\left(\frac{e^{i\theta}}{i^2}\right) \right]_0^{2\pi}$

$= \left[i\theta e^{i\theta} - e^{i\theta} \right]_0^{2\pi}$

$= \left\{ i\,2\pi e^{i2\pi} - e^{i2\pi} \right\} - \left\{ 0 - e^{i(0)} \right\}$

$e^{i0} = \cos 0 + i\sin 0 = 1 + i(0) = 1$

$e^{i2\pi} = \cos 2\pi + i\sin 2\pi = 1 + i(0) = 1$

$= \{ i2\pi - 1 \} - \{ 0 - 1 \} = 2\pi i$

③ Evaluate $\int_c \frac{dz}{(z-4)^{20}}$ where c is $|z-4| = 3$

Sol: put $z - 4 = 3e^{i\theta}$

$dz = 3i e^{i\theta} d\theta$

$I = \int_0^{2\pi} \frac{3ie^{i\theta} d\theta}{(3e^{i\theta})^{20}}$

$= i \int_0^{2\pi} \frac{3e^{i\theta} d\theta}{(3e^{i\theta})^{20}} = i \int_0^{2\pi} (3e^{i\theta})^{-19} d\theta = i\, 3^{-19} \int_0^{2\pi} e^{-i19\theta} d\theta$

$\int e^{-im\theta} = \frac{e^{-im\theta}}{-im}$

$= \frac{i}{3^{19}} \left[\frac{e^{-i19\theta}}{-i19} \right]_0^{2\pi}$

$= \frac{1}{-19 \cdot 3^{19}} \left[e^{-i19\theta} \right]_0^{2\pi} = \frac{-1}{19 \cdot 3^{19}} \left[e^{-i38\pi} - e^{-i0} \right]$

$= \frac{-1}{19 \cdot 3^{19}} \left[(\underset{1}{\cos 38\pi} - i \underset{0}{\sin 38\pi}) - (\underset{1}{\cos 0} - i \underset{0}{\sin 0}) \right]$

$= \frac{-1}{19 \cdot 3^{19}} \left[(1 - i(0)) - (1 - i(0)) \right] = 0$

H.W

Q① Evaluate $\int_C \frac{2z+3}{z} dz$ where c is

ⓐ upper Half of the circle $|z| = 2$

ⓑ Lower Half — " —

ⓒ The whole circle — " —

(c) The whole circle

(2) Evaluate $\int_C (\bar{z}+2z)\,dz$ where C is $x^2+y^2=1$

$|z|=1$

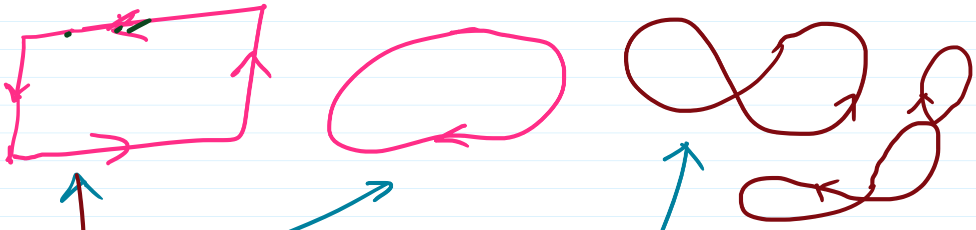
$z=e^{i\theta}$ $\boxed{dz=ie^{i\theta}d\theta}$

$\bar{z}=e^{-i\theta}$

$\bar{z}=$

Simple closed curve :

If a curve does not intersect itself twice then it is called simple curve (Jordan curve)

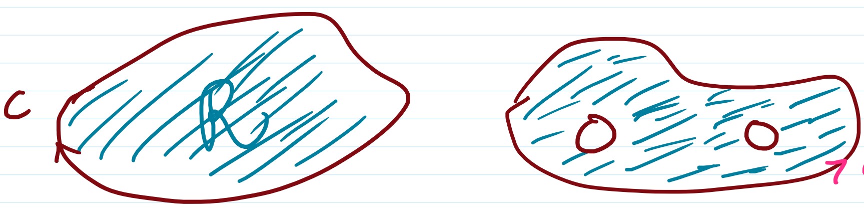


Simple closed curve

Not simple closed curve

Simply connected Region :

A region R is simply connected if every closed curve in the region encloses points of the region R

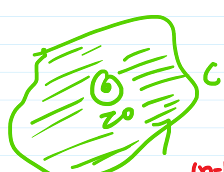


simply connected Region

Not simply connected

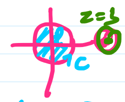
Region without Hole

Region with Hole

<table>
<tr><th>Cauchy's Integral theorem</th><th>Cauchy's Integral formula</th></tr>
<tr><td>If $f(z)$ is analytic inside and on a closed curve c of a simply connected Region R and its derivative $f'(z)$ is continuous then
$$\oint_C f(z)dz = 0$$
</td><td>If $f(z)$ is analytic inside & on a closed curve C of a simply connected Region R and if z_0 is any point within C then
$$\int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$

Note:
$$\int_C \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i f^{(n-1)}(z_0)}{(n-1)!}$$</td></tr>
<tr><td>eg. $\oint \frac{e^z}{\dots} dz$ where c is</td><td>eg. $\oint \frac{e^z}{z-3} dz$ where C is $|z|=4$</td></tr>
</table>

eg. $\oint_C \frac{e^z}{z-3}\,dz$ where c is $|z|=1$

Sol: we have $z-3=0$
$z=3$



$\therefore f(z)$ is analytic inside & on C
$\therefore$ By Cauchy's Integral Thm
$$\oint_C \frac{e^z}{z-3}\,dz = 0$$

eg. $\oint \frac{e^z}{z-3}\,dz$ where C is $|z|=4$

Sol:

We have $z-3=0$
$z=3$

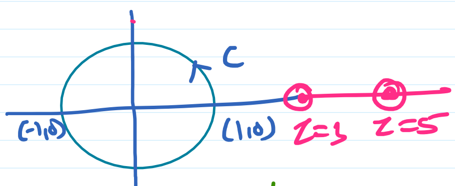
$z=3$ lies inside C
$\therefore$ By Cauchy's Integral formula
$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$
$$\oint_C \frac{e^z}{z-3}\,dz = 2\pi i\, f(3) = 2\pi i\, e^3 = 2\pi e^3 i$$

$$\boxed{\begin{array}{l} f(z)=e^z \\ f(3)=e^3 \end{array}}$$

① Evaluate $\oint_C \frac{z^2+2z}{(z-3)(z-5)}$ where c is ⓐ $|z|=1$
ⓑ $|z|=4$

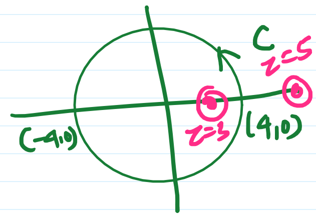
Sol: We have $(z-3)(z-5)=0$
$z=3,5$

ⓐ When c is $|z|=1$



$\therefore z=3,5$ lies outside C
$\therefore f(z)$ is analytic inside and on C
$\therefore$ By Cauchy's Integral Thm.
$$\oint_C f(z)\,dz = 0$$
$$\oint_C \frac{z^2+2z}{(z-3)(z-5)}\,dz = 0$$

ⓑ When c is $|z|=4$



$z=3$ lies inside C
$\therefore f(z)$ is analytic inside and on C
$\therefore$ By Cauchy's Integral Formula
$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$
$$\oint_C \frac{z^2+2z}{(z-3)(z-5)}\,dz = 2\pi i\, f(3)$$
$$\oint_C \frac{\frac{z^2+2z}{z-5}}{z-3}\,dz = 2\pi i\, f(3) = 2\pi i\left(\frac{-15}{2}\right) = -15\pi i$$

Here $f(z) = \frac{z^2+2z}{z-5}$

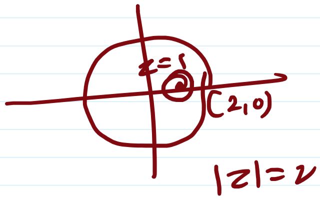
$f(3) = \frac{3^2+2(3)}{3-5} = \frac{-15}{2}$

② Evaluate $\oint \frac{e^{2z}}{\ldots}\,dz$ where c is $|z|=2$

(3) Evaluate $\oint$

Sol: We have $(z-1)^4 = 0$

$z = 1$



$\therefore z=1$ lies inside C

$\therefore f(z)$ is analytic inside & on C

$\therefore$ By Cauchy's Integral formula

$$\oint_C \frac{f(z)\,dz}{(z-z_0)^n} = \frac{2\pi i\, f^{(n-1)}(z_0)}{(n-1)!}$$

$$\oint_C \frac{e^{2z}}{(z-1)^4}\,dz = \frac{2\pi i\, f^{(3)}(1)}{3!} \longrightarrow (*)$$

Now $f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

$f^{(3)}(z) = 8e^{2z}$ $\therefore f^{(3)}(1) = 8e^{2\times 1} = 8e^{2}$

$\therefore$ from $(*)$, we get

$$\oint_C \frac{e^{2z}}{(z-1)^4}\,dz = \frac{2\pi i\, 8e^2}{6} = \frac{8}{3}\pi i e^2$$

(4) Evaluate $\oint_C \frac{\sin^6 z\,dz}{(z-\frac{\pi}{6})^3}$ where c is $|z|=1$

Ans $\frac{21\pi i}{16}$

Sol:

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i\, f^{(n-1)}(z_0)}{(n-1)!}$$

$$\int_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3} = \frac{2\pi i\, f^{(2)}(\frac{\pi}{6})}{2!} = \pi i f^{(2)}\left(\frac{\pi}{6}\right) \longrightarrow *$$

$$\oint_C \frac{\quad}{(z-\frac{\pi}{6})^3} \qquad \frac{\quad}{2!}$$

$$f(z) = \sin^6 z$$

$$f'(z) = 6\sin^5 z \cos z$$

$$f''(z) = 6\left[\sin^5 z(-\sin z) + \cos z\left(5\sin^4 z \cos z\right)\right]$$

$$f''\left(\frac{\pi}{6}\right) = 6\left[\left(\frac{1}{2}\right)^5\left(-\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(5\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)\right)\right]$$

$$= 6\left[-\left(\frac{1}{2}\right)^6 + 5 \times \frac{3}{2^6}\right] = \frac{6}{2^6}\left[-1+15\right] = \frac{6\times 14}{2^6}$$

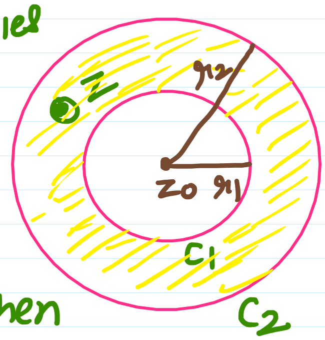
$$= \frac{\cancel{6}^{\,3}\times\cancel{14}^{\,7}}{\cancel{64}_{\,\cancel{32}\,16}} = \frac{21}{16}$$

from ⊛, we get

$$\oint_C \frac{\sin^5 z\, dz}{(z-\frac{\pi}{6})^3} = \pi i\left(\frac{21}{16}\right) = \frac{21\pi i}{16}$$

Taylor's & Laurent's series

If C_1 & C_2 are two concentric circles of radius r_1 & r_2 with centre at z_0 and if $f(z)$ is analytic on C_1 & C_2 and in the annular region R between two circles then for any point z in R



$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Analytic part/ Regular part}} + \underbrace{\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}}_{\text{principal part.}} \quad \text{---} \; *$$

$$= \left[a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots\right]$$

$$+ \left[\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots\right]$$

Some Useful Series

1. $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad , |z| < \infty$

2. $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \quad , |z| < \infty$

3. $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots \quad , |z| < \infty$

4. $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \quad , |z| < \infty$

5. $\cosh z = 1 + z^2 + z^4 + z^6 + \cdots \quad , |z| < \infty$

$$\frac{}{2!} \quad \frac{}{4!} \quad \frac{}{6!}$$

⑥ $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \; |z|<1$

⑦ $\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 - \cdots, \; |z|<1$

⑧ $\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \cdots, \; |z|<1$

① Obtain Laurent's series for $f(z) = z^3 e^{1/z}$ about $z=0$

Sol:

$$f(z) = z^3 e^{1/z}$$

$$= z^3\left[1 + \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^4}{4!} + \frac{\left(\frac{1}{z}\right)^5}{5!} + \cdots\right]$$

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$= z^3\left[1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \frac{1}{24z^4} + \frac{1}{120z^5} + \cdots\right]$$

$$= \underbrace{z^3 + z^2 + \frac{z}{2} + \frac{1}{6}}_{\text{Analytic part}} + \underbrace{\frac{1}{24z} + \frac{1}{120z^2} + \cdots}_{\text{principal part}}$$

② Obtain Laurent's series for $f(z) = \frac{e^{3z}}{(z-1)^3}$ about $z=1$

Sol:

$$f(z) = \frac{e^{3z}}{(z-1)^3}$$

$$= \frac{e^{3(z-1+1)}}{(z-1)^3} = \frac{e^{3(z-1)+3}}{(z-1)^3} = \frac{e^{3(z-1)} \cdot e^3}{(z-1)^3}$$

$$(z-1)^3 \qquad (z-1)^3 \qquad (z-1)^3$$

$$= \frac{e^3}{(z-1)^3} e^{3(z-1)}$$

$$= \frac{e^3}{(z-1)^3}\left[1 + 3(z-1) + \frac{(3(z-1))^2}{2!} + \frac{(3(z-1))^3}{3!} + \frac{(3(z-1))^4}{4!} + \cdots\right]$$

$$= e^3\left[\frac{1}{(z-1)^3} + \frac{3}{(z-1)^2} + \frac{3^2}{2(z-1)} + \frac{3^3}{6} + \frac{3^4(z-1)}{24} + \cdots\right]$$

③ Obtain Laurent's expansion of $f(z) = (z-3)\sin\left(\frac{1}{z+2}\right)$ about $z=-2$.

Sol:

$$f(z) = (z-3)\sin\left(\frac{1}{z+2}\right)$$

$$= (z+2-5)\sin\left(\frac{1}{z+2}\right)$$

$$= (z+2)\sin\left(\frac{1}{z+2}\right) - 5\sin\left(\frac{1}{z+2}\right)$$

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$

$$= (z+2)\left[\frac{1}{z+2} - \frac{\left(\frac{1}{z+2}\right)^3}{3!} + \frac{\left(\frac{1}{z+2}\right)^5}{5!} - \cdots\right]$$

$$-5\left[\frac{1}{z+2} - \frac{\left(\frac{1}{z+2}\right)^3}{3!} + \frac{\left(\frac{1}{z+2}\right)^5}{5!} - \cdots\right]$$

$$= \left[1 - \frac{1}{3!(z+2)^2} + \frac{1}{5!(z+2)^4} - \cdots\right]$$

$$-5\left[\frac{1}{z+2} - \frac{1}{3!(z+2)^3} + \frac{1}{5!(z+2)^5} - \cdots\right]$$

* Residue Theorem *

Consider the Laurent's series Expansion of $f(z)$ at $z=z_0$

① $$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Analytic Part}} + \underbrace{\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots\cdots}_{\text{principle part}}$$

Infinite terms

If principal part contains infinite terms then z_0 is called Isolated essential singularity

② $$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{Analytic part}} + \underbrace{\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}}_{\text{principle part}}$$

finite terms

If principal part contains finite term then z_0 is called pole of order m.

The pole of order 1 is called simple pole

③ $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + (\text{No principal part})$$

z_0 is called removable singularity if it does not contain principal part.

④ **Residue:**

The coefficient of $\frac{1}{z-z_0}$ is called residue in Laurent's series expansion.

Calculation of Residue at Poles

Calculation of Residue at Poles

Pole of order 1 (simple pole)	Pole of order m
Res. of $f(z)$ at $z=z_0$ $= \lim_{z\to z_0}(z-z_0)f(z)$	Res. of $f(z)$ at $z=z_0$ $= \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)$

Cauchy's Residue Theorem:

If $f(z)$ is analytic inside & on a simple closed curve C, except at finite No. of isolated points $z_0, z_1, z_2, \ldots z_n$ then

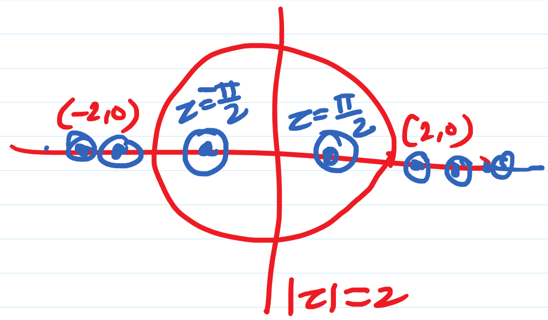
$$\oint_C f(z)\,dz = 2\pi i\,(\text{sum of Residues at } z_1, z_2, \ldots z_n)$$

① Evaluate $\oint_C \frac{dz}{\cos z}$ where C is $|z|=2$ by Residue theorem.

Sol: We have $\cos z = 0$

$z = \cos^{-1} 0$

$z = (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$



$$z = \left\{ \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots \right\}$$

$\therefore$ $z = \pm\frac{\pi}{2}$ are simple poles (pole of order 1)

ⓐ Residue at $z = \pi/2$

$$\text{Res. of } f(z) \text{ at } z = z_0 = \lim_{z \to z_0} (z - z_0) f(z)$$

$$\text{Res. of } f(z) \text{ at } z = \pi/2 = \lim_{z \to \frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) \frac{1}{\cos z}$$

$$= \lim_{z \to \frac{\pi}{2}} \frac{z - \frac{\pi}{2}}{\cos z} \quad \left[\frac{0}{0}\right]$$

(L'Hospital)

$$= \lim_{z \to \frac{\pi}{2}} \frac{1}{-\sin z}$$

$$= \frac{1}{-\sin\frac{\pi}{2}} = \frac{1}{-1} = -1$$

ⓑ Residue of $f(z)$ at $z = -\pi/2$

$$\text{Res. of } f(z) \text{ at } z = z_0 = \lim_{z \to z_0} (z - z_0) f(z)$$

$$\text{Res. of } f(z) \text{ at } z = -\frac{\pi}{2} = \lim_{z \to -\frac{\pi}{2}} \left(z + \frac{\pi}{2}\right) \frac{1}{\cos z}$$

$$= \lim_{z \to -\frac{\pi}{2}} \frac{z + \pi/2}{\cos z} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{z \to -\frac{\pi}{2}} \frac{1}{-\sin z} = \frac{1}{-\sin(-\frac{\pi}{2})} = \frac{1}{1} = 1$$

$\therefore$ By Residue thm.

$$\oint_C f(z)\,dz = 2\pi i\,(\text{sum of Residue})$$

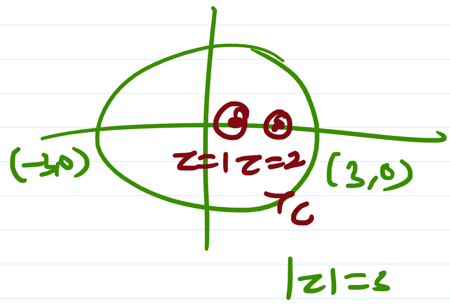
$$\oint_C \frac{dz}{\cos z} = 2\pi i\,(-1+1) = 0$$

② Evaluate by Residue Theorem $\int_C \frac{z^2\,dz}{(z-1)^2(z-2)}$ where C is $|z| = 3$

Sol: We have

$$(z-1)^2(z-2) = 0$$

$$z = 1, 1, \quad z = 2$$



$\therefore$ $z = 1$ is pole of order 2.

Also, $z = 2$ is pole of order 1 (simple pole)

Also, $z = 1, 2$ lies inside C.

ⓐ Residue at simple pole $(z=2)$

$$\text{Res. of } f(z) \text{ at } z = z_0 = \lim_{z \to z_0} (z - z_0) f(z)$$

$$\text{Res. of } f(z) \text{ at } z = 2 = \lim_{z \to 2} \cancel{(z-2)} \frac{z^2}{(z-1)^2 \cancel{(z-2)}}$$

$$= \lim_{z \to 2} \frac{z^2}{(z-1)^2} = \frac{2^2}{(2-1)^2} = 4$$

ⓑ Residue at Pole of order m

$$\text{Res. of } f(z) \text{ at } z = z_0 = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z) \right)$$

Here $m=2$

$$\text{Res. of } f(z) \text{ at } z=1 = \frac{1}{(2-1)!} \lim_{z\to 1} \frac{d}{dz}\left(\cancel{(z-1)^2} \frac{z^2}{\cancel{(z-1)^2}(z-2)}\right)$$

$$= \lim_{z\to 1} \frac{d}{dz}\left(\frac{z^2}{z-2}\right)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \lim_{z\to 1} \frac{(z-2)(2z) - z^2(1)}{(z-2)^2}$$

$$= \frac{(1-2)(2\times 1) - (1)^2(1)}{(1-2)^2}$$

$$= \frac{-2-1}{1} = -3$$

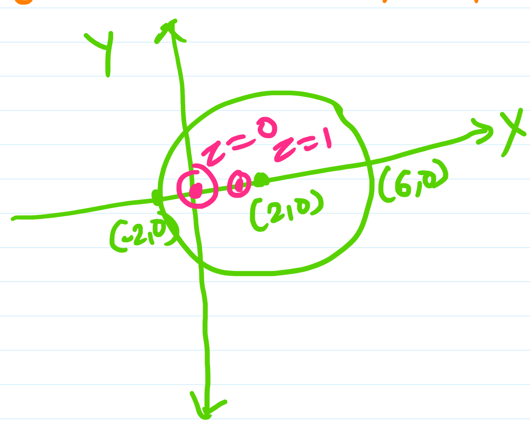
$\therefore$ By Residue thm.

$$\oint_C f(z)\,dz = 2\pi i\,(\text{sum of Residues})$$

$$\oint_C \frac{z^2}{(z-1)^2(z-2)}\,dz = 2\pi i\,(4-3) = 2\pi i$$

③ Using Residue Theorem evaluate

$$\oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-z^2}\,dz \quad \text{where } C \text{ is } |z-2|=4$$



Sol: We have

$$z-z^2=0$$
$$z(1-z)=0$$
$$z=0,\ 1-z=0$$
$$z=0,\ z=1$$

$\therefore z=0,1$ lies inside C

$z = 0, 1$ both are simple pole

Res. of $f(z)$ at simple

$$\underset{\text{at } z=z_0}{\text{Res. of } f(z)} = \lim_{z \to z_0} (z - z_0) f(z)$$

① $$\underset{\text{at } z=0}{\text{Res. of } f(z)} = \lim_{z \to 0} (z-0) \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z - z^2}$$

$$= \lim_{z \to 0} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(1-z)}$$

$$= \frac{\sin(0) + \cos(0)}{1-0} = \frac{0+1}{1} = 1$$

② $$\underset{\text{at } z=1}{\text{Res. of } f(z)} = \lim_{z \to 1} (z-1) \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z - z^2}$$

$$= \lim_{z \to 1} (z-1) \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z(1-z)}$$

$$= \lim_{z \to 1} \cancel{(z-1)} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{-z\cancel{(z-1)}}$$

$$= \lim_{z \to 1} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{-z}$$

$$= \frac{\sin \pi + \cos \pi}{-1} = \frac{0-1}{-1} = 1$$

$\therefore$ By Residue thm.

$$\oint_C f(z)\, dz = 2\pi i \,(\text{sum of Residues})$$

$$\oint \frac{\sin(\pi z^2) + \cos(\pi z^2)}{\ldots^2}\, dz = 2\pi i\,(1+1)$$

$C \qquad z-z \qquad = 4\pi i$