

Mathematical Foundations of Computer Science

Lecture Outline

July 8, 2022

Example Let A_1, A_2, \dots, A_n be sets (where $n \geq 2$). Suppose for any two sets A_i and A_j either $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Prove by induction that one of these n sets is a subset of all of them.

Solution. We will prove the claim using induction on n .

Induction Hypothesis: Assume that the claim is true when $n = k$, for some $k \geq 2$. In other words, assume that if we have sets A_1, A_2, \dots, A_k , where for any two sets A_i and A_j , either $A_i \subseteq A_j$ or $A_j \subseteq A_i$ then one of the k sets is a subset of all of the k sets.

Base Case: $n = 2$. We have two sets A_1, A_2 and we know that $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. Without loss of generality assume that $A_1 \subseteq A_2$. Then A_1 is a subset of A_1 and is also a subset of A_2 , so the claim holds when $n = 2$.

Induction Step: We want to prove the claim when $n = k + 1$. That is, we are given a set $S = \{A_1, A_2, \dots, A_{k+1}\}$ of with the property that for every pair of sets $A_i \in S$ and $A_j \in S$, either $A_i \subseteq A_j$ or $A_j \subseteq A_i$. We want to show that there is a set in S that is a subset of all $k + 1$ sets in S . Let $S' = S \setminus \{A_{k+1}\}$. By induction hypothesis, there is a set $A_p \in S'$ that is a subset of all sets in S' . We now consider the following two cases.

Case 1: $A_p \subseteq A_{k+1}$. Then it follows that A_p is a subset of all sets in S .

Case 2: $A_{k+1} \subseteq A_p$. Since A_p is a subset of all sets in S' and $A_{k+1} \subseteq A_p$, it follows that A_{k+1} is a subset of all sets in S .

Example. For all $n \geq 1$, prove that n lines separate the plane into $(n^2 + n + 2)/2$ regions. Assume that no two of these lines are parallel and no three pass through a common point.

Solution. Let $P(n)$ be the property that n lines, such that no two of them are parallel and no three of them pass through a common point, separate the plane into $(n^2 + n + 2)/2$ regions. We will prove the claim by induction on n .

Induction Hypothesis: Assume that $P(k)$ is true for some $k > 0$.

Base Case: $P(1)$ is true since one line divides the plane into 2 regions which is also given by $(1^2 + 1 + 2)/2$.

Induction Step: To prove that $P(k + 1)$ is true. Consider a set S of $k + 1$ lines such that no two of them are parallel and no three of them pass through a common point. Remove any line ℓ from the set S . Let S' be the resulting set of k lines. By induction hypothesis, the k lines in S' divide the plane into $(k^2 + k + 2)/2$ regions. Now we add the line ℓ to the set S' to obtain the set S . Line ℓ intersects exactly once with each of the k lines in S' . These intersections divide the line ℓ into $k + 1$ line segments. Each of these line segments passes

through a region and hence $k + 1$ additional regions are created. Hence, the total number of regions formed by $k + 1$ lines is given by

$$\frac{k^2 + k + 2}{2} + k + 1 = \frac{k^2 + 3k + 4}{2} = \frac{k^2 + 2k + 1 + k + 3}{2} = \frac{(k + 1)^2 + (k + 1) + 2}{2}$$

Thus $P(k + 1)$ is correct and this completes the proof.

Example. Let n be a non-negative integer. Show that any $2^n \times 2^n$ region with one central square removed can be tiled using L-shaped pieces, where the pieces cover three squares at a time (Figure 1).

Solution. (Attempt 1) Let R_n denote a $2^n \times 2^n$ region. Let $P(n)$ be the property that R_n with one central square removed can be tiled using L-shaped pieces.

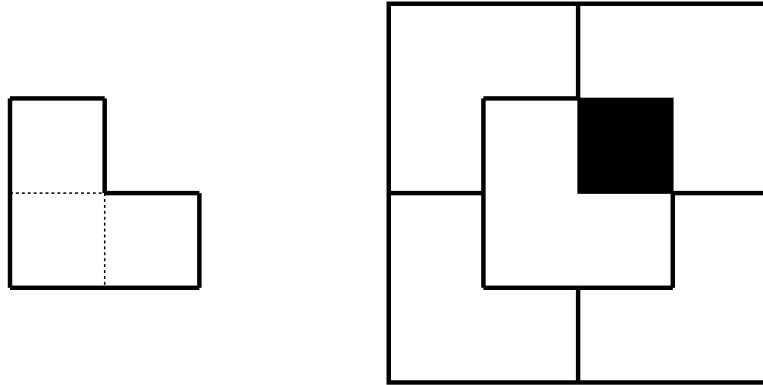


Figure 1: A L-tile and an L-tiling of a $2^2 \times 2^2$ region without a square.

Induction Hypothesis: Assume that $P(k)$ is true for some $k \geq 0$.

Base Case: We want to prove that $P(0)$ is true. This is true because a 1×1 region with one central square removed requires 0 tiles.

Induction Step: We want to prove that $P(k + 1)$ is true, i.e., region R_{k+1} with one central square removed can be tiled using L-shaped pieces.

R_{k+1} can be divided into four regions of size $2^k \times 2^k$. Note that the four central corners of R_{k+1} can be covered using one L-shaped tile and one square hole (Figure 2). Each of the four remaining regions has one hole and is of the size $2^k \times 2^k$. By induction hypothesis, these regions can be covered using L-shaped pieces. Thus, since the four disjoint regions can be covered using L-shaped tiles, R_{k+1} without a central square can also be covered using L-shaped tiles.

Our use of induction hypothesis is incorrect as we have assumed that region R_k without a *central* square (not a *corner* square) can be covered using L-shaped tiles.

Surprisingly, we can get around this obstacle by proving the following stronger claim.

“For all positive integers n , any R_n region with *any* one square removed can be L-tiled.”

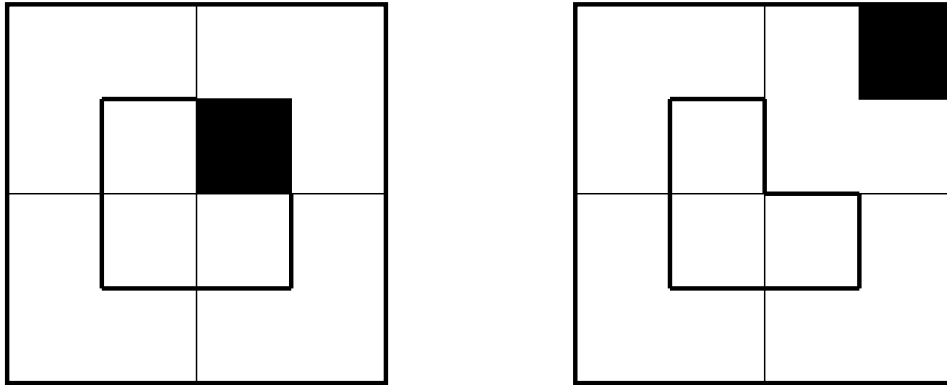


Figure 2: Illustration of the two proof attempts.

Let $P(n)$ be the property that R_n without one square can be L-tiled.

Induction Hypothesis: Assume that $P(k)$ is true for some k .

Base Case: We want to prove that $P(0)$ is true. This is true because a 1×1 region with one square removed requires 0 tiles.

Induction Step: We want to prove that $P(k+1)$ is true, i.e., region R_{k+1} without one square that is located anywhere can be L-tiled. Divide R_{k+1} into four R_k regions. One of the four R_k regions that does not have one square can be L-tiled (using induction hypothesis). Each of the other three R_k regions without the corner square that is located at the center of R_{k+1} can be L-tiled (using induction hypothesis). By using one more L-tile we can cover the three central squares of R_{k+1} .

Strong Induction.

For any property P , if $P(0)$ and $\forall n \in \mathbb{N}, P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$, then $\forall n \in \mathbb{N}, P(n)$.

Example. Prove that if n is an integer greater than 1 then either n is a prime or it can be written as a product of primes.

Solution. Let $P(n)$ be “ n can be written as a product of primes”.

Induction Hypothesis: Assume that $P(j)$ is true for $1 < j \leq k$.

Base Case: We want to show that $P(2)$ is true. This is clearly true as 2 is a prime.

Induction Step: We want to show that $P(k+1)$ is true.

Case I: $k+1$ is prime. In this case we are done.

Case II: $k+1$ is composite. Then,

$$k+1 = a \times b, \quad \text{for some } a \text{ and } b \text{ s.t. } 2 \leq a \leq b < k+1$$

By induction hypothesis, a is a prime or it can be written as a product of primes. The same applies to b . Since $k+1 = a \times b$, it can be written as a product of primes, namely those primes in the factorization of a and those in the factorization of b .

Example. Prove that, for any positive integer n , if x_1, x_2, \dots, x_n are n distinct real numbers, then no matter how the parenthesis are inserted into their product, the number of multiplications used to compute the product is $n - 1$.