Non-Linear Programming Problem

There are 2 methods:

1] Lagrange's method with one equality constraint

1] Optimize $z = f(x_1, x_2, x_3)$ subject to the constraint $g(x_1, x_2, x_3) = b$

2] Let $h = g(x_1, x_2, x_3) - b$ and

$L = F + \lambda h$

To find positive values of x_1, x_2, x_3:

Solve:

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial x_3} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$$

To check whether above values give maximum or minimum, find Hessian matrix H

$$H = \begin{bmatrix} 0 & \partial h/\partial x_1 & \partial h/\partial x_2 & \partial h/\partial x_3 \\ \partial h/\partial x_1 & \partial^2 L/\partial x_1^2 & \partial^2 L/\partial x_1 \partial x_2 & \partial^2 L/\partial x_1 \cdot \partial x_3 \\ \partial h/\partial x_2 & \partial^2 L/\partial x_1 \partial x_2 & \partial^2 L/\partial x_2^2 & \partial^2 L/\partial x_2 \cdot \partial x_3 \\ \partial h/\partial x_3 & \partial^2 L/\partial x_1 \partial x_3 & \partial^2 L/\partial x_2 \partial x_3 & \partial^2 L/\partial x_3^2 \end{bmatrix}$$

Let $m \Rightarrow$ no. of constraints

$n \Rightarrow$ no. of variables

Start with minor of order $2m+1$ and find the last $n-m$ minors. If signs of the minors are alternate, then function will be max, otherwise, it will be minimum.

Ex. 1] Using method of Lagrange's multipliers, solve NLPP.

Optimize $z = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$

subject to $x_1 + x_2 + x_3 = 10$, $x_1, x_2, x_3 \geq 0$

Sol^n: $\Rightarrow$ $h = x_1 + x_2 + x_3 - 10$ and

$L = z + \lambda h$

$L = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23 + \lambda(x_1 + x_2 + x_3 - 10)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 12 - 2x_1 + \lambda = 0$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 8 - 2x_2 + \lambda = 0$

subt.

$4 - 2x_1 + 2x_2 = 0$

$-2x_1 + 2x_2 + 0x_3 = -4$ ... (1)

subt.

$2 - 2x_2 + 2x_3 = 0$

$0x_1 - 2x_2 + 2x_3 = -2$ ... (2)

$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 6 - 2x_3 + \lambda = 0$

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x_1 + x_2 + x_3 - 10 = 0$ ... (3)

(1), (2) and (3) $\Rightarrow$ $\boxed{x_1 = 5}$ $\boxed{x_2 = 3}$ $\boxed{x_3 = 2}$

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}$$

$m = 1 \Rightarrow$ start with

$2m+1 = 3$ i.e., Δ_3

$n = 3 \Rightarrow n - m = 3 - 1 = 2$

$\Rightarrow$ find 2 deltas. $\Rightarrow \Delta_3, \Delta_4$

$$\Delta_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} = 0 - 1(-2-0) + 1(0+2)$$

$= 0 + 2 + 2 = 4$ $\boxed{\Delta_3 = 4}$

$$\Delta_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix} \xrightarrow{\substack{C_3 - C_2 \\ C_4 - C_2}} \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & 2 & 2 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{vmatrix}$$

$$\Delta_4 = -1 \begin{vmatrix} 1 & 2 & 2 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = -\{1(4) - 2(-2) + 2(2)\} = -12$$

$\boxed{\Delta_4 = -12}$

Δ_3 and Δ_4 have alternate signs

$\Rightarrow f$ is maximum at $x_1 = 5,\ x_2 = 3,\ x_3 = 2$

ex. 2 Find the relative maximum or minimum of the function

$$z = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 10x_2 - 14x_3 + 103$$

Solⁿ. $\Rightarrow$ Let $L = z + \lambda h$ But there's no constraint $\therefore h = 0$

$L = \cancel{z}$

$$L = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 10x_2 - 14x_3 + 103$$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 - 6 = 0 \quad \boxed{x_1 = 3}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 - 10 = 0 \quad \boxed{x_2 = 5}$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2x_3 - 14 = 0 \quad \boxed{x_3 = 7}$$

$$\frac{\partial L}{\partial \lambda} = 0$$

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$m = 0$

$n = 3 \quad n - m = 3 - 0 = 3$

$\therefore \Delta_1, \Delta_2, \Delta_3$

$$\left.\begin{aligned} \Delta_1 &= |2| = 2 \\ \Delta_2 &= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \\ \Delta_3 &= \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8 \end{aligned}\right\}$$ Signs same $\Rightarrow$ minimum value of f at $x_1 = 3,\ x_2 = 5,\ x_3 = 7$

Homework:

① Using Lagrange's multipliers, optimize
$z = 2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 196$
subject to $x_1 + x_2 + x_3 = 11$; $x_1, x_2, x_3 \geq 0$.

Ans ⇒ $x_1 = 6,\ x_2 = 2,\ x_3 = 3$

$\Delta_3 = -8$; $\Delta_4 = -48$

$f \Rightarrow$ minimum

② Find maximum or minimum of the function
$z = x_1 + 2x_3 + x_2 \cdot x_3 - x_1^2 - x_2^2 - x_3^2$. (No const).

Ans ⇒ $x_1 = 1/2 \quad x_2 = 2/3 \quad x_3 = 4/3$

$\Delta_1 = -2,\ \Delta_2 = 4 \quad \Delta_3 = -6$

$f \Rightarrow$ maximum.

2] Kuhn-Tucker conditions with only one equality constraint

Optimize $z = f(x_1, x_2, x_3)$ such that $g(x_1, x_2, x_3) \leq b$

Let $h = g(x_1, x_2, x_3) - b$ and $L = f + \lambda h$

Solve $\frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial x_2} = 0,\ \frac{\partial L}{\partial x_3} = 0,\ \lambda h = 0,\ h \leq 0$

Find x_1, x_2, x_3 which will satisfy above equations.

ex. 1] Solve following NLPP by Kuhn-Tucker conditions
$Z_{max} = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$ subject to $2x_1 + x_2 \leq 5$, $x_1, x_2 \geq 0$

Sol.ⁿ ⇒ $h = 2x_1 + x_2 - 5$
$L = F + \lambda h$

Let $L = 10x_1 + 4x_2 - 2x_1^2 - x_2^2 + \lambda(2x_1 + x_2 - 5)$

$\partial L/\partial x_1 = 10 - 2x_1 + 2\lambda = 0 \quad \ldots ①$
$\partial L/\partial x_2 = 4 - 2x_2 + \lambda = 0 \quad \ldots ②$
$\lambda h = 0 \Rightarrow \lambda(2x_1 + x_2 - 5) = 0 \quad \ldots ③$
$h \leq 0 \Rightarrow 2x_1 + x_2 - 5 \leq 0 \quad \ldots ④$

Case I: $\lambda = 0$

① $\Rightarrow$ $10 - 4x_1 = 0$ $\therefore$ $\boxed{x_1 = 2.5}$

② $\Rightarrow$ $4 - 2x_2 = 0$ $\therefore$ $\boxed{x_2 = 2}$

Using these values in eqⁿ IV,

$2(2.5) + 2 - 5 = 2 \nleq 0 \Rightarrow$ Case I fails

Case II: $\lambda \neq 0$ i.e., $2x_1 + x_2 - 5 = 0$

$\Rightarrow$ $2x_1 + x_2 = 5$ ... ⑤

① $10 - 4x_1 - 2\lambda$

2 ② $-$ $8 - 4x_2 - 2\lambda$

$2 - 4x_1 + 4x_2 + 0 = 0$ ... ⑥

$2x_1 + x_2 = 5$

$-4x_1 + 4x_2 = -2$ $\Rightarrow$ $\boxed{x_1 = 1.833} = 11/6$, $\boxed{x_2 = 4/3}$

④ $\Rightarrow$ $2(11/6) + 4/3 - 5 = 11/3 + 4/3 - 5 = 5 - 5 = 0$ ✓

ex. 2] Solve NLPP using Kuhn-Tucker Conditions

$Z_{max} = 8x_1 + 10x_2 - x_1^2 - x_2^2$ such that $3x_1 + 2x_2 \leq 6$, $x_1, x_2 \geq 0$

Solⁿ $\Rightarrow$ let $h = 3x_1 + 2x_2 - 6$

$L = F + \lambda h$

$L = 8x_1 + 10x_2 - x_1^2 - x_2^2 + \lambda(3x_1 + 2x_2 - 6)$

$\partial L/\partial x_1 = 8 - 2x_1 + 3\lambda = 0$ ... ①

$\partial L/\partial x_2 = 10 - 2x_2 + 2\lambda = 0$ ... ②

$\lambda h = \lambda(3x_1 + 2x_2 - 6) = 0$ ... ③

$h \leq 0 \Rightarrow 3x_1 + 2x_2 - 6 \leq 0$ ... ④

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Case 1: $\lambda = 0$

① $8 - 2x_1 = 0$ $\boxed{x_1 = 4}$

② $10 - 2x_2 = 0$ $\boxed{x_2 = 5}$

Using these values in ④ $\Rightarrow$ $3(4) + 2(5) = 12 + 10 - 6 = 16 \neq 0$

$\therefore$ Case 1 fails.

Case 2: $\lambda \neq 0$ i.e., $3x_1 + 2x_2 - 6 = 0$

$\therefore$

$\Rightarrow$ $3x_1 + 2x_2 = 6$ ... ⑤

$$\begin{array}{r} 16 - 4x_1 + 6\lambda = 0 \\ - \quad 30 - 6x_2 + 6\lambda = 0 \\ \hline -14 - 4x_1 + 6x_2 = 0 \end{array}$$

$\Rightarrow -4x_1 + 6x_2 = 14$ ... ⑥

⑤ and ⑥ $\Rightarrow$ $\boxed{x_1 = 4/13}$ $\boxed{x_2 = 33/13}$

④ $\Rightarrow$ $3(4/13) + 2(33/13) - 6 = 12/13 + 66/13 - 6 = 78/13 - 6 = 6 - 6 = 0$ ✓