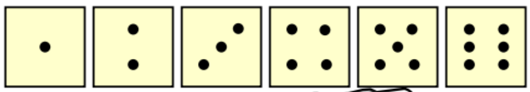
Basic Probability Concepts

- Probability – In general, probability is the chance that some thing will happen. the chance that an uncertain event will occur (always between 0 and 1)
- Event: an event is one or more of the possibly outcomes of doing something.
- Impossible Event – an event that has no chance of occurring (probability = 0)
- Certain Event – an event that is sure to occur (probability = 1)
- Experiment: The activity that produces such an

Sample Space

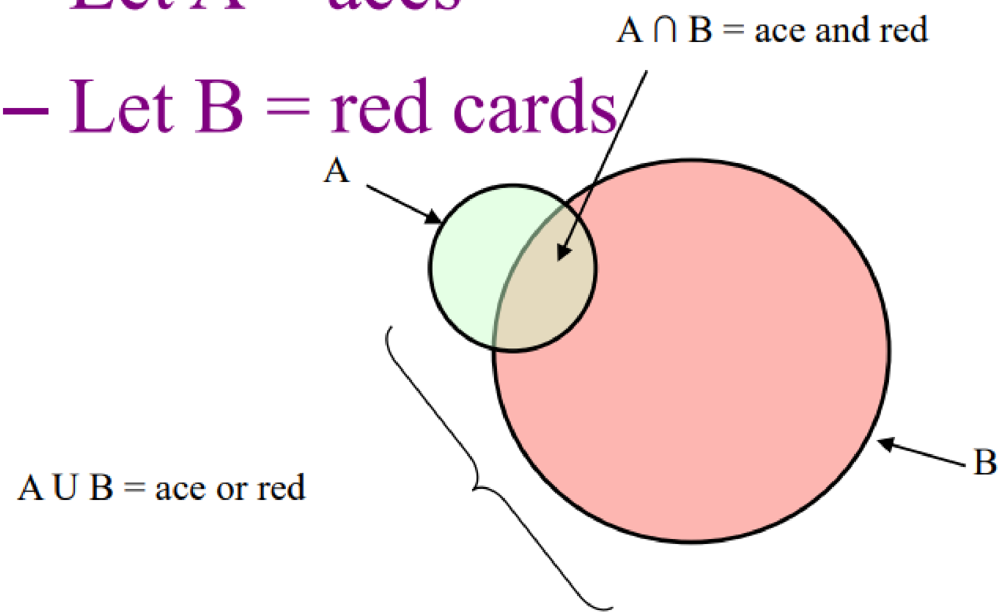
The Sample Space is the collection of all possible events

e.g. **All 6 faces of a <u>die</u>**:



e.g. All 52 cards of a bridge deck:

- Venn Diagrams
  - Let A = aces
  - Let B = red cards



- Simple event
  - An event described by a single **characteristic**
  - e.g., A red card from a deck of cards
- Joint event
  - An event described by two or more characteristics
  - e.g., An **ace that is also red** from a deck of cards
- Complement of an event A (denoted A')
  - All events that are not part of event A
  - e.g., All cards that are not diamonds

Mutually Exclusive Events

- Mutually exclusive events
  - Events that **cannot** occur simultaneously

Example: Drawing one card from a deck of cards

A = queen of diamonds; B = queen of clubs

- Events A and B are mutually exclusive

Collectively Exhaustive Events

- Collectively exhaustive events
  - One of the events must occur
  - The set of events covers the entire sample space

example:

A = aces; B = black cards;
C = diamonds; D = hearts

- Events A, B, C and D are collectively exhaustive (but not mutually exclusive – **an ace may also be a heart**)
- Events B, C and D are collectively exhaustive and also mutually exclusive

Ex: Give a collectively exhaustive list of the possible outcomes of **two dice**.

1,1					1,6
6,1					6,6

Ex: What is the probability for each of the following totals in the rolling of two dice: 1,2,5,6,7,10, and 11.

Answers

P(1)= 0/36

P(2)=1/36

P(5)= 4/36

P(6)= 5/36

P(7)= 6/36

P(10)= 3/36

P(11)= 2/36

Three Types of Probability

1. Classical approach
2. Relative frequency approach
3. Subjective approach

1. Classical approach

Prob of an event = (no. of outcomes where the event occurs)/(total number of possible outcomes)

P(H) = 1/(1+1) = 1/2 Total possible outcomes

P(5) = 1/6 for the dice rolling example

Classical prob is also called a **priori probability** because we don't need to perform experiments.

Example of *a priori* probability

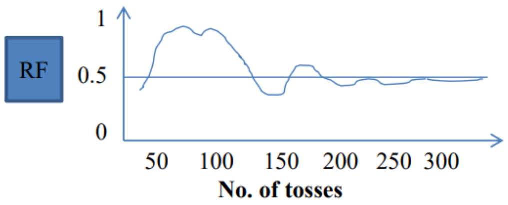
Find the probability of selecting a face card (Jack, Queen, or King) from a standard deck of 52 cards.

$$\text{Probability of Face Card} = \frac{X}{T} = \frac{\text{number of face cards}}{\text{total number of cards}}$$

$$\frac{X}{T} = \frac{12 \text{ face cards}}{52 \text{ total cards}} = \frac{3}{13}$$

2. Relative Frequency (RF) : Live up to 85 yrs, plant near river will substantially kill fish.
We need experiment to answer these.
This method uses the **relative frequencies of past occurrences** as probabilities.
How often something has **happened** in past - we **predict** future

More trails , greater accuracy: Tossing a fair coin for 300 times. In first 100 tosses prob **<u>is far from</u>** 0.5, but approaches 0.5 as we increase number of toss. RF becomes **<u>stable</u>** as no. of tosses **<u>become large</u>**.



Limitation: We need **sufficient no. of experiments** or observations before conclusion.

3. Subjective probability : Based on belief, experience, when event has occurred once or few times.

Because most higher-level social and managerial decisions are concerned with **specific**, **unique** satiations, rather than with a **long series of identical** situation, **decisions makers** use this prob.

Ex: Retirement policy is to be presented to top management. To know the support of the policy a manger conducts a poll.

	Machinists	inspector
Strongly support	9	10
Mildly support	11	3
Undecided	2	2
Mildly oppose	4	8
Strongly oppose	4	7
	30	30

a) What is the prob that a **machinist** randomly selected from the polled group **mildly supports** the package.
b) What is the prob that an **inspector** randomly selected from the polled group **is undecided**.
c) What is the prob that a worker (machinist or inspector) randomly selected from the polled group **strongly or mildly supports** the package.
d) What prob **estimates** are these.

	Machinists	inspector
Strongly support	9	10
Mildly support	11	3
Undecided	2	2
Mildly oppose	4	8
Strongly oppose	4	7
	30	30

a) What is the prob that a **machinist** randomly selected from the polled group **mildly supports** the package =11/30
b)) What is the prob that an **inspector** randomly selected from the polled group **is undecided**=2/30
c)) What is the prob that a worker (machinist or inspector) randomly selected from the polled group **strongly or mildly supports** the package = 9+11+10+3 / 60 = 33/60=11/20
d) What prob **estimates** are these = **Relative frequency**.

2. Classify the following probability estimates as to their type (classical, relative frequency, or subjective):

(a) The probability of scoring on a penalty shot in ice hockey is 0.47. = **RF**

(b) The probability that the current Mayor will resign is 0.85.= **S**

(c) The probability of rolling two sixes with two dice is 1/36.= **C**

(d) The probability that a president elected in a year ending in zero will die in office is 7/10. = **RF**

(e) The probability that you will go Europe this year is 0.14. **S**

Probability rules:

Prob of evet A happening = P(A)

Marginal or unconditional probability: A single prob means that **only one event** can take place. It is called **marginal or unconditional** probability.

Out of 50 students one students is winning free ticket to National Rock Festival

P(w) = 1/50

Probability of one or more ME events:

Addition rule for ME events:

Prob of **either A or B happening**: P(A or B) = P(A) +P(B)

Out a A,B,C,D,E. What is the **prob of A selected**, P(A) = 1/5.

What is the prob of **either A or B** selected = 1/5+1/5 = .4

Example

Number of children	0	1	2	3	4	5	6 or more
Proportion of having this many children	0.05	0.10	0.30	0.25	0.15	0.10	0.05

What is the prob of a randomly chosen family having 4 or more children= P4+P5+P6=0.30

P(A)+P(not A) = 1

What is the prob of a family having 5 or fewer children = 1- 0.05 = 0.95

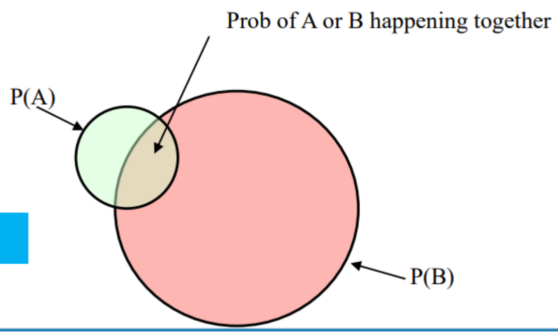
Addition rule for events that are not ME: If two events **are not mutually** exclusive, it is possible for **both events to occur**.

What is the prob of drawing either an *ace* or a *heart* from a deck of cards. *Ace* and *heart* can occur together because we could draw an *ace of heart*. Thus *ace* and *heart* are not ME.

Addition rule for events that are not ME :

P(A or B) = P(A) + P(B) – P(AB) =

P(ace or heart) = (4/52)+(13/52)-(1/52) = 4/13



Let take one more example: The employees have selected five representatives to represent them to management. A spokesperson is to be selected.

Gender	Age
Male	30
Male	32
Female	45
Female	20
Male	40

What is the prob the spokesperson will be *either* female *or* over 35?
P(female *or* over 35) = P(Female)+P (over 35) –P(Feamle and over 35)
= 2/5 + 2/5 -1/5 =3/5

Exapmle: An inspector of the Alaska pipeline has the task of comparing the **reliability** of **two pumping stations**. Each station is susceptible to **two kinds of failure**: **pump** failure and **leakage**. When either (or both) occur, the station must be shut down. The data at hand indicate that the following probabilities prevail:

Station	P (Pump failure)	P (Leakage)	P (Both)
1	0.07	0.10	0
2	0.09	0.12	0.06

Which station has the higher probability of being shut down?

Answer

P (Failure)= P (Pump failure or leakage)

Station 1: 0.07+0.1-0= 0.17

Satation2: 0.09+0.12-0.06=0.15

Thus, **station 1 has** the higher probability of being **shut** down.

Probabilities under conditions of statistical **independence**:

Statistical **independence**: The occurrence of one event has **no** effete on the prob. of occurrence of any **other** event.

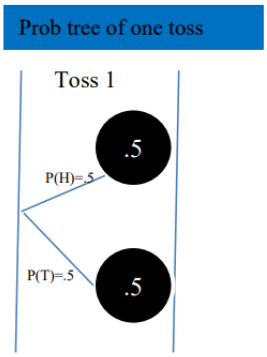
1. Marginal probabilities under statistical independence
2. Joint probabilities under statistical independence
3. Conditional probabilities under statistical independence

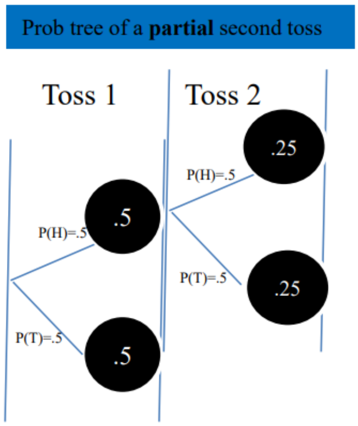
1. Marginal probabilities under statistical independence: Tossing of a fair coin. Outcome of **second toss** is **independent** of outcome of **first** toss. This is true even if the coin is **biased**.

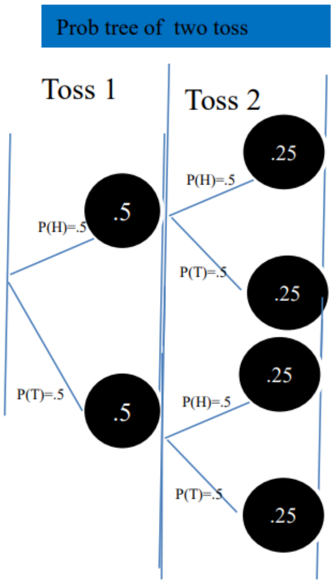
2. Joint probabilities under statistical independence: The prob of ***two or more independent*** events occurring **together or in succession is the product of their** marginal probabilities.

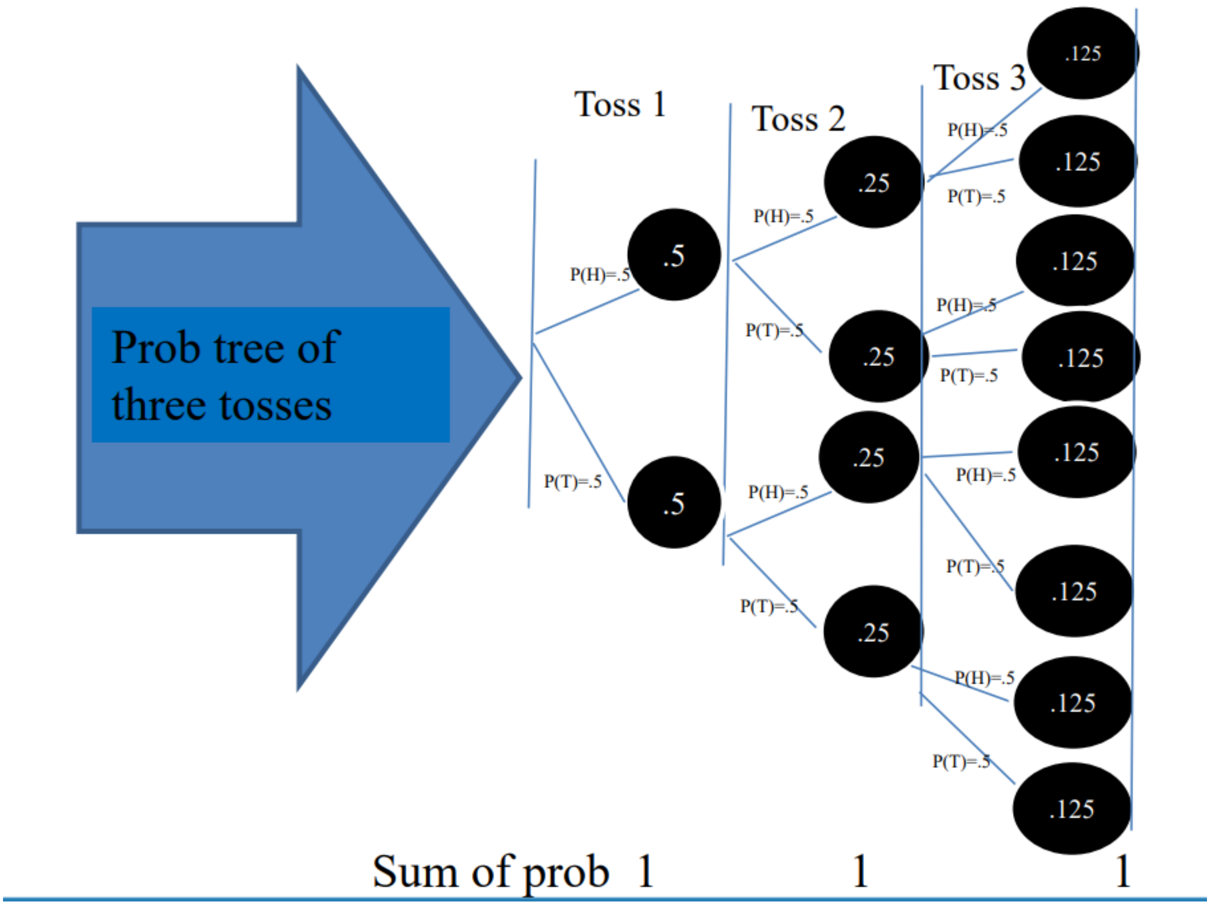
Joint prob. of two independent events:
P(AB) = (PA) X P(B)
P(AB) = Prob of events A and B **occurring together,** this is known as a ***joint prob***.
P(A) = marginal prob of event A occurring
P(B) = marginal prob of event B occurring

P(H1H2) = 0.5*0.5= ***0.25 (this is the prob of heads in two succession tosses***)
Similarly, the prob of heads in **three** in succession tosses = 0.5*0.5*0.5 = 0.125
Even if it is an **unfair coin**, let take P(H) = 0.8 .
Prob of heads in **three** in succession tosses P(H1H2H3)= 0.8*0.8*0.8 = 0.512
Prob of Tails in **three** in succession tosses P(T1T2T3) = 0.2*0.2*0.2 = 0.008
These two don't add up to 1 because the events H1H2H3 and T1T2T3 do **not** constitute a **collectively exhaustive** list. They are ***ME,*** because if one occurs the other cannot.


Prob tree of one toss
Toss 1
P(H)=.5
.5
P(T)=.5
.5


Prob tree of a **partial** second toss
Toss 1
Toss 2
P(H)=.5
.5
P(T)=.5
.5
P(H)=.5
.25
P(T)=.5
.25


Prob tree of two toss
Toss 1
Toss 2
P(H)=.5
.5
P(T)=.5
.5
P(H)=.5
.25
P(T)=.5
.25
P(H)=.5
.25
P(T)=.5
.25


Prob tree of three tosses
Toss 1
Toss 2
Toss 3
P(H)=.5
.5
P(T)=.5
.5
P(H)=.5
.25
P(T)=.5
.25
P(H)=.5
.25
P(T)=.5
.25
P(H)=.5
.125
P(T)=.5
.125
P(H)=.5
.125
P(T)=.5
.125
P(H)=.5
.125
P(T)=.5
.125
P(H)=.5
.125
P(T)=.5
.125
Sum of prob 1 1 1

3. Conditional probabilities under statistical independence: It is written as P(B/A). The prob of event B given A has occurred.

$$P(B/A) = P(B)$$

What is the prob that the **second toss of a fair coin will result in heads**, given that heads **resulted in first toss.** P(H2/H1), we know that **independence means** the first toss's result would not affect the result of second toss.

P(H2/H1) = 0.5

Summary

	Type of prob	Symbol	Formula
Prob. under statistical independence	Marginal	P(A)	P(A)
	Joint	P(AB)	P(A)XP(B)
	Conditional	P(B/A)	P(B)

Ex: What is the probability that in selecting two cards one at a time from a deck with replacement, the second card is:

(a) A face card, given that the first card was red?

(b) An ace, given that the first card was a face card?

(c) A black jack, given that the first card was a red ace?

Answers

(a) $P(Face_2 \mid Red_{1)=}$ 12/52= 3/13

(b) $P(Ace_2 \mid Face_{1)=}$ 4/52= 1/13

(c) $P(Black\ jack \mid Red\ ace_{1)=}$ 2/52= 1/26

Record of 45 years of a jail where prisoners tried to escape.

Attempted Escapes	Winter	Spring	Summer	Fall
0	3	2	1	0
1-5	15	10	11	12
6-10	15	12	11	16
11-15	5	8	7	7
16-20	3	4	6	5
21-25	2	4	5	3
More than 25	2	5	4	2
Total	45	45	45	45

What is the prob that in a year **selected at random**, the number of escapes was **between 16 and 20 during the winter.**
What is the prob that **more than 10** escapes were during **summer**.
What is the prob that **between 11 and 20 escapes** were attempted **during a randomly chosen season**.

What is the prob that in a year selected at random, the number of escapes was **between 16 and 20 during the winter = 3/45**
What is the prob that more than **10 escapes were during summer = 7+6+5+4=22/45**
What is the prob that between **11 and 20 escapes** were attempted during a **randomly chosen season**= 8+12+13+12 = 45/180=1/4.

Probabilities under conditions of statistical **Dependence**:

When prob of some event is **dependent on or affected** by the **occurrence of some other event.**

1. Conditional Probabilities under statistical **dependence**
2. Joint Probabilities under statistical **dependence**
3. Marginal Probabilities under statistical **dependence**

1.Conditional probabilities under statistical Dependence:
Assume a box has 10 balls as follows:

- Three are colored and dotted
- One is colored and striped
- Two are gray and dotted
- Four are gray and stripes

The prob. of drawing one ball is??????

- Three are colored and dotted
- One is colored and striped
- Two are gray and dotted
- Four are gray and stripes

Event	Prob of event	
1	0.1	Colored and dotted
2	0.1	
3	0.1	
4	0.1	Colored and striped
5	0.1	Gray and dotted
6	0.1	
7	0.1	Gray and striped
8	0.1	
9	0.1	
10	0.1	

Ex . If a **colored ball is drawn**.
1.What is the prob that it is **dotted P(D/C)=0.3/0.4**
2. What is the prob that it is **striped P(S/C)=0.1/0.4**

Conditional probabilities for statistical **dependent** events:

P(B/A)= {P(BA)}/P(A)

What is P(D/G)=P(DG)/P(G) =0.2/0.6 =

What is P(S/G) = P(SG)/P(G) =0.4/0.6

What is P(G/D) = P(GD)/P(D)= 0.2/0.5
What is P(C/D) = P(CD)/P(G)= 0.3/0.5

What is P(C/S) = P(CS)/P(S)= 0.1/0.5
What is P(G/S) = P(GS)/P(S)= 0.4/0.5

Event	Prob of event	
1	0.1	colored and dotted
2	0.1	
3	0.1	
4	0.1	colored and striped
5	0.1	gray and dotted
6	0.1	
7	0.1	gray and striped
8	0.1	
9	0.1	
10	0.1	

2. Joint Probabilities under statistical **dependence:**

We know that **conditional** prob under statistical dependence: **P(B/A)= {P(BA)}/P(A)**, we solve for P(BA) as :

P(BA) = P(B/A) * P(A)

Joint prob of events B and A happening together

Prob of event B given that event A has happened.

Prob of events A

We know that :

What is P(D/G) = P(DG)/P(G) =0.2/0.6 =1/3

What is P(S/G) = P(SG)/P(G) = 0.4/0.6=2/3

What is P(G/D) = **P(GD)**/P(D)= 0.2/0.5=0.4

What is P(C/D) = P(CD)/P(G)= 0.3/0.5=0.6

What is P(C/S) = **P(CS)**/P(S)= 0.1/0.5=0.2

What is P(G/S) = **P(GS)**/P(S)= 0.4/0.5=0.8

We can calculate P(CS) = 0.2*0.5=0.1

P(GD)= 0.4*0.5=0.2

P(GS)= 0.5*0.8=0.4

3.Marginal Probabilities under statistical **dependence:** Are computed by **summing** up the prob of **all the joint events** in which the simple event occurs.

Event	Prob of event	
1	0.1	colored and dotted
2	0.1	
3	0.1	
4	0.1	colored and striped
5	0.1	gray and dotted
6	0.1	
7	0.1	gray and striped
8	0.1	
9	0.1	
10	0.1	

P(C) = P(CD)+P(CS) = 0.3+0.1=0.4

P(G) = P(GD)+P(GS) = 0.2+0.4=0.6

P(D)?

P(S)?

Ex: According to a survey, the prob that a family **owns two cars** if its annual income is greater that Rs **35000 is 0.75**.

Of the household surveyed 60 % had income over Rs 35000 and 52 % had two cars.

What is the prob that a family **has two cars** and an income over Rs 35000 a year??.

Answer: Conditional Probabilities for statistical dependent events

Let I = income > 35000 C=2 cars

P(C and I) = P(C/I)/P(I) = 0.75*0.6= 0.45

Ex Friendly's Department store has been the target of **many shoplifters during the past month**, but owing to increased security precautions, **250 shoplifters have been caught**. Each shoplifters **gender** is noted; also noted is whether the perpetrator **was a first time or repeat offender**. The data are summarized in the below table.

Gender	First time offender	Repeat offender
Male	60	70
Female	44	76
Total	104	146

Assuming that an apprehended shoplifter is chosen at random, find:

(a) The probability that the shoplifter is male

(b) The probability that the shoplifter is first time offender, given that the shoplifter is male.

(c) The probability that the shoplifter is female, given that the shoplifter is a repeat offender.

(d) The probability that the shoplifter is female, given that the shoplifter is a first time offender.

(e) The probability that the shoplifter is both male and a repeat offender.

Gender	First time offender	Repeat offender
Male	60	70
Female	44	76
Total	104	146

(a) The probability that the shoplifter is male

(b) The probability that the shoplifter is **first time offender**, **given** that the shoplifter is **male**.

(c) The probability that the shoplifter is **female**, **given** that the shoplifter is a **repeat offender**.

(d) The probability that the shoplifter is **female**, **given** that the shoplifter is **a first time offender.**

(e) The probability that the shoplifter is both **male and a repeat offender**.

Answers

M/W= Shoplifter is male/female; F/R= Shoplifter is first time or repeat offender.

(a) P(M)= (60+70)/250= 0.520

(b) P(F/M)= P(F and M)/ P(M)= (60/250)/(130/250)=0.462

(c) P(W/R)= P(W and R)/ P(R)= (76/250)/(146/250)=0.521

(d) P(W/F)= P(W and F)/ P(F)= (44/250)/(104/250)=0.423

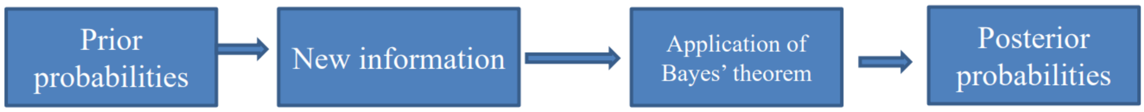
(e) P(M and R)= 70/250=0.280

Revising prior estimates of probabilities: Bayes' theorem

Bayes' theorem is formal procedure that lets decision makers combine **classical probability** theory with their best **intuitive sense** about what is likely to happen.

The basic formula for **conditional** probabilities under statistical **dependence**
P(B/A)= {P(BA)}/P(A)
is called **Bayes' theorem**.

Bayes' theorem offers a powerful statistical method of **evaluating new information** and **revise our prior estimates** (based upon limited information only) for the probability that things are in one state or another.

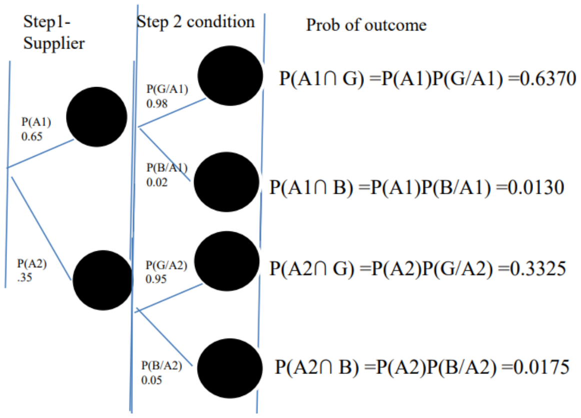


		Good parts %	Quantity %
	Supplier 1	98	65
	Supplier 2	95	35

Machine breakdown due to wrong part.
P(1) and P(2) : Prob that it came from S1 or S2.

Ex

	Good parts %	Quantity %
Supplier 1	98	65
Supplier 2	95	35



P(A1/B)= (0.65*0.02)/ (0.65*0.02)+(.35*0.05) =0.4262
P(A2/B)=(.35*.05)/ (0.65*0.02)+(.35*0.05) =.5738

(1) Events A_i	(2) Prior Probabilities $P(A_i)$	(3) Conditional Probabilities $P(B \mid A_i)$	(4) Joint Probabilities $P(A_i \cap B)$	(5) Posterior Probabilities $P(A_i \mid B)$
A_1	.65	.02	.0130	.0130/.0305 = .4262
A_2	.35	.05	.0175	.0175/.0305 = .5738
	1.00		$P(B)$ = .0305	1.0000

Ex Given the probabilities of three events, A, B, and C, occurring are P(A)=0.35, P(B)= 0.45, P(C)= 0.2. Assuming that A, B, or C has occurred, the probabilities of another event, X, occurring are P(X|A)=0.8, P(X|B=0.65), and P(X|C)= 0.3. Find P(A|X), P(B|X), and P(C|X).

Answer

Event	P (Event)	P (X | Event)	P (X and Event)	P (Event | X)
A	0.35	0.80	0.2800	0.2800/0.6325= 0.4427
B	0.45	0.65	0.2925	0.2925/0.6325=0.4625
C	0.20	0.30	0.0600	0.0600/0.6325= 0.0949
			P(X)= 0.6325	

Thus, P(A|X)= 0.4427, P(B|X)= 0.4625, P(C|X)= 0.0949