Mathematical Foundations of Computer Science
Lecture Outline
January 23, 2022

Mathematical Induction

Example. Prove that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Solution. We will prove the claim using induction on n.
Induction hypothesis: Assume that the claim is true when $n = k$, for some integer $k \geq 1$. In other words assume that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Base Case: $n = 1$. The claim is true for $n = 1$ as both sides of the equation equal to 1.
Induction step: To prove that the claim is true when $n = k+1$. That is, we want to show that

$$\sum_{i=1}^{k+1} = \frac{(k+1)(k+2)}{2}$$

We can do this as follows.

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\
&= \frac{k(k+1)}{2} + k + 1 \qquad\qquad \text{(using induction hypothesis)} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Example. Prove that the sum of the first n positive odd numbers is n^2.

Solution. We want to prove that $\forall$ positive integers $n, P(n)$ where $P(n)$ is the following property.

$$\sum_{i=0}^{n-1} 2i + 1 = n^2$$

Base Case: We want to show that $P(1)$ is true. This is clearly true as

$$\sum_{i=0}^{0} 2i + 1 = 1 = 1^2$$

Induction Hypothesis: Assume $P(k)$ is true for some integer $k \geq 1$.
Induction Step: We want to show that $P(k+1)$ is true, i.e., we want to show that

$$\sum_{i=0}^{k} 2i + 1 = (k+1)^2$$

We can do this as follows.

$$\begin{aligned}\sum_{i=0}^{k} 2i + 1 &= \sum_{i=0}^{k-1} 2i + 1 + 2k + 1 \\ &= k^2 + 2k + 1 \qquad \text{(using induction hypothesis)} \\ &= (k+1)^2\end{aligned}$$

Example. Show that for all integers $n \geq 0$, if $r \neq 1$,

$$\sum_{i=0}^{n} ar^i = \frac{a(r^{n+1} - 1)}{r - 1}$$

Solution. Let r be any real number that is not equal to 1. We want to prove that $\forall$ integers $n, P(n)$, where $P(n)$ is given by

$$\sum_{i=0}^{n} ar^i = \frac{a(r^{n+1} - 1)}{r - 1}$$

Base Case: We want to show that $P(0)$ is true.

$$\sum_{i=0}^{0} ar^i = a = \frac{a(r-1)}{r-1}$$

Induction Hypothesis: Assume that $P(k)$ is true for some $k \geq 0$.
Induction Step: We want to show that $P(k+1)$ is true, i.e., we want to prove that

$$\sum_{i=0}^{k+1} ar^i = \frac{a(r^{k+2}-1)}{r-1}$$

We can do this as follows.

$$\begin{aligned}
\text{L.H.S.} &= \sum_{i=0}^{k+1} ar^i \\
&= \sum_{i=0}^{k} ar^i + ar^{k+1} \\
&= \frac{ar^{k+1}-a}{r-1} + ar^{k+1} \\
&= \frac{a(r^{k+1}-1)}{r-1} + \frac{ar^{k+1}(r-1)}{r-1} \\
&= \frac{a}{r-1}\left(r^{k+1}(1+r-1)-1\right) \\
&= \frac{a}{r-1}\left(r^{k+2}-1\right) \\
&= \frac{a(r^{k+2}-1)}{r-1}
\end{aligned}$$

Example. Prove that $\forall$ non-negative integers n,

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Solution. By setting $a = 1$, $r = 2$ in the result of the previous problem, the claim follows.

Example. Prove that $\forall$ non-negative integers n, $2^{2n} - 1$ is a multiple of 3.

Solution. We want to prove that $\forall$ non-negative integers n, $P(n)$, where $P(n)$ is

$$2^{2n} - 1 = 3k, \text{ for some non-negative integer } k$$

Base Step: $P(0)$ is true as shown below.

$$2^0 - 1 = 0 = 3 \cdot 0.$$

Induction Hypothesis: Assume that $P(x)$ is true for some integer $x \geq 0$, i.e., $2^{2x} - 1 = 3 \cdot k'$, for some $k' \geq 0$.
Induction Step: We want to prove that $P(x+1)$ is true, i.e., we want to show that

$$2^{2(x+1)} - 1 = 3l, \text{ for some non-negative integer } l.$$

We can show this as follows.

$$\begin{aligned}
\text{L.H.S.} &= 2^{2(x+1)} - 1 \\
&= 2^{2x+2} - 1 \\
&= 2^{2x} \cdot 2^2 - 1 \\
&= 2^{2x} \cdot 4 - 1 \\
&= 2^{2x} \cdot (3+1) - 1 \\
&= 3 \cdot 2^{2x} + 2^{2x} - 1 \\
&= 3 \cdot 2^{2x} + 3 \cdot k' \qquad \text{(using induction hypothesis)} \\
&= 3(2^{2x} + k') \\
&= 3l, \qquad \text{where } l = 2^{2x} + k'
\end{aligned}$$

Since x and k' are integers l is also an integer. Hence, $P(x+1)$ is true.

Example. Prove that $\forall n \in \mathbb{N}, n > 1 \rightarrow n! < n^n$.

Solution. Below is a simple direct proof for this inequality.

$$\begin{aligned}
n! &= 1 \times 2 \times 3 \times \cdots \times n \\
&< n \times n \times n \times \cdots \times n \\
&= n^n
\end{aligned}$$

We now give a proof using induction. Let $P(n)$ denote the following property.

$$n! < n^n$$

Induction Hypothesis: Assume that $P(k)$ is true for some integer $k > 1$.
Base Case: We want to prove $P(2)$. $P(2)$ is the proposition that $2! < 2^2$, or $2 < 4$, which is true.
Induction Step: We want to prove $P(k+1)$, i.e., we want to prove that $(k+1)! < (k+1)^{k+1}$.

$$\begin{aligned}
\text{L.H.S.} &= (k+1)! \\
&= k! \times (k+1) \\
&< k^k \times (k+1) \qquad \text{(using induction hypothesis)} \\
&< (k+1)^k \times (k+1) \quad \text{(since } k > 1) \\
&= (k+1)^{k+1}
\end{aligned}$$