Exam 3 : Date & time will be announced soon.

Minimum Spanning Trees (MST)

Input: Undirected connected graph $G = (V, E)$

wts on edges (positive)

Obj : To compute a MST of G.

Assumption: All edge wts are distinct.

Prim's alg.

for each $v \in V$ **do**

$d[v] \leftarrow \infty$

$\pi[v] \leftarrow NIL$

$d[s] \leftarrow 0$ // any vertex can be chosen as

// a source

$S \leftarrow \phi$

while $S \neq V$ **do**

$u \leftarrow$ vertex in $V \setminus S$ with the

smallest $d[\cdot]$

$S \leftarrow S \cup \{u\}$

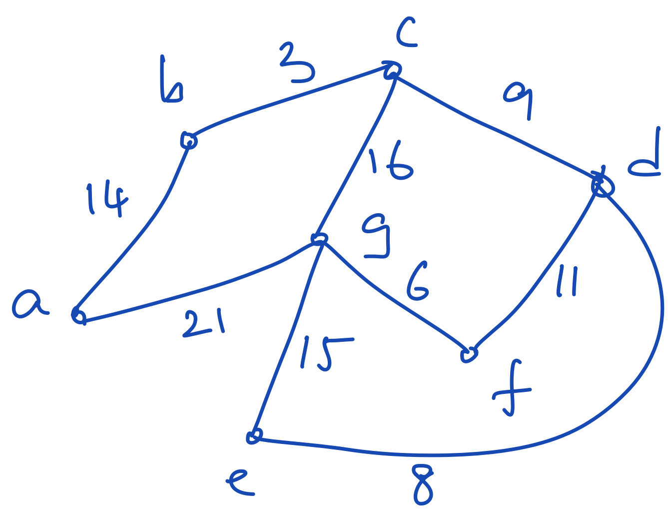
for each $v \in V \setminus S \cap N(u)$ **do**

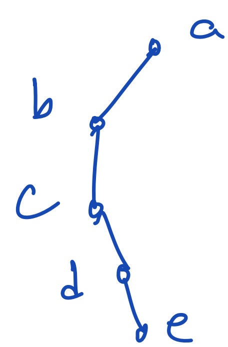
  if $d[v] > w_{uv}$ **then**

    $d[v] \leftarrow w_{uv}$

    $\pi[v] \leftarrow u$

Example





f $\cancel{\infty}$ 11

e $\cancel{\infty}$ 8

g $\cancel{\infty}$ 21

Kruskal's alg.

- Sort the edges in ↗ order of their wts.

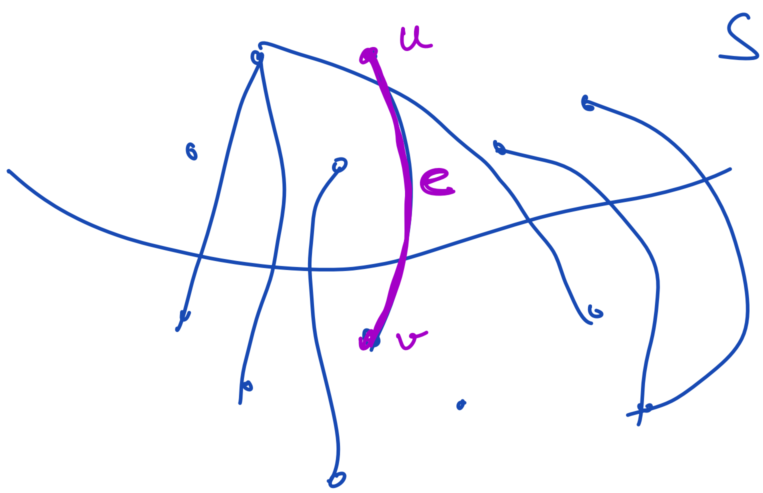
- Process edges in the above order & add an edge to the soln if adding it does not create a cycle

Reverse Delete

- Sort edges in ↘ order of wts.

- Process edges in the above order
  - remove the edge if removing it does not disconnect the graph.

Lemma: Let $S \subset V$, s.t. $S \neq V$, $S \neq \phi$.
Let $e = (u,v)$ be the min wt. edge crossing the cut (partition) $(S, V \setminus S)$. Then e must be in every MST.

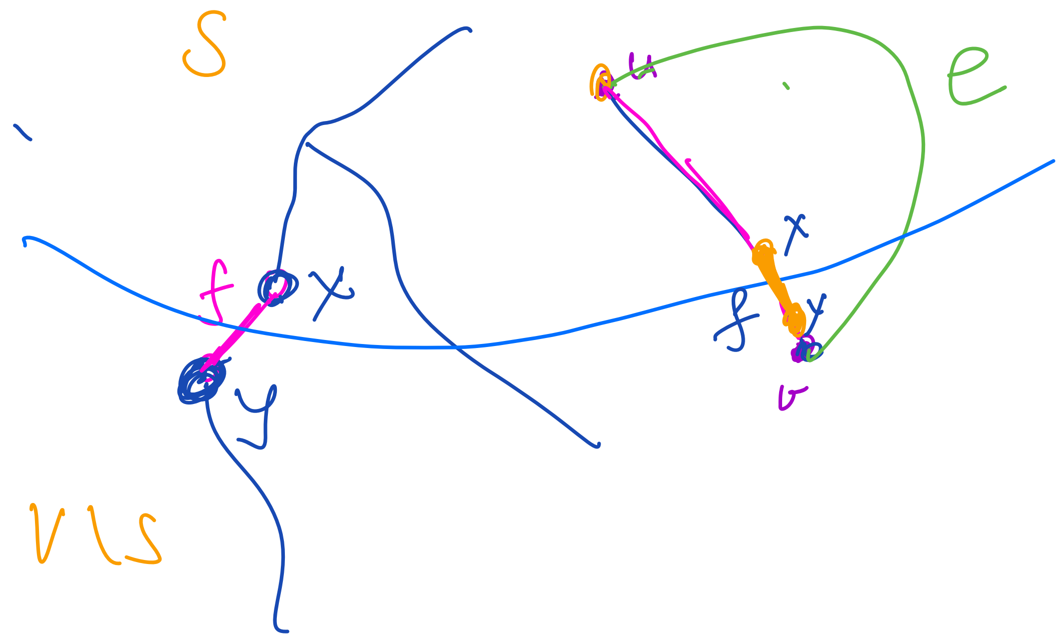


Proof: Assume for contradiction that $e=(u,v)$ does not belong to a MST, say T.

Note that since T is a spanning tree, T must contain a path P between vertices u & v in G. Since u & v are on opposite sides of the partition, there must be an edge f in T that crosses the cut $(S, V \setminus S)$. Since e is the min. wt edge crossing $(S, V \setminus S)$, $w(f) > w(e)$.

Let $T' = T \setminus \{f\} \cup \{e\}$.

Clearly, $wt(T') < wt(T)$, contradicting that T is a MST.



Let f be any edge in P.

$T' = T \setminus \{f\} + \{e\}$.

Since $w_e < w_f$

$wt(T') < wt(T)$, a

contradiction!

Let f be the edge on P

that crosses $(S, V \setminus S)$.

We know that $w_e < w_f$.

$\therefore$ if $T' = T \setminus \{f\} \cup \{e\}$

then $wt(T') < wt(T)$,

a contradiction!

<u>Theorem</u>: Kruskal's algorithm works.

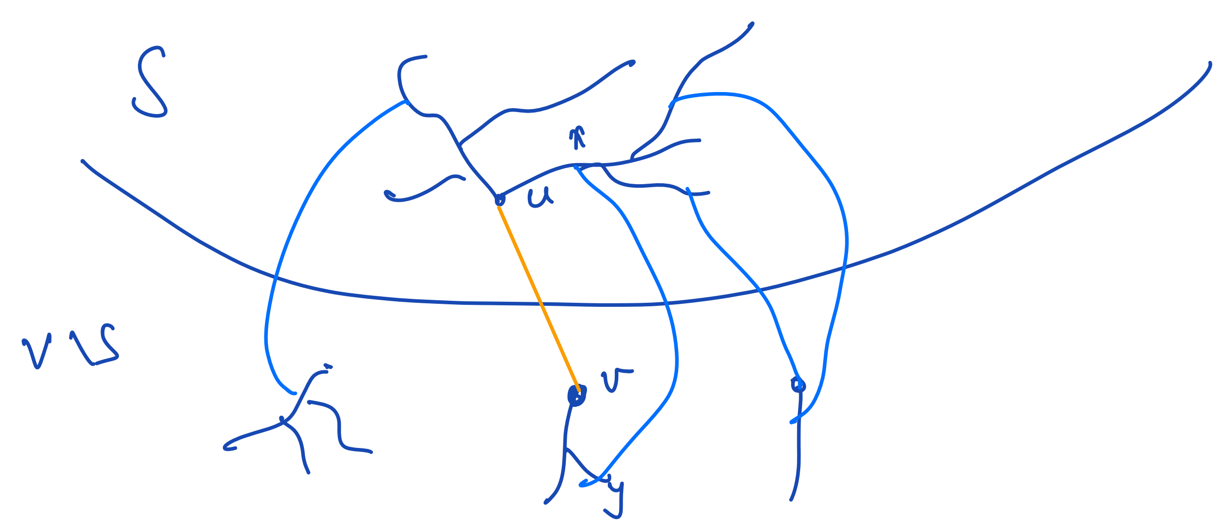
<u>Proof</u>: Let $e = (u, v)$ be an edge about

to be added by Kruskal. We will show

that e must be in every opt. soln.

To prove this, it suffices to show that e is the min. wt edge crossing some cut $(S, V \setminus S)$. What is the cut?

Let S = connected component containing u.



The edge (u,v) crosses the cut $(S, V \setminus S)$.

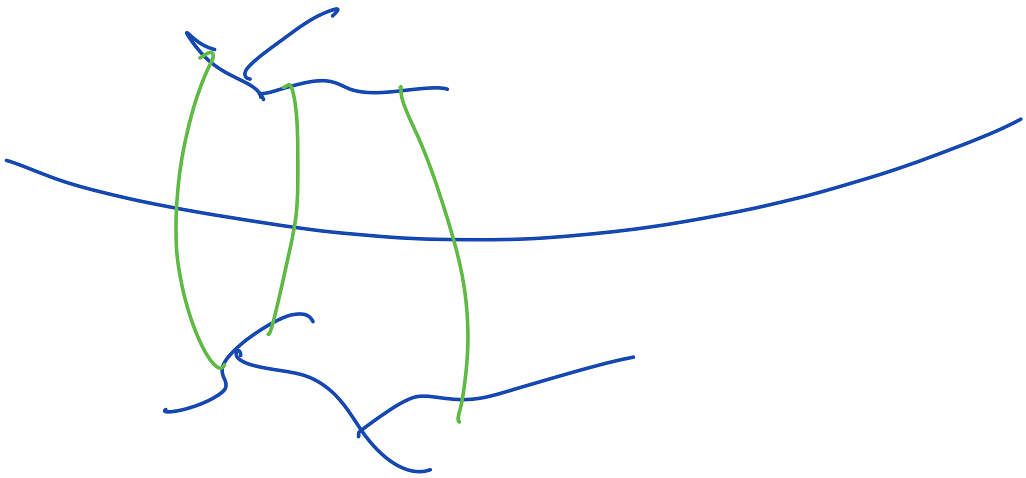
Since Kruskal processes the edges in $\nearrow$ order of their wts, $e = (u, v)$ must be the min. wt. edge among all edges crossing $(S, V \setminus S)$. Thus, by the previous lemma, e must belong to MST.

To complete the proof, we need to show that at the end of the algorithm, we get a spanning tree.

spanning ✓. why? all vertices are included right from the beginning.

<u>acyclic</u>: ✓ Kruskal's makes sure not to add an edge, if it creates a cycle.

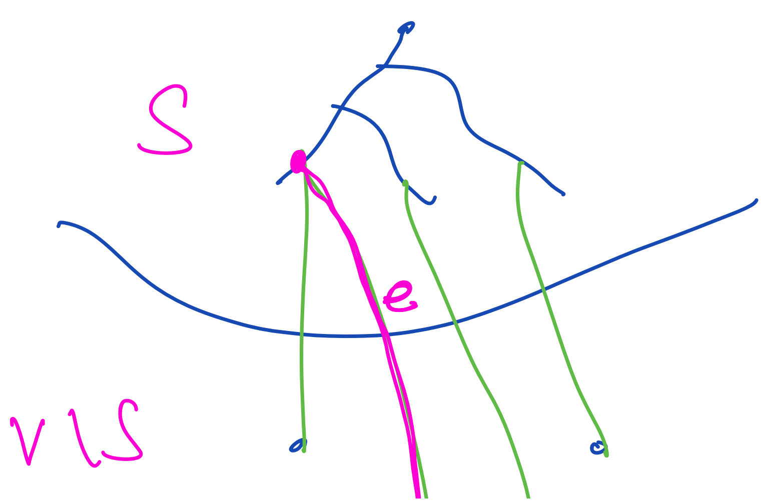
<u>Connected</u> ✓ if the o/p is NOT connected then since the original graph is connected, there must be an edge b/w two cc that is safe & Kruskal would add it.



Theorem : Prim's algorithm works.

Proof : Similar proof to Kruskal.

Here S will be the S in the alg.



$\Delta \geq 1$

4, 4, 8, 12, 12, 12, 12, 13, 16, 16, 18

$4+\epsilon$, $4+2\epsilon$, $8+\epsilon$, $12+\boxed{\epsilon}$, $12+\boxed{2\epsilon}$, $12+\boxed{3\epsilon}$, $12+\boxed{4\epsilon}$ . . .

$$\epsilon = \frac{\Delta}{n^2}.$$