Z-Transform

Sequence :- Arrangement of numbers (real or complex) is called sequence.

Ex: ① $\{1, 3, 5, 7, \cdots\} \equiv \{2n+1\}$, $n = 0, 1, 2, \cdots$

② $\{1, 1/2, 1/4, \cdots\} \equiv \{1/2^n\} = x_n$, $n = 0, 1, 2, \cdots$

Def^n :- Let x_K be the sequence of real numbers. Z-Transform of x_K denoted by $Z\{x_K\}$ and is defined as

$$Z\{x_K\} = \sum_{K=-\infty}^{\infty} x_K z^{-K} \quad \equiv F(z) \text{ say}$$

where $z = x + iy \; \boxed{= r e^{i\omega}, \; \omega \text{ is Frequence.}}$

$\boxed{\text{i.e. } x = r\cos\omega, \quad y = r\sin\omega}$

e.g. If $x_K = \{15, 10, 7, \underset{\uparrow}{4}, 1, -1, 0, 3, 6\}$

$+ \Rightarrow Z\{x_K\} = \sum_{K=-\infty}^{\infty} x_K z^{-K}$

$$Z\{x_K\} = 15z^3 + 10z^2 + 7z + 4 + z^{-1} - z^{-2} + 0 + 3z^{-3} + 6z^{-4}$$

Region of convergence :- (ROC)

we know that Z-Transform is a series. The region in which series is convergent (it has finite value) is called region of convergence.

Ex ① Find Z-Transform of

$$x_K = \begin{cases} 5^K, & K < 0 \\ 3^K, & K \geq 0 \end{cases}$$

Soln :- By defn $Z\{x_K\} = \sum_{K=-\infty}^{\infty} x_K z^{-K}$

$$\Rightarrow Z\{x_K\} = \sum_{K=-1}^{-\infty} 5^{K} z^{-K} + \sum_{K=0}^{\infty} 3^{K} z^{-K}$$

$$= \{5^{-1}z + 5^{-2}z^{2} + 5^{-3}z^{3} + \cdots\} + \{1 + 3z^{-1} + 3^{2}z^{-2} + \cdots\}$$

$$= \underbrace{\left[(z/5) + (z/5)^2 + (z/5)^3 + \cdots\right]}_{\text{G.P.}} + \underbrace{\left[1 + (3/z) + (3/z)^2 + \cdots\right]}_{\text{G.P.}}$$

G.P. $\boxed{\dfrac{a}{1-r}}$ G.P.

a = First term $\quad$ r = common ratio

$$Z\{x_K\} = \frac{z/5}{1 - z/5} + \frac{1}{1 - 3/z}$$

$$\Rightarrow \boxed{Z\{x_K\} = \frac{z}{5-z} + \frac{z}{z-3}}$$

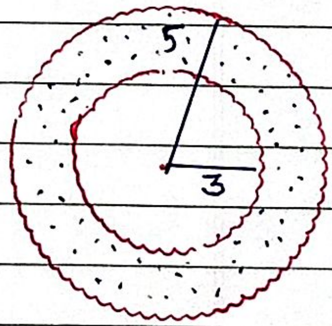
To Find ROC :- G.P. has finite value if $|r| < 1$

$$\left|\frac{z}{5}\right| < 1 \quad \text{and} \quad \left|\frac{3}{z}\right| < 1$$

$$\Rightarrow \frac{|z|}{5} < 1 \quad \text{and} \quad \frac{3}{|z|} < 1$$

$$\Rightarrow |z| < 5 \quad \text{and} \quad 3 < |z|$$

$$\Rightarrow 3 < |z| < 5$$



Note :- $|z| < 5 \Rightarrow |x+iy| < 5 \Rightarrow \sqrt{x^2+y^2} < 5$

$\Rightarrow x^2 + y^2 < 5^2$ circle

and $3 < |z| \Rightarrow 3 < |x+iy| \Rightarrow 3 < \sqrt{x^2+y^2}$

$\Rightarrow 3^2 < x^2 + y^2$ circle

Ex ② Find z-Transform of $(1/2)^{|K|}$

Solution:-

Let $x_K = (1/2)^{|K|} = (2^{-1})^{|K|}$

$$x_K = \begin{cases} (2^{-1})^{-K}, & K<0 \\ (2^{-1})^{K}, & K \geq 0 \end{cases} = \begin{cases} 2^{K}, & K<0 \\ 2^{-K}, & K \geq 0 \end{cases}$$

By defⁿ $Z\{x_K\} = \sum_{K=-\infty}^{\infty} x_K z^{-K}$

$$Z\{x_K\} = \sum_{K=-1}^{-\infty} 2^{K} z^{-K} + \sum_{K=0}^{\infty} 2^{-K} z^{-K}$$

$$= \left[2^{-1}z + 2^{-2}z^{2} + 2^{-3}z^{3} + \cdots\right] + \left[1 + 2^{-1}z^{-1} + 2^{-2}z^{-2} + \cdots\right]$$

$$Z\{x_K\} = \underbrace{\left[(z/2) + (z/2)^2 + (z/2)^3 + \cdots\right]}_{G.P.} + \underbrace{\left[1 + (1/2z) + (1/2z)^2 + \cdots\right]}_{G.P.}$$

$$= \frac{(z/2)}{[1-(z/2)]} + \frac{1}{[1-(1/2z)]}$$

$$\boxed{Z\{x_K\} = \frac{z}{2-z} + \frac{2z}{2z-1}} = f(z) \text{ (say)}$$

For ROC:- $\quad \left|\frac{z}{2}\right| < 1$ and $\left|\frac{1}{2z}\right| < 1$

$$\Longrightarrow \frac{|z|}{2} < 1 \text{ and } \frac{1}{2|z|} < 1$$

$$\Longrightarrow |z| < 2 \text{ and } 1/2 < |z|$$

$$\Longrightarrow \boxed{1/2 < |z| < 2}$$

Ex ③ Find Z-Transform of discrete unit step function

$$U(K) = \begin{cases} 1 & \text{For } K \geq 0 \\ 0 & \text{For } K < 0 \end{cases}$$

Solution :-

$$Z\{U(K)\} = \sum_{K=-\infty}^{\infty} U(K)\, z^{-K}$$

$$= \sum_{K=-1}^{-\infty} 0\, z^{-K} + \sum_{K=0}^{\infty} z^{-K}$$

$$= 1 + z^{-1} + z^{-2} + \cdots$$

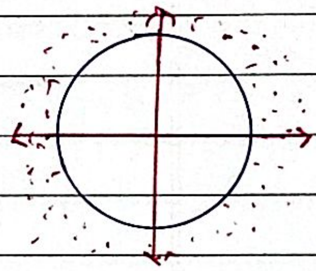
$$= 1 + (1/z) + (1/z)^2 + \cdots \longrightarrow G.P.$$

$$= \frac{1}{1-(1/z)}$$

$$\Longrightarrow \boxed{Z\{U(K)\} = \frac{z}{z-1}}$$

For ROC, $|1/z| < 1 \Longrightarrow \frac{1}{|z|} < 1$

$\Longrightarrow 1 < |z|$



Ex ④ Find Z-Transform of $\sin\alpha K$, $K \geq 0$

solution :- Let $x_K = \sin\alpha K$, $K \geq 0$

By defⁿ $Z\{x_K\} = \sum_{K=-\infty}^{\infty} x_K z^{-K}$

$$= \sum_{K=0}^{\infty} \sin\alpha K\, z^{-K}$$

$$\boxed{= 0 + \sin\alpha \cdot z^{-1} + \sin 2\alpha\, z^{-2} + \cdots \Longrightarrow 420\ \text{volt}}$$

$$= \sum_{K=0}^{\infty} \left[\frac{e^{i\alpha K} - e^{-i\alpha K}}{2i}\right] z^{-K}$$

$$= \frac{1}{2i}\left[\sum_{K=0}^{\infty} e^{i\alpha K} z^{-K} - \sum_{K=0}^{\infty} e^{-i\alpha K} z^{-K}\right]$$

$$= \frac{1}{2i}\left[\sum_{K=0}^{\infty} \left(\frac{e^{i\alpha}}{z}\right)^K - \sum_{K=0}^{\infty} \left(\frac{e^{-i\alpha}}{z}\right)^K\right]$$

$$= \frac{1}{2i}\left[\left[1+\left(\frac{e^{i\alpha}}{z}\right)+\left(\frac{e^{i\alpha}}{z}\right)^2+\cdots\right]-\left[1+\left(\frac{e^{-i\alpha}}{z}\right)+\left(\frac{e^{-i\alpha}}{z}\right)^2+\cdots\right]\right]$$

G.P. G.P.

$$= \frac{1}{2i}\left[\frac{1}{1-(e^{i\alpha}/z)} - \frac{1}{1-(e^{-i\alpha}/z)}\right]$$

$$= \frac{1}{2i}\left[\frac{z}{z-e^{i\alpha}} - \frac{z}{z-e^{-i\alpha}}\right]$$

$$= \frac{1}{2i}\left[\frac{z^2-ze^{-i\alpha}-z^2+ze^{i\alpha}}{z^2-ze^{-i\alpha}-ze^{i\alpha}+1}\right]$$

$$= \frac{1}{2i}\left[\frac{z(e^{i\alpha}-e^{-i\alpha})}{z^2-z(e^{i\alpha}+e^{-i\alpha})+1}\right]$$

$$\boxed{\sin\alpha = \frac{e^{i\alpha}-e^{-i\alpha}}{2i} \qquad \cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2}}$$

$$= \frac{1}{2i}\left[\frac{2zi\sin\alpha}{z^2-2z\cos\alpha+1}\right]$$

$$\Longrightarrow \boxed{Z\{\sin\alpha K\} = \frac{z\sin\alpha}{z^2-2z\cos\alpha+1}}$$

Ex (5) Find Z-Transform of $\cos\alpha K$, $K \geq 0$

Solution :- Let $x_K = \cos\alpha K$, $K \geq 0$

by defⁿ $Z\{x_K\} = \sum_{K=-\infty}^{\infty} x_K z^{-K}$

$$Z\{\cos\alpha K\} = \sum_{K=0}^{\infty} \cos\alpha K \, z^{-K}$$

$$\boxed{= 1+\cos\alpha\, z^{-1} + \cos 2\alpha\, z^{-2} + \cdots} \Longrightarrow 420 \text{ Volt.}$$

it is not G.P.

$$= \sum_{K=0}^{\infty}\left(\frac{e^{i\alpha K}+e^{-i\alpha K}}{2}\right)z^{-K}$$

$$= \frac{1}{2}\left[\sum_{K=0}^{\infty} e^{i\alpha K} z^{-K} + \sum_{K=0}^{\infty} e^{-i\alpha K} z^{-K}\right]$$

$$= \frac{1}{2}\left[\sum_{K=0}^{\infty}\left(\frac{e^{i\alpha}}{z}\right)^K + \sum_{K=0}^{\infty}\left(\frac{e^{-i\alpha}}{z}\right)^K\right]$$

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$$= \frac{1}{2}\left\{\left[1+\left(\frac{e^{i\alpha}}{z}\right)+\left(\frac{e^{i\alpha}}{z}\right)^2+\cdots\right]+\left[1+\left(\frac{e^{-i\alpha}}{z}\right)+\left(\frac{e^{-i\alpha}}{z}\right)^2+\cdots\right]\right\}$$

G.P. G.P.

$$= \frac{1}{2}\left[\frac{1}{1-(e^{i\alpha}/z)}+\frac{1}{1-(e^{-i\alpha}/z)}\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{i\alpha}}+\frac{z}{z-e^{-i\alpha}}\right]$$

$$= \frac{1}{2}\left[\frac{z^2-ze^{-i\alpha}+z^2-ze^{i\alpha}}{z^2-ze^{-i\alpha}-ze^{i\alpha}+1}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2-z(e^{i\alpha}+e^{-i\alpha})}{z^2-z(e^{i\alpha}+e^{-i\alpha})+1}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2-2z\cos\alpha}{z^2-2z\cos\alpha+1}\right]$$

$$\Longrightarrow \boxed{Z\{\cos\alpha k\} = \frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}}$$

Ex ⑥ Find Z-Transform of

i) $\sinh\alpha k$, ii) $\cosh\alpha k$, $k \geq 0$

Solution:- Home Work

(i) $Z\{\sinh\alpha k\} = \dfrac{z\sinh\alpha}{z^2-2z\cosh\alpha+1}$

(ii) $Z\{\cosh\alpha k\} = \dfrac{z(z-\cosh\alpha)}{z^2-2z\cosh\alpha+1}$

Note:- $\sinh\alpha k = \dfrac{e^{\alpha k}-e^{-\alpha k}}{2}$, $\cosh\alpha k = \dfrac{e^{\alpha k}+e^{-\alpha k}}{2}$

Linearity Property :- 1

$Z\{x_k \pm y_k\} = Z\{x_k\} \pm Z\{y_k\}$

Ex 7 Find Z-Transform of $\sin\left(\frac{k\pi}{4}+\alpha\right)$, $k \geq 0$

Soln :- $\sin\left(\frac{k\pi}{4}+a\right) = \sin\frac{k\pi}{4}\cos a + \cos\frac{k\pi}{4}\sin a$

taking Z-Transform we get,

$$Z\left\{\sin\left(\frac{k\pi}{4}+a\right)\right\} = \cos a \, Z\left\{\sin\frac{k\pi}{4}\right\} + \sin a \, Z\left\{\cos\frac{k\pi}{4}\right\}$$

$$= \cos a \cdot \frac{z\sin\pi/4}{[z^2 - 2z\cos\pi/4 + 1]} + \sin a \frac{z[z-\cos\pi/4]}{[z^2 - 2z\cos\pi/4 + 1]}$$

$$\left[\because Z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}\right.$$

$$\left. Z\{\cos\alpha k\} = \frac{z[z-\cos\alpha]}{z^2 - 2z\cos\alpha + 1}\right]$$

$$= \cos a \frac{z/\sqrt{2}}{z^2 - 2z/\sqrt{2} + 1} + \sin a \frac{z[z - 1/\sqrt{2}]}{z^2 - 2z/\sqrt{2} + 1}$$

Ex 8 Find Z-Transform of $\cos(k\pi/4 + 1)$, $k \geq 0$

Soln :- Home work

Ex 1 :-

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Properties Of Z-Transform

1 Linearity :→

Statement:→ If 'a' and 'b' are constants and x_K, y_K are two sequences which can be added, then

$$z(ax_K+by_K) = a\,z(x_K) + b\,z(x_K)$$

Solution:→

by defination $z(ax_K+by_K) = \sum_{K=-\infty}^{\infty} (ax_K+by_K)\, z^{-K}$

$$\therefore z(ax_K+by_K) = \sum_{K=-\infty}^{\infty} a x_K z^{-K} + \sum_{K=-\infty}^{\infty} b y_K z^{-K}$$

$$= a\sum_{K=-\infty}^{\infty} x_K z^{-K} + b\sum_{K=-\infty}^{\infty} y_K z^{-K}$$

$$z(ax_K+by_K) = a\,z(x_K) + b\,z(y_K)$$

Ex Find the z-transform of $x_K = \cos\alpha K \quad K \geq 0$ and α is real;

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Solution:

$$z(x_K) = \sum_{K=-\infty}^{\infty} x_K z^{-K}$$

$$= \sum_{K=-\infty}^{-1} 0 \cdot z^{-K} + \sum_{K=0}^{\infty} \cos\alpha K \cdot z^{-K}$$

$$= \sum_{K=0}^{\infty} \left[\frac{e^{i\alpha K} + e^{-i\alpha K}}{2}\right] z^{-K} = \sum_{K=0}^{\infty} \frac{1}{2}\left[\frac{(e^{i\alpha})^K}{z^K} + \frac{(e^{-i\alpha})^K}{z^K}\right]$$

$$= \frac{1}{2}\sum_{K=0}^{\infty}\left(\frac{e^{i\alpha}}{z}\right)^K + \frac{1}{2}\sum_{K=0}^{\infty}\left(\frac{e^{-i\alpha}}{z}\right)^K$$

$$= \frac{1}{2}\left\{1 + \left(\frac{e^{i\alpha}}{z}\right) + \left(\frac{e^{i\alpha}}{z}\right)^2 + \cdots\right\} + \frac{1}{2}\left\{1 + \left(\frac{e^{-i\alpha}}{z}\right) + \left(\frac{e^{-i\alpha}}{z}\right)^2 + \cdots\right\}$$

G.P. G.P.

$r = \frac{e^{i\alpha}}{z}$, $a=1$, $n=\infty$, $r = \frac{e^{-i\alpha}}{z}$, $a=1$, $n=\infty$

$$z(x_K) = \frac{1}{2}\,\frac{1\left[1-\left(\frac{e^{i\alpha}}{z}\right)^{\infty}\right]}{1-\frac{e^{i\alpha}}{z}} + \frac{1}{2}\,\frac{1\left[1-\left(\frac{e^{-i\alpha}}{z}\right)^{\infty}\right]}{1-\frac{e^{-i\alpha}}{z}} \quad -(1)$$

above series is convergent, $\left|\frac{e^{i\alpha}}{z}\right| < 1$ and $\left|\frac{e^{-i\alpha}}{z}\right| < 1$

$\frac{|e^{i\alpha}|}{|z|} < 1 \Rightarrow |\cos\alpha + i\sin\alpha| < |z|$ and $\frac{|e^{-i\alpha}|}{|z|} < 1$

i.e. $\sqrt{\cos^2\alpha + \sin^2\alpha} < |z|$ and $|\cos\alpha - i\sin\alpha| < |z|$

$1 < |z|$ and $\sqrt{\cos^2\alpha + \sin^2\alpha} < |z|$

i.e. $1 < |z|$

$\therefore$ ROC is $|z| > 1$

$$\therefore (1) \Rightarrow Z(x_K) = \frac{1}{2}\frac{1[1-0]}{[1-\frac{e^{i\alpha}}{z}]} + \frac{1}{2}\frac{1[1-0]}{[1-\frac{e^{-i\alpha}}{z}]}$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{i\alpha}} + \frac{z}{z-e^{-i\alpha}}\right] = \frac{z}{2}\left[\frac{z-e^{-i\alpha}+z-e^{i\alpha}}{z^2-z\,e^{-i\alpha}-z\,e^{i\alpha}+1}\right]$$

$$= \frac{z}{2}\left[\frac{2z-(e^{i\alpha}+e^{-i\alpha})}{z^2-z\,(e^{i\alpha}+e^{-i\alpha})+1}\right]$$

$$= \frac{z}{2}\left\{\frac{2z-2\cos\alpha}{z^2-2z\cos\alpha+1}\right\} \qquad \left[\because \frac{e^{i\alpha}+e^{-i\alpha}}{2} = \cos\alpha\right]$$

$$\Rightarrow Z\{\cos\alpha K\} = \frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}$$

Ex Find Z-transform of $x_K = \sin\alpha K$, $K \geq 0$

Solution: refer above example.

Ans: $Z(x_K) = \dfrac{z\sin\alpha}{z^2-2z\cos\alpha+1}$, ROC : $|z| > 1$

also note that $\sin\alpha = \dfrac{e^{i\alpha}-e^{-i\alpha}}{2i}$

and $\sin\alpha K = \dfrac{e^{i\alpha K}-e^{-i\alpha K}}{2i}$

Ex Find Z-transform of $x_K = \cosh(\alpha K)$, $K \geq 0$

Solution: $Z(x_K) = \sum_{K=-\infty}^{\infty} x_K z^{-K}$

$$= \sum_{K=-\infty}^{-1} 0 + \sum_{K=0}^{\infty} \cosh\alpha K \cdot z^{-K}$$

$$= 0 + \sum_{K=-\infty}^{\infty} \left(\frac{e^{\alpha K}+e^{-\alpha K}}{2}\right) z^{-K} = \sum_{K=0}^{\infty} \frac{1}{2}\left[\frac{(e^{\alpha})^K}{z^K} + \frac{(e^{-\alpha})^K}{z^K}\right]$$

$$= \frac{1}{2}\sum_{K=0}^{\infty}\left(\frac{e^{\alpha}}{z}\right)^K + \frac{1}{2}\sum_{K=0}^{\infty}\left(\frac{e^{-\alpha}}{z}\right)^K$$

$$= \frac{1}{2}\underbrace{\left\{1+\left(\frac{e^{\alpha}}{z}\right)+\left(\frac{e^{i\alpha}}{z}\right)^2+\dots\right\}}_{G.P.} + \frac{1}{2}\underbrace{\left\{1+\left(\frac{e^{-\alpha}}{z}\right)+\left(\frac{e^{-\alpha}}{z}\right)^2+\dots\right\}}_{G.P.}$$

$$Z(x_K) = \frac{1}{2}\left\{\frac{1\left[1-\left(\frac{e^{\alpha}}{z}\right)^{\infty}\right]}{1-\frac{e^{\alpha}}{z}}\right\} + \frac{1}{2}\left\{\frac{1\left[1-\left(\frac{e^{-\alpha}}{z}\right)^{\infty}\right]}{1-\frac{e^{-\alpha}}{z}}\right\} \quad -(1)$$

__For ROC__ :→ $\left|\frac{e^{\alpha}}{z}\right| < 1$ and $\left|\frac{e^{-\alpha}}{z}\right| < 1$

$\Rightarrow |e^{\alpha}| < |z|$ and $|e^{-\alpha}| < |z|$

i.e. $|z| >$ maximum of $|e^{\alpha}|$ and $|e^{-\alpha}|$

$$\therefore (1) \Rightarrow Z(x_K) = \frac{1}{2}\left\{\frac{1-0}{1-\frac{e^{\alpha}}{z}}\right\} + \frac{1}{2}\left\{\frac{1-0}{1-\frac{e^{-\alpha}}{z}}\right\}$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\alpha}} + \frac{z}{z-e^{-\alpha}}\right]$$

$$= \frac{z}{2}\left\{ \frac{z - e^{-\alpha} + z - e^{\alpha}}{z^2 - z\,e^{-\alpha} - z\,e^{\alpha} + 1} \right\} = \frac{z}{2}\left\{ \frac{2z - (e^{\alpha} + e^{-\alpha})}{z^2 - z\,(e^{\alpha} + e^{-\alpha}) + 1} \right\}$$

$$= \frac{z}{2}\left\{ \frac{2z - 2\cosh\alpha}{z^2 - 2\,z\cdot\cosh\alpha + 1} \right\} \qquad \left[\because \cosh\alpha = \frac{e^{\alpha} + e^{-\alpha}}{2} \right]$$

$$= \frac{z(z - \cosh\alpha)}{z^2 - 2\,z\,\cosh\alpha + 1}$$

Ex Find z-transform of $x_K = \sinh\alpha K \; ; K \geq 0$

Solution:- $z(x_K) = \sum_{K=-\infty}^{\infty} x_K z^{-K} = \sum_{K=-\infty}^{-1} 0 \cdot z^{-K} + \sum_{K=0}^{\infty} \sinh\alpha K \cdot z^{-K}$

$$\therefore z(x_K) = \sum_{K=0}^{\infty} \left[\frac{e^{\alpha K} - e^{-\alpha K}}{2} \right] z^{-K} = \frac{1}{2} \sum_{K=0}^{\infty} \left\{ \frac{(e^{\alpha})^K - (e^{-\alpha})^K}{z^K} \right\}$$

$$= \frac{1}{2} \sum_{K=0}^{\infty} \left(\frac{e^{\alpha}}{z} \right)^K - \frac{1}{2} \sum_{K=0}^{\infty} \left(\frac{e^{-\alpha}}{z} \right)^K$$

$$= \frac{1}{2} \underbrace{\left\{ 1 + \left(\frac{e^{\alpha}}{z} \right) + \left(\frac{e^{\alpha}}{z} \right)^2 + \cdots \right\}}_{G.P.} - \frac{1}{2} \underbrace{\left\{ 1 + \left(\frac{e^{-\alpha}}{z} \right) + \left(\frac{e^{-\alpha}}{z} \right)^2 + \cdots \right\}}_{G.P.}$$

$$z(x_K) = \frac{1}{2} \; \frac{1\left[1 - \left(\frac{e^{\alpha}}{z} \right)^{\infty}\right]}{1 - \left(\frac{e^{\alpha}}{z} \right)} - \frac{1}{2} \; \frac{1\left[1 - \left(\frac{e^{-\alpha}}{z} \right)^{\infty}\right]}{1 - \frac{e^{-\alpha}}{z}} \quad -①$$

above series is convergent if $\left| \frac{e^{\alpha}}{z} \right| < 1$ and $\left| \frac{e^{-\alpha}}{z} \right| < 1$

i.e. $|e^{\alpha}| < |z|$ and $|e^{-\alpha}| < |z|$

$\therefore$ ROC is $|z| >$ maximum of $|e^{\alpha}|$ and $|e^{-\alpha}|$

$$\therefore ① \Rightarrow z(x_K) = \frac{1}{2} \frac{(1-0)}{1 - \frac{e^{\alpha}}{z}} - \frac{1}{2} \frac{(1-0)}{1 - \frac{e^{-\alpha}}{z}} = \frac{z}{2(z - e^{\alpha})} - \frac{1}{2} \frac{z}{z - e^{-\alpha}}$$

$$\therefore z(x_K) = \frac{z}{2}\left\{ \frac{z - e^{-\alpha} - z + e^{\alpha}}{z^2 - z e^{-\alpha} - z\, e^{\alpha} + 1} \right\} = \frac{z}{2}\left\{ \frac{(e^{\alpha} - e^{-\alpha})}{z^2 - z(e^{\alpha} + e^{-\alpha}) + 1} \right\}$$

$$= \frac{z}{2}\left\{ \frac{2\sinh\alpha}{z^2 - 2\,z\cosh\alpha + 1} \right\} \qquad \left[\because \frac{e^{\alpha} - e^{-\alpha}}{2} = \sinh\alpha, \quad \frac{e^{\alpha} + e^{-\alpha}}{2} = \cosh\alpha \right]$$

$$\therefore z(x_K) = \frac{z\cdot\sinh\alpha}{z^2 - 2\,z\cosh\alpha + 1}$$

$$z\{\sinh(\alpha K)\} = \frac{z\sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$$

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2 Change of Scale property :→

Thm: State and prove change of scale property.

Solution :→ statement :→ If $z(x_k) = F(z)$ then $z(a^k x_k) = F(\frac{z}{a})$ and if ROC of $z(x_k)$ is $R_1 < |z| < R_2$ then ROC of $z(a^k x_k)$ is $R_1 < |\frac{z}{a}| < R_2$

i.e. $|a|R_1 < |z| < |a|R_2$

Proof :→ given $z(x_k) = F(z)$

$$\Rightarrow \sum_{k=-\infty}^{\infty} x_k z^{-k} = F(z) \quad -①$$

$$\therefore z(a^k x_k) = \sum_{k=-\infty}^{\infty} a^k x_k z^{-k} = \sum_{k=-\infty}^{\infty} x_k \left(\frac{z}{a}\right)^{-k} \quad -②$$

comparing ①, ②, $z(a^k x_k) = F(z/a)$

Further if ROC of $z(x_k)$ is $R_1 < |z| < R_2$ then ROC of $z(a^k x_k)$ from ① will be $R_1 < |\frac{z}{a}| < R_2$ i.e. $|a| R_1 < |z| < |a| R_2$

3 Shifting property:

Thm? Prove that if $z(x_k) = F(z)$ then $z(x_{k+n}) = z^n F(z)$

Proof :→ given $z(x_k) = F(z) \Rightarrow \sum_{k=-\infty}^{\infty} x_k z^{-k} = F(z) \quad -①$

$$\therefore z(x_{k+n}) = \sum_{k=-\infty}^{\infty} x_{k+n} z^{-k} \qquad \text{let } k+n = m$$

$$= \sum_{m=-\infty}^{\infty} x_m z^{-(m-n)}$$

$$= \sum_{m=-\infty}^{\infty} x_m z^{-m} \cdot z^n$$

$$= z^n \sum_{m=-\infty}^{\infty} x_m z^{-m}$$

$$= z^n z(x_k) \qquad [\because \text{by def}^n]$$

$$z(x_{k+n}) = z^n F(z) \qquad [\because \text{by } ①]$$

i.e. $\boxed{z(x_{k+n}) = z^n z(x_k)}$

Note Similarly $z(x_{k-n}) = z^{-n} F(z)$

$$\boxed{z(x_{k-n}) = z^{-n} z(x_k)}$$

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4 Multiplication by K :→

Th^m :→ $Z(Kx_K) = -z \frac{d}{dz} Z(x_K)$

Proof :→ $Z(x_K) = \sum_{K=-\infty}^{\infty} x_K z^{-K}$ — ①

differentiating w.r.t. z, $\frac{d}{dz} Z(x_K) = \sum_{-\infty}^{\infty} x_K (-K) z^{-K-1}$

$= -z^{-1} \sum_{K=-\infty}^{\infty} (Kx_K) z^{-K}$

$= -z^{-1} Z(Kx_K)$ [∵ by def^n

$\therefore Z(Kx_K) = -z \frac{d}{dz} Z(x_K)$ — ②

Simillarly, $Z(K^2 x_K) = Z(K(Kx_K)) = -z \frac{d}{dz} Z(Kx_K)$ [∵ by ②

$= -z \frac{d}{dz}\left[-z \frac{d}{dz} Z(x_K)\right]$ ∵ by ②

$Z(K^2 x_K) = \left(-z \frac{d}{dz}\right)^2 Z(x_K)$

5 Division by K :-

Th^m :→ $Z\left\{\frac{x_K}{K}\right\} = -\int z^{-1} Z(x_K)\, dz$

Proof: we have $Z(x_K) = \sum_{K=-\infty}^{\infty} x_K z^{-K}$ — ①

$\Rightarrow z^{-1} Z(x_K) = \sum_{K=-\infty}^{\infty} x_K z^{-K-1}$ ⇒ by integration $\int z^{-1} Z(x_K) dz$

$= \int \sum_{K=-\infty}^{\infty} x_K z^{-K-1} dz$

$\therefore \int z^{-1} Z(x_K) dz = \sum_{K=-\infty}^{\infty} x_K \frac{z^{-K}}{-K}$ ⇒ $\int z^{-1} Z(x_K) dz = -\sum_{K=-\infty}^{\infty} \left(\frac{x_K}{K}\right) z^{-K}$

∴ by defination, $\int z^{-1} Z(x_K) dz = -Z\left(\frac{x_K}{K}\right)$

$\therefore Z\left(\frac{x_K}{K}\right) = -\int z^{-1} Z(x_K) dz.$

6. Th^m :→ If $Z(x_K) = F(z)$ then $Z(e^{-aK} x_K) = F(e^a z)$

Proof :→ given $Z(x_K) = \sum_{-\infty}^{\infty} x_K z^{-K} = F(z)$ — ①

$\therefore Z(e^{-aK} \cdot x_K) = \sum_{K=-\infty}^{\infty} e^{-aK} x_K z^{-K} = \sum_{K=-\infty}^{\infty} x_K (e^a z)^{-K}$

$Z(e^{-aK} x_K) = \sum_{K=\infty}^{\infty} x_K (e^a z)^{-K}$ — ②

comparing ①, ②, $Z(e^{-aK} x_K) = F(e^a z)$

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Ex Find z-transform of k^2, $k\geq 0$

Solution:- we know z-transform of discrete unit step funct?

$$z(U(k)) = \frac{z}{z-1} \qquad U(k)=\left\{\begin{matrix} 1 & ; k\geq 0 \\ 0 & \text{otherwise}\end{matrix}\right\}$$

i.e. $z(1) = \frac{z}{z-1}$

$$\therefore z(k^2) = Z[k^2U(k)] = \left(-z\frac{d}{dz}\right)^2\left[\frac{z}{z-1}\right] \qquad [\because \text{by property (4) i.e. multiplication by } k]$$

$$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right)$$

$$= \left(-z\frac{d}{dz}\right)(-z)\left[\frac{z-1-z}{(z-1)^2}\right] = -z\frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\}$$

$$= -z\left\{\frac{(z-1)^2\cdot 1 - z\,2(z-1)}{(z-1)^4}\right\} = -z\left\{\frac{z^2-2z+1-2z^2+2z}{(z-1)^4}\right\}$$

$$= \frac{z(z^2-1)}{(z-1)^4} = \frac{z(z-1)(z+1)}{(z-1)^4}$$

$$\therefore z(k^2) = \frac{z(z+1)}{(z-1)^3}$$

Ex. find the z-transform of ke^{-ak}, $k\geq 0$

Solution:- we know that if $U(k)=1$, $k\geq 0$

then $Z(U(k)) = \frac{z}{z-1}$ —(1)

But If $Z(x_k)=F(z)$ then $z(e^{-ak}x_k) = F(e^a z)$,

$$\therefore Z(e^{-ak}U(k)) = \frac{e^a z}{e^a z-1} \qquad \text{where } F(z)=\frac{z}{z-1}$$

$\therefore$ by result (4) i.e. $z[kx_k] = -z\frac{d}{dz}z(x_k)$

$$\therefore z[ke^{-ak}U(k)] = -z\frac{d}{dz}z[e^{-ak}U(k)]$$

$$= -z\frac{d}{dz}\left[\frac{ze^a}{e^a z-1}\right] = -z\left\{\frac{(e^a z-1)e^a - ze^a\cdot e^a}{(e^a z-1)^2}\right\}$$

$$= -z\left\{\frac{e^{2a}z - e^a - ze^{2a}}{(e^a z-1)^2}\right\} = \frac{ze^a}{(e^a z-1)^2}$$

$$\therefore z[ke^{-ak}] = \frac{ze^a}{(e^a z-1)^2}$$

OR

Using direct defination:-

$$z[x_k] = \sum_{k=-\infty}^{\infty} x_k z^{-k}$$

$$\therefore z[ke^{-ak}] = \sum_{k=-\infty}^{-1} 0\cdot z^{-k} + \sum_{k=0}^{\infty} ke^{-ak}z^{-k}$$

$$= 0 + 1e^{-a}z^{-1} + 2e^{-2a}z^{-2} + 3e^{-3a}z^{-3} + 4e^{-4a}z^{-4} + \cdots$$

$$z[ke^{-ak}] = e^{-a}z^{-1} + 2e^{-2a}z^{-2} + 3e^{-3a}z^{-3} + 4e^{-4a}z^{-4} + \cdots \quad \text{—(1)}$$

Let $A = e^{-a}z^{-1} + 2e^{-2a}z^{-2} + 3e^{-3a}z^{-3} + 4e^{-4a}z^{-4} + \dots$

$\Rightarrow e^{-a}z^{-1}A = e^{-2a}z^{-2} + 2e^{-3a}z^{-3} + 3e^{-4a}z^{-4} + \dots$

$\therefore A - Ae^{-a}z^{-1} = [e^{-a}z^{-1} + 2e^{-2a}z^{-2} + 3e^{-3a}z^{-3} + 4e^{-4a}z^{-4} + \dots]$
$- [e^{-2a}z^{-2} + 2e^{-3a}z^{-3} + 3e^{-4a}z^{-4} + \dots]$

$\therefore (1 - e^{-a}z^{-1})A = e^{-a}z^{-1} + e^{-2a}z^{-2} + e^{-3a}z^{-3} + \dots$ which is G.P. $\frac{a[1-r^{\infty}]}{1-r}$

$$(1 - e^{-a}z^{-1})A = \frac{e^{-a}z^{-1}[1 - (e^{-a}z^{-1})^{\infty}]}{1 - e^{-a}z^{-1}} \quad -②$$

For convregenre, $|e^{-a}z^{-1}| < 1 \Rightarrow \left|\frac{e^{-a}}{z}\right| < 1 \Rightarrow |e^{-a}| < |z|$

$$\therefore ② \Rightarrow A = \frac{e^{-a}z^{-1}(1-0)}{(1 - e^{-a}z^{-1})^2}$$

using this in ①, $z(Ke^{-aK}) = \frac{e^{-a}z^{-1}}{(1 - e^{-a}z^{-1})^2} = \frac{e^{-a}z^{-1}}{\left[1 - \frac{1}{e^{a}z}\right]^2}$

$$= \frac{e^{-a}z^{-1}}{[e^{a}z - 1]^2} \cdot e^{2a}z^2$$

$$\Rightarrow z(Ke^{-aK}) = \frac{ze^{a}}{(e^{a}z - 1)^2}$$

Ex Find z-transform of $\frac{2^K}{K} + \frac{3^K}{K}$, $K \geq 0$

Solution: $\frac{2^K}{K} + \frac{3^K}{K} = \frac{2^K + 3^K}{K}$, $K \geq 0$.

Let $x_K = 2^K + 3^K$ $\Rightarrow$ by defination, $z(x_K) = \sum_{-\infty}^{\infty} x_K \cdot z^{-K}$

$\therefore z(2^K + 3^K) = z(2^K) + z(3^K)$

$$= \sum_{K=0}^{\infty} 2^K z^{-K} + \sum_{K=0}^{\infty} 3^K z^{-K}$$

$$= [1 + 2z^{-1} + 4z^{-2} + \dots] + [1 + 3z^{-1} + 9z^{-2} + \dots]$$

$$= \left[1 + \frac{2}{z} + \frac{4}{z^2} + \dots\right] + \left[1 + \frac{3}{z} + \frac{9}{z^2} + \dots\right] \quad \text{G.P.}$$

$$= \frac{1[1 - (\frac{2}{z})^{\infty}]}{1 - \frac{2}{z}} + \frac{1[1 - (\frac{3}{z})^{\infty}]}{1 - \frac{3}{z}}$$

$$= \frac{1}{1 - \frac{2}{z}} + \frac{1}{1 - \frac{3}{z}}$$ where $\left|\frac{2}{z}\right| < 1$, $\left|\frac{3}{z}\right| < 1$

$|2| < |z|$, $|3| < |z|$

$$z(2^K + 3^K) = \frac{z}{z-2} + \frac{z}{z-3}$$ $\Rightarrow |z| > 3$. ROC

by division property $z\left(\frac{x_K}{K}\right) = -\int z^{-1} z(x_K)\, dz$

$$\therefore z\left(\frac{2^K + 3^K}{K}\right) = -\int z^{-1} z(2^K + 3^K)\, dz$$

$$= -\int z^{-1}\left[\frac{z}{z-2}+\frac{z}{z-3}\right]dz$$

$$= -\int\left(\frac{1}{z-2}+\frac{1}{z-3}\right)dz$$

$$\therefore z\left[\frac{2^K}{K}+\frac{3^K}{K}\right] = -\left[\log(z-2)+\log(z-3)\right] \quad \text{where } |z|>3, \text{ ROC}$$

$$= -\log[(z-2)(z-3)]$$

Ex Find z-transform of $K2^K + K3^K$, $K\geq 0$

Solution:→ ~~$K2^K+$~~ by def^n $z(K2^K+K3^K) = z(K2^K)+z(K3^K)$ —①

linearity prop.

by property $z(Kx_K) = -z\frac{d}{dz}z(x_K)$

$$\therefore ① \Rightarrow z(K2^K+K3^K) = -z\frac{d}{dz}z(2^K) - z\frac{d}{dz}z(3^K)$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-2}\right) - z\frac{d}{dz}\left(\frac{z}{z-3}\right) \quad |z|>3 \text{ ROC}$$

$$= -z\left[\frac{(z-2)1-z(1)}{(z-2)^2}\right] - z\left[\frac{(z-3)1-z}{(z-3)^2}\right]$$

For $z(2^K)$, $z(3^K)$ refere above example

$$= \frac{2z}{(z-2)^2} + \frac{3z}{(z-3)^2}$$

Ex Find z-transform of $\sin 5K$; $K\geq 0$

Solution:→ by defination, $z(x_K) = \sum_{K=-\infty}^{\infty} x_K z^{-K}$

$$\therefore z(\sin 5K) = \sum_{-\infty}^{\infty} \sin 5K \cdot z^{-K} = \sum_{K=-\infty}^{-1} 0\cdot z^{-K} + \sum_{K=0}^{\infty} \sin 5K \cdot z^{-K}$$

$$= \sum_{K=0}^{\infty}\left[\frac{e^{i5K}-e^{-i5K}}{2i}\right]z^{-K} = \frac{1}{2i}\sum_{K=0}^{\infty}\left(\frac{e^{5i}}{z}\right)^K - \frac{1}{2i}\sum_{K=1}^{\infty}\left(\frac{e^{-5i}}{z}\right)^K$$

$$= \frac{1}{2i}\left\{1+\frac{e^{5i}}{z}+\left(\frac{e^{5i}}{z}\right)^2+\left(\frac{e^{5i}}{z}\right)^3+\dots\right\} - \frac{1}{2i}\left\{1+\left(\frac{e^{-5i}}{z}\right)+\left(\frac{e^{-5i}}{z}\right)^2+\dots\right\}$$

G.P. G.P.

$$z(\sin 5K) = \frac{1}{2i}\,\frac{1\left[1-\left(\frac{e^{5i}}{z}\right)^{\infty}\right]}{1-\frac{e^{5i}}{z}} - \frac{1}{2i}\,\frac{1\left[1-\left(\frac{e^{-5i}}{z}\right)^{\infty}\right]}{1-\frac{e^{-5i}}{z}} \quad —①$$

For ROC $\left|\frac{e^{5i}}{z}\right|<1$ and $\left|\frac{e^{-5i}}{z}\right|<1$

$\Rightarrow |\cos 5+i\sin 5|<|z|$ and $|\cos 5-i\sin 5|<|z|$

$\sqrt{\cos^2 5+\sin^2 5}<|z|$ and $\sqrt{\cos^2 5+\sin^2 5}<|z|$

$\Rightarrow |z|>1$ ROC

Now $z\{\sin\alpha k\} = \dfrac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$

$\therefore z\{\sin(\frac{k\pi}{3})\} = \dfrac{z\sin(\pi/3)}{z^2 - 2z\cos(\frac{\pi}{3}) + 1} = \dfrac{z\frac{\sqrt{3}}{2}}{z^2 - 2z\frac{1}{2} + 1} = \dfrac{z\sqrt{3}}{2(z^2 - z + 1)}$ —②

also $z\{\cos\alpha k\} = \dfrac{z[z - \cos\alpha]}{z^2 - 2z\cos\alpha + 1}$

$\therefore z\{\cos\frac{k\pi}{3}\} = \dfrac{z[z - \cos\frac{\pi}{3}]}{z^2 - 2z\cos(\frac{\pi}{3}) + 1} = \dfrac{z(z - \frac{1}{2})}{z^2 - 2z(\frac{1}{2}) + 1} = \dfrac{z(2z-1)}{2(z^2 - z + 1)}$ —③

using ②, ③ in ①

$z\{\cos(\frac{k\pi}{3} + \alpha)\} = \cos\alpha\left\{\dfrac{z(2z-1)}{2(z^2 - z + 1)}\right\} - \sin\alpha\left\{\dfrac{z\sqrt{3}}{z^2 - z + 1}\right\}$.

$\boxed{\text{EX}}$ find $Z(k^2 e^{-ak})$; $k \geq 0$ — { EX-09 } 6 mar

Solution:- we know that

if $U(k) = 1$, $k \geq 0$ then $z[U(k)] = \dfrac{z}{z-1}$ —①

we also know that, if $z(x_k) = F(z)$ then $z[e^{-ak} x_k] = F(e^a z)$

$\therefore z\{e^{-ak} U(k)\} = \dfrac{e^a z}{e^a z - 1}$ where $F(z) = \dfrac{z}{z-1}$

i.e. $z\{e^{-ak}\} = \dfrac{e^a z}{e^a z - 1}$ $[\because U(k) = 1]$

again we know the result,

$z(k x_k) = -z\frac{d}{dz} z(x_k)$

$\therefore z\{k e^{-ak}\} = -z\frac{d}{dz} z(e^{-ak})$

$= -z\frac{d}{dz}\left[\dfrac{e^a z}{e^a z - 1}\right] = -z\left\{\dfrac{(e^a z - 1)e^a - e^a z \cdot e^a}{(e^a z - 1)^2}\right\}$

$= -z\left[\dfrac{-e^a}{(e^a z - 1)^2}\right] \Rightarrow z\{k e^{-ak}\} = \dfrac{z e^a}{(e^a z - 1)^2}$

$\therefore z\{k^2 e^{-ak}\} = z\{k(k e^{-ak})\} = -z\frac{d}{dz} z(k e^{-ak})$

$= -z\frac{d}{dz}\left\{\dfrac{z e^a}{(e^a z - 1)^2}\right\} = -z\left\{\dfrac{(e^a z - 1)^2 \cdot e^a - z e^a\, 2(e^a z - 1)\cdot e^a}{(e^a z - 1)^4}\right\}$

$= \dfrac{-z(e^a z - 1)\{(e^a z - 1)e^a - 2z e^a e^a\}}{(e^a z - 1)^4}$

$= -z\left\{\dfrac{-e^a - z e^a e^a}{(e^a z - 1)^3}\right\}$

$\therefore z\{k^2 e^{-ak}\} = \dfrac{z e^a(1 + z e^a)}{(e^a z - 1)^3}$

Ex Find $z\{2^K \sin(3K+2)\}$, $K>0$ $\longrightarrow$ { z - ogi } 6 may4 13

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Solution:- $2^K\sin(3K+2) = 2^K[\sin 3K\cdot\cos 2 + \cos 3K\cdot\sin 2]$

$= \cos 2\cdot 2^K\sin 3K + \sin 2\cdot 2^K\cos 3K$

$\therefore z\{2^K\sin(3K+2)\} = \cos 2\cdot z\{2^K\sin 3K\} + \sin 2\cdot z\{2^K\cos 3K\} \longrightarrow (1)$

to Find $z\{2^K\sin 3K\}$:-

$z\{x_K\} = \sum_{K=-\infty}^{\infty} x_K z^{-K} \Rightarrow z\{2^K\sin 3K\} = \sum_{K=1}^{\infty} 2^K\sin 3K\cdot z^{-K}$

$\therefore z\{2^K\sin 3K\} = \sum_{K=1}^{\infty} 2^K\left\{\frac{e^{3iK}-e^{-3iK}}{2i}\right\}z^{-K} = \frac{1}{2i}\sum_{K=1}^{\infty}\frac{2^K e^{3iK}}{z^K} - \frac{1}{2i}\sum_{K=1}^{\infty}\frac{2^K e^{-3iK}}{z^K}$

$= \frac{1}{2i}\sum_{K=1}^{\infty}\left(\frac{2e^{3i}}{z}\right)^K - \frac{1}{2i}\sum_{K=1}^{\infty}\left(\frac{2e^{-3i}}{z}\right)^K$

$= \frac{1}{2i}\left\{\left(\frac{2e^{3i}}{z}\right) + \left(\frac{2e^{3i}}{z}\right)^2 + \dots\right\} - \frac{1}{2i}\left\{\left(\frac{2e^{-3i}}{z}\right) + \left(\frac{2e^{-3i}}{z}\right)^2 + \dots\right\}$ which are G.P's

$= \frac{1}{2i}\frac{\frac{2e^{3i}}{z}\left[1-\left(\frac{2e^{3i}}{z}\right)^{\infty}\right]}{\left[1-\frac{2e^{3i}}{z}\right]} - \frac{1}{2i}\frac{\frac{2e^{-3i}}{z}\left[1-\left(\frac{2e^{-3i}}{z}\right)^{\infty}\right]}{\left[1-\frac{2e^{-3i}}{z}\right]}$

$= \frac{1}{2i}\frac{2e^{3i}}{z}\frac{[1-0]}{\left(\frac{z-2e^{3i}}{z}\right)} - \frac{1}{2i}\frac{2e^{-3i}}{z}\frac{(1-0)}{\left(\frac{z-2e^{-3i}}{z}\right)}$ For ROC $\left|\frac{2e^{3i}}{z}\right|<1$ and $\left|\frac{2e^{-3i}}{z}\right|<1$

$= \frac{1}{i}\frac{e^{3i}}{(z-2e^{3i})} - \frac{1}{i}\frac{e^{-3i}}{(z-2e^{-3i})} = \frac{1}{i}\left\{\frac{ze^{3i}-2-ze^{-3i}+2}{z^2-2ze^{-3i}-2ze^{3i}+4}\right\}$

$= \frac{1}{i}\left\{\frac{z(e^{3i}-e^{-3i})}{z^2-2z(e^{3i}+e^{-3i})+4}\right\} = \frac{1}{i}\left\{\frac{z\,2i\sin 3}{z^2-2z\,2\cos 3+4}\right\}$

$\because \sin x = \frac{e^{ix}-e^{-ix}}{2i}$, $\cos x = \frac{e^{ix}+e^{-ix}}{2}$

$\therefore \boxed{z\{2^K\sin 3K\} = \frac{2z\sin 3}{z^2-4z\cos 3+4}}$ — (2)

to Find $z\{2^K\cos 3K\}$:-

$z(x_K) = \sum_{K=-\infty}^{\infty} x_K z^{-K}$

$\therefore z(2^K\cos 3K) = \sum_{K=1}^{\infty} 2^K\cos 3K\cdot z^{-K} = \sum_{K=1}^{\infty} 2^K\left(\frac{e^{3iK}+e^{-3iK}}{2}\right)\frac{1}{z^K}$

$= \frac{1}{2}\sum_{K=1}^{\infty}\left(\frac{2e^{3i}}{z}\right)^K + \frac{1}{2}\sum_{K=1}^{\infty}\left(\frac{2e^{-3i}}{z}\right)^K$ which are G.P.'s

$= \frac{1}{2}\left\{\left(\frac{2e^{3i}}{z}\right) + \left(\frac{2e^{3i}}{z}\right)^2 + \dots\right\} + \frac{1}{2}\left\{\left(\frac{2e^{-3i}}{z}\right) + \left(\frac{2e^{-3i}}{z}\right)^2 + \dots\right\}$

$= \frac{1}{2}\frac{\left(\frac{2e^{3i}}{z}\right)[1-0]}{1-\frac{2e^{3i}}{z}} + \frac{1}{2}\frac{\frac{2e^{-3i}}{z}[1-0]}{1-\frac{2e^{-3i}}{z}}$ For ROC, $\left|\frac{2e^{3i}}{z}\right|<1$, $\left|\frac{2e^{-3i}}{z}\right|<1$

$= \frac{e^{3i}}{z-2e^{3i}} + \frac{e^{-3i}}{z-2e^{-3i}} = \frac{ze^{3i}-2+ze^{-3i}-2}{z^2-2ze^{-3i}-2ze^{3i}+4}$

$$= \frac{z(e^{3i}+e^{-3i})}{z^2-2z(e^{3i}+e^{-3i})+4} \qquad \doteq \frac{z\cdot 2\cos 3-4}{z^2-2z(2\cos 3)+4}$$

$$\therefore z(2^K\cos 3K) = \frac{2z\cos 3-4}{z^2-4z\cos 3+4} \quad - (3)$$

using (2), (3) in (1)

$$z\left\{2^K \sin(3K+2)\right\} = \cos 2\cdot\frac{-2z\sin 3}{(z^2-4z\cos 3+4)} + \sin 2\cdot\frac{2z\cos 3-4}{(z^2-4z\cos 3+4)}$$

$$= \frac{2z}{(z^2-4z\cos 3+4)}\left[\cos 2\sin 3+\sin 2\cos 3\right]$$

$$z\left\{2^K\sin(3K+2)\right\} = \frac{2z\sin 5}{z^2-4z\cos 3+4}$$

$\boxed{OR}$

$$\left\{2^K\sin(3K+2)\right\}_{K\geq 0} = \left\{2^K\left[\sin 3K\cos 2+\cos 3K\cdot\sin 2\right]\right\}$$

$$= 2^K\sin 3K\cos 2+2^K\cos 3K\cdot\sin 2$$

$$\therefore z\left\{2^K\sin(3K+2)\right\} = \cos 2\cdot z\left\{2^K\sin 3K\right\}+\sin 2\, z\left\{2^K\cos 3K\right\} \quad -(1)$$

we know that $z\{\sin 3K\} = \dfrac{z\cdot\sin 3}{z^2-2z\sin 3+1}$

and $z\{\cos 3K\} = \dfrac{z[z-\cos 3]}{z^2-2z\cos 3+1}$

Now $z\{a^K x_K\} = F(\frac{z}{a})$ where $F(z)=Z(x_K)$

$$\therefore z\{2^K\sin 3K\} = \frac{\frac{z}{2}\cdot\sin 3}{\frac{z^2}{4}-2\frac{z}{2}\sin 3+1} \quad \text{and } z\{2^K\cos 3K\} = \frac{\frac{z}{2}\left[\frac{z}{2}-\cos 3\right]}{\frac{z^2}{4}-2\frac{z}{2}\cos 3+1}$$

$$\therefore (1) \Rightarrow z\{2^K\sin(3K+2)\} = \cos 2\cdot\frac{\frac{z}{2}\sin 3}{(\frac{z^2}{4}-z\sin 3+1)}+\sin 2\,\frac{\frac{z}{2}\left[\frac{z}{2}-\cos 3\right]}{\frac{z^2}{4}-z\cos 3+1}$$