- Exam 2 : Monday, March 27, 2023.

Graph algorithms

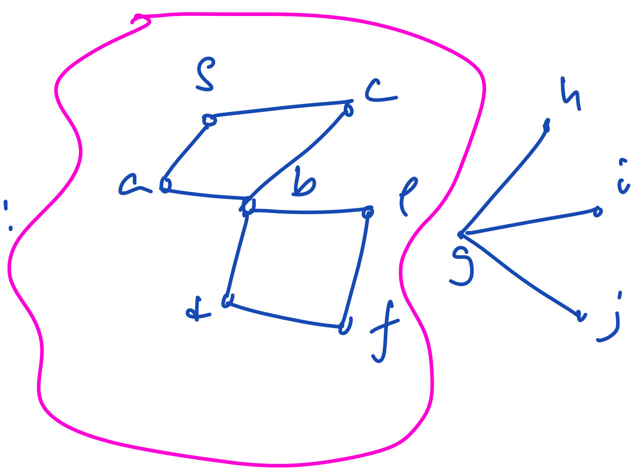
Input: . Undirected graph $G = (V, E)$

. $s \in V$

Objective: To find all vertices reachable from s in G.

In other words, we want to find the connected component containing s in G.

G:



Algorithm

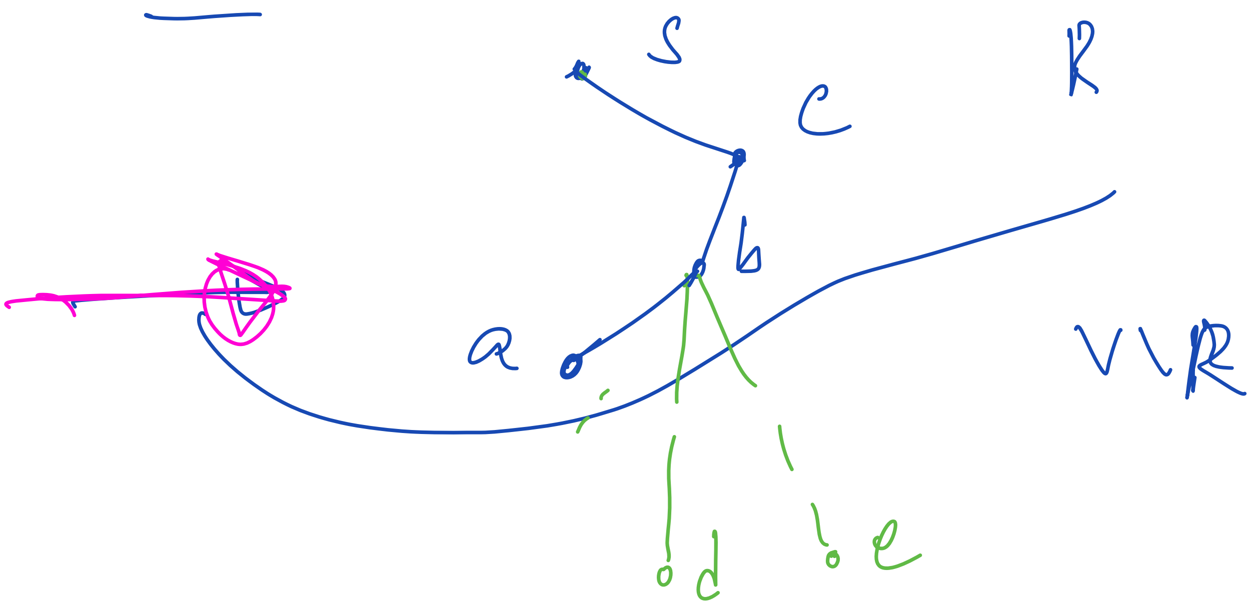
$R \leftarrow \{s\}$

while there exists an edge $e = (u,v)$ s.t. $u \in R$ and $v \in V \setminus R$ **do**

$R \leftarrow R \cup \{v\}$

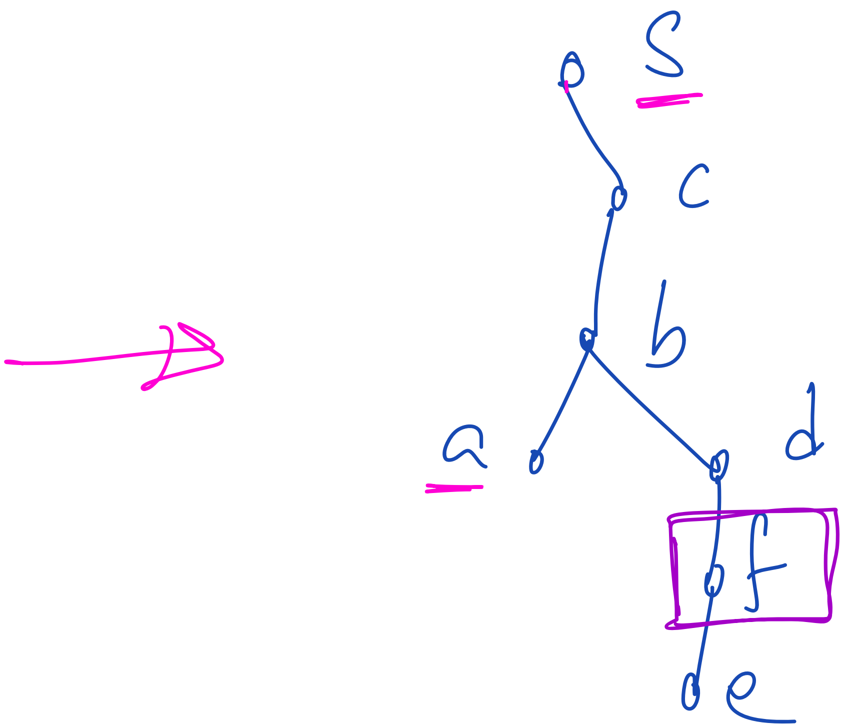
$\pi(v) \leftarrow u$ parent of v is u.

return R



$R = \{s, c, b\}$

$V \setminus R = \{ d, e, f, g, h, i, j \}$

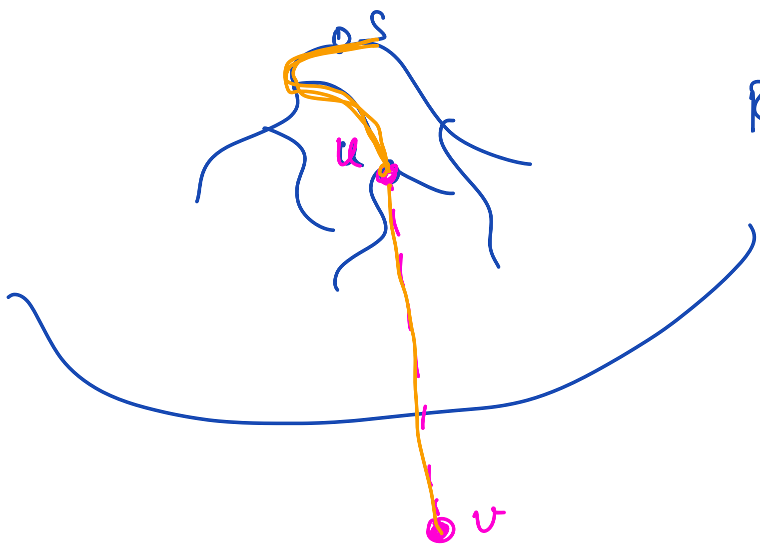


Theorem: Our alg outputs the vertices in the connected component containing s in G.

Proof: We want to prove two claims.

Claim 1: If $v \in R$ then there is a $s \rightsquigarrow v$ path in G. $P \Rightarrow q$: Neg $= p \wedge \bar{q}$.

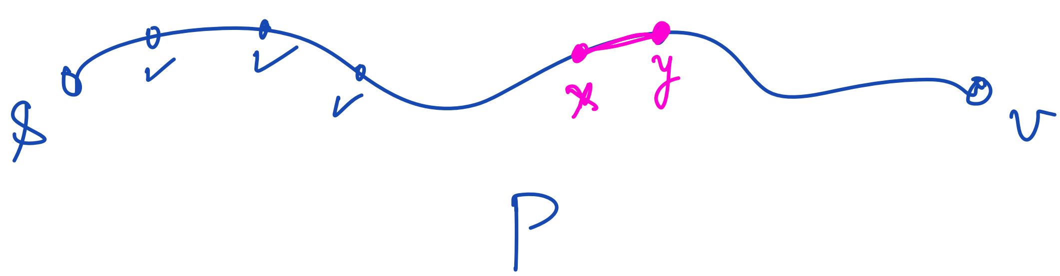
Proof: Assume for contradiction that there are vertices in R, but there is no path from s to those vertices in G. Among all such vertices let v be the $\boxed{\text{first}}$ vertex brought into R s.t there is no $s \rightsquigarrow v$ path in G. Let $u = \pi(v)$. There must be a $s \rightsquigarrow u$ path in G.



Since there is an edge (u,v) in G, the path from $s \rightsquigarrow u$ in G + (u,v) forms the $s \rightsquigarrow v$ path in G. This is a contradiction.

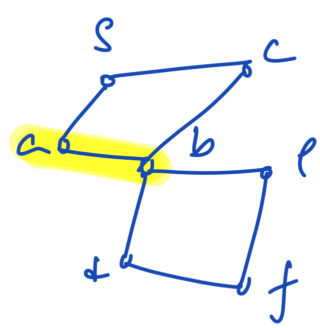
Claim 2: For any vertex v, if there is a $s \leadsto v$ path in G then $v \in R$.

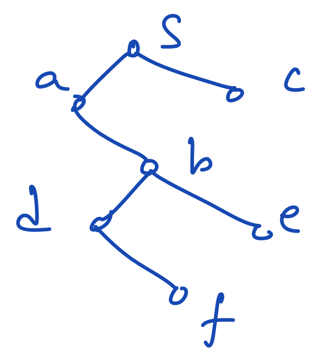
Proof: Assume for contradiction that there exists a vertex v s.t. there is a $s \leadsto v$ path in G, but $v \notin R$. Let P be $s \leadsto v$ path in G.

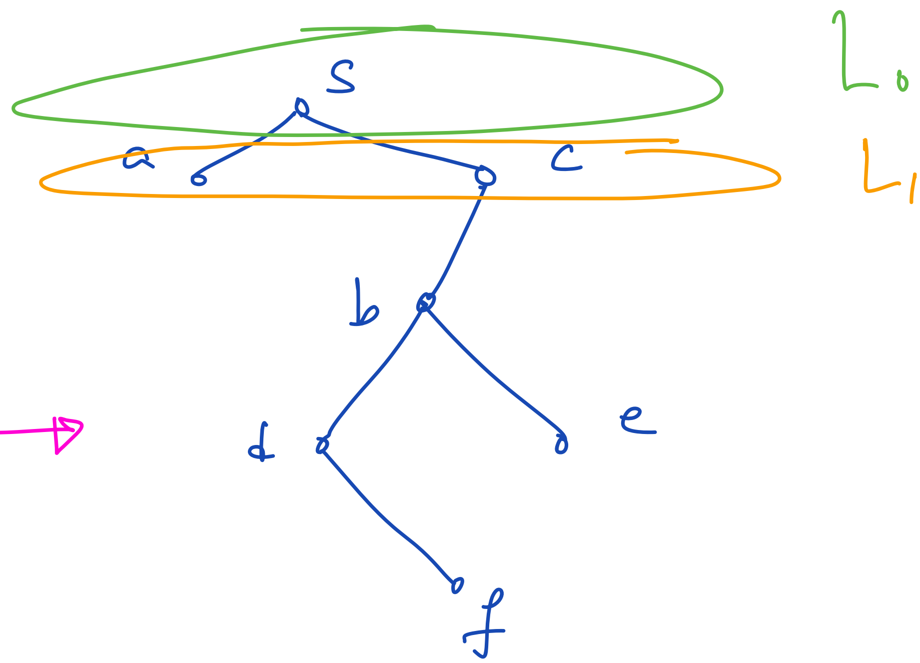


Going from s towards v along P,
let y be the first vertex s.t.
$y \notin R$. Let x be the vertex preceding
y in P. It must be that $x \in R$.
Since $y \notin R$, y must belong to
$V \setminus R$. This means that the edge
(x, y) crosses the cut after is
alg.'s over, a contradiction!

Breadth first Search.







BFS tree

L_0 : contains s

L_1 : contains neighbors of s

L_2 : all vertices that are not discovered yet and that have a neighbor in L_1

⋮

L_{i+1} : all vertices that

- do not belong to $L_0 \cup L_1 \cup \dots \cup L_i$

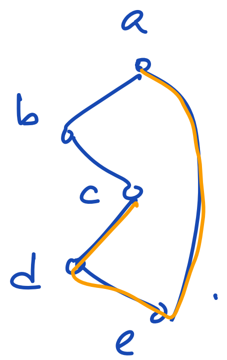
- that have a neighbor in L_i

Properties.

Lemma : Let $v \in L_i$ in the BFS tree.

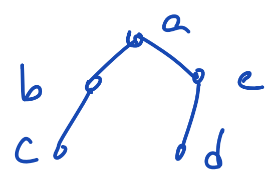
Then $\text{dist}_G(s, v) = \underline{\quad i \quad}$.

↳ smallest # edges in a $s \rightsquigarrow v$ path in G.



$\text{dist}(a,c) = 2$

$\text{dist}(a,e) = 1$



<u>Lemma</u> : Output of BFS is a tree. We call it the BFS tree.

<u>Lemma</u> : Let $e=(u,v)$ be any edge in G. Let $u \in L_i$ and $v \in L_j$ in the BFS tree. Then $|i-j| \leq 1$.

<u>Proof</u> : Without loss of generality, let $i \leq j$.

<u>Case I</u> : $i=j$. Done because $|i-j|=0$.

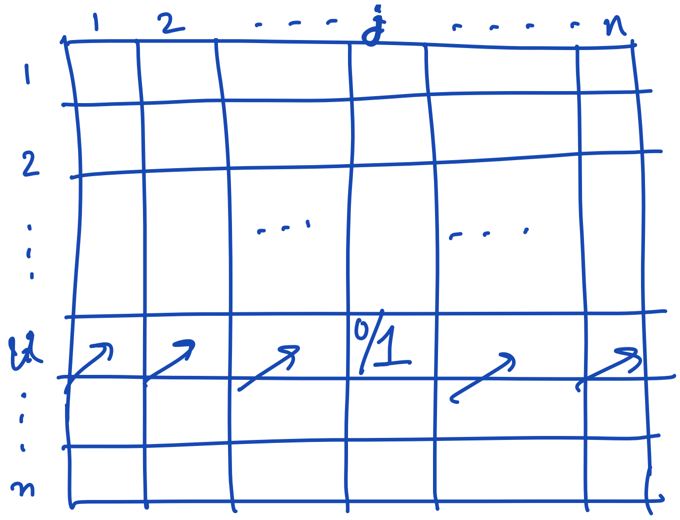
Case II: $i < j$.

To complete the proof, we need to argue that $j = i+1$. Since $v \notin L_0 \cup L_1 \cup \cdots \cup L_i$ and v has a neighbor u in L_i, it must be that $v \in L_{i+1}$.

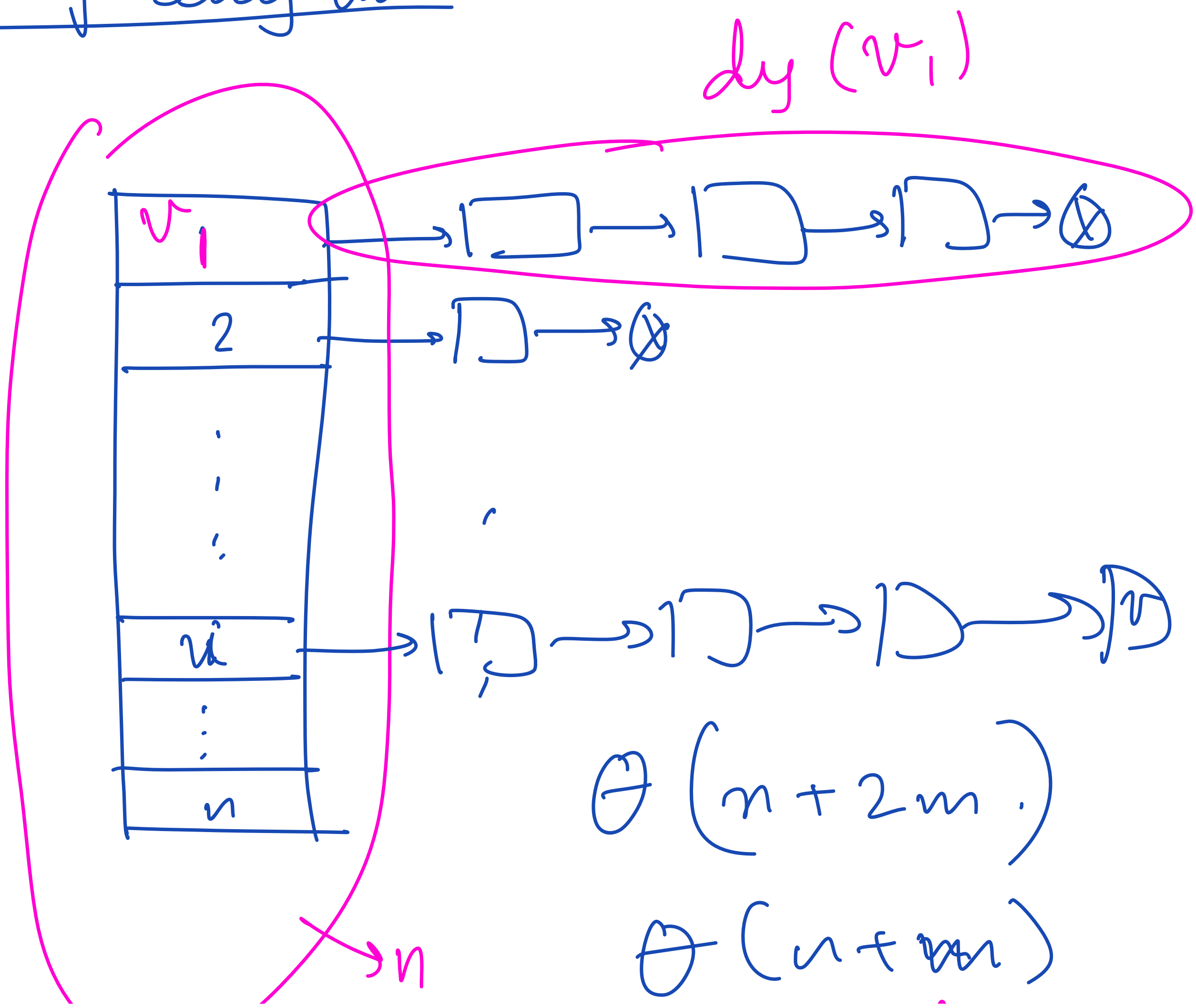
This completes the proof.

Graph representation:

Adjacency matrix



Adjacency list:

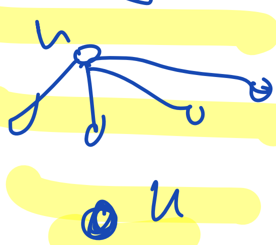


$$\Theta(n+2m)$$

$$\Theta(n+m)$$

Total space $= n + \sum_u dg(u) = n + 2m = O(n+m)$

	Adj Matrix	Adj. list
Space	$\Theta(n^2)$	$\Theta(n+m)$
$(u,v) \in E$	$O(1)$	$\cancel{O(m)}, \cancel{O(1)}$, $O(dg(u))$
Say "hi" to all of u's neighbors.	$\cancel{\Theta(dg(u))}$ $\Theta(n)$	$\Theta(dg(u))$



Unless specified otherwise, we will assume adj. list representation of a graph.