Name ⇒ Manjiri N. Naik
Roll no. ⇒ 21102B0008
Branch ⇒ Computer Engineering (CMPN)
Division ⇒ B   Semester ⇒ III

Good

Tutorial 3 : Module 3 : Fourier Series (Radian Form)

1] Find Fourier series for the function

$$f(x) = 1 + 2x/\pi \quad -\pi \le x \le 0$$
$$ 1 - 2x/\pi \quad 0 \le x \le \pi.$$

Hence prove that $1/1^2 + 1/3^2 + 1/5^2 + \dots = \pi^2/8$.

⇒ $f(x) = 1 + 2x/\pi \quad -\pi \le x \le 0$

$f(-x) = 1 - 2x/\pi \quad -\pi \le -x \le 0$ i.e., $\pi \ge x \ge 0$ i.e., $0 \le x \le \pi$

$\therefore f(-x) = f(x)$ for $-\pi \le x \le 0$ ... ①

$f(x) = 1 - 2x/\pi \quad 0 \le x \le \pi$

$f(-x) = 1 + 2x/\pi \quad 0 \le -x \le \pi$ i.e., $0 \ge x \ge -\pi$ i.e., $-\pi \le x \le 0$

$\therefore f(-x) = f(x)$ for $0 \le x \le \pi$ ... ②

From ① and ②, $f(x)$ is an even function. $\therefore \boxed{b_n = 0}$

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cdot \cos(nx)]$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot dx \Rightarrow a_0 = \frac{1}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cdot dx$$

$$a_0 = \frac{1}{\pi} \left\{ x - \frac{x^2}{\pi} \right\}_0^{\pi} \Rightarrow a_0 = \frac{1}{\pi} [(\pi - \pi) - (0 - 0)]$$

$\therefore \boxed{a_0 = 0}$

$$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos(nx)\cdot dx \Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cdot\cos(nx)\cdot dx$$

$$\Rightarrow a_n = \frac{2}{\pi}\left\{\left(1-\frac{2x}{\pi}\right)\left[\frac{\sin(nx)}{n}\right] - \left(\frac{-2}{\pi}\right)\left[\frac{-\cos(nx)}{n^2}\right]\right\}_0^{\pi}$$

$$\Rightarrow a_n = \frac{2}{\pi}\left\{\frac{2}{\pi n^2}\left[-\cos(nx)\right]_0^{\pi}\right\} \Rightarrow a_n = \frac{4}{\pi^2 n^2}\left[1-\cos(n\pi)\right]$$

For $n = \text{even} \Rightarrow \cos(n\pi) = 1 \quad \therefore 1-1 = 0$

For $n = \text{odd} \Rightarrow \cos(n\pi) = -1 \quad \therefore 1-(-1) = 1+1 = 2$

$$\Rightarrow a_n = \begin{cases} 0 & \text{for } n = 2, 4, 6, \dots \\ \dfrac{8}{\pi^2 n^2} & \text{for } n = 1, 3, 5, \dots \end{cases}$$

Now, $f(x) = a_0 + \sum_{n=1,3,5,\dots}^{\infty}\left[a_n\cdot\cos(nx)\right]$

$$\Rightarrow f(x) = 0 + \sum_{n=1,3,5,\dots}^{\infty}\left[\frac{8}{\pi^2 n^2}\cdot\cos(nx)\right]$$

$$\Rightarrow f(x) = \frac{8}{\pi^2}\sum_{n=1,3,5,\dots}^{\infty}\left[\frac{1}{n^2}\cdot\cos(nx)\right] \Rightarrow \boxed{\left(1+2x/\pi\right)\pi^2/8 = \sum_{n=1,3,5,\dots}^{\infty}\left[\frac{1}{n^2}\cdot\cos(nx)\right]}$$

Put $x = 0$

$$\Rightarrow \pi^2/8 = \sum_{n=1,3,5,\dots}^{\infty}\left(\frac{1}{n^2}\right)(1)$$

$$\therefore \boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots}$$

Hence proved.

2] Find Fourier series for the function

$f(x) = x + \pi/2 \qquad -\pi < x < 0$

$\qquad\quad \pi/2 - x \qquad 0 < x < \pi.$

Hence prove that $1/1^4 + 1/3^4 + 1/5^4 + \ldots\ldots = \pi^4/96.$

Sol^n. ⇒ $f(x) = \pi/2 + x \qquad -\pi < x < 0$

$f(-x) = \pi/2 - x \qquad -\pi < -x < 0$ i.e., $\pi > x > 0$ i.e., $0 < x < \pi$

$\therefore f(-x) = f(x)$ for $-\pi < x < 0 \qquad \ldots\ldots (1)$

$f(x) = \pi/2 - x \qquad 0 < x < \pi$

$f(-x) = \pi/2 + x \qquad 0 < -x < \pi$ i.e., $0 > x > -\pi$ i.e., $-\pi < x < 0$

$\therefore f(-x) = f(x)$ for $0 < x < \pi \qquad \ldots\ldots (2)$

From (1) and (2), $f(x)$ is an even function. $\therefore \boxed{b_n = 0}$

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cdot \cos(nx)]$$

$$\Rightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot dx = \frac{1}{\pi}\int_0^{\pi} f(x)\cdot dx = \frac{1}{\pi}\int_0^{\pi}(\pi/2 - x)\cdot dx$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left[\frac{\pi x}{2} - \frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left\{\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right] - [0-0]\right\} \qquad \therefore \boxed{a_0 = 0}$$

$$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos(nx)\cdot dx \Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi}(\pi/2 - x)\cdot\cos(nx)\cdot dx$$

$$\Rightarrow a_n = \frac{2}{\pi}\left\{(\pi/2 - x)\left[\frac{\sin(nx)}{n}\right] - (-1)\left[\frac{-\cos(nx)}{n^2}\right]\right\}_0^{\pi}$$

$$\Rightarrow a_n = \frac{2}{\pi n^2}\left[-\cos(nx)\right]_0^{\pi} = \frac{2}{\pi n^2}\left[1 - \cos(n\pi)\right] \qquad \therefore \boxed{a_n = \frac{2}{\pi n^2}\left[1 - (-1)^n\right]}$$

For n = even ⇒ $(-1)^n = 1 \quad \therefore 1 - 1 = 0$

For n = odd ⇒ $(-1)^n = -1 \quad \therefore 1 - (-1) = 1 + 1 = 2$

Now, $f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cdot \cos(nx)]$

$\therefore$ For odd, $a_n = \frac{4}{\pi n^2}$

$\therefore f(x) = 0 + \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \left[\frac{1}{n^2} \cdot \cos(nx)\right]$

But $f(x)$ is discontinuous at $x = 0$.

$f(x) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right] = \frac{1}{2}\left[\pi/2 + \pi/2\right] = \pi/2$

Now putting $x = 0$

$\pi/2 = 4/\pi \left[1/1^2 + 1/3^2 + 1/5^2 + \ldots\ldots\right]$

$\therefore \boxed{\pi^2/8 = 1/1^2 + 1/3^2 + 1/5^2 + \ldots\ldots}$

By Parseval's identity,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2 \cdot dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

$$\Rightarrow \frac{1}{\pi}\int_0^{\pi} (\pi/2 - x)^2 \cdot dx = 0 + \frac{1}{2}\sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{4}{\pi n^2}\right)^2$$

$$\Rightarrow \frac{1}{\pi}\int_0^{\pi} (\pi/2 - x)^2 \cdot dx = \frac{8}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{1}{n^4}\right)$$

$$\Rightarrow \frac{1}{\pi}\left[\pi^2/4 - \pi x + x^2\right] \cdot dx = \frac{8}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right]$$

$$\Rightarrow 1/\pi \left[\pi^2 x/4 - \pi x^2/2 + x^3/3\right]_0^{\pi} = 8/\pi^2 \left[1/1^4 + 1/3^4 + 1/5^4 + \ldots\right]$$

$$\Rightarrow 1/\pi \left[\pi^3/4 - \pi^3/2 + \pi^3/3\right] = 8/\pi^2 \left[1/1^4 + 1/3^4 + 1/5^4 + \ldots\right]$$

$$\Rightarrow \pi^4/8 \left[1/4 - 1/2 + 1/3\right] = 1/1^4 + 1/3^4 + 1/5^4 + \ldots\ldots$$

$$\Rightarrow \pi^4/8 \left[1/12\right] = 1/1^4 + 1/3^4 + 1/5^4 + \ldots\ldots$$

$$\therefore \boxed{\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots}$$ Hence proved.

3] Find Fourier series for the function

$f(x) = \sqrt{1-\cos(x)}$ in $(0, 2\pi)$.

Hence deduce that $1/2 = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$.

$\Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cdot dx \Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{1-\cos(x)}\cdot dx.$

$\Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{2\sin^2(x/2)}\cdot dx \Rightarrow a_0 = \frac{1}{\sqrt{2}\pi}\int_0^{2\pi} \sin(x/2)\cdot dx$

$\Rightarrow a_0 = \frac{1}{\sqrt{2}\pi}\left[\frac{-\cos(x/2)}{1/2}\right]_0^{2\pi} \Rightarrow a_0 = \frac{\sqrt{2}}{\pi}\left[\cos(0)-\cos(\pi)\right]$

$\Rightarrow a_0 = \frac{\sqrt{2}}{\pi}\left[1-(-1)\right] = \frac{\sqrt{2}}{\pi}\left[1+1\right] = \frac{2\sqrt{2}}{\pi} \quad \therefore \boxed{a_0 = \frac{2\sqrt{2}}{\pi}}$

$\Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\cos(nx)\cdot dx \Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\cdot\sin(x/2)\cdot\cos(nx)\cdot dx$

$\Rightarrow a_n = \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} \left[\sin(1/2+n)x - \sin(n-1/2)x\right]\cdot dx$

$\Rightarrow a_n = \frac{1}{\sqrt{2}\pi}\int_0^{2\pi} \left[\sin\left(\frac{2nx+x}{2}\right) - \sin\left(\frac{2nx-x}{2}\right)\right]\cdot dx$

$\Rightarrow a_n = \frac{1}{\sqrt{2}\pi}\left[\frac{-\cos(n+1/2)x}{(n+1/2)} + \frac{\cos(n-1/2)x}{(n-1/2)}\right]_0^{2\pi}$

$\Rightarrow a_n = \frac{1}{\sqrt{2}\pi}\left\{\frac{-(-1)^n}{(n+1/2)} + \frac{(-1)^n}{(n-1/2)} + \frac{1}{(n+1/2)} - \frac{1}{(n-1/2)}\right\}$

$\Rightarrow a_n = \frac{1}{\sqrt{2}\pi}\left\{\frac{1-(-1)^n}{(n+1/2)} - \frac{1-(-1)^n}{(n-1/2)}\right\}$

$\Rightarrow a_n = \frac{1}{\sqrt{2}\pi}\left[1-(-1)^n\right]\left\{\frac{n-1/2-n-1/2}{n^2-1/4}\right\}$

$\Rightarrow a_n = \frac{4}{\sqrt{2}\pi}\cdot\frac{1}{(1-4n^2)}\left[1-(-1)^n\right]$

$a_n \xrightarrow{\text{even}} 0 \quad , \; n = 2, 4, 6, \ldots$

$a_n \xrightarrow{\text{odd}} \dfrac{8}{\sqrt{2}\,\pi(1-4n^2)} \quad , \; n = 1, 3, 5, \ldots$

$\Rightarrow \; b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\cdot\sin(nx)\cdot dx \;\Rightarrow\; b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \sqrt{2}\cdot\sin(x/2)\cdot\sin(nx)\cdot dx$

$\Rightarrow \; b_n = \dfrac{\sqrt{2}}{2\pi}\displaystyle\int_0^{2\pi}\left[\cos(n-1/2)x - \cos(n+1/2)x\right]\cdot dx$

$\Rightarrow \; b_n = \dfrac{1}{\sqrt{2}\,\pi}\left\{\cancel{x}\cdot\dfrac{\sin(n-1/2)x}{(n-1/2)} - (1)\,\dfrac{-\sin(n+1/2)x}{(n+1/2)}\right\}_0^{2\pi}$

$\Rightarrow \; b_n = \dfrac{1}{\sqrt{2}\,\pi}\left[0-0\right] \qquad \therefore \boxed{b_n = 0}$

$$f(x) = \frac{2\sqrt{2}}{\pi} + \sum_{n=1,3,5,\ldots}^{\infty}\left[\frac{8}{\sqrt{2}\,\pi(1-4n^2)}\cdot\cos(nx)\right]$$

$\Rightarrow$ Put $x = 0$

$\Rightarrow \; f(0) = \sqrt{1-\cos(0)} = 0$

$\Rightarrow \; 0 = \dfrac{2\sqrt{2}}{\pi} + \displaystyle\sum_{n=1}^{\infty}\dfrac{8}{\sqrt{2}\,\pi(1-4n^2)}$

$\Rightarrow \; \dfrac{2\sqrt{2}}{\pi} = \dfrac{8}{\sqrt{2}\,\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{(4n^2-1)}$

$$\therefore \quad \boxed{1/2 = \sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}}$$

Hence proved.

4] Find Fourier series for $f(x) = x^2 + x$ in $[0, 2\pi]$ with $f(x+2\pi) = f(x)$.

Solⁿ. ⇒ $f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cdot \cos(nx) + b_n \cdot \sin(nx)]$

$\Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cdot dx \Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi}(x^2+x)\cdot dx$

$\Rightarrow a_0 = \frac{1}{2\pi}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^{2\pi} \Rightarrow a_0 = \frac{1}{2\pi}\left[\frac{8\pi^3}{3} + \frac{4\pi^2}{2}\right] = \frac{1}{2\pi}\left[\frac{16\pi^3 + 12\pi^2}{6}\right]$

$\Rightarrow a_0 = \frac{4\pi^2}{3} + \pi \qquad \therefore \boxed{a_0 = \frac{4\pi^2}{3} + \pi}$

$\Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\cos(nx)\cdot dx \Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi}(x^2+x)\cdot\cos(nx)\cdot dx$

$\Rightarrow a_n = \frac{1}{\pi}\left\{(x^2+x)\cdot\frac{\sin(nx)}{n} - (2x+1)\left[\frac{-\cos(nx)}{n^2}\right] + (2)\frac{-\sin(nx)}{n^3}\right\}_0^{2\pi}$

$\Rightarrow a_n = \frac{1}{\pi}\left\{(2x+1)\left[\frac{\cos(nx)}{n^2}\right]\right\}_0^{2\pi} \Rightarrow a_n = \frac{1}{\pi}\left[\frac{4\pi+1}{n^2} - \frac{1}{n^2}\right]$

$\Rightarrow a_n = \frac{1}{\pi n^2}[4\pi] \qquad \therefore \boxed{a_n = 4/n^2}$

$\Rightarrow b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\sin(nx)\cdot dx \Rightarrow b_n = \frac{1}{\pi}\int_0^{2\pi}(x^2+x)\cdot\sin(nx)\cdot dx$

$\Rightarrow b_n = \frac{1}{\pi}\left\{(x^2+x)\cdot\left[\frac{-\cos(nx)}{n}\right] - (2x+1)\left[\frac{-\sin(nx)}{n^2}\right] + (2)\cdot\frac{\cos(nx)}{n^3}\right\}_0^{2\pi}$

$\Rightarrow b_n = \frac{1}{\pi}\left\{\frac{-(4\pi^2+2\pi)}{n} + \frac{2}{n^3} - 0 - \frac{2}{n^3}\right\}$

$\Rightarrow b_n = \frac{-\pi}{\pi}\frac{(4\pi+2)}{n}$

$\therefore \boxed{b_n = \frac{-2(2\pi+1)}{n}}$

Now, $f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cdot \cos(nx) + b_n \cdot \sin(nx)\right]$

$\Rightarrow \quad f(x) = \frac{4\pi^2}{3} + \pi + \sum_{n=1}^{\infty}\left[\frac{4}{n^2} \cdot \cos(nx) - \frac{2(2\pi+1)}{n} \cdot \sin(nx)\right]$

Now, $f(x+2\pi) = f(x)$

$(x+2\pi)^2 + x + 2\pi = x^2 + x$

$x^2 + 4\pi x + 4\pi^2 + 2\pi = x^2$

$4\pi x = -(4\pi^2 + 2\pi)$

$\boxed{x = -(\pi + 1/2)}$

$\therefore f(-\pi - 1/2) = \left[-(\pi+1/2)\right]^2 - (\pi + 1/2)$

$= \pi^2 + \pi + 1/4 - (\pi + 1/2)$

$= \pi^2 - 1/4$

$\therefore \quad \pi^2 - 1/4 = \frac{4\pi^2}{3} + \pi + \sum_{n=1}^{\infty}\left[\frac{4}{n^2} \cdot \cos(nx) - \frac{2(2\pi+1)}{n} \cdot \sin(nx)\right]$

$\therefore \quad \boxed{-\left(\frac{\pi^2}{3} + \pi + \frac{1}{4}\right) = \sum_{n=1}^{\infty}\left[\frac{4}{n^2} \cdot \cos(nx) - \frac{2(2\pi+1)}{n} \cdot \sin(nx)\right]}$

5] Obtain Fourier series for $f(x) = e^{x}$ in $(0, 2\pi)$.

Sol^n. ⇒ $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cdot dx \Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} e^{x}\cdot dx$

⇒ $a_0 = \frac{1}{2\pi}\left[e^{x}\right]_0^{2\pi}$ $\therefore \boxed{a_0 = \frac{e^{2\pi}-1}{2\pi}}$

⇒ $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\cos(nx)\cdot dx \Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} e^{x}\cdot\cos(nx)\cdot dx$

Now, $\int e^{ax}\cdot\cos(bx)\cdot dx = \frac{e^{ax}}{a^2+b^2}\left[a\cdot\cos(bx) + b\cdot\sin(bx)\right]$

⇒ $a_n = \frac{1}{\pi}\left\{\frac{e^{x}}{1+n^2}\left[\cos(nx). + n\cdot\sin(nx)\right]\right\}_0^{2\pi}$

⇒ $a_n = \frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2} - \frac{1}{1+n^2}\right]$ $\therefore \boxed{a_n = \frac{e^{2\pi}-1}{\pi(1+n^2)}}$

⇒ $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\sin(nx)\cdot dx \Rightarrow b_n = \frac{1}{\pi}\int_0^{2\pi} e^{x}\sin(nx)\cdot dx$

⇒ $b_n = \frac{1}{\pi}\left\{\frac{e^{x}}{1+n^2}\left[\sin(nx) - n\cdot\cos(nx)\right]\right\}_0^{2\pi}$

⇒ $b_n = \frac{1}{\pi}\left[\frac{-n\cdot e^{2\pi}}{1+n^2} + \frac{n}{1+n^2}\right] \Rightarrow \boxed{b_n = \frac{n\left[1-e^{2\pi}\right]}{\pi(1+n^2)}}$

Now, $f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cdot\cos(nx) + b_n\cdot\sin(nx)\right]$

⇒ $f(x) = \frac{e^{2\pi}-1}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{(e^{2\pi}-1)}{\pi(1+n^2)}\cdot\cos(nx) + \frac{n(1-e^{2\pi})}{\pi(1+n^2)}\cdot\sin(nx)\right]$

$$\therefore \boxed{f(x) = \frac{e^{2\pi}-1}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{(e^{2\pi}-1)\cdot\cos(nx)}{(1+n^2)} + \frac{n(1-e^{2\pi})}{(1+n^2)}\cdot\sin(nx)\right]}$$

This is the Fourier series for function $f(x) = e^{x}$.

6] Obtain Fourier series for $f(x) = x\cdot\sin(x)$ in $(-\pi, \pi)$.

Soln. ⇒ $f(x) = x\cdot\sin(x)$

$f(-x) = (-x)\cdot\sin(-x) = x\cdot\sin x$

$\therefore f(-x) = f(x)$ $\quad\therefore$ $f(x)$ is an evin function. $\therefore \boxed{b_n = 0}$

$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\cdot dx \Rightarrow a_0 = \frac{1}{\pi}\int_0^{\pi} x\cdot\sin(x)\cdot dx$

$\Rightarrow a_0 = \frac{1}{\pi}\left[x\cdot -\cos(x) - (1) - \sin(x)\right]_0^{\pi}$

$\Rightarrow a_0 = \frac{1}{\pi}\left[-x\cdot\cos(x) + \sin(x)\right]_0^{\pi} \Rightarrow a_0 = \frac{1}{\pi}\left[\sin(x) - x\cdot\cos(x)\right]_0^{\pi}$

$\Rightarrow a_0 = \frac{1}{\pi}\left[0 - (-\pi) - 0 - 0\right] = \frac{\pi}{\pi} \quad \therefore \boxed{a_0 = 1}$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\cos(nx)\cdot dx \Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} x\cdot\sin(x)\cdot\cos(nx)\cdot dx$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} x\left[\cos(nx)\cdot\sin(x)\right]\cdot dx \Rightarrow a_n = \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$

$\Rightarrow a_n = \frac{1}{\pi}\left\{x\cdot\frac{-\cos(n+1)x}{(n+1)} - \frac{-\sin(n+1)x}{(n+1)^2} - x\cdot\frac{-\cos(n-1)}{(n-1)} - (1)\frac{-\sin(n-1)}{(n-1)^2}\right\}_0^{\pi}$

$\Rightarrow a_n = \frac{1}{\pi}\left\{-\frac{\pi(-1)^{n+1}}{n+1} + \frac{\pi(-1)^{n-1}}{n-1}\right\} = \frac{(-1)^{n-1}}{(n-1)} - \frac{(-1)^{n+1}}{n+1}, \quad n \neq 1$

$\Rightarrow a_1 = \frac{2}{\pi}\int_0^{\pi} x\cdot\sin(x)\cdot\cos(x)\cdot dx = \frac{1}{\pi}\int_0^{\pi} x\cdot\sin(2x)\cdot dx$

$\Rightarrow a_1 = \frac{1}{\pi}\left[x\cdot\frac{-\cos(2x)}{2} - (1)\frac{-\sin(2x)}{4}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\sin(2x)}{4} - \frac{x\cdot\cos(2x)}{2}\right]_0^{\pi}$

$\Rightarrow a_1 = 1/\pi\,(-\pi/2) \quad \therefore \boxed{a_1 = -1/2}$

$\therefore f(x) = a_0 + a_1\cdot\cos(x) + \sum_{n=2}^{\infty} a_n\cdot\cos(nx)$

$\therefore \boxed{f(x) = 1 - \frac{\cos(x)}{2} + \sum_{n=2}^{\infty}\left\{\frac{(-1)^{n-1}}{n-1} - \left[\frac{(-1)^{n+1}}{n+1}\right]\cdot\cos(nx)\right\}}$

Name : Manjiri N. Naik
Roll no. : 21102B0008
Branch : Computer Engineering (CMPN)
Division : B Semester ⇒ III

Seen - Good

Tutorial 4 : Module ⇒ Complex Variables

1] Show that given u or v is harmonic function and find corresponding analytic function $f(z) = u + iv$ and their harmonic conjugate functions

i] $u = e^{-x}[y \cdot \sin(y) + x \cdot \cos(y)]$ .... ①

Sol.ⁿ ⇒ Differentiating eqⁿ ① w.r.t x

$\Rightarrow \partial u/\partial x = e^{-x}[\cos(y)] - e^{-x}[y \cdot \sin(y) + x \cdot \cos(y)]$

$\Rightarrow \partial u/\partial x = e^{-x}\{\cos(y) - [y \cdot \sin(y) + x \cdot \cos(y)]\}$

$\Rightarrow \partial u/\partial x = e^{-x}[\cos(y) - y \cdot \sin(y) - x \cdot \cos(y)]$ .... ②

Differentiating eqⁿ ② w.r.t x

$\Rightarrow \partial^2 u/\partial x^2 = e^{-x}[-\cos(y)] - e^{-x}[\cos(y) - y \cdot \sin(y) - x \cdot \cos(y)]$

$\Rightarrow \partial^2 u/\partial x^2 = e^{-x}[-\cos(y) - \cos(y) + y \cdot \sin(y) + x \cdot \cos(y)]$

$\Rightarrow \partial^2 u/\partial x^2 = e^{-x}[-2\cos(y) + y \cdot \sin(y) + x \cdot \cos(y)]$ .... (A)

Differentiating eqⁿ ① w.r.t y

$\Rightarrow \partial u/\partial y = e^{-x}[y \cdot \cos(y) + \sin(y) - x \cdot \sin(y)]$ .... ③

Differentiating eqⁿ ③ w.r.t y

$\Rightarrow \partial^2 u/\partial y^2 = e^{-x}[-y \cdot \sin(y) + \cos(y) + \cos(y) - x \cdot \cos(y)]$

$\Rightarrow \partial^2 u/\partial y^2 = e^{-x}[-y\sin(y) + 2\cos(y) - x \cdot \cos(y)]$ .... (B)

From eq^ns (A) and (B),

$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = e^{-x}[-2\cos(y) + y\cdot\sin(y) + x\cdot\cos(y)] + e^{-x}[-y\cdot\sin(y) + 2\cos(y) - x\cdot\cos(y)]$

$\therefore \boxed{\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0}$ $\therefore$ u is a Harmonic function.

Now, $f(z) = u + iv$

$\Rightarrow f'(z) = \partial u/\partial x + i\cdot\partial v/\partial x$ .... (4)

By CR eq^n (II), $\partial v/\partial x = -\partial u/\partial y$

eq^n (4) $\Rightarrow f'(z) = \partial u/\partial x - i\cdot\partial u/\partial y$ .... (5)

$\Rightarrow \partial u/\partial x = e^{-x}[\cos(y) - y\cdot\sin(y) - x\cdot\cos(y)]$ $\Rightarrow \partial u/\partial y = e^{-x}[y\cdot\cos(y) + \sin(y) - x\cdot\sin(y)]$

$\Rightarrow (\partial u/\partial x)_{x=z, y=0} = e^{-z}[1 - 0 - z]$ $\Rightarrow (\partial u/\partial y)_{x=z, y=0}$ $e^{-z}[0 + 0 - 0] = 0$

eq^n (5) $\Rightarrow f'(z) = e^{-z}(1-z)$

Integrating w.r.t z

$\Rightarrow f(z) = \int (1-z)\cdot e^{-z}\cdot dz + c$ $\Rightarrow f(z) = \{(1-z)[-e^{-z}] - (-1)(e^{-z})\} + c$

$\Rightarrow f(z) = e^{-z}[z - 1 + 1] + c$ $\Rightarrow f(z) = z\cdot e^{-z} + c$

$\Rightarrow f(z) = (x+iy)\cdot e^{-(x+iy)} + c$ $\Rightarrow f(z) = (x+iy)\cdot e^{-x}\cdot e^{i(-y)} + c$

$\Rightarrow f(z) = e^{-x}\cdot(x+iy)[\cos(y) - i\cdot\sin(y)] + c$

$\Rightarrow f(z) = e^{-x}\{[x\cdot\cos(y) - ix\cdot\sin(y)] + i[y\cdot\cos(y) - iy\cdot\sin(y)]\} + c$

$\Rightarrow f(z) = e^{-x}[x\cdot\cos(y) + y\cdot\sin(y)] + i\cdot e^{-x}[y\cdot\cos(y) - x\cdot\sin(y)] + ic_1$

$\Rightarrow f(z) = e^{-x}[x\cdot\cos(y) + y\cdot\sin(y)] + i\{e^{-x}[y\cdot\cos(y) - x\cdot\sin(y)] + c_1\}$

$\therefore \boxed{f(z) = e^{-x}[x\cdot\cos(y) + y\cdot\sin(y)] + i\{e^{-x}[y\cdot\cos(y) - x\cdot\sin(y)] + c_1\}}$

This is the analytic function $f(z) = u + iv$.

The Harmonic conjugate of $u = e^{-x}[y\cdot\sin(y) + x\cdot\cos(y)]$ is

$\boxed{v = e^{-x}[y\cdot\cos(y) - x\cdot\sin(y)] + c_1}$

ii) $v = x^2 - y^2 + \dfrac{x}{x^2+y^2}$

Sol^n $\Rightarrow$ $\partial v/\partial x = 2x - 0 + \dfrac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = 2x + \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$$\partial^2 v/\partial x^2 = 2 + \frac{(x^2+y^2)^2(-2x) - (y^2-x^2)2.(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$\Rightarrow \partial^2 v/\partial x^2 = 2 + \frac{-2x(x^2+y^2) - 4x(y^2-x^2)}{(x^2+y^2)^3}$$

$$\Rightarrow \partial^2 v/\partial x^2 = 2 + \frac{-2x^3 - 2xy^2 - 4xy^2 + 4x^3}{(x^2+y^2)^3}$$

$$\therefore \boxed{\partial^2 v/\partial x^2 = 2 + \frac{(2x^3 - 6xy^2)}{(x^2+y^2)^3}} \quad \ldots (A)$$

$$\Rightarrow \partial v/\partial y = -2y + \frac{(x^2+y^2)(0) - (x)(2y)}{(x^2+y^2)^2} = -2y + \frac{-2xy}{(x^2+y^2)^2}$$

$$\Rightarrow \partial^2 v/\partial y^2 = -2 + \frac{(x^2+y^2)^2(-2x) - (-2xy)2\cdot(x^2+y^2)\cdot(2y)}{(x^2+y^2)^4}$$

$$\Rightarrow \partial^2 v/\partial y^2 = -2 - \frac{2x^3 - 2xy^2 + 8xy^2}{(x^2+y^2)^3}$$

$$\therefore \boxed{\partial^2 v/\partial y^2 = -2 + \frac{(-2x^3 + 6xy^2)}{(x^2+y^2)^3}} \quad \ldots (B)$$

Adding eq^n (A) and (B) $\Rightarrow \partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 = 0$

$\therefore$ $v = x^2 - y^2 + \dfrac{x}{(x^2+y^2)}$ is a Harmonic function.

$f(z) = u + iv.$

$\Rightarrow f'(z) = \partial u/\partial x + i.\partial v/\partial x$

By CR eqn (I) $\Rightarrow \partial u/\partial x = \partial v/\partial y$

$\therefore f(z) = \partial v/\partial y + i.\partial v/\partial x.$

$$(\partial v/\partial x) = 2x + \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$(\partial v/\partial x)_{\substack{x=z \\ y=0}} = 2z - \frac{z^2}{z^4} = 2z - \frac{1}{z^2}$$

$$(\partial v/\partial y) = -2y + \frac{-2xy}{(x^2+y^2)^2}$$

$$(\partial v/\partial y)_{\substack{x=z \\ y=0}} = 0 + 0 = 0$$

$\therefore f'(z) = 0 + i(2z - 1/z^2)$

Integrating w.r.t z

$\Rightarrow f(z) = i\int (2z - 1/z^2).dz$

$\Rightarrow f(z) = i(z^2 + 1/z)$

$\Rightarrow f(z) = i\left[(x+iy)^2\right] + i\left[\frac{1}{(x+iy)}\right] \Rightarrow f(z) = i[x^2 - y^2 + 2ixy] + i\left[\frac{(x-iy)}{x^2+y^2}\right]$

$$\Rightarrow f(z) = \left[-2xy + \frac{y}{(x^2+y^2)}\right] + i\left[(x^2 - y^2 + \frac{x}{(x^2+y^2)}\right]$$

$$\therefore \boxed{f(z) = \left[-2xy + \frac{y}{(x^2+y^2)}\right] + i\left[(x^2-y^2) + \frac{x}{(x^2+y^2)}\right]}$$

This is the analytic function $f(z) = u + iv$.

The Harmonic conjugate of $v = (x^2-y^2) + \frac{x}{(x^2+y^2)}$

is $\boxed{u = -2xy + \frac{y}{(x^2+y^2)}}$

2] Find an analytic function $f(z) = u+iv$ such that
$$u+v = \frac{2\sin(2x)}{e^{2y}+e^{-2y}-2\cos(2x)}.$$

Sol^n ⇒ $f(z) = u+iv$ ⇒ Analytic function .... ①

Multiply by i to both sides

⇒ $i.f(z) = iu - v$ .... ②

Adding eq^{n}s ① and ② ⇒ $(1+i).f(z) = (u-v)+i(u+v)$ ∴ $F(z) = U+iV$

Now, $V = u+v = \frac{2\cdot\sin(2x)}{e^{2y}+e^{-2y}-2\cos(2x)} = \frac{2\sin(2x)}{2\cosh(2y)-2\cos(2x)}$ ∴ $V = \frac{\sin(2x)}{\cosh(2y)-\cos(2x)}$

⇒ $F'(z) = \partial U/\partial x + i\cdot\partial V/\partial x$

By CR eq^n (I) ⇒ $\partial U/\partial x = \partial V/\partial y$

∴ $F'(z) = \partial V/\partial y + i\cdot\partial V/\partial x$ .... ③

⇒ $$\partial V/\partial x = \frac{[\cosh(2y)-\cos(2x)]2\cdot\cos(2x) - \sin(2x)[0+2\cdot\sin(2x)]}{[\cosh(2y)-\cos(2x)]^2}$$

By Milne-Thomson theorem,

$$\left(\partial V/\partial x\right)_{\substack{x=z\\y=0}} = \frac{[1-\cos(2z)]\cdot 2\cos(2z) - 2\sin^2(2z)}{[1-\cos(2z)]^2} = \frac{2\cos(2z)-2\cos^2(2z)-2\sin^2(2z)}{[1-\cos(2z)]^2}$$

⇒ $$\left(\partial V/\partial x\right)_{\substack{x=z\\y=0}} = \frac{2\{\cos(2z)-[\cos^2(2z)+\sin^2(2z)]\}}{[1-\cos(2z)]^2} = \frac{2[\cos(2z)-1]}{[1-\cos(2z)]^2}$$

⇒ $$\left(\partial V/\partial x\right)_{\substack{x=z\\y=0}} = \frac{-2[1-\cos(2z)]}{[1-\cos(2z)]^2} = \frac{-2}{[1-\cos(2z)]} = \frac{-2}{2\sin^2(z)} = \frac{-1}{\sin^2(z)}$$

∴ $\boxed{\left(\partial V/\partial x\right)_{\substack{x=z\\y=0}} = -\operatorname{cosec}^2(z)}$ ... (A)

$$\Rightarrow \partial V/\partial y = \frac{[\cosh(2y)-\cos(2x)](0) - \sin(2x)[2\cdot\sinh(2y)]}{[\cosh(2y)-\cos(2x)]^2} = \frac{-2\cdot\sin(2x)\cdot\sinh(2y)}{[\cosh(2y)-\cos(2x)]^2}$$

By Milne-Thomson theorem,

$$(\partial V/\partial y)_{\substack{x=z\\y=0}} = \frac{-2\sin(2z)(0)}{[1-\cos(2z)]^2} \quad \therefore \boxed{(\partial V/\partial y)_{\substack{x=z\\y=0}} = 0} \quad \ldots\ (B)$$

Putting eq^{n}s (A) and (B) in eq^n (3)

$$\Rightarrow F'(z) = 0 - i.\operatorname{cosec}^2(z)$$

Integrating w.r.t z

$$\Rightarrow \int F'(z).dz = F(z) = -i\int \operatorname{cosec}^2(z).dz + c_1$$

$$\Rightarrow F(z) = i.\cot(z) + c_1$$

$$\Rightarrow (1+i).f(z) = i.\cot(z) + c_1$$

$$\Rightarrow f(z) = \frac{i.\cot(z)}{(1+i)} + \frac{c_1}{(1+i)} \Rightarrow f(z) = \frac{i(1-i).\cot(z)}{2} + c$$

$$\Rightarrow f(z) = \frac{(1+i)}{2}.\cot(z) + c$$

$$\therefore \boxed{f(z) = \frac{(1+i)}{2}\cot(x+iy) + c}$$

This is the required analytic function $f(z) = u+iv$ such that

$$u+v = \frac{2\sin(2x)}{e^{2y}+e^{-2y}-2\cos(2x)}.$$

3] Find an analytic function $f(z) = u + iv$ such that $u + v = \frac{x - y}{x^2 + y^2} + e^x[\cos(y) + \sin(y)]$

Soln. ⇒ $f(z) = u + iv$ ⇒ Analytic function ... ①

Multiply eq i to both sides

⇒ $i.f(z) = iu - v$ ... ②

Adding eqns ① and ② ⇒ $(1+i).f(z) = (u-v) + i(u+v)$

$\therefore \quad F(z) = U + iV$

Now, $V = u + v = \frac{x-y}{x^2+y^2} + e^x[\cos(y) + \sin(y)]$ is given.

⇒ $F'(z) = \partial U/\partial x + i.\,\partial V/\partial x$ ... ③

By CR eqn (I) ⇒ $\partial U/\partial x = \partial V/\partial y$

eqn ③ ⇒ $F'(z) = \partial V/\partial y + i.\,\partial V/\partial x$ ... ④

⇒ $\partial V/\partial x = \frac{(x^2+y^2)(1) - (x-y)(2x)}{(x^2+y^2)^2} + e^x[\cos(y) + \sin(y)]$

⇒ $\partial V/\partial x = \frac{x^2 + y^2 - 2x^2 + 2xy}{(x^2+y^2)^2} + e^x[\cos(y) + \sin(y)]$

⇒ $\partial V/\partial x = \frac{y^2 - x^2 + 2xy}{(x^2+y^2)^2} + e^x[\cos(y) + \sin(y)]$

By Milne-Thomson theorem,

$(\partial V/\partial x)_{\substack{x=z \\ y=0}} = \frac{0 - z^2 + 0}{z^4} + e^z[1 + 0]$

$\therefore \boxed{(\partial V/\partial x)_{\substack{x=z \\ y=0}} = e^z - \frac{1}{z^2}}$ ... Ⓐ

$$\Rightarrow \partial V/\partial y = \frac{(x^2+y^2)(-1)-(x-y)(2y)}{(x^2+y^2)^2} + e^x[-\sin(y)+\cos(y)]$$

$$\Rightarrow \partial V/\partial y = \frac{-x^2-y^2-2xy+2y^2}{(x^2+y^2)^2} + e^x[\cos(y)-\sin(y)]$$

$$\Rightarrow \partial V/\partial y = \frac{y^2-x^2-2xy}{(x^2+y^2)^2} + e^x[\cos(y)-\sin(y)]$$

By Milne - Thomson theorem,

$$(\partial V/\partial y)_{\substack{x=z\\y=0}} = \frac{0-z^2-0}{z^4} + e^z[1-0] \qquad \therefore \boxed{(\partial V/\partial y)_{\substack{x=z\\y=0}} = e^z - \frac{1}{z^2}} \quad \ldots (B)$$

Putting eqⁿ's (A) and (B) in eqⁿ (4)

$$\Rightarrow F'(z) = (e^z - 1/z^2) + i(e^z - 1/z^2)$$

Integrating w.r.t z

$$\Rightarrow \int F'(z).dz = F(z) = \int (e^z - 1/z^2).dz + i\int (e^z - 1/z^2).dz + C_1$$

$$\Rightarrow F(z) = [e^z + 1/z] + i[e^z + 1/z] + C_1$$

$$\Rightarrow (1+i).f(z) = (1+i)(e^z + 1/z) + C_1$$

$$\Rightarrow f(z) = e^z + 1/z + \frac{C_1}{(1+i)} \Rightarrow f(z) = (e^z + 1/z) + iC_2$$

$$\Rightarrow f(z) = e^{(x+iy)} + \frac{1}{(x+iy)} + C \Rightarrow f(z) = e^x . e^{i(y)} + \frac{(x-iy)}{(x^2+y^2)} + iC_2$$

$$\Rightarrow f(z) = e^x[\cos(y) + i\sin(y)] + \frac{(x-iy)}{(x^2+y^2)} + iC_2$$

$$\Rightarrow f(z) = e^x.\cos(y) + e^x.i.\sin(y) + \frac{x}{(x^2+y^2)} - i\frac{y}{(x^2+y^2)} + iC_2$$

$$\Rightarrow \boxed{f(z) = \left[e^x.\cos(y) + \frac{x}{(x^2+y^2)}\right] + i\left[e^x.\sin(y) - \frac{y}{(x^2+y^2)} + C_2\right]}$$

This is the required analytic function $f(z) = u + iv$ such that $u + v = \frac{x-y}{(x^2+y^2)} + e^x[\cos(y)+\sin(y)]$.

4] Find the orthogonal trajectories of the family of curves $u = 2x - x^3 + 3xy^2 = c$.

Soln. ⇒ $f(z) = u + iv$

$f'(z) = \partial u/\partial x + i.\partial v/\partial x$

By C.R. eqn (II), $\partial v/\partial x = -\partial u/\partial y$ ....

$\therefore f'(z) = (\partial u/\partial x) - i(\partial u/\partial y)$ .... (1)

$\partial u/\partial x = 2 - 3x^2 + 3y^2$ $\qquad$ $\partial u/\partial y = 6xy$

$\Rightarrow (\partial u/\partial x)_{x=z, y=0} = 2 - 3z^2$ $\qquad$ $(\partial u/\partial y)_{x=z, y=0} = 0$

eqn (1) ⇒ $f'(z) = 2 - 3z^2$

Integrating w.r.t z

$\Rightarrow \int f'(z).dz = f(z) = \int (2-3z^2).dz + c_1$

$\Rightarrow f(z) = 2z - \frac{3z^3}{3} + c_1$

$\Rightarrow f(z) = 2z - z^3 + c_1$

$\Rightarrow f(z) = 2(x+iy) - (x+iy)^3 + c_1$

$\Rightarrow f(z) = 2x + 2iy - x^3 - 3x^2iy + 3xy^2 + iy^3 + c_1$

$\Rightarrow f(z) = [2x - x^3 + 3xy^2 + c_1] + i[2y - 3x^2y + y^3]$

$\Rightarrow f(z) = [(2x - x^3 + 3xy^2) - c] + i[2y - 3x^2y + y^3]$

$\therefore$ $\boxed{\text{Orthogonal trajectory of } u \text{ is } v = 2y - 3x^2y + y^3}$

5. Find the orthogonal trajectories of the family of curves $u = x^3y - xy^3 = c$.

Sol.ⁿ ⇒ $f(z) = u + iv$

$f'(z) = \partial u/\partial x + i\,\partial v/\partial x$

By C.R. eqⁿ (II), $\partial v/\partial x = -\partial u/\partial y$

$\therefore\ f'(z) = (\partial u/\partial x) - i(\partial u/\partial y) \quad \ldots (1)$

$\partial u/\partial x = 3x^2y - y^3 \qquad \partial u/\partial y = x^3 - 3xy^2$

$\Rightarrow (\partial u/\partial x)_{\substack{x=z\\y=0}} = 0 \qquad (\partial u/\partial y)_{\substack{x=z\\y=0}} = z^3$

eqⁿ (1) $\Rightarrow f'(z) = 0 - iz^3$

Integrating w.r.t z

$\Rightarrow \int f'(z)\,dz = f(z) = i\int -z^3\,dz + c_1$

$\Rightarrow f(z) = \dfrac{-iz^4}{4} + c_1$

$\Rightarrow f(z) = -\dfrac{i}{4}(x+iy)^4 + c_1$

$\Rightarrow f(z) = -\dfrac{i}{4}\left[x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4\right] + c_1$

$\Rightarrow f(z) = -i/4\left[x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4\right] + c_1$

$\Rightarrow f(z) = -i/4\left[x^4 - 6x^2y^2 + y^4\right] + \left[x^3y - xy^3\right] + c_1$

$\Rightarrow f(z) = \left[x^3y - xy^3 + c_1\right] + i\left[\dfrac{-x^4 + 6x^2y^2 - y^4}{4}\right]$

$\Rightarrow f(z) = \left[(x^3y - xy^3) - c\right] + i\left[\dfrac{-x^4 + 6x^2y^2 - y^4}{4}\right]$

$\therefore$ $\boxed{\text{Orthogonal trajectory of } u \text{ is } v = \dfrac{-x^4 + 6x^2y^2 - y^4}{4}}$

6] Find the constants a, b, c, d, e if $f(z) = (ax^4 + bx^2y^2 + dx^2 + cy^4 - 2y^2) + i(4x^3y - exy^3 + 4xy)$ is an analytic function.

Sol^n ⇒ $f(z) = u + iv$ is an analytic function.
∴ CR equations exist.

$u = ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2$
⇒ $\partial u/\partial x = 4ax^3 + 2bxy^2 + 2dx$

$v = 4x^3y - exy^3 + 4xy$
⇒ $\partial v/\partial y = 4x^3 - 3exy^2 + 4x$

By CR. eq^n ①, $\partial u/\partial x = \partial v/\partial y$
⇒ $4ax^3 + 2bxy^2 + 2dx = 4x^3 - 3exy^2 + 4x$
Comparing both sides

⇒ $4ax^3 = 4x^3$
∴ $\boxed{a=1}$

$2bxy^2 = -3exy^2$
$2b = -3e$

$2dx = 4x$ ⇒ $2d = 4$
∴ $\boxed{d=2}$

$\partial u/\partial y = 2bx^2y + 4cy^3 - 4y$

$\partial v/\partial x = 12x^2y - ey^3 + 4y$

By CR eq^n ②, $\partial u/\partial y = -\partial v/\partial x$
⇒ $2bx^2y + 4cy^3 - 4y = -12x^2y + ey^3 - 4y$
Comparing both sides

$2bx^2y = -12x^2y$
$2b = -12$
$\boxed{b=-6}$

$4cy^3 = +ey^3$
$4c = e$
$4c = 4$ ∴ $\boxed{c=1}$

$2b = -3e$
$2(-6) = -3e$
$12 = 3e$ ∴ $\boxed{e=4}$

Ans ⇒ The values of the constants are $a=1, b=-6, c=1, d=2, e=4$.

7. Find a, b, c, d if $f(z) = (x^2 + 2axy + by^2) + i(cx^2 + 2dxy + y^2)$ is an analytic function.

Sol^n. ⇒ $f(z) = u + iv$ is an analytic function.
∴ CR equations exist.

$u = x^2 + 2axy + by^2$ $\qquad$ $v = cx^2 + 2dxy + y^2$

⇒ $\partial u/\partial x = 2x + 2ay + 0$ $\qquad$ ⇒ $\partial v/\partial y = 0 + 2dx + 2y$

⇒ $\partial u/\partial y = 0 + 2ax + 2by$ $\qquad$ ⇒ $\partial v/\partial x = 2cx + 2dy + 0$

By CR eq^n (I), $\partial u/\partial x = \partial v/\partial y$

⇒ $2x + 2ay = 2dx + 2y$

Comparing both sides

⇒ $2x = 2dx$ $\quad$ $\boxed{d = 1}$

$2ay = 2y$ $\quad$ $\boxed{a = 1}$

By CR eq^n (II), $\partial u/\partial y = -\partial v/\partial x$

⇒ $2ax + 2by = -2cx - 2dy$

Comparing both sides

$2ax = -2cx$

$a = -c$

$c = -a$

$\boxed{c = -1}$

$2by = -2dy$

$b = -d$

$\boxed{b = -1}$

Ans ⇒ The values of the constant are $a = 1, b = -1, c = -1, d = 1$.

8] Show that $f(z) = e^{2z} - z$ is an analytic function.

Sol^n. ⇒ $f(z) = e^{2z} - z = e^{2(x+iy)} - (x+iy)$

$f(z) = e^{2x} \cdot e^{i(2y)} - x - iy$

$f(z) = e^{2x}[\cos(2y) + i\sin(2y)] - x - iy$

$f(z) = \{e^{2x}[\cos(2y)] - x\} + i[e^{2x} \cdot \sin(2y) - y]$

$f(z) = U + iV$

$U = e^{2x}[\cos(2y)] - x \qquad V = e^{2x} \cdot \sin(2y) - y$

⇒ $\partial U/\partial x = 2 \cdot e^{2x} \cdot \cos(2y) - 1 \quad \ldots (1)$ ⇒ $\partial V/\partial y = 2 \cdot e^{2x} \cdot \cos(2y) - 1 \quad \ldots (2)$

From (1) and (2), $\partial U/\partial x = \partial V/\partial y$ ⇒ CR eq^n (I) ... (A)

⇒ $\partial U/\partial y = -e^{2x} \cdot 2 \cdot \sin(2y) \quad \ldots (3)$ ⇒ $\partial V/\partial x = e^{2x} \sin(2y) \quad \ldots (4)$

From (3) and (4), $\partial U/\partial y = -\partial V/\partial x$ ⇒ CR eq^n (II) ... (B)

From (A) and (B), we can say that both CR equations exist.

$\therefore \boxed{f(z) = e^{2z} - z \text{ is an analytic function.}}$

Hence proved.

9] Find the constant K if $f(z) = r^2 \cdot \cos(2\theta) + i r^2 \sin(K\theta)$ is an analytic function.

Sol^n ⇒ $f(z) = u + iv$ is an analytic function.

$f(z) = r^2 \cdot \cos(2\theta) + i \cdot r^2 \cdot \sin(K\theta)$

$\therefore \quad u = r^2 \cdot \cos(2\theta) \qquad v = r^2 \cdot \sin(K\theta)$

By C.R. eq^n (I), $\left(\partial u/\partial r\right) = 1/r \cdot \left(\partial v/\partial \theta\right)$ .... (1)

⇒ $\left(\partial u/\partial r\right) = 2r \cdot \cos(2\theta) \qquad \Rightarrow \left(\partial v/\partial \theta\right) = r^2 \cdot \cos(K\theta) \cdot K$

eq^n (1) ⇒ $2r \cdot \cos(2\theta) = \frac{1}{r} \cdot r^2 \cdot \cos(K\theta) \cdot K$

⇒ $2 \cdot \cos(2\theta) = K \cdot \cos(K\theta)$

Comparing both sides

$\boxed{K = 2}$

Ans ⇒ The value of the constant is $K = 2$.

10] Prove that there does not exist an analytic function whose real part is $u = x^2 + 3x - 4y + y^2 + 7$.

Sol^n ⇒

For any function $f(z) = u + iv$ to be an analytic function, real part(u) and imaginary part(v) both must be Harmonic functions. ... Ⓐ

Here, $u = x^2 + 3x - 4y + y^2 + 7$

$\therefore \partial u/\partial x = 2x + 3$

$\partial^2 u/\partial x^2 = 2 \quad .... \quad (1)$

$\therefore \partial u/\partial y = -4 + 2y$

$\partial^2 u/\partial y^2 = 2 \quad .... \quad (2)$

From eq^n (1) and (2),

$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 2 + 2 = 4$

$\therefore \boxed{\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 \neq 0} \quad .... \quad$ Ⓑ

eq^n Ⓑ ⇒ The function u is not Harmonic .... Ⓒ

From eq^n Ⓐ and Ⓒ, we can say that there does not exist an analytic function whose real part is $u = x^2 + 3x - 4y + y^2 + 7$ as the real part is not a Harmonic function.

Name : Manjiri N. Naik
Roll no. : 21102B0008
Branch : Computer Engineering (CMPN)
Division : B Semester : III

21/10/22

51

Tutorial 5 : Module 5 ⇒ Correlation and Regression

1] Find rank correlation coefficient of the following data.

x	y	R_1	R_2	R_1-R_2	$(R_1-R_2)^2$
35	51	9	8	1	1
38	37	8	10	-2	4
43	48	7	9	-2	4
30	62	10	6	4	16
54	93	5	1	4	16
68	73	3	3	0	0
70	56	2	7	-5	25
92	72	1	4	-3	9
44	70	6	5	1	1
56	92	4	2	2	4
					$\Sigma(R_1-R_2)^2 = 80$

$$R = 1 - 6\left[\frac{\Sigma(R_1-R_2)^2}{N^3-N}\right]$$

Here, $N = 10$

$$\Rightarrow R = 1 - 6\left[\frac{80}{1000-10}\right] \Rightarrow R = 1 - \frac{480}{990} = 1 - 0.4848$$

$$\therefore \boxed{R = 0.5151}$$

Ans ⇒ Rank Correlation Coefficient (R) is 0.5151.

2) Find rank correlation coefficient for the following data.

x	y	R_1	R_2	$R_1 - R_2$	$(R_1-R_2)^2$
105	39	9	8	1	1
110	41	7	7	0	0
112	45	5	6	-1	1
108	38	8	9	-1	1
111	48	6	5	1	1
116	58	3	3	0	0
120	60	2	2	0	0
104	35	10	10	0	0
115	54	4	4	0	0
125	69	1	1	0	0
					$E(R_1-R_2)^2 = 4$

$$R = 1 - 6\left[\frac{E(R_1-R_2)^2}{N^3-N}\right]$$

Here, $N = 10$

$$\Rightarrow R = 1 - 6\left[\frac{4}{1000-100}\right] \Rightarrow R = 1 - \frac{24}{990}$$

$$\Rightarrow R = 1 - 0.2424 \qquad \therefore \boxed{R = 0.975}$$

Ans $\Rightarrow$ Rank Correlation Coefficient (R) is 0.975.

3] Find rank correlation coefficient for the following data.

x	y	R_1	R_2	R_1-R_2	$(R_1-R_2)^2$
98	65	8	7.5	0.5	0.25
101	65	7	7.5	-0.5	0.25
104	67	6	6	0	0
107	68	5	3.5	1.5	2.25
113	68	4	3.5	0.5	0.25
120	69	3	1	2	4
125	68	2	3.5	-1.5	2.25
128	68	1	3.5	-2.5	6.25
					$E(R_1-R_2)^2 = 15.5$

m for $68 = 4$
$\therefore m_1 = 4$
m for $65 = 2$
$\therefore m_2 = 2$

$$R = 1 - 6\left[\frac{E(R_1-R_2)^2 + \frac{1}{12}(m_1^3-m_1) + \frac{1}{12}(m_2^3-m_2)}{N^3-N}\right]$$

Here, $N = 8$

$$\Rightarrow R = 1 - 6\left[\frac{15.5 + 1/12(64-4) + 1/12(8-2)}{512-8}\right]$$

$$\Rightarrow R = 1 - 6\left[\frac{15.5 + 5 + 0.5}{504}\right] \Rightarrow R = 1 - \frac{6(21)}{504}$$

$$\Rightarrow R = \frac{504-126}{504} \Rightarrow R = \frac{378}{504} \quad \therefore \boxed{R = 0.75}$$

Ans $\Rightarrow$ Rank correlation coefficient (R) is 0.75.

4] Find i] the lines of regression ii] coefficient of correlation.

x	y	xy	x^2	y^2
10	19	190	100	361
12	22	264	144	484
13	24	312	169	576
16	27	432	256	729
17	29	493	289	841
20	33	660	400	1089
25	37	925	625	1369
Σ 113	191	3276	1983	5449

Here, N = 7.

Solⁿ. ⇒

Regression line y on x

$y = ax + b$

$\Sigma y = a\Sigma x + bN$

$191 = a(113) + b(7) \ldots$ ①

$\Sigma xy = a\Sigma x^2 + b\Sigma x$

$3276 = a(1983) + b(113) \ldots$ ②

Solving eqⁿs ① and ②

$\boxed{a = 1.213}$ $\boxed{b = 7.702}$

$\therefore b_{yx} = 1.213$

$\therefore \boxed{y = 1.213x + 7.702}$

Regression line x on y

$x = ay + b$

$\Sigma x = a\Sigma y + bN$

$113 = a(191) + b(7) \ldots$ ③

$\Sigma xy = a\Sigma y^2 + b\Sigma y$

$3276 = a(5449) + b(191) \ldots$ ④

Solving eqⁿs ③ and ④

$\boxed{a = 0.811}$ $\boxed{b = -6.004}$

$\therefore b_{xy} = 0.811$

$\boxed{x = 0.811y - 6.004}$

Coefficient of correlation $(r) = \sqrt{b_{yx} \cdot b_{xy}} = \sqrt{1.213 \times 0.811}$ $\therefore \boxed{r = 0.991}$

Ans ⇒ $y = 1.213x + 7.702$ and $x = 0.811y - 6.004$ are the lines of regression and $r = 0.991$ is the coefficient of correlation.

5 Calculate r for the following data.

x	y	(x-A)	(y-B)	$(x-A)^2$	$(y-B)^2$	(x-A)(y-B)
57	10	15	-17	225	289	-255
42	26	0	-1	0	1	0
38	41	-4	14	16	196	-56
42	29	0	2	0	4	0
45	27	3	0	9	0	0
42 (A)	27 (B)	0	0	0	0	0
44	19	2	-8	4	64	-16
40	18	-2	-9	4	81	18
46	19	4	-8	16	64	-32
44	31	2	4	4	16	8
43	29	1	2	1	4	2
40	33	-2	6	4	36	-12
		Σ= 19	Σ=-15	Σ= 283	Σ= 755	Σ=-343

Now, $r = \dfrac{\Sigma[(x-A)(y-B)]}{\sqrt{\Sigma(x-A)^2 \cdot \Sigma(y-B)^2}}$

$\Rightarrow \quad r = \dfrac{-343}{\sqrt{283 \times 755}}$

$\therefore \quad \boxed{r = -0.742}$

Ans ⇒ Value of r for the given data is -0.742.

6. By the least square method, find the best values of a, b, c in the second degree curve $y = ax^2 + bx + c$ for the following data.

x	y	x^2	x^3	x^4	xy	x^2y
-2	-3.150	4	-8	16	6.3	-12.6
-1	-1.390	1	-1	1	1.39	-1.39
0	0.620	0	0	0	0	0
1	2.880	1	1	1	2.88	2.88
2	5.378	4	8	16	10.756	21.512
$\Sigma = 0$	$\Sigma = 4.338$	$\Sigma = 10$	$\Sigma = 0$	$\Sigma = 34$	$\Sigma = 21.326$	$\Sigma = 10.402$

Soln. ⇒ Here, $N = 5$

Now, $y = ax^2 + bx + c$ .... ①

$\Sigma y = a\Sigma x^2 + b\Sigma x + cN$

$4.338 = a(10) + b(0) + c(5)$ .... ②

$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$

$21.326 = a(0) + b(10) + c(0)$ .... ③

$\Sigma x^2y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$

$10.402 = a(34) + b(0) + c(10)$ .... ④

Solving eqn's ②, ③ and ④, we get

$\boxed{a = 0.1232}$ $\boxed{b = 2.1326}$ $\boxed{c = 0.621}$

Ans ⇒ ∴ $\boxed{y = 0.1232x^2 + 2.1326x + 0.621}$

7 The height in cms of fathers (x) and that of the eldest sons (y) are given below. Then find the lines of regression. Estimate the height of eldest son if the height of father is 172 cms. and the height of father if height of son is 173 cm. Also find coefficient of correlation.

X	Y	(X-A)	(Y-B)	(X-A)(Y-B)	$(X-A)^2$	$(Y-B)^2$
165	173	7	5	35	49	25
160	168	2	0	0	4	0
170	173	12	5	60	144	25
163	165	5	-3	-15	25	9
173	175	15	7	105	225	49
158 (A)	168 (B)	0	0	0	0	0
178	173	20	5	100	400	25
168	165	10	-3	-30	100	9
173	180	15	12	180	225	144
170	170	12	2	24	144	4
175	173	17	5	85	289	25
180	178	22	10	220	484	100
Σ=2033	Σ=2061	Σ=137	Σ=45	Σ=764	Σ=2089	Σ=415

Solⁿ ⇒ Here, N=12

$\bar{x} = \frac{\Sigma x}{N} = \frac{2033}{12}$ $\therefore$ $\boxed{\bar{x} = 169.4166}$

$\bar{y} = \frac{\Sigma y}{N} = \frac{2061}{12}$ $\therefore$ $\boxed{\bar{y} = 171.75}$

Regression line y on X | Regression line x on y

$\Rightarrow$ $y = ax + b$ | $\Rightarrow$ ~~$x = ay + b$~~

$\Rightarrow$ $\Sigma y = a\Sigma x + bN$ | $\Rightarrow$ $\Sigma xy = a\Sigma y^2 + b\Sigma x$

$\Rightarrow$ $2061 = a(2033) + b(12)$ .... (1) | $\Rightarrow$

$$byx = \frac{\Sigma(x-A)(y-B) - \frac{\Sigma(x-A).\Sigma(y-B)}{N}}{\Sigma(x-A)^2 - \frac{[\Sigma(x-A)]^2}{N}} = \frac{764 - \frac{137\times45}{12}}{2089 - \frac{(137)^2}{12}}$$

$\therefore$ $\boxed{byx = 0.4767}$

$$bxy = \frac{\Sigma(x-A)(y-B) - \frac{\Sigma(x-A).\Sigma(y-B)}{N}}{\Sigma(y-B)^2 - \frac{[\Sigma(y-B)]^2}{N}} = \frac{764 - \frac{137\times45}{12}}{415 - \frac{(45)^2}{12}}$$

$\therefore$ $\boxed{bxy = 1.0162}$

Now, $r^2 = byx \times bxy$ $\therefore$ $r = \sqrt{byx \times bxy}$

$\Rightarrow$ $r = \sqrt{0.4767 \times 1.0162}$ $\therefore$ $\boxed{r = 0.6957}$

Regression line y on x

$\Rightarrow$ $(y - \bar{y}) = byx(x - \bar{x})$

$\Rightarrow$ $(y - 171.75) = 0.4767(x - 169.4166)$

$\Rightarrow$ $y - 171.75 = 0.4767x - 80.7608$

$\therefore$ $\boxed{y = (0.4767)x + 90.9892}$ ... (1)

Regression line X on Y

$\Rightarrow$ $(x - \bar{x}) = bxy(y - \bar{y})$

$\Rightarrow$ $(x - 169.4166) = 1.0162(y - 171.75)$

$\Rightarrow$ $x - 169.4166 = 1.0162y - 174.5323$

$\therefore$ $\boxed{x = (1.0162)y - 5.1157}$ ... (2)

Put $x = 172$ in eq^n 1

$\Rightarrow$ $y = (0.4767)(172) + 90.9892$

$\therefore$ $\boxed{y = 172.9816}$

Put $y = 173$ in eq^n 2

$\Rightarrow$ $x = (1.0162)(173) - 5.1157$

$\therefore$ $\boxed{x = 170.6869}$

Ans $\Rightarrow$ Line of regression y on x $\Rightarrow$ $y = (0.4767)x + 90.9892$

Line of regression x on y $\Rightarrow$ $x = (1.0162)y - 5.1157$

Height of eldest son when father is 172cm $\Rightarrow$ 172.9816

Height of father when eldest son is 173 cm $\Rightarrow$ 170.6869

Coefficient of correlation $(r) = 0.6957$.

8] Find R.

x	y	R_1	R_2	$(R_1-R_2)^2$
85	78	2.5	3.5	1
74	91	6	1	26
85	78	2.5	3.5	1
50	58	10	9	1
65	60	8	8	0
78	72	4	5	1
74	80	6	2	16
60	55	9	10	1
74	68	6	7	1
90	70	1	6	25

$\Sigma(R_1-R_2)^2 = 72.$

Here, $N = 10$

$m_1 = 3 \; ; \; m_2 = 2 \; ; \; m_3 = 2$

$$\Rightarrow R = 1 - 6\left[\frac{\Sigma(R_1-R_2)^2 + 1/12(m_1^3-m_1) + 1/12(m_2^3-m_2) + 1/12(m_3^3-m_3)}{N^3-N}\right]$$

$$\Rightarrow R = 1 - 6\left[\frac{72 + 1/12(24) + 1/12(6) + 1/12(6)}{990}\right]$$

$$\Rightarrow R = 1 - 6\left[\frac{72+2+0.5+0.5}{990}\right] \Rightarrow R = 1 - \frac{6(75)}{990}$$

$$\therefore \boxed{R = 0.5454}$$

Ans $\Rightarrow$ The value of R is 0.5454.

9] A second degree curve (parabola) to the following data and estimate y when $x = 80$.

x	y	$u = \left(\frac{x-40}{10}\right)$	$v = y/10$	u^2	u^3	u^4	uv	u^2v
10	20	-3	2	9	-27	81	-6	18
20	60	-2	6	4	-8	16	-12	24
30	70	-1	7	1	-1	1	-7	7
40	80	0	8	0	0	0	0	0
50	90	1	9	1	1	1	9	9
60	100	2	10	4	8	16	20	40
70	100	3	10	9	27	81	30	90
		$\Sigma=0$	$\Sigma=52$	$\Sigma=28$	$\Sigma=0$	$\Sigma=196$	$\Sigma=34$	$\Sigma=188$

Sol^n $\Rightarrow$ Here, $N = 7$ We know, $v = au^2 + bu + c$ ... (A)

$\Rightarrow \Sigma v = a\Sigma u^2 + b\Sigma u + cN \Rightarrow 52 = a(28) + b(0) + c(7)$ ... (1)

$\Rightarrow \Sigma uv = a\Sigma u^3 + b\Sigma u^2 + c\Sigma u \Rightarrow 34 = a(0) + b(28) + c(0)$ ... (2)

$\Rightarrow \Sigma u^2v = a\Sigma u^4 + b\Sigma u^3 + c\Sigma u^2 \Rightarrow 188 = a(196) + b(0) + c(28)$ ... (3)

Solving (1), (2), (3) $\Rightarrow$ $\boxed{a = -0.2381}$ $\boxed{b = 1.2943}$ $\boxed{c = 8.3809}$

$\therefore v = (-0.2381)u^2 + (1.2943)u + 8.3809$

Now, put $x = 80$ $\therefore u = \left(\frac{80-40}{10}\right) = 4$

$\therefore v = (-0.2381)(16) + (1.2943)(4) + 8.3809$ $\therefore v = 9.7485$

Now, $y = 10v$ $\therefore y = (9.7485)(10)$ $\therefore \boxed{y = 97.485}$

Ans $\Rightarrow$ Value of y is 97.485.

Name : Manjiri N. Naik
Roll no. : 2110280008
Branch : Computer Engineering (CMPN)
Division : B   Semester : III

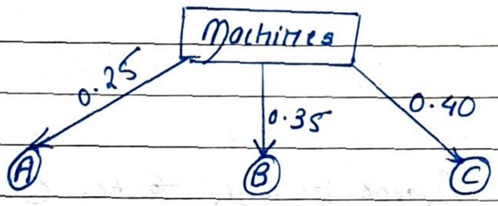
25/10/22

6.1

Tutorial 6 : Module 6 ⇒ Probability and Statistics

1] A factory production line is manufacturing bolts using 3 machines A, B and C. Of the total output, machine A is responsible for 25 %, machine B for 35 % and C for the rest. It is known from previous experience with the machines that 5 % of output from machine A is defective, 4 % from machine B and 2 % from machine C. A bolt is chosen at random from the production line then i] what is the probability selected bolt found to be defective. ii] What is the probability that is bolt is defective then it came from machine A ?

1. ⇒



$P(D/A) = 0.05$ $P(D/B) = 0.04$ $P(D/C) = 0.02$

$P(ND/A) = 1 - 0.05 = 0.95$ $P(ND/B) = 1 - 0.04 = 0.96$ $P(ND/C) = 1 - 0.02 = 0.98$

i] Defective ⇒ Could be from any of the 3 machines

$P(D) = P(A) \cdot P(D/A) + P(B) \cdot P(D/B) + P(C) \cdot P(D/C)$

⇒ $P(D) = (0.25 \times 0.05) + (0.35 \times 0.04) + (0.40 \times 0.02)$

$= 0.0125 + 0.014 + 0.008$

$\therefore P(D) = 0.0345$

ii] Bolt from machine A

$$P(A/D) = \frac{0.25 \times 0.05}{0.0345}$$

$$\therefore \boxed{P(A/D) = 0.3625}$$

Ans ⇒ i] Probability that bolt is defective is 0.0345.
ii] Probability that defective bolt is from machine A is 0.3625.

2] A new constructed flyover is likely to be collapsed. The chance that the design of flyover is faulty is 0.5 and the chance that the flyover will collapse if the design is faulty is 0.95, otherwise it is 0.30. If flyover collapsed, then what is the probability that it collapsed because of faulty design?

Sol^n ⇒ $\boxed{\text{Flyover Collapse}}$

Faulty design ← → Not faulty

$\therefore P(F) = 0.5$ | $P(NF) = 0.5$

↓ Collapse | ↓ Collapse

$P(C/F) = 0.95$ | $P(C/NF) = 0.30$

Now,

$$P(F/C) = \frac{P(F) \cdot P(C/F)}{P(C)} = \frac{0.5 \times 0.95}{P(C)}$$

$\Rightarrow$ $P(C) = P(F) \cdot P(C/F) + P(NF) \cdot P(C/NF)$

$= 0.5 \times 0.95 + 0.5 \times 0.30$

$= 0.475 + 0.15$

$= 0.625$

$$\therefore P(F/C) = \frac{0.5 \times 0.95}{0.625}$$

$\therefore$ $\boxed{P(F/C) = 0.76}$

Ans $\Rightarrow$ Probability that bridge collapsed due to faulty design is 0.76.

3] A random variable X has following PDF. Then find
i] K ii] $V(X)$ iii] $P(X < 5)$

X	1	2	3	4	5	6	7
$P(X=x)$	K	$2K$	$3K$	K^2	K^2+K	$2K^2$	$4K^2$

Sol^n. $\Rightarrow$ $\Sigma P(X_i) = 1$

$\Rightarrow$ $K + 2K + 3K + K^2 + K^2 + K + 2K^2 + 4K^2 = 1$

$\Rightarrow$ $8K^2 + 7K = 1$ $\therefore$ $K = 1/8$ or $K = -1$

But probability cannot be -1. $\therefore$ $\boxed{K = 1/8}$

64

PDF becomes

X	1	2	3	4	5	6	7
$P(X=x)$	1/8	2/8	3/8	1/64	9/64	2/64	4/64

ii] $V(x)$

$\Rightarrow$ $V(x) = E(x^2) - [E(x)]^2$

Now, $[E(x^2)] = \Sigma x^2 \cdot P(x)$

$= (1)(1/8) + (4)(2/8) + (9)(3/8) + (16)(1/64) + (25)(9/64) + (36)(2/64) + (49)(4/64)$

$= 8/64 + 64/64 + 216/64 + 16/64 + 225/64 + 72/64 + 196/64$

$= \frac{797}{64}$

$\therefore$ $\boxed{E(x^2) = 12.453125}$

$E(x) = \bar{x} = \Sigma x \cdot P(x)$

$= 1(1/8) + 2(2/8) + 3(3/8) + 4(1/64) + 5(9/64) + 6(2/64) + 7(4/64)$

$= 1/8 + 4/8 + 9/8 + 4/64 + 45/64 + 12/64 + 28/64$

$= \frac{8 + 32 + 72 + 4 + 45 + 12 + 28}{64}$

$= \frac{201}{64} = 3.140625$ $\therefore \boxed{[E(x)]^2 = 9.863}$

$V(x) = E(x^2) - [E(x)]^2 = 12.453125 - 9.863$

$\therefore$ $\boxed{V(x) = 2.590}$

iii] $P(x<5)$

$\Rightarrow P(x<5) = 1 - P(x>=5)$

$= 1 - [P(x=5) + P(x=6) + P(x=7)]$

$= 1 - [\cancel{(1)}(9/64) + \cancel{(2)}(2/64) + (4/64)]$

$= 1 - 15/64$

$= 49/64$

$\therefore \boxed{P(x<5) = 49/64}$

Ans $\Rightarrow$ i] $K = 1/8$ ii] $V(x) = 2.590$ iii] $P(x<5) = 49/64$

4] A random variable X has following PDF. Then find i] K ii] V(x) iii] $P(x \geq 3)$

X	0	1	2	3	4	5	6	7
$P(X=x)$	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Sol^n $\Rightarrow$ i] K

$\Rightarrow \Sigma P(x_i) = 1$

$\Rightarrow 0+K+2K+2K+3K+K^2+2K^2+7K^2+K = 1$

$\Rightarrow 10K^2 + 9K = 1$

$\therefore \boxed{K = 1/10}$

PDF becomes

X	0	1	2	3	4	5	6	7
$P(X=x)$	0	1/10	2/10	2/10	3/10	1/100	2/100	17/100

ii] $V(x)$

$\Rightarrow$ $V(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \Sigma x^2 \cdot P(x)$

$= [0(0) + 1(1/10) + (4)(2/10) + (9)(2/10) + (16)(3/10) + (25)(1/100) + (36)(2/100) + (49)(17/100)]$

$= \left[\frac{1}{10} + \frac{8}{10} + \frac{18}{10} + \frac{48}{10} + \frac{25}{100} + \frac{72}{100} + \frac{833}{100}\right]$

$= \frac{10+80+180+480+25+72+833}{100} = \frac{1680}{100}$

$\boxed{E(x^2) = 16.8}$

$E(x) = \bar{x} = \Sigma x \cdot P(x)$

$= [(0)(0) + (1)(1/10) + (2)(2/10) + (3)(2/10) + (4)(3/10) + (5)(1/100) + (6)(2/100) + (7)(17/100)]$

$= \left[\frac{1}{10} + \frac{4}{10} + \frac{6}{10} + \frac{12}{10} + \frac{5}{100} + \frac{12}{100} + \frac{119}{100}\right]$

$= \left[\frac{10+40+60+120+5+12+119}{100}\right] = \frac{366}{100}$

$\therefore$ $E(x) = 3.66$ $\therefore$ $\boxed{[E(x)]^2 = 13.3956}$

$V(x) = E(x^2) - [E(x)]^2 = 16.8 - 13.3956$

$\therefore$ $\boxed{V(x) = 3.4044}$

iii] $P(x \geq 3)$

$\Rightarrow$ $P(x \geq 3) = 1 - P(x < 3)$
$= 1 - [P(x=0) + P(x=1) + P(x=2)]$
$= 1 - [0 + 1/10 + 2/10]$
$= 1 - 3/10$

$\therefore$ $\boxed{P(x \geq 3) = 7/10}$

Ans $\Rightarrow$ i] $K = 1/10$ ii] $V(x) = 3.4044$ iii] $P(x \geq 3) = 7/10$

6] A random variable X has following PDF. Then find moment generating function and hence find mean and variance by MGF method.

X	0	1	2	3
$P(X=x)$	1/6	1/3	1/3	1/6

Solⁿ $\Rightarrow$ $M_0(t) = E[e^{tx}] = \sum_{x=0}^{3} e^{tx} P(x) = e^{0t}(1/6) + e^{t}(1/3) + e^{2t}(1/3) + e^{3t}(1/6)$

$M_0(t) = 1/6 + e^{t}/3 + e^{2t}/3 + e^{3t}/6$

$E(x) = \mu_1' = \left\{\frac{d[M_0(t)]}{dt}\right\}_{t=0} = \left[0 + e^{t}/3 + 2 \cdot e^{2t}/3 + 3 \cdot e^{3t}/6\right]_{t=0}$

$\Rightarrow$ $\mu_1' = [0 + 1/3 + 2/3 + 1/2]$ $\therefore$ $\boxed{E(x) = 3/2}$

$$E(x^2) = \mu_2' = \left\{\frac{d^2[M_0(t)]}{dt^2}\right\}_{t=0}$$

$$\Rightarrow \mu_2' = \frac{d}{dt}\left[e^t/3 + 2\cdot e^{2t}/3 + 3\cdot e^{3t}/6\right]_{t=0}$$

$$\Rightarrow \mu_2' = \left[\frac{e^t}{3} + \frac{4\cdot e^{2t}}{3} + \frac{9\cdot e^{3t}}{6}\right]_{t=0}$$

$$\Rightarrow \mu_2' = \left[1/3 + 4/3 + 3/2\right]$$

$$\therefore \boxed{\mu_2' = E(x^2) = 19/6}$$

$$V(x) = E(x^2) - [E(x)]^2 = 19/6 - (3/2)^2 = 19/6 - 9/4 \quad \therefore \boxed{V(x) = 11/12}$$

Ans $\Rightarrow$ i] Mean $= 3/2$ ii] Variance $= 11/12$

7] A random variable X has following PDF.
$P(X=x) = 1/2^x$, $x = 1, 2, 3, \ldots$ Then find moment generating function and hence find mean and variance by MGF method.

Sol^n: $\Rightarrow$ x is ~~continuous~~ discrete random variable (DRV).

~~$M_0(t) = E[e^{tx}] = \int_{x=-1}^{x=1}$~~

PDF of X is

X	1	2	3	4	. . .
$P(X=x)$	$1/2$	$1/2^2$	$1/2^3$	$1/2^4$	. . .

$$\Sigma P(x) = 1/2 + 1/2^2 + 1/2^3 + \ldots\ldots$$

$$= 1/2\left[1 + 1/2 + (1/2)^2 + (1/2)^3 + \ldots\ldots\right]$$

$$\Rightarrow \Sigma P(x) = 1/2 \cdot \frac{1}{(1-1/2)} = 1$$

$$M_0(t) = E\left[e^{tx}\right] = \sum_{x=1}^{\infty} e^{tx} \cdot P(x) = \sum_{x=1}^{\infty} e^{tx}(1/2^x)$$

$$\Rightarrow M_0(t) = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \ldots\ldots\ldots$$

$$= \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \ldots\ldots\ldots\right]$$

$$= \frac{e^t}{2} \cdot \frac{1}{1 - e^t/2} = \frac{e^t}{2} \cdot \frac{2}{2-e^t} = \frac{e^t}{2-e^t} \quad \ldots\ldots (1)$$

$$\text{Mean} = E(x) = \mu_1' = \left\{\frac{d[M_0(t)]}{dt}\right\}_{t=0}$$

$$\Rightarrow \mu_1' = \left[\frac{d}{dt}\left(\frac{e^t}{2-e^t}\right)\right]_{t=0} = \left[\frac{(2-e^t) \cdot e^t - e^t(-e^t)}{(2-e^t)^2}\right]_{t=0} = \left[\frac{2 \cdot e^t}{(2-e^t)^2}\right]_{t=0} \quad \ldots\ldots (2)$$

$$\Rightarrow \mu_1' = \frac{2(1)}{(2-1)^2} \quad \therefore \boxed{\mu_1' = E(x) = 2}$$

$$E(x^2) = \mu_2' = \left\{\frac{d^2[M_0(t)]}{dt^2}\right\}_{t=0}$$

$$\Rightarrow \mu_2' = \left\{\frac{d}{dt}\left[\frac{2e^t}{(2-e^t)^2}\right]\right\}_{t=0} = 2\left[\frac{(2-e^t)^2(e^t) - e^t \cdot 2(2-e^t) \cdot (-e^t)}{(2-e^t)^4}\right]_{t=0}$$

$$\Rightarrow \mu_2' = 2\left[\frac{(2-1)^2(1) + 1(2-1)2(1)}{(2-1)^4}\right] = 2[1+2] \quad \therefore \boxed{\mu_2' = E(x^2) = 6}$$

$V(x) = E(x^2) - [E(x)]^2 = 6 - (2)^2 = 6 - 4$

$\therefore \boxed{V(x) = 2}$

Ans ⇒ i] Mean = 2 ii] Varionce = 2

5] The daily consumption of electric power (in million kwh) in Diwali period is a random variable with

$$PDF\ f(X=x) = \begin{cases} Kx(2-x^2), & 0 \le x \le 1,\ K>0 \\ 0, & \text{otherwise} \end{cases}$$

Then find i] K ii] V(x) iii] $P(1 \le X \le 2)$

Sol^n^ ⇒ $P(x) = Kx(2-x^2)$

$P(0) = K(0)(2-0) = 0$

$P(1) = K(1)(2-1) = K$

x	0	1
$P(X=x)$	0	K

$\Sigma P(x_i) = 1$

$\therefore \boxed{K = 1}$

$V(x) = E(x)^2 - [E(x)]^2$

$= K - K$

$= 0$