Exam 1 : Feb 20.

Saee Patil will hold OH on Tue.

Trees

Ex: For a n-vertex graph G, the following are equivalent and characterize trees with n vertices.

connected, acyclic

(1) G is a tree.

(2) G is connected & has exactly $n-1$ edges

(3) G is minimally connected, i.e., G is connected but $G-e$ is disconnected for every edge $e \in G$.

(4) G contains no cycle but $G+\{x,y\}$

does for any two non-adjacent vertices $x, y \in G$.

(5) Any two vertices in G are linked by a unique path in G.

$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$.

$(2) \Rightarrow (3)$

G is connected & has exactly $n-1$ edges

$\Downarrow$

G is minimally connected, i.e., G is connected but $G - e$ is disconnected for every edge $e \in G$.

<u>Proof</u> G is connected ✓

Let e be an arbitrary but particular edge in G. Let $G' = G - e$. It remains to show that G' is not connected.

G' has n vertices & $\underline{n-2}$ edges. We proved in class that a connected graph with n vertices must have $\geq n-1$ edges. Thus G' is not connected.

$(3) \Rightarrow (4)$

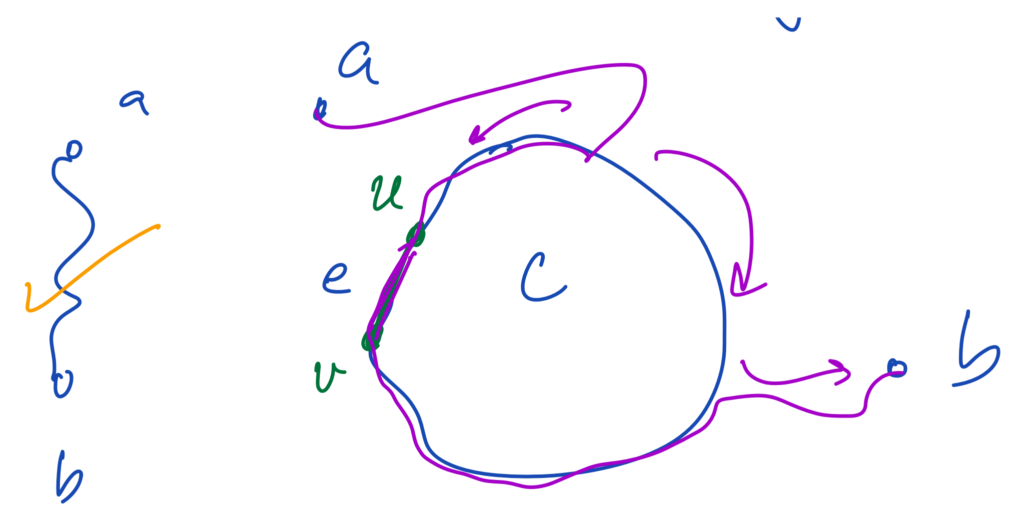
G is minimally connected, i.e., G is connected but $G - e$ is disconnected for every edge $e \in G$.

$$\Downarrow$$

(G contains no cycle) but $G + \{x, y\}$ does for any two non-adjacent vertices $x, y \in G$.

We will first prove that G is connected. We will prove this by proving the contrapositive.

Let C be a cycle in G.

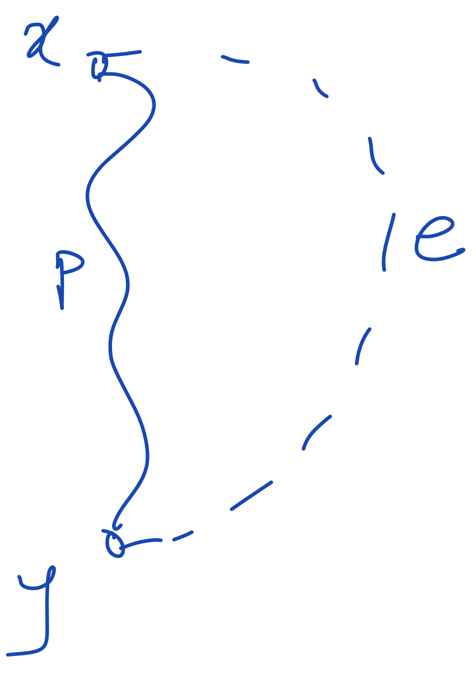


Let $e = (u,v) \in C$. Let $G' = G - e$. G' is <u>connected</u>.

We now want to show that if we add $\{x,y\}$ b/w non-adj. vertices x & y in G then G will contain a cycle.

Let x & y be any pair of non-adj vertices in G.

Since G is connected there must be a path P b/w x & y in G.



$P + e$ forms a cycle.

$(4) \Rightarrow (5)$

G contains no cycle but $\left[G + \{x, y\} \right.$ does for any two non-adjacent vertices $\left. x, y \in G. \right]$

$\Downarrow$

Any two vertices in G are linked by a unique path in G.

T.P.T

≥ 1 path b/w any two vertices in G.

u, v : arb. but part. vertices.

Case I : u & v are adjacent ✓

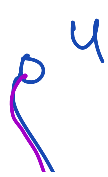
Case II : u & v are non-adj.

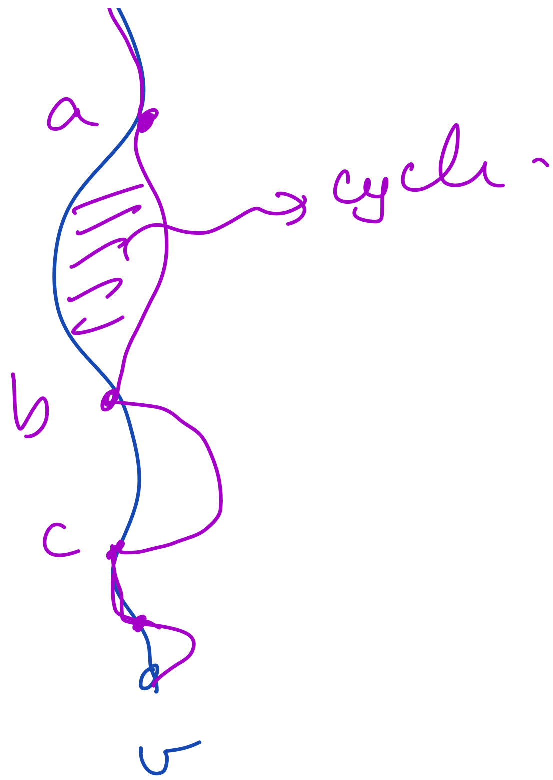
Adding (u, v) creates a cycle.

Thus $u \rightsquigarrow v$ path must exist.

T.P.T exactly one path exists b/w u & v.

- more than one path b/w u & v will create a cycle; thereby proving the contrapositive.





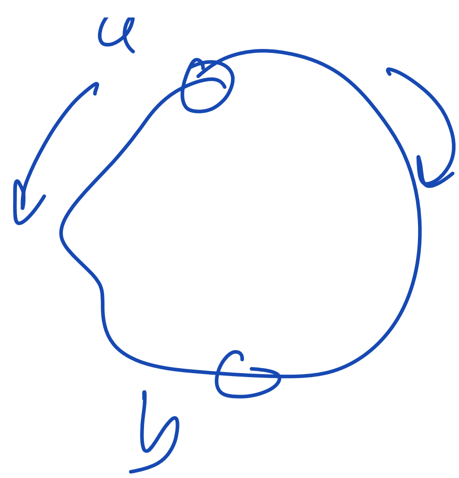
$(5) \Rightarrow (1)$

Any two vertices in G are linked by a unique path in G.

$\Downarrow$

G is a tree.

connected ✓ acyclic.



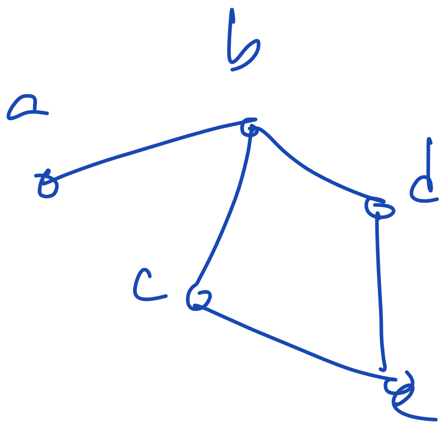
Spanning trees.

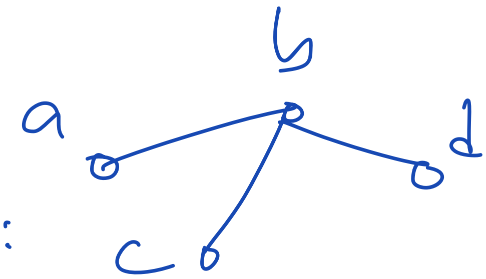
- $H = (V_H, E_H)$ is a <u>spanning subgraph</u> of $G = (V, E)$ if $V_H = V$.

$T = (V_T, E_T)$ is a <u>spanning tree</u> of $G = (V, E)$ if

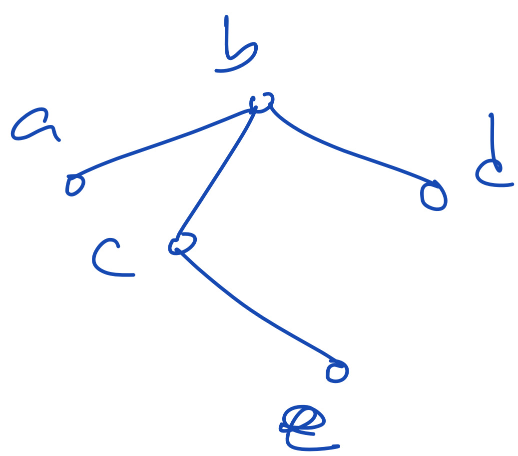
- T is a spanning subgraph of G

- T is a tree.

G 

H: 

not a spanning subgraph of G.

H: 

H: spanning tree of G.

Ex: Every connected graph $G = (V, E)$ contains a spanning tree.

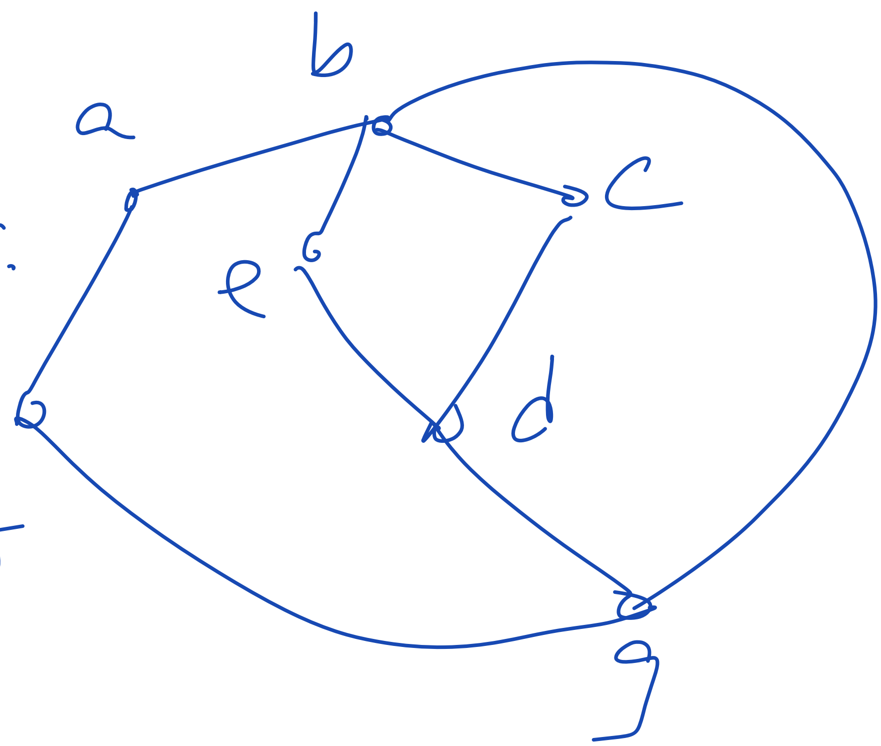
Proof: Let T be a minimally connected spanning subgraph of G. By the equivalence of (i) & (iii) in the previous lemma, T is a tree. Thus T is a spanning tree of G.

It remains to show that

T (minimally connected

spanning subgraph of G exists).

G:



Construct T as follows.

for each edge e in G:

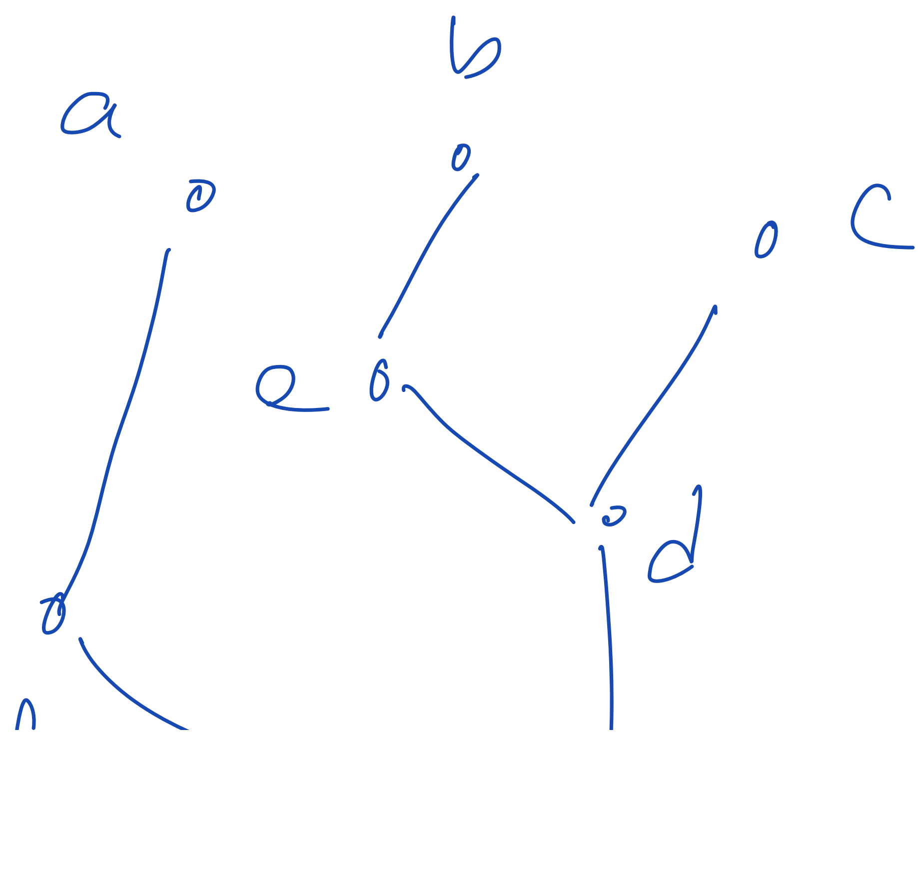
"Do I need you?

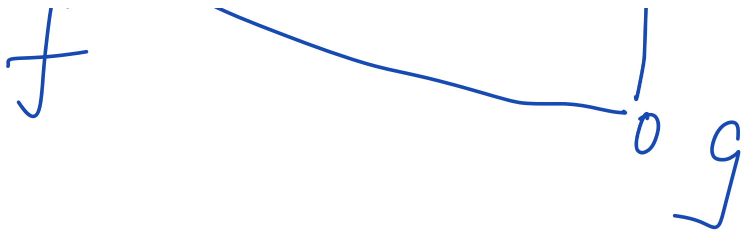
↳ After removing e does the graph become disconnected?

if Yes : keep it.

else

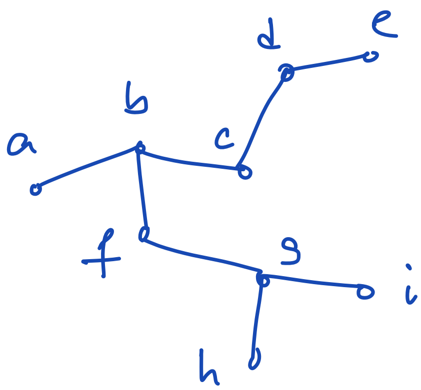
remove it.

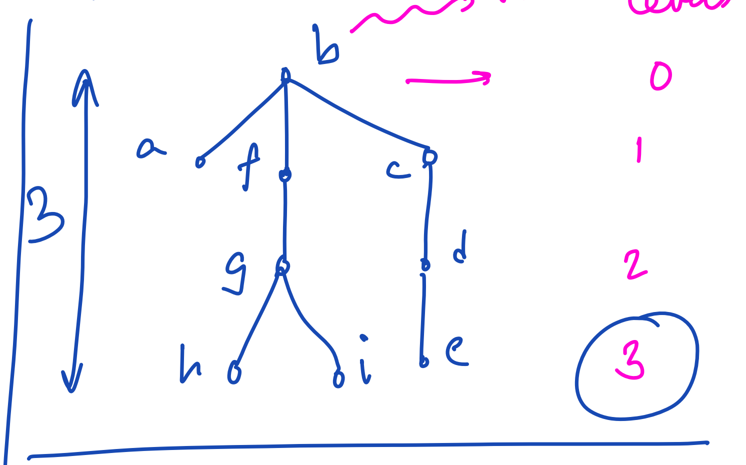


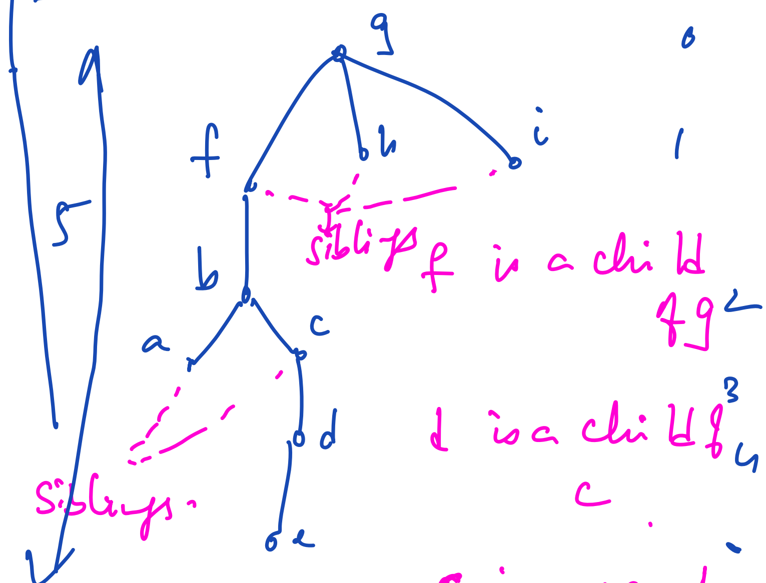


Rooted Trees.

- Tree with a root.







root

children of a node.

parent

ancestor

descendant

siblings

height

f is an ancestor of c,

c is a parent of d

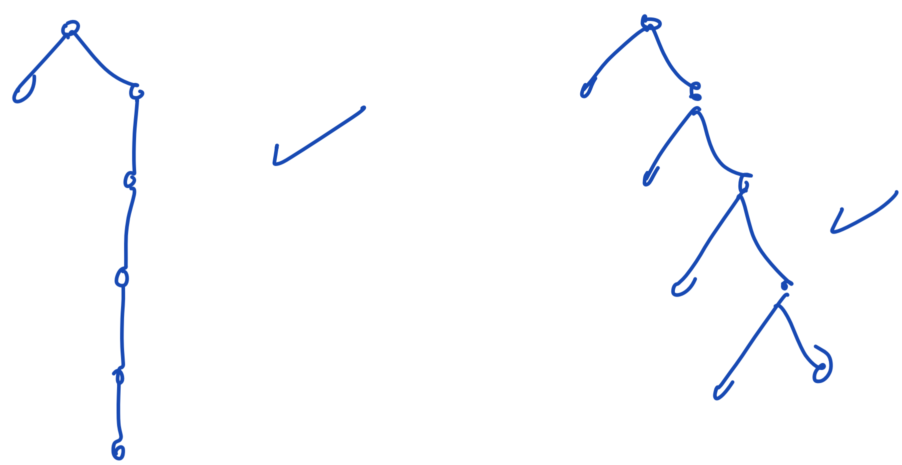
f is a parent

'of b.'

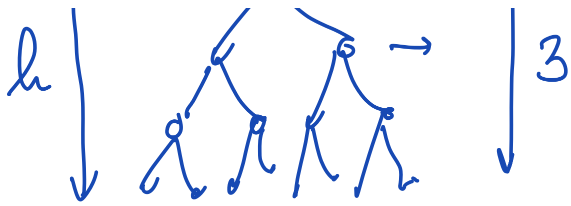
Height of a rooted tree is the length of the longest path from root to a leaf. or Ht = largest level.

<u>Binary tree</u>

- rooted tree in which each node has ≤ 2 children.



$2^0 + 2^1 + 2^2 + 2^3$.

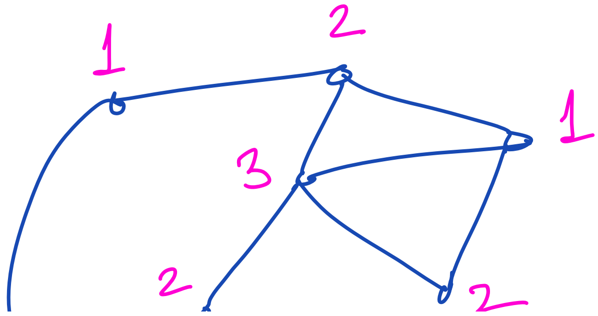


$= 2^4 - 1$

$= 15$.

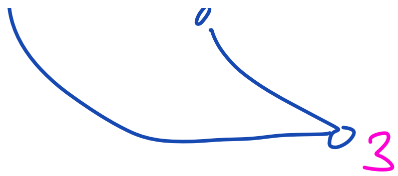
Graph Coloring.

A graph G admits a k-coloring iff there exists a coloring of vertices of G using k colors s.t. $\forall (u,v) \in E$, u & v are colored differently.



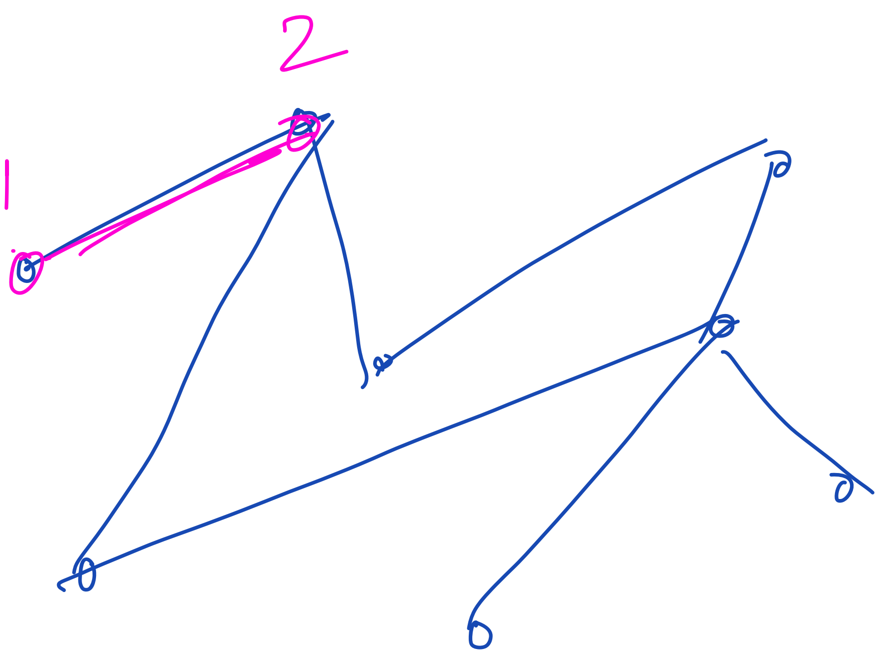
G is 3-colorable.

$\rightarrow \chi(G) = 3$.



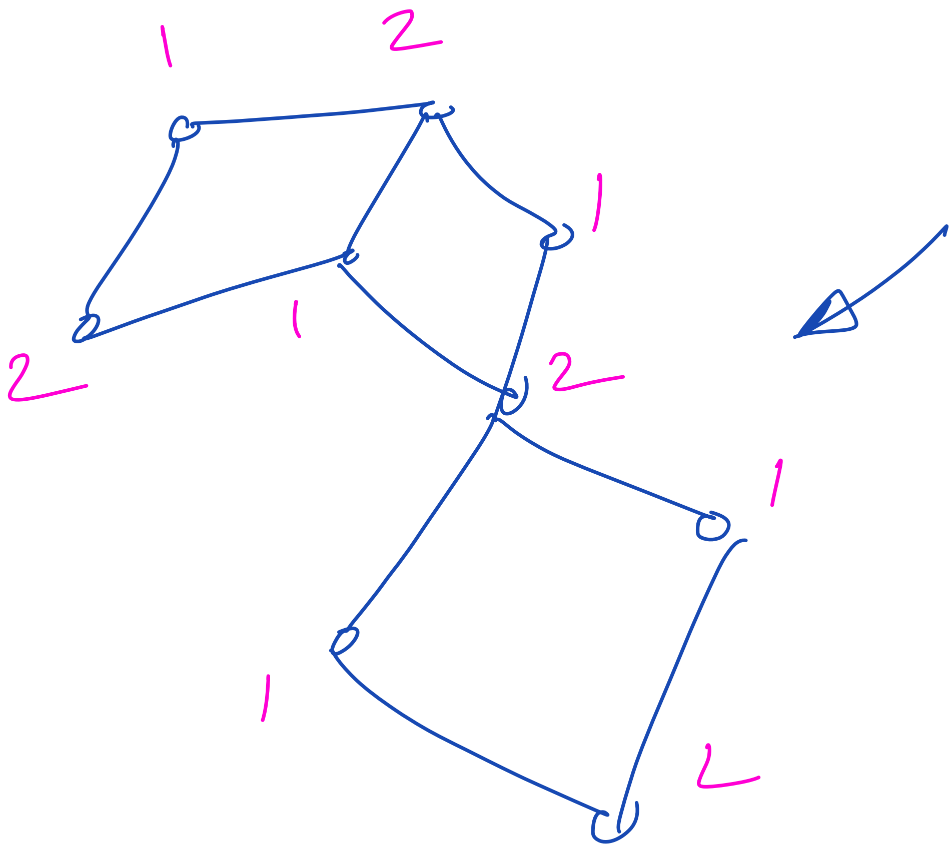
Chromatic index of G ($\chi(G)$) is

the min. # colors needed to color G.



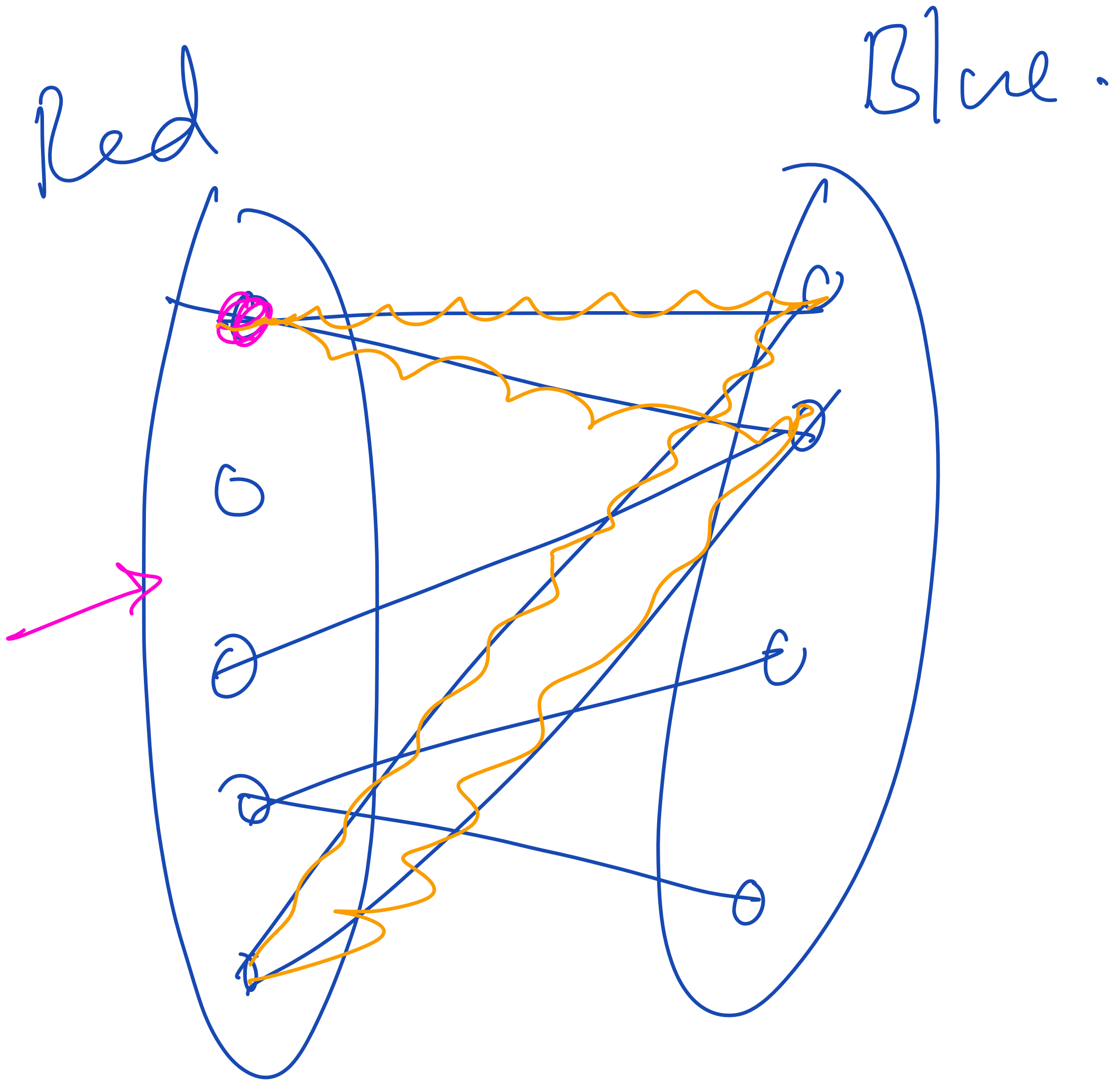
12-2 Mon 3-5 T 2-4 Mon

~~A~~ <u>bipartite</u> graph is a graph that is 2-colorable.



G is bipartite iff G contains no odd

cycles.



Lemma: Let G be a graph with maximum degree Δ. Then G is $(\Delta+1)$-colorable.

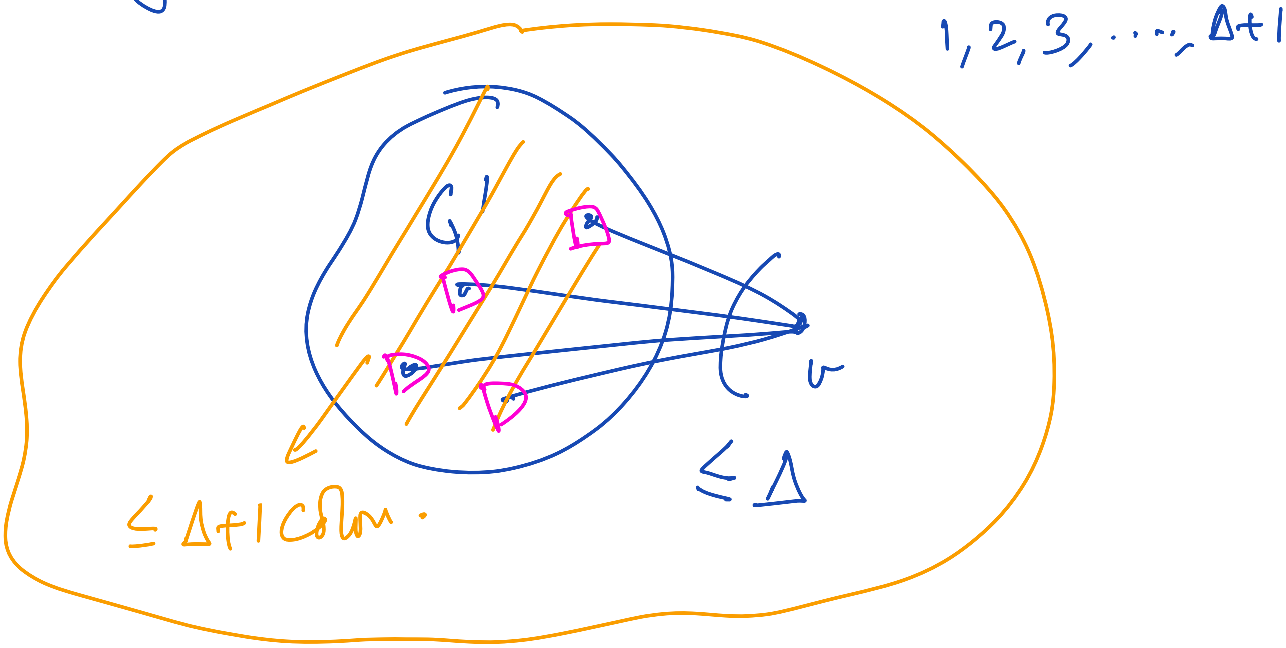
Proof: Induction on n.

IH: Assume that the claim holds when $n = k$, for some int $k \geq 1$.

BC: $\bullet$ 1 color suffices.

IS: Want to prove the claim when $n = k+1$. Let G be a graph on $k+1$ vertices. Let G have a max degree of $\Delta(G)$. Then, we want to show that G is $(\Delta+1)$-colorable.

Let $G' = G - v$, where v is any vertex in G.



By IH: G' can be colored using $\le \Delta(G') + 1$ colors.

$\le \underline{\underline{\Delta(G) + 1}}$ colors.

Add v to G' to obtain G.

$\deg(v) \leq \Delta$.

$\therefore \leq \Delta$ colors are used by neighbors of v

Thus atleast one color is unused by neighbors of v. We can use that color to color v.