NEW SYLLABUS
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Discrete Structures and Graph Theory

(Code : CSC302)

Semester 3 - Computer Engineering

Usha A. Jogalekar

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Discrete Structures and Graph Theory

(Code - CSC302)

Semester III - Computer Engineering (University of Mumbai)

(As per New Syllabus for Academic Year 2020-2021)

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Discrete Structures and Graph Theory

Usha A. Jogalekar

Semester III, Computer Engineering (Mumbai University)

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▶ **Edition :** August 2020

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ISBN 978-93-90376-27-8

This edition is for sale in India, Bangladesh, Bhutan, Maldives, Nepal, Pakistan, Sri Lanka and designated countries in South-East Asia. Sale and purchase of this book outside of these countries is unauthorized by the publisher.

Published by	**Printed at**
Mr. Sachin S. Shah & Mrs. Nayana S. Shah	**Image Offset (Mr. Rahul Shah)**
Permanent Address	Dugane Ind. Area Survey No. 28/25, Dhayari Near Pari Company,
Tech-Neo Publications LLP Sr. No. 38/1, Behind Pari Company, Khedekar Industrial Estate, Narhe, Maharashtra, Pune-411041.	Pune – 411041. Maharashtra State, India. E-mail : rahulshahimage@gmail.com

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407-412, 4th floor, Decision Tower, Above Hotel Tiranga,

Nr. City Pride Theatre, Pune-Satara Road, Pune-411009. Maharashtra State, India.

Email : info@techneobooks.com

Website : www.techneobooks.com

Mobile : 9145531105 / 8668233261

Preface

Dear Students,

I am extremely happy to present the book of **"Discrete Structures and Graph Theory"** for you. It is one of the basic subject of Computer Engineering. I have divided the subject into small chapters so that the topics can be arranged and understood properly. The topics within the chapters have been arranged in a proper sequence to ensure smooth flow of the subject.

I am thankful to Shri. Sachin Shah for the encouragement and support that they have extended. I am also thankful to the staff members of Tech-Neo Publications and others for their efforts to make this book as good as it is. We have jointly made every possible effort to eliminate all the errors in this book. However if you find any, please let me know, because that will help me to improve further.

I am also thankful to my family members and friends for their patience and encouragement.

- Author

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Syllabus...

Mumbai University

Second Year of Computer Engineering (Revise 2019 'C' Scheme)

Discrete Structures and Graph Theory

Course Code	Course Name	Credits
CSC302	Discrete Structures and Graph Theory	3

Prerequisite : Basic Mathematics

Course Objectives

1. Cultivate clear thinking and creative problem solving.
2. Thoroughly train in the construction and understanding of mathematical proofs. Exercise common mathematical arguments and proof strategies.
3. To apply graph theory in solving practical problems.
4. Thoroughly prepare for the mathematical aspects of other Computer Engineering courses

Course Outcomes : On successful completion, of course, learner/student will be able to:

1. Understand the notion of mathematical thinking, mathematical proofs and to apply them in problem solving.
2. Ability to reason logically.
3. Ability to understand relations, functions, Diagraph and Lattice.
4. Ability to understand and apply concepts of graph theory in solving real world problems.
5. Understand use of groups and codes in Encoding-Decoding
6. Analyze a complex computing problem and apply principles of discrete mathematics to identify solutions

Course Contents

Module		Detailed Contents	Hours
1	Logic		6
	1.1	Propositional Logic, Predicate Logic, Laws of Logic, Quantifiers, Normal Forms, Inference Theory of Predicate Calculus, Mathematical Induction. **(Refer chapter 1)**	
2	Relations and Functions		6
	2.1	Basic concepts of Set Theory	
	2.2	**Relations :** Definition, Types of Relations, Representation of Relations, Closures of Relations, Warshall's algorithm, Equivalence relations and Equivalence Classes	
	2.3	**Functions :** Definition, Types of functions, Composition of functions, Identity and Inverse function. **(Refer chapter 2)**	
3	Posets and Lattice		5
	3.1	Partial Order Relations, Poset, Hasse Diagram, Chain and Anti chains, Lattice, Types of Lattice, Sub lattice. **(Refer chapter 3)**	

4	**Counting**		6
	4.1	Basic Counting Principle-Sum Rule, Product Rule, Inclusion- Exclusion Principle, Pigeonhole Principle	
	4.2	Recurrence relations, Solving recurrence relations. **(Refer chapter 4)**	
5	**Algebraic Structures**		8
	5.1	**Algebraic structures with one binary operation:** Semi group, Monoid, Groups, Subgroups, Abelian Group, Cyclic group, Isomorphism	
	5.2	**Algebraic structures with two binary operations:** Ring	
	5.3	**Coding Theory**: Coding, binary information and error detection, decoding and error correction. **(Refer chapter 5)**	
6	**Graph Theory**		8
		Types of graphs, Graph Representation, Sub graphs, Operations on Graphs, Walk, Path, Circuit, Connected Graphs, Disconnected Graph, Components, Homomorphism and Isomorphism of Graphs, Euler and Hamiltonian Graphs, Planar Graph, Cut Set, Cut Vertex. **(Refer chapter 6)**	

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Module I

CHAPTER 1 Logic

Syllabus

Propositional Logic, Predicate Logic, Laws of Logic, Quantifiers, Normal Forms, Inference Theory of Predicate Calculus, Mathematical Induction.

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1.1 PROPOSITIONAL LOGIC

1.1.1 Propositions

Consider two sentences :

(i) It is raining (ii) Is it raining ?

The first sentence is declarative and second one is a question. If it is really raining then the first sentence is considered as true otherwise it is false. The answer to the second sentence is yes or no, depending on the situation.

Consider another sentence "close the door". It is an instruction whereas "door is closed" is declarative. The sentences like "it is raining', "door is closed" are known as propositions. They can be either true or false. It is known as truth value of the proposition.

Definition : A proposition is a declarative sentence which is either true or false.

We generally use lower case alphabets like p, q etc. to represent a proposition.

It is written as

p : Rose is red

True and false are known as truth values of a proposition. A proposition is called tautology if its truth value is always true.

A proposition is called contradiction, if its truth value is always false. A proposition which is neither tautology nor contradiction is called contingency.

Consider the following propositions :

p : There are 31 days in January

q : There are 28 days in February

r : There are 30 days in March.

The proposition p is tautology as it is always true.

The proposition r is contradiction as it is always false.

The proposition q is true if the year is not leap and it is false, if the year is leap. Thus the proposition q is neither tautology nor contradiction.

∴ It is contingency.

1.2 LOGICAL CONNECTIVES OR LOGICAL OPERATIONS

We can combine two or more propositions to form new propositions.

For example let p and q be two propositions given by,

p : It is raining

q : I have an umbrella

With these two sentences, we can form different sentences as follows :

(i) It is raining and I have an umbrella.

(ii) It is raining but I don't have an umbrella.

(iii) I have an umbrella but it is not raining.

(iv) Whenever it is raining, I have an umbrella.

(v) Whenever it rains, I don't have an umbrella.

(vi) Whenever I have an umbrella, it rains.

(vii) Whenever I have an umbrella, it does not rain.

This is the case of formation of sentences in English. In the same way, new propositions can be formed using earlier propositions. To combine the propositions, we use logical connectives. The following subsections explain different logical connectives.

1.2.1 Truth Tables

The connectives can be represented using truth tables, which give truth values for all possible combinations of propositions.

For example, if there is one proposition, then it can have truth values either true or false represented as T or F respectively.

If there are two propositions, they can have 4 combinations of truth values, FF, FT, TF and TT.

If there are three propositions, they can have 8 combinations of truth values, FFF, FFT, FTF, FTT, TFF, TFT, TTF and TTT.

Thus in general, if the number of propositions is n, then the number of combinations of truth values is given by 2^n.

(A) Disjunction (∨) - OR

Definition : The disjunction of two propositions p and q is denoted as $p \vee q$. It is a proposition, which is false, if both p and q are false. Otherwise it is true.

The disjunction connective is also known as OR. Following truth table represents disjunction.

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

This can also represented as,

	∨	q: F	q: T
p	F	F	T
	T	T	T

(B) Conjunction (∧) - AND

Definition : The conjunction of two propositions p and q is denoted as $p \wedge q$. It is a proposition, which is true, if both p and q are true. Otherwise it is false.

The conjunction connective is also known as AND. Following truth table represents conjunction.

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

This can also be represented as,

	∧	q: F	q: T
p	F	F	F
	T	F	T

(C) Negation (– or ¬ or ~) NOT

It is a unary connective. i.e. applied on only one proposition.

Definition : The negation of one proposition p, is denoted as $\bar{p}$ or $\neg p$ or $\sim p$. It is a proposition which is true if p is false and it is false if p is true.

The negation connective is also known as NOT. Following truth table represents negation.

p	$\bar{p}$ or $\neg p$ or $\sim p$
F	T
T	F

It can be observed that $p \vee \bar{p}$ is a tautology and $p \wedge \bar{p}$ is a contradiction.

p	$\bar{p}$	$p \vee \bar{p}$	$p \wedge \bar{p}$
F	T	T	F
T	F	T	F

(D) Exclusive or – (⊕) – EX-OR

Definition : The exclusive or of two propositions p and q is denoted as $p \oplus q$. It is a proposition, which is true, if truth values of p and q are different. Otherwise it is false.

The Exclusive or connective is also known as EX-OR. Following truth table represents exclusive or

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

This can also be represented as,

	⊕	q: F	q: T
p	F	F	T
	T	T	F

It can be noted that $p \oplus q$ is equivalent to $(p \wedge \neg q) \vee (\neg p \wedge q)$

This is shown in the following truth table.

p	q	$\neg p$	$\neg q$	$p \wedge \neg p$	$\neg p \wedge q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$
F	F	T	T	F	F	F
F	T	T	F	F	T	T
T	F	F	T	T	F	T
T	T	F	F	F	F	F

(E) Conditional ($\rightarrow$) or implication

Definition : The proposition $p \rightarrow q$ is read as "if p then q" or "p implies q". $p \rightarrow q$ is false if p is true and q is false. Otherwise it is true.

The truth table is as follows :

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

It can be noted that $p \rightarrow q$ is equivalent to $\neg p \vee q$. This is shown in the following truth table.

p	q	$\neg p$	$\neg p \vee q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	F	T

For the proposition, $p \rightarrow q$ we have corresponding propositions.

The proposition $q \rightarrow p$ is called converse of $p \rightarrow q$

The proposition $\bar{p} \rightarrow \bar{q}$ is called inverse of $p \rightarrow q$

The proposition $\bar{q} \rightarrow \bar{p}$ is called contra positive of $p \rightarrow q$

It can be noted that $p \rightarrow q$ is equivalent to $(\bar{q} \rightarrow \bar{p})$,

This is shown in the following truth table.

p	q	$\bar{p}$	$\bar{q}$	$p \rightarrow q$	$q \rightarrow p$	$\bar{p} \rightarrow \bar{q}$	$\bar{q} \rightarrow \bar{p}$
F	F	T	T	T	T	T	T
F	T	T	F	T	F	F	T
T	F	F	T	F	T	T	F
T	T	F	F	T	T	T	T

F) Biconditional ($\leftrightarrow$, $\leftrightarrows$)

Definition : The proposition $p \leftrightarrow q$ is read as "p if and only if q" or "p double implication q". $p \leftrightarrow q$ is true if both p and q have same truth value.

The truth table is as follows.

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

It can be noted that $p \leftrightarrow q$ is equivalent to $\neg (p \oplus q)$, $(p \rightarrow q) \wedge (q \rightarrow p)$, and $(p \wedge q) \vee (\bar{p} \wedge \bar{q})$.

All these are shown in the following truth table

p	q	$\bar{p}$	$\bar{q}$	$p \oplus q$	$\neg (p \oplus q)$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
F	F	T	T	F	T	T	T	T	F	T	T
F	T	T	F	T	F	T	F	F	F	F	F
T	F	F	T	T	F	F	T	F	F	F	F
T	T	F	F	F	T	T	T	T	T	F	T

1.3 LOGICAL EQUIVALENCES

Two propositions p and q are said to be logically equivalent, if $p \leftrightarrow q$ is a tautology. It is represented as $p \equiv q$. It can also be read as, p tautologically implies q.

In the previous section we have seen 6 different connectives for propositions. Out of them, disjunction, conjunction and negation are called basic connectives. This is because, the remaining three connectives can be represented using these three connectives only.

Thus the following equivalences hold.

$p \rightarrow q \equiv \neg p \vee q$

$p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

If we have two propositions p and q, then there are four combinations of truth values. With these four set of truth values, we can have at the maximum 16 (i.e. 2^4) different propositions.

p	q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
F	F	F	F	F	F	F	F	F	F	T	T	T	T	T	T	T	T
F	T	F	F	F	F	T	T	T	T	F	F	F	F	T	T	T	T
T	F	F	F	T	T	F	F	T	T	F	F	T	T	F	F	T	T
T	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T

1. It is contradiction $p \wedge \bar{p}$ or $q \wedge \bar{q}$
2. $p \wedge q$
3. $\overline{p \rightarrow q}$
4. p
5. $\overline{q \rightarrow p}$
6. q
7. $p \oplus q$
8. $p \vee q$
9. $\overline{p \vee q}$
10. $p \leftrightarrow q$
11. $\neg q$
12. $q \rightarrow p$
13. $\neg p$
14. $p \rightarrow q$
15. $\overline{p \wedge q}$
16. It is tautology $p \vee \bar{p}$ or $q \vee \bar{q}$

Any given proposition with two propositions p and q can be simplified to one of these 16 propositions.

1.3.1 Duality

The connectives disjunction and conjunctions are called duals of each other. The truth values T and F are duals of each other.

> **Definition :** Two propositions A and A* are called duals if one can be obtained by replacing $\vee$ by $\wedge$, $\wedge$ by $\wedge$, T by F and F by T in another statement.

Example

$p \vee q$ is dual of $p \wedge q$, $(p \wedge \bar{q}) \vee r$ is dual of $(p \vee \bar{q}) \wedge r$

> **Note :** Duals are not equal.

If two propositions A and B are equal then their duals A* and B* are also equal.

1.4 PROPERTIES OF LOGICAL CONNECTIVES OR LAWS OF LOGIC

In this section, we study the properties of logical connectives. One can observe the similarities between the properties of logical connectives with that of set operations discussed in section 1. We can find the resemblance between, $\vee$ and $\cup$, $\wedge$ and $\cap$.

The universal set $\cup$, resembles to the truth value true i.e. T and the null set ϕ, resembles to the truth value false i.e. F.

Following are the properties of logical connectives. Let p, q. r be propositions.

1.	Double negation law	$\bar{\bar{p}} \equiv p$
2.	Cumulative law	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$ $p \oplus q \equiv q \oplus p$
3.	Associative law	$p \vee (q \vee r) \equiv (p \vee q) \vee r$ $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ $p \oplus (q \oplus r) \equiv (p \oplus q) \oplus r$
4.	De Morgan's law	$\overline{p \vee q} \equiv \bar{p} \wedge \bar{q}$ $\overline{p \wedge q} \equiv \bar{p} \vee \bar{q}$
5.	Distributive law	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
6.	Idempotent law	$p \vee p \equiv p$ $p \wedge p \equiv p$
7.	Absorption law	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
8.	Operations generating truth value F.	$p \wedge F \equiv F$ – Domination law $p \wedge \bar{p} \equiv F$ – Negation law $p \oplus p \equiv F$ $p \leftrightarrow \bar{p} \equiv F$ All these are contradictions.
9.	Operations generating truth value T	$p \vee T \equiv T$ – Domination law $p \vee \bar{p} \equiv T$ – Negation law $p \oplus \bar{p} \equiv T$ $p \rightarrow p \equiv T$ $F \rightarrow p \equiv T$ $p \leftrightarrow p \equiv T$ $p \rightarrow T \equiv T$ All these are tautologies.
10.	Operations with truth value F	$p \vee F \equiv p$ – identity law $p \wedge F \equiv F$ – Domination law $p \oplus F \equiv p$ $\bar{F} = T$ – Negation law $F \rightarrow p \equiv T$ $p \rightarrow F \equiv \bar{p}$
11.	Operations with truth value T	$p \vee T \equiv T$ – Domination law $p \wedge T \equiv p$ – identity law $p \oplus T \equiv \bar{p}$ $\bar{T} \equiv F$ - Negation law $T \rightarrow p \equiv p$ $p \rightarrow T \equiv T$
12.	Exportation law	$p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

1.5 WELL-FORMED FORMULAE

When we write propositions with many connectives, it becomes difficult to evaluate and simplify them, with the priorities of the connectives. Thus it is necessary to represent a proposition in a standard form. This representation is known as well-formed formula (wff). Here we consider a proposition p without any connectives as a statement variable.

Definition : A well formed formula can be produced using the following rules.

R_1 : A statement variable p is a wff.

R_2 : If p is wff then $\bar{p}$ is also wff.

R_3 : If p and q are wff, then $(p \vee q)$, $(p \wedge q)$, $(p \rightarrow q)$ and $(p \leftrightarrow q)$ are also wff.

R_4 : A string of statement variables, connectives and parentheses i.e. '(' and ')' is said to be wff, if and only if it can be produced by applying rules R_1, R_2 and R_3 finite number of times.

For Example $(\overline{p \wedge q})$, $(p \vee (q \wedge \bar{r}))$, $(p \rightarrow \bar{q})$, $((\bar{p} \leftrightarrow q) \wedge r))$ are wff.

$p \to q$, $(p \vee q) \wedge r$, $p \to (q \vee r)$, $\overline{p \vee q}$ are not wff. These can be converted to wff if they are written as,

$(p \to q)$, $((p \vee q) \wedge r)$, $(p \to (q \vee r))$, $(\overline{p \vee q})$ respectively.

1.5.1 Solved Examples

Ex. 1.5.1 : To describe the various restaurants in the city, we let p denote the statement, "The food is good" q denote the statement, "The service is good", and r denote the statement, "The rating is three-star". Write the following statements in symbolic form.

(a) Either the food is good, or the service is good, or both.

(b) Either the food is good or the service is good, but not both.

(c) The food is good while the service is poor.

(d) It is not the case that both the food is good and the rating is three-star.

(e) If both the food and services are good, then the rating will be three-star.

(f) It is not true that a three-star rating always means good food and good service.

☑ Soln. :

Given :

p : The food is good.

q : The service is good.

r : The rating is three-star.

(a) $p \vee q$

(b) $p \oplus q$

(c) $p \wedge \overline{q}$

(d) $\overline{p \wedge r}$

(e) $(p \wedge q) \to r$

(f) $\overline{r \to (p \wedge q)}$

Ex. 1.5.2 : Let p denote the statement, "The material is interesting", and q denote the statement, "The exercises are challenging", and r denote the statement, "The course is enjoyable". Write the following statements in symbolic form :

(a) The material is interesting and the exercises are challenging.

(b) The material is uninteresting, the exercises are not challenging and the course is not enjoyable.

(c) If the material is not interesting and the exercises are not challenging, then the course is not enjoyable.

(d) The material is interesting means the exercises are challenging, and conversely.

(e) Either the material is interesting or the exercises are not challenging, but not both.

☑ Soln. :

Given :

p : The material is interesting

q : The exercises are challenging

r : The course is enjoyable.

(a) $p \wedge q$ (b) $\overline{p} \wedge \overline{q} \wedge \overline{r}$ (c) $(\overline{p} \wedge \overline{q}) \to \overline{r}$

(d) $q \leftrightarrow p$ (e) $p \oplus \overline{q}$

Ex. 1.5.3 : Write the following statements in symbolic form :

(a) The sun is bright and the humidity is not high.

(b) If I finish my homework before dinner and it does not rain, then I will go to the ball game.

(c) If you do not see me tomorrow, it means I have gone to Chicago.

(d) If the utility cost goes up or the request for additional funding is denied, then a new computer will be purchased if and only if we can show that the current computing facilities are indeed not adequate.

☑ **Soln. :**

Given :

(a) p : Sun is bright q : Humidity is not high

$p \wedge q$

(b) p : I finish my homework before dinner

q : It rains

r : I will go to the ball game

$(p \wedge \bar{q}) \rightarrow r$

(c) p : You see me tomorrow q : I have gone to Chicago

$\bar{p} \rightarrow q$

(d) p : Utility cost goes up

q : Request for additional funding is denied

r : New computer will be purchased

s : Current computing facilities are not adequate

$((p \vee q) \rightarrow r) \leftrightarrow s$

Ex. 1.5.4 : Write the converse, inverse and contra positive of the following :

(a) If you are honest then you will succeed.

p : You are honest q : You will succeed

(b) If you work hard then you will be rich.

p : You work hard q : You will be rich

(c) If you are healthy then you can be wealthy

p : You are healthy

q : You can be wealthy

☑ **Soln. :**

(a) If you are honest then you will succeed.

p : You are honest q : You will succeed

Given : $p \rightarrow q$

Converse : If you succeed then you are honest $(q \rightarrow p)$

Inverse : If you are not honest then you will not succeed $(\bar{p} \rightarrow \bar{q})$

Contra positive : If you do not succeed then you are not honest. $(\bar{q} \rightarrow \bar{p})$

(b) If you work hard then you will be rich.

p : You work hard q : You will be rich

Given : $p \rightarrow q$

Converse : If you are rich then you worked hard. $(q \rightarrow p)$

Inverse : If you do not work hard, then you will not be rich. $(\bar{p} \rightarrow \bar{q})$

Contra positive : If you are not rich then you did not work hard. $(\bar{q} \rightarrow \bar{p})$

(c) If you are healthy then you can be wealthy

p : You are healthy q : You can be wealthy

Given : $p \rightarrow q$

Converse : If you are wealthy then you can be healthy. $(q \rightarrow p)$

Inverse : If you are not healthy then you cannot be wealthy. $(\bar{p} \rightarrow \bar{q})$

Contra positive : If you cannot be wealthy then you are not healthy.

Ex. 1.5.5 : If p is true and q is false, find the truth values of following propositions. p : T q : F

(a) $(\bar{p} \leftrightarrow q) \wedge p$

(b) $(p \wedge \bar{q}) \leftrightarrow p$

(c) $(p \rightarrow q) \leftrightarrow (q \rightarrow p)$

(d) $((p \wedge q) \rightarrow p) \leftrightarrow (\bar{p} \vee q)$

(e) $\sim (p \leftrightarrow q) \wedge p$

(f) $\sim (p \leftrightarrow q) \vee \sim\sim p$

☑ **Soln. :**

(a) $(\bar{p} \leftrightarrow q) \wedge p$

$= (\bar{T} \leftrightarrow F) \wedge T$

$= (F \leftrightarrow F) \wedge T$

$= T \wedge T$

$= T$

(b) $(p \wedge \bar{q}) \leftrightarrow p$

$= (T \wedge \bar{F}) \leftrightarrow T$

$= (T \wedge T) \leftrightarrow T$

$= T \leftrightarrow T$

$= T$

(c) $(p \to q) \leftrightarrow (q \to p)$

$= (T \to F) \leftrightarrow (F \to T)$

$= F \leftrightarrow T$

$= F$

(d) $((p \wedge q) \to p) \leftrightarrow (\bar{p} \vee q)$

$= ((T \wedge F) \to T)\ (\bar{T} \vee F)$

$= (F \to T) \leftrightarrow (F \vee F)$

$= T \leftrightarrow F$

$= F$

(e) $\sim (p \leftrightarrow q) \wedge p$

$= \sim (T \leftrightarrow F) \wedge T$

$= \sim F \wedge T$

$= T \wedge T$

$= T$

(f) $\sim (p \leftrightarrow q) \vee \sim\sim p$

$= \sim (T \leftrightarrow F) \vee \sim\sim T$

$= \sim F \vee T$

$= T \vee T$

$= T$

Ex. 1.5.6 : If p and q are true and r is false then find the truth values of the following statements.

$p : T \quad q : T \quad r : F$

(a) $(p \to q) \leftrightarrow (q \to r)$

(b) $(p \wedge r) \to (q \leftrightarrow r)$

(c) $(q \leftrightarrow \bar{r}) \wedge (p \vee r)$

(d) $(p \to r) \leftrightarrow (p \to \bar{q})$

(e) $\sim (p \leftrightarrow q) \wedge r$

(f) $(p \to q) \leftrightarrow [(p \to \bar{q}) \wedge r]$

Soln. :

(a) $(p \to q) \leftrightarrow (q \to r)$

$= (T \to T) \leftrightarrow (T \to F)$

$= T \leftrightarrow F$

$= F$

(b) $(p \wedge r) \to (q \leftrightarrow r)$

$= (T \wedge F) \to (T \leftrightarrow F)$

$= F \to F$

$= T$

(c) $(q \leftrightarrow \bar{r}) \wedge (p \vee r)$

$= (T \leftrightarrow \bar{F}) \wedge (T \vee F)$

$= (T \leftrightarrow T) \wedge T$

$= T \wedge T$

$= T$

(d) $(p \to r) \leftrightarrow (p \to \bar{q})$

$= (T \to F) \leftrightarrow (T \to \bar{T})$

$= F \leftrightarrow (T \to F)$

$= F \leftrightarrow F$

$= T$

(e) $\sim (p \leftrightarrow q) \wedge r$

$= \sim (T \leftrightarrow T) \wedge F$

$= \sim (T) \wedge F$

$= F \wedge F$

$= F$

(f) $(p \to q) \leftrightarrow [(p \to \bar{q}) \wedge r]$

$= (T \to T) \leftrightarrow [(T \to \bar{T}) \wedge F]$

$= T \leftrightarrow ((T \to F) \wedge F)$

$= T \leftrightarrow (F \wedge F)$

$= T \leftrightarrow F$

$= F$

Ex. 1.5.7 : Let p denote the statement "the weather is nice" and q denote the statement, "we have a picnic". Translate the following in English and simplify if possible.

(a) $p \wedge \bar{q}$ (b) $p \leftrightarrow q$ (c) $\bar{q} \to \bar{p}$

Soln. :

(a) $p \wedge \bar{q}$: The weather is nice and we do not have a picnic.

(b) $p \leftrightarrow q$: If and only if the weather is nice then we have a picnic.

(c) $\bar{q} \rightarrow \bar{p}$: If we do not have a picnic then the weather is not nice.

Ex. 1.5.8 : Write a compound statement that is true, when exactly two of the three statements p, q, r are true.

☑ Soln. :

The compound statement is formed using 3 statements p, q and r. It must be true if exactly any two of these are true.

∴ The compound statement is true if any one of the following are satisfied.

(i) p, q are true and r is false, this gives $p \wedge q \wedge \bar{r}$ as true.

(ii) p, r are true and q is false, this gives $p \wedge \bar{q} \wedge r$ as true.

(iii) q, r are true and p is false. This gives $\bar{p} \wedge q \wedge r$ as true.

∴ The compound statement is,

$$(p \wedge q \wedge \bar{r}) \vee (p \wedge \bar{q} \wedge r) \vee (\bar{p} \wedge q \wedge r)$$

Ex. 1.5.9 : Write a compound statement that is true when none, or one, or two of the three statements p, q, r are true.

☑ Soln. :

The problem says that the compound statement is false, if all statements p, q, r are true.

∴ The compound statement is $\bar{p} \vee \bar{q} \vee \bar{r}$

Ex. 1.5.10 : Given that the value of $p \rightarrow q$ is false, determine the value of $(\bar{p} \vee \bar{q}) \rightarrow q$.

☑ Soln. :

Given : $p \rightarrow q$ is false.

∴ $p = T$ and $q = F$

To find value of $(\bar{p} \vee \bar{q}) \rightarrow q$

$$(\bar{p} \vee \bar{q}) \rightarrow q$$

$$= \left(\overline{\bar{p} \vee \bar{q}}\right) \vee q \qquad \text{from definition of} \rightarrow$$

$$= (p \wedge q) \vee q \qquad \text{De Morgan's law}$$

$$= (T \wedge F) \vee F \qquad \text{Substitute for p and q}$$

$$= F \vee F$$

$$= F$$

Ex. 1.5.11 : Given that the value of $p \rightarrow q$ is true, can you determine the value of $\bar{p} \vee (p \leftrightarrow q)$?

☑ Soln. :

Given : $p \rightarrow q$ is true

To find : Value of $\bar{p} \vee (p \leftrightarrow q)$

$$\bar{p} \vee (p \leftrightarrow q)$$

$$= \bar{p} \vee [(p \rightarrow q) \wedge (q \rightarrow p)] \qquad \text{From definition of} \leftrightarrow$$

$$= \bar{p} \vee [T \wedge (q \rightarrow p)] \qquad \text{Substitute for } p \rightarrow q$$

$$= \bar{p} \vee (q \rightarrow p)$$

$$= \bar{p} \vee (\bar{q} \vee p)$$

$$= (\bar{p} \vee p) \vee \bar{q} \qquad \text{...Associative law}$$

$$= T \vee \bar{q}$$

$$= T$$

Ex. 1.5.12 : Construct the truth tables for the following statements. Find whether they are tautology, contradiction or contingency.

a. $p \rightarrow p$

b. $(p \vee \bar{q}) \vee \bar{p}$

c. $p \leftrightarrow (\bar{p} \vee \bar{q})$

d. $(\bar{q} \rightarrow \bar{p}) \rightarrow (p \rightarrow q)$

e. $(\bar{p} \wedge q) \wedge (q \rightarrow p)$

f. $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow r]$

g. $(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$

☑ Soln. :

a. $p \rightarrow p$

p	$p \rightarrow p$
F	T
T	T

It is tautology.

b. $(p \vee \bar{q}) \vee \bar{p}$

P	q	$\bar{p}$	$\bar{q}$	$p \vee \bar{q}$	$(p \vee \bar{q}) \vee \bar{p}$
F	F	T	T	T	T
F	T	T	F	F	T
T	F	F	T	T	T
T	T	F	F	T	T

It is tautology.

c. $p \leftrightarrow (\bar{p} \vee \bar{q})$

P	q	$\bar{p}$	$\bar{q}$	$\bar{p} \vee \bar{q}$	$p \leftrightarrow (\bar{p} \vee \bar{q})$
F	F	T	T	T	F
F	T	T	F	T	F
T	F	F	T	T	T
T	T	F	F	F	F

It is contingency.

d. $(\bar{q} \rightarrow \bar{p}) \rightarrow (p \rightarrow q)$

p	Q	$\bar{p}$	$\bar{q}$	$\bar{q} \rightarrow \bar{p}$	$p \rightarrow q$	$(\bar{q} \rightarrow \bar{p}) \rightarrow (p \rightarrow q)$
F	F	T	T	T	T	T
F	T	T	F	T	T	T
T	F	F	T	F	F	T
T	T	F	F	T	T	T

It is tautology.

e. $(\bar{p} \wedge q) \wedge (q \rightarrow p)$

p	q	$\bar{p}$	$\bar{q}$	$\bar{p} \wedge q$	$q \rightarrow p$	$(\bar{p} \wedge q) \wedge (q \rightarrow p)$
F	F	T	T	F	T	F
F	T	T	F	T	F	F
T	F	F	T	F	T	F
T	T	F	F	F	T	F

It is contradiction.

f. $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow r]$

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \rightarrow q$	$(p \rightarrow q) \rightarrow r$	$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow r)$
F	F	F	T	T	T	F	F
F	F	T	T	T	T	T	T
F	T	F	F	T	T	F	F
F	T	T	T	T	T	T	T
T	F	F	T	T	F	T	T
T	F	T	T	T	F	T	T
T	T	F	F	F	T	F	T
T	T	T	T	T	T	T	T

It is contingency.

g. $(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$

p	Q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \rightarrow r$	$q \rightarrow (p \rightarrow r)$	$(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$
F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	T	T	F	T	T
T	F	T	T	T	T	T	T
T	T	F	F	F	F	F	T
T	T	T	T	T	T	T	T

It is tautology.

Ex. 1.5.13 : Prove distributive law and De Morgan's law, using truth tables.

☑ **Soln. :** Distributive law.

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

p q r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
F F F	F	F	F	F	F
F F T	F	F	F	T	F
F T F	F	F	T	F	F
F T T	T	T	T	T	T
T F F	F	T	T	T	T
T F T	F	T	T	T	T
T T F	F	T	T	T	T
T T T	T	T	T	T	T
		$\bar{\bar{A}}$			$\bar{\bar{B}}$

Columns A and B are equal ∴ distributive law is proved.

pqr	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
FFF	F	F	F	F	F
FFT	T	F	F	F	F
FTF	T	F	F	F	F
FTT	T	F	F	F	F
TFF	F	F	F	F	F
TFT	T	T	F	T	T
TTF	T	T	T	F	T
TTT	T	T	T	T	T
		$\bar{\bar{C}}$			$\bar{\bar{D}}$

Columns C and D are equal.

∴ Distributive law is proved. With above two truth tables, both equations in the distributive law are proved.

Ex. 1.5.14 : Prove De Morgan's law using truth tables.

Soln. : De Morgan's law is

$$\overline{P \vee q} = \bar{P} \wedge \bar{q}$$

$$\overline{P \wedge q} = \bar{P} \vee \bar{q}$$

pq	$q \wedge r$	$\overline{p \wedge q}$	$\bar{p}$	$\bar{q}$	$\bar{p} \wedge \bar{q}$
FF	F	T	T	T	T
FT	T	F	T	F	F
TF	T	F	F	T	F
TT	T	F	F	F	F
		$\bar{\bar{A}}$			$\bar{\bar{B}}$

Columns A and B are equal.

pq	$q \vee r$	$\overline{p \vee q}$	$\bar{p}$	$\bar{q}$	$\bar{p} \vee \bar{q}$
FF	F	T	T	T	T
FT	F	T	T	F	T
TF	F	T	F	T	T
TT	T	F	F	F	F
		$\bar{\bar{C}}$			$\bar{\bar{D}}$

Columns C and D are equal.

∴ De Morgan's law is proved.

Ex. 1.5.15 : Without constructions truth tables, prove that the following statements are tautologies.

Soln. :

$\bar{p} \rightarrow (p \rightarrow q)$

$= \bar{\bar{p}} \vee (\bar{p} \vee q)$ From definition of $\rightarrow$

$= (p \vee \bar{p}) \vee q$ Associative law.

$= T \vee q$

$= T$

∴ $\bar{p} \rightarrow (p \rightarrow q)$ is a tautology.

Ex. 1.5.16 : Without constructing truth tables. Prove that the following statements are contradictions.

(a) $\neg((p \vee q) \wedge \neg p) \rightarrow q)$

(b) $(\bar{p} \wedge q) \wedge (q \rightarrow p)$

(c) $\overline{p \rightarrow q} \rightarrow p$

(d) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

(e) $(p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r)) \vee (\neg p \wedge \neg q)$

Soln. :

(a) $\neg((p \vee q) \wedge \neg p) \rightarrow q)$

$= \neg(\neg((p \vee q) \wedge \neg p) \vee q)$

$= \neg((\neg(p \vee q) \vee \neg\neg p) \vee q)$

$= \neg((\neg p \wedge \neg q) \vee p \vee q)$

$= \neg((\neg p \vee p \vee q) \wedge (\neg q \vee p \vee q))$

$= \neg(T \wedge T)$

$= \neg(T) = F$

∴ It is contradiction.

(b) $(\bar{p} \wedge q) \wedge (q \rightarrow p)$

$= (\bar{p} \wedge q) \wedge (\bar{q} \vee p)$

$= (\bar{p} \wedge q \wedge \bar{q}) \vee (\bar{p} \wedge q \wedge p)$

$= F \vee F$

$= F$

$\therefore$ It is contradictory.

(c) $\overline{p \to q} \to p$

$= \overline{p \vee q} \to p$ From definition of $\to$

$= (\bar{p} \wedge q) \vee p$ From definition of $\to$

$= \bar{p} \vee (q \vee p)$ Associative law

$= \bar{p} \vee (p \vee q)$ Commutative law

$= (\bar{p} \vee p) \vee q$ Associative law

$= T \vee q$

$= T$

$\therefore \overline{p \to q} \to p$ is a tautology.

(d) $[(p \to q) \wedge (q \to r)] \to (p \to r)$

$= [(\bar{p} \vee q) \wedge (\bar{q} \vee r)] \to (\bar{p} \vee r)$

From definition of $\to$

$= \left(\overline{(\bar{p} \vee q) \wedge (\bar{q} \vee r)}\right) \vee (\bar{p} \vee r)$

From definition of $\to$

$= \overline{(\bar{p} \vee q)} \vee \overline{(\bar{q} \vee r)} \vee (\bar{p} \vee r)$ De Morgan's law

$= (p \wedge \bar{q}) \vee (q \wedge \bar{r}) \vee (\bar{p} \vee r)$

De Morgan's law

$= \left((p \vee q) \wedge (p \vee \bar{r}) \wedge \overset{T}{(\bar{q} \vee q)} \wedge (\bar{q} \vee \bar{r})\right) \vee (\bar{p} \vee r)$

Distributive law

$= (p \vee q \vee \bar{p} \vee p) \wedge (p \vee \bar{r} \vee \bar{p} \vee r) \wedge (\bar{q} \vee \bar{r} \vee \bar{p} \vee r)$

...Distributive law

$= T \wedge T \wedge T = T$

$\therefore [(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology.

(e) $(p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r)) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

$= (p \vee q) \wedge (p \vee \neg(\neg q \vee \neg r)) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ De Morgan's law

$= (p \vee q) \wedge (p \vee (q \wedge r)) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

De Morgan's law

$= [(p \vee q) \wedge p \cancel{\vee} q\,(p \vee q) \wedge (p \vee r)] \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

Distributive law

$= [(\overset{T}{p \cancel{\vee} q} \vee \neg p) \wedge (\overset{T}{p \cancel{\vee} q} \vee \neg q) \wedge (p \cancel{\vee} r \vee \neg p) \wedge (p \vee r) \vee \neg q] \vee (\neg p \wedge \neg r)$

Distributive law

$= (\overset{T}{p \vee r \vee \cancel{\neg} q} \vee \neg q) \wedge (\overset{T}{p \vee r \vee \cancel{\neg} q} \vee \neg r)$

$= T \wedge T = T$

$\therefore$ It is a tautology.

Ex. 1.5.17 : Prove the following without constructing truth tables.

a. $(\bar{p} \vee \bar{q}) \to (p \wedge q \wedge r) = p \wedge q$

b. $\bar{p} \wedge (\bar{q} \wedge r) \vee (q \wedge r) \vee (p \wedge r) = r$

☑ **Soln. :**

a. $(\bar{p} \vee \bar{q}) \to (p \wedge q \wedge r) = p \wedge q$

LHS $= (\bar{p} \vee \bar{q}) \to (p \wedge q \wedge r)$

$= \overline{(\bar{p} \vee \bar{q})} \vee (p \wedge q \wedge r)$

From definition of $\to$

$= (p \wedge q) \vee (p \wedge q \wedge r)$ De Morgan's law

Assume that $p \wedge q = K$.

$\therefore$ LHS $= K \vee (K \wedge r)$ Absorption law

$= K = p \wedge q$

$=$ RHS.

Hence proved.

b. $\bar{p} \wedge (\bar{q} \wedge r) \vee (q \wedge r) \vee (p \wedge r) = r$

LHS $= \bar{p} \wedge (\bar{q} \wedge r) \vee (q \wedge r) \vee (p \wedge r)$

$= ((\bar{p} \wedge \bar{q}) \wedge r) \vee ((q \vee p) \wedge r)$

Distributive law

$= [(\bar{p} \wedge \bar{q}) \vee (q \vee p)] \wedge r$ Distributive law

$= (((\bar{p} \wedge \bar{q}) \vee q) \vee p) \wedge r$ Associative law

T

$$= \left[\left[\left(\bar{p} \vee q\right) \wedge \left(\bar{q} \vee q\right)\right] \vee p\right] \wedge r$$

Distributive law

$$= \left(\left(\bar{p} \vee q\right) \vee p\right) \wedge r = \left(\left(\bar{p} \vee p\right) \vee q\right) \wedge r$$

$$= (T \vee q) \wedge r = T \wedge r = r$$

$$= \text{RHS}$$

Syllabus Topic : Normal Forms

1.6 NORMAL FORMS

In the previous section, we have seen that a proposition can be written in different forms. Thus it very difficult to check the equivalence of two propositions every time. Therefore there is a need for standardizing the propositions. In this section, we see two types of such standardizations. Principal Disjuctive Normal Form (PDNF) and Principal Conjunctive Normal Form (PCNF). But before that, it is necessary to understand few basic concepts of normalization or standardization; they are Disjunctive Normal Form (DNF), Conjunctive Normal Form (CNF), minterm, and maxterm etc. Following subsections explain all these with examples.

1.6.1 Disjunctive Normal Form (DNF)

A proposition, in which if all connectives are either negation or conjunction, then it is known as fundamental conjunction.

For example, p, $p \wedge \neg q$, $p \wedge q$, $p \wedge q \wedge \neg r$, $p \wedge \neg p$, $\neg p$ are all fundamental conjunctions.

> **Definition :** A Proposition is said to be in Disjunctive Normal Form (DNF) if it has fundamental conjunctions connected with disjunctions.

For Example

p, $p \wedge \neg q$, $(p \wedge q) \vee (p \wedge \neg q)$, $p \vee q$ are in disjunctive normal form.

$p \wedge (q \vee r)$, $(p \vee q) \wedge (p \vee r)$ are not in disjunctive normal form, but they can be converted to DNF as follows.

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) \rightarrow \text{DNF}$$

$$(p \vee q) \wedge (p \vee r) = (p \wedge p) \vee (p \wedge r) \vee (q \wedge p) \vee (q \wedge r) \rightarrow \text{DNF}$$

1.6.2 Conjunctive Normal Form (CNF)

A proposition, in which if all connectives are either negation or disjunction, then it is known as fundamental disjunction.

For example : p, $p \vee \neg q$, $p \vee q$, $p \vee q \vee r$, $\neg p$, $p \vee \neg p$ are all fundamental disjunctions.

> **Definition :** A proposition is said to be in Conjunctive Normal Form (CNF), if it has fundamental disjunctions connected with conjunctions.

For example

p, $p \wedge \neg q$, $(p \vee q) \wedge (p \vee \neg q)$, $p \vee q$ are in conjunctive Normal Form.

$p \vee (q \wedge r)$, $(p \wedge q) \vee (p \wedge r)$ are not in Conjunctive Normal Form, but they can be converted to CNF as follows :

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r) \rightarrow \text{CNF}$$

$$(p \wedge q) \vee (p \wedge r) = (p \vee p) \wedge (p \vee r) \wedge (q \vee p) \wedge (q \vee r) \rightarrow \text{CNF.}$$

It can be observed that some propositions are both in DNF and CNF.

For example, p, $p \vee q$, $p \wedge q$, $\bar{P}$ are both CNF and DNF. This is because.

(i) $p \rightarrow p$ is considered as fundamental conjunction.

$\therefore$ it is in DNF.

p is also considered as fundamental disjunction.

$\therefore$ It is in CNF.

(ii) $p \vee q \rightarrow$ $(p) \vee (q)$ is in DNF.

$(p \vee q)$ is in CNF.

(iii) $p \wedge q \rightarrow$ $(p \wedge q)$ is in CNF

$(p) \wedge (q)$ is in DNF.

Thus there is a need for unique standard.

1.6.3 Principal Disjunctive Normal Form (PDNF)

A fundamental conjunction that has all as "min term".

If there are n statements then the number of possible minterms is 2nd.

For example, if p and q are the statements, then there are $2^2 = 4$ minterms. They are

$\bar{p} \wedge \bar{q}$, $\bar{p} \wedge q$, $p \wedge \bar{q}$, $p \wedge q$.

If p, q and r are the statements, then there are $2^3 = 8$ minterms. They are

$\bar{p} \wedge \bar{q} \wedge \bar{r}$, $\bar{p} \wedge \bar{q} \wedge r$, $\bar{p} \wedge q \wedge \bar{r}$, $\bar{p} \wedge q \wedge r$, $p \wedge \bar{q} \wedge \bar{r}$, $p \wedge \bar{q} \wedge r$, $p \wedge q \wedge \bar{r}$, $p \wedge q \wedge r$.

Remember that in all the minterms, the order of statements is same, p followed by q, followed by r

i.e. $p \wedge q \wedge r$ should not be written as $q \wedge r \wedge p$ even though they are equal.

Every minterm is associated with a number, which is equal to the binary representation of the minterm. Every statement is represented as 1 and its negation is represented as 0, thus resulting to a binary number. The minterm is represented as m_k where k is decimal equivalent of that binary number.

For example, consider two statements, p and q. The following table gives the number representation of each minterm.

Minterm	Binary Representation	Decimal Equivalent	Minterm Representation
$\bar{p} \wedge \bar{q}$	00	0	M_0
$\bar{p} \wedge q$	01	1	M_1
$p \wedge \bar{q}$	10	2	M_2
$p \wedge q$	11	3	M_3

Thus for two statements, the decimal equivalents range from 0 to 3.

Similarly if there are n statements, the decimal equivalents range from 0 to $2^n - 1$.

Definition : A proposition is said to be in Principal Disjunctive Normal Form, if and only if it has disjunction of minterms.

For example, the following are in PDNF.

a. $(p \wedge q) \vee (p \wedge \bar{q})$

b. $(p \wedge \bar{q} \wedge r) \vee (\bar{p} \wedge q \wedge r) \vee (\bar{p} \wedge \bar{q} \wedge r)$

As we know that minterms can be represented with decimal numbers. A PDNF is represented as the sum of these numbers, arranged in ascending order.

i.e. $(p \wedge q) \vee (p \wedge \bar{q}) = M_3 \vee M_2 = \Sigma 2, 3.$

1 1 1 0

$(p \wedge \bar{q} \wedge r) \vee (\bar{p} \wedge q \wedge r) \vee (\bar{p} \wedge \bar{q} \wedge r)$

1 0 1 0 1 1 0 0 1

$= N_5 \vee M_3 \vee M_1 = \Sigma 1, 3, 5$

Any given proposition can be converted to its equivalent PDNF. Following are the steps to do so.

(i) Convert the proposition to DNF, i.e. disjunction of conjunction.

(ii) In every fundamental conjunction include the missing statements if any.

i.e. if the fundamental conjunction is p, and there is one more statement 'q' to be included, then p can be converted as follows.

$p = p \wedge (q \vee \bar{q})$ $\quad q \vee \bar{q} = T$ and $p \wedge T = p$

$= (p \wedge q) \vee (P \vee \bar{q})$ $\quad$ Distributive law

$= M_3 \vee M_2$

$= \Sigma 2, 3$

Example : Convert $p \vee (q \wedge r) \vee (p \wedge \bar{q})$ to PDNF

Given proposition is,

$$p \vee (q \wedge r) \vee (p \wedge \bar{q})$$

$$= p \vee (q \wedge r) \vee (p \wedge \bar{q})$$

Here p, $q \wedge r$, $p \wedge \bar{q}$ are fundamental conjunctions. Include (q, r), p, and r in them to convert them to minterms.

$\therefore p = p \wedge (q \vee \bar{q})$ $\quad$ include q.

$= (p \wedge q) \vee (p \wedge \bar{q})$

$= ((p \wedge q) \wedge (r \vee \bar{r})) \vee ((p \wedge \bar{q}) \wedge (r \vee \bar{r}))$ $\quad$ include r

$= (p \wedge q \wedge r) \vee (p \wedge q \wedge \bar{r}) \vee (p \wedge \bar{q} \wedge r) \vee (p \wedge \bar{q} \wedge \bar{r})$

1 1 1 1 1 0 1 0 1 1 0 0

$= M_7 \vee M_6 \vee M_5 \vee M_4$

$= \Sigma\, 4, 5, 6, 7$

Now consider,

$q \wedge r$

$= (q \wedge r) \wedge (P \vee \bar{P})$ include p

$= (q \wedge r \wedge p) \vee (q \wedge r \wedge \bar{P})$

$= (p \wedge q \wedge r) \vee (\bar{p} \wedge q \wedge r)$ arrange in order

1 1 1 0 1 1

$= M_7 \vee M_3$

$= \Sigma\, 3, 7.$

Now consider,

$p \wedge \bar{q}$

$= (p \wedge \bar{q}) \wedge (r \vee \bar{r})$ include r

$= (p \wedge \bar{q} \wedge r) \vee (p \wedge \bar{q} \wedge \bar{r})$

1 0 1 1 0 0

$= M_5 \vee M_4$

$= \Sigma\, 4, 5$

$\therefore$ PDNF for $p \vee (q \wedge r) \vee (p \vee \bar{q})$ is given by

$\Sigma\, 3, 4, 5, 6, 7$

$= (\bar{p} \wedge q \wedge r) \vee (p \wedge \bar{q} \wedge \bar{r}) \vee (p \wedge \bar{q} \wedge r)$
$\vee (p \wedge q \wedge \bar{r}) \vee (p \wedge q \wedge r)$

1.6.4 Principal Conjunctive Normal Form (PCNF)

A fundamental disjunction that has all statements or their negations. Under consideration is known as "Maxterm".

If there are n statements, then the number of possible maxterms is 2^n.

For example, if p and q are the statements, then there are $2^2 = 4$ maxterms. They are,

$\bar{p} \vee \bar{q}$, $\bar{p} \vee q$, $p \vee \bar{q}$, $p \vee q$.

If p, q and r are the statements, then there are $2^3 = 8$ maxterms. They are,

$\bar{p} \vee \bar{q} \vee \bar{r}$, $\bar{p} \vee \bar{q} \vee r$, $\bar{p} \vee q \vee \bar{r}$, $\bar{p} \vee q \vee r$, $p \vee \bar{q} \vee \bar{r}$,

$p \vee \bar{q} \vee r$, $p \vee q \vee \bar{r}$, $p \vee q \vee r$.

Remember that in all maxterms, the order of statements is same. p followed by q, followed by r.

i.e. $p \vee q \vee r$ should not be written as $q \vee p \vee r$ even though they are same.

Every maxterm is associated with a number, which is equal to the binary representation of the maxterm. Every statement is represented as 0 and its negation is represented as 1, thus resulting to a binary number. The maxterm is represented as M_k where k is the decimal equivalent of that binary number.

For example, consider two statements, p and q. The following table gives the number representation of each maxterm.

Minterm	Binary Representation	Decimal Equivalent	Minterm Representation
$\bar{p} \vee \bar{q}$	11	3	M_3
$\bar{p} \wedge q$	10	2	M_2
$p \wedge \bar{q}$	01	1	M_1
$p \wedge q$	00	0	M_0

Thus for two statements, the decimal equivalents range from 0 to 3.

Similarly if these are n statements, the decimal equivalents range from 0 to $2^n - 1$.

Definition : A proposition is said to be in Principal Conjunctive Normal Form (PCNF), if and only if it has conjunction of maxterms.

For example, the following are in PCNF.

a. $(p \vee \bar{q}) \wedge (\bar{p} \vee \bar{q})$

b. $(p \vee q \vee \bar{r}) \wedge (p \vee \bar{q} \vee r) \wedge (\bar{p} \vee q \vee \bar{r})$

As we know that maxterms can be represented with decimal numbers. A PCNF is represented as the product of these numbers, arranged in ascending order.

i.e. $(p \vee \bar{q}) \wedge (\bar{p} \vee \bar{q}) = M_1 \wedge M_3 = \Pi\ 1, 3$

$(p \vee q \vee \bar{r}) \wedge (p \vee \bar{q} \vee r) \wedge (\bar{p} \vee q \vee \bar{r})$

$= M_1 \wedge M_2 \wedge M_5 = \Pi\ 1, 2, 5$

Any given proposition can be converted to its equivalent PCNF. Following are the steps to do so.

(i) Convert the proposition to CNF, i.e. conjunction of disjunctions.

(ii) In every fundamental disjunction include the missing statements if any.

i.e. if the fundamental disjunction is p, and there is one more statement 'q' to be included, then p can be converted as follows :

$$p = p \vee (q \vee \bar{q}) \quad q \wedge \bar{q} = F \text{ and } p \vee F = p$$

$$= (p \vee q) \wedge (p \vee \bar{q}) \qquad \text{Distributive law}$$

$$= M_0 \wedge M_1 = \Pi\ 0, 1$$

Example

Convert $p \vee (q \wedge r) \vee (p \vee \bar{q})$ to PCNF

Given proposition is,

$p \vee (q \wedge r) \vee (p \wedge q)$

$= [\,(p \vee q) \wedge (p \vee r)\,] \vee (p \wedge q)$ Distributive law

$= (p \vee q \vee p) \wedge (p \vee q \vee q) \wedge (p \vee r \vee p) \wedge (p \vee r \vee q)$ Distributive law

$= (p \vee q) \wedge (p \vee q) \vee (p \vee r) \wedge (p \vee q \vee r)$

$= (p \vee q) \wedge (p \vee r) \wedge (p \vee q \vee r)$

Here, $p \vee q$, $p \vee r$, are fundamental disjunctions.

$p \vee q \vee r$ is maxterm.

Include r in $p \vee q$ and q in $p \vee r$ so that they become maxterms.

$\therefore p \vee q = (p \vee q) \vee (r \wedge \bar{r})$ include r

$= (p \vee q \vee r) \wedge (p \vee q \vee \bar{r})$

$= M_0 \wedge M_1$

$= \Pi\ 0, 1$

Now consider,

$p \vee r = (p \vee r) \vee (q \vee \bar{q})$ include q

$= (p \vee r \vee q) \wedge (p \vee \bar{q} \vee r)$ Arrange in order

$= M_0 \wedge M_2$

$= \Pi\ 0, 2$

$\therefore$ PCNF for $p \vee (q \wedge r) \vee (p \wedge \bar{q})$ is given by

$\Pi\ 0, 1, 2 = (p \vee q \vee r) \wedge (p \vee q \vee \bar{r}) \wedge (p \vee \bar{q} \vee r)$

1.6.5 Relationship between PDNF, PCNF, Tautology and Contradiction

In the previous two sections, we have found the PDNF and PCNF for the proposition, $p \vee (q \wedge r) \vee (p \wedge \bar{q})$.

The PDNF $= \Sigma\ 3, 4, 5, 6, 7$ and

The PCNF $= \Pi\ 0, 1, 2.$

It can be observed that the numbers in PCNF are the missing numbers in PDNF.

Thus we can find PCNF if PDNF is known and vice versa, without actually solving.

1.6.6 Solved Examples on Normal Form

Ex. 1.6.1 : If PDNF for a proposition is $(p \wedge q) \vee (p \vee \bar{q}) \vee (\bar{p} \wedge \bar{q})$ then find PCNF.

Soln. :

Given PDNF is,

$(p \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge \bar{q})$

$= M_3 \vee M_2 \vee M_0$

$= \Sigma\ 0, 2, 3.$

There are two statements p and q.

$\therefore$ Numbers range from 0 to 3.

$\therefore$ The missing number in PDNF is 1.

$\therefore$ The PCNF is given by Π_1

$$= M_1 = (p \vee \bar{q})$$

This can be proved with actually converting PDNF to PCNF.

Given PDNF is

$$(p \wedge q) \vee (p \vee \bar{q}) \vee (\bar{p} \wedge \bar{q})$$

$$= [((p \wedge q) \vee p) \wedge ((p \wedge q) \vee \bar{p})] \vee (\bar{p} \wedge \bar{q})$$

distributive law

$$= [(p \vee p) \wedge (p \vee q) \wedge (p \vee \bar{q}) \wedge (q \vee \bar{q})] \vee (\bar{p} \wedge \bar{q})$$

distributive law

$$= [p \wedge (p \vee q) \wedge (p \vee \bar{q})] \vee (\bar{p} \wedge \bar{q})$$

$$= (p \vee \bar{p}) \wedge (p \vee \bar{q}) \wedge (p \vee q \vee \bar{p}) \wedge (p \vee q \vee \bar{q})$$

$$\wedge (p \vee \bar{q} \vee \bar{p}) \wedge (p \vee \bar{q} \vee \bar{q})$$

$$= (p \vee \bar{q}) \wedge (p \vee \bar{q}) = (p \vee \bar{q})$$

$$= M_1$$

$$= \Pi_1$$

Hence the proof.

Consider a proposition, $(p \vee q) \vee (\bar{p} \wedge \bar{q})$.

Let us convert it to PDNF.

$$(p \vee q) \vee (\bar{p} \wedge \bar{q})$$

$$= p \vee q \vee (\bar{p} \wedge \bar{q})$$

$$= p \wedge (q \vee \bar{q}) \vee q \wedge (p \vee \bar{p}) \vee (\bar{p} \wedge \bar{q})$$

Now include the missing statements.

$$= (p \wedge q) \vee (p \vee \bar{q}) \vee (q \wedge p) \vee (q \wedge \bar{p}) \vee (\bar{p} \wedge \bar{q})$$

Removing repeated terms and arranging in order.

$$= M_3 \vee M_2 \vee M_1 \vee M_0$$

$$= \Sigma\ 0, 1, 2, 3.$$

Now try to solve the given proposition

$$(p \vee q) \vee (\bar{p} \wedge \bar{q}) = (p \vee q \vee \bar{p}) \wedge (p \vee q \vee \bar{q})$$

$$= T \wedge T = T$$

$\therefore$ It is a tautology.

Thus it is clear that the PDNF of a tautology has all the terms, and PCNF does not exist.

Consider a proposition $(p \vee q) \wedge (\bar{p} \wedge \bar{q})$

Let us convert it to PCNF

$$(p \vee q) \wedge (\bar{p} \wedge \bar{q}) = (p \vee q) \wedge \bar{p} \wedge \bar{q}$$

$$= (p \vee q) \wedge \bar{p} \vee (q \wedge \bar{q}) \wedge \bar{q} \vee (p \wedge \bar{p})$$

Include missing statements

$$= (p \vee q) \wedge (\bar{p} \vee q) \wedge (\bar{p} \vee \bar{q}) \wedge (\bar{q} \vee p) \wedge (\bar{q} \vee \bar{p})$$

$$= (p \vee q) \wedge (\bar{p} \vee q) \wedge (\bar{p} \vee \bar{q}) \wedge (p \vee \bar{q})$$

Removing repeated terms and arranging in order

$$= M_0 \wedge M_2 \wedge M_3 \wedge M_1$$

$$= \Pi\ 0, 1, 2, 3.$$

Now try to solve the given proposition

$$(p \vee q) \wedge (\bar{p} \wedge \bar{q})$$

$$= (p \wedge \bar{p} \wedge \bar{q}) \vee (q \wedge \bar{p} \wedge \bar{q}) \quad \text{Distributive law}$$

$$= F \vee F$$

$$= F$$

$\therefore$ It is contradiction.

Thus it is clear that the PCNF of a contradiction has all the terms, and PDNF does not exist.

Ex. 1.6.2 : Find PDNF for the following :

a) $(p \rightarrow q) \vee q$

b) $(p \rightarrow q) \vee p$.

c) $[(\bar{p} \wedge (\bar{q} \wedge r)) \vee (q \wedge r) \vee (p \wedge r)] \wedge \bar{r}$

d) $[((p \wedge \bar{q}) \vee r) \vee (p \vee \bar{q})] \wedge r$

☑ **Soln. :**

(a) $(p \rightarrow q) \vee q$

$$= (\bar{p} \vee q) \vee q \quad \text{From definition of} \rightarrow$$

$$= \bar{p} \vee (q \vee q) \quad \text{Associative law}$$

$$= \bar{p} \vee q \quad q \vee q = q.$$

$= [\bar{p} \wedge (q \vee \bar{q})] \vee [q \wedge (p \vee \bar{p})]$

$= (\bar{p} \wedge q) \vee (\bar{p} \wedge \bar{q}) \vee (q \wedge p) \vee (q \vee \bar{p})$

$= (\bar{p} \wedge q) \vee (\bar{p} \wedge \bar{q}) \vee (p \wedge q)$

Removing repeated term and arranging in order.

$= M_1 \vee M_0 \vee M_3$

$= \sum 0, 1, 3$

(b) $(p \rightarrow q) \vee p$.

$= (\bar{p} \vee q) \vee p$ From definition of $\rightarrow$

$= (\bar{p} \vee p) \vee q$ Associative and commutative law.

$= T \vee q$

$= T$

It is a tautology.

$\therefore$ PDNF has all four minterms.

$= \sum 0, 1, 2, 3$

$= (\bar{p} \wedge \bar{q}) \vee (\bar{p} \wedge q) \vee (p \wedge \bar{q}) \vee (p \wedge q)$

(c) $[(\bar{p} \wedge (\bar{q} \wedge r)) \vee (q \wedge r) \vee (p \wedge r)] \wedge \bar{r}$

$= [(\bar{p} \wedge \bar{q} \wedge r) \vee (q \wedge r) \vee (p \wedge r)] \wedge \bar{r}$

$= (\bar{p} \wedge \bar{q} \wedge r \wedge \bar{r}) \vee (q \wedge r \wedge \bar{r}) \vee (p \wedge r \wedge \bar{r})$

$= F \vee F \vee F$

$= F$

$\therefore$ It is a contradiction.

$\therefore$ PDNF does not exist.

(d) $[((p \wedge \bar{q}) \vee r) \vee (p \vee \bar{q})] \wedge r$

$= [[(p \vee r) \wedge (\bar{q} \vee r)] \vee (p \vee \bar{q})] \wedge r$

Distributive law

$= [(p \vee r \vee p \vee \bar{q}) \wedge (\bar{q} \vee r \vee p \vee \bar{p})] \wedge r$

$= [(p \vee \bar{q} \vee r) \wedge (p \vee \bar{q} \vee r)] \wedge r$

$= (p \vee \bar{q} \vee r) \wedge r$

$= (p \vee r) \vee (\bar{q} \wedge r) \vee (r \wedge r)$

Include missing terms

$= [(p \wedge r) \wedge (q \vee \bar{q})] \vee [(\bar{q} \wedge r) \wedge (p \vee \bar{p})] \vee r$

$\wedge (p \vee \bar{p}) \wedge (q \vee \bar{q})$

$= (p \wedge r \wedge q) \vee (p \wedge r \wedge \bar{q}) \vee (\bar{q} \wedge r \wedge p) \vee (\bar{q} \wedge r \wedge \bar{p})$

$\vee (r \wedge p \wedge q) \vee (r \wedge p \wedge \bar{q}) \vee (r \wedge \bar{p} \wedge q) \vee (r \wedge \bar{p} \wedge \bar{q})$

$= (p \wedge q \wedge r) \vee (p \wedge \bar{q} \wedge r) \wedge (\bar{p} \wedge \bar{q} \wedge r) \vee (\bar{p} \wedge q \wedge r)$

Removing repeated terms and arranging in order.

$= M_3 \vee M_5 \vee M_1 \vee M_3$

$= \sum 1, 3, 5, 7$

Ex. 1.6.3 : Find PCNF for the following :

a. $(p \vee q) \rightarrow \bar{p}$

b. $(p \rightarrow q) \leftrightarrow (\bar{q} \rightarrow \bar{p})$

c. $[a \vee (\bar{a} \wedge b)] \wedge (\bar{a} \wedge \bar{b})$

d. $(p \rightarrow q) \vee (r \rightarrow (q \wedge p))$

☑ **Soln. :**

(a) $(p \vee q) \rightarrow \bar{p}$

$= (\overline{p \vee q}) \vee \bar{p}$ From definition of $\rightarrow$

$= (\bar{p} \wedge \bar{q}) \vee \bar{p}$ De Morgan's law

$= (\bar{p} \vee \bar{p}) \wedge (\bar{q} \vee \bar{p}) = \bar{p} \wedge (\bar{p} \vee \bar{q})$

$= [\bar{p} \vee (q \wedge \bar{q})] \wedge (\bar{p} \vee \bar{q})$

Include missing term

$= (\bar{p} \vee q) \wedge (\bar{p} \vee \bar{q}) \wedge (\bar{p} \vee \bar{q})$

$= (\bar{p} \vee q) \wedge (\bar{p} \vee \bar{q})$ Remove repeated term

$= M_1 \wedge M_3$

$= \prod 1, 3.$

(b) $(p \rightarrow q) \leftrightarrow (\bar{q} \rightarrow \bar{p})$

$= (p \rightarrow q) \leftrightarrow (p \rightarrow q)$

$\bar{q} \rightarrow \bar{p}$ is contra positive of $p \rightarrow q$ and they are equal $p \leftrightarrow p$ is T.

$= T$

$\therefore$ It is a tautology.

$\therefore$ PCNF does not exist.

(c) $\left[a \vee (\bar{a} \wedge b)\right] \wedge (\bar{a} \wedge \bar{b})$

$= (a \vee \bar{a}) \wedge (a \vee b) \wedge \bar{a} \wedge \bar{b}$

$= T \wedge (a \vee b) \wedge (\bar{a} \vee (b \wedge \bar{b})) \wedge (\bar{b} \vee (a \wedge \bar{a}))$

Include missing terms

$= (a \vee b) \wedge (\bar{a} \vee b) \wedge (\bar{a} \wedge \bar{b}) \wedge (\bar{b} \vee a) \wedge (\bar{b} \wedge \bar{a})$

$= (a \vee b) \wedge (\bar{a} \vee b) \wedge (\bar{a} \vee \bar{b}) \wedge (a \vee \bar{b})$

$= M_0 \wedge M_2 \wedge M_3 \wedge M_1$

$= \prod 0, 1, 2, 3$

All terms are present.

$\therefore$ It is contradiction.

Let us verify it by simplifying the given proposition

$\left[a \vee (\bar{a} \wedge b)\right] \wedge (\bar{a} \wedge \bar{b})$

$= (a \wedge \bar{a} \wedge \bar{b}) \vee (\bar{a} \wedge b \wedge \bar{b})$

$= F \vee F$

$= F$

$\therefore$ It is contradiction.

(d) $(p \rightarrow q) \vee (r \rightarrow (q \wedge p))$

$= (\bar{p} \vee q) \vee (\bar{r} \vee (q \wedge P))$ From definition of $\rightarrow$

$= (\bar{p} \vee q) \vee \left[(\bar{r} \wedge q) \wedge (\bar{r} \vee p)\right]$

Distributive law

$= (\bar{p} \vee q \vee \bar{r} \vee q) \wedge (\bar{p} \vee q \vee \bar{r} \vee p)$

$= (\bar{p} \vee q \vee \bar{r}) \wedge T = \bar{p} \vee q \vee \bar{r}$

$= M_5 = \prod_5$

UEx. 1.6.4 **MU - Q. 3(a), May 17, 4 Marks**

Show that $(\sim p \wedge (\sim Q \wedge R)) \vee (Q \wedge R) \vee (p \wedge R) \Leftrightarrow R$

☑ **Soln. :**

To prove $(\sim p \wedge (\sim Q \wedge R)) \vee (Q \wedge R) \vee (p \wedge R) \Leftrightarrow R$

☞ **Proof**

LHS $= (\sim p \wedge (\sim Q \wedge R)) \vee (Q \wedge R) \vee (p \wedge R)$

$= [(\sim P \vee (Q \wedge R)) \wedge (\sim Q \vee (Q \wedge R)) \wedge (R \vee (Q \wedge R))] \vee (P \wedge R)$

$= [(\sim P \vee Q) \wedge (\sim P \vee R) \wedge (\sim Q \vee Q) \wedge (\sim Q \vee R) \wedge (R \vee Q) \wedge (R \vee R)] \vee (P \wedge R)$

$= [(\sim P \vee Q) \wedge (\sim P \vee R) \wedge T \wedge (\sim Q \vee R) \wedge (Q \vee R) \wedge (R)] \vee (P \wedge R)$

$= [(\sim P \vee Q) \wedge (\sim P \vee R) \wedge ((\sim Q \wedge Q) \vee R) \wedge R] \vee (P \wedge R)$

$= [(\sim P \vee Q) \wedge (\sim P \vee R) \wedge (F \vee R) \wedge R] \vee (P \wedge R)$

$= [(\sim P \vee Q) \wedge (\sim P \vee R) \wedge R \wedge R] \vee (P \wedge R)$

$= [(\sim P \vee Q) \wedge (\sim P \vee R) \wedge R] \vee (P \wedge R)$

$= (\sim P \vee Q \vee P) \wedge (\sim P \vee Q \vee R) \wedge (\sim P \vee R \vee P) \wedge (\sim P \vee R \vee R) \wedge (R \vee P) \wedge (R \vee R)$

$= T \wedge (\sim P \vee Q \vee R) \wedge T \wedge (\sim P \vee R) \wedge (P \vee R) \wedge R$

$= (\sim P \vee Q \vee R) \wedge (\sim P \vee R) \wedge (P \vee R) \wedge R$

$= (\sim P \vee R) \wedge (P \vee R) \wedge R$

$= [(\sim P \wedge P) \vee (\sim P \wedge R) \vee (R \wedge P) \vee (R \wedge R)] \wedge R$

$= [F \vee (\sim P \wedge R) \vee (P \wedge R) \vee R] \wedge R$

$= (\sim P \wedge R \wedge R) \vee (P \wedge R \wedge R) \vee (R \wedge R)$

$= (\sim P \wedge R) \vee (P \wedge R) \vee R$

$= ((\sim P \vee P) \wedge R) \vee R$

$= (T \wedge R) \vee R$

$= R \vee R = R$

$=$ RHS

Hence the proof.

UEx. 1.6.5 **MU - Q. 5(d), Dec. 13, 4 Marks**

Use the laws of logic to show that $[(p \Rightarrow q) \wedge \neg p \neg q] \Rightarrow$ is a tautology

☑ **Soln. :**

To prove $[(p \rightarrow q) \wedge 7q \rightarrow 7p]$ is a tautology. Proof using laws of logic.

Consider $((p \rightarrow q) \wedge 7q) \rightarrow 7p$

$= ((7p \vee q) \wedge 7q) \rightarrow 7p \quad (p \rightarrow q = 7p \vee q)$

$= (7p \wedge 7q) \rightarrow 7p \quad ((7p \vee q) \wedge 7q = 7p \wedge 7q)$

$= 7(7p \wedge 7q) \vee 7p$

$= (p \vee q) \vee 7p$ (De Morgan's law)

$= (q \vee p) \vee 7p$ (Commutative law)

$= q \vee (p \vee 7p)$ (associate law)

$= q \vee T$

$= T$

$\therefore ((p \rightarrow q) \wedge 7q \rightarrow 7p)$ is tautology

UEx. 1.6.6 **MU - Q. 6(b), May 14, 6 Marks**

Use the laws of logic to determine the following expression as tautology or contradiction. $[p \wedge (p \Rightarrow q)] \Rightarrow q$

Soln. :

Consider $(p \wedge (p \rightarrow q)) \rightarrow q$

$= (p \wedge (7p \vee q)) \rightarrow q$ $[p \rightarrow q = 7p \vee q]$

$= (p \wedge q) \rightarrow q$ $[p \wedge (7p \vee q) \rightarrow p \wedge q]$

$= 7(p \wedge q) \vee q$

$= (7p \vee 7q) \vee q$ [De Morgan's law]

$= 7p \vee (7q \vee q)$ [Associative law]

$= 7p \vee T$

$= T$

$\therefore (p \wedge (p \rightarrow q)) \rightarrow q$ is tautology

UEx. 1.6.7 **MU - Q. 6(c), Dec. 15, 5 Marks**

Show that $(\neg q \wedge (p \Rightarrow q)) \Rightarrow \neg p$ is a tautology

Soln. :

To prove, $(7q \wedge (p \rightarrow q)) \rightarrow 7p$ is a tautology. Proof using laws of logic.

Consider $(7q \wedge (p \rightarrow q)) \rightarrow 7p$

$= (7q \wedge (7p \vee q)) \rightarrow 7p$ $(p \rightarrow q = 7p \vee q)$

$= (7q \wedge 7p) \rightarrow 7p (7q \wedge (7p \vee q) = (7q \wedge 7p))$

$= 7(7q \wedge 7p) \vee 7p$

$= (q \vee p) \vee 7p$ (De Morgan's law)

$= q \vee (p \vee 7p)$ (Associate law)

$= q \vee T$

$= T$

$\therefore (7q \wedge (p \rightarrow q)) \rightarrow 7p$ is a tautology.

1.7 PREDICATE LOGIC

1.7.1 Predicate

Definition : A predicate is a generalization of a proposition. It is denoted by P (x), P (x, y), P (x, y, z) etc. where x, y, z are the arguments. P is either true or flax depending on the values assigned to the arguments.

For example : Consider the following propositions,

"apple is sweet"

"grape is sour"

"Mango is sweet"

"tamarind is sour"

Now let there be a predicate P(x) given as,

P(x) : x is sweet.

Now P(x) is true, if a x is either apple or mango.

If x is grape or tamarind, P(x) is false.

Let the following be predicates.

(i) $P(x) : x + B = 10$

(ii) $Q(x, y) : x + y = 10$

(iii) $R(x, yz) : x + y = z$

The predicate P (x) has only one argument, x, P(x) is true if $x = z$, otherwise it is false.

The predicate Q (x, y) has two arguments, x and y. Assuming that x, y are integers, Q (x, y) is true for 11 combinations of x and y. They are (0, 10), (1, 9), (2, 8) ....., (10, 0).

The predicate R (x, y, z) has three arguments. R (x, y, z) is true for infinite number of combinations of x, y, and z.

Definition : The method of assigning or binding the variables in a predicate is known as quantification. There are two types of quantification; they are universal and existential.

1.7.2 Universal Quantifier

Consider a predicate P(x). If it is true for all values of x within a given boundary, then x is said to be universally quantified.

It is written as $\forall_x$ and read as "for all x", "for every x" or "for each x".

For example

(i) Let there be a predicate

$P(x) : x^2 \geq 0.$

This is true for any value of x.

$\therefore$ we can write

$\forall_x$., P (x) is true.

(ii) Let P (x) : x2 ≤ 100.

This is true if x is in range of – 10 to 10.

$\therefore$ we can write,

$\forall_x$, $-10 \leq x \leq 10$ P(x) is true.

1.7.3 Existential Quantifier

Consider a predicate P(x). If it is not true for all possible values of n, but true for some value of x, then x is said to be existentially quantified.

It is written as $\exists_x$ and read as "for some x" or "there exists a value of x".

For example

Let there be a predicate

$P(x) : x^2 - 1 = 0.$

This is true for $x = \pm 1$.

Thus we can write.

$\exists_x$, P (x) is true.

We can combine both universal and existential quantifiers

Ex. : Let P (x, y) be a predicate, given by

$P(x) : x + y = 8.$

Here we can put any value in x, and find value for y that satisfy the predicate.

i.e. $y = 8 - x$.

Thus it can be written as,

$\forall_x \cdot \exists_x$ P (x, y) is true

or

$\forall_y \cdot \exists_x$ P (x, y) is true

We can negate a quantifier as follows :

$(\forall_x P(x)) = \exists_x (-P(x))$

$(\exists_x P(x)) = \forall_x (-P(x))$

P (x) is read as P (x) is false.

1.7.4 Solved Examples on Predicates

Ex. 1.7.1 : Translate these statements into English, for the given predicates.

A (x) : x is a cat

B(x) : x is an animal.

Soln. :

a. $\forall_x (A(x) \wedge B(x))$

= For all x is a cat and it is an animal

= It is true that every cat is an animal.

c. $\exists_x (A(x) \rightarrow B(x))$

= For some x x is a cat implies that x is as animal.

= There exists a can which is an animal

Ex. 1.7.2 : Let the following be the predicates.

P (x) : x is in the class

Q (x) : x can speak Oriya

R (x) : x is born in India

S (x) : x has been in NCC.

T (x) : x is honest.

Soln. :

a. **Given expression is**

Someone in your class can speak Oria.

$\exists_x (P(x)) \wedge Q(x)$

b. **Given expression is,**

Ever4yone in your class is nonest

$\forall_x (P(x) \wedge T(x))$

c. **Given expression is,**

There is a person in the class who was not born in India.

$\exists_x (P(x) \wedge \sim R(x))$

d. Given expression is,

A student in your class has been in NCC.

$\exists_x (P(x)) \wedge S(x)$

Ex. 1.7.3 : Translate these statements into English, Let Q(x, y) = 'x has sent a letter to y' where universe of discourse for both x and y consist of all students in your class

(a) $\exists_x \exists_y Q(x, y)$ (b) $\exists_x \forall_y Q(x, y)$.

(c) $\forall_x \exists_y Q(x, y)$ (d) $\exists_y \exists_x Q(x, y)$

(e) $\exists_y \forall_x Q(x, y)$ (f) $\forall_y \exists_x Q(x, y)$

☑ **Soln. :** Given Q (x,) = x has sent a letter to y.

a. $\exists_x \exists_y Q(x, y)$

= There exists a student who has sent a letter to another.

b. $\exists_x \forall_y Q(x, y)$.

= There exists a student who has sent a letter to all students.

c. $\forall_x \exists_y Q(x, y)$

= Every student has sent letter to one student.

d. $\exists_y \exists_x Q(x, y)$

= There exists a student who has received a letter from one student.

e. $\exists_y \forall_x Q(x, y)$

= There exists a student who has received a letter from all students.

f. $\forall_y \exists_x Q(x, y)$

= Every student has received a letter from one student.

Syllabus Topic : Principles of Mathematical Induction

Proof techniques

In mathematics, when a theoretical concept is presented, it should be supported with the help of a proof, so that it is universally accepted. There are many techniques to prove a theory or formula. In this section, we study different proof techniques and example of their applications.

1.8 MATHEMATICAL INDUCTION (MI)

- Mathematical induction is one of the widely used proof techniques. It is very simple, generally used to prove a given equation.
- Mathematical induction has following three steps.

▶ **Step 1 : Basis of induction :** In this step, the given equation is proved by substituting the basis value number i.e. the minimum value for the variable, n. This minimum value will be given in the problem statement.

▶ **Step 2 : Induction step :** In this step the given equation is assumed to be true, for a value K of n.

▶ **Step 3 : Inducting hypothesis step :** In this step, the given equation is proved for n = k + 1. To do so, we use the assumption in step 2.

- Hence the given equation is true for $n = n_0$
- If it is true for n = k, then it is true for n = k + 1.
- Thus as per induction hypothesis, the given equation is true for $n \geq n_0$

  i.e. It is true for $n_0, n_0 + 1, n_0 + 2$.....

1.8.1 Solved Examples

Ex. 1.8.1 : Prove that $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$, $n \geq 1$ using MI.

☑ **Soln. :**

Let E be the equation $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

▶ **Step 1 : Basis step :** For n = 1, we have

$$1 = \frac{1 \times (1+1)}{2} = \frac{1 \times 2}{2} = 1$$

∴ E is true for n = 1

▶ **Step 2 : Induction step**

Assume that E is true for n = k

$$\therefore 1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}$$

▶ **Step 3 : Induction Hypothesis**

Prove that E is true for n = k + 1

i.e. $1 + 2 + 3 + \ldots + k + 1 = \frac{(k+1)(k+1+1)}{2}$

Proof : L.H.S $= 1 + 2 + 3 + \ldots + k + 1$

$= 1 + 2 + 3 + \ldots + k + k + 1$

$= \frac{k(k+1)}{2} + k + 1$ (from step 2)

$= \frac{k(k+1) + 2(k+1)}{2}$

$= \frac{(k+1)(k+2)}{2}$

$= \frac{(k+1)(k+1+1)}{2}$

$= \text{RHS}$

$\therefore$ E is true for n = k + 1

Thus as per induction hypothesis E is true for all $n \geq 1$

$\therefore 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}, n \geq 1$

Hence the proof.

Ex. 1.8.2 : Prove that $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$n \geq 1$ by MI.

☑ **Soln. :**

Let E be the equation

$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

▶ **Step 1 : Basis step**

For n = 1, we have

$1 = \frac{1 \times (1+1) \times (2 \times 1 + 1)}{6} = \frac{1 \times 2 \times 3}{6} = 1$

$\therefore$ E is true for n = 1

▶ **Step 2 : Induction step**

Assume that E is true for n = k

$\therefore 1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$

▶ **Step 3 : Induction Hypothesis**

Prove that E is true for n = k + 1

i.e. $1^2 + 2^2 + \ldots + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

☞ **Proof :**

LHS $= 1^2 + 2^2 + \ldots + (k+1)^2$

$= 1^2 + 2^2 + \ldots + k^2 + (k+1)^2$

$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$ (From steps 2)

$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$

$= \frac{(k+1)(k(2k+1) + 6(k+1))}{6}$

$= \frac{(k+1)(2k^2 + k + 6k + 6)}{6}$

$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$ $\left\{ \begin{array}{l} 2k^2 + 7k + 6 \\ = 2k^2 + 4k + 3k + 6 \\ = 2k(k+2) + 3(k+2) \\ = (k+2)(2k+3) \end{array} \right.$

$= \frac{(k+1)(k+2)(2k+3)}{6}$

$= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

$= \text{RHS}$

$\therefore$ E is true for n = k + 1

Thus as per induction hypothesis, E is true for all $n \geq 1$

$\therefore 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}, n \geq 1$

Hence the proof.

Ex. 1.8.3 : Prove by induction that for $n \geq 0$, and $a \neq 1$

$1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a}$

☑ **Soln. :**

Let E be the equation

$1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a}, n \geq 0, a \neq 1$

▶ **Step 1 : Basis step**

For n = 1, we have

$1 = \frac{1 - a^{0+1}}{1 - a} = \frac{1 - a}{1 - a} = 1$

$\therefore$ E is true for n = 1

▶ **Step 2 : Induction step**

Assume that E is true for n = k

$\therefore 1 + a + a^2 + \ldots + a^k = \frac{1 - a^{k+1}}{1 - a}$

▶ **Step 3 : Induction hypothesis**

Prove that E is true for n = k + 1

i.e. $1 + a + a^2 + \ldots + a^{k+1} = \dfrac{1 - a^{k+1+1}}{1 - a}$

☞ **Proof**

$$\begin{aligned}
\text{LHS} &= 1 + a + a^2 \ldots + a^{k+1} \\
&= 1 + a + a^2 \ldots + a^k + a^{k+1} \\
&= \frac{1 - a^{k+1}}{1 - a} + a^{k+1} \quad \text{(From steps 2)} \\
&= \frac{1 - a^{k+1} + a^{k+1}(1 - a)}{1 - a} \\
&= \frac{1 - \cancel{a^{k+1}} + \cancel{a^{k+1}} - a \cdot a^{k+1}}{1 - a} = \frac{1 - a^{k+2}}{1 - a} \\
&= \frac{1 - a^{k+1+1}}{1 - a} = \text{RHS}
\end{aligned}$$

∴ E is true for n = k + 1

Thus as per induction hypothesis, E is true for all $n \geq 1$

$\therefore 1 + a + a^2 + \ldots + a^n = \dfrac{1 - a^{n+1}}{1 - a}, n \geq 0, a \neq 1$

Hence the proof.

Ex. 1.8.4 : Show that $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 = n \dfrac{(n+1)}{2} (-1)^{n-1}, n \geq 1$

☑ **Soln. :**

Let E be the equation

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 = n \frac{(n+1)}{2} (-1)^{n-1}$$

▶ **Step 1 : Basis step**

For n = 1, we have

$$1 = \frac{(1+1)}{2} (-1)^{1-1} = \frac{2}{2} (-1)^0 = 1$$

∴ E is true for n = 1

▶ **Step 2 : Induction step**

Assume that E is true for n = k

$$\therefore 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1} k^2 = \frac{k(k+1)}{2} (-1)^{k-1}$$

▶ **Step 3 : Induction Hypothesis**

Prove that E is true for n = k + 1

i.e. $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^k (k+1)^2 = \dfrac{(k+1)(k+1+1)}{2} (-1)^k$

☞ **Proof**

$$\begin{aligned}
\text{LHS} &= 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^k (k+1)^2 \\
&= 1^2 - 2^2 + 3^2 - 4^2 + \ldots (-1)^{k-1} k^2 + (-1)^k (k+1)^2 \\
&= \frac{k(k+1)}{2} (-1)^{k-1} + (-1)^k (k+1)^2 \quad \text{(From steps 2)} \\
&= \frac{k(k+1)(-1)^{k-1} + 2(-1)^k (k+1)^2}{2} \\
&= \frac{(-1)^k (k+1)}{2} [-k + 2(k+1)] \\
&= \frac{(-1)^k (k+1)(-k + 2k + 2)}{2k} \\
&= \frac{(-1)^k (k+1)(k+1+1)}{2} = \text{RHS}
\end{aligned}$$

∴ E is true for n = k + 1

Thus as per induction hypothesis, E is true for all $n \geq 1$

$\therefore 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 = (-1)^{n-1} \dfrac{n(n+1)}{2}$, $n \geq 1$

Hence the proof.

Ex. 1.8.5 : Show that $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$

☑ **Soln. :**

Let E be the equation

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

In this case n_0 is not given.

By observing the equation we can find that $n_0 = 1$

▶ **Step 1 : Basis step**

For n = 1, we have

$$1^2 = \frac{1(2 \times 1 - 1)(2 \times 1 + 1)}{3} = \frac{1 \times 1 \times 3}{3} = 1$$

∴ E is true for n = 1

▶ **Step 2 : Induction step**

Assume that E is true for n = k

$$\therefore 1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$$

▶ **Step 3 : Induction Hypothesis**

Prove that E is true for n = k + 1

$$1^2+3^2+5^2+\ldots+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3}$$

i.e. $1^2+3^2+5^2+\ldots+(2(k-1)-1)^2=\frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$

☞ **Proof :**

$$\text{LHS} = 1^2+3^2+5^2+\ldots+(2(k+1)-1)^2$$

$$= 1^2+3^2+5^2+\ldots+(2k-1)^2+(2(k+1)-1)^2$$

$$= \frac{k(2k-1)(2k+1)}{3}+(2(k+1)-1)^2$$

(From steps 2)

$$= \frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}$$

$$= \frac{(2k+1)[k(2k-1)+3(2k+1)]}{3}$$

$$= \frac{(2k+1)[2k^2-k+6k+3]}{3}$$

$$= \frac{(2k+1)(2k^2+5k+3)}{3}$$

$$\left\{\begin{array}{l} 2k^2+5k+3 \\ = 2k^2+2k+3k+3 = 2k(k+1)+3(k+1) \\ = (k+1)(2k+3) \end{array}\right.$$

$$= \frac{(2k+1)(k+1)(2k+3)}{3}$$

$$= \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

$$= \text{RHS}$$

$\therefore$ E is true for $n = k+1$

Thus as per induction hypothesis, E is true for all $n \geq 1$

$$1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3} \quad n \geq 1$$

Hence the proof.

Ex. 1.8.6 : Show that $1^3+2^3+\ldots+n^3=(1+2+3+\ldots+n)^2, n \geq 1$

☑ **Soln. :**

Let E be the equation

$$1^3+2^3+\ldots+n^3=(1+2+3+\ldots+n)^2$$

▶ **Step 1 : Basis step**

For $n = 1$, we have

$$1^3 = 1^2$$

$\therefore$ E is true for $n = 1$

▶ **Step 2 : Induction step**

Assume that E is true for $n = k$

$$\therefore 1^3+2^3+\ldots+k^3=(1+2+3+\ldots+k)^2$$

We know that, $1+2+3+\ldots+=\frac{n(n+1)}{2}$

$\therefore$ we get

$$1^3+2^3+\ldots+k^3=\left[\frac{k(k+1)}{2}\right]^2$$

▶ **Step 3 : Induction Hypothesis**

Prove that E is true for $n = k+1$

i.e. $1^3+2^3+\ldots+(k+1)^3=\left[\frac{(k+1)(k+1+1)}{2}\right]^2$

☞ **Proof**

$$\text{LHS} = 1^3+2^3+\ldots+(k+1)^3$$

$$= 1^3+2^3+\ldots+k^3+(k+1)^3$$

$$= \left[\frac{k(k+1)}{2}\right]^2+(k+1)^3 \quad \text{(From steps 2)}$$

$$= \frac{k^2(k+1)^2+4(k+1)^3}{4}=\frac{(k+1)^2[k^2+4(k+1)]}{4}$$

$$= \frac{(k+1)^2}{4}(k^2+4k+4)$$

$$= \frac{(k+1)^2}{4}(k+2)^2=\left[\frac{(k+1)(k+1+1)}{2}\right]^2$$

$$= [1+2+3+\ldots+k+1]^2 = \text{RHS}$$

$\therefore$ E is true for $n = k+1$

Thus as per induction hypothesis, E is true for all $n \geq 1$

$$1^3+2^3+\ldots+n^3 = (1+2+3+\ldots+n)^2, \; n \geq 1$$

Hence the proof.

Ex. 1.8.7 : Show that $\frac{1^2}{1.3}+\frac{2^2}{3.5}+\ldots+\frac{n^2}{(2n-1)(2n+1)}=\frac{n(n+1)}{2(2n+1)}, n \geq 1$

☑ **Soln. :** Let E be the equation

$$\frac{1^2}{1.3}+\frac{2^2}{3.5}+\ldots+\frac{n^2}{(2n-1)(2n+1)}=\frac{n(n+1)}{2(2n+1)}$$

▶ **Step 1 : Basis step**

For $n = 1$, we have

$$\text{LHS} = \frac{1^2}{1.3}=\frac{1}{3}$$

$$\text{RHS} = \frac{1(1+1)}{2(2\times 1+1)} = \frac{2}{2\times 3} = \frac{1}{3} = \text{LHS}$$

$\therefore$ E is true for n = 1

▶ **Step 2 : Induction step**

Assume that E is true for n = k

$$\therefore \frac{1^2}{1.3} + \frac{2^2}{3.5} + \ldots + \frac{k^2}{(2k-1)(2k+1)} = \frac{k(k+1)}{2(2k+1)}$$

▶ **Step 3 : Induction Hypothesis**

Prove that E is true for n = k + 1

$$\text{i.e. } \frac{1^2}{1.3} + \frac{2^2}{3.5} + \ldots + \frac{(k+1)^2}{[2(k+1)-1][2(k+1)+1]} = \frac{(k+1)(k+1+1)}{2(2(k+1)+1)}$$

☞ **Proof**

$$\text{LHS} = \frac{1^2}{1.3} + \frac{2^2}{3.5} + \ldots + \frac{(k+1)^2}{(2(k+1)-1)(2(k+1)+1)}$$

$$= \frac{1^2}{1.3} + \frac{2^2}{3.5} + \ldots + \frac{k^2}{(2k-1)(2k+1)} + \frac{(k+1)^2}{(2(k+1)-1)(2(k+1)+1)}$$

$$= \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2(k+1)-1)(2(k+1)+1)}$$

(From steps 2)

$$= \frac{k(k+1)}{2(2k+1)} + \frac{(k+1)^2}{(2k+1)(2k+3)}$$

$$= \frac{k(k+1)(2k+3) + 2(k+1)^2}{2(2k+1)(2k+3)}$$

$$= \frac{(k+1)[k(2k+2) + 2(k+1)]}{2(2k+1)(2k+3)}$$

$$= \frac{(k+1)(2k^2+3k+2k+2)}{2(2k+1)(2k+3)} \quad \begin{cases} 2k^2+3k+2k+2 \\ 2k^2+4k+k+2 \\ 2k(k+2)+1(k+2) \\ (k+2)(2k+1) \end{cases}$$

$$= \frac{(k+1)(2k+1)(k+2)}{2(2k+1)(2k+3)}$$

$$= \frac{(k+1)(k+1+1)}{2(2(k+1)+1)} = \text{RHS}$$

$\therefore$ E is true for n = k + 1

Thus as per induction hypothesis, E is true for all $n \geq 1$

$$\therefore \frac{1^2}{1.3} + \frac{2^2}{3.5} + \ldots + \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}, n \geq 1$$

Hence the proof.

Ex. 1.8.8 : Show that $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ $n \geq 1$

☑ **Soln. :**

Let E be the equation

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

▶ **Step 1 : Basis step**

For n = 1, we have

$$\text{LHS} = \frac{1}{1.2} = \frac{1}{2}$$

$$\text{RHS} = \frac{1}{1+1} = \frac{1}{2} = \text{LHS}$$

$\therefore$ E is true for n = 1

▶ **Step 2 : Induction step**

Assume that E is true for n = k

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

▶ **Step 3 : Induction Hypothesis**

Prove that E is true for n = k + 1

$$\text{i. e. } \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{(k+1)(k+1+1)} = \frac{k+1}{k+1+1}$$

☞ **Proof**

$$\text{LHS} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{(k+1)(k+1+1)}$$

$$= \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+1+1)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+1+1)} \quad \text{(From step 2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} = \frac{k+1}{k+1+1} = \text{RHS}$$

$\therefore$ E is true for n = k + 1

Thus as per induction hypothesis, E is true for all $n \geq 1$

$$\therefore \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}, n \geq 1$$

Hence the proof.

Ex. 1.8.9 : Show that $\frac{1}{1.3}+\frac{1}{3.5}+\ldots\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$ $n \geq 1$

Soln. :

Let E be the equation

$$\frac{1}{1.3}+\frac{1}{3.5}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

Step 1 : Basic step

For n = 1, we have

$$\text{LHS} = \frac{1}{1.3}=\frac{1}{3}$$

$$\text{RHS} = \frac{1}{2\times 1+1}=\frac{1}{3}=\text{LHS}$$

$\therefore$ E is true for n = 1

Step 2 : Induction step

Assume that E is true for n = k

$$\therefore \frac{1}{1.3}+\frac{1}{3.5}+\ldots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

Step 3 : Induction Hypothesis

Prove that E is true for n = k + 1

i.e. $\frac{1}{1.3}+\frac{1}{3.5}+\ldots\ldots+\frac{1}{[2(k+1)-1][2(k+1)+1]}$

$$=\frac{k+1}{2(k+1)+1}$$

Proof

$$\text{LHS} = \frac{1}{1.3}+\frac{1}{3.5}+\ldots\ldots+\frac{1}{[2(k+1)-1][2(k+1)+1]}$$

$$= \frac{1}{1.3}+\frac{1}{3.5}+\ldots\ldots+\frac{1}{(2k-1)(2k+1)}$$

$$+\frac{1}{(2(k+1)-1)+(2(k+1)+1)}$$

$$= \frac{k}{2k+1}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

(From steps 2)

$$= \frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$= \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}=\frac{k+1}{(2(k+1)+1)}=\text{RHS}$$

$2k^2+3k+1$

$$= 2k^2+2k+k+1$$

$$= 2k(k+1)+(k+1)=(2k+1)(k+1)$$

$\therefore$ E is true for n = k + 1

Thus as per induction hypothesis, E is true for all $n \geq 1$

$$\therefore \frac{1}{1.3}+\frac{1}{3.5}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1} \quad n \geq 1$$

Hence the proof.

Ex. 1.8.10 : Show that $1.2.3+2.3.4+3.4.5+\ldots.+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}$, $n \geq 1$

Soln. :

Let E be the equation

$$1.2.3+2.3.4+3.4.5+\ldots.+n(n+1)(n+2)$$

$$=\frac{n(n+1)(n+2)(n+3)}{4}$$

Step 1 : Basis step

For n = 1, we have

$$\text{LHS} = 1.2.3=6$$

$$\text{RHS} = \frac{1\times 2\times 3\times 4}{4}=6=\text{LHS}$$

$\therefore$ E is true for n = 1

Step 2 : Induction step

Assume that E is true for n = k

$$\therefore \quad 1.2.3+2.3.4+3.4.5+\ldots.+k(k+1)(k+2)$$

$$=\frac{k(k+1)(k+2)(k+3)}{4}$$

Step 3 : Induction Hypothesis

Prove that E is true for n = k + 1

i.e. $1.2.3+2.3.4+3.4.5+\ldots.+(k+1)(k+1+1)(k+1+2)$

$$=\frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}$$

Proof

$$\text{LHS} = 1.2.3+2.3.4+3.4.5+\ldots.+(k+1)(k+1+1)(k+1+2)$$

$$=1.2.3+2.3.4+3.4.5+\ldots,+k(k+1)(k+2)+(k+1)(k+2)(k+3)$$

$$=\frac{k(k+1)(k+2)(k+3)}{4}+(k+1)(k+2)(k+3)$$

(From steps 2)

$$=(k+1)(k+2)(k+3)\left[\frac{k}{4}+1\right]$$

$$=\frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

$$=\frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}=\text{RHS}$$

$\therefore$ E is true for $n = k + 1$

Thus as per induction hypothesis, E is true for all $n \geq 1$

$\therefore$ $1.2.3 + 2.3.4 + 3.4.5 + ... + n(n+1)(n+2)$

$$= \frac{n(n+1)(n+2)(n+3)}{4}, n \geq 1$$

Hence the proof.

Ex. 1.8.11 : Show that $n^3 + 2n$ is divisible by 3 for all $n \geq 1$ by induction.

Soln. :

We need to prove $n^3 + 2n$ is divisible by 3

▶ **Step 1 : Basis step**

For $n = 1$, we have

$n^3 + 2n = 1^3 + 2 \times 1 = 3$, divisible by 3

$\therefore$ $n^3 + 2n$ is divisible by 3 for $n = 1$

▶ **Step 2 : Induction step**

Assume that $n^3 + 2n$ is divisible by 3 for $n = k$

$$\therefore k^3 + 2k = 3x$$

▶ **Step 3 : Induction Hypothesis**

Consider $n^3 + 2n$ for $n = k + 1$

i. e. $(k+1)^3 + 2(k+1)$

$= k^3 + 1 + 3k(k+1) + 2k + 2$

$= k^3 + 2k + 3 + 3k^2 + 3k$ (Rearranging terms)

$= 3x + 3k^2 + 3k + 3$ (From step 2)

$= 3(x + k^2 + k + 1)$

$\therefore$ $(k+1)^3 + 2(k+1)$ is divisible by 3

Thus as per induction hypothesis, $n^3 + 2n$ is divisible by 3 for all $n \geq 1$

Ex. 1.8.12 : Show that $n^4 - 4n^2$ is divisible by 3 for all $n \geq 2$ by induction

Soln. :

We need to prove $n^4 - 4n^2$ is divisible by 3 here $n_0 = 2$.

▶ **Step 1 : Basis step**

For $n = 2$, we have

$2^4 - 4 \times 2^2$, $16 - 16 = 0$, divisible by 3

$n^4 - 4n^2$ is divisible, by 3 for $n = 2$

▶ **Step 2 : Induction step**

Assume that $n^4 - 4n^2$ is divisible by 3 for $n = k$

$$\therefore k^4 - 4k^2 = 3x$$

$$\therefore k^4 = 3x + 4k^2$$

▶ **Step 3 : Induction Hypothesis**

Consider $n^4 - 4n^2$ for $n = k + 1$

i.e. $(k+1)^4 - 4(k+1)^2$

$= (k^2 + 1 + 2k)(k^2 + 1 + 2k) - 4(k^2 + 1 + 2k)$

$= k^4 + k^2 + 2k^3 + k^2 + 1 + 2k + 2k^3$
$+ 2k + 4k^2 - 4k^2 - 4 - 8k$

$= k^4 + 4k^3 + 2k^2 - 4k - 3$

$= 3x + 4k^2 + 4k^3 + 2k^2 - 4k - 3$ (From step 2)

$= 4k^3 - 4k + 6k^2 + 3x - 3$

$= 4k(k^2 - 1) + 6k^2 + 3x - 3$

$= 4k(k+1)(k-1) + 6k^2 + 3x - 3$

$= 4(3y) + 6k^2 + 3x - 3$

$\because$ $(k-1)(k)(k+1)$ is product of 3 consecutive numbers, which is divisible by 3. Assume that $(k-1)k(k+1) = 3y$

$= 3(4y + 2k^2 + x - 1)$

$\therefore$ $(k+1)^4 - 4(k+1)^2$ is divisible by 3.

Thus as per induction hypothesis, $n^4 - 4n^2$ is divisible by3 for all $n \geq 2$

Ex. 1.8.13 : Show that $2^n \times 2^n - 1$ is divisible by 3 for all $n \geq 1$ by induction.

Soln. :

We need to prove that $2^n \times 2^n - 1$ is divisible by 3

▶ **Step 1 : Basis step**

For $n = 1$, we have

$2^1 \times 2^1 - 1 = 4 - 1 = 3$, divisible by 3

$\therefore$ $2^n \times 2^n - 1$ is divisible by 3 for $n = 1$

▶ **Step 2 : Induction step**

Assume that $2^n \times 2^n - 1$ is divisible by 3 for $n = k$

$$\therefore 2^k \times 2^k - 1 = 3x$$

$\therefore \quad 2^k \times 2^k = 3x + 1$

▶ **Step 3 : Induction Hypothesis**

Consider $2^n \times 2^n - 1$ for $n = k + 1$

i.e. $2^{k+1} + 2^{k+1} - 1$

$= 2^k \times 2^1 \times 2^k \times 2^1 - 1 = 2^k \times 2^k \times 2 \times 2 - 1$

$= (3x + 1) \times 4 - 1$ (from step 2)

$= 12x + 4 - 1$

$= 12x + 3 = 3(4x + 1)$

$\therefore 2^{k+1} + 2^{k+1} - 1$ is divisible by 3

Thus as per induction hypothesis,

$2^n \times 2^n - 1$ is divisible by 3 for all $n \geq 1$.

Ex. 1.8.14 : Show that sum of cubes of three consecutive integers is divisible by q, using induction.

☑ **Soln. :**

To prove the sum of cubes of three consecutive integers is divisible by 9, we first consider, three such integers, $(n - 1)$, n and $(n + 1)$

(we can also take n, n + 1, n + 2 but requires more calculations) $\therefore (n-1)^3 + n^3 (n+1)^3$ is the required product and we should prove that it is divisible by 9.

Consider $(n-1)^3 + n^3 (n+1)^3$

$= n^3 - 1 - 3n^2 + 3n + n^3 + n^3 + 1 + 3n^2 + 3n$

$= 3n^3 + 6n = 3(n^3 + 2n)$

$\therefore (n-1)^3 + n^3 + (n+1)^3 = 3(n^3 + 2n)$

$\therefore$ Sum of cubes of three consecutive integers is divisible by 3. To prove that it is divisible by 9, we have to prove that $n^3 + 2n$ is divisible by 3. This is shown in example Ex. 11

$\therefore (n-1)^3 + n^3 + (n+1)^3$ is divisible by 9

Hence the proof.

Ex. 1.8.15 : Show that $n < 2^n$ for all $n \geq 0$.

☑ **Soln. :** We have to prove that for all $n \geq 0$, $n < 2^n$.

▶ **Step 1 : Basis step**

For $n = 0$, $0 < 2^0 \Rightarrow 0 < 1$

$\therefore n < 2^n$ is true for $n = 0$

▶ **Step 2 : Induction step**

Assume that $n < 2^n$ for $n = k$

i.e. $k < 2^k$

▶ **Step 3 : Induction Hypothesis**

We know that,

$k < 2^k$ (from step 2)

$\therefore k + 1 < 2^k + 1$

$\therefore k + 1 < 2^k + 2^k$ Inequality is true if you increase right hand side

$\therefore k + 1 < 2.2^k$

$\therefore k + 1 < 2^{k+1}$

$\therefore n < 2^n$ is true for $n = k + 1$.

$\therefore$ From induction hypothesis, $n < 2^n$ for all $n \geq 0$.

Ex. 1.8.16 : Show that $n^2 < n!$ for $n \geq 4$.

☑ **Soln. :**

We have to prove that $2^n < n!$ for $n \geq 4$.

In this case, $n_0 = 4$. (Check for $n = 3$, $2^3 > 3!$)

▶ **Step 1 : Basis step**

Consider $n = 4$ for 2^n and $n!$

$2^4 = 16$, $4! = 24$

$\therefore 2^4 < 4!$

$\therefore 2^n < n!$ for $n = 4$

▶ **Step 2 : Induction step**

Assume that $2^n < n!$ is true for $n = k$

$\therefore 2^k < k!$

▶ **Step 3 : Induction hypothesis**

We know that

$2^k < k!$ (from step 2)

$\therefore 2.2^k < 2 \times k!$

$\therefore 2^{k+1} < 2 \times k!$

$\therefore 2^{k+1} < (k+1) \times k!$ (Inequality is true if you increase RHS)

$\therefore 2^{k+1} < (k+1)!$

$\therefore 2^n < n!$ for $n = k + 1$

$\therefore$ From induction hypothesis, $2^n < n!$ for all $n \geq 4$

UEx.1.8.17 **MU - Q. 2(a), Dec. 13, 4 Marks**

Prove by mathematical induction $x^n - y^n$ is divisible by $x - y$.

Soln. :

To prove : $x^n - y^n$ is divisible by $x - y$. Proof using mathematical induction.

Let P(n) be $x^n - y^n$ is divisible by $x - y$.

(i) Basis step : Consider n = 1

$\therefore x^1 - y^1 = x - y$ is divisible by $x - y$

$\therefore$ P(1) is true.

(ii) Induction step

Assume that P(k) is true.

$\therefore x^k - y^k$ is divisible by $x - y$.

$\therefore (x^k - y^k) = a \cdot (x - y)$ where $a \in I$ and $a \neq 0$ ...(I)

(iii) Induction hypothesis

Consider $x^{k+1} - y^{k+1}$

$$= x \cdot x^k - y \cdot y^k$$

$$= x\,[a\,(x - y) + y^k] - y \cdot y^k \text{ from I}$$

$$= ax^2 - axy + x \cdot y^k - y \cdot y^k$$

$$= ax\,(x - y) + y^k (x - y)$$

$$= (ax + y^k)\,(x - y)$$

$\therefore x^{k+1} - y^{k+1}$ is divisible by $x - y$

$\therefore$ P(k + 1) is true.

$\therefore$ From the principle of mathematical induction, P(n) is true for $n \geq 1$.

$\therefore x^n - y^n$ is divisible by $x - y$.

UEx. 1.8.18 **MU - Q. 1(a), May 14, 5 Marks**

Prove that $8^n - 3^n$ is a multiple of 5 by mathematical induction, $n \geq 1$.

Soln. :

To prove $8^n - 3^n$ is a multiple of 5. Proof using mathematical induction.

Let P(n) be, $8^n - 3^n$ is multiple of 5.

(i) Basis step : Consider n = 1

$\therefore 8^1 - 3^1 = 5$ is multiple of 5.

$\therefore$ P(1) is true.

(ii) Induction step

Assume that P(k) is true

$\therefore 8^k - 3^k$ is multiple of 5

$\therefore 8^k - 3^k = 5a$ where $a \in I$ and $a \neq 0$ ...(I)

(iii) Induction hypothesis

Consider $8^{k+1} - 3^{k+1}$

$$= 8.8^k - 3.3^k$$

$$= 8\,(5a + 3^k) - 3.3^k$$

$$= 40\,a + 8.3^k - 3.3^k$$

$$= 40a + 3^k\,(8 - 3)$$

$$= 40\,a + 5.3^k$$

$$= 5\,(8a + 3^k)$$

$\therefore 8^{k+1} - 3^{k+1}$ is multiple of 5,

$\therefore$ P(k + 1) is true

$\therefore$ From the principle of mathematical induction, P(n) is true for $n \geq 1$

$\therefore 8^n - 3^n$ is multiple of 5.

UEx. 1.8.19 **MU - Q. 1(b), May 16, 6 Marks**

By mathematical induction prove that $1 + a + a^2 + \ldots a^n = \dfrac{1 - a^{n+1}}{1 - a}$ where $n \geq 0$.

Soln. : Let P(n) be

$$1 + a + a^2 + \ldots + a^n = \frac{(1 - a^{n+1})}{1 - a} \text{ where } n \geq 0$$

Proof of P(n) using mathematical induction.

(i) Basis step : Let n = 0

Then LHS of P(1) = 1

RHS of $P(1) = \dfrac{1 - a^{0+1}}{1 - a} = \dfrac{1 - a}{1 - a} = 1$

$\therefore$ P(1) is true.

(ii) Induction step

Assume that P(k) is true.

i.e. $1 + a + a^2 + \ldots + a^k = \dfrac{1 - a^{k+1}}{1 - a}$

(iii) Induction hypothesis

Consider P(k + 1)

i.e. $1 + a + a^2 + \ldots + a^k + a^{k+1}$

$$= \frac{1-a^{k+1}}{1-a} + a^{k+1}$$
$$= 1 - a^{k+1} + a^{k+1} - a^{k+2}$$
$$= 1 - a^{(k+1)+1}$$

$\therefore$ P(k + 1) is true.

$\therefore$ From the principle of mathematical induction, P(n) is true for $n \geq 0$

$$\therefore 1 + a + a^2 + \ldots + a^n = \frac{1-a^{n+1}}{1-a}$$

Hence the proof.

UEx. 1.8.20 MU - Q. 2(b), Dec. 16, 6 Marks

Show that $n(n^2 - 1)$ is divisible by 24, where n is any odd positive integer.

☑ Soln. :

Let P(n) be $n(n^2 - 1)$ is divisible by 24 where n is any odd positive number.

Proof by mathematical induction.

(i) Basis step : Consider n = 1. (least odd positive number) then $P(1) = 1(1^2 - 1) = 0$ is divisible by 24.

(ii) Induction step

Assume that P(k) is true.

i.e. $k(k^2 - 1)$ is divisible by 24, where k is an odd positive integer.

$\therefore k(k^2 - 1) = 24a$ where $a \geq 1$

(iii) Induction hypothesis

Consider P(k + 2) (k + 2 is next odd number of k)

$$= (k+2)((k+2)^2 - 1)$$
$$= (k+2)(k^2 + 4 + 4k - 1)$$
$$= (k+2)(k^2 + 4k + 3)$$
$$= (k+2)(k+3)(k+1)$$
$$= (k+2)(k+1)(k+3)$$
$$= k(k+1)(k+3) + 2(k+1)(k+3)$$

[Distributive law]

$$= k(k+1)(k-1+4) + 2(k+1)(k+3)$$
$$= k(k+1)(k-1) + 4k(k+1) + 2(k+1)(k+3)$$
$$= k(k^2 - 1) + 2(k+1)(2k+k+3)$$
$$= 24a + 2(k+1) \cdot 3(k+1)$$
$$= 24a + 6(k+1)^2$$

We know that k is an odd number.

$\therefore$ k + 1 is an even number.

Let $k + 1 = 2x$

$\therefore$ Our equation becomes

$$24a + 6(2x)^2$$
$$= 24a + 6 \times 4x^2 = 24(a + x^2)$$

$\therefore$ P (k + 2) is divisible by 24.

$\therefore$ From the principle of mathematical induction, $n(n^2 - 1)$ is divisible by 24 where n is an odd positive integer.

UEx. 1.8.21 MU - Q. 1(a), May 17, 5 Marks

Prove using Mathematical Induction

$2 + 5 + 8 + \ldots + (3n - 1) = n(3n + 1)/2$

☑ Soln. :

Let P(n) be $2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n+1)}{2}$

Proof by mathematical induction.

(i) Basis step

Consider n = 1

Then LHS of P(1) $= 3 \times 1 - 1 = 2$

$$\text{RHS} = \frac{1(3 \times 1 + 1)}{2} = \frac{4}{2} = 2$$

$\therefore$ LHS = RHS

$\therefore$ P(n) is true for n = 1

(ii) Induction step

Assume that P(n) is true for n = k

i.e. $2 + 5 + 8 + \ldots + (3k - 1) = \frac{k(3k+1)}{2}$

(iii) Induction hypothesis

Consider p(k + 1)

$2 + 5 + 8 + \ldots + (3k - 1) + (3(k+1) - 1)$

$$= \frac{(k+1)(3(k+1)+1)}{2}$$

$$\text{LHS} = \frac{k(3k+1)}{2} + (3k + 3 - 1)$$
$$= \frac{k(3k+1)}{2} + (3k+2)$$
$$= \frac{3k^2 + k + 6k + 4}{2} = \frac{3k^2 + 7k + 4}{2}$$
$$= \frac{3(k+1)(3k+4)}{2}$$
$$= \frac{(k+1)(3(k+1)+1)}{2} = \text{RHS}$$

From the principle of mathematical induction, p(n) is true for $n \geq 1$

$$\therefore 2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n+1)}{2}$$

Chapter Ends...

□□□

Note

Module II

CHAPTER 2 Relations and Functions

Syllabus

Basic concepts of Set Theory,

Relations : Definition, Types of Relations, Representation of Relations, Closures of Relations, Warshall's algorithm, Equivalence relations and Equivalence Classes,

Functions : Definition, Types of functions, Composition of functions, Identity and Inverse function

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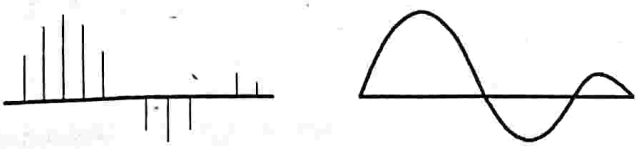
UEx. 2.21.36 **(MU - Q. 2(d), Dec. 17, 4 Marks)** 2-66
UEx. 2.21.37 **(MU - Q. 3(c), Dec. 17, 4 Mark)** 2-66

• Chapter Ends... 2-66

Introduction

This chapter deals with fundamentals of discrete structures. The word discrete indicates individual, separate and distinct. It is opposite to the word continuous. Fig. 1 shows the representation of discrete and continuous.



Discrete Continuous

Fig. 1

Syllabus Topic : Sets

2.1 SETS

property form
list form

2.1.1 Definition

Set is a collection of distinct elements or objects.

The word 'distinct' is important, because a set cannot have two copies of same object. For example, we cannot have set of 3 pens. But if we name each pen distinctly, then we can call the same collection as a set i.e. collection of 3 pens is not a set, whereas collection of pen 1, pen 2 and pen 3 is a set.

It is not necessary that there must be always a relation between the objects in a set. There can be a set which consists of an alphabet A, number 5, a symbol $, a name 'Ganga' etc.

2.1.2 Representation of Sets

A set can be represented in different ways, using different notations. The simplest way is to write the objects within braces. i.e. { }. The sets discussed in section 1.1.1 are represented as follows :

{pen 1, pen 2, pen 3}

{A, 5, $, Ganga}

Every set can be given a name, usually an uppercase alphabet. It is written as,

A = {a, b, c, d}

B = {1, 2, 3, 4, 5}

This method of set representation is known as listing method.

If the elements of a set share common characteristics, then they can be represented using either statement form or set-builder notation.

In statement form, a set is represented using a sentence that describes the common characteristics of elements.

For example

(i) Set A consists of all prime numbers less than 20.

(ii) Set B consists of divisors of 60.

Using listing method, the above sets are represented as,

(i) A = {2, 3, 5, 7, 11, 13, 17, 19}

(ii) B = {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}

In a set builder notation, a set is represented as {x | P(x)}, where x is considered as an element in the set, and P(x) is the property of x, which is common to all elements in the set. The symbol | is read as such that or where. The sets A and B discussed above are represented as,

(i) A = {x | x < 20 and x is prime}

(ii) B = {x | 60 is divisible by x}

Syllabus Topic : Countable

2.1.3 Types of Sets

Depending on the number of elements present in a set, a set can be one of the five different types. (Refer Fig. 2.1.1)

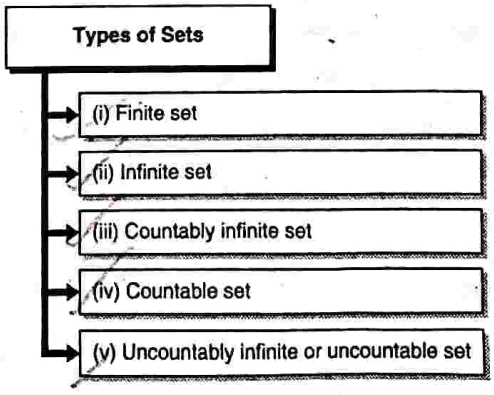


Fig. 2.1.1

Module II

➜ **(i) Finite set**

A set that consists of fixed number of elements.

Ex. : A = {2, 5, 7, 9, 12}

➜ **(ii) Infinite set**

When there is no limit to the number of elements in a set, it is known as infinite set.

Ex. N = {1, 2, 3, 4, ...}

The three dots ... indicate tending to infinity, the set has no end. This example gives a special set, known as set of natural numbers represented by N.

➜ **(iii) Countably infinite set**

It is an infinite set, but there exists a one to one correspondence (or mapping) between the elements of the set and the elements of N, set of natural numbers.

Ex. 1 : X = {– 1000, – 100, – 10, 0, 10, 100, 1000, ...}

Though this set is infinite, we can map it to N as follows :

X		N				
				100	→	6
– 1000	→	1		1000	→	7
– 100	→	2		10000	→	8
– 10	→	3		⋮		⋮
0	→	4		.		.
10	→	5		.		.

Ex. 2 : Y = {0, 1, 4, 9, 16, 25, 36, ...}

Is countably infinite because of the following mapping.

Y		N
0	→	1
1	→	2
4	→	3
9	→	4
16	→	5
25	→	6
36	→	7
⋮		⋮

➜ **(iv) Countable set**

A set is called countable, if it is finite or countably infinite.

➜ **(v) Uncountably infinite or uncountable set**

It is an infinite set, that cannot be mapped with set of natural numbers.

Ex. 1 : Z = {... – 2, – 1, 0, 1, 2, 3, ...}

We cannot have one to one correspondence between Z and N, because both first and last elements of Z tend to infinity.

Ex. 2 : M = {x | 0 < x < 1 and x is real number }

In this set, x can be any real number between 0 and 1.

i.e. M = {0.1, 0.01, 0.001, 0.2, 0.3, ...}

or M = {0.00001, 0.02, 0.0003, 0.2134, ...}

Thus, there is no fixed order for the elements and hence there cannot be a mapping with N. Therefore, M is an uncountable set.

Note :

Every finite set is countable but not vice versa.

Every uncountable set is infinite but not vice versa.

A countable set can be finite or infinite.

An infinite set can be countable or uncountable.

2.1.4 Number Sets

There are some standard sets of numbers. They are represented by unique names. These sets are infinite sets. They are as follow :

N – Set of natural numbers.

N = {1, 2, 3, ...}

W – Set of whole numbers

W = {0, 1, 2, 3, ...}

Z – Set of integers

Z = {..., – 3, –2, – 1, 0, 1, 2, 3, ...}

Q – Set of rational numbers

Q = {p / q | p, q are integers and q ≠ 0}

R – Set of real numbers.

$$R = \{\ldots, -0.1, \ldots, 0, 0.001, \ldots\}$$

The set names may have superscript + or –, indicating positive and negative numbers.

For example,

Q^+ is set of positive rational numbers.

R^- is set of negative real numbers.

2.1.5 Symbols Used in Sets

(i) Belongs to - $\in$

It is used to represent that an element belongs to a set.

For example consider a set A = {1, 2, 3, 4} then we can write,

$1 \in A$, $2 \in A$, $3 \in A$ and $4 \in A$.

(ii) Does not belong to - $\notin$

It is used to represent that an element does not belong to a set. For set A, considered in previous example we can write,

$5 \notin A$

(iii) For all - $\forall$

It is used to represent all elements in a set.

When we write $\forall\ a \in A$, it is read as for all a's that belong to A, a represents all the elements in A.

(iv) There exists or for some - $\exists$

It is used to represent some element in a set. When we write $\exists\ x \in A$, it is read as there exists an element x that belongs to A. Here x represents at least one element in A.

(v) Subset - $\subseteq$

When every element of set A is also an element of set B, then we say that set A is subset of set B. it is represented as $A \subseteq B$.

(vi) Not subset - $\nsubseteq$

If there exists at least one element in set A such that it is not present in set B, then we say that set A is not a subset of set B. It is represented as $A \nsubseteq B$.

(vii) Equal sets S - =

If the element in set A and set B are same then, we say that they are equal. It is represented as A = B. When two sets are equal, we can write A = B, $A \subseteq B$ and $B \subseteq A$.

(viii) Proper subset - $\subset$

A set A is called proper subset of set B, if A is subset of B and A is not equal to B. It is represented as $A \subset B$.

(ix) Unequal sets - $\neq$

If sets A and B have at least one element such that it is present in one set but not in other, they are called unequal sets. It is represented as $A \neq B$.

(x) Cardinality of a set - ||

Cardinality of a set represents the number of elements present in a set.

For example, if A = {5, 4, 3, 2} then cardinality of set A is 4. It is represented as $|A| = 4$.

(xi) Universal set - U

It consists of all possible elements under consideration. Any set A is always a subset of universal set. i.e. $A \subseteq U$.

(xii) Null set of empty set - ϕ

It is the set which does not contain any element. Cardinality of ϕ is 0.

$$\text{i.e. } \phi = \{\ \}$$

$$|\phi| = 0$$

If we have a set $A = \{\phi\}$ then, A is a set that has one element which is a null set.

$$\text{Now } |A| = 1$$

Null set is subset of any set A. i.e. $\phi \subseteq A$.

(xiii) Complement of a set

$\bar{A}$ has elements which are not present in A but present in the universal set.

(xiv) Power set of a set - P(A) or 2^A

If A is a set then the power set of A represents a set that consists of all possible subsets of A.

Example 1 : If $A = \{a, b, c\}$ then

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

$|P(A)| = 2^{|A|}$

In this case $|A| = 3$, $|P(A)| = 2^3 = 8$

Example 2

Consider the following sets

$A = \{1, 2, 3, 4, 5\}$

$B = \{2, 3, 4\}$

$C = \{4, 5, 6, 7\}$

$D = \{4, 2, 3\}$

$E = \{1, 2, 3, 4\}$

$F = \{4, 5\}$

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is universal set.

Following are true with respect to above sets.

$1 \in A$, $1 \in E$, $6 \notin A$, $5 \notin B$

$\forall a \in B$ we have $a \in A$,

$\exists a \in C$ we have $a \in B$,

$B \subseteq A$, $B \subset A$, $C \nsubseteq A$, $E \subseteq A$, $F \subseteq C$,

$C \neq F$, $B = D$, $D \subseteq B$, $|A| = 5$, $|F| = 2$

$|C| = |E|$

$\bar{A} = \{6, 7, 8, 9, 10\}$

$\bar{C} = \{1, 2, 3, 8, 9, 10\}$

Few more definitions

(i) Singleton set

A set containing only one element is known as singleton set.

Ex. : $A = \{1\}$ is singleton

(ii) Super set

If set A is subset of set B, then set B is called super set of A.

Ex. : Let $A = \{1, 2, 3\}$, $B = \{1, 3\}$ then A is superset of B.

(iii) Disjoint sets

If two sets have no common elements, then they are called disjoint sets.

Ex. : $A = \{1, 2, 3\}$ and $B = \{4, 5\}$ are disjoint sets.

> **Note :** $A \neq B$ does not mean that A and B are disjoint.
> But if A and B are disjoint, then $A \neq B$.
> **Example :** $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$
> In this case, $A \neq B$ but they are not disjoint.

Syllabus Topic : Venn Diagram

2.2 VENN DIAGRAM

Use : Venn diagram is used to represent a set pictorially.

Symbols used in a Venn diagram

(i) A rectangle : To represent universal set.

(ii) Ellipse or circle : To represent a set.

Example

Consider the following sets

Universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A = \{1, 3, 5, 7\}$ $B = \{2, 4, 6\}$

$C = \{1, 2, 3, 4\}$ $D = \{1, 3, 4, 5, 7\}$

The sets A, B, C, D are individually represented in Fig. 2.2.1.

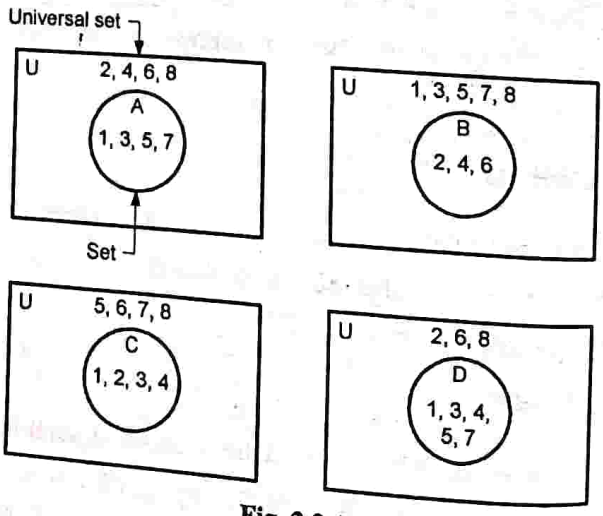


Fig. 2.2.1

To represent all the above sets in a single Venn diagram, we have to consider the following :

$1 \in A, C, D$

$2 \in B, C$

$3 \in A, C, D$

$4 \in B, C, D$

$5 \in A, D$

$6 \in B$

$7 \in A, D$

$8 \notin A, B, C, D$

It is clear from the above sets that, $A \subseteq D$ and A and B are disjoint sets. The following Venn diagram represents all these sets.

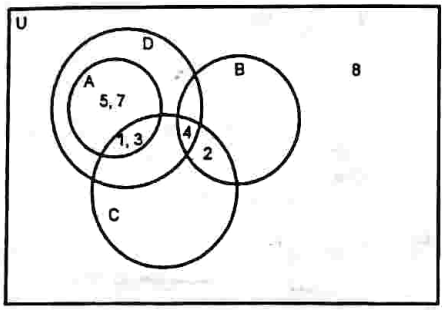


Fig. 2.2.2

2.3 SET OPERATIONS

A set can be considered as an operand, and we have different operations that can be performed on them. Following subsections discuss on these operations, their representation using Venn diagrams with examples.

2.3.1 Union of Sets - ∪

all together.

Union of two or more sets results into a set that includes all elements of those sets. i.e. if $a \in A$ or $a \in B$ then $a \in A \cup B$.

If an element belongs to both sets, then it is present only once in the union set.

Fig. 2.3.1 shows the Venn diagram for union of two sets. The shaded area represents the union.

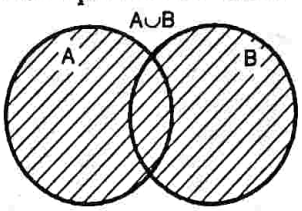


Fig. 2.3.1

Example

$$\text{Let } A = \{1, 2, 3, 4, 5\}$$
$$B = \{2, 4, 6, 8, 10\}$$
$$A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$$

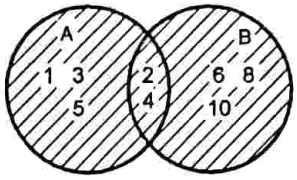


Fig. 2.3.2

Note that $A \cup B = B \cup A$

Consider n different sets $A_1, A_2, A_3, \ldots, A_n$. The union of these sets is represented as

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$$

2.3.2 Intersection of Sets - ∩

only intersection

Intersection of two or more sets results into a set that includes all elements that are present in all the sets.

i.e. if $a \in A$ and $a \in B$ then $a \in A \cap B$.

Following (Fig. 2.3.3) is the Venn diagram for intersection of two sets. The shaded area represents the intersection.

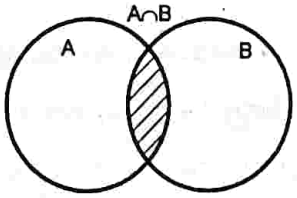


Fig. 2.3.3

Example

$$\text{Let } A = \{1, 2, 3, 4, 5\}$$
$$B = \{2, 4, 6, 8, 10\}$$
$$A \cap B = \{2, 4\}$$

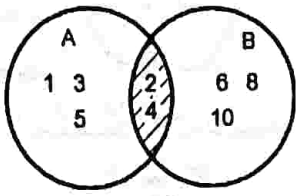


Fig. 2.3.4

Consider n different sets $A_1, A_2, A_3, \ldots, A_n$.

The intersection of these sets is represented as

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$$

Note that $A \cap B = B \cap A$

Two sets A and B are said to be disjoint sets, if $A \cap B = \phi$.

Ex. : $A = \{1, 2, 4, 6\}$, $B = \{3, 5\}$

2.3.3 Complement of a Set Or

The complement of a set is a set that includes all elements that are not present in the given set. Here we should consider the universal set for remaining elements.

i.e. if $a \notin A$ then $a \in \bar{A}$ or $a \in A'$

Fig. 2.3.6, Venn diagram represents the complement set.

Note that $\bar{\bar{A}} = (A')' = A$

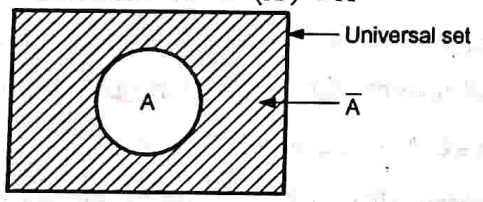


Fig. 2.3.5

Example

Le A be a set of names of months ending with 'y'.

i.e. A = {January, February, May, July}

Here universal set is names of all months.

$\therefore$ $\bar{A}$ = {March, April, June, August, September, October, November, December}

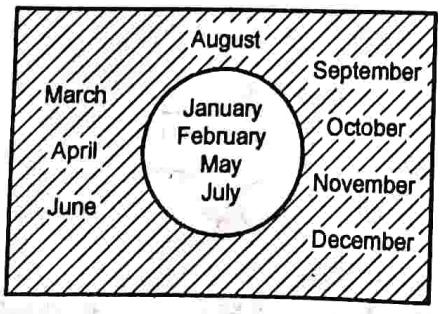


Fig. 2.3.6

2.3.4 Set Difference (–)

Consider two sets A and B. The difference operation A – B is a set that includes all elements in A that are not in B.

i.e. $A - B = \{x \mid x \in A \text{ and } x \notin B\} = A \cap \bar{B}$

Note : If $A \neq B$ then $A - B \neq B - A$

if $A = B$ then $A - B = B - A = \phi$

Following Fig. 2.3.8 is the Venn diagram that represents A – B

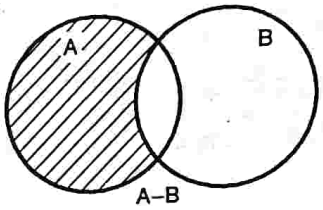


Fig. 2.3.7

Example

Let A = {1, 2, 3, 4, 5}

B = {2, 4, 6, 8, 10}

element present only in A

A – B = {1, 3, 5}

B – A = {6, 8, 10}

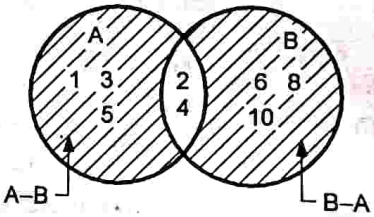


Fig. 2.3.8

2.3.5 Symmetric Difference - ⊕ or Boolean Sum

The symmetric difference of two sets A and B is the set that contains elements present either in A or in B but not in both sets. This is also known as **Boolean sum.**

i.e. $A \oplus B = \{x \mid (x \in A \text{ or } x \in B)\}$ and $(x \notin A \cap B)$

Thus, we can write that

$$A \oplus B = (A - B) \cup (B - A) = (A \cap \bar{B}) \cup (B \cap \bar{A})$$

Fig. 2.3.10 is the Venn diagram represents $A \oplus B$.

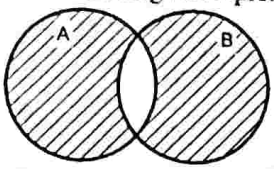


Fig. 2.3.9

Example

$A = \{1, 2, 3, 4, 5\}$

$B = \{2, 4, 6, 8, 10\}$

$A \oplus B = \{1, 3, 5, 6, 8, 10\}$

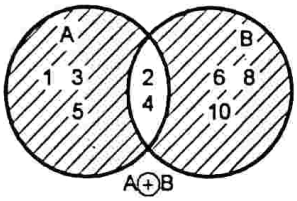


Fig. 2.3.10

Syllabus Topic : Cartesian Product

2.3.6 Cross Product or Cartesian Product : ×

Consider two sets A and B. The cross product or Cartesian product $A \times B$ is the set of all ordered pairs such that first element in the ordered pair is from A and second element is from B.

An ordered pair is pair of element represented as $< p, q >$. Remember that $< p, q > \neq < q, p >$.

$A \times B = \{< p, q > \mid p \in A \text{ and } q \in B\}$

It is clear that $|A \times B| = |A| \times |B|$

i.e. Number of elements in $A \times B$ is equal to the product of number of elements in A and B.

Example

Let $A = \{1, 2, 3, 4\}$

$B = \{2, 4, 6\}$

$A \times B = \{<1,2>, <1,4>, <1,6>, <2,2>, <2,4>, <2,6>, <3,2>, <3,4>, <3,6>, <4,2>, <4,4> <4,6>\}$

Here $|A| = 4$, $|B| = 3$ and $|A \times B| = 12$

Note that $A \times B \neq B \times A$ if $A \neq B$

2.3.7 Properties of Set Operations or Laws of Set Theory

Till now, we have considered 6 set operations. They are union (∪), intersection (∩), complement (′) or (–), set difference (–), symmetric difference or Boolean sum (⊕), and Cartesian or cross product (×.)

Following are the properties of these set operations.

A, B, C are the sets.

(i) Double negation law

$$\bar{\bar{A}} = A$$

It is also known as involution.

(ii) Commutative law

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

$$A \oplus B = B \oplus A$$

(iii) Associative law

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \oplus (B \oplus C) = (A \oplus B) \oplus C$$

(iv) De Morgan's law

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

(v) Distributive law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

(vi) Idempotent law

$$A \cup A = A$$

$$A \cap A = A$$

Module II

(vii) Absorption law

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

$$A - \bar{A} = A$$

If C is sub set of A, then

$$A \cup C = A$$

$$A \cap C = C$$

(viii) Operations generating Null set ϕ

$A \cap \phi = \phi$ - Domination law

$A \cap \bar{A} = \phi$ - Inverse law

$A - A = \phi$

$A \oplus A = \phi$

$A - U = \phi$

$\phi - A = \phi$

$A \times \phi = \phi$

(ix) Operations generating universal set U

$A \cup U = U$ - Domination law

$A \cup \bar{A} = U$ - Inverse law

$A \oplus \bar{A} = U$

(x) Operations with Null set ϕ

$A \cup \phi = A$ - Indentify

$A \cap \phi = \phi$ - Domination law

$A - \phi = A$

$\phi - A = \phi$

$A \oplus \phi = A$

$\bar{\phi} = U$ - Inverse law

(xi) Operations with universal set U

$A \cup U = U$ - Domination law

$A \cap U = A$ - Identity law

$A - U = \phi$

$U - A = \bar{A}$

$A \oplus U = \bar{A}$

$\bar{U} = \phi$ - Inverse law

(xii) If $A \subseteq B$ then $A \cap B = A$, $A \cup B = B$ And $\bar{B} \subseteq \bar{A}$

2.3.8 Venn Diagram for Set Operations Properties

Many properties of set operations are straight forward, but few need Venn diagrams for proof. In this section we discuss the same.

2.3.9 De Morgan's Law

It states the following

(i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$ (ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Following table illustrates the same

(i) Left Hand side **Right hand side**

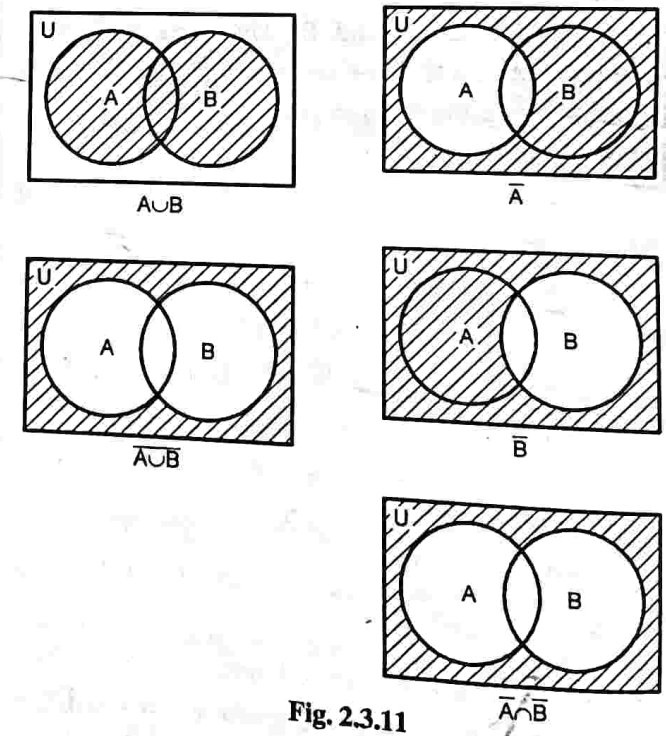
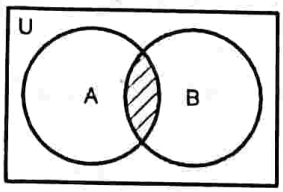
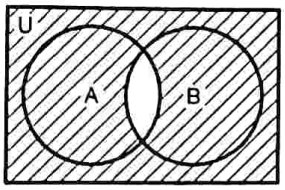


Fig. 2.3.11

(ii) Left hand Side



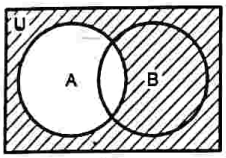
$A \cap B$



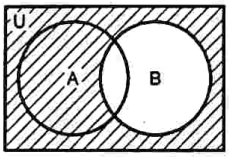
$\overline{A \cap B}$

Fig. 2.3.12(a)

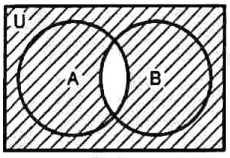
Right hand side



$\overline{A}$



$\overline{B}$



$\overline{A} \cup \overline{B}$

Fig. 2.3.12(b)

2.3.10 Distributive Law

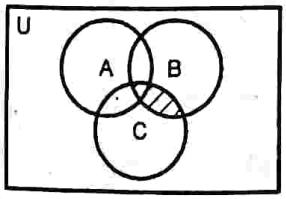
It states that

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

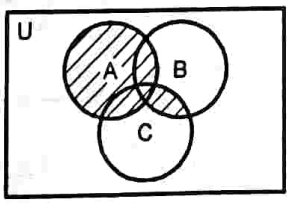
(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Following table illustrates the same

(i) LHS



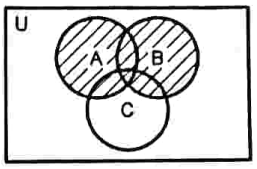
$B \cap C$



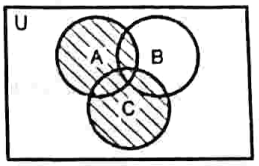
$A \cup (B \cap C)$

Fig. 2.3.13(a)

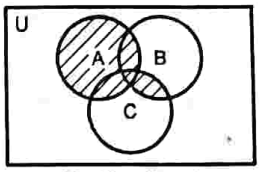
RHS



$A \cup B$



$A \cup C$

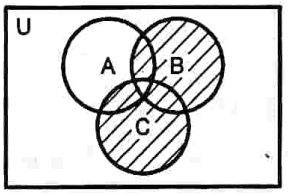


$(A \cup B) \cap (A \cup C)$

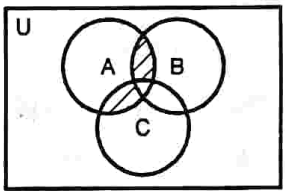
Fig. 2.3.13(b)

(ii)

LHS



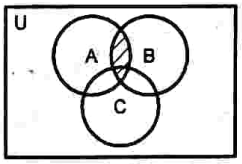
$B \cup C$



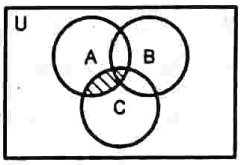
$A \cap (B \cup C)$

Fig. 2.3.14(a)

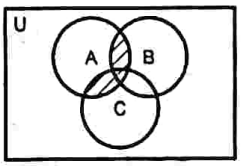
RHS



$A \cap B$



$A \cap C$



$(A \cap B) \cup (A \cap C)$

Fig. 2.3.14(b)

2.4 SOLVED EXAMPLES ON SETS

Ex. 2.4.1 Determine whether each of the following statements is true or false. Justify your answer.

(a) $\phi \subseteq \phi$ (b) $\phi \in \phi$

(c) $\phi \in \{\phi\}$ (d) $\phi \subseteq \{\phi\}$

(e) $\{\phi\} \subseteq \phi$

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(f) $\{a, b\} \subseteq \{a, b, c, \{a, b, c\}\}$

(g) $\{a, b\} \in \{a, b, c, \{a, b, c\}\}$

(h) $\{a, b, c\} \in \{a, b, c, \{a, b, c\}\}$

(i) $\{a, \phi\} \subseteq \{a, \{a, \phi\}\}$

(j) $\{a, \phi\} \in \{a, \{a, \phi\}\}$

Soln. :

(a) $\phi \subseteq \phi$: true - Every set is subset of itself

(b) $\phi \in \phi$: false - A set cannot be member of itself

(c) $\phi \in \{\phi\}$: True - ϕ is a member of set $\{\phi\}$

(d) $\phi \subseteq \{\phi\}$: True - ϕ is a subset of every set

(e) $\{\phi\} \subseteq \phi$: False - $\{\phi\}$ is the set which has null set member.

(f) $\{a, b\} \subseteq \{a, b, c, \{a, b, c\}\}$: True - Elements of set $\{a, b\}$ belong to $\{a, b, c, \{a, b, c\}\}$.

(g) $\{a, b\} \in \{a, b, c \{a, b, c\}\}$: False - $\{a, b\}$ is not a member of $\{a, b, c, \{a, b, c\}\}$

(h) $\{a, b, c\} \in \{a, b, c, \{a, b, c\}\}$: True

(i) $\{a, \phi\} \subseteq \{a, \{a, \phi\}\}$: False - The elements of $\{a, \phi\}$ are a and ϕ. They are not members of $\{a, \{a, \phi\}\}$

(j) $\{a, \phi\} \in \{a, \{a, \phi\}\}$: True - The set $\{a, \phi\}$ is member of $\{a, \{a, \phi\}\}$

Ex. 2.4.2 Determine the following Sets

(a) $\phi \cup \{\phi\}$ (b) $\phi \cap \{\phi\}$

(c) $\{\phi\} \cup \{a, \phi, \{\phi\}\}$ (d) $\{\phi\} \cap \{a, \phi, \{\phi\}\}$

(e) $\phi \oplus \{a, \phi \{\phi\}\}$ (f) $\{\phi\} \oplus \{a, \phi, \{\oplus\}\}$

Soln. :

(a) $\phi \cup \{\phi\} = \{\phi\}$

(b) $\phi \cap \{\phi\} = \phi$

(c) $\{\phi\} \cup \{a, \phi, \{\phi\}\} = \{a, \phi, \{\phi\}\}$

(d) $\{\phi\} \cap \{a, \phi, \{\phi\}\} = \{\phi\}$

(e) $\phi \oplus \{a, \phi, \{\phi\}\} = \{a, \phi, \{\phi\}\}$

(f) $\{\phi\} \oplus \{a, \phi, \{\phi\}\} = \{a, \{\phi\}\}$

Ex. 2.4.3 Let A, B, C be sets. Draw Venn diagrams for the given conditions.

(a) $(A \cup B) \subseteq B$ and $B \subseteq A$

(b) $A \subseteq B$, $A \subseteq C$, $(B \cap C) \subseteq A$, and $A \subseteq (B \cap C)$

(c) $A \cap B \cap C = \phi$, $(A \cap B) \neq \phi$

(d) $C \subseteq A \cup B$, $C \nsubseteq A$, $C \nsubseteq B$

(e) $A \cup B = A \cup C$, $B \neq C$

(f) $A \cap B = A \cap C$, $B \neq C$

Soln. :

(a) $(A \cup B) \subseteq B$ and $B \subseteq A$

This indicates the $A = B$

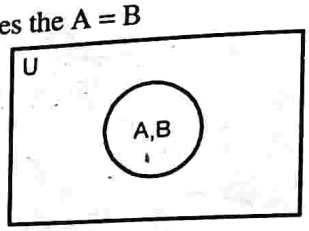


Fig. P. 2.4.3(a)

(b) $A \subseteq B$, $A \subseteq C$, $(B \cap C) \subseteq A$, and $A \subseteq (B \cap C)$

It is given that $(B \cap C) \subseteq A$ and $A \subseteq B \cap C$

$\therefore A = B \cap C$

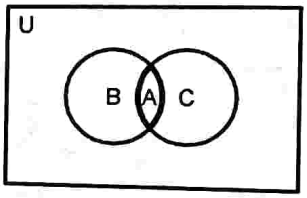


Fig. P. 2.4.3(b)

(c) $A \cap B \cap C = \phi$, $(A \cap B) \neq \phi$, $A \cap C \neq \phi$ and $B \cap C \neq \phi$

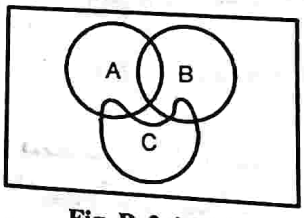


Fig. P. 2.4.3(c)

(d) $C \subseteq A \cup B$, $C \nsubseteq A$, $C \nsubseteq B$

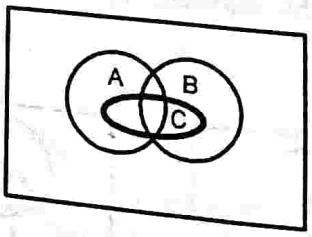


Fig. P. 2.4.3(d)

(e) $A \cup B = A \cup C, B \neq C$

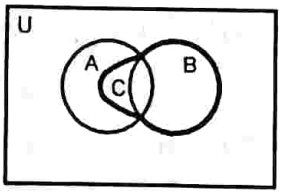


Fig. P. 2.3.4(e)

Example

$A = \{1, 2, 3, 4, 5\}$

$B = \{3, 4, 5, 6, 7\}$

$C = \{2, 3, 6, 7\}$

(f) $A \cap B = A \cap C, \ B \neq C$

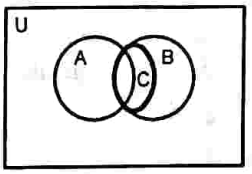


Fig. P. 2.3.4(f)

Example

$A = \{1, 2, 3, 4, 5\}$

$B = \{3, 4, 5, 6, 7\}$

$C = \{3, 4, 5, 6\}$

Ex. 2.4.4 Prove the following statements without using Venn diagrams.

a. $(A \cap B) \cup \left(B \cap ((C \cap D) \cup (C \cap \bar{D}))\right) = B \cap (A \cup C)$

b. $(A - B) \cap (B - A) = \phi$

c. $A \oplus B = (A \cup B) \cap (\bar{A} \cup \bar{B})$

d. $(A - B) \cup (A \cap B) = A$

e. $A \cup (A \cap B) = A$

f. $(A - B) - C = A - (B \cup C)$

g. $(A - B) - C = (A - C) - B$

h. $(A - B) - C = (A - C) - (B - C)$

☑ **Soln. :**

a. $(A \cap B) \cup \left(B \cap ((C \cap D) \cup (C \cap \bar{D}))\right) = B \cap (A \cup C)$

L.H.S. $= (A \cap B) \cup \left(B \cap ((C \cap D) \cup (C \cap \bar{D}))\right)$

$= (A \cap B) \cup \left(B \cap (C \cap (D \cup \bar{D}))\right)$ ...Distributive law

$= (A \cap B) \cup \left(B \cap (C \cap \cup)\right)$ ...Set complement

$= (A \cap B) \cup (B \cap C)$ ...Universal set property

$= (B \cap A) \cup (B \cap C)$ ...Commutative law

$= B \cap (A \cup C)$ ...Distributive law

$=$ R.H.S.

b. $(A - B) \cap (B - A) = \phi$

L.H.S. $= (A - B) \cap (B - A)$

$= (A \cap \bar{B}) \cap (B \cap \bar{A})$ ...Set difference

$= (A \cap (\bar{B} \cap B)) \cap \bar{A}$ ...Associative law

$= (A \cap \phi) \cap \bar{A}$ ...Set complement

$= \phi \cap \bar{A}$ ...Null set property

$= \phi$ ...Null set property

$=$ R.H.S.

c. $A \oplus B = (A \cup B) \cap (\bar{A} \cup \bar{B})$

L.H.S. $= A \oplus B$

$= (A - B) \cup (B - A)$ ...Symmetric difference

$= (A \cap \bar{B}) \cup (B \cap \bar{A})$ ...Set difference

$= [(A \cap \bar{B}) \cup B] \cap [(A \cap \bar{B}) \cup \bar{A}]$ ...Distributive law

$= [(A \cup B) \cap (\bar{B} \cup B)] \cap [(A \cup \bar{A}) \cap (\bar{B} \cup \bar{A})]$ ....Distributive law

$= [(A \cup B) \cap \cup] \cap [\cup \cap (\bar{B} \cup \bar{A})]$ ...Set complement

$= (A \cup B) \cap (\bar{B} \cup \bar{A})$ ...Universal set property

$= (A \cup B) \cap (\bar{A} \cup \bar{B})$ ...Commutative law

$=$ R.H.S.

d. $(A - B) \cup (A \cap B) = A$

L.H.S $= (A - B) \cup (A \cap B)$

Module II

$= (A \cap \bar{B}) \cup (A \cap B)$ ...Set difference

$= A \cap (\bar{B} \cup B)$ ...Distributive law

$= A \cap \cup$ ...Set complement

$= A$ ...Universal set property

e. $A \cup (A \cap B) = A$

Let $x \in A \cup (A \cap B)$

$\therefore$ $x \in A$ or $x \in A \cap B$

$\therefore$ $x \in A$ or ($x \in A$ and $x \in B$)

$\therefore$ $x \in A$

Thus $\forall x \in A \cup (A \cap B)$, $x \in A$

$\therefore$ $A \cup (A \cap B) = A$

f. $(A - B) - C = A - (B \cup C)$

L.H.S. $= (A - B) - C$

$= (A \cap \bar{B}) \cap \bar{C}$ ...Set difference

$= A \cap (\bar{B} \cap \bar{C})$ ...Associative law

$= A \cap (\overline{B \cup C})$ ...De Morgans law

$= A - (B \cup C)$ ...Set difference

= R.H.S.

g. $(A - B) - C = (A - C) - B$

L.H.S. $= (A - B) - C$

$= (A \cap \bar{B}) \cap \bar{C}$ ...Set difference

$= A \cap (\bar{B} \cap \bar{C})$ ...Associative law

$= A \cap (\bar{C} \cap \bar{B})$ ...Commutative law

$= (A \cap \bar{C}) \cap \bar{B}$ ...Associative law

$= (A - C) - B$ ...Set difference

= R.H.S.

h. $(A - B) - C = (A - C) - (B - C)$

R.H.S. $= (A - C) - (B - C)$

$= (A \cap \bar{C}) - (B \cap \bar{C})$ ...Set difference

$= (A \cap \bar{C}) \cap (\overline{B \cap \bar{C}})$ ...Set difference

$= (A \cap \bar{C}) \cap (\bar{B} \cup C)$ ...De-Morgan's law

$= (A \cap \bar{C} \cap \bar{B}) \cup (A \cap \bar{C} \cap C)$ ...Distributive law

$= (A \cap \bar{C} \cap \bar{B}) \cup (A \cap \phi)$ ...Set complement

$= (A \cap \bar{C} \cap \bar{B}) \cup \phi$ ...Null set property

$= (A \cap \bar{C} \cap \bar{B})$ ...Null set property

$= (A \cap \bar{B}) \cap \bar{C}$ ...Associative law

$= (A - B) - C$ ...Set difference

= L.H.S.

Ex. 2.4.5 Let $A = \{\phi, b\}$ construct the following sets

a. $A - \phi$ b. $\{\phi\} - A$

c. $A - \{\phi\}$

d. $A \cup P(A)$ P (A) is power set of A

e. $A \cap P(A)$ f. $P(A) - P(\{\phi\})$

☑ **Soln. :**

a. $A - \phi$

$= \{\phi, b\} - \phi$

$= \{\phi, b\}$

$= A$

b. $\{\phi\} - A$

$= \{\phi\} - \{\phi, b\}$

$= \phi$

c. $A - \{\phi\}$

$= \{\phi, b\} - \{\phi\}$

$= \{b\}$

d. $A \cup P(A)$ P (A) is power set of A.

$= \{\phi, b\} \cup \{\{\phi\}, \{b\}, \{\phi, b\}, \phi\}$

$= \{\phi, b, \{\phi\}, \{b\}, \{\phi, b\}\}$

e. $A \cap P(A)$

$= \{\phi, b\} \cup \{\phi, \{\phi\}, \{b\}, \{\phi, b\}\}$

$= \{\phi\}$

f. $P(A) - P(\{\phi\})$

$= \{\phi, \{\phi\}, \{b\}, \{\phi, b\}\} - \{\phi, \{\phi\}\}$

$= \{\{b\}, \{\phi, b\}\}$

Ex. 2.4.6 Find the power sets of following sets and verify $|P(A)| = 2^{|A|}$

a. $A = \{1, 2, 3\}$

b. $A = \{\phi\}$

c. $A = \phi$

d. $A = \{a, \phi, \{\phi\}\}$

e. $A = \{1, 2\} \times \{a, b\}$

Soln. :

a. $A = \{1, 2, 3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$

$|A| = 3 \quad |P(A)| = 2^3 = 8$

b. $A = \{\phi\}$

$P(A) = \{\phi, \{\phi\}\}$

$|A| = 1 \quad |P(A)| = 2^1 = 2$

c. $A = \phi$

$P(A) = \{\phi\}$

$|A| = 0 \quad |P(A)| = 2^0 = 1$

d. $A = \{a, \phi, \{\phi\}\}$

$P(A) = \{\phi, \{a\}, \{\phi\}, \{\{\phi\}\}, \{a, \phi\}, \{\phi, \{\phi\}\}, \{a, \{\phi\}\}, \{a, \phi, \{\phi\}\}\}$

$|A| = 3 \quad |P(A)| = 2^3 = 8$

e. $A = \{1, 2\} \times \{a, b\}$

$P(A) = P(\{1, 2\} \times \{a, b\})$

$= P(\{<1, a>, <1, b> <2, a>, <2, b>\})$

$= \{\phi, \{<1, a>\}, \{<1, b>\}, \{<2, a>\}, \{<2, b>\}$
$\{<1, a>, <1, b>\}, \{<1, a>, <2, a>\},$
$\{<1, a>, <2, b>\}, \{<1, b>, <2, a>\},$
$\{<1, b>, <2, b>\}, \{<2, a>, <2, b>\}$
$\{<1, a>, <1, b>, <2, a>\},$
$\{<1, a>, <1, b>, <2, b>\}$
$\{<1, a>, <2, a>, <2, b>\},$
$\{<1, b>, <2, a>, <2, b>\}$
$\{<1, a>, <1, b>, <2, a>, <2, b>\}\}$

$|A| = |\{1, 2\}| \times |\{a, b\}| = 2 \times 2 = 4$

$|P(A)| = 2^{|A|} = 2^4 = 16$

Ex. 2.4.7 Prove the following :

a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

b) $A \times (B \cap C) = (A \times B) \cap (A \cap C)$

c) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Soln. :

a. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

L.H.S $= A \times (B \cup C)$

$= \{ <x, y> | x \in A \text{ and } y \in (B \cup C) \}$

$= \{ <x, y> | x \in A \text{ and } (y \in B \text{ or } y \in C) \}$

$= \{ <x, y> | (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C)\}$

$= \{ <x, y> | x \in A \text{ and } y \in B\} \cup \{ <x, y> | x \in A \text{ and } y \in C \}$

$= (A \times B) \cup (A \times C)$

$=$ R.H.S.

b. $A \times (B \cap C) = (A \times B) \cap (A \times C)$

L.H.S. $= A \times (B \cap C)$

$= \{<x, y> | x \in A \text{ and } y \in (B \cap C)\}$

$= \{ <x, y> | x \in A \text{ and } (y \in B \text{ and } y \in C)\}$

$= \{ <x, y> | (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C)$

$= \{ <x, y> | x \in A \text{ and } y \in B \} \cap \{ <x, y> | x \in A \text{ and } y \in C \}$

$= (A \times B) \cap (A \times C)$

$=$ R.H.S.

c. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

L.H.S. $= (A \times B) \cap (C \times D)$

$= \{<x, y> | x \in A \text{ and } y \in B \} \cap \{<x, y> | x \in C \text{ and } y \in D \}$

$= \{<x, y> | x \in A \text{ and } x \in C \text{ and } y \in B \text{ and } y \in D \}$

$= \{<x, y> | x \in A \cap C \text{ and } y \in B \cap D\}$

$= (A \cap C) \times (B \cap D)$

$=$ R.H.S.

Module II

UEx. 2.4.8

MU - Q. 6(d), May 18, 4 Marks, Q. 3(b), Dec. 18, 6 Marks

For all sets A, X and Y show that

$A \times (X \cap Y) = (A \times X) \cap (A \times Y)$

Soln. :

To prove $A \times (X \cap Y) = (A \times X) \cap (A \times Y)$

$$\begin{aligned}
\text{LHS} &= A \times (X \cap Y) \\
&= \{(x, y)\ x \in A \text{ and } y \in X \text{ n } Y\} \\
&= \{(x, y) \mid x \in A \text{ and } y \in X \text{ and } y \in Y\} \\
&= \{(x, y) \mid (x \in A \text{ and } y \in X) \text{ and } (x \in A \text{ and } y \in Y)\} \\
&= \{(x, y) \mid (x \in A \text{ and } y \in X) \cap (x \in A \text{ and } y \in Y)\} \\
&= (A \times X) \cap (A \times Y) \\
&= \text{RHS.}
\end{aligned}$$

Hence proved.

UEx. 2.4.9 **MU - Q. 6(a), May 15, 8 Marks**

Show that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$

Soln. :

To prove $A - (B - C) = (A - B) \cup (A \cap B \cap C)$

$$\begin{aligned}
\text{RHS} &= (A - B) \cup (A \cap B \cap C) \\
&= \left(A \cap \bar{B}\right) \cup (A \cap B \cap C) && \text{[Definition of –]} \\
&= A \cap \left(\bar{B} \cup (B \cap C)\right) && \text{[Distributive law]} \\
&= A \cap \left(\bar{B} \cup B\right) \cap \left(\bar{B} \cup C\right) && \text{[Distributive law]} \\
&= A \cap \left(\bar{B} \cup C\right) && \text{[Universal set identity]} \\
&= A \cap \left(\overline{B \cap \bar{C}}\right) && \text{[De Morgan's law]} \\
&= A \cap \left(\overline{B - C}\right) && \text{[Definition of –]} \\
&= A - (B - C) && \text{[Definition of –]} \\
&= \text{LHS}
\end{aligned}$$

Hence the proof.

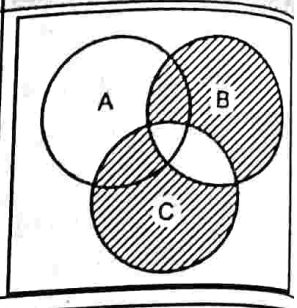
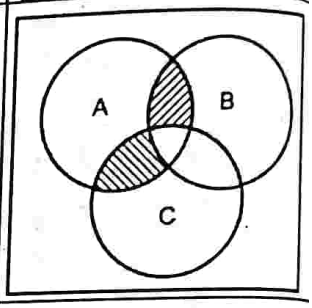
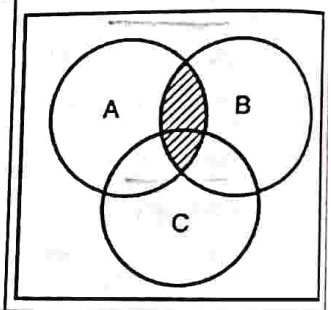
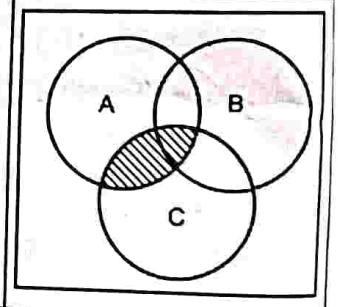
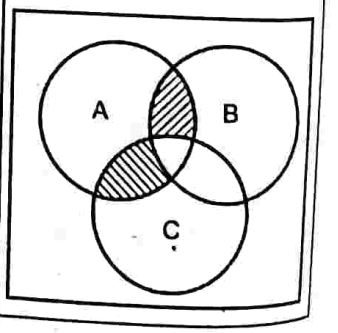
UEx. 2.4.10 **MU - Q. 3(a), Dec. 14, 4 Marks**

Show that $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

Soln. :

To prove $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

The following table gives venn diagram for the sets.

Sr. No.	Set	Venn diagram
1.	$B \oplus C$	
2.	$A \cap (B \oplus C)$	
3.	$A \cap B$	
4.	$A \cap C$	
5.	$(A \cap B) \oplus (A \cap C)$	

From 2 and 5, it is clear that,

$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$$

Hence the proof.

UEx. 2.4.11 **MU - Q. 4(b), Dec. 16, 6 Marks**

Prove the following $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

Soln. :

To prove $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

$$\begin{aligned}
LHS &= (A - B) \cup (B - A) \\
&= (A \cap \bar{B}) \cup (B \cap \bar{A}) \text{ [Definition of –]} \\
&= (A \cup B) \cap (A \cup \bar{A}) \cap (\bar{B} \cup \bar{B}) \cap (\bar{B} \cup \bar{A}) \\
&\quad \text{[Distributive law]} \\
&= (A \cup B) \cap (\bar{B} \cup \bar{A}) \text{ [Universal set identity]} \\
&= (A \cup B) \cap (\bar{A} \cup \bar{B}) \text{ [Commutative law]} \\
&= (A \cup B) \cap (\overline{A \cap B}) \quad \text{[De Morgan's law]} \\
&= (A \cup B) - (A \cap B) \text{ [Definition of –]} \\
&= RHS
\end{aligned}$$

Hence proved.

Ex. 2.4.12 Find under what conditions, the following equations are true

a. $(A - B) \cup (A - C) = A$

b. $(A - B) \cup (A - C) = \phi$

c. $(A - B) \cap (A - C) = \phi$

d. $(A - B) \oplus (A - C) = \phi$

e. $A - B = B$

f. $A - B = B - A$

g. $A \oplus B = A \cup B$

Soln. :

(a) $(A - B) \cup (A - C) = A$

$\therefore \quad (A \cap \bar{B}) \cup (A \cap \bar{C}) = A$ ...Set difference

$\therefore \quad [(A \cap \bar{B}) \cup A] \cap [(A \cap \bar{B}) \cup \bar{C}] = A$

...Distributive law

$\therefore \quad (A \cup A) \cap (\bar{B} \cup A) \cap (A \cup \bar{C}) \cap (\bar{B} \cup \bar{C}) = A$

...Distributive law

$\therefore \quad A \cap (\bar{B} \cup A) \cap (A \cup \bar{C}) \cap (\bar{B} \cup \bar{C}) = A$ ...(i)

It is obvious that

$A \subseteq \bar{B} \cup A, \quad A \subseteq A \cup \bar{C}$

To satisfy Equation (i) we should have

$A \subseteq \bar{B} \cup \bar{C}$

$\therefore \quad A \subseteq \overline{B \cap C}$ ...De Morgan's law

i.e. if $x \in A$ then $x \in \overline{B \cap C}$

$\therefore \quad x \notin B \cap C$

$\therefore \quad x \notin B$ and $x \notin C$

$\therefore$ If $x \in A$ then $x \notin B$ and $x \notin C$

$\therefore \quad A \cap B = \phi$ and $A \cap C = \phi$

Thus the given equation $(A - B) \cup (A - C) = A$ is true if $A \cap B = \phi$ and $A \cap C = \phi$.

(b) $(A - B) \cup (A - C) = \phi$

Union of two sets is null set if two sets are null.

$\therefore \quad A - B = \phi$ and $A - C = \phi$

If $A = \phi$ then $A - B = \phi$ and $A - C = \phi$

If $A \neq \phi$ then to have $A - B = \phi$ we should have

$A = B$ or $A \subseteq B$

Similarly for $A - C = \phi$, $A = C$ or $A \subseteq C$.

$\therefore$ Thus the given Equation $(A - B) \cup (A - C) = \phi$

If any one of the following conditions are true.

(i) $A = \phi$

(ii) $A \subseteq B$ and $A \subseteq C$

(c) $(A - B) \cap (A - C) = \phi$

$\therefore \quad (A \cap \bar{B}) \cap (A \cap \bar{C}) = \phi$ ...Set difference

$\therefore \quad A \cap \bar{B} \cap \bar{C} = \phi$...Associative law and $A \cap A = A$

$\therefore \quad A \cap (\bar{B} \cap \bar{C}) = \phi$ ...Associative law

$\therefore \quad A \cap (\overline{B \cup C}) = \phi$ ...De-Morgan's law

$\therefore$ If $x \in A$ then $x \notin \overline{B \cup C}$

$\therefore \quad x \in B \cup C$

$\therefore \quad A \subseteq B \cup C$

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$\therefore$ Thus the given equation $(A-B) \cap (A-C) = \phi$ if

$A \subseteq B \cup C$

(d) $(A-B) \oplus (A-C) = \phi$

Symmetric difference of two sets is null set, if two sets are equal.

$$\therefore \quad A-B = A-C$$

$$A \cap \bar{B} = A \cap \bar{C} \quad \text{...Set difference}$$

$\therefore$ Thus the given Equation $(A-B) \oplus (A-C) = \phi$

if $A \cap \bar{B} = A \cap \bar{C}$.

(e) $A - B = B$

The $A - B$ includes all elements which are present in A and not in set B.

$$\text{i.e.} \quad A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

But it is given that $A - B = B$.

This is possible only if $A = B = \phi$

Thus given equation $A - B = B$ is true if $A = B = \phi$

(f) $A - B = B - A$

This is possible only if

$$A - B = B - A = \phi$$

$\therefore \quad A - B = \phi$ and $B - A = \phi$

$A \subseteq B$ and $B \subseteq A$

$A = B$.

Thus the given equation $A - B = B - A$ is true if $A = B$.

(g) $A \oplus B = A \cup B$

$\therefore \quad (A-B) \cup (B-A) = A \cup B$

$\therefore (A \cap \bar{B}) \cup (B \cap \bar{A}) = A \cup B$

$\therefore \quad [(A \cap \bar{B}) \cup B] \cap [(A \cap \bar{B}) \cup \bar{A}] = A \cup B$

...Distributive law

$\therefore \quad (A \cup B) \cap (\bar{B} \cup B) \cap (A \cup \bar{A}) \cap (\bar{B} \cup \bar{A}) = A \cup B$

$\therefore \quad (A \cup B) \cap U \cap U \cap (\bar{B} \cup \bar{A}) = A \cup B$

...Union of complements is universal set

$\therefore \quad (A \cup B) \cap (\bar{B} \cup \bar{A}) = A \cup B$ ...Universal set property

$\therefore \quad A \cup B \subseteq \bar{B} \cup \bar{A}$ $\quad \because$ if $x \cap y = x$ then $x \subseteq$

$\therefore$ If $x \in A \cup B$ then $x \in \bar{B} \cup \bar{A}$

$\therefore \quad x \in \overline{B \cap A}$

$\therefore$ If $x \in A$ or $x \in B$ then $x \in \bar{B}$ and $x \in \bar{A}$

This is true only if $A \cap B = \phi$

Thus the given equation $A \oplus B = A \cup B$ if the sets A and B are disjoint i.e. $A \cap B = \phi$.

2.5 PARTITION OF A SET

Definition : Partition of a set A is defined as the set of subsets of A with the following properties

$P(A) = S1, S2, ..., Sn\}$

Where $S1 \cup S2 \cup ... \cup Sn = A$

For all $1 \le i, j \le n$ $Si - \cap Sj = \phi$

Ex. Following are few partitions of a set $A = \{1,2,3,4,5\}$

$P(A) = \{1\},\{2\},\{3\},\{4\},\{5\}\}$

$P(A) = \{\{1,2\},\{3,4\},\{5\}\}$

$P(A) = \{1,2,5\},\{3,4\}\}$

$P(A) = \{\{2,3,5\},\{4,1\}\}$

2.6 INTRODUCTION TO RELATIONS

- In many situations, the computational objects are related, or they share common characteristics.
- For example, C and C++. Both are programming languages. C++ has some extra features than C.
- The binary number 10110 is one's complement of another binary number 01001. It is a relation.
- The numbers 1, 2, 3, 4 are related to the numbers 1, 4, 9, 16 respectively, and the relation is given as the former numbers are positive square roots of the latter numbers.
- Here the word positive is important. Otherwise we could have $-1, -2, -3, -4$, as well.
- In discrete mathematics, we generally consider relation between two elements, thus it is known as binary relation.

- A relation indeed is a set which has pairs of elements that are related.
- To be specific, if <a, b> is a pair in a relation, then we say that a is related to b, but not the otherwise.
- In the previous unit, we have seen that an ordered pair is a member of Cartesian product.

Definition : A relation is a subset of a Cartesian product. If an element x is related to y with a relation R, then we can write, $<x, y> \in R$ or $x R y$.

For example

Let A = {1, 2, 3} and B = {1, 3, 5, 7} be two sets. Let R be relation such that an element a from set A is related to an element in set B if a < b. The relation is given as

$$R = \{<a, b> \mid a \in A \text{ and } b \in A \text{ and } a < b\}$$
$$= \{<1, 3>, <1, 5>, <1, 7>, <2, 3>, <2, 5>, <2, 7>, <3, 5>, <3, 7>\}$$

Note : $R \subseteq A \times B$

2.6.1 Representation of a Relation

- A relation can be represented in different ways. One way we have already seen, i.e. by explicitly writing the ordered pairs or the members of the relation.
- The other ways are relation matrix, and directed graph or diagraph. A relation matrix M_R is defined as follows.

$$M_R = [m_{ij}] \text{ where } m_{ij} \text{ is } i^{th} \text{ and } j^{th} \text{ column element}$$
$$m_{ij} = 1 \text{ if } <a_i, b_j> \in R$$
$$m_{ij} = 0 \text{ if } <a_i, b_j> \notin R$$

- A relation is represented using a diagraph, whose nodes represent the elements in the sets.
- If the relation is between two different sets, then the elements are arranged in two columns, if the relation is within a set, then the elements are arranged in a circular manner. If an element a is related to element b, then there is a directed edge from node representing a to the node representing b.
- Following matrix and diagraph represent the relation discussed in the section

$$M_r = A \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} B \\ \begin{array}{cccc} 1 & 3 & 5 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Ex. 2.6.1 Let A = {1,2,3,4,5,6} Represent a relation R, such that xRy if x, y ∈ A and x\y (x divides y).

Soln. :

Here the set is A = {1, 2, 3, 4, 5, 6}. The relation R is on the same set. The relation elements are as follows.

$$R = \{<x, y> \mid x, y \in A \text{ and } x \mid y\}$$
$$= \{<1, 1>, <1, 2>, <1, 3>, <1, 4>, <1, 5>, <1, 6>, <2, 2>, <2, 4>, <2, 6>, <3, 3>, <3, 6>, <4, 4>, <5, 5>, <6, 6>\}$$

The matrix representation is as follows

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The graphical representation is as shown in Fig. P. 2.6.1.

It is a digraph, i.e. every edge is directed. The path is represents a relation. The terms path and digraph are studied in detail in chapter on graphs.

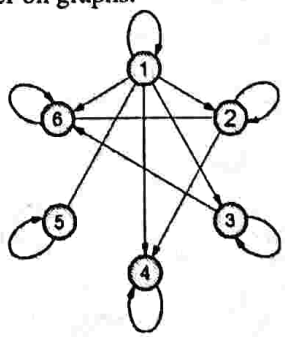


Fig. P. 2.6.1

2.7 SOLVED EXAMPLES ON RELATION

Ex. 2.7.1 : Consider the following sets

$A = \{1, 2, 3, 4, 5\}$

$B = \{-2, -1, 0, 1, 2\}$

Find the elements of the following relation

$R = \{<x, y>, x \in A \text{ and } y \in B \text{ and } x + y = 4\}$

Soln. :

$R = \{<2, 2>, <3, 1>, <4, 0>, <5, -1>\}$

Ex. 2.7.2 : Write the matrix and draw graph for the relation in Fig. P.2.6.1(a).

Soln. :

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The graph representing the relation is

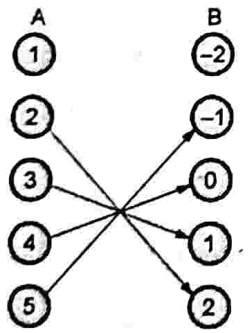


Fig. P. 2.7.2(a)

Ex. 2.7.3 For the graph shown in Fig. P.2.7.3 below, write the relation in set notation form.

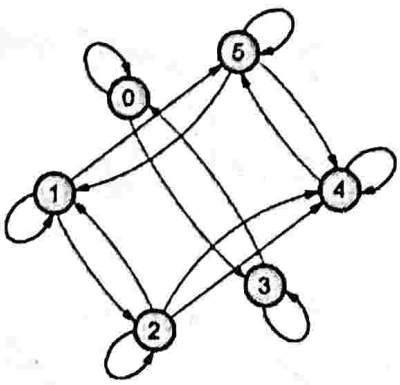


Fig. P.2.7.3

Soln. :

The relation is given by.

$R = \{<0, 0>, <1, 1>, <2, 2>, <3, 3>, <4, 4>, <5, 5>, <0, 3>, <3, 0>, <1, 2>, <2, 1>, <1, 5>, <5, 1>, <2, 4>, <4, 2>, <4, 5>, <5, 4>\}$

It can be observed that two numbers are related if their sum is multiple of 3 or divisible by 3.

$\therefore$ The set representation is as follows.

Let $A = \{0, 1, 2, 3, 4, 5\}$

$R = \{<x, y> \mid x, y \in A \text{ and } (x + y) \bmod 3 = 0\}$

Ex. 2.7.4 Consider the set $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ Write the elements of relation R such that for x, $y \in A$, xRy if x/y and $x \neq y$.

Soln. :

The relation R is given by,

$R = \{<2, 4>, <2, 6>, <2, 8>, <2, 10>, <3, 6>, <3, 9>, <4, 8>, <5, 10>\}$

$x \mid y$ indicates 'x divides y' i.e. y is divisible by x.

Ex. 2.7.5 : Consider the set

$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Find the cardinality of relation.

$R = \{<x, y> \mid x, y \in A \text{ and } x \mid y\}$

Soln. : Here the relation is 'x divides y'. We need to find the cardinality the relation gives the number of relations.

Table

x	y	z
1	1, 2, ..., 10	10
2	2, 4, ..., 10	5
3	3, 6, 9	3
4	4, 8	2
5	5, 10	2
6	6	1
7	7	1
8	8	1
9	9	1
10	10	1
		Total = 27

$\therefore |R| = 27$

Ex. 2.7.6 Consider the set A {1, 2, 3, 4, 5}

Find the cardinality of relation

$R = \{<x, y> \mid x, y \in A \text{ and } x \leq y\}$

Write the matrix and draw the graph representing R.

Soln. :

The relation matrix is given, by

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The graph is given in Fig. 2.7.6(a)

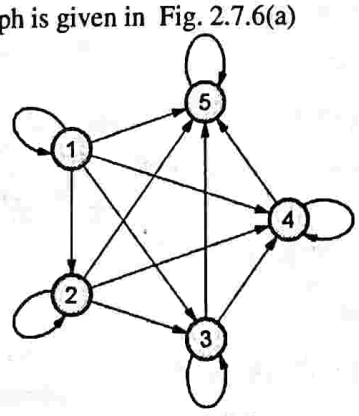


Fig. 2.7.6(a)

The cardinality can be calculated as the Number of 1's in the matrix or number of edges in the graph.

$\therefore |R| = 15$

UEx. 2.7.7 **MU - Q. 5(c), May 18,4 Marks**

Let A = {a, b, c, d, e, f, g, h}. Consider the following subsets of A

A1 = {a, b, c, d} A2 = {a, c, e, g, h}

A3 = {a, c, e, g} A4 = {b, d} A5 = {f, h}

Determine whether following is partition of A or not. Justify your answer.

(i) (A1, A2) (ii) (A3, A4, A5)

Soln. :

Given : A = {a, b, c, d, e, f, g, h}

A1 = {a, b, c, d}

A2 = {a, c, e, g, h}

A3 = {a, c, e, g} A4 = {b, d}, A5 = {f, h}

(i) Consider {A1, A2}

A1 ∪ A2 = {a, b, c, d, e, g, h} ≠ A

∴ {A1, A2} is not a partition of A.

(ii) Consider {A3, A4, A5}

A3 ∪ A4 ∪ A5 ={a, c, e, g, b, d, f, h} = A

A3 ∩ A4 = ϕ , A3 ∩ A5 = ϕ, A4 ∩ A5 = ϕ

∴ {A3, A4, A5} is a partition of A.

UEx. 2.7.8 **MU - Q. 2(a), May 17, 4 Marks**

Let A = {a, b, c, d, e, f, g, h}. Consider the following subsets of A

A_1 = {a, b, c, d }

A_2 = {a, c, e, g, h}

A_3 = {a, c ,e, g}

A_4 = {b, d}

A_5 = {f, h}

Determine whether following is partition of A or not. Justify your answer.

i) $\{A_1, A_2\}$ ii) $\{A_3, A_4, A_5\}$

Soln. :

Given : A = {a, b, c, d, e, f, g, h}

Following are the subsets of A,

A_1 = {a, b, c, d}, A_2 = {a, c, e, g, h}

A_3 = {a, c, e, g}, A_4 = {b, d}

A_5 = {f, h}

(i) Consider the set, $\{A_1, A_2\}$

$A_1 \cup A_2$ = {a, b, c, d, e, g, h} ≠ A

∴ $\{A_1, A_2\}$ is not a partition of A.

(ii) Consider the set, $\{A_1, A_5\}$

$A_1 \cup A_5$ = {a, b, c, d, f, h} ≠ A

∴ $\{A_1, A_5\}$ is not a partition of A

(iii) Consider the set, $\{A_3, A_4, A_5\}$

$A_3 \cup A_4 \cup A_5$ = {a, c, e, g, b, d, f, h} = A

$A_3 \cap A_4 \cap A_5 = \phi$

∴ $\{A_3, A_4, A_5\}$ is a partition of A

Module II

UEx. 2.7.9 **MU - Q. 1(c), May 16, 8 Marks**

Let A = {1,2,3,4,5} and R be the relation defined by a R b if and only if a < b. Compute R, R^2 and R^3. Draw digraph of R, R^2 and R^3.

Soln. : Given A = {1, 2, 3, 4, 5}

R is the relation defined by,

aRb if and only if a < b.

To find R, R^2 and R^3 and their digraphs.

$$R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$$

$$R^2 = R \bullet R$$

$$= \{(1, 3), (1, 4), (1, 5), (2, 4), (2, 5), (3, 5)\}$$

$$R^3 = R^2 \bullet R = \{(1, 4), (1, 5), (2, 5)\}$$

The digraphs are as follows (Refer Fig. P. 2.7.10) :

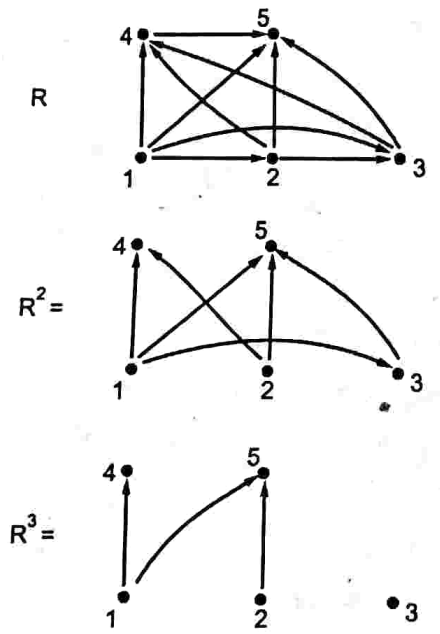


Fig. P. 2.7.10

2.8 TERMS IN RELATIONS

Consider a relation R_1 on sets A and B $\exists$ x $\in$ A, $\exists$ y $\in$ B, xRy. Following are few terms regarding this relation.

2.8.1 Domain

Domain $D(R_1)$ a set of all elements in A, such that every element in the domain, is related to at least one element in B.

i.e. $D(R_1) \subseteq A$

$D(R_1) = \{x \mid x \in A \text{ and } \exists y \in B, x R_1 y\}$

2.8.2 Range

Definition : Range $R(R_1)$ is a set of all elements in B, such that every element in the range, is related with at least one element in A.

i.e. $R(R_1) \subseteq B$

$R(R_1) = \{y \mid y \in B \text{ and } \exists x \in A, xR_1y\}$

Example

Let A = {2, 4, 6, 8, 10, 12, 14}

B = {3, 6, 9, 12, 15, 18, 21, 24, 27, 30}

Let S be a relation such that

xSy, x$\in$A and y$\in$B and x | y

S = {<2, 6>, <2, 12>, <2, 18>, <2,24>, <2, 30>, <4, 12>, <4, 24>, <6, 6>, <6, 12>, <6, 18,>, <6, 24>, <6, 30>,<8, 24>, <10, 30>, <12, 24>}

$\therefore$ Range of the relation is given by

R(S) = {6, 12, 18, 24, 30}

Domain of the relation is given by

D(S) = {2, 4, 6, 8, 10, 12}

2.8.3 Universal Relation

Definition : Consider a set A. We know that any relation R on set A is a subset of A × A. Thus A × A. This A × A is a also a relation. This is known as universal relation.

Example

Find universal relation on set A = {a, b, c}

The universal set is A × A

$\therefore$ A × A = {< a, a >, < a, b >, < a, c >, < b, a >, < b, b >,< b, c >, < c, a >, < c, b >, < c, c >}

This is the universal set.

2.8.4 Void Relation

An empty set ϕ is a subset of $A \times A$. Thus it is also a relation. This is known as void relation.

Ex. 2.8.1 Find all possible relations on set A {1, 2}. Define them with a set notation. Find the relation between cardinality of A and number of relations.

Soln. :

Given set is $A = \{1, 2\}$

We know that every relation is subset of $A \times A$

$A \times A = \{<1, 1>, <1, 2>, <2, 1>, <2, 2>\}$

Thus, following are the possible relations.

1. $R_1 = \phi$ $R_1 = \{<x, y> \mid x, y \in A \text{ and } x, y > 3\}$
2. $R_2 = \{<1, 1>\}$
   $R_2 = \{<x, y> \mid x, y \in A \text{ and } x = y = 1\}$
3. $R_3 = \{<1, 1>, <1, 2>\}$
   $R_3 = \{<x, y> \mid x, y \in A \text{ and } x = 1\}$
4. $R_4 = \{<1, 1>, <2, 1>\}$
   $R_4 = \{<x, y> \mid x, y \in A \text{ and } y = 1\}$
5. $R_5 = \{<1, 1>, <2, 2>\}$
   $R_5 = \{<x, y> \mid x, y \in A \text{ and } x = y\}$
6. $R_6 = \{<1, 2>, <2, 1>\}$
   $R_6 = \{<x, y> \mid x, y \in A \text{ and } x \neq y\}$
7. $R_7 = \{<1, 2>, <2, 2>\}$
   $R_7 = \{<x, y> \mid x, y \in A \text{ and } y = z\}$
8. $R_8 = \{<2, 1>, <2, 2>\}$
   $R_8 = \{<x, y> \mid x, y \in A \text{ and } x = z\}$
9. $R_9 = \{<1, 1>, <1, 2>, <2, 1>\}$
   $R_9 = \{<x, y> \mid y \in A \text{ and } (x \neq 2 \text{ or } y \neq 2)\}$
10. $R_{10} = \{<1, 1>, <1, 2>, <2, 2>\}$
    $R_{10} = \{<x, y> \mid x, y \in A \text{ and } x \leq y\}$
11. $R_{11} = \{<1, 1>, <2, 1>, <2, 2>\}$
    $R_{11} = \{<x, y> \mid x, y \in A \text{ and } x \geq y\}$
12. $R_{12} = \{<1, 2>, <2, 1>, <2, 2>\}$
    $R_{12} = \{<x, y> \mid x, y \in A \text{ and } (x \neq 1 \text{ or } y \neq 1)\}$
13. $R_{13} = \{<1, 1>, <1, 2>, <2, 1>, <2, 2>\}$
    $R_{13} = \{<x, y> x \mid y \in A\}$
14. $R_{14} = \{<1, 2>\}$ $R_{14} = \{<x, y> \mid x\ y \in A \text{ and } x < y\}$
15. $R_{15} = \{<2, 1>\}$
    $R_{15} = \{<x, y> \mid x, y \in A \text{ and } x < y\}$
16. $R_{16} = \{<2, 2>\}$
    $R_{16} = \{<x, y> \mid x, y \in A \text{ and } x = y = z\}$

- The total number of possible relations is 16, i.e. the number of possible subsets of $A \times A$
- We know that if cardinality of any set is n, then cardinality of its cross product is n^2.
- If cardinality of any set is K, then number of possible subsets is 2^k
- Thus if cardinality of any set is n, then the number of possible relations is given by $2^{(n^2)}$. In this example, the set $A = \{1, 2\}$

$$\therefore |A| = 2$$

$$\therefore |A \times A| = 2^2 = 4$$

$\therefore$ Number of relations $= 2^4 = 16$

- If $A = \{1, 2, 3\}$, then number of possible relations $= 2^{(3^2)} = 2^9 = 512$.

2.9 SOLVED EXAMPLES ON RELATION

Ex. 2.9.1 Given the sets $A = \{x \mid x \in I \text{ and } -5 \leq x \leq 5\}$
$B = \{x \mid x \in I \text{ and } 0 \leq x \leq 10\}$
Find the following relations
(i) $R_1 = \{<x, y> \mid x \in A \text{ and } y \in B \text{ and } x > y\}$
(ii) $R_2 = \{<x, y> \mid x \in A \text{ and } y \in B \text{ and } y = x^2\}$
(iii) $R_3 = \{<x, y> \mid x \in A \text{ and } y \in B \text{ and } x + y < 0\}$

Soln. :

Given sets are

$A = \{x, \mid x \in I \text{ and } -5 \leq x \leq 5\}$

$A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$

$B = \{x \mid x \in I \text{ and } 0 \le x \le 10\}$

$\therefore B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(i) $\therefore R_1 = \{<x, y> \mid x \in A \text{ and } y \in B \text{ and } x > y\}$

$\therefore R_1 = \{<1, 0>, <2, 0>, <2, 1>, <3, 0>, <3, 1>, <3, 2>, <4, 0>, <4, 1>, <4, 2>, <4, 3>, <5, 0>, <5, 1>, <5, 2>, <5, 3>, <5, 4>\}$

(ii) $R_2 = \{<x, y> \mid x \in A \text{ and } y \in B \text{ and } y > x^2\}$

$\therefore R_2 = \{<-3, 9>, <-2, 4>, <-1, 1>, <0, 0>, <1, 1>, <2, 4>, <3, 9>\}$

(iii) $R_3 = \{<x, y> \mid x \in A \text{ and } y \in B \text{ and } x + y < 0\}$

$\therefore R3 = \{<-5, 0>, <-5, 1>, <-5, 2>, <-5, 3>, <-5, 4>, <-4, 0>, <-4, 1>, <-4, 2>, <-4, 3>, <-3, 0>, <-3, 1>, <-3, 2>, <-2, 1>, <-1, 0>\}$

Ex. 2.9.2 Given two relations, R_1 and R_2, prove that $D(R_1 \cup R_2) = D(R_1) \cup D(R_2)$

☑ **Soln. :**

☞ **Proof :**

$D(R_1)$ = set of first elements of ordered pairs in R_1.

Now consider $D(R_1 \cup R_2)$

It consists of first elements of ordered pairs in either R_1 or R_2

i.e. if $x \in D(R_1)$ or $x \in D(R_2)$

then $x \in D(R_1 \cup R_2)$

$\therefore D(R_1 \cup R_2) = D(R_1) \cup D(R_2)$

Ex. 2.9.3 Given two relations R_1 and R_2.
Prove that $R(R_1 \cap R_2) \subseteq R(R_1) \cap R(R_2)$.

☑ **Soln. :**

☞ **Proof**

- $R(R_1)$ is a set of second elements of ordered pairs in R_1.
- $R(R_2)$ set of second elements of ordered pairs in R_2
- $R(R_1 \cap R_2)$ is a set of second elements of ordered pairs in $R_1 \cap R_2$
- Consider the relation $R_1 \cap R_2$. It has the ordered pairs that are present in both R_1 and R_2.
- i.e. if $x \in R(R_1 \cap R_2)$ these $x \in R(R_1)$ and $x \in R(R_2)$
- But we may have as element $y \in R(R_1)$ and $y \in R(R_2)$, but $y \notin R(R_1 \cap R_2)$.

For example Let

$R_1 = \{<1, 2>, <2, 3>\}$

$R_2 = \{<1, 2>, <1, 3>\}$

$R_1 \cap R_2 = \{<1, 2>\}$

Here $3 \in R(R_1)$, $3 \in R(R_2)$ and $3 \notin R(R_1 \cap R_2)$

$\therefore R(R_1 \cap R_2) \subseteq R(R_1) \cap R(R_2)$

Ex. 2.9.4 Let $R_1 = \{<1, 2>, <2, 4>, <3, 3>\}$
$R_2 = \{<1, 3>, <2, 4>, <4, 2>\}$
Find
$R_1 \cup R_2$, $R_1 \cap R_2$, $D(R_1)$, $D(R_2)$, $D(R_1 \cup R_2)$, $D(R_1 \cap R_2)$, $R(R_1)$, $R(R_2)$, $R(R_1 \cup R_2)$, $R(R_1 \cap R_2)$

☑ **Soln. :**

$R_1 \cup R_2 = \{<1, 2>, <2, 4>, <3, 3>, <1, 3>, <4, 2>\}$

$R_1 \cap R_2 = \{<2, 4>\}$, $R(R_1) = \{2, 4, 3\}$

$D(R_1) = \{1, 2, 3\}$, $R(R_2) = \{3, 4, 2\}$

$D(R_2) = \{1, 2, 4\}$, $R(R_1 \cup R_2) = \{2, 4, 3\}$

$D(R_1 \cup R_2) = \{1, 2, 3, 4\}$, $R(R_1 \cap R_2) = \{4\}$

$D(R_1 \cap R_2) = \{2\}$

Ex. 2.9.5 Find ranges of relations
$S = \{<x, x^2> \mid x \in N\}$ and
$T = \{<x, 2x> \mid x \in N\}$
Where $N = \{0, 1, 2, ...\}$
Find $T \cup S$ and $T \cap S$ also

☑ **Soln. :**

Given sets :

$S = \{<x, x^2> \mid x \in N\}$ $T = \{<x, 2x> \mid x \in N\}$

$N = \{0, 1, 2, ...\}$

Let us rewrite the sets.

$S = \{<0, 0>, <1, 1>, <2, 4>, <3, 9>, <4, 16>, ...\}$

$T = \{<0,0>, <1, 2>, <2, 4>, <3, 6>, <4, 8>, \ldots\}$

$\therefore R(S) = \{0, 1, 4, 9, \ldots\}$

$= \{x \mid \sqrt{x} \in N\}$

$R(T) = \{0, 2, 4, 6, \ldots\}$

$= \{x \mid x \in N \text{ and } x \bmod 2 = 0\}$

$T \cup S = \{<x, y> \mid x, y \in N \text{ and } (y = x^2 \text{ or } y = 2x)\}$

It is an infinite set

$T \cap S = \{<0, 0>, <2, 4>\}$

It is a finite set.

Ex. 2.9.6 Let L denote the relation, "less than or equal to" and D denote the relation "divides". The relations L and D are defined on the set {1, 2, 3, 6}. Write L and D as sets. Find $L \cap D$.

☑ **Soln. :**

Given set is {1, 2, 3, 6}.

The relation L is "less than or equal to"

$\therefore L = \{<1, 1>, <1, 2>, <1, 3>, <1, 6>, <2, 2>, <2, 3>, <2, 6>, <3, 3>, <3, 6>, <6, 6>\}$

The relation D is "divides".

$\therefore D = \{<1, 1>, <1, 2>, <1, 3>, <1, 6>, <2, 2>, <2, 6>, <3, 3>, <3, 6>, <6, 6>\}$

$L \cap D = \{<1, 1>, <1, 2>, <1, 3>, <1, 6>, <2, 2>, <2, 6>, <3, 3>, <3, 6>, <6, 6>\} = D$

Note : If A, B are sets, and R is relation on A to B then

$R \subseteq A \times B$

$A \times B$ is universal relation

$\varnothing$ is void relation

If R_1 and R_2 are relations, then

$D(R_1 \cup R_2) = D(R_1) \cup D(R_2)$

$R(R_1 \cup R_2) = R(R_1) \cup R(R_2)$

$D(R_1 \cap R_2) \subseteq D(R_1) \cap D(R_2)$

$R(R_1 \cap R_2) \subseteq R(R_1) \cap R(R_2)$

2.10 PROPERTIES AND TYPES OF A BINARY RELATION

- There are six properties of a relation, in a set. They are (i) reflexive (ii) irreflexive (iii) symmetric (iv) antisymmetric (v) asymmetric (vi) transitive.
- A relation may process zero or more properties. Note that these properties are considered only when a relation is in a set. i.e. both elements of the ordered pair belong to the same set.

Module II

1. Reflexive Relation

- Let A be a set. A relation R on set A is said to be reflexive, if every element in A is related to itself.

  i.e. $\forall x \in A, x R x$ x divides y

  Example : Let A = {1, 2, 3, 4, 5} R be the relation divides.

  $\therefore R = \{<1, 1>, <1, 2>, <1, 3>. <1, 4>, <1, 5>, <2, 2>, <2, 4>, <3, 3>, <4, 4>, <5, 5>\}$

  This is reflexive, as every element is related to itself. ✓

2. Irreflexive Relation

- Let A be a set. A relation R on set A is said to be irreflexive, if every element in A is not related to itself.

  i.e. $\forall x \in A, x R x$ ~~x R x~~

  or $\forall x \in A, <x, x> \notin R$

  Example : Let A = {1, 2, 3, 4, 5} and R be the relation "less than".

  $\therefore R = \{<1, 2>, <1, 3>, <1, 4>, <1, 5>, <2, 3>, <2,4>, <2, 5>, <3, 4>, <3, 5>, <4, 5>\}$

- This is irreflexive, as every element is not related to itself.

3 Symmetric Relation

- Let A be a set. A relation R on set a is said to be symmetric, if an element x is related to y, then y is also related to x.

  i.e. if xRy then yRx

Example : Let A = {1, 2, 3, 4, 5} and R be the relation defined as follows.

xRy if x + y is even

$\therefore$ R = {<1, 1>, <1, 3>, <1, 5>, <2, 2>, <2, 4>, <3, 1>, <3, 3>, <3, 5>, <4, 2>, <4, 4>, <5, 1>, <5, 3>, <5, 5>}

- This relation is symmetric. The relation is reflexive as well.

4. Antisymmetric Relation

Let A be a set. A relation R on set A is said to be antisymmetric, if we have xRy and yRx then x must be equal to y i.e. if $x \neq y$ then either xRy or xRx but not both.

Example

- Let A = {1, 2, 3, 4, 5} and R be a relation such that xRy if $y = x^2$

  $\therefore$ R = {<1, 1>, <2, 4>}

- This is antisymmetric because if we consider 2 and 4, $2R_4$ and $4R_2$. | R | allowed in case of antisymmetric.

5. Asymmetric Relation

- Let A be a set. A relation R on set A said to be asymmetric, if for any x, y, if we have xRy then yRx
- i.e. if xRy then yRx.
- The difference between asymmetric and antisymmetric relation is, ordered pair <x, x> is valid in antisymmetric not in asymmetric.

Example

- Let A = {1, 2, 3, 4, 5} and R be the relation 'greater than'

  $\therefore$ R = {<2, 1>, <3, 1>, <3, 2>, <4, 1>, <4, 2>, <4, 3>, <5, 1>, <5, 2>, <5, 3>, <5, 4>}

  This is asymmetric.

- Note that the relation is antisymmetric as well as irreflexive.

Syllabus Topic : Transitivity Relation

6. Transitive Relation

- Let A be a set. A relation R on set A is said to be transitive, if for any x, y, z $\in$ A, we have xRy and yRz then xRz.

  i.e. $\forall$ x, y, z, $\in$ A, if xRy and yRz then xRz.

Example

Let A = {1, 2, 3, 4, 5} and R be the relation, "less than or equal to".

$\therefore$ R = {<1, 1>, <1, 2>, <1, 3>, <1, 4>, <1, 5>, <2, 2>, <2, 3>, <2, 4>, <2, 5>, <3, 3>, <3, 4>, <3, 5>, <4, 4>, <4, 5>, <5, 5>}

- This relation is transitive because if $x \leq y$ and $y \leq z$ then $x \leq z$. Note that the relation is reflexive, and antisymmetric as well.

Note : Every reflexive relation is always not irreflexive and vice versa.

A relation can be neither reflexive nor irrflexive.

A relation can be both symmetric and antisymmetric.

Every asymmetric relation is antisymmetric.

Every symmetric relation is always not asymmetric and vice versa.

If a relation is transitive and symmetric need not be reflexive.

7. Equivalence Relation

A relation is said to be equivalence, if it is reflexive, symmetric and transitive.

Example

Consider the set A = {1, 2, 3, 4, 5}

Let R be the relation given by,

$\therefore$ R = {<x, y> | x, y $\in$ A and x + y is even}

$\therefore$ R = {<1, 1>, <1, 3>, <1, 5>, <2, 2>, <2, 4>, <3, 1>, <3, 3>, <3, 5>, <4, 2>, <4, 4>, <5, 1>, <5, 3>, <5, 5>}

This is an equivalence relation.

8. Partial Order Relation

A relation is said to be partial order, if it is reflexive, antisymmetric and transitive.

Example

Consider the set A = {1, 2, 3, 4}

Let R be the relation given by

$$R = \{<x, y> \mid x, y \in A \text{ and } x \le y\}$$

$$\therefore R = \{<1, 1>, <1, 2>, <1, 3>, <1, 4>, <2, 2>, <2, 3>, <2, 4>, <3, 3>, <3, 4>, <4, 4>\}$$

This is a partial order relation

2.11 SOLVED EXAMPLES OF TYPES OF RELATIONS

Ex. 2.11.1 Give as example of a relation which is neither reflexive nor irreflexive.

Soln. :

Let $A = \{1, 2, 3, \ldots, 10\}$

$R = \{<x, y> \mid x, y \in A \text{ and } y = x^2\}$

$R = \{<1, 1>, <2, 4>, <3, 9>\}$

- R is not reflexive because not every element is related to itself i.e. 2 R2, 3 R3 etc.
- R is not irreflexive because | R |

Ex. 2.11.2 Give as example of a relation which is both symmetric and antisymmetric.

Soln. :

Let $A = \{1, 2, 3, 4\}$

$R = \{<x, y> \mid x, y \in A \text{ and } x = y\}$

$\therefore R = \{<1, 1>, <2, 2>, <3, 3>, <4, 4>\}$

R is both symmetric and antisymmetric.

Note

= is the only relation which is both symmetric and antisymmetric. The relation can also be given as follows :

$R = \{<x, y> \mid x - y = 0\}$

$R = \{<x, y> \mid \frac{x}{y} = 1)\}$

Ex. 2.11.3 Let R be a relation on the set of all positive integers, such that R = {<a, b> | a – b is an odd positive integer}. Is R reflexive ? irreflexive ? Symmetric ? Antisymmetric ? transitive ?

Soln. :

$R = \{<a, b> \mid a - b \text{ is an odd positive integer}\}$

Let x is a positive integer.

Consider x – x = 0, not a positive integer

i.e. $\forall x \in I^+, x \not R x$

$\therefore$ R is not reflexive

R is irreflexive

Let $x, y \in I^+$ such that x – y = odd positive integer

Assume that x – y = K

$\therefore <x, y> \in R$

Now consider y – x = – k, a negative number

$\therefore <y, x> \notin R$

$\therefore$ R is not symmetric

R is a symmetric and hence antisymmetric

Consider $x, y, z \in I^+$ such that xRy and yRz.

$\therefore x - y = K \quad y - z = p$

where K and p are positive odd integers.

By adding both equations, we get,

$$x - y + y - z = K + p$$

$$\therefore x - z = K + p$$

K + p must be positive even integer because sum of two odd integers is even.

$\therefore x - z$ = even integer

$\therefore x \not R z$

Module II

i.e. if xRy and yRz then $x \not R z$

$\therefore$ R is not transitive.

Ex. 2.11.4 Let R be a symmetric and transitive relation on set A. Show that if for every a in A there exists to in A such that (a, b) is in R, then R is an equivalence relation.

Soln. :

Given that relation R is symmetric and transitive, we need to prove that R is reflexive, so that it is equivalence.

It is given that for every a in A. there exist in A such that < a, b > is in R

i.e. $\forall_a, \exists_b, <a, b> \in R$

If $<a, b> \in R$, then $<b, a> \in R$ because R is symmetric

If $<a, b> \in R$ and $<b, a> \in R$ then $<a, a> \in R$ because R is transitive.

$\therefore \forall a \in A, <a, a> \in R$

$\therefore$ R is reflexive.

$\therefore$ R is equivalence.

Ex. 2.11.5 Let A be a set with 10 distinct elements.

(a) How many different relations on A are there ?

(b) How many of them are reflexive ?

(c) How many of them are symmetric ?

Soln. :

A relation is subset of $A \times A$

The number of ordered pairs in $A \times A$ is $10 \times 10 = 100$

i.e. $|A| = 10 \quad |A \times A| = 100$

(a) Number of relations on A is given by the number of subsets of $A \times A$.

$A \times A$ has 100 elements.

$\therefore$ Number of subsets $= 2^{|A \times A|} = 2^{100}$

$\therefore$ There are 2^{100} relations on A.

(b) A reflexive relation should have the ordered pair $<a, a> \forall a \in A$

i.e. minimum number of ordered pairs is 10.

$\therefore$ Remaining ordered pairs are 90.

The 10 ordered pairs can be combined with zero or more of remaining ordered pairs to form a reflexive relation.

This turns to be 2^{90}

UEx. 2.11.6 **MU - Q. 2(b). Dec.13. 6 Marks**

Let m be the positive integers greater than 1. Show that the relation R = {(a, b) I a R b (mod m)}, i.e. a R b if and only if m divides a-b, is an equivalence relation on the set of integers.

Soln. :

Given relation is, R = {(a, b) I a = b (mod m)} where m is positive integer greater than 1.

To prove : R is equivalence

Proof : A relation is said to be equivalence if it is reflexive, symmetric and transitive.

$a R b \Rightarrow a = b \pmod m \Rightarrow (a - b) \bmod m = 0$

(i) A relation is reflexive, if x Rx for all x.

Let us check whether x Rx

i.e. $(x - x) \bmod m = 0 \bmod m = 0$

$\therefore$ x is related to x.

$\therefore$ R is reflexive.

(ii) A relation is said to be symmetric, if for all x, y if x R y then y R x.

Consider $x R_y$

$\therefore (x - y) \bmod m = 0$

$\therefore (x - y) = mz.$ Where $Z \in I$

$\therefore y - x = -mz$

$\therefore (y - x) \bmod m = 0$

$\therefore$ y Ry

$\therefore$ R is symmetric

(iii) A relation is said to be transitive if for all x, y, z if x Ry and y Rz then xRz.

Consider x Ry

$\therefore (x - y) \bmod m = 0$

$\therefore x - y = ma$ where $a \in I$ ...(i)

Consider y Rz

$\therefore (y - z) \bmod m = 0$

$\therefore y - z = mb$ where $b \in I$ ...(ii)

Adding (i) and (ii) we get,

$x - z = m(a + b)$

$\therefore (x - z) \bmod m = 0$

$\therefore$ x Rz

$\therefore$ R is transitive

$\therefore$ R is equivalence

Hence the proof.

UEx. 2.11.7 MU - Q. 1(b), May 14, 5 Marks

Show that if a relation on set A is transitive an irreflexive, then it is asymmetric.

☑ Soln. :

To prove, if a relation on set A is transitive and irreflexive then it is asymmetric.

☞ Proof :

Let R be the relation on set A which is transitive and irreflexive.

i.e. $\forall x \in A$, $x \not R x$ {irreflexive}

$\forall x, y, z \in A$, if x Ry and y Rz then xRz {transitive}

Assume that R is symmetric.

Consider any $x, y \in A$, and xRy

$\therefore$ yRy as per the assumption.

Now we have, xRy and yRy

$\therefore$ xRx as R is transitive.

But $x \not R x$ because, R is irreflexive

This violates the assumption of R is symmetric.

$\therefore$ R is asymmetric.

Hence the proof.

UEx. 2.11.8 MU - Q. 2(b), May 14, 8 Marks

Let R be a relation on the set of integers Z defined by a R b if and only if $a \equiv m \pmod 5$. Prove that R is an equivalence relation. Find Z/R.

☑ Soln. :

Given relation R on set of integers Z, aRb if and only if $a \equiv b \pmod 5$

To prove R is equivalence.

☞ Proof :

Given relation R is as follows,

if $a, b \in z$, aRb if $a \equiv b \pmod 5$

i.e. $(a - b) \bmod 5 = 0$

(i) Consider $\forall a \in Z$,

$(a - a) \bmod 5 = 0 \bmod 5 = 0$

$\therefore$ aRa

$\therefore$ R is reflexive

(ii) Consider $\forall a, b \in Z$,

if aRb then, $(a - b) \bmod 5 = 0$

$\therefore (b - a) \bmod 5 = 0$

$\therefore$ bRa

$\therefore$ R is symmetric.

(iii) Consider for $a, b, c \in Z$, let aRb and bRc then

$(a - b) \bmod 5 = 0$ and $(b - c) \bmod 5 = 0$

$\therefore a - b = 5x$ and $b - c = 5y$

where $x, y \in I$

$\therefore$ By adding above equations we get,

$$a - b + b - c = 5x + 5y$$

$$\therefore a - c = 5(x + y)$$

$$\therefore (a - c) \bmod 5 = 0$$

$\therefore$ aRc

$\therefore$ R is transitive

$\therefore$ R is equivalence

Z | R is a set of partitions of Z, such that the elements in one partition are related by R.

Ex. Consider $A_0 = \{0, 5, 10, \ldots\}$, $A_1 = \{1, 6, 11, 16, \ldots\}$, $A_2 = \{2, 7, 12, 17, \ldots\}$, $A_3 = \{3, 8, 13, 18, \ldots\}$, $A_4 = \{4, 9, 14, 19, \ldots\}$ are the partitions. To explain it, let us consider the set A_1, $\forall a, b \in A_1$, aRb

ex. 1R6, 1R11, 6R11, 6R16 etc

Module II

$Z \mid R = \{\{0, 5, \ldots, 5n\}, \{1, 6, 11, \ldots, 5n+1\}, \{2, 7, 12, \ldots, 5n+2\}, \{3, 8, 13, \ldots, 5n+3\}, \{4, 9, 14, \ldots, 5n+4\}\}$

UEx. 2.11.9 **MU - Q. 1(c). Dec. 14. 5 Marks**

Show that a relation is reflexive and circular if and only if it is an equivalence relation.

Soln. :

A relation R is said to be circular, if for any a, b, c we have aRb and bRc then cRa.

To prove :

A relation is reflexive and circular if and only if it is an equivalence relation.

Proof :

Let a relation R be reflexive and circular.

$\therefore \forall a$, we have aRa

and if aRb then we have

bRb (reflexive) and bRa (Circular)

i.e. aRb and bRb $\Rightarrow$ bRa

$\therefore$ if aRb then bRa

$\therefore$ R is symmetric.

Now consider if aRb and bRc then we have cRa (circular)

$\therefore$ aRc (Symmetric)

i.e. if aRb and bRc $\Rightarrow$ aRc

$\therefore$ R is transitive.

$\therefore$ R is equivalence.

$\therefore$ If R is reflexive and circular, then R is equivalence

Hence the proof

UEx. 2.11.10 **MU - Q. 6(c). Dec. 14. 8 Marks**

R be a relation on set of integers Z defined by

$R = \{(x, y) \mid x - y \text{ is divisible by } 3\}$

Show that R is an equivalence relation and describe the equivalence classes.

Soln. :

Given relation is, $R = \{(x, y) \mid (x - y) \text{ is divisible by } 3\}$

To prove : R is equivalence

Proof :

Given relation R is as follows,

if $a, b \in z$, aRb if $a \equiv b \pmod 3$

i.e. $(a - b) \bmod 3 = 0$

(i) Consider $\forall a \in Z$,

$(a - a) \bmod 3 = 0 \bmod 3 = 0$

$\therefore$ aRa

$\therefore$ R is reflexive

(ii) Consider $\forall a, b \in Z$,

if aRb then, $(a - b) \bmod 3 = 0$

$\therefore (b - a) \bmod 3 = 0$

$\therefore$ bRa

$\therefore$ R is symmetric.

(iii) Consider for $a, b, c \in Z$, let aRb and bRc then

$(a - b) \bmod 3 = 0$ and $(b - c) \bmod 3 = 0$

$\therefore a - b = 3x$ and $b - c = 3y$

where $x, y \in I$

$\therefore$ By adding above equations we get,

$$a - b + b - c = 3x + 3y$$

$$\therefore a - c = 3(x + y)$$

$$\therefore (a - c) \bmod 3 = 0$$

$\therefore$ aRc

$\therefore$ R is transitive

$\therefore$ R is equivalence

$Z \mid R$ is a set of partitions of Z, such that the elements in one partition are related by R.

ex. Consider $A_0 = \{0, 3, 10, \ldots\}$, $A_1 = \{1, 6, 11, 16, \ldots\}$, $A_2 = \{2, 7, 12, 17, \ldots\}$, $A_3 = \{3, 8, 13, 18, \ldots\}$, $A_4 = \{4, 9, 14, 19, \ldots\}$ are the partitions. To explain it, let us consider the set A_1,

$\forall a, b \in A_1$, aRb

ex. 1R6, 1R11, 6R11, 6R16 etc

$Z \mid R = \{\{0, 3, \ldots, 3n\}, \{1, 6, 11, \ldots, 3n+1\}, \{2, 7, 12, \ldots, 3n+2\}, \{3, 8, 13, \ldots, 3n+3\}, \{4, 9, 14, \ldots, 3n+4\}\}$

UEx. 2.11.11 MU - Q. 2(c), May 15, 8 Marks

Define Equivalence relation with an example. Let m be a positive integer other than 1. Show that the relation $R = \{(a, b) \mid a = b \pmod m\}$ i.e. m divides a-b is an equivalence relation on the set of integers.

Soln. :

Given relation is, $R = \{(a, b) \mid a = b \pmod m\}$ where m is positive integer greater than 1.

To prove :

R is equivalence

Proof :

A relation is said to be equivalence if it is reflexive, symmetric and transitive.

$a R b \Rightarrow a = b \pmod m \Rightarrow (a - b) \bmod m = 0$

(i) A relation is reflexive, if x Rx for all x.

Let us check whether x Rx

i.e. $(x - x) \bmod m = 0 \bmod m = 0$

$\therefore$ x is related to x.

$\therefore$ R is reflexive.

(ii) A relation is said to be symmetric, if for all x, y if x R y then y R x.

Consider x Ry

$\therefore (x - y) \bmod m = 0$

$\therefore (x - y) = mz$ Where $Z \in I$

$\therefore y - x = -mz$

$\therefore (y - x) \bmod m = 0$

$\therefore$ y Rx

$\therefore$ R is symmetric.

(iii) A relation is said to be transitive if for all x, y, z if x Ry and y Rz then xRz.

Consider x Ry

$\therefore (x - y) \bmod m = 0$

$\therefore x - y = ma$ where $a \in I$ ...(i)

Consider y Rz

$\therefore (y - z) \bmod m = 0$

$\therefore y - z = mb$ where $b \in I$ ...(ii)

Adding (i) and (ii) we get,

$x - z = m(a + b)$

$\therefore (x - z) \bmod m = 0$

$\therefore$ x Rz

$\therefore$ R is transitive.

$\therefore$ R is equivalence.

Hence the proof.

UEx. 2.11.12 MU - Q. 2(c), Dec. 15, 6 Marks

Let S = {1,2,3,4,5} and A- S × S. Define the following relation R on A : (a,b) R (c,d) if and only if ad = bc. Show that R is an equivalence relation and compute A/R.

Soln. :

Given : $S = \{1, 2, 3, 4, 5\}$, $A = S \times S$

The relation R on A is given by,

(a, b) R (c, d) if and only if ad = bc

To prove :

R is equivalence relation.

Proof :

(i) Consider $\forall$ a, b, $\in$ S, (a, b) R (a, b)

as ab = ab

$\therefore$ R is reflexive.

(ii) Consider if (a, b) R (c, d) then,

We have ad = bc

$\therefore$ cb = da

$\therefore$ (c, d) R (a, b)

$\therefore$ R is symmetric

(iii) Consider if (a, b) R (c, d) and (c, d) R (e, f) then we have,

ad = bc and cf = de

$\therefore \frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$

Module II

$$\therefore \frac{a}{b} = \frac{e}{f}$$

$$\therefore af = be$$

$\therefore$ (a, b) R (e, f)

$\therefore$ R is transitive.

$\therefore$ R is equivalence.

We have to compute A/R.

A/R = {{(1,1), (2, 2), (3, 3), (4, 4), (5, 5)}, {(1, 2), (2, 4)}
{(1, 3)}, {(1, 4)}, {(1, 5)}, {(2, 1), (4, 2)}
{(2, 3)}, {(2, 5)}, {(3, 1)}, {(3, 2)}, {(3, 4)}, {(3, 5)}
{(4, 1)}, {(4, 3)}, {(4, 5)}, {(5, 1)}, {(5, 2)} {(5, 3)}
{(5, 4)} }

UEx. 2.11.13 **MU - Q. 5(c), May 17, 8 Marks**

Let A be a set of integers, let R be a relation on AXA defined by (a,b) R (c,d) if and only if a + d = b + c. Prove that R is an equivalence Relation.

☑ Soln. :

Given set is A - set of integers.

R is relation on $A \times A$ given by,

(a, b) R (c, d) if and only if $a + d = b + c$

☞ To prove :

R is equivalence.

(i) Consider

$$a + b = a + b$$

$$\therefore a + b = b + a$$

$\therefore$ (a, b) R (a, b)

$\therefore$ R is reflexive.

(ii) Consider if (a, b) R (c, d)

$$\therefore a + d = b + c$$

$$\therefore d + a = c + b$$

$$\therefore c + b = d + a$$

$$\therefore (c, d) \ R \ (a, b)$$

$\therefore$ R is symmetric.

(iii) Consider (a, b) R (c, d) and (c, d) R (e, f)

$\therefore a + d = b + c$ and $c + f = d + e$

$\therefore a - b = c - d$ and $c - d = e - f$

$\therefore a - b = e - f$

$\therefore a + f = b + e$

$\therefore$ (a, b) R (e, f)

$\therefore$ R is transitive.

$\therefore$ R is equivalence.

UEx. 2.11.14 **MU - Q. 1(c), May 2014, 5 Marks**

Show that if a relation on set A is transitive and irreflexive, then it is asymmetric.

☑ Soln. :

Given : The relation R is transitive and irreflexive.

i.e. . □a∈A, (a,a) ∉ R and

if (a,b) ∈ R and (b,c) ∈ R then (a,c) ∈ R

Assume that the relation R is symmetric.

Then if (a,b) ∈ R then (b,a) ∈ R.

But R is transitive.

Therefore, (a,a) ∈ R.

But R is irreflexive. And this becomes contradiction.

Therefore R is asymmetric.

Ex. 2.11.14(A) Find f^{-1}, if $f(x) = \frac{(4x+3)}{(5x-2)}$

☑ Soln. :

$$\text{Given } f(x) = \frac{(4x+3)}{(5x-2)}$$

To find f^{-1}.

$$\text{Let } f(x) = y = \frac{4x+3}{5x-2}$$

$$\therefore f^{-1}(y) = x$$

Now let us solve for x in terms of y.

$$\therefore y = \frac{4x+3}{5x-2}$$

$$\therefore 5xy - 2y = 4x + 3$$

$$\therefore 5xy - 4x = 3 + 2y$$

$$\therefore x(5y - 4) = 3 + 2y$$

$$\therefore x = \frac{3+2y}{5y-4}$$

Writing in terms of x, we get,

$$\therefore f^{-1}(x) = \frac{2x+3}{5x-4}$$

UEx. 2.11.15 MU - Q. 5(c), May 18,4 Marks

Let A = {a, b, c, d, e, f, g, h}. Consider the following subsets of A

A1 = {a, b, c, d} A2 = {a, c, e, g, h}

A3 = {a, c, e, g} A4 = {b, d} A5 = {f, h}

Determine whether following is partition of A or not. Justify your answer.

(i) (A1, A2) (ii) (A3, A4, A5)

Soln. :

Given : A = {a, b, c, d, e, f, g, h}

A1 = {a, b, c, d}

A2 = {a, c, e, g, h}

A3 = {a, c, e, g} A4 = {b, d}, A5 = {f, h}

(i) Consider {A1, A2}

A1 ∪ A2 = {a, b, c, d, e, g, h} ≠ A

∴ {A1, A2} is not a partition of A.

(ii) Consider {A3, A4, A5}

A3 ∪ A4 ∪ A5 ={a, c, e, g, b, d, f, h} = A

A3 ∩ A4 = ϕ , A3 ∩ A5 = ϕ, A4 ∩ A5 = ϕ

∴ {A3, A4, A5} is a partition of A.

UEx. 2.11.16 MU – Q. 4(a). Dec. 18,6 Marks

Let R is a binary relation. Let S = {(a, b) | (a, c) ∈ R and (c, b) ∈ R for some c }. Show that if R is an equivalence relation then S is also an equivalence relation.

Soln. :

Given : R is binary relation.

S = {(a, b) | (a, c) ∈ R and (c, b) ∈ R for some c}

To show :

If R is equivalence then S is also equivalence.

Proof :

R is equivalence.

∴ R is reflexive, symmetric and transitive.

(i) Let us test whether (a, a) ∈ S or not.

(a, a) ∈ R as R is reflexive.

∴ We can write a = a, b = a and c = a.

∴ If (a, c) ∈ R and (c, b) ∈ R ∴ (a, b) ∈ S.

∴ S is reflexive.

(ii) Let us test whether S is symmetric or not.

Consider (a, b) ∈ S

∴ We have a c. Such that, (a, c) ∈ R and (c, b) ∈ R

As R is symmetric, we have

(c, a) ∈ R and (b, c) ∈ R

i.e. (b, c) ∈ R and (c, a) ∈ R

∴ (b, a) ∈ S

∴ S is symmetric.

(iii) Let us test whether S is transitive or not.

Let (a, b) ∈ S and (b, d) ∈ S

∴ We have c and e such that,

(a, c) ∈ R and (c, b) ∈ R

(b, e) ∈ R and (e, d) ∈ R,

As R is transitive, (c, e) ∈ R, (c, d) ∈ R

∴ (a, c) ∈ R, (c, d) ∈ R

∴ (a, d) ∈ S

∴ We get (a, b) ∈ S, (b, d) ∈ S → (a, d) ∈ S.

∴ S is transitive.

∴ S is equivalence.

UEx. 2.11.17 MU - Q. 3(b), May 18,8 Marks

Let A = {1, 2, 3, 4, 5}, let R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)} and S = {(1, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (4, 5)} be the relations on A. Find the smallest equivalence relation containing the relation R and S.

Soln. :

Given A = {1, 2, 3, 4, 5} R and S are relations on A.

R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)}

S = {(1, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (4, 5)}

To find : Smallest equivalence relation containing R and S.

∴ R ∪ S = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (4, 5), (5, 4), (5, 5)}

Module II

$\forall\ a \in A,\quad (a, a) \in R \cup S$

$\therefore\ R \cup S$ is reflexive.

$\forall\ (a, b) \in R \cup S$, we have $(b, a) \in R \cup S$.

$\therefore\ R \cup S$ is symmetric.

if $(a, b) \in R \cup S$ and $(b, c) \in R \cup S$ then $(a, c) \in R \cup S$.

$R \cup S$ is transitive.

$\therefore\ R \cup S$ is the required equivalence relation.

UEx. 2.11.18 **MU - Q. 2(c), Dec. 13, 6 Marks**

Let S = {1, 2, 3, 4} and A= S x S. Define the following relation :

R on A : (a, b) R (a′, b′) if and only if $a + b = a' + b'$.

(i) Show that R is an equivalence relation.

Soln. :

Given :

$$S = \{1, 2, 3, 4\} \quad \text{and} \quad A = S \times S$$

R is given by,

$(a, b)\ R\ (a', b')$ if and only if $a + b = a' + b'$

To prove : R is equivalence

(i) Consider $a + b = a + b$

$\therefore (a, b)\ R\ (a, b)$

$\therefore$ R is reflexive.

(ii) Consider, if (a, b) R (c, d) and (c, d) R (e, f) then $a + b = c + d$ and $c + d = e + f$

$\therefore a + b = e + f$

$\therefore (a, b)\ R\ (e, f)$

$\therefore$ R is transitive.

(iii) Let (a, b) R (c, d)

then $a + b = c + d$

$\therefore c + d = a + b$

$\therefore (c, d)\ R\ (a, b)$

$\therefore$ R is symmetric.

$\therefore$ R is equivalence relation.

2.12 OPERATIONS ON RELATIONS

2.12.1 Converse of a Relation

Definition : The converse of a relation R is denoted as R^{-1}, it is given by,

If $< x, y > \in R$ then $< y, x > \in R^{-1}$. It is also represented as $\tilde{R}$.

If R is reflexive then R^{-1} is also reflexive.

If R is symmetric, then R^{-1} is also symmetric.

If R is transitive, then R^{-1} is also transitive.

Example 1

Find he converse of relation given by

$$R = \{<1, 2>, <2, 3>, <2, 2>, <3, 1>\}$$

$$R^{-1} = \{<2, 1>, <3, 2>, <2, 2>, <1, 3>\}$$

Example 2

Find the converse of relation given by

$$R = \{<a, b> \mid a, b \in N \text{ and } b = 2a\}$$

$$R^{-1} = \{<a, b> \mid a, b \in N \text{ and } a = 2b\}$$

Closure of Relations

Sometimes it is necessary to have a particular type of relation. In such cases we can get that relation by including few ordered pairs into the give relation such relation is known as closure of relation.

2.12.2 Reflexive Closure

Definition : Reflexive closure of a relation is the relation that is obtained by including ordered pairs that are missing in the original relation so that it becomes reflexive.

Example

Find reflexive closure of the relation.

$$R = \{<1, 2>, <2, 2>, <2, 3>, <4, 2>, <4, 1>, <4, 4>\}$$

This is not a reflexive relation because of the missing ordered pairs $< 1, 1 >$ and $< 3, 3 >$

$\therefore$ The reflexive closure is

$R_{reflexive} = \{<1, 1>, <3, 3>, <1, 2>, <2, 2>, <2, 3>, <4, 2>, <4, 1>, <4, 4>\}$

2.12.3 Symmetric Closure

Definition : Symmetric closure of a relation is the relation that is obtained by including ordered pairs that are missing in the original relation so that it becomes symmetric.

Example

Find symmetric closure of the relation

$R = \{<1, 2>, <2, 2>, <2, 3>, <4, 2>, <4, 1>, <4, 4>\}$

In this case, we have to include the missing ordered pairs so that the relation becomes symmetric.

For example, for ordered pair <1, 2> include <2, 1>

Thus symmetric closure is given by

$R_{symmetric} = \{<1, 2>, <2, 2>, <2, 3>, <4, 2>, <4, 1>, <4, 4>, <2, 1>, <3, 2>, <2, 4>, <1, 4>\}$

2.12.4 Transitive Closure

Definition : Transitive closure of a relation is the relation that is obtained by including ordered pairs that are missing in the original relation so that it becomes transitive.

Example

Find transitive closure of the relation

$R = \{<1, 2>, <2, 2>, <2, 3>, <4, 2>, <4, 1>, <4, 4>\}$

It is difficult (not impossible) to find transitive closure just by looking at the ordered pairs.

One easier method is to draw graph and check the paths. The graph for the given relation is shown in Fig. 2.12.1

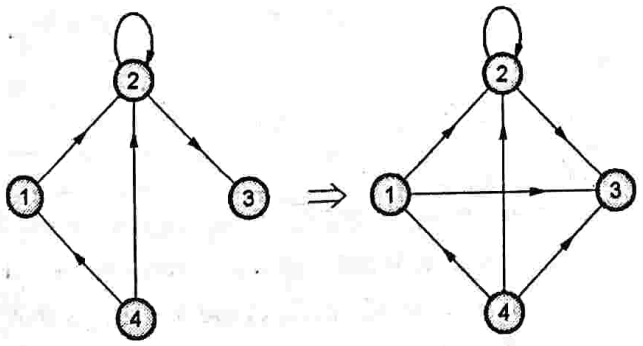


Fig. 2.12.1

It can be observed that, there is a path from 1 to 2 and 2 to 3, so include 1 to 3. There is a path from 4 to 2 and 2 to 3, so include 4 to 3

Thus transitive closure is given by,

$R_{transitive} = \{<1, 2>, <2, 2>, <2, 3>, <4, 2>, <4, 1>, <4, 4>, <1, 3>, <4, 3>\}$

2.12.5 Complement of a Relation

Definition : For a relation R, there exists a complement relation R′ such that $\forall <a, b> \notin R$, $<a, b> \in R'$. i.e. it is a collection of ordered pairs that are not present in the relation.

Example

Let $R = \{<1, 2>, <2, 2>, <2, 3>, <3, 1> <3, 2>\}$ be a relation on $A = \{1,2,3\}$ then

$R' \{<1, 1>, <1, 3>, <2, 1>, <3, 3>\}$

Note :

1. If R is reflexive then R′ is irreflexive and vice versa.
2. If R is symmetric then R′ is also symmetric
3. If R is antisymmetric, then R′ need not be antisymmetric.
4. If R is a relation from set A to B then $R \cup R' = A \times B$

2.12.6 Composition of Relations

Definition : Let R_1 be a relation from set A to B, R_2 be a relation from set B to C then composition of relation R_1, R_2 represented as $R_1 \circ R_2$ is given by, $R_1 \circ R_2 = \{<a, c> \mid a \in A, c \in C$ such that $\exists\, b \in B$, $<a, b> \in R_1$ and $<b, c> \in R_2\}$

Example

Let $R_1 = \{<1, 2>, <1, 3>, <3, 2>, <3, 3>, <2, 4>, <4, 1>\}$

$R_2 = \{<2, 2>, <2, 1>, <1, 3>, <4, 3>, <3, 1>, <3, 3>, <4, 2>\}$

Module II

Then,

$R_1 \circ R_2 = \{<1, 2>, <1, 1>, <1, 3>, <3, 2>, <3, 1>, <3, 3>, <2, 3>, <2, 2>, <4, 3>\}$

This can be explained as follows :

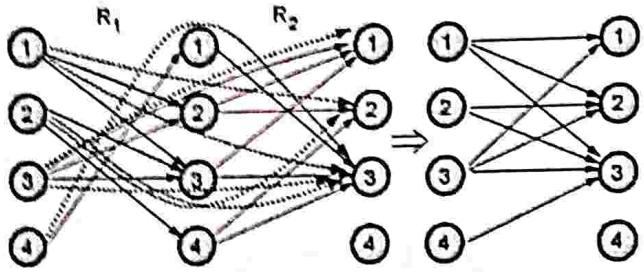


Fig. 2.12.2

2.13 PROPERTIES OF RELATION MATRICES AND GRAPHS OF DIFFERENT TYPES OF RELATIONS

1. The relation matrix of a reflexive relation has all its principal diagonal elements as 1. The graph of a reflexive relation has self loop for every vertex.
2. The relation matrix of a symmetric relation is symmetric.

   The graph of a symmetric relation has either zero or 2 edges between any two nodes.
3. The graph of a transitive relation has a path from node a to c, if there are paths from node a to b and from b to c.

Thus by looking at the matrix and graph, the property of the relation can be recognized.

Ex. 2.13.1 Write the relation matrix and draw graph of following relations.

$R_1 = \{<1, 1>, <2, 2>, <2, 3>, <3, 3>, <3, 1>, <4, 4>, <4, 2>\}$

☑ **Soln. :** It is a reflexive relation

Matrix

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 1 \end{array}$$

Graph

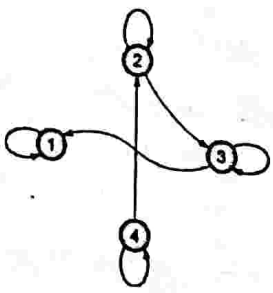


Fig. (a)

$R_2 = \{<1, 2>, <2, 1>, <3, 2>, <2, 3>, <4, 3>, <3, 4>, <4, 1>, <1, 4>\}$

It is a symmetric relation

Matrix

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 1 & 0 & 1 & 0 \end{array}$$

Graph

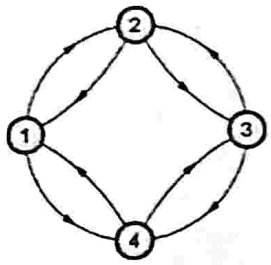


Fig. (b)

$R_2 = \{<1, 2>, <2, 2>, <2, 4>, <1, 4>, <4, 3>, <1, 3>, <2, 3>\}$

It is a transitive relation

Matrix

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

Graph

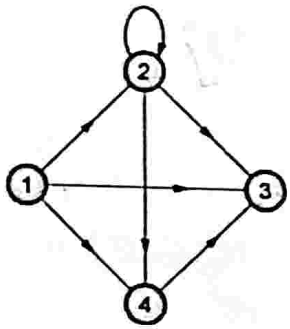


Fig. (c)

Ex. 2.13.2 State whether the following statements are True or False.

1. The relation ⊥ (perpendicular) on the set L of lines in the plane is reflexive.
2. Among 13 people; there are at least two, who are born in the same month is an example of Dirichlet drawer principle.
3. London is in Denmark is a proposition.

☑ **Soln. :**

1. False 2. False 3. True

Ex. 2.13.3 Let A = {1, 2, 3,}, B = {a, b,c}, and C = {x, y, z}. Consider the following relations R and S from A to B and from B to C, respectively.

R = {1, b}, (2, a), (2, c) and S = {(a, y), (b, x), (c, y), (c, z)}

(a) Find the composition relation R∘S.

(b) Find the matrices M_R, M_s, and $M_{R\circ s}$ of the respective relations R, S, and R∘S, and compare $M_{R\circ s}$ to the product $M_R M_s$.

Soln. :

Given A = {1, 2, 3},

B = {a, b, c}, C = {x, y, z}

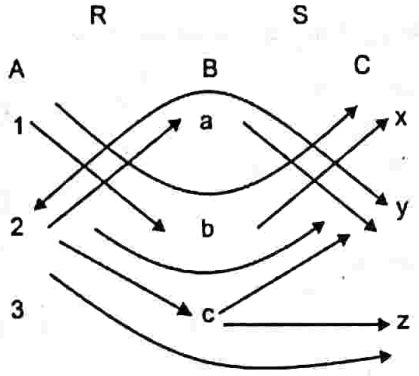
R is relation from A to B

S is relation from B to C

R = {(1, b), (2, a), (2, c) }

S = {(a, y), (b, x), (c, y), (c, z) }

(a) Composition of relation R∘S is given as follows.



R∘S = {(1, x), (2, y), (2, z)}

(b) Following are matrix representations.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$M_S = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$M_{R\circ S} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$M_R M_S = \begin{bmatrix} 0\times0+1\times1+0\times0 & 0\times0+1\times0+0\times1 & 0\times1+1\times0+0\times1 \\ 1\times0+0\times1\times0 & 1\times0+0\times0+1\times1 & 1\times1+0\times0+1\times1 \\ 0\times0+0\times1+0\times0 & 0\times0+0\times0+0\times1 & 0\times1+0\times0+0\times1 \end{bmatrix}$$

In this case, × is and operation

+ is or operation.

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore M_{R\circ S} = M_R \cdot M_S$

Ex. 2.13.4 Consider the following five relations on the set A = {1, 2, 3}:

R = {(1, 1), (1, 2), (1, 3), (3, 3)}, ∅ = empty relation

S = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}, A × A = universal relation

T = {(1, 1), (1, 2), (2, 2), (2, 3)}

Determine whether or not each of the above relations on A is : (a) reflexive; (b) symmetric; (c) transitive; (d) antisymmetric.

Soln. :

Given set A = {1, 2, 3}

Given relations

R = {(1, 1), (1, 2), (1, 3), (3, 3) }

ϕ = empty relation

S = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}

A × A = Universal relation

T = {(1, 1), (1, 2), (2, 2), (2, 3)}

a) Reflexive relations - S, A × A

b) Symmetric relations - S, A × A, ϕ

c) Transitive relations - R, ϕ, S, A × A

d) Antisymmetric relations – T

Ex. 2.13.5 Consider the relation R = {(1, 3), (1, 4), (3, 2), (3, 3), (3, 4)} on A = {(1, 2, 3, 4)}.

(a) Find the matrix M_R of R.

(b) Find the domain and range of R.

(c) Find R^{-1}.

(d) Draw the directed graph of R.

Soln. :

Given relation

R = {(1, 3), (1, 4), (3, 2), (3, 3), (3, 4) }

Set A = {1, 2, 3, 4}

(a)

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(b) Domain of R = {1, 3}

Range of R = {2, 3, 4}

(c) R^{-1} = {(3, 1), (4, 1), (2, 3), (3, 3), (4, 3)}

(d) Directed graph of R

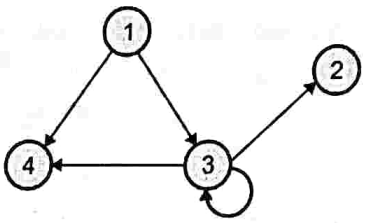


Fig. P. 2.13.5

2.14 WARSHALL'S ALGORITHM

In the previous section, it has been observed that finding a transitive closure of a relation is a crucial task Stephen Warshall proposed an algorithm to find the transitive closure of a relation.

Following are the steps in the algorithm.

1. Find relation matrix for the relation. W, let w_{ij} represent i^{th} row, j^{th} column element
2. For i = 1 to n where n is number of vertices repeat the steps 3, 4.
3. Consider i^{th} row and i^{th} column
4. $\forall_{p,q}$ where $w_{ip} = 1$ and $w_{qi} = 1$, assign $w_{qp} = 1$
5. The resultant matrix represents transitive closure

Ex. 2.14.1 Find transitive closure for the relation

R = {< 1, 2 >,< 2, 2 >,< 2, 3 >,< 4, 2 >, <4, 1 >, <4, 4 >}

Soln. :

The relation matrix of the given relation is

$$W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Step 1 : Consider first row and first column

$\omega_{12} = 1$ and $\omega_{41} = 1$

$\therefore$ ω_{42} must be made 1

It is already 1 in this case so no change

$$W_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \Rightarrow W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Convert to 1

Step 2 : Consider second row and second column

$\omega_{23} = 1$ and $\omega_{12} = 1$ $\therefore$ $\omega_{13} = 1$

$\omega_{23} = 1$ and $\omega_{42} = 1$ $\therefore$ $\omega_{43} = 1$

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \Rightarrow W_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Convert to 1

Convert to 1

Step 3 : Consider third row and third column. No change in the matrix as third row does not contain any 1.

$$W_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow W_3 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Step 4 : Consider fourth row and fourth column. No change in the matrix as fourth column has a 1 in the fourth row, so no intersection point could be found.

$$\overset{W_3}{\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}} \Rightarrow \overset{W_4}{\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}}$$

$R_{transitive}$ = {< 1, 2 >,< 1, 3 >,< 2, 2 >,< 2, 3 >, < 4, 1 >,< 4, 2 >,< 4, 3 >, < 4, 4 >}

C Program for Warshall's Algorithm

```
# include <stdio.h>
void main ( )
{
   int r[10] [10], i, j, k n ;
   printf("How many rows in the relation matrix?\n");
   scanf ("%d &n");
   printf ("enter the relation matrix \n");
   for(i = 0; i < n; i ++)
      for(j = 0; j < n; j + +)
         scanf("% d", &r[i] [j]);
   for(k = 0; k < n; k++)
      for(i = 0; i < n; i++)
         for(j = 0; j < n; i++)
         r[i][j] = r[i][j] | | (r[i][k] && r[k] [j]);}
         printf("relation matrix representing
               transitive closure \n");
   for(i = 0; i < n; i++)
   {
      for(j = 0; j  < n; j++)
      printf("% %d", r[i][j]);
      printf ('\n');
   }
}
```

2.15 SOLVED EXAMPLES

Ex. 2.15.1 Find the converse of the relation on set N

$R = \{< a, b > | a, b \in N \text{ and } b = a + 2\}$

Soln. : The converse of R is given by

$R^{-1} = \{< a, b > | a, b \in N \text{ and } b = a - 2\}$

i.e. $R = \{< 1, 3 >, < 2, 4 >, < 3, 5 >, \ldots\}$

$R^{-1} = \{< 3, 1 >, < 4, 2 >, < 5, 3 >\}$

Ex. 2.15.2 Write the relation matrix for relations R and R^{-1} Where

R = {<1, 1>, <1, 3>, <1, 4>, <2, 1>, <2, 4>, <3, 2>, <3, 4>, <4, 2>, <4, 3>}

Soln. :

The relation matrix for R is given by

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The relation matrix for the relation R^{-1} is the transpose of matrix for R.

i.e. the matrix is,

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

R^{-1} = {< 1, 1 >, < 1, 2 >, < 2, 3 >, < 2, 4 >, < 3, 1 > m< 3, 4 >, < 4, 1 > , < 4, 2 >, < 4, 3 >}

Ex. 2.15.3 Let A = {0, 1, 2, 3} Let R be a relation A, such that

R = {< a, b >} a, b ∈ A and

(a + b + 1) mod 3 = 1}

Find elements of R and R^{-1}. What are the properties of R and R^{-1}. What is the relation between R and R^{-1}.

Soln. :

A = {0, 1, 2, 3}

R = {< a, b > | a, b ∈ A and (a + b + 1) mod 3 = 0}

∴ R = {< 0, 2 >, < 1, 1 >, < 2, 0 >, < 2, 3 >, < 3, 2 >}

R^{-1} = {< 2, 0 >, < 1, 1 >, < 0, 2 >, < 3, 2 >, < 2, 3 >}

R = R^{-1}

R is symmetric.

Thus if R is symmetric, then R = R^{-1}

Module II

Ex. 2.15.4 Find transitive closure of

$R = \{<1, 2>, <1, 3>, <2, 1>, <2, 3>, <2, 4>, <3, 2>, <4, 1>, <4, 3>\}$ using Warshall's algorithm.

Soln. :

For the given relation, the matrix is

$$W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Step 1 : Consider first row and first column

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} = W_1$$

Step 2 : Consider second row and second column

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = W_2$$

As all elements are 1, no need to proceed with next steps.

$\therefore R_{transitive} = \{<1, 1>, <1, 2>, <1, 3>, <1, 4>, <2, 1>, <2, 2>, <2, 3>, <2, 4>, <3, 1>, <3, 2>, <3, 3>, <3, 4>, <4, 1>, <4, 2>, <4, 3>, <4, 4>\}$

Ex. 2.15.5 Find transitive closure of the relation R, using Warshall's algorithm. $R = \{<1, 2>, <2, 3>, <3, 4>, <4, 2>\}$

Soln. :

For the given relation, the matrix is,

$$W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Step 1 : Consider first row and first column

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{No change}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = W_1$$

Step 2 : Consider second row and second column

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = W_2$$

Step 3 : Consider third row and third column

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = W_3$$

Step 4 : Consider fourth row and fourth column

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = W_4$$

$\therefore R_{transitive} = \{<1, 2>, <1, 3>, <1, 4>, <2, 2>, <2, 3>, <3, 2>, <3, 3>, <3, 4>, <4, 2>, <4, 3>, <4, 4>\}$

Ex. 2.15.6 Find transitive closure of R using Warshall's algorithm

$\therefore R_{transitive} = \{<1, 1>, <1, 2>, <2, 5>, <3, 1>, <3, 2>, <3, 4>, <4, 5>, <5, 2>, <5, 3>\}$

Soln. : Relation matrix of R is given by,

$$W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

➪ **Step 1 : Consider first row and first column**

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \xRightarrow{\text{No change}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} = W_1$$

➪ **Step 2 : Consider second row and second column**

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} = W_2$$

➪ **Step 3 : Consider third row and third column**

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = W_3$$

➪ **Step 4 : Consider fourth row and fourth column.**

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \xRightarrow{\text{No change}} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = W_4$$

➪ **Step 5 : Consider fifth row and fifth column.**

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = W_5$$

$\therefore R_{transitive}$ = {<1,1>, <1,2>, <1,3>, <1,4>, <1,5>, <2,1>, <2,2>, <2,3>, <2,4>, <2,5>, <3,1>, <3,2>, <3,3>, <3,4>, <3,5>, <4,1>, <4,2>, <4,3>, <4,4>, <4,5>, <5,1>, <5,2>, <5,3>, <5,4>, <5,5>}

Ex. 2.15.7 Using Warshall's algorithm, find transitive closure of

R = {<1,1>, <1,4>, <2,2>, <2,3>, <3,1>, <4,3>, <4,4>}

☑ **Soln. :** Relation matrix of R is given by,

$$W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

➪ **Step 1 : Consider first row and first column**

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = W_1$$

➪ **Step 2 : Consider second row and second column.**

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = W_2$$

➪ **Step 3 : Consider third row and third column.**

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = W_3$$

➪ **Step 4 : Consider fourth row and fourth column.**

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = W_4$$

$\therefore R_{transitive}$ = {<1,1>, <1,3>, <1,4>, <2,1>, <2,2>, <2,3>, <2,4>, <3,1>, <3,3>, <3,4>, <4,1>, <4,3>, <4,4>}

Ex. 2.15.8 What is the complement of the relation on N

R = {<1,1>, <2,2>, <3,3>, ...}

☑ **Soln. :**

The relation R contains all ordered pairs in which each element is related to itself.

$\therefore$ Complement of relation is given by,

$R' = \{\langle x,y\rangle \mid x, y \in N \text{ and } x \neq y\}$

Module II

Ex. 2.15.9 Prove that if a relation R is symmetric, then its complement R′ is also symmetric.

☑ **Soln. :** R is symmetric

$\therefore$ $\forall$ x, y if <x, y> $\in$ R then <y, x> $\in$ R

Let <a, b> $\notin$ R

$\therefore$ <a, b> $\in$ R′

As <a, b> $\notin$ R $\Rightarrow$ <b, a> $\notin$ R

$\therefore$ <b, a> $\in$ R′

$\therefore$ $\forall$ a, b if <a, b> $\in$ R′, <b, a> $\in$ R′

$\therefore$ R′ is symmetric

Ex. 2.15.10 Give an example of a relation on set A = {1,2,3,4} such that R is antisymmetric and R′ is not antisymmetric.

☑ **Soln. :**

A = {1,2,3,4}

R = {<1,1>, <1,2>, <3,4>, <2,2>, <3,3>, <4,4>}

R is antisymmertic

R′ = {<1,3>, <1,4>, <2,1>, <2,3>, <2,4>, <3,1>, <3,2>,<4,1>, <4,2>, <4,3>}

R′ is not antisymmertic

Another example

A = {1,2,3,4}

R = {<x,y> | x, y $\in$ A and x < y (x + y) mod 3 = 0 }

$\therefore$ R = {<1,2>, <2,4>, <3,3>}

R is antisymmertic

R′ = {<1,1>, <1,3>, <1,4>, <2,1>, <2,2>, <2,3>, <3,1>,<3,4>, <4,1>, <4,2>, <4,3>,<4,4>}

R′ is not antisymmertic.

Ex. 2.15.11 Give an example of a relation on set A = {1,2,3,4} such that R is antisymmetric and R′ is also antisymmetric.

☑ **Soln. :**

A = {1,2,3,4}

R = {<1,1>, <1,2>, <2,2>, <2,4>, <3,1>, <3,2>, <4,1>, <4,3>}

R is antisymmertic

R′ = {<1,3>, <1,4>, <2,1>, <2,3>, <3,3>, <3,4>, <4,2>, <4,4>}

R′ is not antisymmertic

Another example :

A = {1,2,3,4}

R = {<x,y> | x, y $\in$ A and x $\geq$ y }

$\therefore$ R = {<1,2>, <2,1>, <2,2>, <3,1>, <3,2>, <3,3>, <4,1>, <4,2>, <4,3>,<4,4>}

R is antisymmertic

R′ = {<1,2>, <1,3>, <1,4>, <2,3>, <2,4>, <3,4>}

R′ is not antisymmertic .

Ex. 2.15.12 Let R_1 = {<1,1>,<1,2>,<1,3>,<2,2>,<2,4>, <3,1>,<3,3>,<3,4>,<4,1>,<4,2>}

R_2 = <1,2>,<1,4>,<2,1>,<2,3>,<2,4>, <3,2>,<3,3>,<3,4>,<4,2>,<4,3>,<4,4>}

Find (a) $R_1 \circ R_1$, (b) $R_1 \circ R_2$, (c) $R_2 \circ R_1$, (d) $R_2 \circ R_2$

☑ **Soln. :**

a. To find $R_1 \circ R_1$

For 1 :

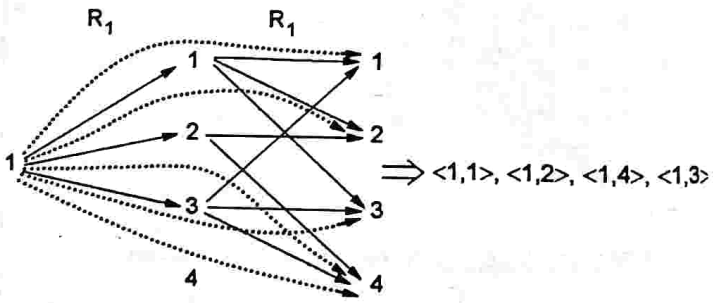


Fig. P. 2.15.12(a)

$\Rightarrow$ <1,1>, <1,2>, <1,4>,<1,3>

For 2 :

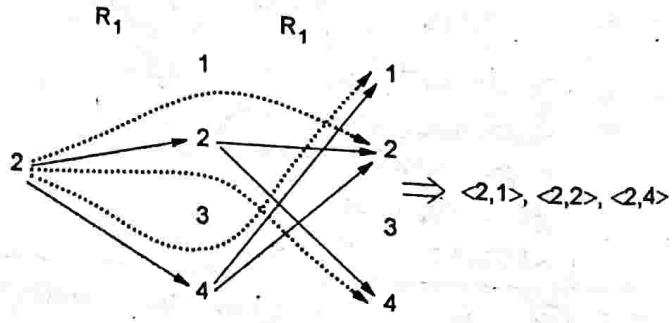


Fig. P. 2.15.12(b)

$\Rightarrow$ <2,1>, <2,2>, <2,4>

For 3 :

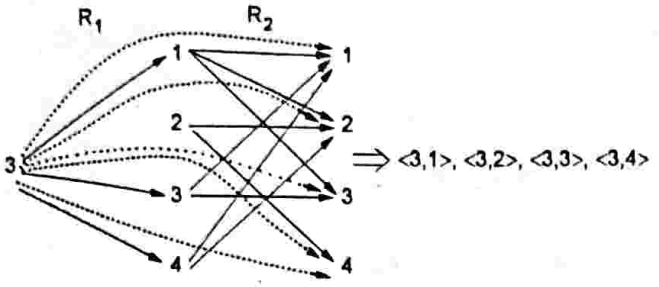


Fig. P. 2.15.12(c)

$\Rightarrow$ <3,1>, <3,2>, <3,3>,<3,4>

For 4 :

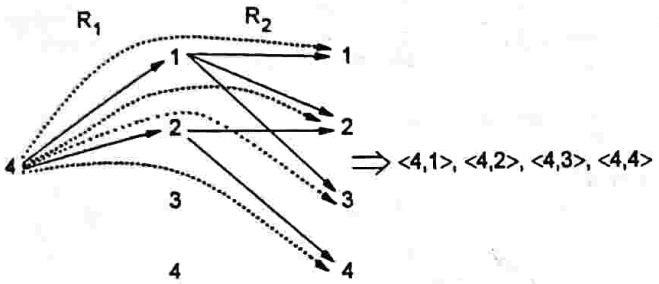


Fig. P. 2.15.12(d)

$\Rightarrow$ <4,1>, <4,2>, <4,3>,<4,4>

$\therefore R_1 \circ R_1 =$ {<1,1>,<1,2>,<1,3>,<1,4>,<2,1>,<2,2>,<2,4>, <3,1>, <3,2>, <3,3>, <3,4>, <4,1>,<4,2>, <4,3>, <4,4>}

b. To find $R_1 \circ R_2$

For 1 :

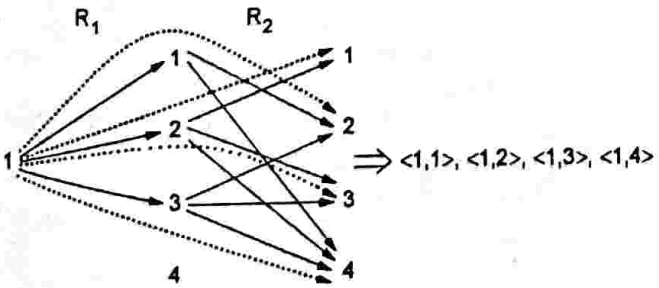


Fig. P. 2.15.12(e)

$\Rightarrow$ <1,1>, <1,2>, <1,3>,<1,4>

For 2 :

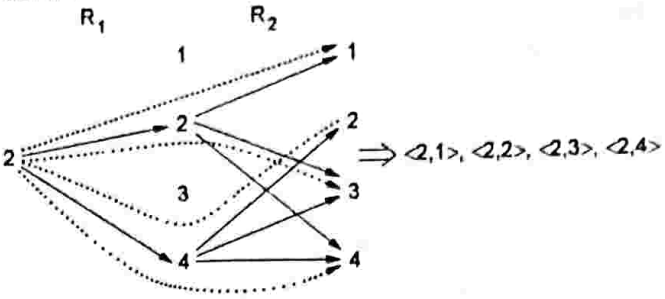


Fig. P. 2.15.12(f)

$\Rightarrow$ <2,1>, <2,2>, <2,3>, <2,4>

For 3 :

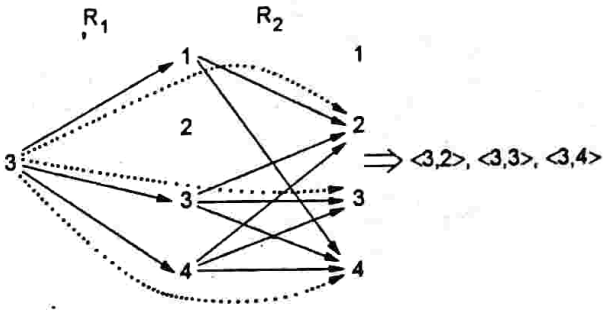


Fig. P. 2.15.12(g)

$\Rightarrow$ <3,2>, <3,3>, <3,4>

For 4 :

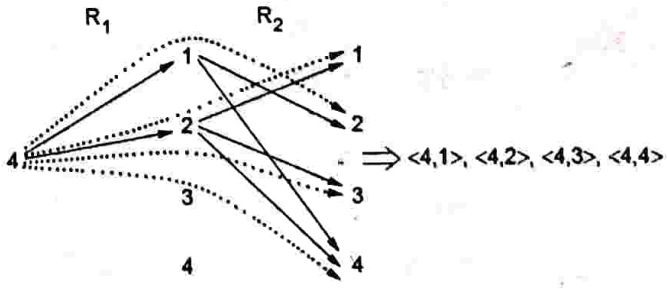


Fig. P. 2.15.12(h)

$\Rightarrow$ <4,1>, <4,2>, <4,3>,<4,4>

$\therefore R_1 \circ R_1 =$ <1,1>,<1,2>,<1,3>,<1,4>,<2,1>,<2,2>,<2,3>, <2,4>,<3,2>,<3,3>,<3,4>,<4,1>,<4,2>,<4,3>,<4,4>}

Module II

c. To find $R_2 \circ R_1$

For 1 :

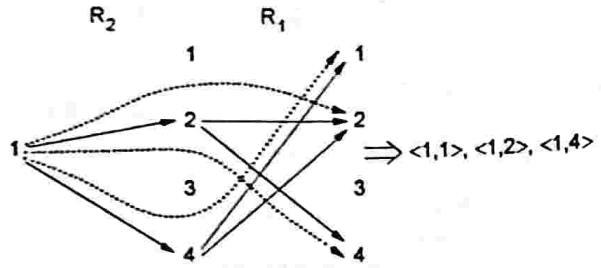


Fig. P. 2.15.12(l)

$\Rightarrow$ <1,1>, <1,2>, <1,4>

For 2 :

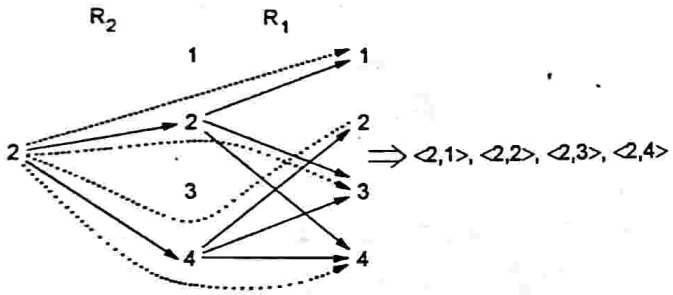


Fig. P. 2.15.12(j)

$\Rightarrow$ <2,1>, <2,2>, <2,3>, <2,4>

For 3 :

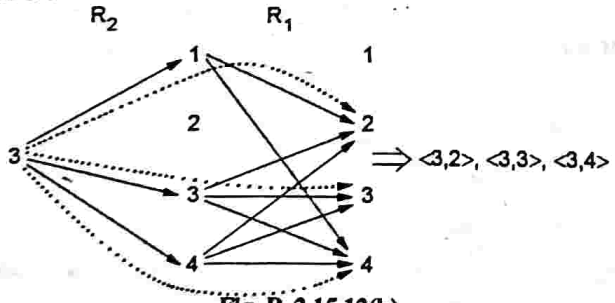


Fig. P. 2.15.12(k)

$\Rightarrow$ <3,2>, <3,3>,<3,4>

For 4 :

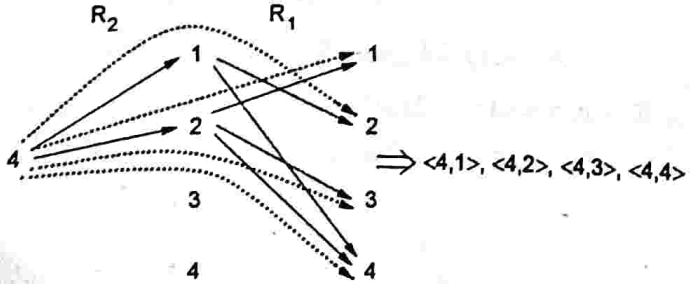


Fig. P. 2.15.12(l)

$\Rightarrow$ <4,1>, <4,2>, <4,3>,<4,4>

$\therefore R_1 \circ R_2 =$
{<1,1>,<1,2>,<1,3>,<1,4>,<2,1>,<2,2>,<2,3>, <2,4>, <3,2>, <3,3>, <3,4>, <4,1>,<4,2>, <4,3>, <4,4>}

d. To find $R_2 \circ R_2$

For 1 :

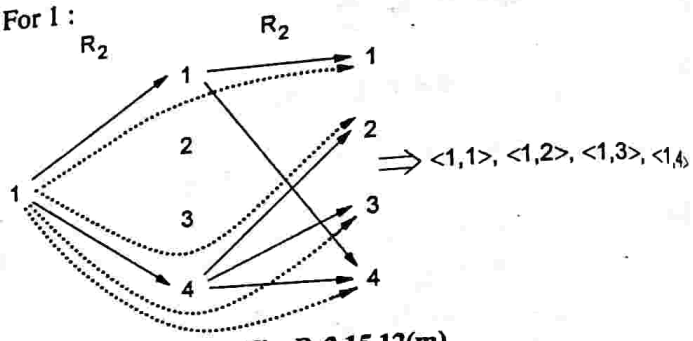


Fig. P. 2.15.12(m)

$\Rightarrow$ <1,1>, <1,2>, <1,3>, <1,4>

For 2 :

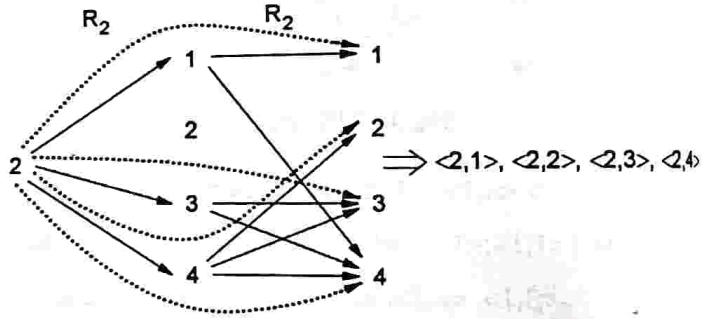


Fig. P. 2.15.12(n)

$\Rightarrow$ <2,1>, <2,2>, <2,3>, <2,4>

For 3 :

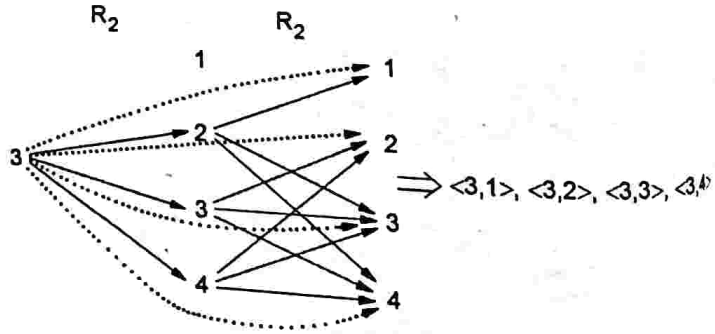


Fig. P. 2.15.12(o)

$\Rightarrow$ <3,1>, <3,2>, <3,3>, <3,4>

For 4 :

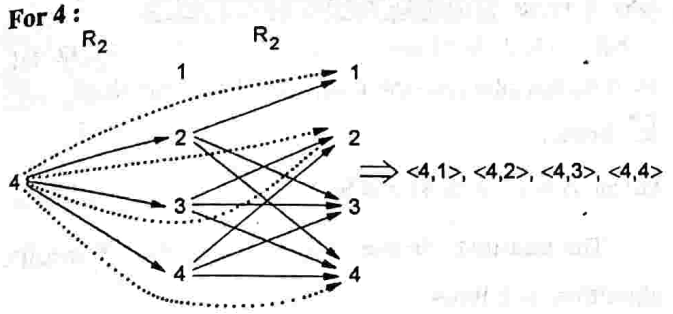


Fig. P. 2.15.12(p)

$\Rightarrow$ <4,1>, <4,2>, <4,3>,<4,4>

$\therefore R_2 \circ R_2$ = {<1,1>, <1,2>, <1,3>, <1,4>, <2,1>, <2,2>, <2,3>, <2,4>, <3,1>, <3,2>, <3,3>, <3,4>, <4,1>, <4,2>, <4,3>, <4,4>}

Ex. 2.15.13 Let R_1 and R_2 be relations on set of integers I, defined as follows.

$R_1 = \{<x,y> \mid x, y \in I \text{ and } (x + y) \bmod 2 = 0\}$

$R_2 = \{<x,y> \mid x, y \in I \text{ and } x \geq y)\}$

Find (a) $R_1 \circ R_1$, (b) $R_1 \circ R_2$,

(c) $R_2 \circ R_1$, (d) $R_2 \circ R_2$

☑ Soln. :

Consider relation R_1

$R_1 = \{<x,y> \mid x, y \in I \text{ and } (x + y) \bmod 2 = 0\}$

$\therefore x R_1 y$ if both x and y are even or x and y are odd.

$\therefore$ Every even number is related to all even numbers and every odd number is related to all odd numbers.

(a) Now consider $R_1 \circ R_1$

If $x R_1 y$ and $y R_1 z$ then $x (R_1 \circ R_1) z$

$\therefore$ If x is even then y, z are also even.

If x is odd then y, z are also odd.

$\therefore R_1 \circ R_1 = R_1$

(b) Now consider $R_1 \circ R_2$

If $x R_1 y$ and $y R_2 z$ then $x (R_1 \circ R_2) z$

$\therefore$ If x is even then y is even and $y \geq z$

If x is odd then y is odd and $y \geq z$

There is no relation between x an z.

$\therefore$ For any x, z can be any number.

$\therefore R_1 \circ R_2 = \{<x,y> \mid x, y \in I\}$

(c) Now consider $R_2 \circ R_1$

If $x R_2 y$ and $y R_1 z$ then $x (R_2 \circ R_1) z$

$\therefore x \geq y$ and y and z are both either even or odd.

$\therefore$ No relation between x and z.

$\therefore$ For any x, z can be any number.

$\therefore R_2 \circ R_1 = \{<x,y> \mid x, y \in I\}$

$= R_1 \circ R_2$

(d) Now consider $R_2 \circ R_2$

If $x R_2 y$ and $y R_2 z$ then $x (R_2 \circ R_2) z$

$\therefore x \geq y$ and $y \geq z$ then $x \geq z$

$\therefore R_2 \circ R_2 = R_2$

Ex. 2.15.14 Consider two sets

$A = \{1, 2, 3, 4\}$

$B = \{a, b, c\}$

and two relations from A to B

R_1 = {<1,b>, <1,c>, <2,a>, <3,b>, <4,a>, <4,c>}

R_2 = {<1,a>, <1,b>, <2,b>, <2,c>, <3,b>, <4,b>, <4,c>}

Find the following.

(a) $(R_1 \cup R_2)^{-1}$ (b) $R_1^{-1} \cap R_2^{-1}$

☑ Soln. :

(a) $R_1 \cup R_2$ = {<1,a>, <1,b>, <1,c>, <2,a>, <2,b>, <2,c>, <3,b>, <4,a>, <4,b>, <4,c>}

$(R_1 \cup R_2)^{-1}$ = {<3,a>, <3,c>}

(b) R_1^{-1} = {<1,a>, <2,b>, <2,c>, <3,a>, <3,c>, <4,b>}

R_2^{-1} = {<1,c>, <2,a>, <3,a>, <3,c>, <4,a>}

$R_1^{-1} \cap R_2^{-1}$ = {<3,a>, <3,c>}

$= (R_1 \cup R_2)^{-1}$

Ex. 2.15.15 If R_1 and R_2 are relations from A to B, then prove that $(R_1 \cup R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$

☑ Soln. :

Consider the relation $R_1 \cup R_2$

$<x,y> \in R_1 \cup R_2$ if $<x,y> \in R_1$ or $<x,y> \in R_2$

Module II

Consider $(R_1 \cup R_2)^{-1}$

If $<x, y> \notin R_1 \cup R_2$ then $<x, y> \in (R_1 \cup R_2)^{-1}$

$\therefore$ If $<x, y> \notin R_1$ and $<x, y> \notin R_2$ then $<x, y> \in (R_1 \cup R_2)^{-1}$

$\therefore$ If $<x, y> \in R^{-1}$ and $<x, y> \in R_2^{-1}$ then $<x, y> \in (R_1 \cup R_2)^{-1}$

$\therefore$ If $<x, y> \in R_1^{-1} \cap R_2^{-1}$ then $<x, y> \in (R_1 \cup R_2)^{-1}$

$\therefore$ $R_1^{-1} \cap R_2^{-1} = (R_1 \cup R_2)^{-1}$

Ex. 2.15.16 For relations in Ex. 13, find

(a) $(R_1 \cap R_2)^{-1}$ (b) $R_1^{-1} \cup R_2^{-1}$

☑ Soln. :

(a) $R_1 \cap R_2 = \{<1,b>, <3,b>, <4,c>\}$

$(R_1 \cap R_2)^{-1} = \{<1,a>, <1,c>, <2,a>, <2,b>, <2,c>, <3,a>, <3,c>, <4,a>, <4,b>\}$

(b) $R_1^{-1} = \{<1,a>, <2,b>, <2,c>, <3,a>, <3,c>, <4,b>\}$

$R_2^{-1} = \{<1,c>, <2,a>, <3,a>, <3,c>, <4,a>\}$

$\therefore R_1^{-1} \cup R_2^{-1} = \{<1,a>, <1,c>, <2,a>, <2,b>, <2,c>, <3,a>, <3,c>, <4,a>, <4,b>\}$

$= (R_1 \cap R_2)^{-1}$

Ex. 2.15.17 If R_1 and R_2 are relations from A to B, then prove that

$(R_1 \cap R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$

☑ Soln. :

Consider the relation $R_1 \cap R_2$

$<x,y> \in R_1 \cap R_2$ if $<x,y> \in R_1$ and $<x,y> \in R_2$

Consider $(R_1 \cap R_2)^{-1}$

If $<x, y> \notin R_1 \cap R_2$ then $<x, y> \in (R_1 \cap R_2)^{-1}$

$\therefore$ If $<x, y> \notin R_1$ or $<x, y> \notin R_2$ then $<x, y> \in (R_1 \cap R_2)^{-1}$

$\therefore$ If $<x, y> \in R_1^{-1}$ or $<x, y> \in R_2^{-1}$ then $<x, y> \in (R_1 \cap R_2)^{-1}$

$\therefore$ If $<x, y> \in R_1^{-1} \cup R_2^{-1}$ then $<x, y> \in (R_1 \cap R_2)^{-1}$

$\therefore$ $R_1^{-1} \cup R_2^{-1} = (R_1 \cap R_2)^{-1}$

UEx. 2.15.18 **MU - Q. 3(a), May 14, 6 Marks**

Let A = {1, 2, 3, 4} and R = {(1, 2), (2, 3), (3, 4), (2, 1)}. Find the transitive closure using Warshall's algorithm.

☑ Soln. :

Given A = {1, 2, 3, 4} and R = {(1, 2), (2, 3), (3, 4), (2, 1)}

The transitive closure of R is found using Warshall's algorithm, as follows.

The matrix representation of R is

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Now we shall write the Warshall matrices W_i where $i = 1, 2, 3, 4$. In each matrix we consider i^{th} row and i^{th} column. Next step is to mark the 1's in them and their corresponding rows and columns. If the intersecting element of such row and column is 0, then convert it to 1.

We know that, $W_1 = M_R$ and W_4 represents transitive closure.

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Row, column 1

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Row, column 2

$$W_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & \boxed{0} \\ 1 & 1 & 1 & \boxed{0} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$ Row, column 3

$$W_4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$ Row, column 4

No change

$\therefore$ The transitive closure represented by W_4 is given by,

$R_T = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4)\}$

UEx. 2.15.19 **MU - Q. 3(c), Dec. 14, 8 Marks**

Let R be a relation on set A = {1, 2, 3, 4}, given as

R = {(1, 1), (1, 4), (2, 2), (2, 3), (3, 2), (3, 3), (4, 1), (4, 4)}

Find transitive closure using Warshall's Algorithm.

Soln. :

Given set is A = {1, 2, 3, 4} and the relation is R = {(1, 1), (1, 4), (2, 2), (2, 3), (3, 2), (3, 3), (4, 1), (4, 4)}

Transitive closure.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & \boxed{1} \end{bmatrix} \end{array}$$

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & \boxed{1} & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$W_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$W_4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$\therefore$ Transitive closure of R = R

UEx. 2.15.20 **MU - Q. 3(c), May 15, 8 Marks**

Define Reflexive closure, Symmetric closure along with a suitable example. Let R be a relation on set

S = {a, b, c, d, e}, given as

R = { (a, a) ,(a, d) , (b, b), (c, d), (c, e), (d, a), (e,b), (e, e)}

Find transitive closure using Marshall's Algorithm.

Soln. :

Given set and relation

S = {a, b, c, d, e}

R = {(a, a), (a, d), (b, b), (c, d), (c, e), (d, a), (e, b), (e, e)}

To find transitive closure using Warshall's algorithm.

$$W_0 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$W_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array} = W_2$$

$$W_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array} = W_3$$

No change

$$W_3 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$W_4 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$W_5 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

W_5 represents the transitive closure.

$\therefore$ R_T = {(a, a), (a, d), (b, b), (c, a), (c, b), (c, d), (c, e), (d, a), (d, d), (e, b), (e, e)}

UEx. 2.15.21 **MU - Q. 5(b), May 16. 6 Marks**

Let A = {1,2,3,4}and let R={(1,2) (2,3)(3,4)(2,1)}. Find transitive closure of R by using Warshall's algorithm.

☑ **Soln. :**

Given A = {1, 2, 3, 4} R = {(1, 2), (2, 3), (3, 4), (2, 1)}

To find : Transitive closure of R using Warshall's algorithm.

$$W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\therefore$ Transitive closure of R.

= {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4)}

UEx. 2.15.22 **MU - Q. 2(a), Dec. 16. 8 Marks**

Given S = {1,2,3,4} and a Relation R on S given by R = {(4,3),(2,2),(2,1),(3,1),(1,2)}

(i) Show that R is not transitive

(ii) Find transitive closure of R by Warshall's algorithm

☑ **Soln. :**

Given S = {1, 2, 3, 4}

R = {(4, 3), (2, 2), (2, 1), (3, 1), (1, 2)}

(i) **To prove :** R is not transitive.

Proof : Consider the following pairs,

(4, 3), (3, 1) ∈ R.

But (4, 1) ∉ R.

∴ R is not transitive.

(ii)

$$W_0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

$$W_1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

$$W_2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

$$W_3 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 \\ 4 & 1 & 1 & 1 & 0 \end{array}$$

$$W_4 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 \\ 4 & 1 & 1 & 1 & 0 \end{array}$$

$$W_4 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 \\ 4 & 1 & 1 & 1 & 0 \end{array}$$

W_4 represents transitive closure of R.

∴ R_T = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)}

UEx. 2.15.23 **MU - Q. 6(a), May 17, 4 Marks**

Define reflexive closure and symmetric closure of a elation. Also find reflexive and symmetric closure of R.

A = {1,2,3,4}

R = {(1,1), (1,2), (1,4), (2,4), (3,1), (3,2), (4,2), (4,3), (4,4)}

☑ **Soln. :**

Define reflexive closure and symmetric closure.

Refer

Given set is A = {1, 2, 3, 4}

Given relation on A is,

R = {(1, 1), (1, 2), (1, 4), (2, 4), (3, 1), (3, 2), (4, 2), (4, 3), (4, 4)}

Reflexive closure of R

= {(1, 1), (1, 2), (1, 4), (2, 4), (3, 1), (3, 2), (4, 2), (4, 3), (4, 4), (2, 2), (3, 3)}

Symmetric closure of R

= {(1, 1), (1, 2), (1, 4), (2, 4), (3, 1), (3, 2), (4, 2), (4, 3), (4, 4), (2, 1), (4, 1), (1, 3), (2, 3), (3, 4)}

UEx. 2.15.24 **MU - Q. 2(b), Dec. 17, 6 Marks**

Let A = {a1, a2, a3, a4, a5} and let R be a relation on A whose matrix is

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find M_R^* by Warshall's algorithm.

☑ **Soln. :**

Given : A = {a1, a2, a3, a4, a5}

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The relational matrix R is

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Warshall's algorithm to find transitive closure M_R^*

➪ **Step I : Consider 1st row and column.**

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

➪ **Step II : Consider 2nd row and column.**

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$ No change

➪ **Step III : Consider 3rd row and column.**

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$ No change

➪ **Step IV :Consider 4th row and column.**

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

➪ **Step V : Consider 5th row and column.**

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad M_R^* = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

⏭ 2.16 FUNCTIONS

In the previous section, we have discussed relations.

A function is a special type of relation. Before studying functions, let us revise few concepts of relations.

- Let A and B be two sets.
- R be a relation from A to B
- $R \subseteq A \times B$.
- Domain of R is set of first elements in the ordered pairs present in R.
- Range of R is set of second elements in the ordered pairs present in R.
- Set B is called co-domain.

❑ **Definition :** A function f from set A to set B is a relation, such that domain of the relation is set A, and every element in A is related to exactly one element in B.

It is represented as, $f : A \rightarrow B$

As f is a relation, the following points can be noted.

- If $\subseteq A \times B$.
- If $<x, y> \in f$, it is represented as $f(x) = x \xrightarrow{f} y$ (y or x)
- x is called an argument and y is called image of x, and it is unique. y is also called value of f at x.

  $|f| = |A|$
- A function is also called mapping.

☞ **Example**

Let $A = \{1, 2, 3, 4\}$ $B = \{0,1\}$

f be a function from A to B given as follows.

$$f = \{<x, y> \mid x \in A \text{ and } y \in B \text{ and } y = x \bmod 2\}$$
$$= \{<1,1>, <2,0>, <3,1>, <4,0>\}$$

This can also be represented as,

$$y = f(x) = x \bmod 2$$

$$f(1) = 1, f(2) = 0, f(3) = 1, f(4) = 0$$

2.17 REPRESENTATION OF FUNCTIONS

As every function is a relation, it can be represented using matrix and graph. In the previous example we have seen that,

$$f = \{<1,1>, <2,0>, <3,1>, <4,0>\}$$

This is represented as follows.

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \\ 4 & 1 & 0 \end{array}$$

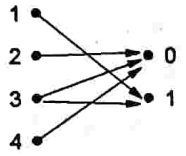

Fig. 2.17.1

It can be seen that, in matrix representation, every row has single entry as 1, and other entries are zeros. Thus, we can use a tabular form to represent a function as follows.

Ex. 2.17.1 Let A = {0,1, 2, 3, 4} and $f : A \to A$

Such that f (x) = (x + 1) mod 5. Write the function in the form of a set, and represent using tabular and graphical notation.

Soln. :

$$f(x) = (x + 1) \bmod 5$$

$$f = \{<0,1>, <1,2>, <2,3>, <3,4>, <4,0>\}$$

Tabular form

	f
0	1
1	2
2	3
3	4
4	0

Graphical notation

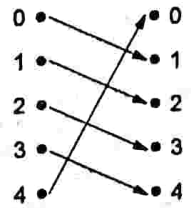

Fig. P. 2.17.1

If the relation is on one set, then we represent the elements in the set as nodes of the graphs, but in case of function, the elements are represented as if the elements belong to two different sets.

Thus, the graphical representation is not recommended as follows.

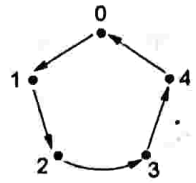

Fig. P. 2.17.1(a)

Example

Let f and g be two functions from set of real numbers R to set of integers I, given as follows.

$f : R \to I, \quad f(x) = \lfloor x \rfloor$ where

$\lfloor x \rfloor$ is greatest integer less than or equal to x.

It is known as floor function.

$g : R \to I$

$g(x) = \lceil x \rceil$ where

$\lceil x \rceil$ is smallest integer greater than or equal to x.

It is known as floor function.

Ex.

$$f(7.32) = \lfloor 7.32 \rfloor = 7$$

$$f(7.9999) = \lfloor 7.9999 \rfloor = 7$$

$$g(7.32) = \lceil 7.32 \rceil = 8$$

$$g(7.9999) = \lfloor 7.9999 \rfloor = 8$$

$$f(-2.3) = \lfloor -2.3 \rfloor = -3$$

$$g(-2.3) = \lceil -2.3 \rceil = -4.2$$

2.18 TYPES OF FUNCTIONS

There are three different types of functions. This differentiation is as per the number of elements in the sets, and the way the elements are related.

If a function is given as $f : A \to B$, then A is called the domain, B is co-domain, and the range of the function, represented as R_f is the sub set of B with elements that are images of the function.

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Example

Let A = {0,1, 2, 3, 4, 5, 6} B = {2, 3, 4, 5}

Let f : A → B,

Such that f (x) = number of alphabets when x is written in words.

∴ f = {<0,4>, <1,3>, <2,3>, <3,5>, <4,4>, <5,4>, <6,3>}

∴ Domain = A

Co-domain = B

Range of function = $R_f = \{3, 4, 5\} \subseteq B$.

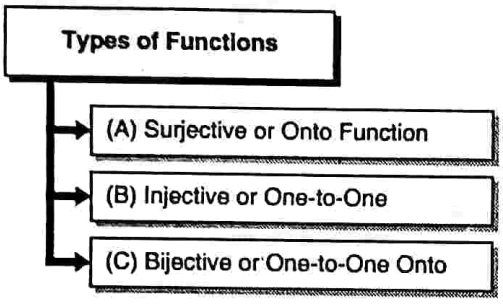

Fig. 2.18.1 : Types of Functions

Syllabus Topic : Surjective Function

2.18.1 Surjective or Onto Function

Definition : A function f : A → B is called onto or surjective, if the range of function f, R_f = B.

Example range equal to codomain

Let A = {1, 2, 3, 4, 5} B = {0, 1} and f be a function given as,

f : A → B, where f (x) = x mod 2.

∴ f = {<1,1>, <2,0>, <3,1>, <4,0>, <5,1>}

∴ R_f = {0,1} ⊆ B.

∴ f is onto.

Note that, if f : A → B is onto, then | A | ≥ | B |.

If a function is not onto, then it is called into.

Syllabus Topic : Injective Function

2.18.2 Injective or One-to-One

Definition : A function f : A → B is called one-to-one or injective, if every element in A is related to distinct element in B.

i.e. ∀ x, y ∈ A, if x ≠ y then f (x) ≠ f (y).

| R_f | = | A |.

Example

Let A = {1, 2, 3, 4, 5} and f be a function Given as,

f : A → N, where f (x) = x^2.

∴ f = {<1,1>, <2,4>, <3,9>, <4,16>, <5,25>}

∴ R_f = {1, 4, 9, 16, 25}

∴ | R_f | = | A | = 5

∴ f is one-to-one.

Note that, if f : A → B is one-to-one, then | A | ≤ | B |.

Syllabus Topic : Bijective Function

2.18.3 Bijective or One-to One Onto

Definition : A function f : A → B is called one-to-one onto or bijective, if f is both onto and one-to-one.

Example

Let A = {0, 1, 2, 3, 4} and f be a function,

Given as,

f : A → A, where f (x) = (x + 1) mod 5.

∴ f = {<0,1>, <1,2>, <2,3>, <3,4>, <4,0>}

∴ R_f = {0, 1, 2, 3, 4} = A

∴ f is one-to-one onto

Note that, if f : A → B is one-to-one onto, then | A | = | B |.

Syllabus Topic : Composition of Functions

2.19 COMPOSITION OF FUNCTIONS

We have seen composition of relations in section 2.9 composition of functions is similar to it. But the representation and order of computation is different.

Example 1

Let $f : A \to B$, and $g : B \to C$, be two functions, then composition function $g \circ f$ is given by,

$g \circ f : A \to C$, such that,

$\forall x \in A, g \circ f(x) = g(f(x))$.

Example 2

Let $A = \{0, 1, 2, 3, 4\}$

$B = \{1, 2, 3, 4, 5\}$ and $C = \{0, 1, 2\}$

Let $f : A \to B$, $g : B \to C$ be two functions given as,

$f(x) = (x + 1)$, $g(x) = x \bmod 3$.

Find $g \circ f$.

Let us represent the functions in tabular form.

A	f
0	1
1	2
2	3
3	4
4	5

B	g
1	1
2	2
3	0
4	1
5	2

$\therefore g \circ f(0) = g(f(0)) = g(1) = 1$

$g \circ f(1) = g(f(1)) = g(2) = 2$

$g \circ f(2) = g(f(2)) = g(3) = 0$

$g \circ f(3) = g(f(3)) = g(4) = 1$

$g \circ f(4) = g(f(4)) = g(5) = 2$

$\therefore g \circ f = \{<0,1>, <1,2>, <2,0>, <3,1>, <4,2>\}$

In this example, the function f is one-to-one onto or bijective, the functions g and $g \circ f$ are onto or surjective.

Syllabus Topic : Inverse of a Function

2.20 INVERSE OF A FUNCTION

In section 2.12.1, we have seen converse of a relation.

Definition : As function is also a relation, thus it has its converse.

A function $f : A \to B$ is said to have inverse function $f - 1$ if its converse is also a function. The inverse function if it exists, is given by

$f^{-1} : B \to A$, such that,

$f^{-1}(x) = y$ if $f(y) = x$.

Note that every function cannot have an inverse function. A function has an inverse function if and only if it is bijective i.e. one-to-one onto. The function which has inverse is called as invertible function.

Example

Let $A = \{1, 2, 3, 4\}$ $B = \{a, b, c, d\}$ and function

$f : A \to B$ is given by,

$f = \{<1,a>, <2,b>, <3,c>, <4,d>\}$

This function is one-to-one onto.

$\therefore$ It has an inverse function.

$f^{-1} : B \to A$, where

$f^{-1} = \{<a,1>, <b,2>, <c,3>, <d,4>\}$

Ex. 2.20.1 Let $A = \{0, 1, 2, 3, 4\}$ and f be a function $f : A \to A$ where $f(x) = (x + 1) \bmod 5$. Find f^{-1} if it exists.

☑ **Soln. :**

f is bijective.

$\therefore$ It has an inverse function.

Given that $f(x) = (x + 1) \bmod 5$ we can write.

$f(0) = 1$, $f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $f(4) = 0$.

$\therefore f^{-1}(0) = 4$, $f^{-1}(1) = 0$, $f^{-1}(2) = 1$, $f^{-1}(3) = 2$, $f^{-1}(4) = 3$.

Ex. 2.20.2 Let $A = \{0, 1, 2, 3\}$ and f be a function $f : A \to A$ where $f(x) = x \bmod 2$. Find f^{-1} if it exists.

☑ **Soln. :**

$f(x) = x \bmod 2$.

$\therefore f = \{<0,0>, <1,1>, <2,0>, <3,1>\}$

This is not one-to-one onto.

$\therefore$ Inverse function does not exists.

Module II

2.21 SOLVED EXAMPLES

Ex. 2.21.1 Let N be the set of natural numbers including zero. Determine which of the following functions are one-to-one, which are onto and which are one-to-one onto.

(a) $f : N \to N \quad f(j) = j^2 + 2$

(b) $f : N \to N \quad f(j) = j \bmod 3$

(c) $f : N \to N \quad f(j) = \begin{cases} 1 & j \text{ is odd} \\ 0 & j \text{ is even} \end{cases}$

(d) $f : N \to \{0, 1\} \; f(j) = \begin{cases} 1 & j \text{ is odd} \\ 0 & j \text{ is even} \end{cases}$

Soln. :

(a) $f : N \to N \qquad f(j) = j^2 + 2$

$f = \{<0, 2>, <1, 3>, <2, 6>, <3, 11>, ...\}$

Every element in N has distinct element

$\therefore$ f is one-to-one.

(b) $f : N \to N \qquad f(j) = j \bmod 3$

$f = \{<0, 0>, <1, 1>, <2, 2>, <3, 0>, <4, 1>, <5, 2> ...\}$

f is neither one-to-one, nor onto.

(c) $f : N \to N \qquad f(j) = \begin{cases} 1 & j \text{ is odd} \\ 0 & j \text{ is even} \end{cases}$

$f = \{<0, 0>, <1, 1>, <2, 0>, <3, 1>, ... \}$

f is neither one-to-one, nor onto.

(d) $f : N \to \{0, 1\} \quad f(j) = j \begin{cases} 1 & j \text{ is odd} \\ 0 & j \text{ is even} \end{cases}$

$f = \{<0, 0>, <1, 1>, <2, 0>, <3, 1>, ... \}$

f is onto.

Ex. 2.21.2 Let I be the set of integers, I be the set of positive integers and $I_p = \{0, 1, 2, ... p - 1\}$. Determine which of the following functions are one-to-one, onto or one-to-one onto.

(a) $f : I \to I$

$f(j) = \begin{cases} j/2 & j \text{ is even} \\ (j-1)/2 & j \text{ is odd} \end{cases}$

(b) $f : I+ \to I+ \quad f(x) = \text{greatest integer} \le \sqrt{x}$

(c) $f : I7 \to I7 \quad f(x) = 3x \bmod 7$

(d) $f : I4 \to I_4 \quad f(x) = 3x \bmod 4$

Soln. :

(a) $f : I \to I \quad f(i) = \begin{cases} j/2 & j \text{ is even} \\ (j-1)/2 & j \text{ is odd} \end{cases}$

$f = \{... <-1, -1> <0, 0> <1, 0>, <2, 1> <3, 1>, ... \}$

Range of f = I

$\therefore$ f is onto

(b) $f : I_+ \to I_+ \qquad f(x) = \text{greatest integer} \le \sqrt{x}$

$f = \{ <0, 0> <1, 1>, <2, 1> <3, 1> <4, 2>, ... \}$

Range of $f = I_+$

$\therefore$ f is onto.

(c) $f : I_7 \to I_7 \qquad f(x) = 3x \bmod 7$

$f = \{<0, 0>, <1, 3>, <2, 6>, <3, 2>, <4, 5> <5, 1>, <6, 4>\}$

f is one-to-one onto.

(d) $f : I_4 \to I_4 \quad f(x) = 3x \text{ and } 4$

$f = \{<0, 0> <1, 3>, <2, 2> <3, 1> \}$

f is one-to-one onto.

Ex. 2.21.3 Which of the following sets are functions,

(a) $\{<1, <2, 3>>, <2, <3, 4>>, <3, <1, 4>>, <4, <1, 4>> \}$

(b) $\{<1, <2, 3>>, <2, <3, 4>>, <3, <3, 2>> \}$

(c) $\{<1, <2, 3>>, <2, <3, 4>>, <1, <2, 4>> \}$

(d) $\{<1, <2, 3>>, <2, <2, 3>>, <3, <2, 3>>\}$

Soln. :

a, b, d are functions. c is not a function because I is related to both <2, 3> and <2, 4>.

Ex. 2.21.4 List all possible functions from X = {a, b, c} to Y {0, 1} and indicate in each case whether the function is one-to-one, is onto, and is one-to-one onto.

Soln. : Functions from X = {a, b, c} to Y = {0, 1} cannot be one-to-one because $|X| > |Y|$.

Following are the possible functions represented using graphs. Total number of functions $= |Y|^{|X|} = 2^3 = 8$

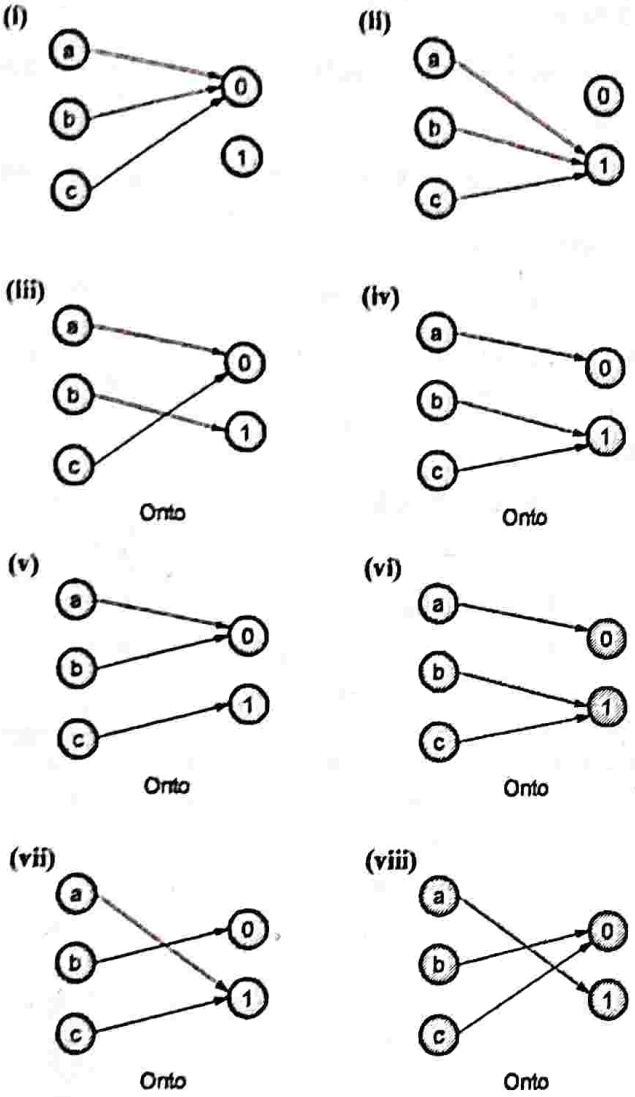

Fig. P. 2.21.4

Ex. 2.21.5 Show that the functions f and g from $N \times N$ to N given by $f(<x, y>) = x + y$ and $g(<x, y>) = xy$ are onto but not one-to-one.

☑ **Soln. :**

Consider the function $f : N \times N \rightarrow N$

$f(<x, y>) = x + y$

Now $f(<y, x>) = y + x = x + y = f(<x, y>)$

$\therefore$ f cannot be one-to-one.

Let $x \in N$, x can be written as sum of any two numbers.

$\therefore$ Range of $f = N$

$\therefore$ f is onto.

Consider the function $g : N \times N \rightarrow N$

$$g(<x, y>) = xy$$

Now $g(<y, x>) = yx = xy = g(<x, y>)$

$\therefore$ g cannot be one-to-one.

Let $x \in N$, x can be written as sum of any two numbers.

$\therefore$ Range of $f = N$

$\therefore$ f is onto.

Consider the function $g : N \times N \rightarrow N$

$$g(<x, y>) = xy$$

Now $g(<y, x>) = xy$

Now $g(<y, x>) = yx = xy = g(<x, y>)$

$\therefore$ g can not be one-to-one.

Let $x \in N$, x can be written as product of any two numbers. Though x is prime, $x = 1 \times x$.

$\therefore$ Range of $g = N$

$\therefore$ g is onto.

Ex. 2.21.6 Let $f : R \rightarrow R$ and $g : R \rightarrow R$, where R is the set of real numbers. Find $f \circ g$ and $g \circ f$ where $f(x) = x^2 - 2$ and $g(x) = x + 4$ State whether these functions are surjective, injective and bijective.

☑ **Soln. :**

Given : $f : R \rightarrow R, g : R \rightarrow R$

$$f(x) = x^2 - 2, g(x) = x + 4$$

Now consider fog

i.e. $f \circ g(x) = f(g(x)) = f(x + 4) = (x + 4)^2 - 2$

$= x + 8x + 16 - 2$

$= x + 8x + 14$

$\therefore$ For $\forall x \in R$, we have distinct $f \circ g(x)$

$\therefore$ $f \circ g$ is injective or one-to-one.

It is not onto as $\forall x \in R$, x is not an image.

Now consider $g \circ f$

i.e. $g \circ f(x) = g(f(x)) = g(x^2 - 2)$

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$= x^2 - 2 + 4 = x^2 + 2$

$\therefore$ $\forall x \in R$, we have distinct $g \circ f(x)$

$\therefore$ $g \circ f$ is objective or one-to-one.

It is not onto as $\forall x \in R$, n is not an image.

Ex. 2.21.7 If $f : x \to y$ and $g : Y \to Z$ and both f and g are onto, show that $g \circ f$ is also onto.

☑ Soln. :

Given : $f : X \to Y$, $g : Y \to Z$

F and g are onto.

i.e. range of f = Y

range of g = Z

Now consider, $g \circ f$

$g \circ f : X \to Z$

Consider, $\forall x \in X$, $g \circ f(x) = g(f(x))$

$= g(\forall_{x \in X} f(x))$

$= g(y)$ where $y \in Y$ because f is onto.

$= x$ where $z \in Z$ because g is onto.

$\therefore$ $\forall_{x \in X}$, $g \circ f(x) = Z$, where $z \in Z$.

$\therefore$ $g \circ f$ is onto.

Ex. 2.21.8 Let $f : R \to R$ be given by $f(x) = x^3 - 2$. Find f^{-1}.

☑ Soln. :

Given : $f(x) = x^3 - 3$

Let $f(x) = y$ $\therefore$ $y = x^3 - 3$

$\therefore$ $x^3 = y + 3$

$x = \sqrt[3]{(y+3)}$

This is writing x in terms of f(x)

$\therefore$ $x = \sqrt[3]{(f(x)+3)}$

We know that, if $f(x) = y$ then $f^{-1}(y) = x$

$\therefore$ $f^{-1}(y) = \sqrt[3]{(y+3)}$

The general term will be,

$f(x) = x^3 - 3$

$f^{-1}(x) = \sqrt[3]{(x+3)}$

Ex. 2.21.9 What is a function $I_x : X \to X$? Give an example.

☑ Soln. :

A function $I_x : x \to x$ is known as identity function. It is given by,

$I_x : \{<x, x> \mid x \in X\}$

Example $A = \{a, b, c, d\}$

$I_x = \{<a, a>, <b, b>, <c, c>, <d, d>\}$

Ex. 2.21.10 If is a function given as, $f(x) = (x + 1) \bmod s$ in a set $A = \{0, 1, 2, 3, 4\}$ Find f^{-1}, $f \circ f^{-1}$, $f^{-1} \circ f$.

Soln. :

$A = \{0, 1, 2, 3, 4\}$

$f = \{<0, 1>, <1, 2>, <2, 3>, <3, 4> <4, 0>\}$

$f^{-1} = \{<1, 0>, <2, 1>, <3, 2>, <4, 3> <0, 4>\}$

$f \circ f^{-1} = \{<0, 0>, <1, 1>, <2, 2>, <3, 3> <4, 4>\}$

$f^{-1} \circ f = \{<1, 1>, <2, 2>, <3, 3> <4, 4><0, 0>\}$

Ex. 2.21.11 Let $X = \{1, 2, 3, 4\}$. Define a function $f : X \to X$, suc h that $f \neq I_x$ and is one-to-one. Find $f \circ f = f^2$ $f^3 = f \circ f^2$, f^{-1} and $f \circ f^{-1}$. Can you find another function $g : X \to X$. Such that $g \neq I_x$ but $g \circ g = I_x$.

☑ Soln. :

$X = \{1, 2, 3, 4\}$

$f : X \to X$

$f = \{<1, 2>, <2, 3>, <3, 4> <4, 1>\}$

$f^2 = f \circ f = \{<1, 3>, <2, 4>, <3, 1> <4, 2>\}$

$f^3 = f \circ f^2 = \{<1, 4>, <2, 1>, <3, 2> <4, 3>\}$

$f^{-1} = \{<2, 1>, <3, 2>, <4, 3> <1, 4>\}$

$f \circ f^{-1} = \{<1, 1>, <2, 2>, <3, 3> <4, 4>\}$

A function $g \neq I_x$. Such that $g \circ g = I_x$ is possible, if and only if, $g = g^{-1}$.

$\therefore$ $g = \{<1, 2>, <2, 1>, <3, 4> <4, 3>\}$ can be an example for $g^{-1} = g$.

$g \circ g = \{\{<1, 1>, <2, 2>, <3, 3> <4, 4>\} = I_x$

Few more examples could be,

$g = \{<1, 3>, <3, 1>, <2, 4> <4, 2>\}$

$g = \{\{<1, 4>, <4, 1>, <2, 3> <3, 2>\}\}$

$g = \{\{<1, 1>, <2, 2>, <3, 4> <4, 3>\}\}$

2.21.1 SOLVED EXAMPLES ON FUNCTIONS

Ex. 2.21.12 Let the functions $f: A \to B$, $g: B \to C$, $h: C \to D$ be defined by Fig. . Determine if each functions is;

(a) onto, (b) inverible.

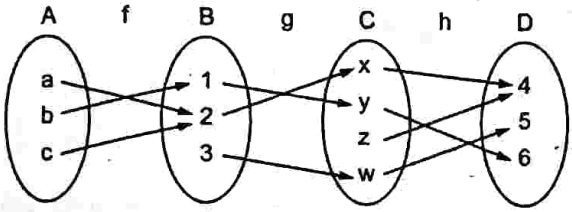

Fig. P. 2.21.12

Soln. : Given functions

$f: A \to B, \quad g: B \to C, \quad h: C \to D$

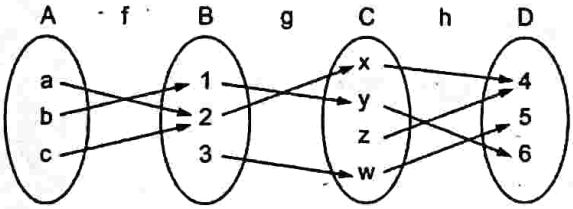

Fig. P. 2.21.12(a)

(a) onto functions - h

(b) Invertible functions - none of them.

Ex. 2.21.13 Fill in the blanks.

1. Number of elements in largest equivalence relation of a finite set A with n elements is ——.
2. A partial ordering relation is reflexive, ——, and transitive.

Soln. :

1. n^2 2. Antisymmetric

Ex. 2.21.14 Given f, g and h are functions on $X = \{1,2,3\}$

$f = \{(1,2), (2,3), (3,1)\}$

$g = \{(1,2), (2,1), (3,1)\}$

$h = \{(1,1), (2,2), (3,1)\}$

Compute : (i) $f \circ g$ (ii) $g \circ f$ (iii) $f \circ r \circ h$

Soln. :

Given : $X = \{1, 2, 3\}$

$f = \{(1, 2), (2, 3), (3, 1)\}$

$g = \{(1, 2), (2, 1), (3, 3)\}$

$h = \{(1, 1), (2, 2), (3, 1)\}$

(i) $f \circ g = \{f(g(1)), f(g(2)), f(g(3))\}$

$= \{f(2), f(1), f(3)\}$

$= \{3, 2, 1\}$

(ii) $g \circ f = \{g(f(1)), g(f(2)), g(f(3))\}$

$= \{g(2), g(3), g(1)\}$

$= \{1, 3, 2\}$

(iii) $f \circ g \circ h = \{f(g(h(1))), f(g(h(2))), f(g(h(3)))\}$

$= \{f(g(1)), f(g(2)), f(g(1))\}$

$= \{f(2), f(1), f(2)\}$

$= \{3, 2, 3\}$

Ex. 2.21.15 Given $A = \{1,2,3,4,5,6,7,8\}$. Let xRy whenever y is divisible by x. Find and write relation R as a set of ordered pairs. Is R so found reflexive, symmetric or transitive? Is R an equivalence relation or partial ordering? Justify with a valid argument.

Soln. :

Given : $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$x R y \Rightarrow y$ is divisible by x.

$R = \{<1, 1>, <1, 2>, <1, 3>, <1, 4>, <1, 5>, <1, 6>, <1, 7>, <1, 8>, <2, 2>, <2, 4>, <2, 6>, <2, 8>, <3, 3>, <3, 6>, <4, 4>, <4, 8>, <5, 5>, <6, 6>, <7, 7>, <8, 8>\}$

It can be observed that,

(i) $\forall_x \in A, \quad x R x$

$\therefore$ R is reflexive

(ii) $\forall x, y \in A$, if $x R y$ then $y \not R x$ where $x \neq y$.

Module II

(iii) If y is divisible by x and z is divisible by y then z is divisible by x.

i.e. if x Ry and y R2 $\Rightarrow$ x Rz

$\therefore$ R is transitive

$\therefore$ The relation R is partial ordering.

UEx. 2.21.16 **MU - Q. 1(c), Dec. 16, 6 Marks**

Suppose that A is non empty set and f is a function that has A as it's domain. Let R be the relation on A consisting of all ordered pairs (x, y) where f (x) = f(y). Show that R is an equivalence relation on A.

☑ **Soln. :**

Given : A is non empty set

f is a function whose domain is A.

$R = \{(x, y) \mid f(x) = f(y)\}$

☞ **To prove :**

R is equivalence relation.

☞ **Proof :**

(i) Consider $\forall x \in A$, $f(x) = f(x)$,

$\therefore$ x R x

$\therefore$ R is reflexive.

(ii) Consider if x R y then,

$f(x) = f(y)$

$\therefore f(y) = f(x)$

$\therefore$ y R x

$\therefore$ R is symmetric.

(iii) Consider if x Ry and y Rz then,

$f(x) = f(y)$ and $f(y) = f(z)$.

$\therefore f(x) = f(z)$

$\therefore$ x R z.

$\therefore$ R is transitive.

$\therefore$ R is equivalence.

UEx. 2.21.17 **MU – Q. 2(d), Dec. 13**

If $f \cdot A \to B$ is one-to-one and onto then prove that f^{-1} is also one-to-one and onto.

☑ **Soln. :**

Given $f : A \to B$ is one-to-one and onto.

To prove : $f^{-1} : B \to A$ is also one-to-one and onto.

☞ **Proof :**

$f : A \to B$ is one-to-one and onto.

$\therefore \forall x \neq y$, we have $f(x) \neq f(y)$.

$\forall b \in B$, we have one $a \in A$ such that,

$$f(a) = b$$

Now consider $f^{-1} : B \to A$,

Let there exist $a, b \in B$ such that, $a \neq b$.

$$\text{And } f^{-1}(a) = f^{-1}(b) = c$$

$$\therefore f(c) = a \text{ and } f(c) = b$$

This is not possible.

$$\therefore f^{-1}(a) \neq f^{-1}(b)$$

$\therefore f^{-1} : B \to A$ is one-to-one.

Consider $f^{-1} : B \to A$

We know that $\forall a \in A$, there exist exactly one $b \in B$, such that $f(a) = b$.

$$\therefore f^{-1}(b) = a$$

$\therefore \forall a \in A$, we have one $b \in B$, such that,

$$f^{-1}(b) = a$$

$\therefore f^{-1} : B \to A$ is onto.

Hence the proof.

UEx. 2.21.18 **MU – Q. 3(a), May 14**

If $f(x) = x + 2$, $g(x) = x - 2$, $h(x) = 3x$, for all $x \in R$., find (i) fogoh (ii) hogof (iii) fofof

☑ **Soln. :**

Given : $f(x) = x + 2$, $g(x) = x - 2$, $h(x) = 3x$, for all $x \in R$.

To find : (i) fogoh (ii) hogof (iii) fofof

The composition of functions is given by,

$$fog(x) = f(g(x))$$

(i) fogoh (x) $= f(g(h(x))) = f(g(3x))$

$= f(3x - 2) = 3x - 2 + 2$

$= 3x$

(ii)hogof (x) $= h(g(f(x))) = h(g(x + 2))$

$= h(x + 2 - 2) = h(x)$

$= 3x$

(iii) fofof (x) $= f(f(f(x))) = f(f(x+2))$

$= f(x+2+2) = f(x+4)$

$= x+4+2$

$= x+6$

UEx. 2.21.19 **MU – Q.5(b), Dec.14**

If $f: R \to R$, $f(x) = 2x+3$

$g: R \to R$, $g(x) = 3x+4$

$h: R \to R$, $h(x) = 4x$

find gof, fog, foh, hof, gofoh

Soln. :

Given functions are

$f: R \to R$, $f(x) = 2x+3$

$g: R \to R$, $g(x) = 3x+4$

$h: R \to R$, $h(x) = 4x$

To find, gof, fog, foh, hof, gofoh

(i) $gof(x) = g(f(x)) = g(2x+3) = 3(2x+3)+4$

$= 6x+13$

(ii) $fog(x) = f(g(x)) = f(3x+4) = 2(3x+4)+3$

$= 6x+11$

(iii) $foh(x) = f(h(x)) = f(4x) = 2(4x)+3 = 8x+3$

(iv) $hof(x) = h(f(x)) = h(2x+3) = 4(2x+3)$

$= 8x+12$

(v) $gofoh(x) = g(f(h(x))) = g(f(4x))$

$= g(2(4x)+3) = g(8x+3)$

$= 3(8x+3)+4$

$= 24x+13$

UEx. 2.21.20 **MU – Dec. 08, Dec. 15**

If $f: A \to B$ and $g: B \to C$ are both one-one and onto, the, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Soln. :

Given : $f: A \to B$ and $g: B \to C$, are one-to-one and onto,

To prove : $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof :

Given that both f and g are one - to - one and onto. we can write the following

Let $f(x) = k$ $g(k) = m$

$\therefore f^{-1}(k) = x, g^{-1}(m) = k$

Now consider $g \circ f(x)$

$= g(f(x)) = g(k) = m$

$\therefore (g \circ f)^{-1}(m) = x$

Now consider $f^{-1} \circ g^{-1}$

i.e. $(f^{-1} \circ g^{-1})(m) = f^{-1}(g^{-1}(m))$

$= f^{-1}(k)$

$= x$

$\therefore (g \circ f)^{-1}(m) = (f^{-1} \circ g^{-1})(m) = x$

$\therefore (g \circ f)^{-1} = f^{-1} \circ g^{-1}$

UEx. 2.21.21 **MU – Dec. 05, May 08, May 09, May 12**

If A = B = C = R where R is set of real number and $f: A \to B$, $g: B \to C$ are functions defined by $f(x) = x+1$, $g(x) = x^2+2$, then find $(g \circ f)(x)$ and $(f \circ g)(2)$.

Soln. :

Given A = B = C = R, R is set of real numbers.

$f: A \to B$, $g: B \to C$ and

$f(x) = x+1$, $g(x) = x^2+2$

(i) $(g \circ f)(x) = g(f(x)) = g(x+1)$

$(x+1)^2+2 = x^2+2x+1+2 = x^2+2x+3$

(ii) $(f \circ g)(2) = f(g(2))$

$= f(2^2+2) = f(6) = 6+1$

$= 7$

UEx. 2.21.22 **MU – May 09**

If A = B = C = R where R is set of real number and $f: A \to B$, $g: B \to C$ are functions defined by $f(x) = x+1$, $g(x) = x^2+2$, then find $(g \circ f)(x)$ and $(f \circ g)(2)$.

Soln. :

Given : $A = \{0, -1, 1\}$, $B = \{0, 1\}$

$f: A \to B$, $f(a) = a|$

To find : Whether f is onto

$f = \{<0, 0>, <-1, 1>, <1, 1>\}$

$\therefore \forall b \in B$, we have at least one $a \in A$, such that $f(a = b)$

$\therefore$ f is onto.

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UEx. 2.21.23 **MU – Dec. 10**

Let $f : R \to R$ $f(x) = x^3$, $g : R \to R$ $g(x) = 4x^2 + 1$, $h : R \to R$ $h(x) = 7x - 2$

Find (i) $g \circ (h \circ g)$; (ii) $g \circ (h \circ f)$

☑ **Soln. :**

Given :

$f : R \to R, \quad f(x) = x^3$

$g : R \to R, \quad g(x) = 4x^2 + 1$

$h : R \to R, \quad h(x) = 7x - 2$

To find : $h \circ g$ of $g \circ (h \circ f)$

$$h \circ g(x) = h(g(x)) = h(4x^2 + 1)$$
$$= 7(4x^2 + 1) - 2 = 28x^2 + 7 - 2$$
$$= 28x^2 + 5$$

$$g \circ (h \circ f) = g(h(f(x))) = g(h(x^3))$$
$$= g(7x^3 - 2) = 4(7x^3 - 2) + 1$$
$$= 4(49x^6 + 4 - 28x^3) + 1$$
$$= 196x^6 + 16 - 112x^3 + 1$$
$$= 196x^6 - 112x^3 + 17$$

$$h \circ g \text{ of } g \circ (h \circ f) = h \circ g(g \circ (h \circ f))$$
$$= 28[196x^6 - 112x^3 + 17]^2 + 5$$
$$= 28[38{,}416x^{12} + 12544x^6 + 289 - 43{,}904x^3 - 3808x^3 + 6664x^6] + 5$$
$$= 1075648x^{12} - 1229312x^9 + 537824x^6 - 106624x^3 + 8097$$

UEx. 2.21.24 **MU – Dec. 11**

A function $f : R - \left\{\frac{7}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ is defined as : $f(x) = \frac{4x-5}{3x-7}$

Prove that 'f' is bijective and find the rule for f^{-1}.

☑ **Soln. :**

Given : $f : R - \left\{\frac{7}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$

$$f(x) = \frac{4x-5}{3x-7}$$

☞ **To prove :**

f is bijective

To find : f^{-1}

Let n_1, n_2 be two numbers and $f(n_1) = f(n_2)$

$$\frac{4n_1 - 5}{3n_1 - 7} = \frac{4n_2 - 5}{3n_2 - 7}$$

$$\therefore \quad n_1 = n_2$$

$$\therefore \quad \text{If } f(n_1) = f(n_2) \text{ then } n_1 = n_2$$

$\therefore$ If is one to one

Consider any $n \in R$, such that $f(x) = n$; where $x \in R$.

$$\therefore \quad n = \frac{4x-5}{3x-7}$$

$\therefore$ For every $n \in R$, we have a $x \in R$, such that $f(x) = n$

$\therefore$ f is onto.

$\therefore$ f is bijective.

$$\text{Let } f(x) = \frac{4x-5}{3x-7} = k$$

$$\therefore \quad f^{-1}(k) = x$$

$$\therefore \quad \frac{4x-5}{3x-7} = k$$

$$\therefore \quad 4x - 5 = 3kx - 7k$$

$$\therefore \quad (4 - 3k) = 5 - 7k$$

$$x = \frac{5-7x}{4-3x}$$

$$\therefore \quad f^{-1}(x) = \frac{5-7x}{4-3x}$$

UEx. 2.21.25 **MU – Dec. 12**

Let $f : R \to R$, $f(x) = x^2 - 1$, $g(x) = 4x^2 + 2$ find (i) $f \circ (g \circ f)$ (ii) $g \circ (f \circ g)$

☑ **Soln. :**

Given : $f : R \to R$, $f(x) = x^2 - 1$, $g(x) = 4x^2 + 2$

To find : (i) $f \circ (g \circ f)$, (iii) $g \circ (f \circ g)$.

$f \circ (g \circ f)(x)$

$f(g(f(x)))$

$$= f(g(x^2 - 1))$$
$$= f(4(x^2 - 1)^2 + 2)$$

$= f(4(x^4 + 1 - 2x^2) + 2)$

$= f(4x^4 - 8x^2 + 6)$

$= (4x^4 - 8x^2 + 6)^2 = 16x^8 + 64x^4 + 36 - 64x^6 - 96x^2 + 48x^4 - 1$

$= 16x^8 - 64x^6 + 112x^4 - 96x^2 + 35$

(ii) $g \circ (f \circ g)(x) = g(f(g(x)))$

$= g(f(4x^2 + 2))$

$= g((4x^2 + 2)^2 - 1)$

$= g(16x^4 + 4 + 16x^2 - 1)$

$= g(16x^4 + 16x^2 + 3)$

$= 4(16x^4 + 16x^2 + 3)^2 + 2$

$= 4(256x^8 + 256x^4 + 9 + 512x^6 + 96x^2 + 96x^4) + 2$

$= 1024x^8 + 2048x^6 + 1408x^4 + 384x^2 + 38$

UEx. 2.21.26 **MU – Dec. 05, May 08, May 09, May 12**

Define injective, surjective and bijective functions. If $f : R \to R$ and $g : R \to R$ are defined by the formulas $f(x) = x + 2$ and $g(x) = x^2$

Find (i) $f.g.f$; (ii) $g.f.g$.

Given : $f : R \to R$, $g : R \to R$

$f(x) = x + 2$, $g(x) = x^2$

To find : (i) $f \circ g \circ f$; (ii) $g \circ f \cdot g$

☑ **Soln. :**

(i) $f \circ g \circ f(x)$

$= f(g(f(x))) = f(g(x + 2))$

$= f((x + 2)^2) = f(x^2 + 4x + 4)$

$= x^2 + 4x + 6$

(ii) $g \circ f \circ g(x)$

$= g(f(g(x))) = g(f(x^2))$

$= g(x^2 + 2) = (x^2 + 2)^2 = x^4 + 4x^2 + 4$

UEx. 2.21.27 **MU – May 14**

Function $f(x) = \dfrac{(4x + 3)}{(5x - 2)}$. Find f^{-1}

☑ **Soln. :**

Given : $f(x) = \dfrac{4x + 3}{5x - 2}$

To find : f^{-1}

$$\text{Let } f(x) = \frac{4x + 3}{5x - 2} = k$$

$$f^{-1}(x) = x$$

$$\therefore k = \frac{4x + 3}{5x - 2}$$

$$\therefore 5kx - 2k = 4x + 3$$

$$\therefore (5k - 4)x = 3 + 2k$$

$$\therefore x = \frac{2k + 3}{5k - 4}$$

$$\therefore f^{-1}(x) = \frac{2x + 3}{5x - 4}$$

UEx. 2.21.28 **MU – May 14**

Let $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$ for all $x \in R$. (R is the set of real number)

Find (i) $f \circ g \circ h$ (ii) $h \circ g \circ f$ (iii) $f \circ f \circ f$

Given : $f(x) = x + 2$, $g(x) = x - 2$, $h(x) = 3x$

To find : (i) $f \circ g \circ h$ (i) $h \circ g \circ f$ (iii) $f \circ f \circ f$

☑ **Soln. :**

(i) $f \circ g \circ h(x)$

$= f(g(h(x))) = f(g(3x))$

$= f(3x - 2) = 3x - 2 + 2$

$= 3x$

(ii) $h \circ g \circ f(x)$

$= h(g(f(x))) = h(g(x + 2))$

$= h(x + 2 - 2) = h(x) = 3x$

(iii) $f \circ f \circ f(x) = f(f(f(x)))$

$= f(f(x + 2)) = f(x + 2 + 2)$

$= x + 2 + 2 + 2 = x + 6$

UEx. 2.21.29 **MU – Q. 5(b), Dec. 15,**

If $f : R \to R$, $f(x) = 2x + 3$

$g : R \to R$, $g(x) = 3x + 4$

$h : R \to R$, $h(x) = 4x$

find : (i) $g \circ f$ (ii) $f \circ g$, (iii) $f \circ h$ (iv) $h \circ f$ (v) $g \circ f \circ h$

Given : $f : R \to R$, $f(x) = 2x + 3$

$g : R \to R$, $g(x) = 3x + 4$

Module II

$h : R \to R, \quad h(x) = 4x$

To find : (i) $g \circ f$ (ii) $f \circ g$,

(iii) $f \circ h$ (iv) $h \circ f$ (v) $g \circ f \circ h$

☑ **Soln. :**

(i) $g \circ f(x)$

$$g \circ f(x) = g(f(x)) = g\ (2x + 3)$$
$$= 3\ (2x + 3) + 4$$
$$= 6\,x + 13$$

(ii) $f \circ g\ (x) = f(g(x))$

$$= f\ (3x + 4) = 2\ (3x + 4) + 3$$
$$= 6x + 11$$

(iii) $f \circ h\ (x) = f\ (h\ (x))$

$$= f\ (4x) = 2(4x) + 3$$
$$= 8x + 3$$

(iv) $h \circ f\ (x) = h(f(x))$

$$= h\ (2x + 3) = 4\ (2x + 3)$$
$$= 8x + 12$$

(v) $g \circ f \circ g\ h \circ (x)$

$$= g\ (f\ (h(x))) = 9\ (8x + 3)$$
$$= 3\ (8x + 3) + 4 = 24x + 13$$

UEx. 2.21.30 MU – Dec. 05. May 08. May 09. May 12

Functions f, g, h are defined on a set, X = {1,2,3} as f = {(1,2) (2,3) (3,1)} g = {(1,2} (2,1) {3,3)} h = (1, 1) (2,2) (3,1)}.

(i) Find $f \circ g$, $g \circ f$ are they equal ?

(ii) Find $f \circ g \circ h$ and $f \circ h \circ g$.

☑ **Soln. :**

Given : X = {1, 2, 3}

f, g and h are functions defined on set X.

$$f = \{(1, 2), (2, 3), (3, 1)\}$$
$$g = \{(1, 2), (2, 1), (3, 3)\}$$
$$h = ((1, 1), (2, 2), (3, 1)\}$$

To find : $f \circ g$, $g \circ f$, $f \circ g \circ h$, $f \circ h \circ g$

$$f \circ g(x) = f\ (g\ (x))$$
$$f \circ g = \{f\ (g\ (1)), f\ (g(2)), f(g(3))\}$$
$$= \{f(2), f(1), f(3)\}$$
$$= \{3, 2, 1\}$$
$$= \{(1, 3), (2, 2), (3, 1)\}$$

The composite function can be calculated using graphical representation as follows

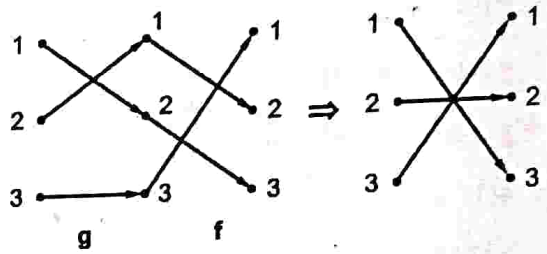

Fig. P. 2.21.30

$$\therefore\ f \circ g = \{(1, 3), (2, 2), (3, 1)\}$$

(ii) $g \circ f$ is calculated as follows

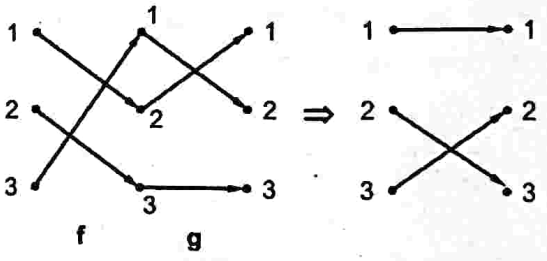

Fig. P. 2.21.30(a)

$$\therefore\ g \circ f = \{(1, 1), (2, 3), (3, 2)\}$$

$\therefore$ $f \circ g$ and $g \circ f$ are not equal

(iii)

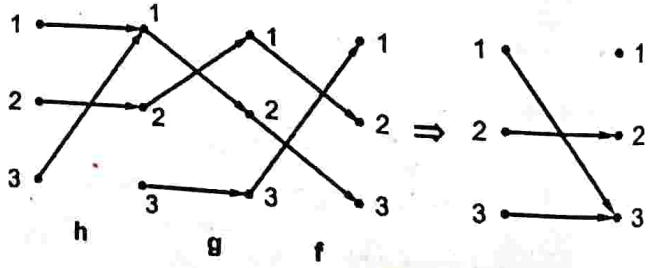

Fig. P. 2.21.30(b)

$$\therefore\ f \circ g \circ h = \{(1, 3), (2, 2), (3, 3)\}$$

(iv)

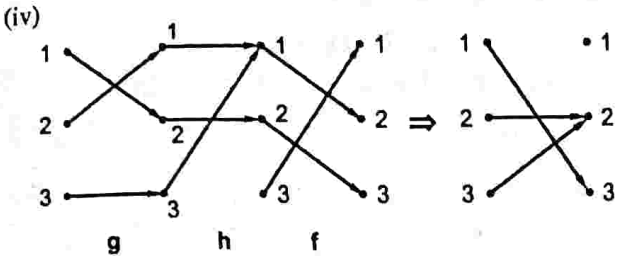

Fig. P. 2.21.30(c)

$\therefore \quad f \circ h \circ g = \{(1, 1), (2, 2), (3, 2)\}$

UEx. 2.21.31 **MU – May 17**

Let the function $f : R \to R$, $f(x) = 2x - 3$

Find $f^2 = f \circ f$, $f^3 = f \circ f \circ f$

Soln. :

Given : $f(x) = 2x - 3$

To find : (i) $f^2 = f \circ f$ (ii) $f^3 = f \circ f \circ f$

(i) $f \circ f(x) = f(f(x))$

$= f(2x - 3)$

$= 2(2x - 3) - 3 = 4x - 9$

(ii) $f^3 = f \circ f \circ f(x) = f(f(f(x)))$

$= f(f(2x - 3))$

$= f(4x - 9)$

$= 2(4x - 9) - 3$

$= 8x - 21$

UEx. 2.21.32 **MU - Q. 3(c), Dec. 18, 8 Marks**

Let $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$ for $x \in R$, where R = set of real numbers.

Find $(g \circ f)$, $(f \circ g)$, $(f \circ f)$, $(g \circ g)$, $(f \circ h)$, $(h \circ g)$, $(h \circ f)$, $(f \circ h \circ g)$.

Soln. :

Given : $f(x) = x + 2$, $g(x) = x - 2$ $h(x) = 3x$

$x \in R$

(i) $g \circ f(x) = g(f(x)) = g(x + 2) = (x + 2 - 2) = x$

(ii) $f \circ g(x) = f(g(x)) = f(x - 2) = (x - 2 + 2) = x$

(iii) $f \circ f(x) = f(f(x)) = f(x + 2) = x + 4$

(iv) $g \circ g(x) = g(g(x)) = g(x - 2) = x - 4$

(v) $f \circ h(x) = f(h(x)) = f(3x) = 3x + 2$

(vi) $h \circ g(x) = h(g(x)) = h(x - 2) = 3x - 6$

(vii) $h \circ f(x) = h(f(x)) = h(x + 2) = 3x + 6$

(viii) $f \circ h \circ g(x) = f(h(g(x))) = f(h(x - 2))$

$= f(3x - 6) = 3x - 4.$

UEx. 2.21.33 **MU - Q. 3(c), May 18, 4 Marks**

Test whether the following function is one-to-one, onto or both.

$f : Z \to Z$, $f(x) = x^2 + x + 1$

Soln. :

Given function is,

$f : z \to z$, $f(x) = x^2 + x + 1$

for every $x \in z$, we have distinct $f(x)$

$\therefore$ f is one-to-one.

Each element y in z may not have a corresponding x in z.

For eg. Let $y = f(x) = 3$

$\therefore \quad x^2 + x + 1 = 3$

$\therefore \quad x^2 + x = 2$

$\therefore \quad x(x + 1) = 2 \qquad \therefore \quad x \notin z$

$\therefore$ f(x) is not onto.

UEx. 2.21.34 **MU - Q. 6(b), May 18, 8 Marks**

Let the functions f, g and h defined as follows :

$f : R \to R$, $f(x) = 2x + 3$

$g : R \to R$, $g(x) = 3x + 4$

$h : R \to R$, $h(x) = 4x$

Find $g \circ f$, $f \circ g$, $f \circ h$, $g \circ f \circ h$.

Soln. :

Given :

$f : R \to R$, $f(x) = 2x + 3$

$g : R \to R$, $g(x) = 3x + 4$

$h : R \to R$, $h(x) = 4x$

$\therefore$ $g \circ f$ is given by,

$g \circ f(x) = g(f(x))$

$= g(2x + 3) = 3(2x + 3) + 4 = 6x + 13$

$g \circ f : R \to R$, $g \circ f(x) = 6x + 13$

Module II

(ii) $f \circ g$ is given by,

$$f \circ g(x) = f(g(x)) = f(3x+4)$$
$$= 2(3x+4)+3 = 6x+11$$

$\therefore$ $f \circ g : R \to R$, $f \circ g(x) = 6x+11$

(iii) $f \circ h$ is given by,

$$f \circ h(x) = f(h(n)) = f(4x)$$
$$= 2(4x)+3 = 8x+3$$

$\therefore$ $f \circ h : R \to R$, $f \circ h(x) = 8x+3$

(iv) $g \circ f \circ h$ is given by,

$$g \circ f \circ h(x) = g(f(h(x))) = g(f(4x))$$
$$= g(2(4x)+3) = g(8x+3)$$
$$= 3(8x+3)+4$$
$$= 24x+13$$

$\therefore$ $g \circ f \circ h : R \to R$, $g \circ f \circ h(x) = 24x+13$

UEx. 2.21.35 MU – Q. 1(d). Dec. 17. 5 Marks

Comment whether the function f is one to one or onto.

Consider function : $f : N \to N$ where N is set of natural numbers including zero.

$f(j) = j^2 + 2$

☑ **Soln. : Given :** $f : N \to N$

$$f(j) = j^2 + 2$$

$\forall j \in N$, $f(j)$ is distinct.

$\therefore$ f is one to one.

$\exists x \in N$, we may not have a j such that

$$f(j) = x$$

Ex. : Let $f(j) = 5$

$\therefore$ $j^2 + 2 = 5$

$\therefore$ $j^2 = 3$

$\therefore$ $j = \sqrt{3} \notin N$

$\therefore$ f is not onto.

UEx. 2.21.36 MU - Q. 2(d). Dec. 17. 4 Marks

Let $f : R \to R$ defined as $f(x) = x^3$ and

$g : R \to R$ defined as $g(x) = 4x^2 + 1$

Find out $g \circ f$, $f \circ g$, f^2, g^2

☑ **Soln. :**

Given : $f(x) = x^3$, $g(x) = 4x^2 + 1$

(i) $g \circ f$ is given by,

$$g \circ f(x) = g(f(x)) = g(x^3)$$
$$= 4(x^3)^2 + 1 = 4x^6 + 1$$

$\therefore$ $g \circ f : R \to R$ is defined as $g \circ f(x) = 4x^6 + 1$

(ii) $f \circ g$ is given by,

$$f \cdot g(x) = f(g(x)) = f(4x^2+1) = (4x^2+1)^3$$

$\therefore$ $f \circ g = R \to R$ is defined as $f \circ g(x) = (4x^2+1)^3$

(iii) f^2 is given by,

$$f \circ f = f(f(x)) = f(x^3) = (x^3)^3 = x^9$$

$\therefore$ $f^2 : R \to R$ is defined as $f^2(x) = x^9$

(iv) g^2 is given by,

$$g \circ g = g(g(x)) = g(4x^2+1) = 4(4x^2+1)^2 + 1$$
$$= 64x^4 + 32x^2 + 5$$

$\therefore$ $g^2 : R \to R$ is defined as $g^2(x) = 64x^4 + 32x^2 + 5$

UEx. 2.21.37 MU - Q. 3(c). Dec. 17. 4 Marks

Let A = B = R, the set of real numbers

Let $f : A \to B$ be given by the formula $f(x) = 2x^3 - 1$ and

Let $g : B \to A$ be given by

$$g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}$$

Show that f is a bijection between A and B and g is a bijection between B and A.

☑ **Soln. :**

Given : A = B = R

$f : A \to B$, $f(x) = 2x^3 - 1$

$g : B \to A$, $g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}$

To prove : f and g are bijective.

Consider $k = f(x) = 2x^3 - 1$

$\forall x \in A$, k is distinct.

$\therefore$ f is one-to-one.

Now $k = 2x^3 - 1$

$2x^3 = k + 1$

$x^3 = \frac{k+1}{2}$ $\quad x = \sqrt[3]{\frac{k}{2} + \frac{1}{2}}$

$\therefore$ $\forall k \in B$, x is distinct.

$\therefore$ f is onto.

$\therefore$ f is bijective.

Similarly we can prove that g is bijective.

It can be noted that $f^{-1} = g$

Chapter Ends...

□□□

Module III

CHAPTER 3 Posets and Lattice

Syllabus

Partial Order Relations, Poset, Hasse Diagram, Chain and Anti chains, Lattice, Types of Lattice, Sub lattice

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3.1 POSET AND HASSE DIAGRAMS:

UQ. 3.1.1 Explain the terms : Poset.

MU- Q. 1(c), Dec. 2017, 5 Marks

Hasse diagram is used to represent a partially ordered set. A partially ordered set (or POSET) is represented as $(P, \leq)$, where P is a set and $\leq$ is a partial order relation on P.

Ex. : $(\{1,2,3,4,5,6,7,8\}, \leq)$ is a POSET.

Following are the rules to draw a hasse diagram :

(i) Every element in the set are represented by a dot.

(ii) If for any two elements a and b are related as $a \leq b$, then dot representing a is drawn below b and they are joined.

(iii) If for any three elements a, b, and c, if $a \leq b$, $b \leq c$, and $a \leq c$(transitivity), then the dots a-b and b-c are connected, not a and c.

(iv) The dots on the same level in the diagram represent non related elements.

Ex. : Draw the hasse diagram for the following POSETs.

(i) $(\{1,2,3,4,5\}, \leq)$, $\leq$ is less than or equal to relation [Refer Fig. 3.1.1]

(ii) $(\{1,2,3,4,5,6,7,8\}, |)$, $|$ is divides relation. [Refer Fig. 3.1.2]

(iii) $A=\{1,2,3\}$, $(2^A, \subseteq)$, 2^A is power set of A, $\subseteq$ is subset relation. [Refer Fig. 3.1.3]

(iv) $(G_{30}, |)$, G_{30} is the set of divisors of 30, $|$ is divides relation [Refer Fig. 3.1.4]

(v) $(G_{45}, |)$, G_{45} is the set of divisors of 30, $|$ is divides relation. [Refer Fig. 3.1.5]

Soln.:

(i)

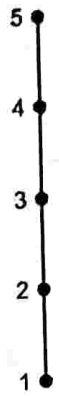

Fig. 3.1.1

(ii)

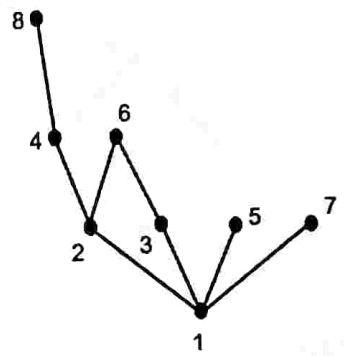

Fig. 3.1.2

(iii)

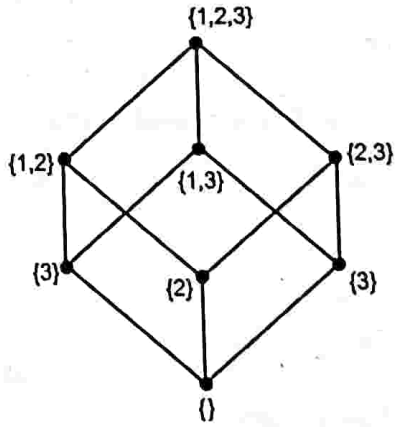

Fig. 3.1.3

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(iv)

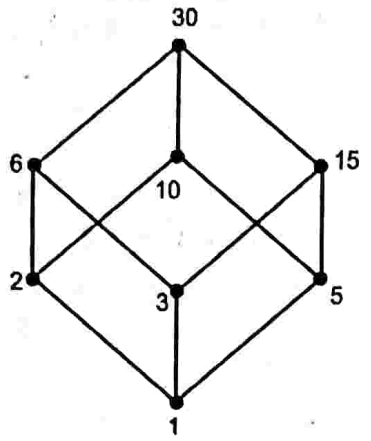

Fig. 3.1.4

(v)

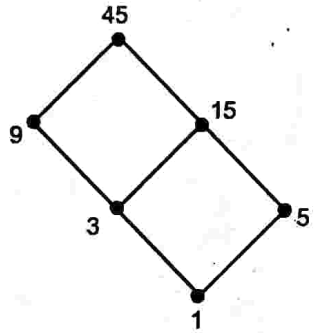

Fig. 3.1.5

3.1.1 Upper Bound

Definition : Let S be a sub set of P and (P, ≤) is a POSET. An element a∈P is called upper bound of S, if

∀x∈ S, x≤a.

3.1.2 Least Upper Bound (LSB)

Definition : Let S be a sub set of P and (P, ≤) is a POSET. Let UB be a set of all upper bounds of S. There exists an unique element k called least upper bound (LSB) of S, if

k∈ UB and ∀x∈ S, a≤x.

3.1.3 Lower Bound

Definition : Let S be a sub set of P and (P, ≤) is a POSET. An element a∈P is called lower bound of S, if

∀x∈ S, a≤x.

3.1.4 Greatest Lower Bound (GLB)

Definition : Let S be a sub set of P and (P, ≤) is a POSET. Let LB be a set of all lower bounds of S. There exists an unique element k called greatest lower bound (GLB) of S, if

k∈ LB and ∀x∈ S, x≤k.

Ex. 3.1.1 : Consider the POSET (G_{40},|), where G_{40} is the set of divisors of 40, | is divides relation. The equivalent hasse diagram is as follows .

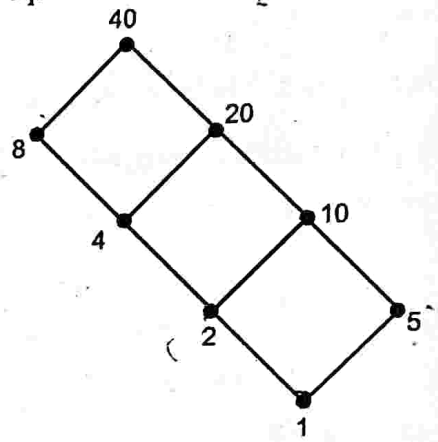

Fig. Ex. 3.1.1

Following table gives few subsets and their upper bounds, lower bounds, LUB and GLB.

Subset	Upper bound	LUB	Lower bound	GLB
{2,4,10}	{20,40}	20	{1,2}	2
{5,10}	{10,20,40}	10	{1,5}	5
{1,5,20,8}	{40}	40	{1}	1

Ex. 3.1.2 : Consider the POSET ({2,4,6,8,10,12,14,16,18,20},|). Following is the Hasse diagram for the POSET.

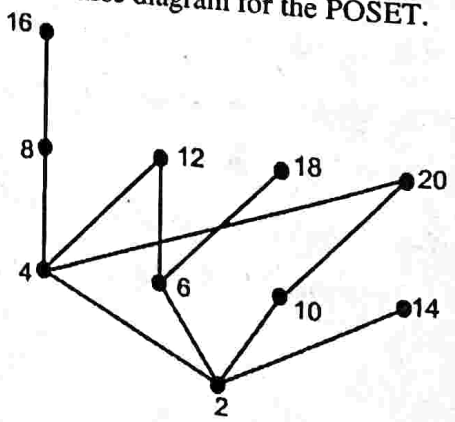

Fig. Ex. 3.1.2

Following table gives few subsets and their upper bounds, lower bounds, LUB and GLB.

Subset	Upper bound	LUB	Lower bound	GLB
{4,6}	{12}	12	{2}	2
{16,20}	Nil	Nil	{2,4}	4
{4,8}	{8,16}	8	{1,4}	4

3.2 LATTICE

UQ. 3.2.1 Explain the terms : (i) Lattice

MU- Q. 1(c), Dec. 2017, 5 Marks

Definition : A lattice is a POSET (P, ≤) in which every pair of two elements in P has LUB and GLB.

i.e. ∀a,b∈P, {a,b} has LUB and GLB.

Example

(i) ({1,2,3,4,5},≤) , ≤ is less than or equal to relation

(ii) A={1,2,3}, (2^A, ⊆), 2^A is power set of A , ⊆ is subset relation.

(iii) (G_{30},|), G_{30} is the set of divisors of 30, | is divides relation

(iv) (G_{45},|), G_{45} is the set of divisors of 30, | is divides relation

3.3 CHAIN

Subset

Let < P, ≤ > be a poset. A set C ⊆ P is called chain, if ∀a,b∈ C, such that a≠b and either a≤b or b≤a.

Example

Consider the Poset< P, ≤ >, where P={1,2,3,4,5,6,8} and a≤b if a divides b. Following the equivalent Hasse diagram.

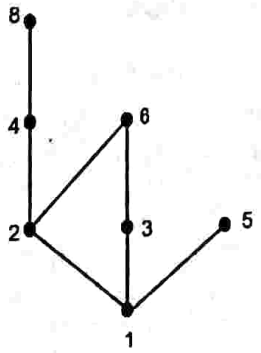

Fig. 3.3.1

Following sets are the few examples of chains with respect to poset< P, ≤ >.

{1,2,4,8}, {1,2,6}, {1,3,6}, {1,5}

3.4 ANTICHAIN

Let < P, ≤ > be a poset. A set K ⊆ P is called antichain, if ∀a,b∈ K, such that a≠b and neither a≤b nor b≤a.

Example

Consider the Poset< P, ≤ >, where P={1,2,3,4,5,6,8} and a≤b if a divides b. Following the equivalent Hasse diagram.

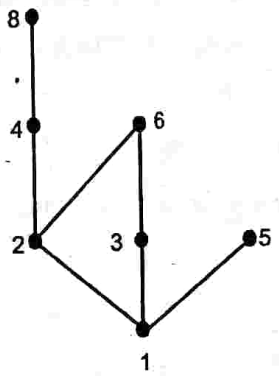

Fig. 3.4.1

Following sets are the few examples of antichains with respect to poset< P, ≤ >.

{2,3,5}, {4,5,6}, {3,4,5}

3.5 TYPES OF LATTICE

3.5.1 Bounded Lattice

A Lattice is called bounded lattice, if it has greatest element 1 and least element 0.

Example

Consider the (2^A, ⊆), 2^A is power set of A , ⊆ is subset relation. If A={1,2,3} then the equivalent Hasse diagram is as follows.

Module III

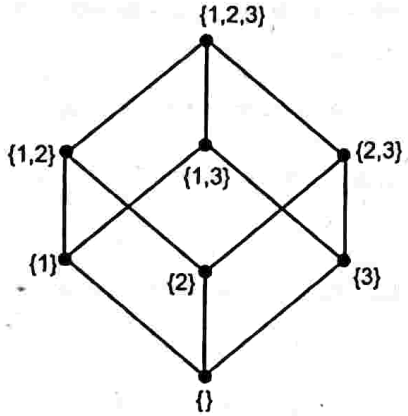

Fig. 3.5.1

This is a lattice. It can be seen that {} is the least element and {1,2,3} is the greatest element.

Thus$(2^A, \subseteq)$ is a bounded lattice.

Properties of bounded lattice:

If L is a bounded lattice, then $\forall$ a $\in$ L, we have

1. $a \vee 1 = 1$
2. $a \wedge 1 = a$
3. $a \vee 0 = a$
4. $a \wedge 0 = 0$

3.5.2 Isomorphic Lattices

Two lattices L_1 and L_2 are called isomorphic lattices if there is a bijection from L_1 toL_2

i.e., $f: L_1 \rightarrow L_2$, such that $f(a \wedge b) = f(a) \wedge f(b)$

and $f(a \vee b) = f(a) \vee f(b)$

Example

Consider the lattices <A, |> where A={1,2,3,6} and $<2^B, \subseteq>$ where B={a,b}

The equivalent Hasse diagrams are as follows.

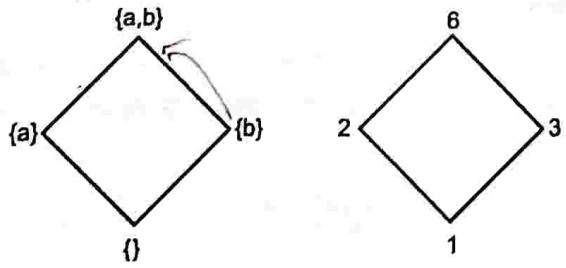

Fig. 3.5.2

Consider the mapping f = {({},1), ({a},2), ({b},3), ({a,b},6)}..

A	B	a V b	f (a V b)	f(a)	f(b)	f(a)V f(b)
{}	{a}	{a}	2	1	2	2
{}	{b}	{b}	3	1	3	3
{}	{a,b}	{a,b}	6	1	6	6
{a}	{b}	{a,b}	6	2	3	6
{a}	{a,b}	{a,b}	6	2	6	6
{b}	{a,b}	{a,b}	6	3	6	6

Thus we can say that f (a V b) =f(a)V f(b)

Thus the lattices <A, |> where A={1,2,3,6} and $<2^B, \subseteq>$ where B={a,b} are isomorphic.

3.5.3 Distributive Lattice

A lattice L is called distributive lattice if for any elements a, b and c of L,it satisfies following distributive properties:

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

If the lattice L does not satisfies the above properties, it is called a non-distributive lattice.

Example

Consider the lattices <A, |> where A={1,2,3,6}

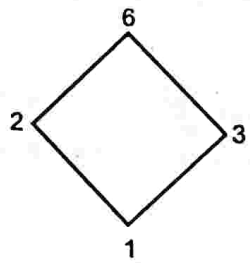

Fig. 3.5.3

a	b	C	(b ∧ c)	a V (b ∧ c)	(a V b)	(a V c)	(a V b) ∧ (a V c)
1	2	3	1	1	2	3	1
1	2	6	2	2	2	6	2
2	3	6	3	6	6	6	6

Thus it can be seen that a V (b ∧ c)= (a V b) ∧ (a V c)

Therefore it is a distributive lattice.

3.5.4 Complemented Lattices

Let L be a bounded lattice with lower bound 0 and upper bound 1.

Let a be an element in L. An element x in L is called a complement of a, if a V x = 1 and a ∧ x = 0. The complement of a is represented as a'.

A lattice L is said to be complemented if L is bounded and every element in L has a complement.

Consider the set P={1,2,3,6,9,18} and the partial order relation divides.

The poset<P, | > is a lattice. The equivalent Hasse diagram is as follows.

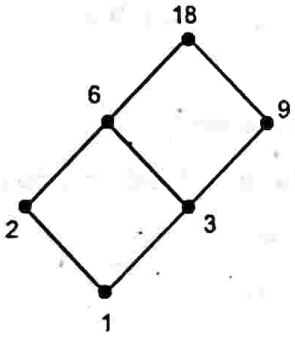

Fig. 3.5.4

As per the definition of complement of an element we can write, 1'=18, 2'=9, 3'=6, 6'=3, 9'=2 and 18'=1.

Thus it is a complemented lattice.

3.5.5 Modular Lattice

A lattice (L, ∧,V) is called a modular lattice if a V (b ∧ c) = (a V b) ∧ c whenever a ≤ c.

3.6 SUB-LATTICES

Consider a non-empty subset L_1 of a lattice L. Then L_1 is called a sub-lattice of L if L_1 itself is a lattice i.e., the operation of L i.e., a V b ∈ L_1 and a ∧ b ∈ L_1 whenever a ∈ L_1 and b ∈ L_1.

Example 1

Consider the lattice of all +ve integers I_+ under the operation of divisibility. The lattice D_n of all divisors of n > 1 is a sub-lattice of I_+.

Example 2

The sub-lattices of D_{30} = {1,2,3,5,6,10,15,30} that contain at least four elements, are as follows

1. {1, 2, 6, 30}
2. {1, 2, 3, 30}
3. {1, 5, 15, 30}
4. {1, 3, 6, 30}
5. {1, 5, 10, 30}
6. {1, 3, 15, 30}
7. {2, 6, 10, 30}

3.7 SOLVED EXAMPLES

UEx. 3.7.1 MU - Q. 3(b), Dec. 2013. 6 Marks

Let L_1 and L_2 be lettices shown below :

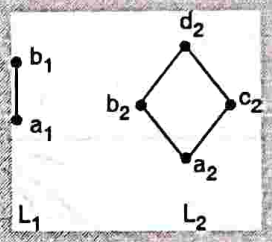

Fig. Ex. 3.7.1

Draw the Hasse diagram of $L_1 \times L_2$ with product partial order..

Soln. : Given lattices are,

For L_1, R = $\{(a_1, a_1), (b_1, b_1), (a_1, b_1)\}$

For L_2, R = $(a_2, a_2), (b_2, b_2), (c_2, c_2), (d_2, d_2), (a_2, d_2), (a_2, b_2), (a_2, c_2), (b_2, d_2), (c_2, d_2)$

With the Hasse diagrams,

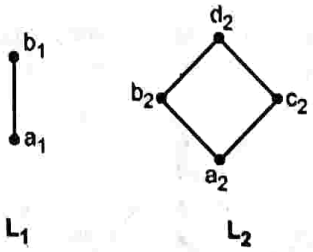

Fig. Ex. 3.7.1(a)

Now, for

$L_1 \times L_2 = \{(a_1, a_2), (a_1, b_2), (a_1, c_2), (a_1, d_2), (b_1, a_2), (b_1, b_2), (b_1, c_2), (b_1, d_2)\}$

Module III

R = {((a$_1$ a$_2$), (a$_1$ a$_2$)), ((a$_1$, a$_2$) (b$_1$, b$_2$)) ((a$_1$ a$_2$), (b$_1$, c$_2$)),
((a$_1$ a$_2$), (a$_1$, b$_2$)), ((a$_1$, a$_2$), (a$_1$, c$_2$)), ((a$_1$, a$_2$), (a$_1$, a$_2$)),
((a$_1$, a$_2$), (b$_1$, d$_2$)), ((a$_1$, b$_2$), (a$_1$, b$_2$)), ((a$_1$, b$_2$), (b$_1$, d$_2$))
((a$_1$, b$_2$), (a$_1$, d$_2$)), ((a$_1$, c$_2$), (a$_1$, d$_2$))
((a$_1$, c$_2$), (a$_1$, c$_2$)), ((a$_1$, c$_2$), (b$_1$, d$_2$)), ((a$_1$, d$_2$), (a$_1$, d$_2$)),
(a$_1$, d$_2$), (b$_1$, d$_2$)}
((b$_1$, a$_2$), (b$_1$, a$_2$)), ((b$_1$, a$_2$), (b$_1$, b$_2$)), ((b$_1$, a$_2$), (b$_1$, c$_2$))
((b$_1$, a$_2$), (b$_1$, d$_2$)), ((b$_1$, b$_2$), (b$_1$, b$_2$)), ((b$_1$, b$_2$), (b$_1$, d$_2$))
((b$_1$, c$_2$), (b$_1$, c$_2$)), ((b$_1$, c$_2$), (b$_1$, d$_2$)), ((b$_1$, d$_2$), (b$_1$, d$_2$))

The diagraph of the above relation is as shown in Fig. Ex. 3.7.1(b).

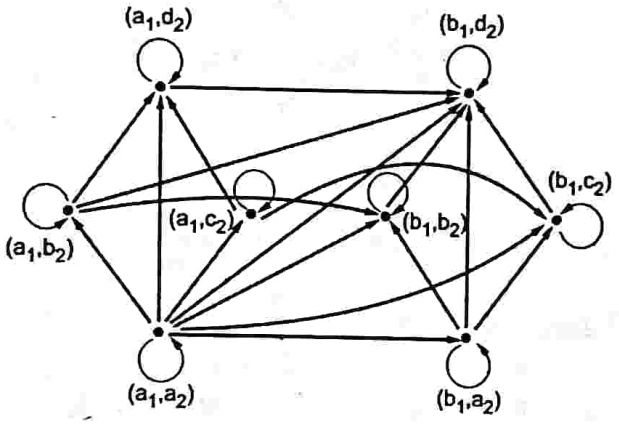

Fig. Ex. 3.7.1(b)

To draw the equivalent lattice, eliminate the reflexive and transitive representations. The lattice is as show in Fig. Ex. 3.7.1(c).

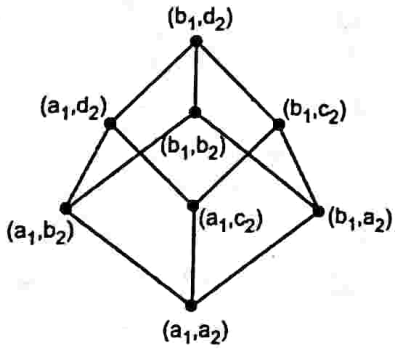

Fig. Ex. 3.7.1(c)

UEx. 3.7.2 **MU - Q. 3(c), Dec. 2013, 4 Marks**

Let A = {a, b, c}. Show that (P(A), ⊆) is a poset. Draw its Hasse diagram. P(A) is the power set of A.

☑ Soln. :

Given, A = {a, b, c}

To prove (P(A), ⊆) is a poset.

☞ Proof

(P(A), ⊆) is poset if, ⊆ is reflexive anti symmetric and transitive w.r.t. P(A)

P(A) = {ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}, {a, b, c}}

Consider the relation ⊆

∀ x ∈ P(A), x ⊆ x

∴ ⊆ is reflexive,

∀x, y ∈ P(A), if x ⊆ y then

either x = y or y ⊈ x

∴ ⊆ is antisymmetric

∀ x, y, z ∈ P(a), if x ⊆ y and y ⊆ z, then x ⊆ z

∴ ⊆ is transitive

∴ (P(A), ⊆) is a poset.

The relation ⊆ is given as follows.

⊆ = {(ϕ, ϕ), (ϕ, {a}), (ϕ, {b}), (ϕ, {c}), (ϕ,{a, b}),
(ϕ, {b, c}), (ϕ, {a, c}), (ϕ, {a, b, c}), ({a}, {a, b})
({a}, {a}), ({a}, {a, c}), ({a}, {a, b, c}),
({b},{b}), ({b}, {a, b}), ({b}, {b, c}),
({b}, {a, b, c}), ({c}, {c}), ({c}, {a, c}), ({c},
{b, c}), ({c}, {a, b, c}), ({a, b}, {a, b}), ({a, b},
{a, b, c}), ({b, c},{b, c}), ({b, c}, {a, b, c}),
({a, c}, {a, c}), ({a, c}, {a, b, c}), ({a, b, c},
{a, b, c})}

The diagraph is as shown in Fig. Ex. 3.7.2.

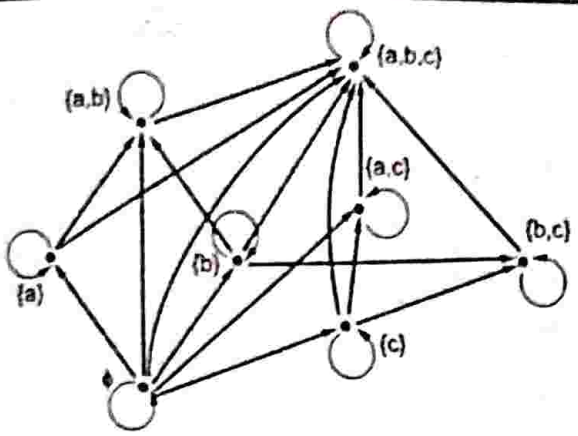

Fig. Ex. 3.7.2

By eliminating the reflexive and transitive representations, the lattice is as shown in Fig. Ex. 3.7.2(a).

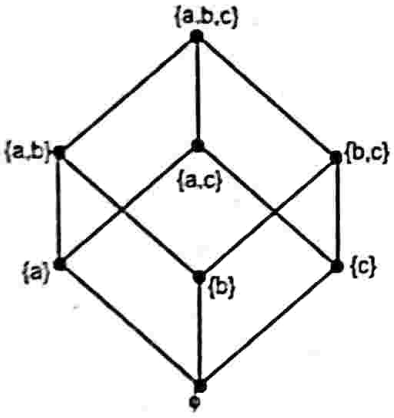

Fig. Ex. 3.7.2(a)

UEx. 3.7.3 **MU - Q. 3(b), May 2014, 7 Marks**

Consider the lattices L1 = {1, 2, 4}, L2 = {1, 3, 9} under divisibility. Draw the lattice L1 × L2.

☑ **Soln. :**

Given : Chains of divisors of 4 and 9.

$L_1 = \{1, 2, 4\}$

$L_2 = \{1, 3, 9\}$

Partial order relation of division on L_1, L_2 are as follows

$R_1 = \{(1, 1), (1, 2), (1, 4), (2, 2), (2, 4), (4, 4)\}$

$R_2 = \{(1, 1), (1, 3), (1, 9), (3, 3), (3, 9), (9, 9)\}$

Now $L_1 \times L_2 = \{(1, 1), (1, 3), (1, 9), (2, 1), (2, 3), (2, 9), (4, 1), (4, 3), (4, 9)\}$

Now R_3 is the relation of division on $L_1 \times L_2$ given as follows.

$R_3 = \{((1, 1), (1, 1)), ((1, 1), (1, 3)), (1, 1), (1, 9)), ((1, 1), (2, 1)), ((1, 1), (2, 3)), ((1, 1), (2, 9)),$

$((1, 1), (4, 1)), ((1, 1), (4, 3)), ((1, 1), (4, 9)),$
$((1, 3), (1, 3)), ((1, 3), (1, 9)), ((1, 3), (2, 3)),$
$((1, 3), (2, 9)), ((1, 3), (4, 3)), ((1, 3), (4, 9)),$
$((1, 9), (1, 9)), ((1, 9), (2, 9)), ((1, 9), (4, 9)),$

$((2, 1), (2, 1)), ((2, 1), (2, 3)), ((2, 1), (2, 9)),$
$((2, 1), (4, 1)), ((2, 1), (4, 3)), ((2, 1), (4, 9)),$
$((2, 3), (2, 3)), ((2, 3), (2, 9)), ((2, 3), (4, 3)),$
$((2, 3), (4, 9)), ((2, 9), (2, 9)), ((2, 9), (4, 9)),$
$((4, 1), (4, 1)), ((4, 1), (4, 3)), ((4, 1), (4, 9)),$
$((4, 3), (4, 3)), ((4, 3), (4, 9)), ((4, 9), (4, 9))\}$

The lattice is as follows.

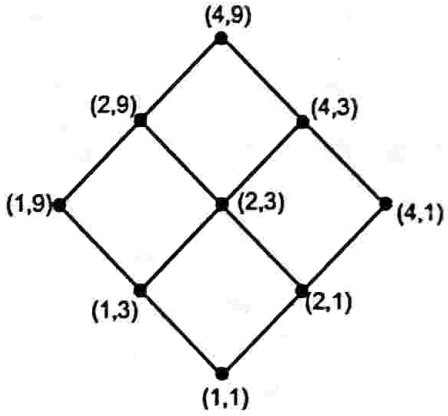

Fig. Ex. 3.7.3

UEx. 3.7.4 **MU - Q. 6(c), May 2014, 6 Marks**

Draw the Hasse Diagram of the following :

(a) D_{105} (b) D_{72}

☑ **Soln. :**

Given

a. $D_{105} = \{1, 3, 5, 7, 15, 21, 35, 105\}$

The Hasse diagram is,

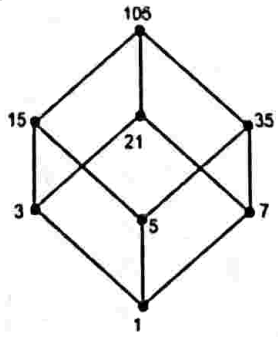

Fig. Ex. 3.7.4

Module III

b. $D_{72} = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$

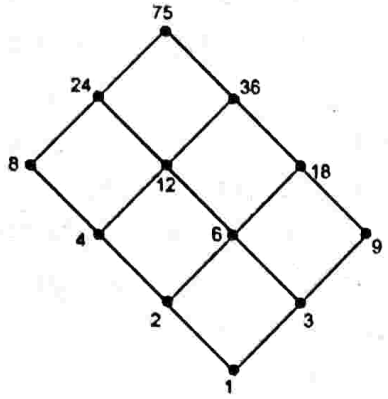

Fig. Ex. 3.7.4(a)

UEx. 3.7.5 **MU - Q. 5(a), Dec. 2014, 8 Marks**

Define Distributive Lattice along with one appropriate example.

☑ **Soln. :** A lattice is said to be distributive, if ∀a, b, c we have

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

and $$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Consider the following Hasse diagram for D_{12}

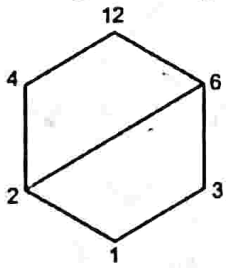

Fig. Ex. 3.7.5

☑ **Soln. :** Following table illustrates the distributive law.

a	B	c	b ∨ c	a ∧ (b ∨ c)	A ∧ b	a ∧ c	(a ∧ b) ∨ (a ∧ c)
1	2	3	6	1	1	1	1
2	3	6	6	2	1	2	2
3	4	6	12	3	1	3	3
4	6	12	12	4	2	4	4

a	b	c	b ∧ c	a ∨ (b ∧ c)	a ∨ b	a ∨ c	(a ∨ b) ∧ (a ∨ c)
1	2	3	1	1	2	3	1
2	3	4	1	2	6	4	2
3	4	6	2	6	12	6	6
4	6	12	6	12	12	12	12

∴ D_{12} is a distributive lattice

UEx. 3.7.6 **MU - Q. 1(A), May 2015, 5 Marks**

Let A = {a, b, c}. Show that (P (A), C) is a poset. Draw its Hasse Diagram. P (A) is the power set of A.

☑ **Soln. :**

Given, $A = \{a, b, c\}$

To prove $(P(A), \subseteq)$ is a poset.

☞ **Proof :**

$(P(A), \subseteq)$ is poset if, $\subseteq$ is reflexive anti symmetric and transitive w.r.t. P(A)

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

Consider the relation $\subseteq$

$\forall x \in P(A), x \subseteq x$

∴ $\subseteq$ is reflexive,

$\forall x, y \in P(A)$, if $x \subseteq y$ then

either $x = y$ or $y \nsubseteq x$

∴ $\subseteq$ is antisymmetric

$\forall x, y, z \in P(a)$, if $x \subseteq y$ and $y \subseteq z$, then $x \subseteq z$

∴ $\subseteq$ is transitive

∴ $(P(A), \subseteq)$ is a poset.

The relation $\subseteq$ is given as follows.

$\subseteq = \{(\phi, \phi), (\phi, \{a\}), (\phi, \{b\}), (\phi, \{c\}), (\phi, \{a, b\}),$
$(\phi, \{b, c\}), (\phi, \{a, c\}), (\phi, \{a, b, c\}), (\{a\}, \{a, b\})$
$(\{a\}, \{a\}), (\{a\}, \{a, c\}), (\{a\}, \{a, b, c\}),$
$(\{b\}, \{b\}), (\{b\}, \{a, b\}), (\{b\}, \{b, c\}),$
$(\{b\}, \{a, b, c\}), (\{c\}, \{c\}) (\{c\}, \{a, c\}),$
$(\{c\}, \{b, c\}), (\{c\}, \{a, b, c\}), (\{a, b\}, \{a, b\}),$
$(\{a, b\}, \{a, b, c\}), (\{b, c\}, \{b, c\}),$
$(\{b, c\}, \{a, b, c\}), (\{a, c\}, \{a, c\}),$
$(\{a, c\}, \{a, b, c\}), (\{a, b, c\}, \{a, b, c\})\}$

The diagraph is as shown in Fig. Ex. 3.7.6.

(8 point will form cube
multiple of 3 points make windows)

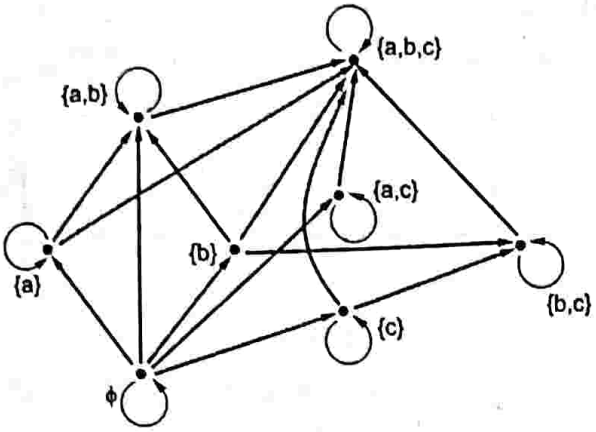

Fig. Ex. 3.7.6

By eliminating the reflexive and transitive representations, the lattice is as shown in Fig. Ex. 3.7.6(a).

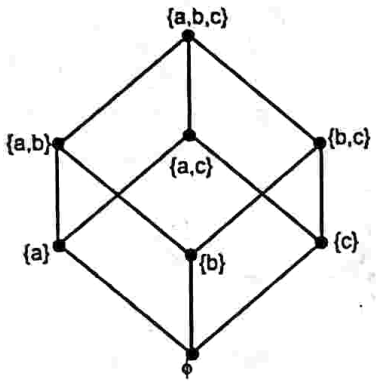

Fig. Ex. 3.7.6(a)

UEx. 3.7.7 MU - Q. 3(a), May 2015, 4 Marks
Show that the set of all divisors of 70 forms a lattice.

☑ **Soln. :**

To prove : All divisors of 70 forms a lattice.

☞ **Proof**

The set is $D_{70} = \{1, 2, 5, 7, 10, 14, 35, 70\}$

The Hasse diagram is as shown in Fig. Ex. 3.7.7.

A poset is said to be lattice, if for every pair of elements there is greatest lower bound and least upper bound. Following table illustrates this with few examples.

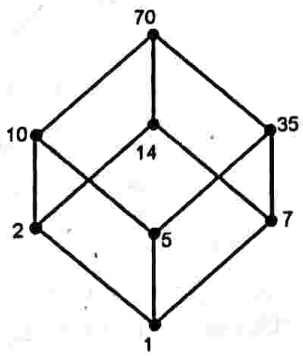

Fig. 3.7.7

a	b	GLB	LUB
1	2	1	2
2	5	1	10
5	7	1	35
7	10	1	70
10	14	2	70
14	35	7	70
35	70	35	70

UEx. 3.7.8 MU - Q. 1(d), Dec. 2015, 5 Marks
Draw the Hasse diagram of D_{60}. Also find whether it is a lattice.

☑ **Soln. :** Given set D_{60}

$D_{60} = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$

The Hasse diagram is as shown in Fig. 3.7.8.

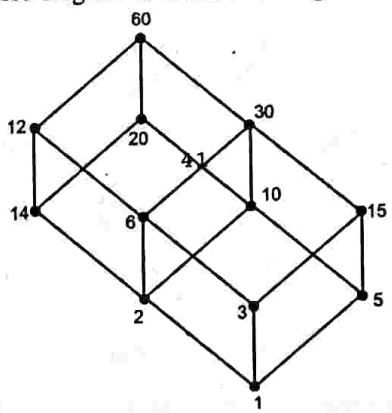

Fig. 3.7.8

a	b	GLB	LUB
1	2	1	2
2	3	1	6
3	5	1	15
4	6	2	12
6	10	2	30
12	15	3	60
20	30	10	60
10	60	10	60
3	20	1	60
5	12	1	60

Module III

UEx. 3.7.9 MU - Q. 3(c), Dec. 2015, 6 Marks

Determine whether the below Hasse diagram represents a lattice.

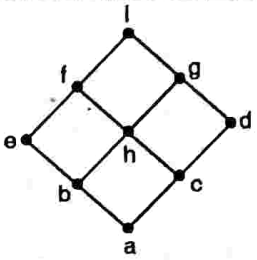

Fig. Ex. 3.7.9

Soln. :

Given Hasse diagram shown in Fig. 3.7.9(a)

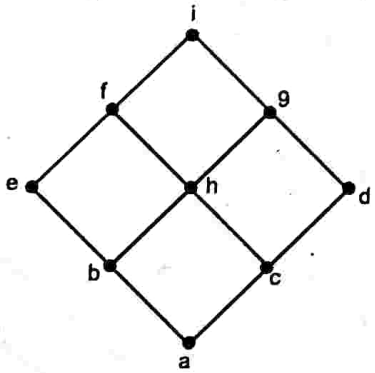

Fig. Ex. 3.7.9(a)

Now, consider any two elements x_1, x_2 there is one GLB and the LUB. The following table illustrates few of them.

x_1	x_2	GLB	LUB
a	b	a	b
b	c	a	h
d	e	a	i
f	g	h	i
h	i	h	i
d	f	c	i

$\therefore$ It is a lattice.

UEx. 3.7.10 MU - Q. 5(b), Dec. 2015, 7 Marks

Consider the chain of divisors of 4 and 9, i.e. $L_1 = \{1, 2, 4\}$ and $L_2 = \{1, 3, 9\}$

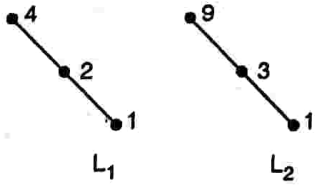

Fig. Ex. 3.7.10

Find the Hasse shown in Fig. Ex. 3.7.10 of $L_1 \times L_2$

Soln. :

Given chains are $L_1 = \{1, 2, 4\}$, $L_2 = \{1, 3, 9\}$

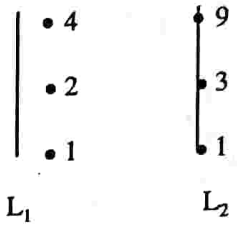

Fig. Ex. 3.7.10(a)

$L_1 \times L_2 = \{(1, 1), (1, 3), (1, 9), (2, 1), (2, 3), (2, 9), (4, 1), (4, 3), (4, 9)\}$

The digraph is as shown in Fig. Ex. 3.7.10(b)

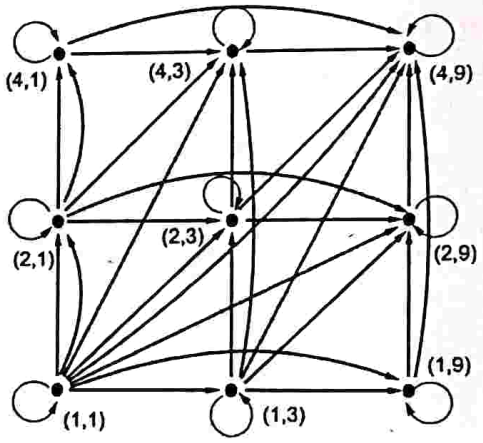

Fig. Ex. 3.7.10(b)

By eliminating reflexive and transitive edges, the Hasse shown in Fig. Ex. 3.7.10(c).

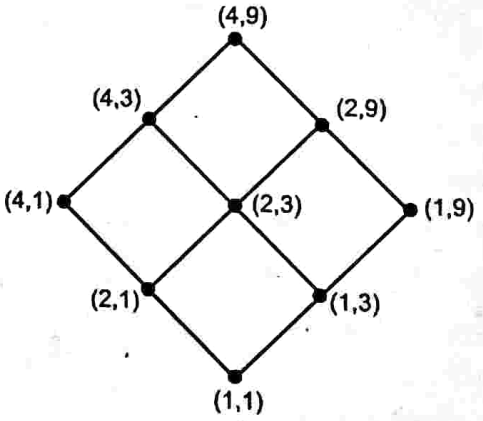

Fig. Ex. 3.7.10(c)

UEx. 3.7.11 : MU - Q. 5(a), Dec. 2016, 8 Marks

Determine the matrix of the partial order of divisibility on the set A = {11,3,5,15,30 1}. Draw the Hasse diagram of the poset. Indicate whether it is a chain or not?

☑ **Soln. :**

Given set

(i) A = {2, 4, 12, 24}

(ii) A = {1, 3, 5, 15, 30}

(i) **Partial ordering relation of divisibility on set A is,**

R= {(2, 2), (2, 4), (2, 12), (2, 24), (4, 4), (4, 12), (4, 24), (12, 12), (12, 24), (24, 24)}

The diagraph of R is as shown in Refer Fig 3.7.11 (a)

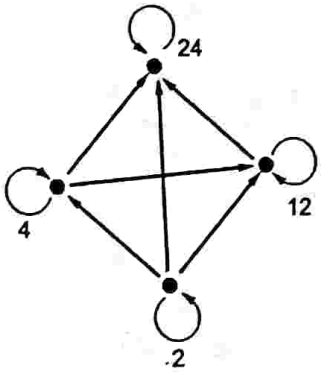

Fig. Ex. 3.7.11

The equivalent Hasse diagram is,

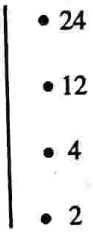

Fig. Ex. 3.7.11(a)

This is a chain, as ∀a, b, ∈ A, we have either aRb or bRa.

(ii) **Partial order relation of divisibility on A is,**

R = {(1, 1), (1, 3), (1, 5), (1, 15), (1, 30), (3, 15), (3, 3), (3, 30), (5, 5), (5, 15), (5, 30), (15, 15), (15, 30), (30, 30)}

The equivalent diagraph is as shown in Fig. Ex. 3.7.11(b)

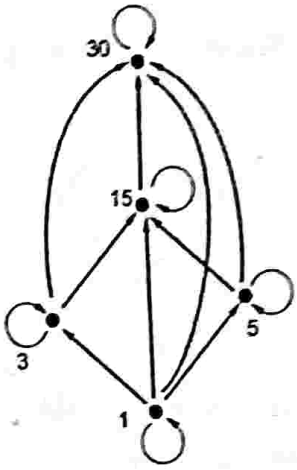

Fig. Ex. 3.7.11(b)

The Hasse diagram is as shown in Fig. Ex. 3.7.11(c)

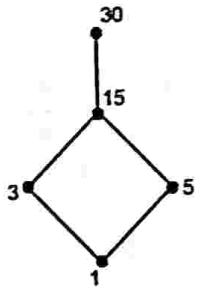

Fig. Ex. 3.7.11(c)

This is not a chain, because for two elements 3 and 5, we have 3 ℝ 5 and 5 ℝ 3.

UEx. 3.7.12 MU - Q. 6(a). Dec. 2016. 8 Marks

Determine whether the following posets are Boolean algebras. Justify your answers.

(i) A – {1, 2, 3, 6} with divisibility

(ii) D20 : divisors of 20 with divisibility

☑ **Soln. :**

(i) **Given set is A = {1, 2, 3, 6}**

Partial order relation of divisibility on A is,

R = {(1, 1), (1, 2), (1, 3), (1, 6), (2, 2), (2, 6), (3, 3), (3, 6), (6, 6)}

Module III

The Hasse diagram is as shown in Fig. Ex. 3.7.12.

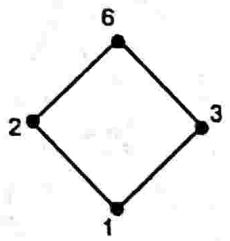

Fig. Ex. 3.7.12

A poet is said to be Boolean algebra, if it is a lattice, should have a greatest element and least element. It must also be a distributive and complemented lattice.

Now consider the following tables with respect to given poset.

LUB

∨	1	2	3	6
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

GLB

∧	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

We can see that every pair has GLB and LUB.

∴ A is a lattice.

The least element = 1,

The greatest element = 6.

Following table gives complement of every element.

a	a^1
1	6
2	3
3	2
6	1

∴ A is complemented lattice.

Now consider the distributive law.

Check $2 \vee (2 \wedge 3) = (2 \vee 2) \wedge (2 \vee 3)$

$$\text{LHS} = 2 \vee (2 \wedge 3) = 2 \vee 1 = 2$$

$$\text{RHS} = (2 \vee 2) \wedge (2 \vee 3) = 2 \wedge 6 = 2$$

∴ ∨ and ∧ are distributive.

∴ A under divisibility is a Boolean algebra.

(ii) **Given set is $D_{20} = \{1, 2, 4, 5, 10, 20\}$**

The lattice is as shown in Fig. Ex. 3.7.12(a).

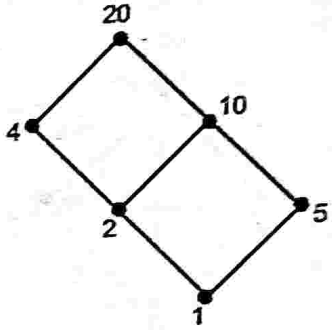

Fig. Ex. 3.7.12(a)

GLB

∧	1	2	4	5	10	20
1	1	1	1	1	1	1
2	1	2	2	1	2	2
4	1	2	4	1	2	4
5	1	1	1	5	5	5
10	1	2	2	5	10	10
20	1	2	4	5	10	20

LUB

∨	1	2	4	5	10	20
1	1	2	4	5	10	20
2	2	2	4	10	10	20
4	4	4	4	20	20	20
5	5	10	20	5	10	20
10	10	10	20	10	10	20
20	20	20	20	20	20	20

∴ D_{20} is a lattice.

The least element = 1

The greatest element = 20

a	a^1
1	20
2	Does not exist
4	5
5	4
10	Does not exist
20	1

∴ D_{20} is not a Boolean algebra.

UEx. 3.7.13 MU - Q. 1(d), May 2017, 5 Marks

Find the complement of each element in D_{30}.

Soln. :

Given set is $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

The Hasse diagram is as shown in Fig. Ex. 3.7.13

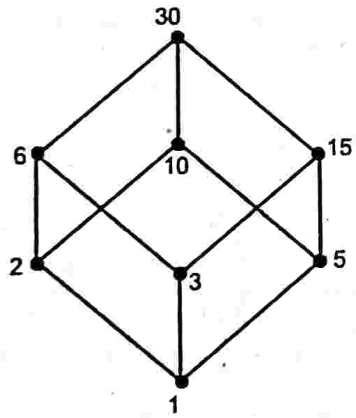

Fig. 3.7.13

Greatest element = 30

Least element = 1

$a \vee a^1$ = GE

$a \wedge a^1$ = LE

The complement of each element is calculated as follows :

∨	1	2	3	5	6	10	15	30
1	1	2	3	5	6	10	15	30
2	2	2	6	10	6	10	30	30
3	3	6	3	15	6	30	15	30
5	5	10	15	5	30	10	15	30
6	6	6	6	30	6	30	30	30
10	10	10	30	10	30	10	30	30
15	15	30	15	15	30	30	15	30
30	30	30	30	30	30	30	30	30

∧	1	2	3	5	6	10	15	30
1	1	1	1	1	1	1	1	1
2	1	2	1	1	2	2	1	2
3	1	1	3	1	3	1	3	3
5	1	1	1	5	1	5	5	5
6	1	2	3	1	6	2	3	6
10	1	2	1	5	2	10	5	10
15	1	1	3	5	3	5	15	15
30	1	2	3	5	6	10	15	30

Complement of each element in D_{30} is given by,

a	a^{-1}
1	30
2	15
3	10
5	6
6	5
10	3
15	2
30	1

UEx. 3.7.14 MU - Q. 2(c), May 2017, 8 Marks

Explain distributive Lattice. Show that following diagrams represent non-distributive lattice.

Soln. :

Explain distribution lattice.

Given distributive are :

(a)

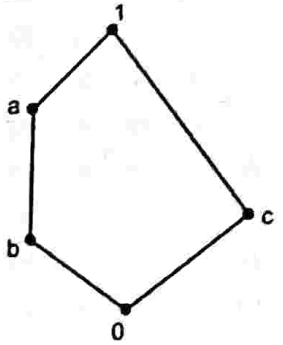

Fig. Ex. 3.7.14

LUB

∨	a	b	c
A	a	a	1
B	a	b	1
C	1	1	c

GLB

∧	a	b	c
a	a	b	0
b	b	b	0
c	0	0	c

$\forall x, x \vee 0 = x, x \wedge 1 = x, x \wedge 0 = 0, x \vee 1 = 1$

Consider the following table of distributive law

x	y	z	(x ∧ y)	(x ∧ z)	(x ∧ y) ∨ (x ∧ z)	(y ∨ z)	x ∧ (y ∨ z)
a	b	c	b	0	B	1	a
a	c	b	0	b	B	1	a
b	a	c	b	0	b	1	b
b	c	a	0	b	b	1	b
c	a	b	0	0	0	a	0
c	b	a	0	0	0	a	0

$\therefore \; x \wedge (y \vee z) \neq (x \wedge y) \vee (x \wedge z)$

$\therefore$ It is not distributive lattice.

(b)

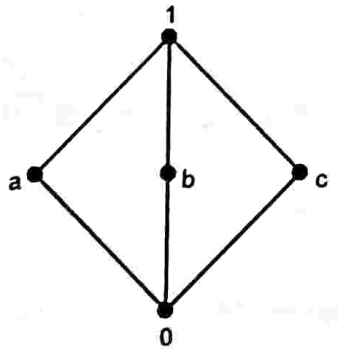

Fig. Ex. 3.4.14(a)

LUB

∨	a	b	c
a	a	1	1
b	1	b	1
c	1	1	c

GLB

∧	a	b	c
a	a	0	0
b	0	b	0
c	0	0	c

$\forall x, x \vee 0 = x, \; x \wedge 1 = x, \; x \wedge 0 = 0, \; x \vee 1 = 1$

X	y	z	(x ∧ y)	(x ∧ z)	(x ∧ y) ∨ (x ∧ z)	(y ∨ z)	x ∧ (y ∨ z)
A	b	c	0	0	0	1	a
A	c	b	0	0	0	1	a
B	a	c	0	0	0	1	b
B	c	a	0	0	0	1	b
C	a	b	0	0	0	1	c
C	b	a	0 -	0	0	1	c

$\therefore \; x \wedge (y \vee z) \neq (x \wedge y) \vee (x \wedge z)$

$\therefore$ It is non distributive lattice.

UEx. 3.7.15 **MU - Q.1(c), Dec. 2018, 8 Marks**

Determine the matrix of the partial order of divisibility on the set A. Draw the Hasse diagram of the Poset. Indicate those which are chains

(1) $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$

(2) $A = \{3, 6, 12, 36, 72\}$

☑ **Soln. :**

1. $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Hasse diagram

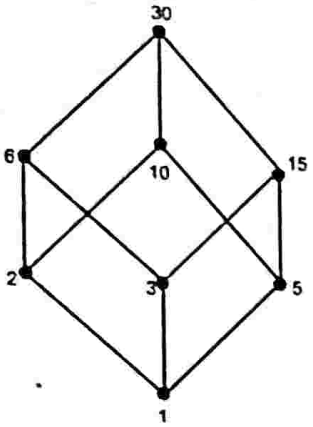

Fig. Ex. 3.7.15

This is not a chain.

2. $A = \{3, 6, 12, 36, 72\}$

Hasse diagram

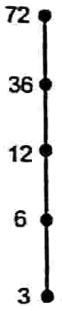

Fig. Ex. 3.7.15(a)

This is a chain.

UEx. 3.7.16 MU - Q. 2(a). Dec. 2018. 6 Marks

Find the complement of each element in D_{42}.

☑ **Soln. :**

$D_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$

Hasse diagram is

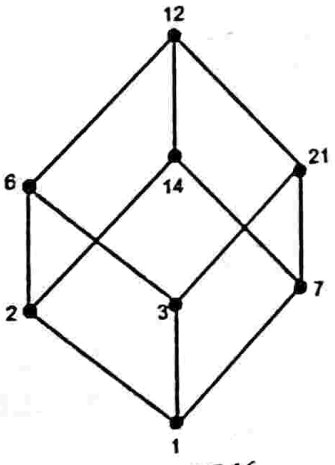

Fig. Ex. 3.7.16

$1' = 42,\ 2' = 21, 3' = 14, 7' = 6, 6' = 7, 14' = 3,$

$21' = 2,\ 42' = 1$

UEx. 3.7.17 MU - Q. 5(c). Dec. 2018. 8 Marks

Find the greatest lower bound and least upper bound of the set {3, 9, 12} and {1, 2, 4, 5, 10} if they exists in the poset (z+, /). Where / is the realtion of divisibility.

☑ **Soln. :**

Given : Poset is $(z^+, /)$

Least upper bound of {3, 9, 12} and {1, 2, 4, 5, 10} is 60 and greatest lower bound is 1.

UEx. 3.7.18 MU - Q. 1(d). May 2018. 5 Marks

Find the complement of each element in D_{30}.

☑ **Soln. :**

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ (Divisiors of 30)

The Hasse diagram is as shown in Fig. Ex. 3.7.18.

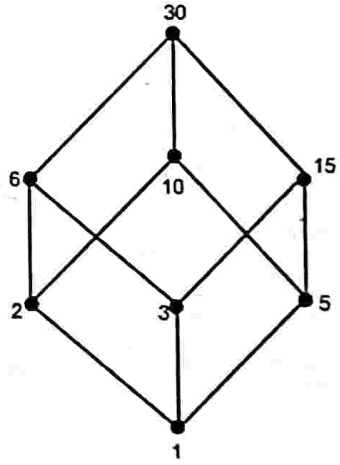

Fig. Ex. 3.7.18.

Greatest lower bound = 1

Least upper bound = 30

The complements are as follows :

$1' = 30$	$2' = 15$	$3' = 10$
$5' = 6$	$6' = 5$	$10' = 3$
$15' = 2$	$30' = 1$	

UEx.3.7.19 MU - Q. 6(a)May 2018. 8 Marks

Draw the Hasse diagram of the following sets under the partial order relation divides and indicate which are chains. Justify your answers.

(i) $A = \{2, 4, 12, 24\}$

(ii) $A = \{1, 3, 5, 15, 30\}$

☑ **Soln. :**

(i) **A = {2, 4, 12, 24}**

Hasse diagram for the relation divides is as shown in Fig. Ex. 3.7.19.

Fig. Ex. 3.7.19

It is a chain because, ∀ a, b ∈ A, we have either a ≤ b or b ≤ a.

i.e. $2 \leq 4 \leq 12 \leq 24$.

(ii) **A = {1, 3, 5, 15, 30}**

Hasse diagram for the relation divides is as shown in Fig. Ex. 3.7.19(a)

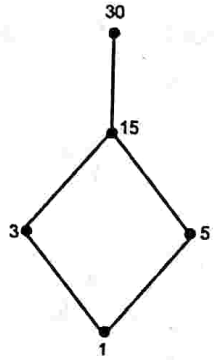

Fig. Ex. 3.7.19(a)

It is not a chain, because $3 \not\leq 5$ or $5 \not\leq 3$

UEx. 3.7.20 **MU - Q. 1(b), Dec. 2017, 5 Marks**

Let A = {a, b, c}. Show that (P (A), ⊆) is a poset and draw its Hasse diagram.

☑ **Soln. : (Refer Section 3.1)**

UEx. 3.7.21 **MU- Q. 5(c), Dec. 2017, 4 Marks**

From the following digraphs, write the relation a set of ordered pairs. Are the relations equivalence relations ?

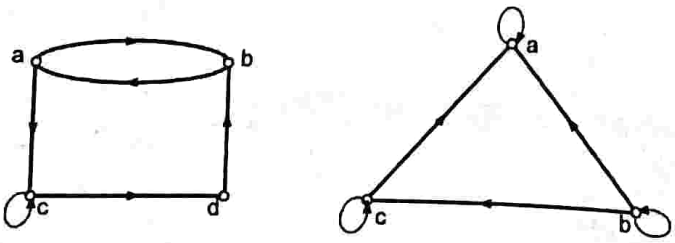

Fig. Ex. 3.7.21

☑ **Soln. :**

Relation for first diagraph.

R_1 = {(a, b), (b, a), (a, c), (b, d), (c, c), (c, d)}

It is not equivalence.

Relation for second diagraph.

R_2 = {(a, a), (a, b), (b, b), (b, c), (c, c), (c, a)}

It is not equivalence, but it is reflexive.

UEx. 3.7.22 **MU - Q. 5(d), Dec. 2017, 4 Marks**

For the set X = {2, 3, 6, 12, 24, 36}, a relation ≤ is defined as x ≤ y if x divides y.

Draw the Hasse diagram for (X, ≤). Answer the following :

(i) What are the maximal and minimal elements ?

(ii) Give one example of chain and antichain.

(iii) Is the poset a lattice ?

☑ **Soln. :**

Hasse diagram is

(i) **There is no minimal and maximum element.**

(ii) **Chain = 2, 6, 12, 24**

Antichain = 2, 3

(iii) **It is not a lattice.**

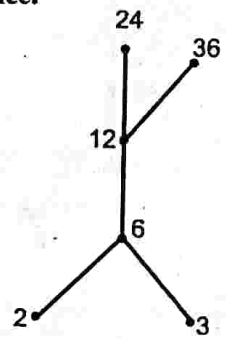

Fig. Ex. 3.4.22

Chapter Ends...

□□□

Module IV

CHAPTER 4 Counting

Syllabus

Basic Counting Principle-Sum Rule, Product Rule, Inclusion-Exclusion Principle, Pigeonhole Principle Recurrence relations, Solving recurrence relations

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Syllabus Topic : Counting

4.1 INTRODUCTION

Counting is one of the basic concepts in Discrete Mathematics. It falls under the area of combinatorics, which has an important role when a big data is dealt.

In principle, counting is as simple as that of counting few things, or objects. But the complexity increases as the combination and number of objects. For example, the number of two letter words in English that are possible are $26 \times 26 = 676$. But if the restriction is not to repeat the letters, then it is

$$26 \times 26 - 26 = 26 \times 25 = 650$$

Let us consider another example. An identifier in a programming language may have an alphabet (both capital or small) or a digit. How many single length identifiers are possible? Here possible numbers of identifiers are $26 + 26 + 10 = 62$.

The above examples explain two basic rules in counting. The product rule and the sum rule.

4.1.1 Two Product Rule

Definition : When a procedure P is broken into a sequence of two tasks P_1 and P_2, such that there are n_1 ways to do P_1 and n_2 ways to do P_2 then there are $n_1 \times n_2$ ways to do the procedure P.

4.1.2 The Sum Rule

Definition : When a procedure P can be done in two different procedures P_1 and P_2 such that there are n_1 ways to do P_1 and n_2 ways to do P_2, then there are $n_1 + n_2$ ways to do produce P.

4.2 SOLVED EXAMPLES

For the following example, check whether sum rule or product rule is applicable and why? Also solve the examples.

Ex. 4.2.1 : In an auditorium, the chairs are to be numbered with an alphabet followed by a positive integer not exceeding 100. Find the maximum number of chairs that can be placed in the auditorium.

Soln. :

Chairs are numbered with alphabet (P_1) in 26 ways along with positive integers (P_2) in 100 ways.

Thus numbering is done in combination of P_1 and P_2.

$\therefore$ The product rule is used.

$\therefore$ The maximum number of chairs that can be placed in the auditorium is $26 \times 100 = 2600$.

Ex. 4.2.2 : How many different binary numbers can be generated if the length of the number is 5 ?

Soln. :

Generating a binary number with 5 bits

Thus the procedure is divided into 5 parts $P_1, P_2, \ldots P_5$. Each bit may have 2 values i.e. 0 or 1. Thus, each of $P_1, P_2, \ldots, P_5$ are done is 2 ways. It is a product rule problem.

$\therefore$ Binary numbers that can be generated $= 2 \times 2 \times \ldots \times 2 = 2^5 = 32$.

Ex. 4.2.3 : A student can select a project from 3 lists where the lists contain 22, 17 and 19 projects. How many projects are there for a student to select ?

Soln. :

A student select one project from any of the lists.

$\therefore$ We should use the sum rule.

$\therefore$ Student can select a project from

$22 + 17 + 19 = 58$ projects

Ex. 4.2.4 : From example Ex. 3.1.3, if the student has to select two projects, from two different lists, then in how many ways the selection can be done.

Soln. :

The student has following three choices. Let the lists be L_1, L_2, L_3.

(i) Select one project each from L_1 and L_2.

(ii) Select one project each from L_2 and L_3.

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(iii) Select one project each from L_1 and L_3.

Here we should use sum rule. But selecting two projects from two lists requires the product rule.

∴ It is a combination of product rule and sum rule.

∴ The number of ways in which two projects can be selected is given as follows.

$22 \times 17 + 17 \times 19 + 22 \times 19 = 1115$ ways

Ex. 4.2.5 : In example Ex. 3.1.3, if three projects are to be selected, one from each list, then how many ways there for selection.

☑ **Soln. :**

Selecting one project from one list, and we have 3 such lists.

∴ We should use product rule.

∴ Number of ways for selection $= 22 \times 17 \times 19 = 7106$ ways

Ex. 4.2.6 : A computer has a password which can be formed with upper case letter or digits. The length of the password can be 6, 7 or 8. The password must start with an alphabet and must contain atleast one digit. How many such passwords are possible ?

☑ **Soln. :** The length of the password can be 6, 7 or 8.

Let us consider the passwords as P_6, P_7 and P_8 with length 6, 7 and 8 respectively.

∴ Total number of passwords = number of passwords of type P_6 + number of passwords of type P_7 + number of passwords of type P_8.

It's a sum rule.

Now, let us consider passwords of type P_6. There one 6 characters, first character is always an alphabet.

∴ There are 26 ways to select the first character.

Out of remaining 5 characters, one must be digit.

∴ There are 10 ways to select one digit.

Now the situation is as follows.

The password is like

Alpha $x_1\ x_2\ x_3\ x_4\ x_5$

One of x_1, x_2, x_3, x_4, x_5 is a digit.

∴ There are 5 ways to choose a place for the digit.

Now remaining characters are 4, which can be either alphabets or digits i.e. (26 + 10) possibilities are 36^4.

∴ Total number of possible passwords with length 6

$$n_6 = 26 \times 5 \times 10 \times 36^4 = 2,183,500,800$$

Similarly,

∴ Total number of possible passwords with length 7

$$n_7 = 26 \times 6 \times 10 \times 36^5 = 94,327,234,560$$

∴ Total number of possible passwords with length 8

$$n_8 = 26 \times 7 \times 10 \times 36^6 = 3,961,743,851,520$$

∴ Total number of possible passwords

$$= n_6 + n_7 + n_8$$

$$= 4,058,254,586,880$$

Ex. 4.2.7 : In how many ways can two integers be select from the integers 1,2,...,100 so that their difference is exactly seven?

☑ **Soln. :**

This is a problem of simple counting.

The given integers are 1,2,...,100.

Let us consider the set of two integers such that their difference is 7.

(1, 8), (2, 9), (3,10), ..., (93, 100)

∴ There are 93 ways to select two numbers such that their difference is 7.

Ex. 4.2.8 : In Ex. 3.1.7, in how many ways two integers can be selected such that their difference is seven or less.

Soln. :

Now consider the set of two integers as follows.

(1,8), (1,7), (1,6), (1,5), (1,4), (1,3), (1,2) → 7 count
(2,9), (2,8), ... , (2,3) → 7 count
(3,10), (3,9), ... , (3,4) → 7 count
⋮
(93,100), (93,99), ... , (93,94), → 7 count
(94,100), (94,99), ... , (94,95), → 6 count
(95,100), (95,99), ... , (95,96), → 5 count
(96,100), (96,99), ... , (96,97), → 4 count
(97,100), (97,99), (97,98), → 3 count
(98,100), (98,99), → 2 count
(99,100), → 1 count

∴ Total number of ways of select two integers so that their difference is seven or less.

$$= 93 \times 7 + 6 + 5 + 4 + 3 + 2 + 1$$
$$= 672$$

Ex. 4.2.9 : What is the value of k at the end of the following part of C program.

```
k = 0;
for (i = 1 ; i <= 10 ; i++)
{
    for (j = 1 ; j <= 5 ; j++)
    k++;
}
```

Soln. :

In the given C program, the first for loop is executed 10 times. The inner for loop is executed 5 times, with one value of i.

∴ The statement k++ is executed $10 \times 5 = 50$ times.

∴ Value of k = 50 at the end.

Ex. 4.2.10 : Find how many times the statement x is executed in the following code.

```
for (i = 1 ; i <= 10 ; i++)
{
    for (j = i ; j <= 20 ; j++)
    x
}
```

Soln. :

In the given code, the first for loop is executed 10 times.

The inner for loop is executed 20 – i + 1 times, with one value of i.

∴ If i = 1, inner for loop is executed 20 times

∴ If i = 2, inner for loop is executed 19 times

⋮

∴ If i = 10, inner for loop is executed 11 times

The statement x is present in inner for loop.

∴ x is executed 11 + 12 + ... + 20 times

∴ x is executed 155 times.

Ex. 4.2.11 : What is a tree diagram?

Soln. :

A tree diagram is one of the ways to solve counting problems. A tree consists of a root and branches leaving the roots. The nodes are present at every level of tree. The nodes at the last level are called leaf nodes. The solution for the counting problem is the path from root node to leaf node.

Example

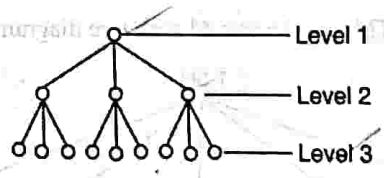

Fig. Ex. 4.2.11

This is a tree where every node has 3 branches or children, The maximum level of tree is 3.

Ex. 4.2.12 : Draw a tree to generate binary numbers of length 4 find the following.

(i) How many numbers end with 11.

(ii) How many numbers have no consecutive 1's.

(iii) How many numbers do not start with 1.

Soln. : The required tree is as follows for every bit one level is extended. Few examples of binary numbers are shown in the Fig. P. 3.1.12.

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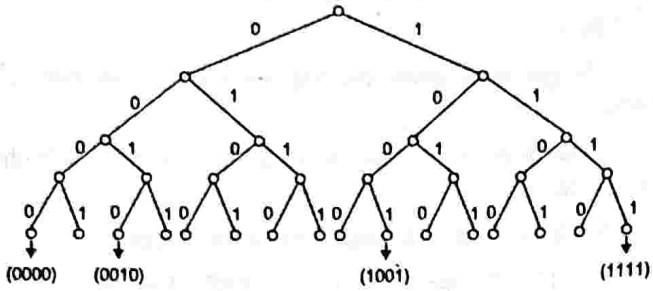

Fig. Ex. 4.2.12

(i) Consider the last two levels, for 1.

∴ 4 binary numbers end with 11.

(ii) Consider all paths in the tree for consecutive 1's.

∴ 8 binary numbers have no consecutive 1's.

(iii) Consider the first level that start with 0.

∴ 8 binary numbers do not start with 1.

Ex. 4.2.13 : Let there be a T-shirt that comes in 5 different sizes, S, M, L, XL and XXL. If XL comes in three colors red, green and black, XXL comes in green and black, and remaining come in white, green, red and black, then find the minimum number of T-shirts to be kept in a store, so that each variety is available.

☑ **Soln. :** This can be solved with tree diagram as follows.

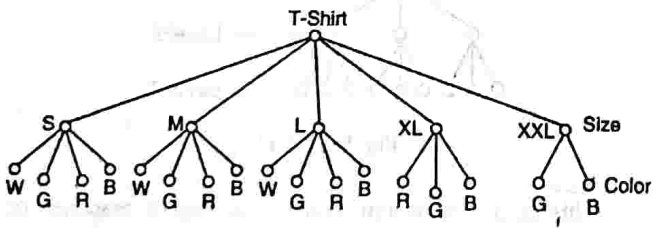

Fig. Ex. 4.2.13

The total number of leaf nodes = 17

∴ The minimum number of T-shirts to be kept in the store = 17

Ex. 4.2.14 : A multiple-choice test contains 10 questions. There are four possible answers for each question.

a. How many ways can a student answer the questions on the test if the student answer every question?

b. How many ways can a student answer the questions on the test if the student can leave answers blank?

☑ **Soln. :**

One question can be answered in 4 ways.

Total number of questions = 10.

a. If the student answers every question, then the number of ways that he can answer = 4^{10} = 10,48,576

b. If the student can leave answer blank, then the number of ways he can answer a question will be 5.

∴ Total number of ways that the can answer = 5^{10} = 97,65,625.

Ex. 4.2.15 : How many different three letter initials can people have?

☑ **Soln. :**

Number of alphabets = 26

Possible three-letter initials = 26^3 = 17,576

Ex. 4.2.16 : How many license plates can be made using either three letters followed by three digits or four letters followed by two digits.

☑ **Soln. :**

Let P_1 = License plate with 3 letters and 3 digits.

P_2 = License plate with 4 letters and 2 digits.

Total number of license plates = $P_1 + P_2$

Consider P_1.

Let K_1 = Number of ways in which 3 letters are organized.

K_2 = Number of ways in which 3 digits are organized.

$P_1 = K_1 \times K_2$.

$K_1 = 26^3$

$K_2 = 10^3$

∴ $P_1 = 26^3 \times 10^3 = 1,75,76,000$

Consider P_2,

Let K_3 = Number of ways in which 4 letters are organized.

K_4 = Number of ways in which 2 digits are organized.

$P_2 = K_3 \times K_4$

$K_3 = 26^4$

$K_4 = 10^2$

$\therefore \quad P_2 = 26^4 \times 10^2 = 45697600$

$\therefore$ Total number of license plates $= P_1 + P_2$

$= 17576000 + 45697600 = 63273600$

Ex. 4.2.17 : How many license plates can be made using either two or three letters followed by either two or three digits?

Soln. :

We can breakup the problem as follows.

P_1 = 2 letters followed by 2 digits.

P_2 = 2 letters followed by 3 digits.

P_3 = 3 letters followed by 2 digits.

P_4 = 3 letters followed by 3 digits.

$\therefore$ The numbers of license plates $= P_1 + P_2 + P_3 + P_4$

$P_1 = 26^2 \times 10^2 = 67600$

$P_2 = 26^2 \times 10^3 = 67600$

$P_3 = 26^3 \times 10^2 = 1757600$

$P_4 = 26^3 \times 10^3 = 17576000$

$\therefore$ Total number of license plates

$= 67,600 + 6,76,000 + 17,57,600 + 1,75,76,000 = 2,00,77,200$

Ex. 4.2.18 : How many strings of eight English letters are there.

a. If letters can be repeated?

b. If no letter can be repeated?

c. That start with x, if letters can be repeated?

d. That start with x, if no letter can be repeated?

e. That starts and ends with x, if letters can be repeated?

f. That start with the letters BO (in that order), if letter can be repeated.

g. That start and end with letters BO (in that order), if letter can be repeated.

h. That start or end with letters BO (in that order), if letter can be repeated.

Soln. :

The string contains 8 letters.

a) If letters can be repeated,

Number of strings $= 26^8 = 2, 08, 82, 70, 64, 576$

b) If no letter can be repeated.

Number of strings $= 26 \times 25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19$

$= 62,99,09,28,000$

c) If string starts with x, then we have 7 letters left, that can be repeated.

$\therefore$ Number of strings $= 26^7 = 8,03,18,10,176$

d) If string starts with x, then we have 7 letters left, that are not to be repeated.

$\therefore$ Number of strings $= 25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19$

$= 2,42,27,28,000$

e) If string start and ends with x, then we have 6 letters left, that can be repeated.

$\therefore$ Number of strings $= 26^6 = 30,89,15,776$

f) If string starts with letters BO (in that order) then we have 6 letters left, that can be repeated.

$\therefore$ Number of strings $= 26^6 = 30,89,15,776$

g) If string starts and end with letters BO (in that order) then we are left with 4 letters, that can be repeated,

$\therefore$ Number of string $= 26^4 = 4,5,6,976$

h) If the strings starts or ends with letters BO (in that order) then we can use principle of inclusion and exclusion.

$\therefore$ Number of strings = number of strings starting with BO

+ number of strings ending with BO

– number of strings and ending with BO

$= 26^6 + 26^6 - 26^4$ order), if letter can be repeated.

$= 61,73,74,576$

Module IV

Ex. 4.2.19 : How many strings of three decimal digits.

a. Do not contain the same digit three times?

b. Begin with as odd digit?

c. Have exactly two digits that are 4's ?

☑ Soln. :

Total number of strings with three decimal digits = 10^3 = 1000.

i.e. from 000 to 999.

Here we consider them as strings not as decimal numbers.

a. The string with same digit three times are 000, 111, 222, ..., 999. (total 10 strings).

∴ Number of string that do not contain the same digit three times = 1000 – 10 = 990.

b. Strings that begin with an odd digit are as follows :

100 to 199 (total 100)

300 to 399 (total 100)

⋮

900 to 999 (total 100)

This is for all 5 odd numbers 1,3,5,7,9.

∴ Number of strings that begin with odd digit

$= 100 \times 5 = 500.$

c. If the string has exactly two digit that are 4's then the string has one of the following forms.

$P_1 = x\,44$

$P_2 = 44\,x$

$P_3 = 4\,x\,4$

Here x can be any digit, from 0 to 9 expect 4.

∴ Number of P_1, P_2, P_3 are each 9.

∴ Total number of strings which have exactly two 4's are 27.

Ex. 4.2.20 : Use of a tree diagram to determine the number of subsets of {3, 7, 9, 11, 24} with the properly that sum of the elements in the subset is less than 28.

☑ Soln. :

The given set is {3, 7, 9, 11, 24}

The number of subsets of this set = $2^5 = 32$.

We need to find the number of subsets, such that the sum of elements in the subset is less than 28.

This is done using tree diagram.

The tree diagram is not completely drawn for all possible subsets, we draw it for only subset such that sum of elements in the subset is less than 28.

The diagram is as follows.

There are 17 subsets which have the sum of their elements as less than 28.

They are

$s_1 = \{\} = \phi$

$s_2 = \{3\}$

$s_3 = \{3, 7\}$

$s_4 = \{3, 7, 9\}$

$s_5 = \{3, 7, 11\}$

$s_6 = \{3, 9\}$

$s_7 = \{3, 9, 11\}$

$s_8 = \{3, 11\}$

$s_9 = \{3, 24\}$

$s_{10} = \{7\}$

$s_{11} = \{7, 9\}$

$s_{12} = \{7, 9, 11\}$

$s_{13} = \{7, 11\}$

$s_{14} = \{9\}$

$s_{15} = \{9, 11\}$

$s_{16} = \{11\}$

$s_{17} = \{24\}$

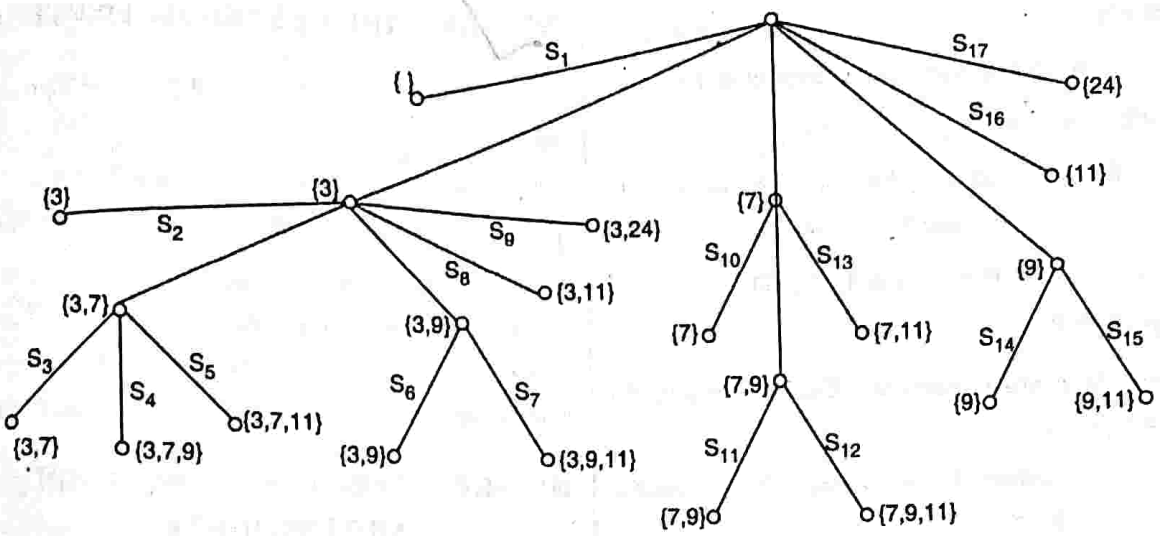

Fig. Ex. 4.2.20

Ex. 4.2.21 : How many subsets of a set with 100 elements have more than one elements?

☑ **Soln. :** The number of elements in the set = 100

∴ Total number of subsets = 2^{100}

One subsets is ϕ i.e. set with no elements.

100 subsets have one element each.

∴ Number of subsets with more than one. Element

$$= 2^{100} - 1 - 100$$

$$= 1.26765 \times 10^{30}$$

Ex. 4.2.22 : A palindrome is a string whose reversal is identical to the string. How many bit strings of length n are palindrome ?

☑ **Soln. :** Length of the string = n.

∴ n/2 bits are repeated in the reverse order.

Each bit can have 2 values i.e. 0 and 1.

∴ Total number of possible strings that are palindrome = $2^{\frac{n}{2}}$

Ex. 4.2.23 : How many bit strings of length seven either begin with two 0's or end with three 1's ?

☑ **Soln. :**

Let P_1 = Bit strings begin with two 0's.

P_2 = Bit strings end with three 1's

Number $(P_1) = 2^5 = 32$

Number $(P_2) = 2^4 = 16$

Number of strings that either begin with two 0's or end with three 1's is given by

number $(P_1 \cup P_2)$ = number (P_1) + number (P_2) + number $(P_1 \cap P_2)$

$P_1 \cap P_2$ = Strings that start with two 0's and end with three 1's

∴ number $(P_1 \cap P_2) = 2^2 = 4$

∴ number $(P_1 \cup P_2) = 32 + 16 - 4 = 44$

∴ 44 bit strings of length seven either start with two 0's or end with three 1's.

Ex. 4.2.24 : How many bit strings of length eight contain either three consecutive 0's or five consecutive 1's?

☑ **Soln. :**

Consider P_1 = Strings that contain three consecutive 0's.

These three consecutive 0's are considered as one bit B.

Thus we have 8 – 3 + 1 = 6 bits, out of which 5 bits b_1, b_2, b_3, b_4, b_5 can be anything.

Now, the situation is, we have string

$x_1\ b_1\ x_2\ b_2\ x_3\ b_3\ x_4\ b_4\ x_5\ b_5\ x_6$ and B can take place anywhere in $x_1, x_2, \ldots, x_6$. i.e. in 6 ways.

∴ Number of strings with consecutive three 0's

$$= 6 \times 2^5 = 192$$

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Similarly,

Consider P_2 = strings that contain five consecutive 1's.

$\therefore$ Number $(P_2) = 4 \times 2^3 = 32$

$P_1 \cap P_2$ = Strings which have three consecutive 0's and 5 consecutive 1's.

= 00011111 or 11111000

$\therefore$ Number $(P_1 \cap P_2) = 2$

$\therefore$ Number of strings with either three consecutive 0's or 5 consecutive 1's

= number (P_1) + number (P_2) − number $(P_1 \cap P_2)$

$= 192 + 32 - 2 = 222$

UEx. 4.2.25 **MU - Q. 2(a), Dec. 2017. 6 Marks**

Find the number of ways a person can distribute Rs. 601 as pocket money to his three sons, so that no son should receive more than the combined total of the other two. (Assume no fraction of a rupee is allowed).

Soln. :

Let A, B, C be the pocket money received by 3 sons.

$\therefore \quad A + B + C = 601$

$A + B \geq C$

$A + C \geq B$

$B + C \geq A$

Consider $\quad A + B \geq C$

$\therefore \quad A + B + C \geq C + C$

$\therefore \quad 601 \geq 2C$

$\therefore \quad C \leq 300$

Similarly $A \leq 300$, $B \leq 300$.

Assume that A = 300, then B and C will be pairs of (300, 1), (299, 2) ....(1, 300) – total 300.

If A = 299, B, C pair will be (301, 1), (300, 2), ..., (1, 301) 301 pairs.

If A = 301, B, C pair will be (299, 1), (298, 2), ... (1, 299) 299 pairs.

The value for A can vary from 1 to 301.

$\therefore$ Total number of possible ways are **301!**

4.3 THE PIGEONHOLE PRINCIPLE

It is a very simple concept of occupying holes by pigeons.

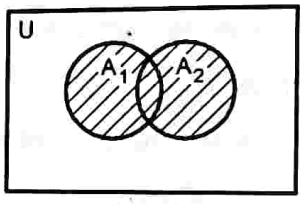
Definition : If there are n pigeonholes and n + 1 pigeons, then at least one pigeonhole is occupied by two or more pigeons.

Here pigeon or pigeonhole is concepts. It is as simple as n + 1 students sitting on n benches. In this case, we have at least one bench where two or more students are sitting.

4.4 PRINCIPLE OF INCLUSION AND EXCLUSION

We know that the number of elements present in a set A is known as cardinality of A and it is represented as $|A|$.

Let us consider the Fig. 4.4.1 Venn diagram.



$A_1 \cup A_2$

Fig. 4.4.1

It represents union of two sets A_1 and A_2,

The number of elements present in $A_1 \cup A_2$ is given by,

$|A_1 \cup A_2|$ = Number of elements present in either A_1 or A_2 or both.

Thus if we add $|A_1|$ and $|A_2|$ then the common elements i.e. $A_1 \cap A_2$ will be counted twice. Therefore this must be subtracted to get the count.

$$\therefore \quad |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Consider three sets A_1, A_2 and A_3 represented in the Fig. 4.4.2 Venn diagram.

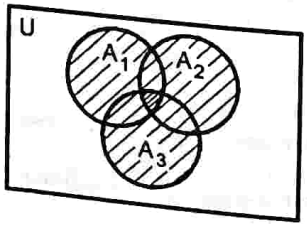


Fig. 4.4.2

To find the number of elements present in any one of the three sets, we have to add the number of elements present in each of the sets.

With this the elements present in more than one set are added twice, so we have to subtract. With this the elements present in all three sets are subtracted thus they are to be added.

Thus we get the equation,

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cup A_3|$$

This is known as principle of inclusion and exclusion.

Note the following equations.

(i) $|A_1 \cup A_2| \leq |A_1| + |A_2|$

(ii) $|A_1 \cap A_2| \leq \min(|A_1|, |A_2|)$

(iii) $|A_1 - A_2| \leq |A_1|$

(iv) $|A_1 - A_2| \geq |A_1| - |A_2|$

(v) $|A_1 \oplus A_2| = |A_1| + |A_2| - 2|A_1 \cap A_2|$

(vi) $|\bar{A}| = |U| - |A|$

(vii) $|A_1 - A_2| = |A_1| - |A_1 \cap A_2|$

(viii) $|A_1 - A_2 - A_3| = |A_1| - |A_1 \cap A_2| - |A_1 - A_3| + |A_1 \cap A_2 \cap A_3|$

Fig. 4.4.3, Venn diagrams represent different operations with the formula to calculate the cardinality of the shaded area.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

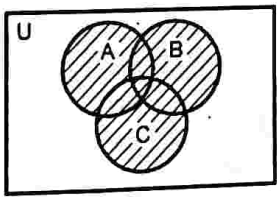


Fig. 4.4.3

$$|A \cup B| = |A| + |B| - |A \cap B|$$

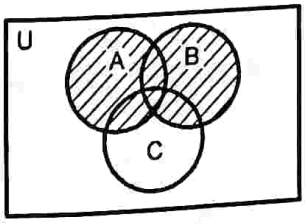


Fig. 4.4.4

Number of elements present at least in two sets.

$$= |A \cap B| + |A \cap C| + |B \cap C| - 2|A \cap B \cap C|$$

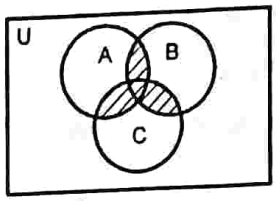


Fig. 4.4.5

Number of elements present only in A

$$= |A - B - C|$$

$$= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

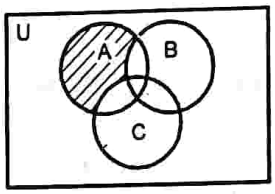


Fig. 4.4.6

Number of elements present exactly in one set.

$$= |A \oplus B \oplus C|$$

$$= |A| + |B| + |C| - 2|A \cap B| - 2|A \cap C| - 2|B \cap C| + 3|A \cap B \cap C|$$

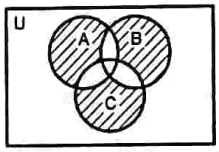


Fig. 4.4.7

Number of elements present exactly in two sets.

$$= |A \cap B \cap \bar{C}| + |A \cap C \cap \bar{B}| + |B \cap C \cap \bar{A}|$$

$$= |A \cap B| - |A \cap B \cap C| + |A \cap C| - |A \cap B \cap C| + |B \cap C| - |A \cap B \cap C|$$

$$= |A \cap B| + |A \cap C| + |B \cap C| - 3|A \cap B \cap C|$$

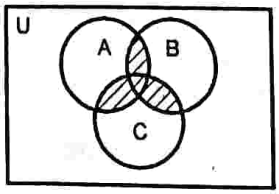


Fig. 4.4.8

Module IV

Number of elements present in either of two sets but not in third.

$= |(A \cup C) \cap \bar{B}| = |(A \cap \bar{B}) \cup (C \cap \bar{B})|$

$= |A \cap \bar{B}| + |C \cap \bar{B}| - |A \cap \bar{B} \cap C|$

$= |A| - |A \cap B| + |C| - |C \cap B| - [|A \cap C| - |A \cap B \cap C|]$

$= |A| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

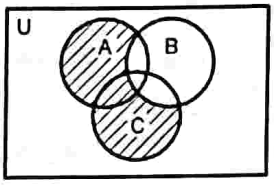


Fig. 4.4.9

4.5 SOLVED EXAMPLES

Ex. 4.5.1 : Out of 200 students, 50 of them take the course Discrete Mathematics, 140 of them take the course Economics, and 24 of them take both the courses. Since both courses have scheduled examinations for the following day, only students who are not in either one of these courses will be able to go to the party, the night before. Find how many students will be at the party.

☑ **Soln. :** Consider the following sets :

U = Set of all students $|U| = 200$

A = Set of students who take discrete mathematics $|A| = 50$

B = Set of students who take Economics $|B| = 140$

$A \cap B$ = Set of students who take both. $|A \cap B| = 24$

To find : Number of students who can attend the party, i.e. those who have not taken any of the subjects.

$\therefore$ We need to find $|\overline{A \cup B}|$

$|A \cup B| = |A| + |B| - |A \cap B| = 50 + 140 - 24$

$= 166$

$|\overline{A \cup B}| = |U| - |A \cup B| = 200 - 166 = 34$

$\therefore$ 34 students can attend the party.

Ex. 4.5.2 : Among the integers 1-300, how many of them are not divisible by 3, nor by 5, nor by 7 ?. How many of them are divisible by 3, but not by 5 nor by 7 ?

☑ **Soln. :**

Consider the following sets.

U = Set of integers concerned $|U| = 300$

A = Set of integers divisible by 3 $|A| = \left|\frac{300}{3}\right| = 100$

B = Set of integers divisible by 5 $|B| = \left|\frac{300}{5}\right| = 60$

C = Set of integers divisible by 7 $|C| = \left|\frac{300}{7}\right| = 42$

To find

(i) Number of integers not divisible by 3, nor by 5, nor by 7.

$\therefore$ We need to find $|\bar{A} \cap \bar{B} \cap \bar{C}|$

Consider the set

$|\bar{A} \cap \bar{B} \cap \bar{C}| = |\overline{A \cup B \cup C}|$ ....De-Morgan's law.

$A \cap B$ - set of integers divisible by both 3 and 5

$|A \cap B| = \left|\frac{300}{15}\right| = 20$

$A \cap C$ - set of integers divisible by both 3 and 7

$|A \cap C| = \left|\frac{300}{21}\right| = 14$

$B \cap C$ - set of integers divisible by both 5 and 7

$|B \cap C| = \left|\frac{300}{35}\right| = 8$

$A \cap B \cap C$ = set of integers divisible by both 3, 5 and 7

$|A \cap B \cap C| = \left|\frac{300}{105}\right| = 2$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 100 + 60 + 42 - 20 - 14 - 8 + 2 = 162$

$|\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C|$

$= 300 - 162 = 38$

$\therefore$ 38 integers are not divisible by 3, 5 and 7.

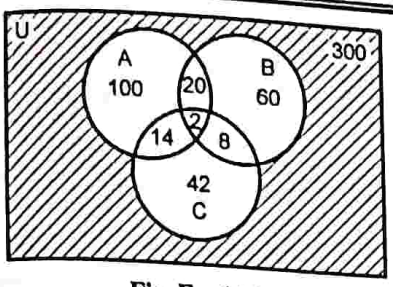


Fig. Ex. 4.5.2

(ii) Number of integers that are divisible by 3, but not by 5 nor by 7.

i.e. we need to find $|A \cap \bar{B} \cap \bar{C}|$

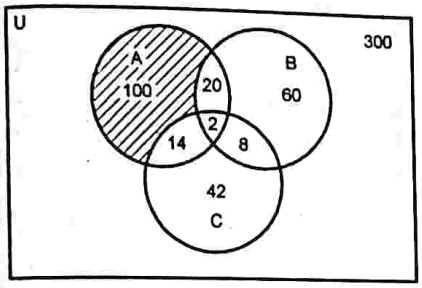


Fig. Ex. 4.5.2(a)

$$|A \cap \bar{B} \cap \bar{C}| = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

$$= 100 - 20 - 14 + 2 = 68$$

$\therefore$ 68 integers are divisible by 3 but not by 5 and nor by 7.

Ex. 4.5.3 : A survey was conducted among 1000 people. Of these 595 are democrats, 595 wear glasses, and 550 like ice cream; 395 of them are Democrats who wear glasses, 350 of them are Democrats who like ice cream, and 400 of them wear glasses and like ice cream; 250 of them are Democrats who wear glasses and like ice cream. How many of them who are not Democrats, do not wear glasses, and do not like ice cream ? How many of them are Democrats who do not wear glasses, and do not like ice cream ?

☑ **Soln. :** Consider the following sets.

U = Set of people surveyed $|U| = 1000$

A = Set of people who are democrats $|A| = 595$

B = Set of people who wear glasses $|B| = 595$

C = Set of people who like ice cream $|C| = 550$

$A \cap B$ = Set of people who are democrats and wear glass $|A \cap B| = 395$

$A \cap C$ = Set of people who are democrats and like ice cream

$|A \cap C| = 350$

$B \cap C$ = Set of people who wear glass and like ice cream $|B \cap C| = 400$

$A \cap B \cap C$ = Set of people who democrats, wear glass and like ice cream $|A \cap B \cap C| = 250$

To find :

(i) How many people are not democrats, do not wear glasses and do not like ice cream.

i.e. we need to find $|\bar{A} \cap \bar{B} \cap \bar{C}|$

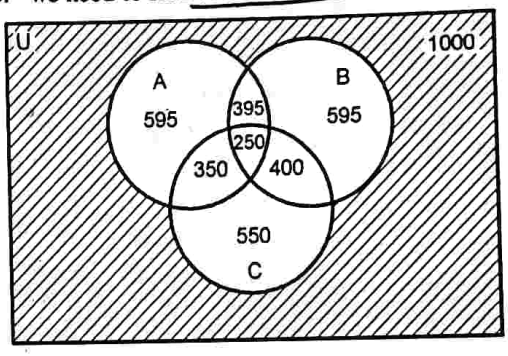


Fig. Ex. 4.5.3

First find $|A \cup B \cup C|$

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 595 + 595 + 550 - 395 - 350 - 400 + 250 = 845$$

Now,

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = |\overline{A \cup B \cup C}| \quad \text{De-Morgans law}$$

$$= |U| - |A \cup B \cup C|$$

$$= 1000 - 845 = 155$$

$\therefore$ 155 people are not democrats, do not wear glasses and do not like ice cream.

(ii) How many people are democrats who do not wear glasses and do not like ice cream ?

i.e. we need to find $|A \cap \bar{B} \cap \bar{C}|$

Module IV

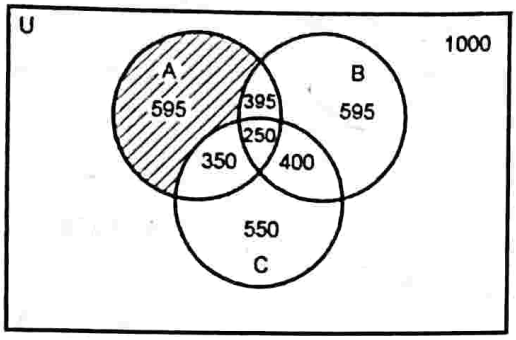


Fig. Ex. 4.5.3(a)

$|A \cap \bar{B} \cap \bar{C}| = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

$= 595 - 395 - 350 + 250 = 100$

$\therefore$ 100 people are democrats who do not wear glasses and do not like ice cream.

Ex. 4.5.4 : It is known that at the university 60 percent of the professors play tennis, 50 percent of them play bridge, 70 percent jog, 20 percent play tennis and bridge, 30 percent play tennis and jog, and 40 percent play bridge and jog. If someone claimed that 20 percent of the professors jog and play bridge and tennis, would you believe this claim ? Why ?

☑ Soln. :

Consider the following sets

U = Set of all professors $|U| = 100$

A = Set of all professors who play tennis $|A| = 60$

B = Set of all professors who play bridge $|B| = 50$

C = Set of all professors who jog $|C| = 70$

$A \cap B$ = Set of all professors who play tennis and bridge $|A \cap B| = 20$

$A \cap C$ = Set of all professors who play tennis and jog $|A \cap C| = 30$

$B \cap C$ = Set of all professors who play bridge and jog $|B \cap C| = 40$

To find : Percent of professors who play tennis bridge and jog.

i.e. $|A \cap B \cap C|$

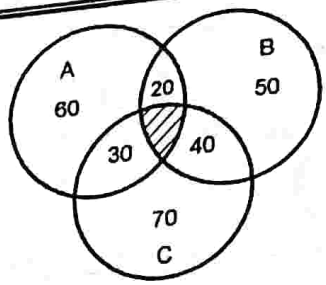


Fig. Ex. 4.5.4

In this case Universal set $U = A \cup B \cup C$

$|U| = |A \cup B \cup C| = 100$

$\therefore |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$100 = 60 + 50 + 70 - 20 - 30 - 40 + |A \cap B \cap C|$

$\therefore |A \cap B \cap C| = 10$

$\therefore$ Only 10 percent of professors play tennis and bridge and jog.

Ex. 4.5.5 : Out of a total of 130 students, 60 are wearing hats to class, 51 are wearing scarves, and 30 are wearing both hats and scarves. Of the 54 students who are wearing sweaters, 26 are wearing hats, 21 are wearing scarves, and 12 are wearing both hats and scarves. Everyone wearing neither a hat nor a scarf is wearing gloves.

(a) How many students are wearing gloves ?

(b) How many students not wearing a sweater are wearing hats but not scarves ?

(c) How many students not wearing a sweater are wearing neither a hat nor a scarf ?

☑ Soln. : Consider the following sets.

U = Set of all students $|U| = 130$

A = Set of all students wearing hats $|A| = 60$

B = Set of all students wearing scarves $|B| = 51$

$|A \cap B|$ = Set of all students wearing both $|A \cap B| = 30$

C = Set of all students wearing sweaters $|C| = 54$

$|A \cap C|$ = Set of all students wearing sweaters and hats $|A \cap C| = 26$

$|B \cap C|$ = Set of all students wearing scarf and hat $|B \cap C| = 21$

$|A \cap B \cap C|$ = Set of all students wearing all $|A \cap B \cap C| = 12$

Those who are not wearing a hat nor scarf is wearing gloves.

To find :

(a) How many students are wearing gloves

i.e. we need to find $|\bar{A} \cap \bar{B}|$

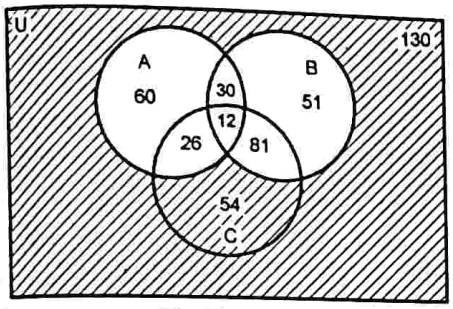


Fig. Ex. 4.5.5

Let us first find

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A \cup B| = 60 + 51 - 30 = 81$

$|\bar{A} \cap \bar{B}| = |\overline{A \cup B}|$ ...De-Morgan's law

$\therefore |\bar{A} \cap \bar{B}| = |U| - |A \cup B| = 130 - 81 = 49$

$\therefore$ 49 students wear gloves.

(b) How many students not wearing a sweater are wearing hats but not scarfs.

i.e. we need to find $|\bar{C} \cap A \cap \bar{B}|$

$\therefore |\bar{C} \cap A \cap \bar{B}| = |A| - |A \cap C| - |A \cap B|, + |A \cap B \cap C|$

$= 60 - 26 - 30 + 12 = 16$

$\therefore$ 16 students wear hats, but not sweater and not scarfs.

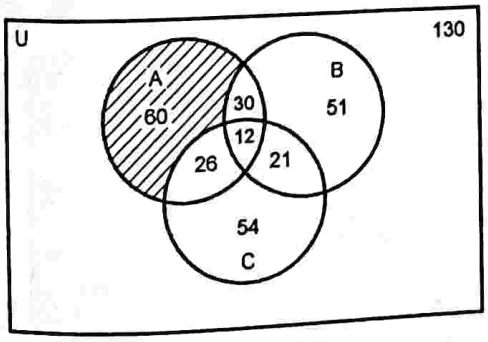


Fig. Ex. 4.5.5(a)

(c) How many students not wearing a sweater are wearing neither a hat nor a scarf ?

i.e. we need to find $|\bar{A} \cap \bar{B} \cap \bar{C}|$

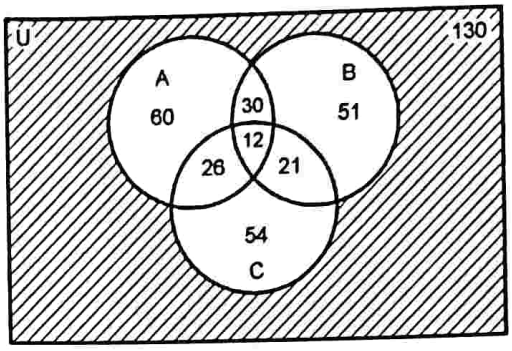


Fig. Ex. 4.5.5(b)

$|\bar{A} \cap \bar{B} \cap \bar{C}| = |\overline{A \cup B \cup C}|$ ...De-Morgan's law

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C|$

$- |B \cap C| + |A \cap B \cap C|$

$= 60 + 51 + 54 - 30 - 26 - 21 + 12 = 100 \therefore |$

$\bar{A} \cap \bar{B} \cap \bar{C}| = |U| - |A \cup B \cup C| = 130 - 100 = 30$

$\therefore$ 30 students do not wear sweaters, not hats and not scarf.

Ex. 4.5.6 : Among 100 students, 32 study mathematics, 20 study physics, 45 study biology, 15 study mathematics and biology. 7 study mathematics and physics, 10 study physics and biology, and 30 do not study any of the three subjects.

(a) Find the number of students studying all three subjects.

(b) Find the number of students studying exactly one of the three subjects.

Module IV

☑ **Soln. :**

Consider the following sets.

U = Set of all students $|U| = 100$

A = Set of all students who study Mathematics $|A| = 32$

B = Set of all students who study Physics $|B| = 20$

C = Set of all students who study biology $|C| = 45$

$A \cap C$ = Set of all students who study Mathematics and biology

$|A \cap C| = 15$

$A \cap B$ = Set of students who study mathematics and physics

$|A \cap B| = 7$

$B \cap C$ = Set of students who study physics and biology

$|B \cap C| = 10$

$\bar{A} \cap \bar{B} \cap \bar{C}$ = Set of students who do not study any of three subjects.

$|\bar{A} \cap \bar{B} \cap \bar{C}| = 30$

To find :

a) Number of students studying all subjects i.e. $|A \cap B \cap C|$

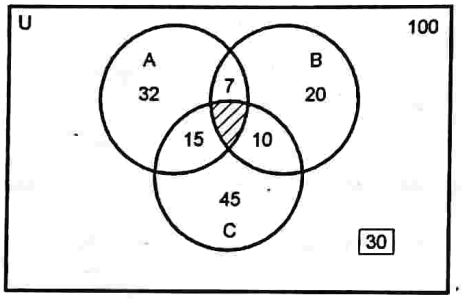


Fig. Ex. 4.5.6

$$|A \cup B \cup C| = |\overline{\bar{A} \cap \bar{B} \cap \bar{C}}|$$

$$|A \cup B \cup C| = |U| - |\bar{A} \cap \bar{B} \cap \bar{C}| = 100 - 30 = 70$$

$$|A \cap B \cap C| = |A \cup B \cup C| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C|$$

$$= 70 - 32 - 20 - 45 + 7 + 15 + 10 = 5$$

$\therefore$ 5 students study all three subjects.

b) Number of students who study exactly one subject.

i.e. $A \oplus B \oplus C$

$$|A \oplus B \oplus C| = |A| + |B| + |C| - 2|A \cap B| - 2|A \cap C| - 2|B \cap C| + 3|A \cap B \cap C|$$

$$= 32 + 20 + 45 - 2 \times 7 - 2 \times 15 - 2 \times 10 + 3 \times 5$$

$$= 48$$

$\therefore$ 48 students study exactly one subject.

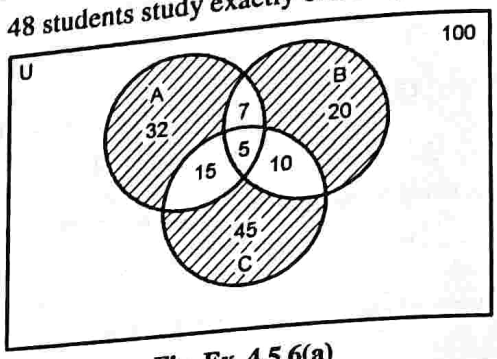


Fig. Ex. 4.5.6(a)

Ex. 4.5.7 : Seventy-five children went to an amusement park where they can ride on the merry-go-round, roller coaster and ferrie wheel. It is known that 20 of them have taken all three rides, and 55 of them have taken at least two of the three rides. Each ride costs \$ 0.50, and the total receipt of the amusement park was \$ 70. Determine the number of children who did not try any of the rides.

☑ **Soln. :** Consider the following sets.

U = Set of total number of children $|U| = 75$

A = Set of children who ride merry-go-round

B = Set of children who ride roller coaster

C = Set of children who ride ferris wheel $|A \cap B \cap C| = 20$

55 children ride at least two of three rider.

i.e. Number of children who have taken exactly two rides

$$= 55 - 20 = 35$$

Cost of each ride = \$ 0.50

Total cost = \$ 70

To find : Number of children who did not try any of the rides.

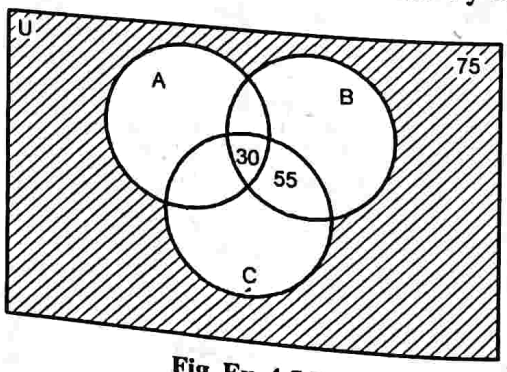


Fig. Ex. 4.5.7

Total number of rides $= \frac{70}{0.5} = 140$

20 children take all three rides $= 20 \times 3 = 60$ rides

35 children take 2 rides $= 35 \times 2 = 70$ rides

$\therefore$ Number of children who took single ride.

$= 140 - 60 - 70 = 10$

Total number of children = Number of children with single ride + Number of children with two rides + Number of children with 3 rides +Number of children with no rides.

$\therefore$ Number of children with no rides

$= 75 - 10 - 35 - 20 = 10$

$\therefore$ 10 children did not try any of the rides.

Ex. 4.5.8 : A college record gives the following information : 119 students enrolled in Introductory Computer Science, of their 96 took Data Structures, 53 took Foundations, 39 took Assembly Language, 31 took both foundations and Assembly language, 32 took both Data Structures and Assembly language, 38 took Data Structures and Foundations and 22 took all the three courses. Is the information consistent ? Why ?

☑ **Soln. :** Consider the following sets :

U = Set of total number of students $|U| = 119$

A = Set of students who took Data structures $|A| = 96$

B = Set of students who took foundations $|B| = 53$

C = Set of students who took Assembly language $|C| = 39$

$B \cap C$= Set of students who took foundations and assembly language

$|B \cap C| = 31$

$A \cap C$ = Set of students who took Data structures and assembly language

$|A \cap C| = 32$

$A \cap B$ = Set of students who took Data structures and foundation

$|A \cap B| = 38$

$A \cap B \cap C$ = Set of students who took all three courses

$|A \cap B \cap C| = 22$

As per the principle of inclusion and exclusion.

We have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 96 + 53 + 39 - 38 - 32 - 31 + 22 = 109$$

But as per the given data, total number of students is 119.

$\therefore$ There are 10 students who have not taken any of the courses.

$\therefore$ The given data is not consistent.

Ex. 4.5.9 : A construction company has to appoint 30 Civil engineers on road construction and 25 Civil engineers on building construction of these 15 will be asked to work on both types of construction. How many Civil engineers the company must appoint ?

☑ **Soln. :** Consider the following sets.

A = Set of civil engineers for road construction $|A| = 30$

B = Set of civil engineers for building construction $|B| = 25$

$A \cap B$ = Set of civil engineers for both $|A \cap B| = 15$

To find : Total number of civil engineers.

i.e. $|A \cup B| = |A| + |B| - |A \cap B|$

$= 30 + 25 - 15 = 40$

$\therefore$ The company must appoint 40 civil engineers.

Ex. 4.5.10 : In a group of 500 persons 40% drink tea, 50% drink coffee and 20% drink both. Find the number of persons who drink (i) only tea, (ii) only coffee, (iii) neither tea nor coffee.

☑ **Soln. :**

Consider the following sets.

U = Set of all persons $|U| = 500$

A = Set of people who drink tea $|A| = 50\% = 250$

B = Set of people who drink coffee $|B| = 50\% = 250$

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$A \cap B$ = Set of people who drink both.

$|A \cap B| = 20\% = 100$

To find :

(i) Number of people who drink only tea.

i.e. $|A| - |A \cap B| = 250 - 100 = 150$

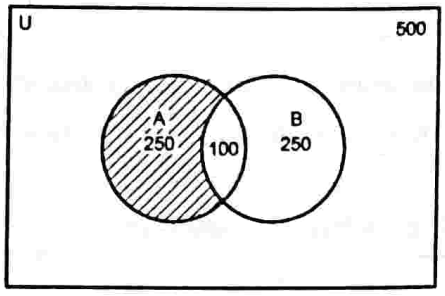

Fig. Ex. 4.5.10

(ii) Number of people who drink only coffee

i.e. $|B| - |A \cap B| = 250 - 100 = 150$

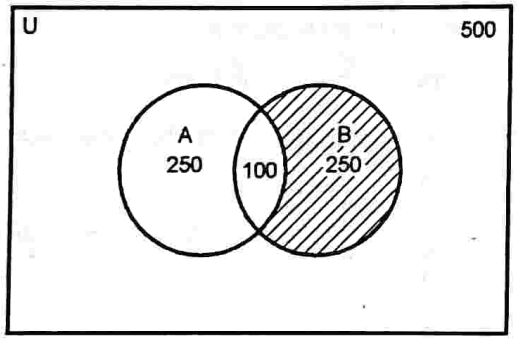

Fig. Ex. 4.5.10(a)

(iii) Number of people who neither drink tea nor coffee.

i.e. $|\bar{A} \cap \bar{B}| = |\overline{A \cup B}|$

$= |U| - |A \cup B|$

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 250 + 250 - 100 = 400$

$\therefore \quad |\bar{A} \cap \bar{B}| = 500 - 400 = 100$

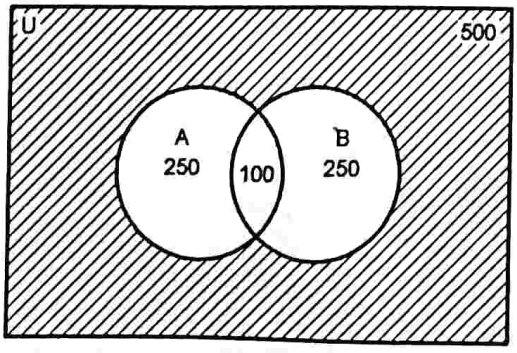

Fig. Ex. 4.5.10(b)

$\therefore$ 100 People do not drink tea or coffee.

Ex. 4.5.11 : In a group of 500 persons 100 speak English only and 350 speak Marathi only. Find the number of persons who speak (i) Marathi, (ii) English, (iii) both ?

☑ **Soln. :** Consider the following sets.

A = Set of people who can speak English

B = Set of people who can speak Marathi.

Total number of people $= |A \cup B| = 500$

Number of people who can speak English only = 100

$= |A| - |A \cap B|$

Number of people who can speak Marathi only = 350

$= |B| - |A \cap B|$

To find :

(i) Number of people who can speak English i.e. $|A|$

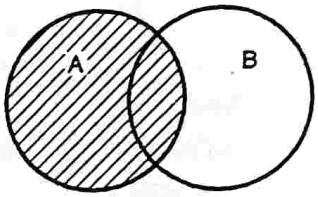

Fig. Ex. 4.5.11

We know that,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$500 = |A| + 350$$

$$\therefore \quad |A| = 500 - 350 = 150$$

$\therefore$ 150 people can speak English.

(ii) Number of people who can speak Marathi i.e. $|B|$

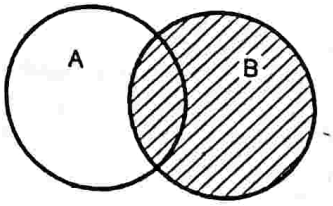

Fig. Ex. 4.5.11(a)

We know that,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= |B| + |A| - |A \cap B|$$

$$\therefore \quad 500 = |B| + 100$$

$$\therefore \quad |B| = 500 - 100$$

$$= 400$$

$\therefore$ 400 people can speak Marathi.

(iii) Number of people who can speak both i.e. $|A \cap B|$

$$|A \cap B| = |A| + |B| - |A \cup B|$$
$$= 400 + 150 - 500$$
$$= 50$$

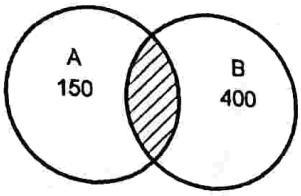

Fig. Ex. 4.5.11(b)

$\therefore$ 50 people can speak both English and Marathi.

Ex. 4.5.12 : Consider the following sets

U = Set of all integers $|U| = 200$

A = Set of all integers divisible by 2.

$$|A| = \left|\frac{200}{2}\right| = 100$$

B = Set of all integers divisible by 3.

$$|B| = \left|\frac{200}{3}\right| = 66$$

$A \cap B$ = Set of all integers divisible by 2 and 3

$$|A \cap B| = \left|\frac{200}{2 \times 3}\right| = 33$$

☑ **Soln. :**

To find : Number of integers ≤ 200 that are divisible by 2 or 3.

i.e. $|A \cup B| = |A| + |B| - |A \cap B|$
$= 100 + 66 - 33 = 133$

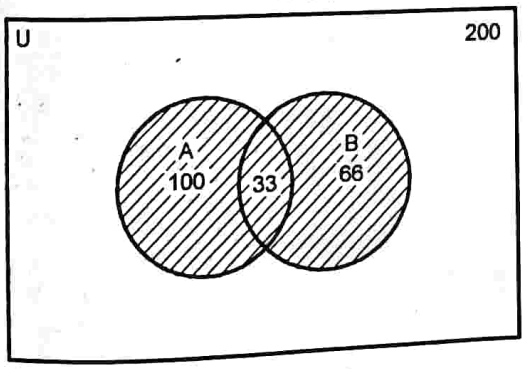

Fig. Ex. 4.5.12

Ex. 4.5.13 : Consider the following sets.

U - Set of all students $|U| = 100$

A - Set of students who like blue color $|A| = 53$

B - Set of students who like yellow color $|B| = 47$

C - Set of students who like red color $|C| = 41$ $A \cap B$ - Set of students who like blue and yellow $|A \cap B| = 23$

$B \cap C$ - Set of students who like yellow and red $|B \cap C| = 18$

$A \cap C$ - Set of students who like blue and red $|A \cap C| = 20$

$A \cap B \cap C$ - Set of students who like all three colors $|A \cap B \cap C| = 8$

☑ **Soln. :**

To find

(i) Number of students who like blue but not red.

i.e. $|A \cap \bar{C}| = |A| - |A \cap C| = 53 - 20 = 33$

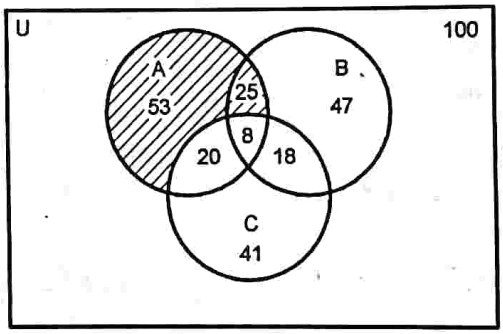

Fig. Ex. 4.5.13

(ii) Number of students who like blue and red but not yellow

i.e. $|A \cap C \cap \bar{B}| = |A \cap C| - |A \cap B \cap C|$
$= 20 - 8 = 12$

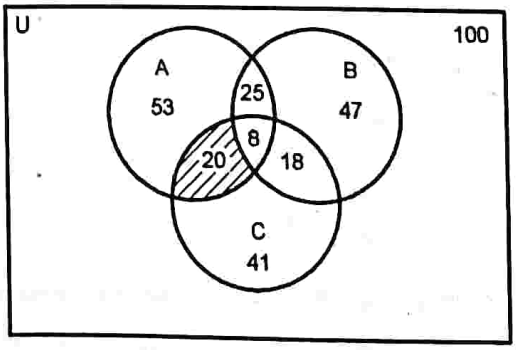

Fig. Ex. 4.5.13(a)

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(iii) Number of students who like blue or red but not yellow :

i.e. $(A \cup C) \cap \overline{B}$

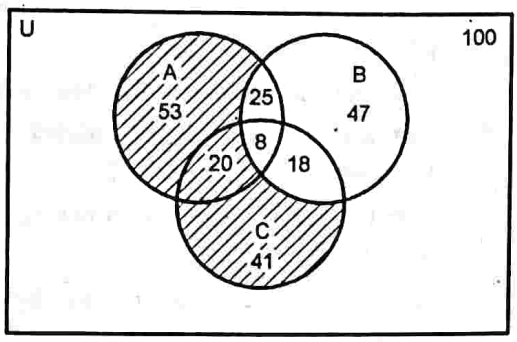

Fig. Ex. 4.5.13(b)

$$|(A \cup C) \cap \overline{B}| = |A| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 53 + 41 - 25 - 20 - 18 + 8 = 39$$

UEx. 4.5.14 **MU - Q. 1(c), Dec. 13, 5 Marks**

Find the number of integers between 1 and 1000 which are

(i) Divisible by 2, 3 or 5.

(ii) Divisible by 3 only but not by 2 nor by 5.

☑ **Soln. :** Let A, B and C be the sets with integers between 1 and 1000, which are divisible by 2, 3 and 5 respectively.

$$\therefore |A| = \frac{1000}{2} = 500, |B| = \frac{1000}{3} = 333, |C| = \frac{1000}{5} = 200$$

The set $A \cap B$ has integers between 1 and 1000, which are divisible by 2 and 3.

$$\therefore |A \cap B| = \frac{1000}{2 \times 3} = 166$$

Similarly we can write the cardinalities of the sets as follows :

i.e. $|A \cap C| = \dfrac{1000}{2 \times 5} = 100,$

$|B \cap C| = \dfrac{1000}{3 \times 5} = 66$

$|A \cap B \cap C| = \dfrac{1000}{2 \times 3 \times 5} = 33$

(i) We have to find the number of integers between 1 and 1000, which are divisible by 2, 3 or 5.

i.e. $|A \cup B \cup C|$

The corresponding Venn diagram is as follows.

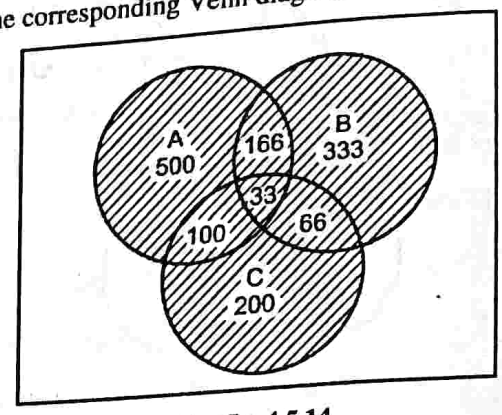

Fig. Ex. 4.5.14

Using the principle of inclusion and exclusion, we can write,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 500 + 333 + 200 - 166 - 100 - 66 + 33$$

$$= 7341$$

(ii) We have to find the number of integers between 1 and 1000, which are divisible by 3 only but not by 2 nor by 5.

i.e. $|\overline{A} \cap B \cap \overline{C}|$

It can be represented by the following Venn diagram.

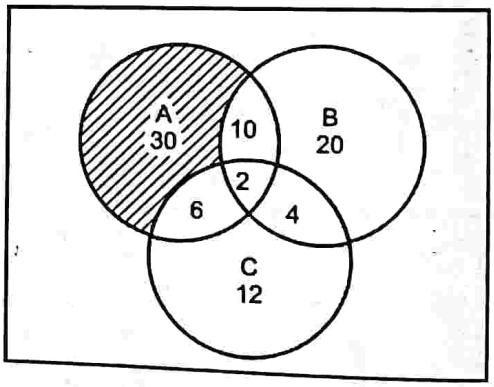

Fig. Ex. 4.5.14(a)

$$\therefore |\overline{A} \cap B \cap \overline{C}| = |B| - |A \cap B| - |B \cap C| + |A \cap B \cap C|$$

$$= 333 - 166 - 66 + 33$$

$$= 134$$

UEx. 4.5.15 **MU - Q. 2(b)(i), Dec. 14, 4 Marks**

Find how many integers between 1 to 60 are not divisible by 2 nor by 3 and nor by 5

Soln. :

Let A, B and C be the sets with integers between 1 and 60 which are divisible by 2, 3 and 5 respectively.

$$\therefore \quad |A| = \frac{60}{2} = 30, \quad |B| = \frac{60}{3} = 20, \quad |C| = \frac{60}{5} = 12$$

The set $A \cap B$ has integers between 1 and 60, which are divisible by, 2 and 3.

$$\therefore \quad |A \cap B| = \frac{60}{2 \times 3} = 10$$

Similarly we can write the cardinalities of the sets as follows :

$$\text{i.e.} \quad |A \cap C| = \frac{60}{2 \times 5} = 6$$

$$|B \cap C| = \frac{60}{3 \times 5} = 4$$

$$|A \cap B \cap C| = \frac{60}{2 \times 3 \times 5} = 2$$

The set which has integers between 1 and 60, that are not divisible by 2, nor by 3 and nor by 5 is $\bar{A} \cap \bar{B} \cap \bar{C}$.

We know that, $\bar{A} \cap \bar{B} \cap \bar{C}$

$$= \overline{A \cup B \cup C}. \text{ [De Margin's law]}$$

It is represented by the following Venn diagram.

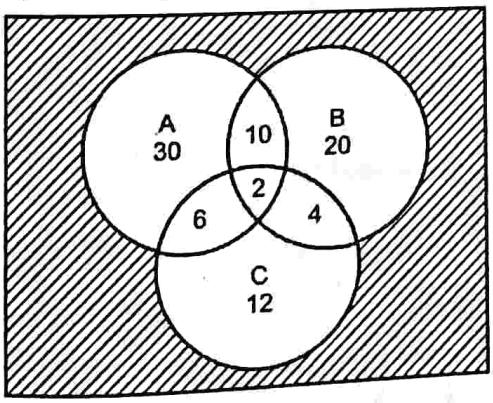

Fig. Ex. 4.5.15

Using the principle of Inclusion and exclusion,

we can write,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| \, |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 30 + 20 + 12 - 10 - 6 - 4 + 2 = 44$$

We know that,

$$\left|\overline{A \cup B \cup C}\right| = |U| - |(A \cup B \cup C)$$

U - is the universal set, i.e. integers between 1 and 60.

$$\therefore \quad |U| = 60$$

$$\therefore \left|\overline{A \cup B \cup C}\right| = 60 - 44 = \mathbf{16}$$

UEx. 4.5.16 **MU - Q. 2(b)(ii), Dec. 14, 4 Marks**

Find how many integers between 1 to 60 are Divisible by 2 not by 3 and nor by 5.

Soln. :

Let A, B and C be the sets with integers between 1 and 60 which are divisible by 2, 3 and 5 respectively.

$$\therefore \quad |A| = \frac{60}{2} = 30, \quad |B| = \frac{60}{3} = 20, \quad |C| = \frac{60}{5} = 12$$

The set $A \cap B$ has integers between 1 and 60, which are divisible by 2 and 3.

$$\therefore \quad |A \cap B| = \frac{60}{2 \times 3} = 10$$

Similarly we can write the cardinalities of the sets as follows :

$$\text{i.e.} \quad |A \cap C| = \frac{60}{2 \times 5} = 6$$

$$|B \cap C| = \frac{60}{3 \times 5} = 4$$

$$|A \cap B \cap C| = \frac{60}{2 \times 3 \times 5} = 2$$

The set which has integers between 1 and 60 that are divisible by 2 not by 3 and nor by 5 is given by $A \cap \bar{B} \cap \bar{C}$

It is represented by the following Venn diagram.

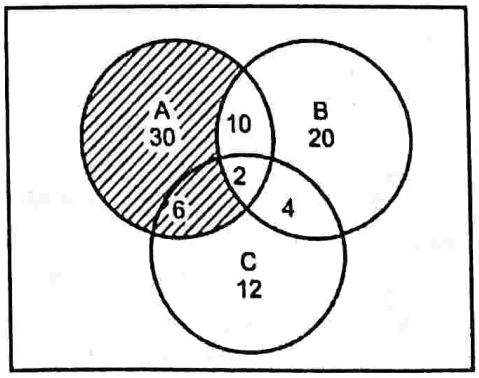

Fig. Ex. 4.5.16

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$\therefore \quad |A \cap \bar{B} \cap \bar{C}| = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

$= 30 - 10 - 6 + 2$

$= \mathbf{16}$

UEx. 4.5.17 **MU - Q. 4(b), May 15, 8 Marks**

A survey of 500 television watchers produced the following information: 285 watch football games, 195 watch hockey games, 115 watch basket ball games, 45 watch football and basketball games, 70 watch football and hockey games, 50 watch basketball and hockey games. 50 do not watch any three kinds of games. Find :

(i) How many in the survey watch all 3 kinds of games ?

(ii) How many watch exactly one of the sports languages ?

(iii) Draw Venn diagram showing results of the survey.

☑ **Soln. :**

Given :

Number of television watchers surveyed = 500

Number of people who watch football = 285

Number of people who watch hockey = 195

Number of people who watch basketball = 115

Number of people who watch football and basket ball = 45

Number of people who watch football and hockey = 70

Number of people who watch basketball and hockey = 50

Number of people who watch donot watch any game = 50

Let A, B and C be the sets of people who watch foot ball, hockey and basketball respectively. With the given data, we can write,

$|A| = 285, \quad |B| = 195,$

$|C| = 115$

$|A \cap C| = 45,$

$|A \cap B| = 70,$

$|B \cap C| = 50$

$\bar{A} \cap \bar{B} \cap \bar{C} = 50,$

$|U| = 500$

(i) To find the number of people who watch all three kinds of games.

$\therefore \quad |A \cap B \cap C|$

This is represented by the following Venn diagram.

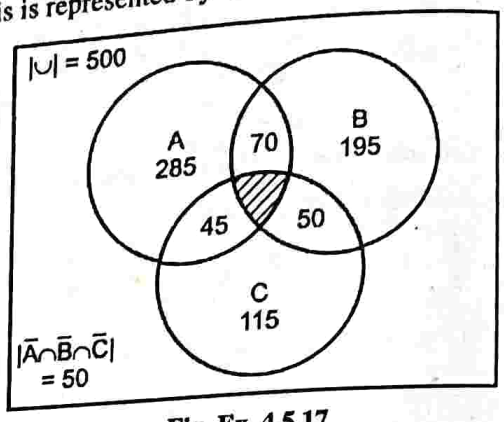


Fig. Ex. 4.5.17

We know that,

$|A \cup B \cup C| = \left(\overline{\overline{A \cup B \cup C}}\right) = \left|\overline{\bar{A} \cap \bar{B} \cap \bar{C}}\right|$

$= |U| - |\bar{A} \cap \bar{B} \cap \bar{C}|$

$\therefore |A \cup B \cup C| = 500 - 45 = 455$

As per the principle of inclusion and exclusion,

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B|$

$- (A \cap C) - (B \cap C) + |A \cap B \cap C|$

$\therefore |A \cap B \cap C| = |A \cup B \cup C| - |A| - |B| - |C| + |A \cap B|$

$+ (A \cap C) + |B \cap C|$

$= 455 - 285 - 195 - 115 + 45 + 70 + 50$

$= 25$

(ii) To find the number of people who watch exactly one sport.

i.e. $|A \oplus B \oplus C|$

It is represented by the following Venn diagram.

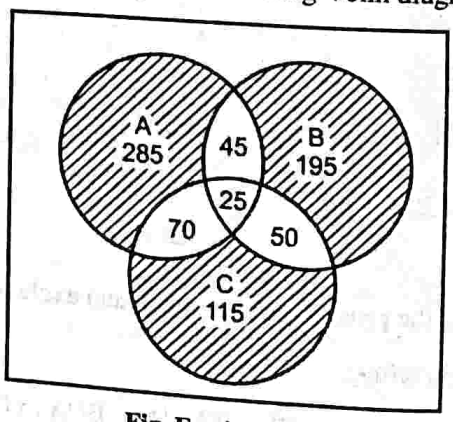


Fig. Ex. 4.5.17(a)

$$A \oplus B \oplus C = |A| + |B| + |C| - 2 \times |A \cap B| - 2 \times |A \cap C| - 2 \times |B \cap C| + 3 \times |A \cap B \cap C|$$

$$= 285 + 195 + 115 - 2 \times 45 - 2 \times 70 - 2 \times 50 + 3 \times 25$$

$$= \mathbf{340}$$

(iii) Venn diagrams are shown in the examples.

UEx. 4.5.18 **MU - Q. 1(b), Dec. 16, 6 Marks**

Determine the number of integers between 1 and 250 that are divisible by any of the integers 2, 3, 5 and 7.

Soln. :

Let A, B, C and D be the sets of integers between 1 and 250 which are divisible by 2, 3, 5 and 7.

$$\therefore \quad |A| = \frac{250}{2} = 125, \qquad |B| = \frac{250}{3} = 83$$

$$|C| = \frac{250}{5} = 50, \qquad |D| = \frac{250}{7} = 35$$

The set $A \cap B$ has the integers between 1 and 250, which are divisible by 2 and 3.

$$\therefore \quad |A \cap B| = \frac{250}{2 \times 3} = 41$$

Similarly, cardinalities of the different sets can be written as follows.

$$|A \cap C| = \frac{250}{2 \times 5} = 25$$

$$|A \cap D| = \frac{250}{2 \times 7} = 17$$

$$|B \cap C| = \frac{250}{3 \times 5} = 16$$

$$|B \cap D| = \frac{250}{3 \times 7} = 11$$

$$|C \cap D| = \frac{250}{5 \times 7} = 7$$

$$|A \cap B \cap C| = \frac{250}{2 \times 3 \times 5} = 8$$

$$|A \cap B \cap D| = \frac{250}{2 \times 3 \times 7} = 5$$

$$|B \cap C \cap D| = \frac{250}{3 \times 5 \times 7} = 2$$

$$|A \cap C \cap D| = \frac{250}{2 \times 5 \times 7} = 3$$

$$|A \cap B \cap C \cap D| = \frac{250}{2 \times 3 \times 5 \times 7} = 1$$

We have to find the number of integers between 1 and 250, which are divisible by any of the integers 2, 3, 5 and 7.

i.e. $|A \cup B \cup C \cup D|$

It can be represented by the following Venn diagram.

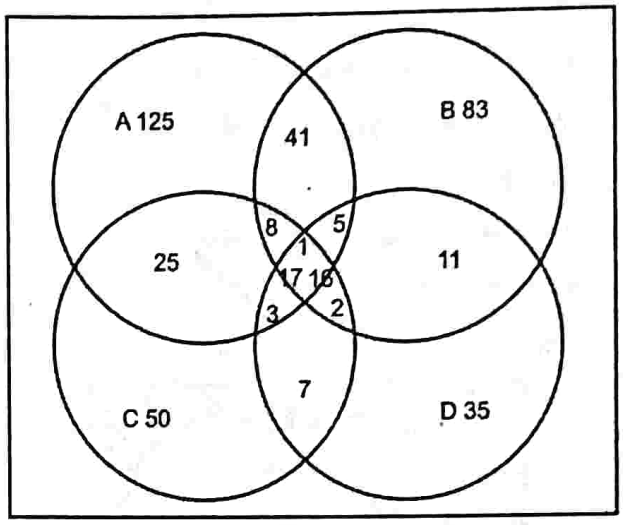


Fig. Ex. 4.5.18

Using the principle of inclusion and exclusion, we can write,

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

$$= 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7 + 8 + 5 + 3 + 2 - 1$$

$$= \mathbf{193}$$

UEx. 4.5.19 **MU - Q. 4(c), May 17, 8 Marks**

In a class of students undergoing a computer course the following were observed. Out of a total of 50 students :

30 know Pascal, 18 know Fortran, 26 know COBOL,

9 know both Pascal and Fortran, 16 know both Pascal and COBOL, 8 know both Fortran and COBOL,

47 know at least one of the three Languages.

From this we have to determine

a. How many students know none of these languages?

b. How many students know all three languages?

c. How many students know exactly one language?

Module IV

☑ **Soln. :**

Given : Total number of students = 50

Let it be the universal set U.

Following are the other sets, given

A - Students who know Pascal, $|A| = 30$

B - Students who know Fortran, $|B| = 18$

C - Students who know COBOL, $|C| = 26$

$|A \cap B| = 9$, $|A \cap C| = 16$, $|B \cap C| = 8$,

$|A \cup B \cup C| = 47$

(a) The set that has students who know none of the languags is $\overline{A \cup B \cup C}$.

It is represented by the Venn diagram.

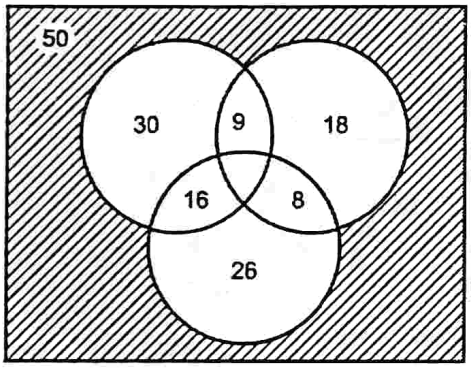


Fig. Ex. 4.5.19

We know that, $\left|\overline{A \cup B \cup C}\right| = |U| - |A \cup B \cup C|$

$$\therefore \left|\overline{A \cup B \cup C}\right| = 50 - 47 = 3$$

(b) The set which has students who know all three languages is $A \cap B \cap C$. It is represented by the following Venn diagram.

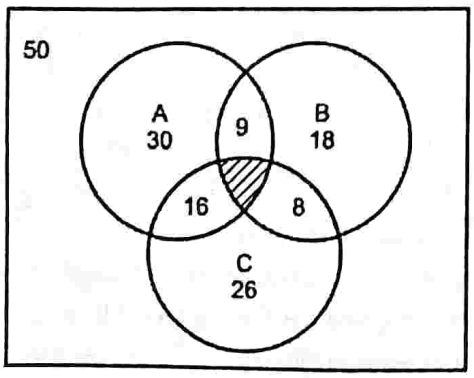


Fig. Ex. 4.5.19(a)

Using the principle of Inclusion and exclusion, we can write,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |\bar{B} \cap C| + |A \cap B \cap C|$$

$$\therefore |A \cap B \cap C| = 47 - 30 - 18 - 26 + 9 + 16 + 8$$

$$= 6$$

(c) The set that represents students who know exactly one language is $A \oplus B \oplus C$. It is represented by the following Venn diagram.

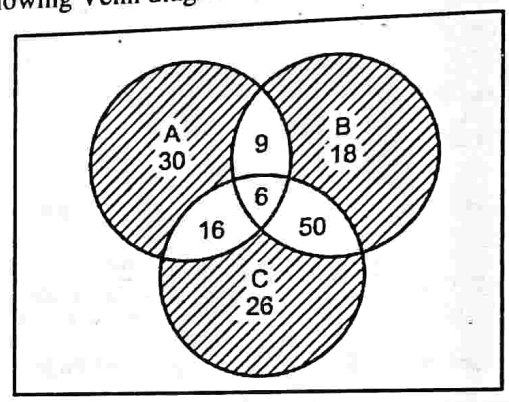


Fig. Ex. 4.5.19(b)

We know that, from the Venn diagram.

$$|A \oplus B \oplus C| = |A| + |B| + |C| - 2 \times |A \cap B| - 2 \times |A \cap C| - 2 \times |B \cap C| + 3\,|A \cap B \cap C|$$

$$= 30 + 18 + 26 - 2 \times 9 - 2 \times 16 - 2 \times 8 + 3 \times 6$$

$$= \mathbf{26}$$

UEx. 4.5.20 **MU - Q. 2(b), Dec. 15, 8 Marks**

It was found that in a class, 80 students are passed in English, 60 in Science and 50 in Mathematics. If was also found that 30 students passed in both English and Science, 15 students passed in both English and Mathematics and 20 students passed in both Mathematics and Science, 10 students passed in all three subjects. If there are 150 students in the class, find.

(i) How many students passed in at least one subject ?

(ii) How many students passed in English only ?

(iii) How many students failed in all three subjects ?

☑ **Soln. :**

Given : Total number of students = 150

Let, A, B, C be the sets of students who have passed in English, Science and Mathematics respectively.

$\therefore \quad |A| = 80, \ |B| = 60, \ |C| = 50$

$\therefore$ The set $A \cap B$ represents the set of students who have passed in English and Science.

$\therefore \ |A \cap B| = 30$

Similarly the following sets can be written

$|A \cap C| = 15, \ |B \cap C| = 20, \ |A \cap B \cap C| = 10$

(i) The set of students who passed in at least one subject is given by, $A \cup B \cup C$. It is represented by the following venn diagram.

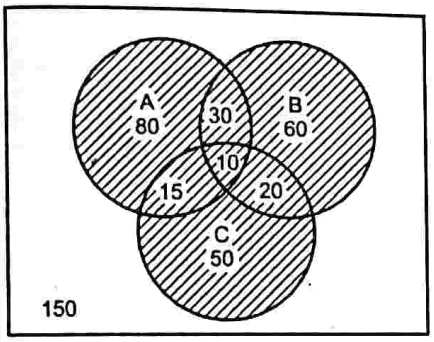


Fig. Ex. 4.5.20

From the principle of inclusion and exclusion, we can write

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + (A \cap B \cap C)$$

$$= 80 + 60 + 50 - 30 - 15 - 20 + 10$$

$$= \mathbf{135}$$

(ii) The set that represents students who passed in English only is represented by the following Venn diagram.

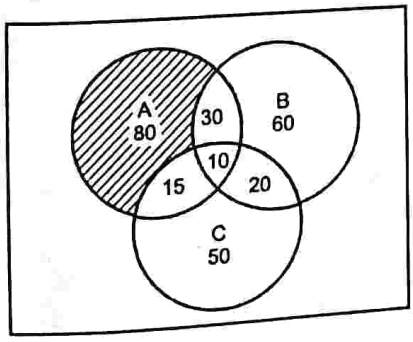


Fig. Ex. 4.5.20(a)

It is equal to $|A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

$$= 80 - 30 - 15 + 10$$

$$= 45$$

(iii) The set that **represents students** who failed in all subjects is given by $A \cup B \cup C$.

It is given by the following Venn diagram.

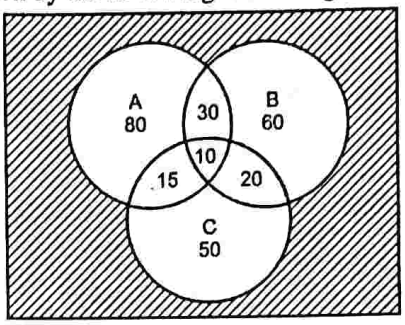


Fig. Ex. 4.5.20(b)

It is equal to $|\cup| - |A \cup B \cup C|$

$$= 150 - 135$$

$$= 15$$

UEx. 4.5.21 **MU - Q. 5(a), Dec. 2018,6 Marks**

Determine the number of positive integers n where $1 \le n \le 100$ and n is not divisible by 2, 3 or 5.

☑ **Soln. :**

Number of positive integers n where $1 \le n \le 100$ and n is not divisible by 2, 3 or 5.

Let P = Number of integers divisible by $2 = \frac{100}{2} = 50$

Q = Number of integers divisible by $3 = \frac{100}{3} = 33$

R = Number of integers divisible by $5 = \frac{100}{5} = 20$

$\therefore \ |P \cap Q| = \frac{100}{6} = 16, \ |Q \cap R| = \frac{100}{15} = 6$

$\therefore \ |P \cap R| = \frac{100}{10} = 10, \ |P \cap Q \cap R| = \frac{100}{30} = 3$

$$|X| = |\bar{P} \cap \bar{Q} \cap \bar{R}| = 100 - |P \cup Q \cup R|$$

$$|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |P \cap R| - |Q \cap R| + |P \cap Q \cap R|$$

$$= 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74$$

$$\therefore \ P(x) = \frac{|x|}{100} = \frac{100 - 74}{100} = \frac{26}{100} = \frac{13}{50}$$

Module IV

UEx. 4.5.22 **(MU - Q. 5(a), Dec. 18, 6 Marks)**

Determine the number of positive integers n where $1 \le n \le 100$ and n is not divisible by 2, 3 or 5.

Soln. :

Number of positive integers n where $1 \le n \le 100$ and n is not divisible by 2, 3 or 5.

Let P = Number of integers divisible by $2 = \frac{100}{2} = 50$

Q = Number of integers divisible by $3 = \frac{100}{3} = 33$

R = Number of integers divisible by $5 = \frac{100}{5} = 20$

$\therefore \ |P \cap Q| = \frac{100}{6} = 16, \ |Q \cap R| = \frac{100}{15} = 6$

$\therefore \ |P \cap R| = \frac{100}{10} = 10, \ |P \cap Q \cap R| = \frac{100}{30} = 3$

$$|X| = |\bar{P} \cap \bar{Q} \cap \bar{R}| = 100 - |P \cup Q \cup R|$$

$$|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |P \cap R| - |Q \cap R| + |P \cap Q \cap R|$$

$$= 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74$$

$$\therefore \ P(x) = \frac{|x|}{100} = \frac{100 - 74}{100} = \frac{26}{100} = \frac{13}{50}$$

UEx. 4.5.23 **(MU - Q. 4(a)(i)(ii), Dec. 17, 6 Marks)**

(i) Among 50 students in a class, 26 got an A in the first examination and 21 got an A in the second examination. If 17 students did not get an A in either examination, how many students got an A in both

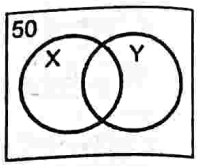
(ii) If the number of students who got an A in the first examination is equal to that in the second examination, if the total number of students who got an A in exactly one examination is 40 and if 4 students did not get an A in either examination, then determine the number of students who got an A in the first examination only, who got an A in the second examination only, and who got an A in both the examination.

Soln. :

(i) Total number of students = 50

$|X|$ = students with A in exam 1 = 26

$|Y|$ = students with A in exam 2 = 21



$|\bar{X} \cap \bar{Y}| = 17$

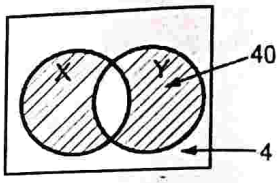
$|X \cup Y| = 50 - 17 = 33$

Students with A in both exams.

$$|X \cap Y| = -|X \cup Y| + |X| + |Y| = -33 + 26 + 21$$

$$= \mathbf{14}$$

(ii)



$|X| = |Y|$

$|X \cup Y| - |X \cap Y| = 40$

$|\bar{X} \cap \bar{Y}| = 4$

$|X \cup Y| = 50 - 4 = 46$

$\therefore \ |X \cap Y| = 46 - 40 = 6$

We know that $|X \cup Y| = |X| + |Y| - |X \cap Y|$

$46 = 2|x| - 6$

$\therefore \ 2|X| = 40$

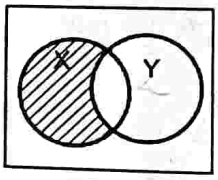
$|X| = 20$

Students with A in first exam only is given by,

$|X| - |X \cap Y|$

$= |X| - 6$

$= 20 - 6 = 14$



$\therefore$ Students with A in first exam = students with A in second exam = 14

$\therefore$ Students with A in both exams = 6.

UEx. 4.5.24 **(MU - Q. 2(b), Dec. 15, 8 Marks)**

It was found that in a class, 80 students are passed in English, 60 in Science and 50 in Mathematics. If was also found that 30 students passed in both English and Science, 15 students passed in both English and Mathematics and 20 students passed in both Mathematics and Science, 10 students passed in all three subjects. If there are 150 students in the class, find.

(i) How many students passed in at least one subject ?

(ii) How many students passed in English only ?

(iii) How many students failed in all three subjects ?

☑ **Soln. :**

Given : Total number of students = 150

Let, A, B, C be the sets of students who have passed in English, Science and Mathematics respectively.

$\therefore \quad |A| = 80, \ |B| = 60, \ |C| = 50$

$\therefore$ The set $A \cap B$ represents the set of students who have passed in English and Science.

$\therefore |A \cap B| = 30$

Similarly the following sets can be written

$|A \cap C| = 15, \ |B \cap C| = 20, \ |A \cap B \cap C| = 10$

(i) The set of students who passed in at least one subject is given by, $A \cup B \cup C$. It is represented by the following Venn diagram.

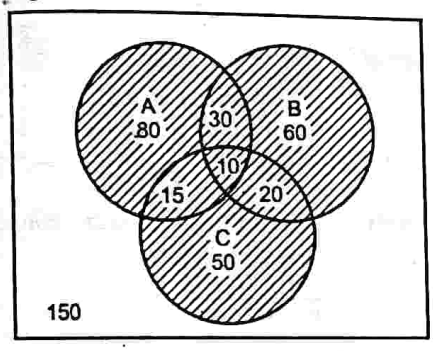


Fig. Ex. 4.5.24

From the principle of inclusion and exclusion, we can write

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + (A \cap B \cap C)$$

$$= 80 + 60 + 50 - 30 - 15 - 20 + 10$$

$$= \mathbf{135}$$

(ii) The set that represents students who passed in English only is represented by the following Venn diagram.

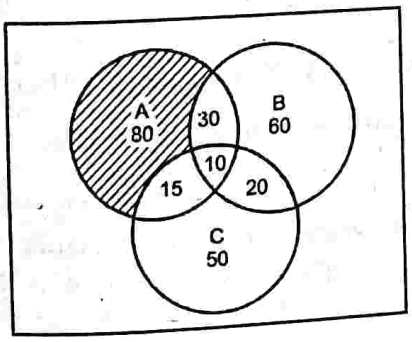


Fig. Ex. 4.5.24(a)

It is equal to $|A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

$$= 80 - 30 - 15 + 10$$

$$= 45$$

(iii) The set that **represents students** who failed in all subjects is given by $A \cup B \cup C$.

It is given by the following Venn diagram.

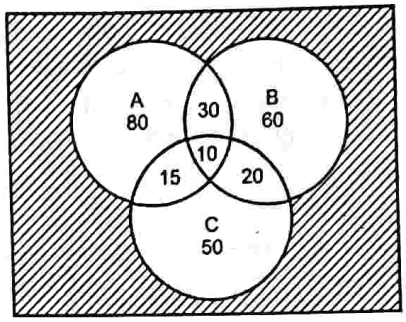


Fig. Ex. 4.5.24(b)

It is equal to $|\cup| - |A \cup B \cup C|$

$$= 150 - 135$$

$$= 15$$

4.6 THE GENERALIZED PIGEONHOLE PRINCIPLE

Module IV

❑ **Definition :** If there are n pigeons and k pigeonholes, where $n > k$, then there is at least one pigeonhole containing at least $\lceil n/k \rceil$ pigeons.

4.7 SOLVED EXAMPLES ON PIGEONHOLE PRINCIPLE

Ex. 4.7.1 Among a group of 13 people, at least two are born in the same month.

OR If there are 27 English words, then atleast two words begin with the same alphabet.

OR How many students must be there in a class room, so that atleast two students get same marks. The marks are in the range 0 to 100.

☑ **Soln. :**

The range of marks is 0 to 100.

$\therefore$ 101 possible marks.

∴ According to pigeonhole principle, there must be 102 students in the class, so that atleast two of them score same marks.

Ex. 4.7.2 : Among a group of 100 people, there are atleast $\lceil 100/12 \rceil = 9$ people who are born in the same month.

OR

What is the minimum number of students required in a class such that atleast 6 of them have the same grade and the possible grade are A, B, C, D, and E ?

☑ Soln. :

Here the number of grades = 5

$\therefore k = 5$

It is given that at least 6 of the students should have same grade.

∴ As per generalized pigeonhole principle,

$$\left\lceil \frac{n}{5} \right\rceil = 6$$

We must find a minimum value for n.

$$\therefore \left\lceil \frac{n}{5} \right\rceil = 6$$

$$\therefore \frac{n-1}{5} = (6-1)$$

$$\therefore n - 1 = 5 \times 5 = 25$$

$$\therefore n = 26$$

∴ Atleast 26 students must be there in the class.

Ex. 4.7.3 : How many cards must be selected from a deck of 52 cards to guarantee that at least three cards are of same suit ?

☑ Soln. :

Given : Total number of cards = 52

Number of suits in deck = 4

Three cards are of same suit.

$$\therefore k = 3$$

∴ As per generalized pigeonhole principle,

$$\left\lceil \frac{n}{4} \right\rceil = 3$$

We must find a minimum value for n

$$\therefore \left\lceil \frac{n}{4} \right\rceil = 3$$

$$\therefore \frac{n-1}{4} = (3-1)$$

$$\therefore n = 1 = 2 \times 4 = 8$$

$$\therefore n = 9$$

∴ Atleast 9 cards must be selected so that atleast 3 cards are of same suit.

Ex. 4.7.4 : Show that if there are 30 students in a class, then atleast two have last names that begin with same letter.

☑ Soln. :

Number of students = 30

$\therefore n = 30$

Number of letters = 26

$\therefore k = 26$

∴ As per generalized pigeonhole principle, we have $\left\lceil \frac{n}{k} \right\rceil$ have their last names starting with same letter

$$\therefore \left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{30}{26} \right\rceil = 2$$

∴ At least 2 student have last names that begin with same letter.

Ex. 4.7.5 : A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

a) How many balls must she select to be sure of having at least three balls of the same color?

b) How many balls must she select to be sure of having at-least there blue balls?

☑ Soln. :

Given : Bowl contains 10 red balls and 10 blue balls.

a. There are two types of balls, red and blue.

To have 3 balls of same colour, we should apply, generalized pigeonhole principle as follows.

$$\left\lceil \frac{n}{2} \right\rceil = 3$$

To get minimum value for n, we have

$$\therefore \frac{n-1}{2} = (3-1)$$

$$\therefore n-1 = 2\times 2 = 4$$

$$\therefore n = 5$$

$\therefore$ The woman should pick atleast 5 balls to be sure of having atleast 3 balls of the same color.

b. To have atleast 3 blue balls, to be sure, we may find that there can be all red balls picked along with 3 blue balls.

$\therefore$ The women should select atleast 13 balls to be sure of having atleast 3 blue balls.

Ex. 4.7.6 : A survey among 1000 people, 595 are democrats, 595 wear glasses and 550 like ice-cream. 395 of them are democrats who wear glasses, 350 of them are democrats who like ice-cream. 400 of them wear glasses and like ice-cream and 250 all the three.

(a) How many of them are not democrats, do not wear glasses and do not like ice-cream?

(b) How many of them are democrats who do not wear glasses and do not like ice-cream?

☑ **Soln. :** Let the following be given sets.

U = All people surveyed, $|U| = 1000$

A = Democratic people, $|A| = 595$

B = People who wear glasses, $|B| = 595$

C = People who like ice-cream, $|C| = 550$

$A \cap B$ = Democrats who wear glasses, $|A \cap B| = 395$

$A \cap C$ = Democrats who like ice-cream, $|A \cap C| = 350$

$B \cap C$ = People who wear glass and like ice-cream, $|B \cap C| = 400$

$A \cap B \cap C$ = Democrats who wear glasses and like ice-cream, $|A \cap B \cap C| = 250$

Following Venn diagram illustrates this

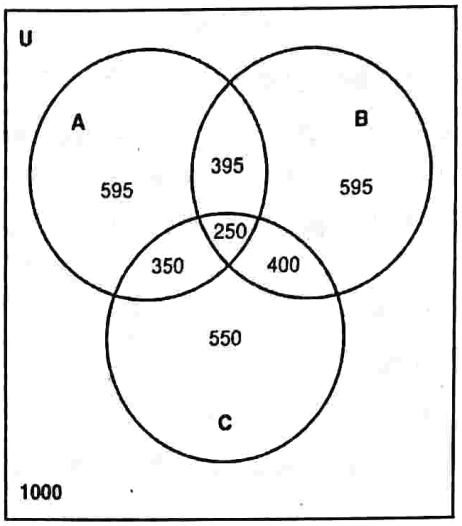


Fig. Ex. 4.7.6

(a) Number of people who are not democrats, do not wear glasses and do not like ice-cream.

$$= \left[\overline{A \cup B \cup C}\right] = |U| - |A \cup B \cup C|$$

$$= |U| - [|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|]$$

$$= 1000 - (595 + 595 + 550 - 395 - 350 - 400 + 250)$$

$$= 1000 - 845 = 155$$

(b) Number of people who are democrats, do not wear glasses and do not like ice-cream.

$$= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

$$= 595 - 395 - 350 + 250 = 100$$

UEx. 4.7.7 (MU - Q. 2(a), Dec. 17, 6 Marks)

Find the number of ways a person can distribute Rs. 601 as pocket money to his three sons, so that no son should receive more than the combined total of the other two. (Assume no fraction of a rupee is allowed).

☑ **Soln. :**

Let A, B, C be the pocket money received by 3 sons.

$$\therefore A + B + C = 601$$

$$A + B \geq C$$

$$A + C \geq B$$

$$B + C \geq A$$

Consider $A + B \geq C$

$$\therefore A + B + C \geq C + C$$

Module IV

$\therefore \quad 601 \geq 2C$

$\therefore \quad C \leq 300$

Similarly $A \leq 300$, $B \leq 300$.

Assume that A = 300, then B and C will be pairs of (300, 1), (299, 2) ....(1, 300) – total 300.

If A = 299, B, C pair will be (301, 1), (300, 2), ..., (1, 301) 301 pairs.

If A = 301, B, C pair will be (299, 1), (298, 2), ... (1, 299) 299 pairs.

The value for A can vary from 1 to 301.

$\therefore$ Total number of possible ways are **301!**

UEx. 4.7.8 (MU – Q. 3(b), Dec. 14, 2 Marks)

If 4 numbers are selected from set {1,2,3,4,5,6} the at least one pair add upto 7.

☑ **Soln. :** Given set is {1, 2, 3, 4, 5, 6}.

The pairs which add upto 7 are, (3, 4), (2, 5), (1, 6).

The pairs which do not add upto 7 are (1, 2), (1, 3), (2, 3).

$\therefore$ We must have atleast 4 numbers so that atleast one pair must add upto 7.

UEx. 4.7.9 MU - May 05, Dec. 08, Dec. 15

If 11 numbers are chosen from a set = {1, 2,.....20}. Prove that one of them is multiple of other.

☑ **Soln. :** Given set {1, 2,...,20}

To prove :

If 11 numbers are selected, then one of them will be multiple of another.

Proof :

From the given set, we have 10 even numbers and 10 odd numbers.

Now the number of pigeon be 11.

If we have all odd numbers as pigeonholes, one number will be even, which is divisible by 2. Hence the proof.

UEx. 4.7.10 MU - Dec. 05, May 08, May 09, May 12

Prove that if any 14 integers from 1 to 25 are chosen, then one of them is a multiple of another.

☑ **Soln. :**

Given set : {1, 2, ..., 25}

To prove :

If 14 numbers are chosen, then one of them is multiple of another.

Proof :

From the given set, we have 12 even numbers and 13 odd numbers.

Now the number of pigeon be 14.

If we have all odd numbers as pigeonholes, one number will be even, which is divisible by 2.

Hence the proof.

UEx. 4.7.11 MU - Dec. 09, Dec. 12

Show that if any five integers from 1 to 8 are selected, then the sum of at least two of them will be 9.

☑ **Soln. :**

Given set : {1, 2, ..., 8}

To prove :

If any five numbers are chosen, then two of them add up to 9.

Proof :

Consider the following sets of two numbers such that sum of two numbers in each set is 9.

$S_1 = \{1, 8\}$, $S_2 = \{2, 7\}$, $S_3 = \{3, 6\}$, $S_4 = \{4, 5\}$

Now when we select 5 numbers, then at least two will be from the same set.

$\therefore$ Two number from 5 selected will add to 9. Hence the proof.

UEx. 4.7.12 MU - May 10

Define a pigeonhole principle. Show that if seven colours are used to paint 50 bicycles, at least 8 bicycles will be of same colour.

☑ **Soln. :**

Given : Number of bicycles = 50

Number of colors = 7

As per the extended pigeon hole principle, we have

$$\left\lfloor \frac{(50-1)}{7} \right\rfloor + 1 = 8$$

$\therefore$ 8 of the bicycles will have the same color.

UEx. 4.7.13 MU - May 10

How many numbers must be selected from the set {1,2,3,4,5,6} to Guarantee that at least one pair of these numbers add up to 7 ?

☑ **Soln. :** Given set = {1, 2, 3, 4, 5, 6}

Consider the following sets of two numbers such that sum of two numbers in each set is 7. n+1

$$S_1 = \{1, 6\}, S_2 = \{2, 5\}, S_3 = \{3, 4\} + 1$$

$\therefore$ We need to have at least 4 numbers to be chosen, so that two of them add upto 7.

UEx. 4.7.14 MU - May 10

How many friends must you have to guarantee that at least five of them will have birthdays in same month ?

☑ **Soln. :** 12

To find : The number of friends so that at least five of them have birthday in the same month

Let n be the number of friends

Now as per the extended pigeonhole principle,

we have $\boxed{\left\lfloor \frac{(n-1)}{12} \right\rfloor + 1} = 5$, 12 months per year

$$\therefore \left\lfloor \frac{(n-1)}{12} \right\rfloor = 4$$

$$\therefore n - 1 = 48$$

$$\therefore n = 49$$

UEx. 4.7.15 MU - Dec. 10

What is the minimum number of students required in Discrete Structure class to be sure that at least six will receive the same grade. If there are five possible grades A, B,C,D,E?

☑ **Soln. :**

Given : Total number of grades = 5 (A, B, C, D, E). Minimum number of students required so that atleast 6 should have same grade is calculated using extended pigeonhole Principle.

let it be n.

$$\therefore \left\lfloor \frac{(n-1)}{5} \right\rfloor + 1 = 6$$

$$\therefore \left\lfloor \frac{(n-1)}{5} \right\rfloor = 5$$

$$\therefore n - 1 = 25$$

$$\therefore n = 26$$

$\therefore$ 26 students are required to be sure that at least 6 will receive the same grade.

UEx. 4.7.16 MU - May 12

Prove that If any 14 integer from 1 to 25 are chosen, then one of them is multiple of another.

☑ **Soln. :**

To prove :

If any 14 integers from 1 to 25 are chosen, then one of them is a multiple of another.

Proof :

Following are the sets of two numbers from 1 to 25, such that none of them divides the other.

{2, 3}, {4, 5}, {6, 7}, {8, 9}, {10, 11}, {12, 13}, {14, 15}, {16, 17}, {18, 19}, {20, 21}, {22, 23}, {24, 25}

These are 12 in number.

Following are the sets in which, one number divides all other and all sets are mutually exclusive.

{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24}

{3, 9, 15, 21}

{5, 25}, 9860643703

They are 3 in number.

Following set has numbers such that none of them divides the other.

{7, 11, 13, 17, 19, 23}

By considering all the above we can say that if any 14 numbers are selected any one will divide another.

UEx. 4.7.17 (MU - Q. 4(c), May 2018, 4 Marks)

How many friends must you have to guarantee that at least five of them will have birthdays in the same month ?

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Soln. :

Number of friends required so that atleast 5 of them will have birthdays in the same month.

$$= 4 \times 12 + 1 = 49$$

4.8 PERMUTATIONS

Definition : A permutation of a set of n objects is an ordered arrangement of r objects. It is written as ${}^nP_r$ and is given by,

$${}^nP_r = \frac{n!}{(n-r)!}$$

It is also represented as P (n, r) or r – permutation.

4.8.1 Solved Examples on Permutations

Ex. 4.8.1 : How many ways are there to select, first, second and third prize winners from a group of 100 people, in a contest ?

Soln. :

Total number people = n = 100

Number of prizes = r = 3

∴ Number of ways of selecting 3 prizes

$$= {}^{100}P_3 = P(100,3) = \frac{100!}{(100-3)!} = \frac{100!}{97!}$$

$$= \frac{100 \times 99 \times 98 \times \cancel{97!}}{\cancel{97!}} = 970200$$

Ex. 4.8.2 : List all the permutations of {a, b, c}.

Soln. :

Following are the permutations of {a, b, c}:

(i) a b c	(ii) a c b	(iii)	b a c
(iv) b c a	(v) c a b	(vi)	c b a

In this case n = 3 and r = 3

$$\therefore {}^nP_r = {}^3P_r = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3 \times 2 \times 1}{1} = 6$$

Ex. 4.8.3 : How many permutation of {a, b, c, d, e, f, g} end with a?

Soln. : The given set is {a, b, c, d, e, f, g}

∴ n = 7

The permutations should end with a.

∴ The permutations have arrangements of {a, b, c, d, e, f, g}and end with a.

i.e. n = 6, r = 6

∴ Number of permutations endings with a

$$= {}^6P_6 = \frac{6!}{(6-6)!} - \frac{6!}{0!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{1}$$

$$= 720$$

Ex. 4.8.4 : Let S = {1,2,3,4,5}

List all 2-permutations of 5.

Soln. :

Given : S = {1,2,3,4,5}

We need to find 2 – permutations of S.

∴ n = 5 , r = 2

$$\therefore {}^nP_r = {}^5P_2 = \frac{5!}{(5-2)!} - \frac{5!}{3!} = \frac{5 \times 4 \times \cancel{3 \times 2 \times 1}}{\cancel{3 \times 2 \times 1}} = 20$$

∴ There are 20, 2 – permutations of S.

They are listed as follows.

1. 12	2. 13	3. 14	4. 15
5. 21	6. 23	7. 24	8. 25
9. 31	10. 32	11. 34	12. 35
13. 41	14. 42	15. 43	16. 45
17. 51	18. 52	19. 53	20. 54

Ex. 4.8.5 : Find the number of 5 – permutations of a set of nine elements

Soln. :

Here n = 9, r = 5

∴ Number of 5 – permutations of a set of nine elements

$$= {}^9P_5 = \frac{9!}{(9-5)!}$$

$$= \frac{9 \times 8 \times 7 \times 6 \times 5 \times \cancel{4 \times 3 \times 2} \times 1}{\cancel{4 \times 3 \times 2} \times 1}$$

$$= 15120$$

Ex. 4.8.6 How many bit strings of length 10 contain

a) exactly four 1's?

b) at most four 1's?

c) at least four 1's?

d) an equal number of 0's and 1's?

Soln. :

Length of the string = 10

Total number of possible strings = $2^{10} = 1024$

a) Number of strings with exactly four 1's is given by,

$$= \frac{{}^{10}P_4}{4!}$$

$$= \frac{10!}{(10-4)!\,4!}$$

$$= \frac{10!}{6!\,4!}$$

$$= \frac{\cancel{10} \times \cancel{9} \times \cancel{8} \times 7 \times \cancel{6!}}{6! \times \cancel{4 \times 3 \times 2} \times 1}$$

$$= 210$$

It is considered as arranging 4 1's in 10 places i.e. ${}^{10}P_4$, But all there four 1's are same. So their internal permutations must be avoided.

∴ It is divided by 4!

b) Number of strings with at most four 1's.

= Number of strings with no 1 + Number of strings with one 1

+ Number of strings with two 1's

+ Number of strings with three 1's

+ Number of strings with four 1's

$$= \frac{{}^{10}P_0}{0!} + \frac{{}^{10}P_1}{1!} + \frac{{}^{10}P_2}{2!} + \frac{{}^{10}P_3}{3!} + \frac{{}^{10}P_4}{4!}$$

$$= \frac{10!}{(10-0)!\,0!} + \frac{10!}{(10-1)!\,1!} + \frac{10!}{(10-2)!\,2!} + \frac{10!}{(10-3)!\,3!} + \frac{10!}{(10-4)!\,4!}$$

$$= \frac{\cancel{10!}}{\cancel{10!} \times 0!} + \frac{10!}{9!\,1!} + \frac{10!}{8!\,2!} + \frac{10!}{7!\,3!} + \frac{10!}{6!\,4!}$$

$$= 1 + \frac{10 \times \cancel{9!}}{\cancel{9!}} + \frac{10 \times 9 \times \cancel{8!}}{\cancel{8!} \times 2 \times 1}$$

$$+ \frac{10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!} \times 3 \times 2 \times 1} + \frac{10 \times 9 \times 8 \times 7 \times \cancel{6!}}{\cancel{6!} \times 4 \times 3 \times 2 \times 1}$$

$$= 1 + 10 + 45 + 120 + 210$$

$$= 386$$

c) Number of strings with atleast four 1's.

= Total number of possible strings

– number of strings with at most four 1's

+ number of strings with exactly four 1's.

$$= 1024 - 386 + 210$$

$$= 848$$

d) Number of strings with equal number of 0's and 1's.

= Number of strings with five 0's and five 1's.

$$= \frac{{}^{10}P_5}{5!} = \frac{10!}{5!\,5!} = \frac{10 \times \cancel{9 \times 8} \times 7 \times \cancel{6} \times 5!}{5! \times \cancel{5 \times 4 \times 3 \times 2} \times 1} = 252$$

Ex. 4.8.7 : How many permutations of the letters A B C D E F G contain

a. the strings BCD?

b. the strings CFGA?

c. the strings BA and GF?

d. the strings ABC and DE?

Soln. :

The given letters are A B C D E F G, 7 letters

Total number of possible permutations = 7!

a) Number of permutations that contain string BCD is calculated as follows.

Consider BCD as a single letter.

∴ Number of letters = (BCD, A, E, F, G) = 5

∴ Number of permutations with string BCD = 5! = 120

b) Number of permutations that contain CFGA is calculated as follows.

Consider CFGA as a single letter.

∴ Number of letters = (CFGA, B, D, E) = 4

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$\therefore$ Number of permutation with string CFGA = 4! = 24

c) Consider strings BA and GF as single letters

$\therefore$ Number of letters = (BA, GF, C, D, E) = 5

$\therefore$ Number of permutations with strings BA as GF = 5! = 120

d) Consider strings ABC and DF as single letter

$\therefore$ Number of letters = (ABC, DE, F, G) = 4

$\therefore$ Number of permutations with strings ABC and DE = 4! = 24

4.9 COMBINATIONS

Definition : A combination of a set of n objects is a selection of r objects. It is written as ${}^nC_r$ or c (n, r) and is given by,

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

It is also known as r –combination.

$${}^nC_r = \frac{n!}{(n-r)!} \times \frac{1}{r!}$$

$$\therefore {}^nC_r = \frac{{}^nP_r}{r!}$$

$$\therefore {}^nP_r = {}^nC_r . r!$$

$$\therefore {}^nP_r = {}^nC_r . {}^rP_r$$

$[\therefore {}^rP_r = r!]$

4.9.1 Solved Examples on Combinations

Ex. 4.9.1 : How many ways are there to select five players from a 10 – member tennis team.

Soln. :

The problem is to select 5 players from a team of 10.

$\therefore n = 10, r = 5$

The number of ways to select

$$= {}^{10}C_5$$

$$= \frac{10!}{5!\,(10-5)!} = \frac{10!}{5!\,5!}$$

$$= \frac{\cancel{10} \times \cancel{9} \times \cancel{8} \times 7 \times 6 \times \cancel{3!}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1 \times \cancel{5!}}$$

$$= 252$$

Ex. 4.9.2 : Prove that ${}^nC_r = {}^nC_{n-r}$

Soln. :

$$\text{LHS} = {}^nC_r = \frac{n!}{(n-r)!\,r!}$$

$$\text{RHS} = {}^nC_{n-r}$$

$$= \frac{n!}{(n-(n-r))!\,(n-r)!} = \frac{n!}{r!\,(n-r)!} = {}^nC_r$$

$$= \text{LHS}$$

Hence proved.

Ex. 4.9.3 : How many bit strings of length 12 contain

a. exactly three 1's?

b. at most three 1's?

c. at least three 1's?

d. an equal number of 0's and 1's?

Soln. :

Length of the strings is 12.

$\therefore$ Total number of possible strings $= 2^{12} = 4096$

a) Number of strings with exactly three 1's

$$= {}^{12}C_3 = \frac{12!}{(12-3)!\,3!}$$

$$= \frac{\overset{2}{\cancel{12}} \times 11 \times 10 \times \cancel{9!}}{\cancel{9!} \times \cancel{3} \times \cancel{2} \times 1} = 220$$

b) Number of strings with at most three 1's

$$= {}^{12}C_0 + {}^{12}C_1 + {}^{12}C_2 + {}^{12}C_3$$

$$= \frac{12!}{(12-0)!\,0!} + \frac{12!}{(12-1)!\,1!} + \frac{12!}{(12-2)!\,2!} + \frac{12!}{(12-3)!\,3!}$$

$$= \frac{\cancel{12!}}{\cancel{12!}} + \frac{12 \times \cancel{11!}}{\cancel{11!}} + \frac{\overset{6}{\cancel{12}} \times 11 \times \cancel{10!}}{\cancel{10!} \times 2 \times 1} + \frac{12 \times 11 \times 10 \times \cancel{9!}}{\cancel{9!} \times 3 \times 2 \times 1}$$

$$= 1 + 12 + 66 + 220$$

$$= 299$$

c) Number of strings with atleast three 1's.

= Total number of strings – number of strings with atmost three 1's + number of strings with exactly three 1's.

$= 4096 - 299 + 220 = 3577$

d) Number of strings with equal number of 1's and 0's.

$$= {}^{12}C_6$$

$$= \frac{12!}{(12-6)!\,6!}$$

$$= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6!}{6! \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$= 1848$$

Ex. 4.9.4 : In how many ways can a set of two positive integers less than 100 be chosen?

Soln. :

n = Positive integers less than 100 = 99

r = 2

∴ Number of ways to choose two positive less than 100 = ${}^{99}C_2$

$$= \frac{99!}{(99-2)!\,2!} = \frac{99 \times 98 \times 97!}{97! \times 2 \times 1} = 4851$$

Ex. 4.9.5 : A coin is flipped eight times, where each comes up either heads or tails. How many possible outcomes

(a) are there in total?

(b) contain exactly three heads?

(c) contain atleast three heads?

(d) contain the same number of heads and tails?

Soln. :

(a) Possible outcomes with one flip = 2 (heads or tails)

∴ Possible outcomes with flips = $2^8 = 256$

(b) Possible outcomes with exactly 3 heads

$$= {}^8C_3 = \frac{8!}{(8-3)!\,3!} = \frac{8 \times 7 \times 6 \times 5!}{5! \times 3 \times 2 \times 1} = 56$$

(c) Possible outcomes with atleast three heads

$$= {}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6 + {}^8C_7 + {}^8C_8$$

$$= \frac{8!}{(8-3)!\,3!} + \frac{8!}{(8-4)!\,4!} + \frac{8!}{(8-5)!\,5!} + \frac{8!}{(8-6)!\,6!} + \frac{8!}{(8-7)!\,7!} + \frac{8!}{(8-8)!\,8!}$$

$$= \frac{8 \times 7 \times 6 \times 5!}{5! \times 3 \times 2 \times 1} + \frac{8 \times 7 \times 6 \times 5 \times 4!}{4! \times 4 \times 3 \times 2 \times 1} + \frac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 5!} + \frac{8 \times 7 \times 6!}{2 \times 1 \times 6!} + \frac{8 \times 7!}{1 \times 7!} + \frac{8!}{8!}$$

$$= 56 + 70 + 56 + 28 + 8 + 1 = 219$$

(d) Possible outcomes with same number of heads and tails

$$= {}^8C_4 = \frac{8!}{(8-4)!\,4!}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times 4!}{4! \times 4 \times 3 \times 2 \times 1} = 70$$

Ex. 4.9.6 : In how many ways can a set of five letters be selected from the English alphabet?

Soln. :

n = number of alphabets = 26

r = number of letters selected at a time = 5

∴ Number of ways to select five letters

$$= {}^{26}C_5 = \frac{26!}{(26-5)!\,5!}$$

$$= \frac{26 \times 25 \times 24 \times 23 \times 22 \times 21!}{21! \times 5 \times 4 \times 3 \times 2 \times 1} = 65780$$

Ex. 4.9.7 : How many different words can be formed using the letters in the word SUCCESS?

Soln. : The given word is SUCCESS.

There are 7 alphabets 3S, 2C, U and E. If all alphabets were different, then the number of words formed would be 7!. But now few alphabets are repeated.

$$\therefore \text{Number of words formed} = \frac{7!}{3!\,2!} = \frac{7 \times 6 \times 5 \times 4 \times 3!}{3! \times 2 \times 1}$$

$$= 420$$

Ex. 4.9.8 : How many different words can be formed using the letters in the word MISSISSIPPI?

Soln. : The given word is MISSISSIPPI.

There are 11 alphabets. 4S, 4I, 2P, and M. If all alphabets were different, then the number of words formed would be 11!, But now few alphabets are repeated.

$$\therefore \text{Number of words formed} = \frac{11!}{4!\,4!\,2!}$$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4! \times 4 \times 3 \times 2 \times 1 \times 2 \times 1}$$

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4.10 BINOMIAL COEFFICIENTS

When an expression of the type $(a + b)^n$ is expanded, the terms like ${}^nC_r$ will be the coefficients of powers of a, b. thus ${}^nC_r$ is called binomial coefficient.

4.10.1 Binomial Theorem

Definition : It states that, if x and y are variables, and n is a non negative integer then,

$$(x+y)^n = \sum_{i=0}^{n} {}^nC_i x^{n-i} y^i$$

$$= {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \dots + {}^nC_r x^{n-r} y^r + \dots + {}^nC_n y^n$$

4.10.2 Solved Examples on Binomial Coefficients

Ex. 4.10.1 : What is the coefficient of $x^{12} y^{13}$ in the expansion of $(x + y)^{25}$?

Soln. :

As per binomial theorem, the coefficient of $x^{12} y^{13}$ in the expansion of $(x + y)^{25}$ is calculated as follows.

Here the term is $x^{12} y^{13}$.

$$\therefore \quad x^{n-i} y^i = x^{12} y^{13} = x^{25-13} y^{13}$$

$$\therefore \quad i = 13$$

$$\therefore \text{ The coefficient } = {}^nC_i = {}^{25}C_{13}$$

$$= \frac{25!}{(25-13)!\,13!}$$

$$= 5200300$$

Ex. 4.10.2 : Prove that $\sum_{k=0}^{n} {}^nC_k = 2^n$

Soln. : Binomial theorem states that,

$$(x+y)^n = \sum_{k=0}^{n} {}^nC_k x^{n-k} \cdot y^k$$

The given equation is

$$\sum_{k=0}^{n} {}^nC_k = 2^n$$

Consider LHS, if it is compared with RHS in the Binomial theorem, we can see that, $x = y = 1$.

$$\therefore \text{ LHS } = \sum_{k=0}^{n} {}^nC_k = \sum_{k=0}^{n} {}^nC_k 1^{n-k} 1^k$$

$$= (1+1)^n \text{ From Binomial theorem}$$

$$= 2^n = \text{RHS}$$

Hence the proof.

Ex. 4.10.3 : If n is a positive integer, then prove that

$$\sum_{k=0}^{n} (-1)^k {}^nC_k = 0.$$

Soln. :

Binomial theorem states that,

$$(x+y)^n = \sum_{k=0}^{n} {}^nC_k x^{n-k} \cdot y^k$$

The given equation is

$$\sum_{k=0}^{n} (-1)^k {}^nC_k = 0$$

Consider LHS, if it is compared with RHS in the Binomial theorem, we can see that, $x = 1$ and $y = -1$. This is because, alternate terms are negative.

$$\therefore \text{ LHS } = \sum_{k=0}^{n} (-1)^k {}^nC_k = \sum_{k=0}^{n} {}^nC_k 1^{n-k} (-1)^k$$

$$= (1-1)^n = 0$$

$$= \text{RHS}$$

Hence the proof.

Ex. 4.10.4 : Find the expansion of $(x + y)^4$

Soln. :

As per Binomial theorem,

$$(x+y)^4 = \sum_{k=0}^{4} {}^4C_k x^{4-k} y^k$$

$$= {}^4C_0 x^{4-0} y^0 + {}^4C_1 x^{4-1} y^1 + {}^4C_2 x^{4-2} y^2 + {}^4C_3 x^{4-3} y^3 + {}^4C_4 x^{4-4} y^4$$

$$= x^4 + 4x^3 y + 6x^2 y^2 + 4x y^3 + y^4$$

Ex. 4.10.5 : What is the coefficient of $x^8 y^9$ in the expansion of $(3x + 2y)^{17}$?

Soln. :

The given expression is $(3x + 2y)^{17}$

We have to find coefficient of $x^8 y^9$

The coefficient as per Binomial theorem is ${}^nC_k\, a^{n-k}\, b^k$ for $(a + b)^n$.

Here $k = 9,\ a = 3x,\ b = 2y,\ n = 17$

$\therefore$ The coefficient of $x^8 y^9$ is

$$
\begin{aligned}
{}^{17}C_9\,(3x)^{17-9}\,(2y)^9 &= \frac{17!}{(17-9)!\,9!} \times (3x)^8\,(2y)^9 \\
&= \frac{17!}{8!\,9!} \times 3^8 \times x^8 \times 2^9 \times y^9 \\
&= 81,66,29,29,920
\end{aligned}
$$

Ex. 4.10.6 : Give a formula for the coefficient of x^k in the expansion of $\left(x + \frac{1}{x}\right)^{100}$, where k is an integer.

Soln. : Given expression is $\left(x + \frac{1}{x}\right)^{100}$

The Binomial theorem states that,

$$(a + b)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, b^r$$

When it is applied to the given expression,

$$
\begin{aligned}
\left(x + \frac{1}{x}\right)^n &= \sum_{r=0}^{n} {}^nC_r\, x^{n-r} \left(\frac{1}{x}\right)^r \\
&= \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\,(x)^{-r} \\
&= \sum_{r=0}^{n} {}^nC_r\, x^{n-2r}
\end{aligned}
$$

Now, we have to find the coefficient for x^k.

$$\therefore\ x^{n-2r} = x^k$$

$$\therefore\ n - 2r = k$$

$$\therefore\ r = \frac{n-k}{2}$$

$\therefore$ The coefficient of $x^k = {}^nC_{\frac{n-k}{2}}$

Ex. 4.10.7 : Show that ${}^{2n}C_{n+1} + {}^{2n}C_n = {}^{2n+2}C_{n+1}$ where n is a positive integer.

Soln. :

Given expression is ${}^{2n}C_{n+1} + {}^{2n}C_n = {}^{2n+1}C_{n+1}$

$$
\begin{aligned}
\text{LHS} &= {}^{2n}C_{n+1} + {}^{2n}C_n \\
&= \frac{2n!}{(2n-n-1)!\,(n+1)!} + \frac{2n!}{(2n-n)!\,n!} \\
&= \frac{2n!}{(n-1)!\,(n+1)!} + \frac{2n!}{n!\,n!} \\
&= \frac{2n!}{(n-1)!\,(n+1)\,(n)!} + \frac{2n!}{n \times (n-1)!\,n!} \\
&= \frac{2n! \times n + 2n! \times (n+1)}{(n-1)!\,n!\,n \cdot (n+1)} \\
&= \frac{2n!\,(n+n+1)}{n!\,(n+1)!} \\
&= \frac{2n!\,(2n+1)}{n!\,(n+1)!} \\
&= \frac{(2n+1)!}{n!\,(n+1)!} \\
&= \frac{(2n+1)!}{[2n+1-(n+1)]!\,(n+1)!} \\
&= {}^{2n+1}C_{n+1}
\end{aligned}
$$

Syllabus Topic : Recurrence Relations

Module IV

4.11 RECURRENCE RELATIONS

Recursion is an important concept in computer program. A recursive function calls itself. It is a very easy way of writing a program. Many complex algorithms can be easily written using recursion. Recursion is based on the mathematical concept of recurrence relation. Though recursion has many advantages, it has a major drawback of space utilization.

In every call of recursive function, space of stack will be used, and may result into stack overflow. Thus, it is necessary to convert a recursive function to iterative. Similarly, we need to solve a recurrence relation.

Definition : A recurrence relation R_1 is a function from set of natural numbers including zero to set of real numbers.
i.e. $R_1 : N \rightarrow R$

The integers 0, 1, 2, ... are related to the real numbers $a_0, a_1, a_2, \ldots$

The relation is given by,

$$a_n = f(a_{n-1}, a_{n-2}, \ldots)$$

It indicates that the value of a_n depends on its previous terms like $a_{n-1}, a_{n-2}, \ldots$. The relation comes along with the initial values of a. i.e. $a_0, a_1, \ldots$

Ex. Let R be a recurrence relation given by $a_n = 2a_{n-1}$ and $a_0 = 1$.

$\therefore$ We can find the values of a, as follows

$a_0 = 1, \quad a_1 = 2 \times a_0 = 2, \quad a_2 = 2 \times a_1 = 4, \ldots$

By looking at these values we can simply say that $a_n = 2^n$.

This is known as solving a recurrence relation.

Definition : Solving a recurrence relation is to eliminate the dependency in calculating value of a_n on its previous values.

The simple way to solve a recurrence relation, where a_n depends on a_{n-1} only, is just by substituting a_{n-1}, a_{n-2} ..., a_0. After substitutions and simplifying the equations we get, the solution for the recurrence relation.

4.11.1 Solved Examples on Recurrence Relations

Ex. 4.11.1 : Solve the following recurrence relation

$$a_n = 3\,a_{n-1} + 1$$
$$a_0 = 1$$

Soln. :

Given : $a_0 = 1$ and $a_n = 3\,a_{n-1} + 1$

$$a_1 = 3a_0 + 1 = 3 \times 1 + 1 = 4$$
$$a_2 = 3\,a_1 + 1 = 3 \times 4 + 1 = 13$$
$$\vdots$$

Now consider

$$\begin{aligned} a_n &= 3\,a_{n-1} + 1 \\ &= 3\,[3\,a_{n-2} + 1] + 1 \\ &= 3^2\,a_{n-2} + 3 + 1 \\ &= 3^2\,[3\,a_{n-3} + 1] + 3 + 1 \\ &= 3^3\,a_{n-3} + 3^2 + 3 + 1 \\ &\vdots \end{aligned}$$

With this we can write

$$\begin{aligned} a_n &= 3^n\,a_{n-n} + 3^{n-1} + 3^{n-2} + \ldots + 3^1 + 3^0 \\ &= 3^n + 3^{n-1} + 3^{n-2} + \ldots + 3^1 + 3^0 \\ &= \frac{(3^{n+1} - 1)}{(3-1)} = \frac{3^{n+1} - 1}{2} \end{aligned}$$

$$\left[a^0 + a^1 + a^2 + \ldots + a^n = \frac{a^n - 1}{a - 1}\right]$$

$\therefore \; a_n = 3^{n+1} - 1$ is the solution

Verify the answer as follows

$$a_1 = \frac{3^{1+1} - 2}{2} = \frac{3^2 - 1}{2} = \frac{8}{2} = 4$$

$$a_2 = \frac{3^{2+1} - 1}{2} = \frac{3^3 - 1}{2} = \frac{26}{2} = 13$$

Ex. 4.11.2 : Solve the following recurrence relation

$$a_n = -2\,a_{n-1}, \quad a_0 = -1$$

Soln. :

Given : $a_n = -2\,a_{n-1}, \quad a_0 = -1$

$$a_1 = -2 \times a_0 = -2 \times -1 = 2$$
$$a_2 = -2 \times a_1 = -2 \times 2 = -4$$
$$\vdots$$

Now consider,

$$\begin{aligned} a_n &= -2\,a_{n-1} = -2\,[-2\,a_{n-2}] = +2^2\,a_{n-2} \\ &= -2\,[2^2\,a_{n-3}] = -2^3\,a_{n-3} \end{aligned}$$

With this we can write,

$$\begin{aligned} a_n &= (-1)^n\,2^n\,a_{n-n} = (-1)^n\,2^n \times -1 \\ &= (-1)^{n+1}\,2^n \end{aligned}$$

$\therefore \; a_n = (-1)^{n+1}\,2^n$ is the solution.

Verification

$$a_1 = (-1)^{1+1} 2^1 = 2$$

$$a_2 = (-1)^{2+1} 2^2 = -4$$

Ex. 4.11.3 : Let $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2, \ldots$

(a) Find a_0, a_1, a_2, a_3 and a_4

(b) Show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$ and $a_4 = 5a_3 - 6a_2$

(c) Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers with $n \geq 2$.

☑ Soln. :

Given : $a_n = 2^n + 5.3^n$

(a)
$$a_0 = 2^0 + 5 \cdot 3^0 = 1 + 5 = 6$$
$$a_1 = 2^1 + 5 \cdot 3^1 = 2 + 5 \times 3 = 17$$
$$a_2 = 2^2 + 5 \cdot 3^2 = 4 + 5 \times 9 = 49$$
$$a_3 = 2^3 + 5 \cdot 3^3 = 8 + 5 \times 27 = 143$$
$$a_4 = 2^4 + 5 \cdot 3^4 = 16 + 5 \times 81 = 421$$

(b) We have to prove that

$$a_2 = 5a_1 - 6a_0$$
$$a_3 = 5a_2 - 6a_1$$
$$a_4 = 5a_3 - 6a_2$$

Consider

$$5a_1 - 6a_0 = 5 \times 17 - 6 \times 6 = 85 - 36 = 49 = a_2$$
$$5a_2 - 6a_1 = 5 \times 49 - 6 \times 17 = 143 = a_3$$
$$5a_3 - 6a_2 = 5 \times 143 - 6 \times 49 = 421 = a_4$$

(c) We have to prove that,

$$a_n = 5a_{n-1} - 6a_{n-2}$$

Consider RHS

$$5a_{n-1} - 6a_{n-2}$$
$$= 5(2^{n-1} + 5 \cdot 3^{n-1}) - 6(2^{n-2} + 5 \cdot 3^{n-2})$$
$$= 5 \cdot 2^{n-1} + 25 \cdot 3^{n-1} - 6 \cdot 2^{n-2} + 30 \cdot 3^{n-2}$$
$$= 5 \cdot 2^{n-1} - 3 \cdot 2^{n-1} + 25 \cdot 3^{n-1} + 10 \cdot 3^{n-1}$$
$$= (5 \cdot 3) 2^{n-1} + (25 - 10) \cdot 3^{n-1}$$
$$= 2 \cdot 2^{n-1} + 15 \cdot 3^{n-1}$$
$$= 2^n + 5 \cdot 3^n$$
$$= a_n$$
$$= \text{LHS}$$

Hence the proof.

Ex. 4.11.4 Find the first 6 terms of the sequence defined by the following recurrence relations

(a) $a_n = a_{n-1} - a_{n-2}$, $a_1 = -1$, $a_0 = 2$

(b) $a_n = n\, a_{n-1} + a_{n-2}^2$, $a_0 = -1$, $a_1 = 0$

(c) $a_n = a_{n-1} - a_{n-2} + a_{n-3}$, $a_0 = 1$, $a_1 = 1$, $a_2 = 2$

☑ Soln. :

(a) **Given :** $a_n = a_{n-1} - a_{n-2}$

$a_1 = -1$, $a_0 = 2$

First six terms are

$$a_0 = 2$$
$$a_1 = -1$$
$$a_2 = a_1 - a_2 = -1 - 2 = -3$$
$$a_3 = a_2 - a_1 = -3 - (-1) = -2$$
$$a_4 = a_3 - a_2 = -2 - (-3) = 1$$
$$a_5 = a_4 - a_3 = 1 - (-2) = 3$$

(b) **Given :** $a_n = n\, a_{n-1} + a_{n-2}^2$

$a_0 = -1$, $a_1 = 0$

First six terms are

$$a_0 = -1$$
$$a_1 = 0$$
$$a_2 = 2 \times a_1 + a_0^2 = 2 \times 0 + (-1)^2 = 1$$
$$a_3 = 2 \times a_2 + a_1^2 = 2 \times 1 + 0^2 = 2$$
$$a_4 = 2 \times a_3 + a_2^2 = 2 \times 2 + 1^2 = 5$$
$$a_5 = 2 \times a_4 + a_3^2 = 2 \times 5 + 2^2 = 14$$

(c) **Given :** $a_n = a_{n-1} - a_{n-2} + a_{n-3}$

$a_0 = 1$, $a_1 = 1$, $a_2 = 2$

First six terms are

$$a_0 = 1$$
$$a_1 = 1$$
$$a_2 = 2$$
$$a_3 = a_2 - a_1 + a_0 = 2 - 1 + 1 = 2$$
$$a_4 = a_3 - a_2 + a_1 = 2 - 2 + 1 = 1$$
$$a_5 = a_4 - a_3 + a_2 = 1 - 2 + 2 = 1$$

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Ex. 4.11.5 : Solve the following recurrence relation.

$$a_n = -a_{n-1} + n - 1, \quad a_0 = 7$$

☑ Soln. :

Given : $a_n = -a_{n-1} + n - 1, \quad a_0 = 7$

$$\therefore \quad a_1 = -7 + 1 - 1 = -7$$

$$a_2 = -(-7) + 2 - 1 = 7 + 1 = 8$$

$$a_3 = -8 + 3 - 1 = -6$$

$$a_4 = -(-6) + 4 - 1 = 9$$

$$a_5 = -9 + 5 - 1 = -5$$

$$a_6 = -(-5) + 6 - 1 = 10$$

$$\vdots$$

Now consider,

$$\begin{aligned}
a_n &= -a_{n-1} + n - 1 \\
&= -[-a_{n-2} + n - 2] + n - 1 \\
&= a_{n-2} - n + 2 + n - 1 \\
&= a_{n-2} + 1 \\
&= -a_{n-3} + n - 3 + 1 \\
&= -a_{n-3} + n - 2 \\
&= -(-a_{n-4} + n - 4) + n - 2 \\
&= a_{n-4} - n + 4 + n - 2 \\
&= a_{n-4} + 2 \\
&= -a_{n-5} + n - 5 + 2 \\
&= -a_{n-5} + n - 3 \\
&= -[-a_{n-6} + n - 6] + n - 3 \\
&= a_{n-6} - n + 6 + n - 3 \\
&= a_{n-6} + 3 \\
&\vdots
\end{aligned}$$

With this we can write,

if n is even

$$\begin{aligned}
a_n &= a_{n-n} + \frac{n}{2} \\
&= a_0 + \frac{n}{2} = 7 + \frac{n}{2}
\end{aligned}$$

if n is odd

$$\begin{aligned}
a_n &= -a_{n-n} + n - \left(\frac{n+1}{2}\right) \\
&= -a_0 + \frac{2n - n - 1}{2} \\
&= -7 + \frac{(n-1)}{2}
\end{aligned}$$

$\therefore$ The solution is

$$a_n = \begin{cases} 7 + \dfrac{n}{2} & \text{if n is even} \\ -7 + \dfrac{(n-1)}{2} & \text{if n is odd} \end{cases}$$

Verify this on follows

$$a_0 = 7 + \frac{0}{2} = 7$$

$$a_1 = -7 + \frac{1-1}{2} = -7$$

$$a_2 = 7 + \frac{2}{2} = 8$$

$$a_3 = -7 + \frac{3-1}{2} = -6$$

$$a_4 = 7 + \frac{4}{2} = 9$$

$$a_5 = -7 + \frac{5-1}{2} = -5$$

$$a_6 = 7 + \frac{6}{3} = 10$$

Ex. 4.11.6 : Solve the following recurrence relation

$$a_n = 2n\, a_{n-1}, \quad a_0 = 1$$

☑ Soln. :

Given : $a_n = 2n\, a_{n-1}, \quad a_0 = 1$

$$\therefore \quad a_1 = 2 \times 1 \times a_0 = 2 \times 1 = 2$$

$$a_2 = 2 \times 2 \times a_1 = 4 \times 2 = 8$$

$$a_3 = 2 \times 3 \times a_2 = 6 \times 8 = 48$$

$$a_4 = 2 \times 4 \times a_3 = 8 \times 48 = 384$$

Now, consider,

$$\begin{aligned}
a_n &= 2n\, a_{n-1} = 2n\,[2(n-1)\, a_{n-2}] \\
&= 2^2\, n\,(n-1)\, a_{n-2} \\
&= 2^2\, n\,(n-1)\,[2\,(n-2)\, a_{n-3}] \\
&= 2^3\, n\,(n-1)\,(n-2)\, a_{n-3}
\end{aligned}$$

With this we can write,

$$a_n = 2^n n(n-1)(n-2)\ldots\ldots(n-(n-1))\, a_{n-n}$$
$$= 2^n n(n-1)(n-2)\ldots.$$
$$= 2^n n!$$

The solution is,

$$a_n = 2^n \cdot n!$$

Verify this as follows

$$a_0 = 2^0\, 0! = 1$$
$$a_1 = 2^1\, 1! = 2$$
$$a_2 = 2^2\, 2! = 8$$
$$a_3 = 2^3\, 3! = 48$$
$$a_4 = 2^4\, 4! = 384$$

UEx. 4.11.7 : MU - Q. 6(c), Dec. 2018, 8 Marks

Find the complete solution of the recurrence relation $a_n + 2a_{n-1} = n + 3$ for $n \geq 1$ and with $a_0 = 3$.

Soln. :

Given recurrence relation is,

$$a_n + 2a_{n-1} = n + 3 \quad \text{for} \quad n \geq 1 \quad \text{and} \quad a_0 = 3$$
$$\therefore \quad a_n = n + 3 - 2a_{n-1}$$
$$\therefore \quad a_1 = 1 + 3 - 2 \times 3 = 4 - 6 = -2$$

$\therefore$ We can write

$$a_n = n + 3 - 2a_{n-1}$$
$$= n + 3 - 2\,[(n-1) + 3 - 2a_{n-2}]$$
$$= n + 3 - 2\,(n - 1 + 3 - 2a_{n-2})$$
$$= n + 3 - 2n - 4 + 4a_{n-2}$$
$$= -n - 1 + 4\,a_{n-2}$$
$$= -n - 1 + 4\,[n - 2 + 3 - 2a_{n-3}]$$
$$= -n - 1 + 4\,[n + 1 - 2a_{n-3}]$$
$$= -n - 1 + 4n + 4 - 8a_{n-3}$$
$$= 3n + 3 - 8a_{n-3}$$
$$= 3n + 3 - 8\,[n - 3 + 3 - 2a_{n-4}]$$
$$= 3n + 3 - 8n + 16a_{n-4}$$
$$= -5n + 3 + 16a_{n-4}$$

4.12 LINEAR RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

A recurrence relation is called k-th order linear recurrence relations with constant coefficients if it is of the form,

$$C_0\, a_r + C_1\, a_{r-1} + C_2\, a_{r-2} + \ldots\ldots + C_k\, a_{r-k} = f(r)$$

Where C_i s are constants and f(r) is a function.

Example

Consider the following linear recurrence relations with constant coefficients.

(i) $2a_r + 3a_{r-1} = 2^r$

(ii) $3a_r - 5a_{r-1} + 2a_{r-2} = r^2 + 5$

(iii) $a_r + 7a_{r-2} = 0$

(i) is first order relation.

(ii) and (ii) are second order relations.

4.12.1 Homogeneous Solutions

In this method, we can find the solution for linear recurrence relations. The method is as follows.

- **Step 1 :** Let the recurrent relation be,

  $$C_0\, a_r + C_1\, a_{r-1} + \ldots\ldots + C_k\, a_{r-k} = f(r)$$

  in which the values of $a_0, a_1, \ldots\ldots, a_{r-(k-1)}$ are given.

- **Step 2 :** Convert the recurrence relation to a characteristic equation as,

  $$C_0\, \alpha + C_1\, \alpha^{K-1} + \ldots + C_{K-1}\, \alpha + C_K\, f(r) = 1$$

- **Step 3 :** Solve this equation, for α.

- **Step 4 :** Let $\alpha = \alpha_1, \alpha_2, \ldots \alpha_K$

  $\therefore$ The equation becomes

  $$a_r = A_1\, \alpha_1^r + A_2\, \alpha_2^r + \ldots + A_K\, \alpha_K^r$$

- **Step 5 :** By substituting values of $a_0, a_1, \ldots\ldots a_{r-(k-1)}$ in a_r find the values of $A_1, A_2, \ldots\ldots A_K$

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▶ **Step 6 :** Thus we get the solution for recurrence relation.

4.12.2 Solved Examples

Ex. 4.12.1 : Solve the following recurrence relation.

$$a_r - 7a_{r-1} + 10a_{r-2} = 0,\ a_0 = 0,\ a_1 = 3.$$

Soln. :

Given recurrence relation is,

$$a_r - 7a_{r-1} + 10a_{r-2} = 0,\ a_0 = 0,\ a_1 = 3$$

The characteristics equation is,

$$\alpha^2 - 7\alpha + 10 = 0$$

$$\therefore \alpha^2 - 5\alpha - 2\alpha + 10 = 0$$

$$\therefore \alpha(\alpha - 5) - 2(\alpha - 5) = 0$$

$$\therefore (\alpha - 5)(\alpha - 2) = 0$$

$$\therefore \alpha = 5 \text{ or } 2$$

$$\therefore \alpha_1 = 5,\ \alpha_2 = 2$$

$\therefore$ The solution is given by

$$a_r = A_1 \alpha_1^r + A_2 \alpha_2^r$$

$$= A_1 5^r + A_2 2^r$$

Now get two equations by substituting values of a_0 and a_1

$$\therefore a_0 = A_1 \alpha_1^0 + A_2 \alpha_2^0$$

$$\therefore 0 = A_1 + A_2 \quad \text{...(i)}$$

$$a_1 = A_1 \alpha_1^1 + A_1 \alpha_2^1$$

$$\therefore 3 = 5A_1 + 2A_2 \quad \text{...(ii)}$$

Solving (i) and (ii) we get

$A_1 = 1$ and $A_2 = -1$

$\therefore$ The solution for the recurrence relation is

$$a_r = 5^r - 2^r$$

Verify the solution as follows :

Given equation is

$$a_r - 7a_{r-1} + 10a_{r-2} = 0$$

$$a_r = 7a_{r-1} - 10a_{r-2}$$

$$a_2 = 7a_1 - 10a_0$$

$$= 7 \times 3 - 10 \times 0$$

$$= 21$$

Consider the solution.

$$a_r = 5^r - 2^r$$

$$a_0 = 5^0 - 2^0 = 0$$

$$a_1 = 5^1 - 2^1 = 3$$

$$a_2 = 5^2 - 2^2 = 21$$

Ex. 4.12.2 : Solve the recurrence relation $a_r - 4a_{r-1} + 4a_{r-2} = 0$ given that $a_0 = 1,\ a_1 = 6$.

Soln. :

Given recurrence relation is

$$a_r - 4a_{r-1} + 4a_{r-2} = 0,\ a_0 = 1,\ a_1 = 6$$

The characteristic equation is

$$\alpha^2 - 4\alpha + 4 = 0$$

$$\therefore \alpha^2 - 2\alpha - 2\alpha + 4 = 0$$

$$\therefore (\alpha - 2)(\alpha - 2) = 0$$

$$\therefore \alpha = -2$$

$\therefore$ The solution is given by

$$a_r = A_1 \alpha_1^r + A_2 \alpha_2^r$$

$$= A_1 (-2)^r + A_2 (-2)^r$$

Hence the values of α_1 and α_2 are same.

$\therefore$ We do not get simultaneous equation.

Thus the recurrence relation cannot be solved. We must use generating functions methods for this type of relations.

Ex. 4.12.3 : Solve the following recurrence relation.

$$a_r - a_{r-1} - a_{r-2} = 0,\ a_0 = 1,\ a_1 = 1.$$

Soln. :

The given recurrence relation is,

$$a_r - a_{r-1} - a_{r-2} = 0,\ a_0 = 1,\ a_1 = 1$$

The characteristic equation is,

$$\alpha^2 - \alpha - 1 = 0$$

$$\therefore \alpha = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$\therefore$ The solution is given by,

$$\alpha_r = A_1 \left(\frac{1+\sqrt{5}}{2}\right)^r + A_2 \left(\frac{1-\sqrt{5}}{2}\right)^r$$

Substitute the values of a_0 and a_1 to get simultaneous equations.

$$\therefore a_0 = A_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + A_2\left(\frac{1-\sqrt{5}}{2}\right)^0$$

$$\therefore 1 = A_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + A_2\left(\frac{1-\sqrt{5}}{2}\right)^0$$

$$= A_1 + A_2 \quad \ldots(i)$$

$$\therefore a_1 = A_1\left(\frac{1+\sqrt{5}}{2}\right) + A_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\therefore 1 = A_1\left(\frac{1+\sqrt{5}}{2}\right) + A_2\left(\frac{1-\sqrt{5}}{2}\right) \quad \ldots(ii)$$

Solving Equations (i) and (ii) we get,

$$A_1 + A_2 = 1 \quad (\text{for } a_0 = 1)$$

$$\therefore A_1 = 1 - A_2$$

$$A_1\left(\frac{1+\sqrt{5}}{2}\right) + A_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \quad (\text{for } a_1 = 1)$$

$$\therefore \frac{(1-A_2)(1+\sqrt{5})}{2} + \frac{A_2(1-\sqrt{5})}{2} = 1$$

$$\therefore \frac{1+\sqrt{5}-\cancel{A_2}-\sqrt{5}A_2+\cancel{A_2}-\sqrt{5}A_2}{2} = 1$$

$$\therefore \frac{1+\sqrt{5}-2\sqrt{5}A_2}{2} = 1$$

$$\therefore 1+\sqrt{5}-2\sqrt{5}A_2 = 2$$

$$\therefore 2\sqrt{5}A_2 = \sqrt{5}-1$$

$$\therefore A_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

$$A_2 = 1 - \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right) = \frac{2\sqrt{5}-\sqrt{5}+1}{2\sqrt{5}}$$

$$= \frac{\sqrt{5}+1}{2\sqrt{5}}$$

The solution for the recurrence relation is

$$a_r = \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^r + \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^r$$

Verify the result as follows.

From the given relation,

$$a_2 - a_1 - a_0 = 0$$

$$\therefore a_2 = a_1 + a_0 = 2$$

From the solution,

$$a_2 = \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2 + \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2$$

$$= 2$$

Note : This recurrence relation represents Fibonacci sequence.

Ex. 4.12.4 : Solve the following recurrence relation.

$$a_r + 6a_{r-1} + 6a_{r-2} = 0,\ a_0 = 0,\ a_1 = 1.$$

Soln. : Given recurrence relation is,

$$a_r + 6a_{r-1} + 6a_{r-2} = 0,\ a_0 = 0,\ a_1 = 1$$

The characteristic equation is

$$\alpha^2 + 6\alpha + 6 = 0$$

$$\therefore \alpha^2 + 6\alpha + 6 = 0$$

$$\therefore \alpha = \frac{-6 \pm \sqrt{36-24}}{2} = \frac{-6 \pm \sqrt{12}}{2}$$

$$= \frac{-6 \pm 2\sqrt{3}}{2} = -3 \pm \sqrt{3}$$

$\therefore$ The solution is given by,

$$\therefore a_r = A_1(-3+\sqrt{3})^r + A_2(-3-\sqrt{3})^r$$

Substitute the values of a_0 and a_1 to get the simultaneous equations.

$$\therefore a_0 = A_1(-3+\sqrt{3})^0 + A_2(-3-\sqrt{3})^0$$

$$\therefore 0 = A_1 + A_2 \quad \therefore A_1 = -A_2$$

$$\therefore a_1 = A_1(-3+\sqrt{3})^1 + A_2(-3-\sqrt{3})^1$$

$$\therefore 1 = -3A_1 + \sqrt{3}A_1 - 3A_2 + \sqrt{3}A_2$$

$$1 = 3A_2 - \sqrt{3}A_2 - 3A_2 - \sqrt{3}A_2$$

(Substitute $A_1 = -A_2$)

$$1 = -2\sqrt{3}A_2$$

$$\therefore A_2 = \frac{-1}{2\sqrt{3}}$$

$$\therefore A_1 = \frac{1}{2\sqrt{3}}$$

$\therefore$ The solution is

$$a_r = \frac{1}{2\sqrt{3}}(-3+\sqrt{3})^r - \frac{1}{2\sqrt{3}}(-3-\sqrt{3})^r$$

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Verify the solution as follows.

Given relation for a_2 is,

$$a_2 + 6a_1 + 6a_0 = 0$$

$$a_2 = 0 - 6 \times 1 - 9 \times 0 = 0 - 6$$

$$= -6$$

From the solution we get,

$$\therefore a_2 = \frac{1}{2\sqrt{3}}(-3+\sqrt{3})^2 - \frac{1}{2\sqrt{3}}(-3-\sqrt{3})^2$$

$$= \frac{1}{2\sqrt{3}}(9+3-6\sqrt{3}) - \frac{1}{2\sqrt{3}}(9+3+6\sqrt{3})$$

$$= \frac{\cancel{12} - 6\sqrt{3} - \cancel{12} - 6\sqrt{3}}{2\sqrt{3}} = -6$$

UEx. 4.12.5 **MU - Dec. 13**

Explain the linear recurrence relations with constant co-efficients.

Soln. :

Given recurrence relation is

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$$

$$\therefore a_n - 5a_{n-1} + 6a_{n-2} = 7^n$$

The particular solution is of the form $P.7^n$

$\therefore$ Using the recurrence relation and the solution, we get,

$$a_n = P.7^n$$

and $P.7^n - 5.P.7^{n-1} + 6.P.7^{n-2} = 7^n$

$$\therefore 7^n\left(P - \frac{5P}{7} + \frac{6P}{7^2}\right) = 7^n$$

$$\therefore 49P - 35P + 6P = 49$$

$$\therefore 20P = 49$$

$$\therefore P = 2.45$$

$\therefore a_n = 2.45 \times 7^n$ is the required solution

No let us verify this solution.

$$a_0 = 2.45 \times 7^0 = 2.45$$

$$a_1 = 2.45 \times 7^1 = 17.15$$

$$a_2 = 2.45 \times 7^2 = 120.05$$

Comparing with the given recurrence relation we get,

$$a_2 = 5a_1 - 6a_0 + 7^2$$

$$= 5 \times 17.15 - 6 \times 2.45 + 7^2$$

$$= 120.05$$

$\therefore$ The solution for the given recurrence relation is verified

$\therefore$ The solution is

$$a_n = 2.45 \times 7^n$$

UEx. 4.12.6 **MU - May 14**

Solve the recurrence relation $a_n = -3(a_{n-1} + a_{n-2}) - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$ and $a_2 = 15$.

Soln. :

Given recurrence relation is

$$a_n = -3(a_{n-1} + a_{n-2}) - a_{n-3}$$

$$a_0 = 5, a_1 = -9, \quad a_2 = 15$$

We can write the equation as,

$$a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$$

$\therefore$ The characteristic equation is

$$\alpha^3 + 3\alpha^2 + 3\alpha + 1 = 0$$

Solving this we get

$$(\alpha + 1)^3 = 0$$

$$\therefore \alpha = -1, -1, -1$$

$\therefore$ The solution can be written as,

$$a_n = (A_1 n^2 + A_2 n + A_3)(-1)^n$$

Substituting it using initial values, we get

$$a_0 = (A_1 \times 0^2 + A_2 \times 0 + A_3)(-1)^0$$

$$= A_3 = 5$$

$$a_1 = (A_1 \times 1^2 + A_2 \times 1 + 5)(-1)^1$$

$$= -(A_1 + A_2 + 5) = -9$$

$$\therefore A_1 + A_2 = 4 \quad \text{...(i)}$$

$$a_2 = (A_1 \times 2^2 + A_2 \times 2 + 5)(-1)^2$$

$$= 4A_1 + 2A_2 + 5 = 15$$

$$\therefore 4A_1 + 2A_2 = 10$$

$$\therefore 2A_1 + A_2 = 5 \quad \text{...(ii)}$$

Solving (i) and (ii) we get,

$A_1 = 1, A_2 = 3$

$\therefore$ The solution is $a_n = (n^2 + 3n + 5)(-1^n)$

Now let us verify it

$a_2 = (2^2 + 3 \times 2 + 5)(-1)^2 = 4 + 6 + 6 = 15$

$\therefore$ The solution for the recurrence relation is,

$a_n = (n^2 + 3n + 5)(-1^n)$

UEx. 4.12.7

MU - May 16, Q. 5(b), May 2018, May 16, 8 Marks

Solve the following recurrence relation :

$a_n - 7a_{n-1} + 10\, a_{n-2} = 0$ with initial condition $a_0 = 1, a_2 = 6$.

☑ **Soln. :** Given recurrence relation is,

$a_n - 7a_{n-1} + 10\, a_{n-2} = 0$ with $a_0 = 1, a_1 = 6$

The characteristic equation is,

$$\alpha^2 - 7\alpha + 10 = 0$$

$$\therefore \quad (\alpha - 5)(\alpha - 2) = 0$$

$$\therefore \quad \alpha = 5, 2$$

$\therefore$ The solution of the recurrence relation is,

$$a_n = A_1 (5)^n + A_2 (2)^n$$

Let $n = 0$,

$$\therefore \quad a_0 = A_1 (5)^0 + A_2 (2)^0$$

$$= A_1 + A_2 = 1 \qquad \ldots(i)$$

Let $n = 1$

$$\therefore \quad a_1 = A_1 (5)^1 + A_2 (2)^1$$

$$= 5A_1 + 2A_2 = 6 \qquad \ldots(2)$$

Solving (i) and (ii) we get,

$$A_1 = \frac{4}{3} \quad \text{and} \quad A_2 = \frac{-1}{3}$$

$\therefore$ The solution is,

$$a_n = \frac{4}{3} \times 5^n - \frac{1}{3} \times 2^n$$

Now let us verify this solution.

Consider $a_2 = \frac{4}{3} \times 5^2 - \frac{1}{3} \times 2^2 = 3^2$

$$\text{Now} \quad a_2 = 7a_1 - 10a_0$$

$$= 7 \times 6 - 10 \times 1 = 32$$

$$\therefore \quad a_n = \frac{4}{3} \times 5^n - \frac{1}{3} \times 2^n$$

is the required solution.

UEx. 4.12.8 **MU - Dec. 16**

Find the solution of $a_{r+2} + 2a_{r+1} - 3_{ar} = 0$ that satisfies $a_0 = 1, a_1 = 2$

☑ **Soln. :**

Given recurrence relation is,

$a_{r+2} + 2a_{r+1} - 3_{a_r} = 0$ and $a_0 = 1, a_1 = 2$

The characteristic equation is,

$$\alpha^2 + 2\alpha - 3 = 0$$

$$\therefore \quad \alpha^2 + 3\alpha - \alpha - 3 = 0$$

$$\therefore \quad \alpha(\alpha + 3) - 1(\alpha + 3) = 0$$

$$\therefore \quad (\alpha + 3)(\alpha - 1) = 0$$

$$\therefore \quad \alpha = -3, 1$$

$\therefore$ The solution of the recurrence relation is,

$$a_n = A_1 (-3)^n + A_z (1)^n$$

$$= A_1 (-3)^n + A_2$$

Let $n = 0$

$$\therefore \quad a_0 = A_1 (-3)^0 + A_2$$

$$= A_1 + A_2 = 1 \qquad \ldots(i)$$

Let $n = 1$

$$\therefore \quad a_0 = A_1 (-3)^1 + A_2$$

$$= -3A_1 + A_2 = 2 \qquad \ldots(ii)$$

Solving (i) and (ii), we get

$$A_1 = \frac{-1}{4} \quad \text{and} \quad A_2 = \frac{5}{4}$$

$\therefore$ The solution is,

$$a_n = \frac{-1}{4} \times (-3)^2 + \frac{5}{4}$$

Now let us verify this solution.

$$\text{Consider} \quad a_2 = \frac{-1}{4} \times (-3)^n + \frac{5}{4}$$

$$= \frac{-9}{4} + \frac{5}{4}$$

$$= -1$$

Module IV

Now $a_2 = -2a_1 + 3a_0$

as per the given relation.

$$= -2 \times 2 + 3$$

$$= -1$$

$$\therefore \quad a_n = \frac{-1}{4}(-3^n) + \frac{5}{4}$$

is the required solution.

UEx. 4.12.9 **MU - Q. 2(c), Dec. 2017, 4 Marks**

Find the complete solution of the recurrence relation : $a_n + 2a_{n-1} = n + 3$ for $n \geq 1$ and with $a_0 = 3$.

☑ **Soln. :**

Given recurrence relation is,

$$a_n + 2a_{n-1} = n + 3 \quad \text{for } n \geq 1 \text{ and } a_0 = 3$$

$$\therefore \quad a_n = n + 3 - 2a_{n-1}$$

$$\therefore \quad a_1 = 1 + 3 - 2 \times 3 = 4 - 6 = -2$$

$\therefore$ We can write

$$a_n = n + 3 - 2a_{n-1}$$

$$= n + 3 - 2[(n-1) + 3 - 2a_{n-2}]$$

$$= n + 3 - 2(n - 1 + 3 - 2a_{n-2})$$

$$= n + 3 - 2n - 4 + 4a_{n-2}$$

$$= -n - 1 + 4a_{n-2}$$

$$= -n - 1 + 4[n - 2 + 3 - 2a_{n-3}]$$

$$= -n - 1 + 4[n + 1 - 2a_{n-3}]$$

$$= -n - 1 + 4n + 4 - 8a_{n-3}$$

$$= 3n + 3 - 8a_{n-3}$$

$$= 3n + 3 - 8[n - 3 + 3 - 2a_{n-4}]$$

$$= 3n + 3 - 8n + 16a_{n-4}$$

$$= -5n + 3 + 16a_{n-4}$$

4.12.3 Particular Solutions

A recurrence relation is considered as a difference equation, when the initial values are not mentioned. To solve such relations, we use the method of particular solutions. The method is as follows.

- **Step 1 :** Let $C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} = f(r)$ be the difference equation.
- **Step 2 :** Assume that the general form of the particular solution is $P_1 r^2 + P_2 r + P_3$ Where P_1, P_2 and P_3 are constants.
- **Step 3 :** Substituting this solution in the given equation we get,

$$C_0(P_1 r^2 + P_2 r + P_3) + C_1(P_1(r-1)^2 + P_2(r-2) + P_3)$$
$$+ C_2(P_1(r-2)^2 + P_2(r-2) + P_3) = f(r).$$

- **Step 4 :** Solve the above equation, to find the values of P_1, P_2 and P_3.
- **Step 5 :** Thus we get the solution of the given recurrence relation.

4.12.4 Solved Examples

Ex. 4.12.10 Solve the following recurrence relation.

$$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2.$$

☑ **Soln. :** Given recurrence relation is,

$$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$$

Let the solution be $P_1 r^2 + P r^2 + P_3$ substituting the solution in the given equation we get,

$$(P_1 r^2 + P_2 r + P_3) + 5(P_1(r-1)^2 + P_2(r-2) + P_3)$$
$$+ 6(P_1(r-2)^2 + P_2(r-2) + P_3) = 3r^2$$

$$\therefore \quad P_1 r^2 + P_2 r + P_3 + 5P_1 r^2 + 5P_1 - 10P_1 r + 5P_2 r$$
$$- 5P_2 + 5P_3 + 6P_1 r^2 - 24P_1 r - 24P_1 + 6P_2 r$$
$$- 12P_2 + 6P_3 = 3r^2$$

$$\therefore \quad 12P_1 r^2 + (+24P_1 + 12P_2)^r + (29P_1 - 17P_2 + 12P_3) = 3r^2$$

By comparing coefficients of r^2, r, and constant on both sides of the equation, we get

$$12P_1 = 3$$

$$-24P_1 + 12P_2 = 0$$

$$29P_1 - 17P_2 + 12P_3 = 0$$

By solving above equations, we get,

$$P_1 = \frac{1}{4}.$$

$$P_2 = \frac{17}{24}$$

$$P_3 = \frac{115}{228}$$

$\therefore$ The solution for the given recurrence relation is,

$$a_r = \frac{1}{4} r^2 + \frac{17}{24} r + \frac{115}{228}$$

To verify the solution calculate a_0, a_1 and a_2 from the solution, and compare it with the result of recurrence relation.

$$a_0 = \frac{1}{4} \times 0^2 + \frac{17}{24} \times 0 + \frac{115}{288}$$

$$\therefore a_0 = \frac{115}{288}$$

$$a_1 = \frac{1}{4} \times 1^2 + \frac{17}{24} \times 1 + \frac{115}{288}$$

$$\therefore a_1 = \frac{72 + 204 + 115}{288} = \frac{391}{288}$$

$$a_2 = \frac{1}{4} \times 2^2 + \frac{17}{24} \times 2 + \frac{115}{288}$$

$$\therefore a_2 = \frac{288 + 408 + 115}{288} = \frac{811}{288}$$

Given equation is

$$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$$

$$\therefore a_2 = 3 \times 2^2 - 5 \times a_1 - 6 \times a_0$$

$$= 12 - 5 \times \frac{391}{288} - 6 \times \frac{115}{288}$$

$$= \frac{3456 - 1955 - 690}{288}$$

$$= \frac{811}{288}$$

Ex. 4.12.11 : Solve the following difference equation.

$$a_r - 5a_{r-1} + 6a_{r-2} = 1$$

☑ Soln. :

The given equation is

$$a_r - 5a_{r-1} + 6a_{r-2} = 1$$

In this case f(r) is a constant.

$\therefore$ The solution will also be a constant.

Let the solution be P

Substituting it in the given equation, we get,

$$P - 5P + 6P = 1$$

$$\therefore P = ½$$

$\therefore$ The solution for the given recurrence relation is

$$a_r = \frac{1}{2}$$

Verification.

$$a_r = 1 + 5a_{r-1} - 6\,a_{r-2}$$

$$= 1 + 5 \times \frac{1}{2} - 6 \times \frac{1}{2} = \frac{1}{2}$$

Ex. 4.12.12 : Solve the following difference equation.

$$a_r + 5a_{r-1} + 6a_{r-2} = 42.4^r.$$

☑ Soln. :

Given equation is,

$$a_r + 5a_{r-1} + 6a_{r-2} = 42.4^r$$

Here $f(r) = 42.4^r$

$\therefore$ The solution will be of the form, $P4^r$

Substituting it in the given equation, we, get,

$$P4^r + 5.\,P4^{r-1} + 6.P.\,4^{r-2} = 42.\,4^r$$

$$\therefore P.\,4^{r-2}[4^2 + 5 \times 4 + 6] = 42.\,4^r$$

$$\therefore P.\,4^{r-2} \times 42 = 42.\,4^r$$

$$\therefore P = 4^2 = 16$$

$\therefore$ The solution is

$$a_r = 16.\,4^r$$

Verify the solution as follows.

$$a_0 = 16 \times 4^0 = 16$$

$$a_1 = 16 \times 4^1 = 64$$

$$a_2 = 16 \times 4^2 = 256$$

From the given equation,

$a_r + 5a_{r-1} + 6a_{r-2} = 42.4^r$ we get,

$$a_2 = 42 \times 4^2 - 5 \times a_1 - 6\,a_0$$

$$= 42 \times 16 - 5 \times 64 - 6 \times 16$$

$$= 256$$

Module IV

Ex. 4.12.13 : Solve the following difference equation.

$$a_r + a_{r-1} = 3r.\ 2^r$$

☑ Soln. :

The given recurrence relation is

$$a_r + a_{r-1} = 3r.\ 2^r$$

Here $\quad f(r) = 3r.\ 2^r$

$\therefore$ The particular solution is of the form $(P_1 r + P_2)\ 2^r$.

Substituting it in the given equation, we get,

$$(P_1 r + P_2)\ 2^r + (P_1 (r-1) + P_2)\ 2^{r-1} = 3r.\ 2^r$$

$$\therefore P_1 r + P_1 + \frac{P_1 r}{2} - \frac{P_1}{2} + \frac{P_2}{2} = 3r$$

$$\therefore \frac{3P_1}{2} - \frac{P_1}{2} + \frac{3P_2}{2} = 3r$$

Comparing the coefficients on both sides, we get

$$\frac{3P_1}{2} = 3$$

$$\therefore P_1 = 2$$

$$\frac{-P_1}{2} + \frac{3P_2}{2} = 0$$

$$\therefore -1 + \frac{3P_2}{2} = 0$$

$$\therefore \frac{3P_2}{2} = 1$$

$$P_2 = \frac{2}{3}$$

$\therefore$ The solution is

$$a_r = \left(2r + \frac{2}{3}\right) 2^r$$

Verify the solution as follows.

$$a_0 = \left(2 \times 0 + \frac{2}{3}\right) 2^0 = \frac{2}{3}$$

$$a_1 = \left(2 \times 1 + \frac{2}{3}\right) 2^1 = \frac{16}{3}$$

$$a_0 = \left(2 \times 2 + \frac{2}{3}\right) 2^2 = \frac{56}{3}$$

From the given equation $a_r + a_{r-1} = 3r.\ 2^r$ we get,

$$a_r = 3r.\ 2^r - a_{r-1}$$

$$a_1 = 3 \times 1 \times 2^1 - a_0$$

$$= 6 - \frac{2}{3} = \frac{16}{3}$$

$$a_2 = 3 \times 2 \times 2^2 - \frac{16}{3}$$

$$= 24 - \frac{16}{3} = \frac{56}{3}$$

Ex. 4.12.14 : Find the solution for the following difference equation.

$$a_r + 4a_{r-1} + 4a_{r-2} = (r+1)2^r.$$

☑ Soln. :

The given difference equation is,

$$a_r + 4a_{r-1} + 4a_{r-2} = (r+1)2^r$$

Here, the characteristic equation has 2 roots because we have a_{r-2} in the equation and $f(r) = (r+1)2^r$

$\therefore$ The solution is of the type,

$$r^2 (P_1 r + P_2)\ 2^r$$

Substituting this in the given equation we get,

$$r^2 (P_1 r + P_2)\ 2^r - 4\ [(r-1)^2 (P_1 (r-1) + P_2) 2^{r-1} + 4\ [(r-2)^2 (P_1 (r-2) + P_2)\ 2^{r-2} = (r+1)\ 2^r$$

$$\therefore r^2 (P_1 r + P_2)\ 2^r - 4\ [(r^2 + 1 - 2r)\ (P_1 r - P_1 + P_2) 2^{r-1})]\ [r^2 + 4 - 4r)\ (P_1 r - 2P_1 + P_2)\ 2^{r-2}] = (r+1)\ 2^r$$

$$[(r^2 + 4 - 4r)\ (P_r r - 2P_1 + P_2)\ 2^{r-2}] = (r+1)\ 2^r$$

$$\therefore r^2 (P_1 r + P_2)\ 2^r - 2.2^r\ [P_1 r^3 - P_1 r^2 + P_2 r^2 + P_1 r - P_1 + P_2 - 2P_1 r^2 + 2P_1 r - 2P_2 r) + 2^r\ [P_1 r^3 - 2P_1 r^2 + P_2 r^2 + 4P_1 r - 8P_1 + 4P_2 - 4P_1 r^2 + 8P_1 r - 4P_2 r] = (r+1)\ 2^r$$

$$\therefore r^2 (P_1 r + P_2)\ 2^r - 2.2^r\ [P_1 r^3 - 3P_1 r^2 + P_2 r^2 + 3P_1 r - P_1 + P_2 - 2P_2 r] + 2^r\ [P_1 r^3 - 6P_1 r^2 + P_2 r^2 + 12P_1 r - 8P_1 + 4P_2 - 4P_2 r] = (r+1)\ 2^r$$

$$\therefore \cancel{P_1 r^3} + \cancel{P_2 r^2} - \cancel{2P_1 r^3} + \cancel{6P_1 r^2} - \cancel{2P_2 r^2} - 6P_1 r + 2P_1 - \cancel{2P_2} + \cancel{4P_2 r} + \cancel{P_1 r^3} - \cancel{6P_1 r^2} + P_2 r^2 + 12P_1 r - 8P_1 + \cancel{4P_2} - 4P_2 r = r + 1$$

$$\therefore +6P_1 r - 6P_1 + 2P_2 = r + 1$$

$$\therefore 6P_1 = 1 \text{ and } -6P_1 + 2P_2 = 1$$

$$\therefore P_1 = \frac{1}{6} \text{ and } -6 \times \frac{1}{6} + 2P_2 = 1$$

$$\therefore\ 2P_2 = 2$$

$$\therefore\ P_2 = 1$$

$$\therefore\ P_1 = \frac{1}{6}P_2 = 1$$

$\therefore$ The solution for the given recurrence relation is

$$a_r = r^2\left(\frac{1}{6}r + 1\right)2^r$$

Verify the solution as follows from the solution we get

$$a_0 = 0^2\left(\frac{1}{6}\times 0 + 1\right)2^0 = 0$$

$$a_1 = 1^2\left(\frac{1}{6}\times 1 + 1\right)2^1 = \frac{7}{6}\times 2 = 7/3$$

$$a_2 = 2^2\left(\frac{1}{6}\times 2 + 1\right)2^2$$

$$= 4\times\frac{4}{3}\times 4 = \frac{64}{3}$$

From the given difference equation we get,

$$a_2 = (2+1)2^2 + 4\times a_1 - 4\times a_0$$

$$= 3\times 4 + 4\times\frac{7}{3} - 0$$

$$= \frac{36+28}{3} = \frac{64}{3}$$

Ex. 4.12.15 : The solution of the recurrence relation

$a_r = Aa_{r-1} + w\,B.\,3^r \quad r \geq 1$ is

$a_r = C.\,2^r + D.\,3^{r+1} \quad r \geq 0$

Given that $a_0 = 19$ and $a_1 = 50$, determine the constants A, B, C, and D.

☑ **Soln. :**

Given recurrence relation is

$$a_r = Aa_{r-1} + B.\,3^r$$

Its solution is

$$a_r = C.\,2^r + D.\,3^{r+1}$$

and $a_0 = 19, \quad a_1 = 50$

By substituting a_0, and a_1 in the solution, we get,

$$a_0 = C.\,2^0 + D\,3^1 a_1 = \qquad C.2^1 + D\,3^2$$

$$\therefore 19 = C + 3D \quad 50 = 2C + 9D$$

Solving these equations, we get

$C = 7$ and $D = 4$

$$\therefore\ a_2 = 7\times 2^2 + 4\times 3^2 = 136$$

Now consider recurrence relation

$$a_r = Aa_{r-1} + B.\,3^r$$

$$\therefore\ a_1 = Aa_0 + B.\,3^1 \text{ and } a_2 = Aa_1 + B.\,3^2$$

$$\therefore\ 50 = 19A + 3B \text{ and } 136 = 50A + 9B$$

Solving these equations, we get

$\therefore A = 2$ and $B = 4$

$\therefore A = 2 \quad B = 4 \quad C = 7 \quad D = 4$

Verify the result as follows, from recurrence relation,

$$a_3 = Aa_2 + B.\,3^3$$

$$\therefore\ a_3 = 2\times 136 + 4\times 27$$

$$= 380$$

From solution of recurrence relation,

$$a_3 = C.2^3 + D.\,3^4$$

$$= 7\times 8 + 4\times 81$$

$$= 380$$

Ex. 4.12.16 : Given that $a_0 = 0$, $a_1 = 1$, $a_2 = 4$ and $a_3 = 12$ satisfy the recurrence relation

$a_r + c_1 a_{r-1} + c_2 a_{r-2} = 0.$

Determine a_r.

☑ **Soln. :** Given recurrence relation is,

$$a_r = -c_1 a_{r-1} - c_2 a_{r-2}$$

$$\therefore\ a_2 = -c_1\times a_1 - c_2\times a_0$$

$$\therefore\ 4 = -c_1\times 1 - c_2\times 0 = -c_1$$

$$\therefore\ c_1 = -4$$

Similarly,

$$a_3 = -c_1\times a_2 - c_2\times a_1$$

$$12 = -(-4)\times 4 - c_2\times 1$$

$$\therefore\ c_2 = 4$$

Given recurrence relation now becomes

$$a_r - 4a_{r-1} + 4a_{r-2} = 0$$

$\therefore$ The characteristic equation is,

$$r^2 - 4r + 4 = 0$$

$$\therefore\ (r-2)^2 = 0$$

$$\therefore\ r = 2$$

Modul IV

The characteristic equation has single root.

∴ It cannot be solved.

∴ It should be solved using generating functions.

Syllabus Topic : Generating Functions

4.13 GENERATING FUNCTIONS

Definition : A generating function is an infinite series of the type

$$\sum_{i=0}^{\infty} a_r z^r = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_r z^r + \ldots$$

where z is known as a formal variable, the powers of z are known as indicators in the infinite series. The coefficient of z^r is the value of the numeric function at r.

This infinite series is written as A (z). The following table gives different forms of A (z) and their corresponding values of a_r.

Sr. No.	A (z)	a_r
s_1	$\frac{p}{1-qz}$	$p q^r$
s_2	$\frac{p-z}{(1-z)^2}$	r
s_3	$\frac{1}{1-z}$	1
s_4	$\frac{p}{(1-qz)^2}$	$p(r+1)q^r$
s_5	$\frac{pz}{(1-qz)^2}$	$p r q^{r-1}$

A recurrence relation can be solved using generating functions as follows.

- **Step 1 :** Convert the recurrence relation to the characteristic function.
- **Step 2 :** Multiply the equation by z^r.
- **Step 3 :** Take the summation of all terms from K to ∞ where K is order of character equation,

i.e. a term x in the characteristic equation becomes.

$$\sum_{r=K}^{\infty} x.z^r$$

- **Step 4 :** We know that $\sum_{r=0}^{\infty} a_r z^r = A(z)$

  Represent each term using A (z).

- **Step 5 :** Consider $A(z) = a_0 + a_1 z + a_2 z^2 + \ldots$

$$\therefore A(z) = \sum_{r=0}^{\infty} a_r z^r$$

$$\therefore \sum_{r=1}^{\infty} a_r z^r = A(z) - a_0$$

$$\therefore \sum_{r=2}^{\infty} a_r z^r = A(z) - a_0 - a_1 z$$

etc.

- **Step 6 :** Solve for A (z).
- **Step 7 :** Convert A (z) to a_r.
- **Step 8 :** a_r is the solution to the given recurrence relation.

4.14 SOLVED EXAMPLES

Ex. 4.14.1 : Solve the recurrence relation,

$$a_n = a_{n-1} + 1, \; a_0 = 0$$

Soln. : Given recurrence relation is

$$a_n = a_{n-1} + 1, \; a_0 = 0$$

$$\therefore a_n - a_{n-1} - 1 = 0$$

∴ The characteristic equation is

$$a_r - a_{r-1} - 1 = 0$$

Multiply by z^r and take summation from 1 to ∞ as the equation is of first order.

$$\therefore \sum_{r=1}^{\infty} a_r z^r - \sum_{r=1}^{\infty} a_{r-1} z^r - \sum_{r=1}^{\infty} z^r = 0 \quad \ldots(i)$$

Consider $\sum_{r=1}^{\infty} a_r z^r = \sum_{r=0}^{\infty} a_r z^r - a_0$

$$= A(z) - a_0$$

$$= A(z) - 0$$

$$= A(z) \quad \ldots(ii)$$

Consider $\sum_{r=1}^{\infty} a_{r-1} z^r = z \sum_{r=1}^{\infty} a_{r-1} z^{r-1}$

$$= z \sum_{r=0}^{\infty} a_r z^r$$

$$= z A(z) \quad \ldots(iii)$$

Consider $\sum_{r=1}^{\infty} z^r = \sum_{r=0}^{\infty} z^r - 1$

$$= \frac{1}{1-z} - 1 \quad \text{(from } S_3\text{)}$$

$$= \frac{1-1+z}{1-z}$$

$$= \frac{z}{1-z} \quad \ldots(iv)$$

Substituting (ii), (iii) and (iv) in equation (i) we get,

$$A(z) - z A(z) \frac{z}{1-z} = 0$$

$$\therefore A(z)[1-z] = \frac{z}{1-z}$$

$$\therefore A(z) = \frac{z}{(1-z)^2}$$

$$\therefore a_r = r$$

$\therefore$ Solution of the recurrence relation

$a_n = a_{n-1} + 1$, is $a_r = r$

Ex. 4.14.2 : Find the solution for the recurrence relation, $a_r = 3a_{r-1} + 2,\ a_0 = 1$

☑ Soln. :

Given relation is

$$a_r = 3a_{r-1} + 2,\ a_0 = 1$$

$$\therefore a_r - 3a_{r-1} - 2 = 0$$

Multiply by z^r and take summation from 1 to ∞

$$\therefore \sum_{r=1}^{\infty} a_r z^r - \sum_{r=1}^{\infty} 3a_{r-1} z^r - \sum_{r=1}^{\infty} 2z^r = 0 \quad \ldots(i)$$

Consider $\sum_{r=1}^{\infty} a_r z^r = \sum_{r=0}^{\infty} a_r z^r - a_0$

$$= A(z) - a_0$$

$$= A(z) - 1 \quad \ldots(ii)$$

Consider $\sum_{r=1}^{\infty} 3a_{r-1} z^r = 3z \sum_{r=1}^{\infty} a_{r-1} z^{r-1}$

$$= 3z \sum_{r=0}^{\infty} a_r z^r$$

$$= 3z A(z) \quad \ldots(iii)$$

Consider $\sum_{r=1}^{\infty} 2z^r = 2 \sum_{r=1}^{\infty} z^r$

$$= 2\left[\sum_{r=0}^{\infty} z^r - 1\right]$$

$$= 2\left[\frac{1}{1-z} - 1\right] = 2\left[\frac{1-1+z}{1-z}\right]$$

$$= \frac{2z}{1-z} \quad \ldots(iv)$$

Substituting (ii), (iii) and (iv) in equation (i) we get

$$A(z) - 1 - 3z A(z) - \frac{2z}{1-z} = 0$$

$$\therefore A(z)(1-3z) = \frac{2z}{1-z} + 1$$

$$\therefore A(z) = \frac{2z+1-z}{(1-z)(1-3z)}$$

$$= \frac{z+1}{(1-z)(1-3z)}$$

Consider $\frac{z+1}{(1-z)(1-3z)} = \frac{A}{(1-z)} + \frac{B}{(1-3z)}$

$$= \frac{A - 3Az + B - Bz}{(1-z)(1-3z)}$$

$$= \frac{(-3A-B)z + (A+B)}{(1-z)(1-3z)}$$

$\therefore -3A - B = 1$, and $A + B = 1$

Solving these equations we get

$A = -1$ and $B = 2$

$$B = 1 - A$$

$$-3A - 1 + A = 1$$

$$-2A = 2$$

$$A = -1$$

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$$B = 2$$

$$\therefore A(z) = \frac{2}{1-3z} - \frac{1}{1-z}$$

$$\therefore a_r = 2.3^r - 1$$

Verify the solution as follows.

From the recurrence relation, we get,

$$a_1 = 3a_0 + 2 = 3 \times 1 + 2 = 5$$

$$a_2 = 3a_1 + 2 = 3 \times 5 + 2 = 17$$

From the solution, we get

$$a_1 = 2 \times 3^1 - 1 = 5$$

$$a_2 = 2 \times 3^2 - 1 = 17$$

Ex. 4.14.3 : Solve the following recurrence relation

$$a_r + 6a_{r-1} + 9a_{r-2} = 3, a_0 = 0, a_1 = 1$$

Soln. : Given recurrence relation is

$$a_r + 6a_{r-1} + 9a_{r-2} = 3, a_0 = 0, a_1 = 1$$

$$\therefore a_r + 6a_{r-1} + 9a_{r-2} - 3 = 0$$

Multiply by z^r and take summation from 2 to ∞.

$$\therefore \sum_{r=2}^{\infty} a_r z^r + \sum_{r=2}^{\infty} 6a_{r-1} z^r + \sum_{r=2}^{\infty} 9a_{r-2} z^r - \sum_{r=2}^{\infty} 3z^r = 0. \quad \ldots(i)$$

$$\text{Consider } \sum_{r=0}^{\infty} a_r z^r = \sum_{r=2}^{\infty} a_r z^r - a_0 - a_1 z$$

$$= A(z) - 0 - 1 \times z$$

$$= A(z) - z \quad \ldots(ii)$$

$$\text{Consider } \sum_{r=2}^{\infty} 6a_{r-1} z^r = 6z\left[\sum_{r=2}^{\infty} a_{r-1} z^{r-1}\right]$$

$$= 6z \sum_{r=1}^{\infty} a_r z^r = 6z\left[\sum_{r=0}^{\infty} a_r z^r a_0\right]$$

$$= 6z A(z) \ldots(iii)$$

$$\text{Consider } \sum_{r=2}^{\infty} 9a_{r-2} z^r = 9z^2\left[\sum_{r=2}^{\infty} a_{r-2} z^{r-2}\right]$$

$$= 9z^2\left[\sum_{r=0}^{\infty} a_r z^r\right]$$

$$= 9z^2 A(z) \quad \ldots(iv)$$

Consider

$$\therefore \sum_{r=2}^{\infty} 3z^r = 3\sum_{r=2}^{\infty} z^r = 3z^2 \sum_{r=2}^{\infty} z^{r-2} = 3z^r \sum_{r=0}^{\infty} z^r = \frac{3z^2}{1-z} \quad \ldots(v)$$

Substituting Equations (ii), (iii) (iv) and (v) in Equation (i) we get

$$A(z) - z + 6zA(z) + 9z^2 A(z) - \frac{3z^2}{1-z} = 0$$

$$\therefore A(z)[9z^2 + 6z + 1] = \frac{3z^2}{1-z} + z$$

$$\therefore A(z) = \frac{3z^2 + z - z^2}{(9z^2 + 6z + 1)(1-z)}$$

$$= \frac{2z^2 + z}{(9z^2 + 6z + 1)(1-z)}$$

$$\text{Consider } \frac{2z^2 + z}{(3z+1)^2(1-z)} = \frac{A}{(1-z)} + \frac{B}{(3z+1)} + \frac{C}{(Bz+1)^2}$$

$$= 9z^2 A + 6zA + A - 3z^2 B + 2zB + B + C - Cz$$

$$= (9A - 3B)z^2 + (6A + 2B - C)z + (A + B + C)$$

$$\therefore 9A - 3B = 2,\ 6A + 2B - C = 1,\ A + B + C = 0$$

Solving these equations we get,

$$A = \frac{3}{16},\ B = -\frac{5}{48},\ C = -\frac{1}{12}$$

$$\therefore A(z) = \frac{3}{16}\left[\frac{1}{1-z}\right] - \frac{5}{48}\left[\frac{1}{3z+1}\right] - \frac{1}{12}\left[\frac{1}{(3z+1)^2}\right]$$

$$\therefore a_r = \frac{3}{16} - \frac{5}{48}(-3)^r - \frac{1}{12}(r+1)(-3)^r$$

This is the solution for the given recurrence relation

Verify the result as follows.

From the recurrence relation we get,

$$a_2 = 3 - 6a_1 - 9a_0 = 3 - 6 \times 1 - 9 \times 0 = -3$$

From the solution we get,

$$a_2 = \frac{3}{16} - \frac{5}{48}(-3)^2 - \frac{1}{12}(2+1)(-3)^2$$

$$= \frac{3}{16} - \frac{5}{48} \times 9 - \frac{1}{12} \times 3 \times 9$$

$$= -3$$

Ex. 4.14.4 : Solve the recurrence relation that represents the Fibonacci sequence,

$$a_r = a_{r-1} + a_{r-2},\ a_0 = 1,\ a_1 = 1$$

Soln. : Given recurrence relation is,

$$a_r = a_{r-1} + a_{r-2},\ a_0 = 1,\ a_1 = 1$$

$$\therefore\ a_r - a_{r-1} - a_{r-2} = 0$$

Multiply by z^r and take summation from 2 to ∞.

$$\therefore \sum_{r=2}^{\infty} a_r z^r - \sum_{r=2}^{\infty} a_{r-1} z^r - \sum_{r=2}^{\infty} a_{r-2} z^r = 0 \qquad \text{...(i)}$$

$$\text{Consider } \sum_{r=2}^{\infty} a_r z^r = \sum_{r=0}^{\infty} a_r z^r - a_0 - a_1 z$$

$$= A(z) - 1 - z \qquad \text{...(ii)}$$

$$\text{Consider } \sum_{r=2}^{\infty} a_{r-1} z^r = z \sum_{r=2}^{\infty} a_{r-1} z^{r-1} = z \sum_{r=1}^{\infty} a_r z^r$$

$$= z\left[\sum_{r=0}^{\infty} a_r z^r - a_0\right]$$

$$= z(A(z) - 1)$$

$$= z A(z) - z \qquad \text{...(iii)}$$

$$\text{Consider } \sum_{r=2}^{\infty} a_{r-2} z^r = z^2 \sum_{r=2}^{\infty} a_{r-2} z^{r-2} = z^2 \sum_{r=0}^{\infty} a_r z^r$$

$$= z^2 A(z) \qquad \text{...(iv)}$$

Substituting Equations (ii), (iii) and (iv) in Equation (i) we get

$$A(z) - 1 - z - z A(z) + z - z^2 A(z) = 0$$

$$\therefore\ A(z)(1 - z - z^2) - = 0$$

$$\therefore\ A(z) = \frac{1}{1 - z - z^2}$$

Now consider,

$$\frac{1}{1 - z - z^2} = \frac{A}{z - \left(\frac{1+\sqrt{5}}{-2}\right)} + \frac{B}{z - \left(\frac{1-\sqrt{5}}{-2}\right)}$$

$$= \frac{-2A}{-2z - 1 - \sqrt{5}} + \frac{-2B}{-2z - 1 + \sqrt{5}}$$

$$= \frac{4Az + 2A - 2\sqrt{5}A + 4Bz + 2B + 2\sqrt{5}B}{(-2z - 1 - \sqrt{5})(-2z - 1 + \sqrt{5})}$$

$$= \frac{(4A + 4B)z + 2A + 2B - 2\sqrt{5}A + 2\sqrt{5}B}{(-2z - 1 - \sqrt{5})(-2z - 1 + \sqrt{5})}$$

$$\frac{1}{1 - z - z^2} = \frac{(4A + 4B)z + (2A + 2B - 2\sqrt{5}A + 2\sqrt{5}B)}{-4(1 - z - z^2)}$$

$$\frac{-4}{1 - z - z^2} = \frac{(4A + 4B)z + (2A + 2B - 2\sqrt{5}A + 2\sqrt{5}B)}{1 - z - z^2}$$

$$\therefore\ 4A + 4B = 0$$

$$2A + 2B - 2\sqrt{5}A + 2\sqrt{5}B = -4$$

Solving these equations, we get

$$A = \frac{1}{\sqrt{5}},\ B = \frac{-1}{\sqrt{5}}$$

$$\therefore \frac{1}{1 - z - z^2} = \frac{\frac{1}{\sqrt{5}}}{z - \left(\frac{1+\sqrt{5}}{-2}\right)} - \frac{\frac{1}{\sqrt{5}}}{z - \left(\frac{1-\sqrt{5}}{-2}\right)}$$

$$= \frac{1}{\sqrt{5}}\left[\frac{\frac{2}{1+\sqrt{5}}}{z\left(\frac{2}{1+\sqrt{5}}\right) + 1} - \frac{\frac{2}{1-\sqrt{5}}}{z\left(\frac{2}{1-\sqrt{5}}\right) + 1}\right]$$

$$= \frac{2}{\sqrt{5}}\left[\frac{1}{(1+\sqrt{5})}\left(\frac{1}{1 - \left(\frac{-2}{1+\sqrt{5}}\right)z}\right) - \frac{1}{(1-\sqrt{5})}\left(\frac{1}{1 - \left(\frac{-2}{1-\sqrt{5}}\right)z}\right)\right]$$

$$\therefore a_r = \frac{2}{\sqrt{5}}\left(\frac{1}{(1+\sqrt{5})}\left(\frac{-2}{1+\sqrt{5}}\right)^r - \frac{1}{(1-\sqrt{5})}\left(\frac{-2}{1-\sqrt{5}}\right)^r\right)$$

This is the solution for the given recurrence relation.

> **Note :** Example Ex. 3.6.9 has the same example, and the two solutions seen to be different. But both of them are equal. Students can verify it by simplifying them.

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Verify the solution as follows.

From the recurrence relation, we get

$$a_2 = a_0 + a_1$$

$$= 1 + 1 = 2$$

From the solution, we get

$$a_2 = \frac{2}{\sqrt{5}}\left(\frac{1}{(1+\sqrt{5})}\left(\frac{-2}{1+\sqrt{5}}\right)^2 - \frac{1}{(1-\sqrt{5})}\left(\frac{-2}{1-\sqrt{5}}\right)^2\right)$$

$$= 2$$

Ex. 4.14.5 : Solve the recurrence relation

$$a_r + 4\,a_{r-1} + 4\,a_{r-2} = 0,\ a_0 = 1,\ a_1 = 6$$

☑ **Soln. :** Given recurrence relation is

$$a_r + 4\,a_{r-1} + 4\,a_{r-2} = 0,\ a_0 = 1,\ a_1 = 6$$

Multiply by z^r and take summation from 2 to ∞

$$\therefore \sum_{r=2}^{\infty} a_r z^r - \sum_{r=2}^{\infty} 4\,a_{r-1} z^r + \sum_{r=2}^{\infty} 4\,a_{r-2} z^r = 0 \qquad \ldots(i)$$

Consider
$$\sum_{r=2}^{\infty} a_r z^r = \sum_{r=0}^{\infty} a_r z^r - a_0 - a_1 z$$
$$= A(z) - 1 - 6z \qquad \ldots(ii)$$

Consider
$$\sum_{r=2}^{\infty} 4\,a_{r-1} z^r = 4z \sum_{r=2}^{\infty} a_{r-1} z^{r-1}$$
$$= 4z \sum_{r=1}^{\infty} a_r z^r$$
$$= 4z\left(\sum_{r=0}^{\infty} a_r z^r - a_0\right)$$
$$= 4z\,A(z) - 4z \qquad \ldots(iii)$$

Consider
$$\sum_{r=2}^{\infty} 4\,a_{r-2} z^r = 4z^2 \sum_{r=2}^{\infty} a_{r-2} z^{r-2}$$
$$= 4z \sum_{r=0}^{\infty} a_r z^r$$
$$= 4z^2 A(z) \qquad \ldots(iv)$$

Substituting Equations (ii), (iii) and (iv) in Equation (i) we get

$$A(z) - 1 - 6z - 4z\,A(z) + 4z + 4z^2 A(z) = 0$$
$$\therefore A(z)(1 - 4z + 4z^2) - 1 - 2z = 0$$
$$\therefore A(z) = \frac{1+2z}{1-4z+4z^2} = \frac{1+2z}{(1-2z)^2}$$
$$= \frac{1}{(1-2z)^2} + \frac{2}{(1-2z)^2}$$
$$\therefore a_r = (r+1)\,2^r + 2r.\,2^{r-1}$$
$$= (r+1)\,2^r + r.\,2^r = 2^r(r+1+r)$$
$$= 2^r(2r+1)$$

$\therefore a_r = 2^r(2r+1)$ is the solution for the recurrence relation.

Verify the result as follows.

Using the recurrence relation,

$$a_2 = 4\,a_1 - 4\,a_2 = 4 \times 6 - 4 \times 1 = 20$$

Using the solution,

$$a_2 = 2^2(2 \times 2 + 1) = 20$$

Ex. 4.14.6 : Solve the recurrence relation

$$a_r - 4\,a_{r-1} + 4\,a_{r-2} = 0,\ \text{for } a_0 = 0,\ a_1 = 1$$

☑ **Soln. :** Given recurrence relations is

$$a_r - 4\,a_{r-1} + 4\,a_{r-2} = 0,\ a_0 = 0,\ a_1 = 1$$

Multiply by z^r and take summation from 2 to ∞

$$\therefore \sum_{r=2}^{\infty} a_r z^r - \sum_{r=2}^{\infty} 4\,a_{r-1} z^r + \sum_{r=2}^{\infty} 4\,a_{r-2} z^r = 0 \qquad \ldots(i)$$

Consider
$$\sum_{r=2}^{\infty} a_r z^r = \sum_{r=0}^{\infty} a_r z^r - a_0 - a_1 z$$
$$= A(z) - 0 - z = A(z) - z \ldots(ii)$$

Consider
$$\sum_{r=2}^{\infty} 4\,a_{r-1} z^r = 4z \sum_{r=2}^{\infty} a_{r-1} z^{r-1}$$
$$= 4z\left[\sum_{r=1}^{\infty} a_r z^r\right]$$
$$= 4z\left[\sum_{r=0}^{\infty} a_r z^r - a_0\right]$$
$$= 4z\,A(z) \qquad \ldots(iii)$$

Consider
$$\sum_{r=2}^{\infty} 4\,a_{r-2} z^r = 4z^2 \sum_{r=2}^{\infty} a_{r-2} z^{r-2}$$
$$= 4z^2 \sum_{r=0}^{\infty} a_r z^r$$
$$= 4z^2 A(z) \qquad \ldots(iv)$$

Substituting Equations (ii), (iii) and (iv) in Equation (i), we get

$$A(z) - z - 4z\,A(z) + 4z^2 A(z) = 0$$
$$\therefore A(z)(1 - 4z + 4z^2) - z = 0$$

$$\therefore A(z) = \frac{z}{1-4z+4z^2} = \frac{z}{(1-2z)^2}$$

$$\therefore a_r = r\,2^{r-1}$$

This is the solution for the given recurrence relation.

Verify the result as follows.

From the recurrence relation, we get

$$a_2 = 4a_1 - 4a_0 = 4 \times 1 - 4 \times 0 = 4$$

From the solution we get,

$$a_2 = 2 \times 2^{r-1} = 4$$

Ex. 4.14.7 : Find generating function for the sequence 1, 2, 3, 4, 5, 6.

☑ **Soln. :** Given sequence is, 1, 2, 3, 4, 5, 6,...

∴ The generating function is,

$$a(x) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z + \ldots$$

where $a_0 = 1, a_1 = 2, a_2 = 3, \ldots, a_n + n + 1$

$$= n + 1$$

$$\therefore a(x) = 1 + 2z + 3Z^2 + \ldots + nz^{n-1} + \ldots$$

$$\therefore a(z) = \frac{1}{(1-z)^2}$$

Ex. 4.14.8 : Find the ordinary generating functions for the given sequences :

(i) {0,1,2,3,4,...} (ii) {1,2,3,4,....}

(iii) {2,2,2,2,....}

☑ **Soln. :**

(i) Given set is {0, 1, 2, 3, 4, ...}

The generating function is

$$a(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

Where $a_0 = a_1 = 1, a_2 = 2, \ldots$

$$\therefore a(z) = 0 + z + 2z^2 + 3z^3 + \ldots + nz^n + \ldots$$

$$\therefore a(z) = \frac{z}{(1-z)^2}$$

(ii) Given set is {1, 2, 3, 4, 5...}

The generating function is

$$a(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

where $a_0 = 1, \quad a_1 = 1, \quad a_2 = 2, \ldots$

$$\therefore a_n = n + 1$$

$$\therefore a(z) = 1 + 2z + 3z^2 + \ldots + nz^{n-1} + \ldots$$

$$\therefore a(z) = \frac{1}{(1-z)^2}$$

Given set is {2, 2, ...}.

The generating function is

$$a(z) = a_0 + a_1 + a_2 z^2 + \ldots + a_n x^n + \ldots$$

where $a_0 = a_1 = a_2 = \ldots = a_n = 2$

$$\therefore a(z) = 2 + 2z + 2z^2 + \ldots + 2z^n$$

$$\therefore a(z) = \frac{2}{1-z}$$

(iv) Given set is {0, 0, 0, 1, 1, 1, 1, ...}

The generating function is

$$a(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

where $a_0 = 0, a_1 = 0, a_2 = 0, a_3 = 1, \ldots$

$$\therefore a(z) = 0 + 0 + 0 + z^3 + z^4 + \ldots$$

$$\therefore a(z) = \sum_{n=0}^{\infty} z^n - 3$$

$$= \frac{1}{1-z} - 3 = \frac{1-3+3z}{1-z}$$

$$= \frac{3z-2}{1-z}$$

UEx. 4.14.9 MU - Dec. 14

Solve the recurrence relation $a_{r+2} - a_{r+1} - 6a_r = 4$

☑ **Soln. :** Given recurrence relation is

$$a_{r+2} - a_{r+1} - 6a_r = 4$$

The particular solution is of the form P.

∴ Using the recurrence relation and the solution, we get,

$$a_n = P$$

and $P - P - 6P = 4$

$$\therefore P = \frac{-2}{3}$$

$$\therefore a_n = \frac{-2}{3} \text{ is the solution.}$$

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Now let us verify it

$$a_0 = \frac{-2}{3}$$

$$a_1 = \frac{-2}{3}$$

$$a_2 = \frac{-2}{3}$$

Comparing with the given recurrence relation we get

$$a_2 = 4 + a_1 + 6a_2$$

$$= 4\frac{-2}{3} - \frac{2}{3} \times 6 = \frac{-2}{3}$$

$\therefore$ The solution $a_2 = \frac{-2}{3}$ is verified.

UEx. 4.14.10 **MU - Dec. 14**

Find the generating function for the following sequence.

(i) 1, 2, 3, 4, 5, 6.... (ii)

Soln. :

(i) Refer Ex. 4.14.7

(ii) Refer Ex. 4.14.10

Replace 2 by 3

$$\therefore a(z) = \frac{3}{1-z}$$

UEx. 4.14.11 **MU - Dec. 15**

Find the generating functions for the following sequence

$6, -6, 6, -6, 6, -6, 6 - \ldots\ldots\ldots\ldots$

☑ **Soln. :** Given set is $\{6, -6, 6, -6 \ldots a_n\}$

The generating function is

$$a(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

Where $a_0 = 6, a_1 = -6, a_2 = 6, \ldots$

$$\therefore a_n = 6 \times (-1)^n$$

$$\therefore a(z) = 6 - 6z + 6z^2 - 6z^3 + \ldots + 6 \times (-1)^n z^n + \ldots$$

$$\therefore a(z) = \frac{6}{1+z}$$

UEx. 4.14.12 **MU - Dec. 16**

Find the ordinary generating functions for the given sequences :

(i) $\{0,1,2,3,4,\ldots\}$ (ii) $\{1,2,3,4,\ldots\}$

(ii) $\{0,3,3^2,3^3,\ldots\}$ (iv) $\{2,2,2,2,\ldots\}$

☑ **Soln. :**

(i), (ii) and (iv) Refer Ex. 4.14.8

(iii) Given set is $\{0, 3, 3^2, 3^3, \ldots\}$

The generating function is,

$$a(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

where $a_0 = 0, \quad a_1 = 3, a_2 = 3^2, \ldots$

$$\therefore a(z) = 0 + 3z + 3^2 z^2 + \ldots + 3^n z^n$$

$$\therefore a(z) = \sum_{n=0}^{\infty} 3^n z^n - 1$$

$$= \frac{1}{1-3z} - 1$$

$$= \frac{1 - 1 + 3z}{1-3z}$$

3, 3, 3, 3, 3...

$$= \frac{3z}{1-3z}$$

UEx. 4.14.13 **MU - Q. 4(b), Dec. 2018, 6 Marks**

Determining the generating function of the numeric function a_r where

(i) $a_r = 3^r + 4^{r+1}, \quad r \geq 0$

(ii) $a_r = 5 \quad , r \geq 0$

☑ **Soln. :**

(i) Given $a_r = 3^r + 4^{r+1}, \quad r \geq 0$

Generating function $= \frac{1}{1-3z} + \frac{z}{1-4z}$

(ii) Given $a_r = 5$

Generating function $= \frac{5}{1-z}$

UEx. 4.14.14 **MU - Q. 1(b), May 2018, 5 Marks**

Find the generating function for the following finite sequences.

(i) 2,2,2,2,2,2 (ii) 1,1,1,1,1,1

☑ **Soln. :**

(i) 2,2,2,2,2,2

The generating function for the sequence 2, 2, 2, 2, 2, 2 is as follows :

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

Here $a_i = 2, \quad 0 \leq i \leq 5$

$2 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5$ is the required generating function.

(ii) The generating function for the sequence 1, 1, 1, 1, 1, 1 is as follows :

$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$

Here $a_i = 1, \quad 0 \le i \le 5$

$\therefore \; 1 + x + x^2 + x^3 + x^4 + x^5$ is the required generating function.

UEx. 4.14.15 MU - Q. 6(b), Dec. 2017, 6 Marks

Given a generating function, find out corresponding sequence.

(i) $\dfrac{1}{3-6x}$ (ii) $\dfrac{x}{1-5x+6x^2}$

Soln. :

(i) Given generating function is $\dfrac{1}{3-6x} = \dfrac{1/3}{1-2x}$

Comparing this with standard G.F. $\dfrac{p}{1-q}$ with $a_r = pq^r$

We get the sequence as $a_r = \left(\dfrac{1}{3} \times 2^r\right)$

$\therefore$ Sequence $= a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots..$

$$= \frac{1}{3} + \frac{2}{3}x + \frac{4}{3}x^2 + \frac{8}{3}x^3 + \ldots\ldots.$$

(ii) Given generating function is

$$\frac{x}{1-5x+6x^2}$$

$$= \frac{x}{(1-2x)(1-3x)} \quad \text{\{solution is in box\}}$$

$$= \frac{1}{1-3x} - \frac{1}{1-2x}$$

$\therefore$ The sequence function is $3^r - 2^r$

$\therefore$ The sequence is,

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots..$$

$$= 1 + x + 5x^2 + 19x^3 + \ldots.$$

$$1 - 5x + 6x^2$$

$$= 1 - 2x - 3x + 6x^2$$

$$= (1-2x) - 3x(1-2x)$$

$$= (1-2x)(1-3x)$$

$$\frac{x}{(1-2x)(1-3x)} = \frac{A}{1-2x} + \frac{B}{1-3x}$$

$$= \frac{A - 3Ax + B - 2Bx}{(1-2x)(1-3x)}$$

$\therefore \; A + B = 0, \quad A = -B$

$-3A - 2B = 1, \quad 3B - 2B = 1,$

$B = 1$ and $A = -1$

Module IV

Chapter Ends...

□□□

Note

Module V

CHAPTER 5 Algebraic Structures

Syllabus

- **Algebraic structures with one binary operation :** Semi group, Monoid, Groups, Subgroups, Abelian Group, Cyclic group, Isomorphism
- **Algebraic structures with two binary operations :** Ring
- **Coding Theory :** Coding, binary information and error detection, decoding and error correction

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5.1 ALGEBRAIC SYSTEM

A system that consists of a set and one or more n-ary operations on the set, is known as algebraic system. It is represented within angular brackets, the set followed by operators separated by commas.

Ex. $< N, +, \times >$ is an algebraic system where N is the set of natural numbers, + and $\times$ and addition and multiplication operators respectively.

> **Note :** The operator in the algebraic system must satisfy the closure property. i.e. the set must be closed under the given operations.
>
> i.e. If $<A, f_1, f_2, \ldots, f_n>$ is an algebraic system where $A = \{a_1, a_2, \ldots\}$ and f_k is an operator, $1 \le k \le n$.
>
> then, $\forall\ a_i, a_j \in A,\ a_i\, f_k\, a_j \in A$ where $1 < k < n$.

Ex. $< N, - >$ cannot be an algebraic system because for any two numbers $a_1, a_2 \in N$, if $a_1 < a_2$ then $a_1 - a_2 \notin N$.

i.e. the set N is not closed under the subtraction operation.

Ex. $<I, - >$ is an algebraic system.

5.2 ALGEBRAIC STRUCTURE

> **Definition :** An **algebraic system** is said to be as algebraic structure, if along with the operators, it includes relations on set. There can be more than one set in the algebraic structure.

Here we can describe the relations as rules or properties the operators should satisfy.

5.3 PROPERTIES OF ALGEBRAIC SYSTEMS

Let us consider an algebraic system with two binary operators $\oplus$ and $\ominus$. Let the algebraic system be $< A, \oplus, \ominus >$.

Following are the properties that any algebraic system may have.

$\forall\ a, b, c, \in A.$

(i) Associativity

$$(a \oplus b) \oplus c = (a \oplus (b \oplus c))$$

$$(a \ominus b) \ominus c = a \ominus (b \ominus c)$$

(ii) Commutativity

$$a \oplus b = b \oplus a$$

$$a \ominus b = b \ominus a$$

(iii) Identity element

If there exist two elements i_1 and i_2 in A such that

$$a \oplus i_1 = i_1 \oplus a = a \quad \text{and}$$

$$a \ominus i_2 = i_2 \ominus a = a$$

then i_1 and i_2 are called identity elements with respect to operators $\oplus$ and $\ominus$.

(iv) Inverse element

If there exist two elements a_1 and a_2 with respect to an element a, such that $a, a_1, a_2 \in A$ and

$$a \oplus a_1 = a_1 \oplus a = i_1$$

$$a \ominus a_2 = a_2 \ominus a = i_2$$

where i_1 and i_2 are identity elements for operators $\oplus$ and $\ominus$respectively, then a_1 is said to be inverse element of a with respect to the operator $\oplus$ and a_2 is said to be inverse element of a with respect to the operator $\ominus$.

(v) Distributivity

$a \oplus (b \ominus c) = (a \oplus b) \ominus (a \oplus c) \rightarrow \oplus$ distributes over $\ominus$

$a \ominus (b \oplus c) = (a \ominus b) \oplus (a \ominus c) \rightarrow \ominus$ distributes over $\oplus$

(vi) Cancellation property

$$a \oplus b = a \oplus c \Rightarrow b = c$$

$$a \ominus b = a \ominus c \Rightarrow b = c$$

E.g. Consider the algebraic system $< I, +, \times >$, where I is set of integers, + is addition, operator and x is multiplication operator.

Let $a, b, c \in I$,

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Following are the properties of the algebraic system :

(i) Both + and × are associative

i.e. $a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$

(ii) Commutativity

Both + and × are commutative

i.e. $a + b = b + a$ and

$a \times b = b \times a$

(iii) Identity element

0 and 1 are identity elements for + and × respectively.

i.e. $0 + a = a + 0 = a$

$1 \times a = a \times 1 = a$

(iv) Inverse element

For an element a, – a is inverse element with respect to +.

i.e. $a + (-a) = (-a) + a = 0$

There is no inverse element for ×.

(v) Distributivity

× distributes over +

i.e. $a \times (b + c) = (a \times b) \times (a \times c)$

+ does not distribute over ×

i.e. $a + (b \times c) \neq (a + b) \times (a + c)$

(vi) Cancellation property

It is true for both + and ×

i.e. $a + b = a + c \Rightarrow b = c$ and

$a \times b = a \times c \Rightarrow b = c$

Example

Let A = {1, 2, 3, 4, 5}. Consider the algebraic system < P (A), ∪, ∩ >, where P(A) is power set of A, ∪ is union operation and ∩ is intersection operation.

Let $x, y, z \in P(A)$. i.e. $x, y, z \subseteq A$.

Following are the properties of the algebraic system :

(i) Associativity

Both ∪ and ∩ are associative.

i.e. $(x \cup y) \cup z = x \cup (y \cup z)$ and

$(x \cap y) \cap z = x \cap (y \cap z)$

(ii) Commutativity

Both ∪ and ∩ are commutative

i.e. $x \cup y = y \cup x$ and

$x \cap y = y \cap x$

(iii) Identity element

ϕ and A are identity elements of ∪ and ∩ respectively.

i.e.

$x \cup \phi = \phi \cup x = x$ and

$x \cap A = A \cap x = x \quad \because x \subseteq A.$

(iv) Inverse element

There is no inverse element for both ∪ and ∩.

(v) Distributivity

Both ∪ and ∩ distribute over other.

i.e. $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$ and

$x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$

(vi) Cancellation property

It is not true for both ∪ and ∩

For Example let x = {1, 2, 3}, y = {2, 3, 4}, z = {1, 4}

$x \cup y = \{1, 2, 3, 4\}$

$x \cup z = \{1, 2, 3, 4\}$

$x \cup y = x \cup z \not\Rightarrow y = z$

For Example let x = {1, 2, 3}, y = {3, 4}, z = {3, 5}

$x \cap y = \{3\}$

$x \cap z = \{3\}$

$\therefore x \cap y = x \cap z \not\Rightarrow y = z$

Syllabus Topic : Algebraic Structures with One Binary Operation

5.4 ALGEBRAIC STRUCTURES WITH SINGLE OPERATOR

This is the simplest algebraic system, which has only one operator. The algebraic system may have five properties. i.e. all properties except distributivity. If the set is finite, then the operation is represented in the form of a table.

Example : Let A = {00, 01, 10, 11} and the operation is ⊕ (exclusive – or). The algebraic system < A, ⊕ > is represented as follows.

⊕	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

It can be noted that ⊕ is associative, commutative. The identity element is 00. The inverse element for any element is it self.

i.e. $\forall x \in A$, $x \oplus x = 00$ the cancellation property is true for ⊕.

Syllabus Topic : Semigroups

5.4.1 Semigroups

Definition : An algebraic structure <A, ∘> where ∘ is a binary operator is called semigroup, if ∘ is associative.

i.e. $\forall x, y, z \in A$, we have

$(x \circ y) \circ z = x \circ (y \circ z)$

Syllabus Topic : Monoids

5.4.2 Monoids

Definition : An algebraic structure < A, ∘ > where ∘ is a binary operator is called monoid, if ∘ is associative and there exists an identity element

i.e. $\forall x \in A$, we have

$(x \circ e) = e \circ x = x$

Note : Every monoid is a semigroup.

Syllabus Topic : Groups

5.4.3 Groups

UQ. 5.4.1 Explain the terms : group

MU - Dec. 17, 5 Marks

Definition : An algebraic structure < A, ∘ > where ∘ is binary operator is called a group. If ∘ is a associative – ve, there exists an identity element and every element in A has an inverse element e.

i.e. $\forall a \in A$ we have an element $a^{-1} \in A$

such that

$a \circ a^{-1} = a^{-1} \circ a = e$ where e is the identity element.

The inverse element of a is represented as a^{-1}.

Note : Every group is also a semigroup and monoid.

5.4.4 Abelian Group

Definition : An algebraic structure < A, ∘ > where ∘ is a binary operator is called an abelian group. If ∘ is associative and commutative and there exists an identity element and every element in A has an inverse element.

i.e. $\forall x, y, z \in A$, we have

$(x \circ y) \circ z = x \circ (y \circ z)$

i.e. $\forall x \in A$, we have

$(x \circ e) = e \circ x = x$

i.e. $\forall a \in A$ we have an element $a^{-1} \in A$

such that

i.e. $\forall a, b \in A$ we have

$a \circ b = b \circ a$

Note : Every abelian group is also a semigroup, a monoid and also a group.

5.5 SOLVED EXAMPLE

Ex. 5.5.1 : Let A = {0, 1, 2, 3, 4, 5} and τ is an unary operator on A, is given by,

$\tau(x) = (x + 1) \bmod 6$

Discuss the properties of $< A, \tau >$

Soln. :

The algebraic system is $< A, \tau >$. The unary operator τ is given as follows :

x	τ (x)
0	1
1	2
2	3
3	4
4	5
5	0

Graphical representation

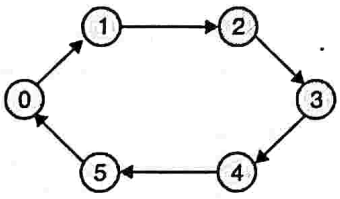

Fig. Ex. 5.5.1

As it can be seen in the graphical representation of the operator τ, the algebraic system is called a "clock algebra".

Ex. 5.5.2 : Show that the set N of natural numbers is a semigroup under the operation

$x * y = \max\{x, y\}$. Is it a monoid ?

Soln. :

Given algebraic system is $< N, * >$ where $*$ is given by, $x * y = \max\{x, y\}$.

Now consider $\forall x, y, z \in N$,

$$(x * y) * z = \max\{\max\{x, y\}, z\}$$
$$= \max\{x, y, z\}$$
$$\text{and } x * (y * z) = \max\{x, \max\{y, z\}\}$$
$$= \max\{x, y, z\}$$
$$\therefore x * (y * z) = (x * y) * z$$

$\therefore$ $*$ is associative.

$\therefore$ $< N, x >$ is a semigroup.

Let us assume that there exists an identity element e in N such that,

$\forall x \in N$, we have

$x * e = e * x = x$

i.e. $\max\{x, e\} = \max\{e, x\} = x$.

$\therefore$ e must be the minimum number among N.

$\therefore$ $e = 1$.

i.e. $\max\{x, 1\} = \max\{1, x\} = x$

$\therefore$ $x * 1 = 1 * x = x$

$\therefore$ $< N, * >$ has identity element.

$\therefore$ $< N, * >$ is a monoid.

Ex. 5.5.3 : If A = {00, 01, 10, 11} then discuss the properties of the algebraic system $< A, \wedge >$ where $\wedge$ is bitwise and operation.

Soln. :

Given A = {00, 01, 10, 11}

The algebraic system is $< A, \wedge >$ where $\wedge$ is bitwise and operation.

The following table represents the algebraic system.

$\wedge$	00	01	10	11
00	00	00	00	00
01	00	01	00	01
10	00	00	10	10
11	00	01	10	11

$\wedge$ operator is associative.

Consider the last row or last column in the table. We have the same column or row as the input, respectively.

i.e. $\forall x \in A$, we have

$x \wedge 11 = 11 \wedge x = x$

$\therefore$ 11 is the identity element.

It can be observed in the table that the first 3 rows do not have 11 in them. This indicates that, there is no inverse element.

$\therefore$ $< A, \wedge >$ is a monoid.

Ex. 5.5.4 : Prove that $< A, \circ >$ is an abelian group, where $A = \{f_0, f_1, f_2, f_3\}$ and $\circ$ is composition operation, f_0, f_1, f_2, f_3 are functions on set $X = \{1, 2, 3, 4\}$ given by,

$f_0 = \{<1, 2>, <2, 3>, <3, 4>, <4, 1>\}$

$f_1 = \{<1, 3>, <2, 4>, <3, 1>, <4, 2>\}$

$f_2 = \{<1, 4>, <2, 1>, <3, 2>, <4, 3>\}$

$f_3 = \{<1, 1>, <2, 2>, <3, 3>, <4, 4>\}$

☑ **Soln. :**

Given : $X = \{1, 2, 3, 4\}$

$f_i : X \to X \quad \text{for } 0 \le i \le 3$

$f_0 = \{<1, 2>, <2, 3>, <3, 4>, <4, 1>\}$

$f_1 = \{<1, 3>, <2, 4>, <3, 1>, <4, 2>\}$

$f_2 = \{<1, 4>, <2, 1>, <3, 2>, <4, 3>\}$

$f_3 = \{<1, 1>, <2, 2>, <3, 3>, <4, 4>\}$

$A = \{f_0, f_1, f_2, f_3\}$

The algebraic system is $< A, \circ >$ where $\circ$ is composition operation.

The algebraic system is represented as follows,

$\circ$	f_0	f_1	f_2	f_3
f_0	f_1	f_2	f_3	f_0
f_1	f_2	f_3	f_0	f_1
f_2	f_3	f_0	f_1	f_2
f_3	f_0	f_1	f_2	f_3

The operation $\circ$ is associative.

By observing the table we understand that f_3 is the identity element.

$f_0^{-1} = f_2,\ f_1^{-1} = f_1,\ f_2^{-1} = f_0,\ f_3^{-1} = f_3$

Though the operation $\circ$ is not commutative in general, in this example, it behaves commutative.

$\therefore$ All properties are satisfied.

$\therefore$ $< A, \circ >$ is an abelian group.

Ex. 5.5.5 : Let $A = \{0, 1, 2, 3\}$ and $< A, * >$ be an algebraic system where $\forall\, a, b, \in A, a * b = (a + b) \bmod 4$. Find $\forall\, a \in A,\ a^2, a^3, a^4$.

☑ **Soln. :**

Given : $A = \{0, 1, 2, 3\}$

$\forall\, a, b \in A, a * b = (a + b) \bmod 4.$

We know that $a^n = a * a * \ldots * a$ n times

The following, table gives a^2, a^3, a^4 for all $a \in A$.

a	a^2	a^3	a^4
0	0	0	0
1	2	3	0
2	0	2	0
3	2	1	0

Ex. 5.5.6 : If $< G, * >$ is an abelian group, then $a, b \in G$, show that $(a * b)^n = a^n * b^n$.

☑ **Soln. :**

Given : $< G, * >$ is an abelian group.

$\therefore$ $*$ is associative and commutative.

Let e be the identity element.

To prove : $(a * b)^n = a^n * b^n$

$$\text{LHS} = (a * b)^n$$

$$= (a * b) * (a * b) * \ldots * (a * b) \quad \text{n times}$$

$$= a * (b * a) * b \times \ldots * (b * a) * b \quad \text{n times}$$

($\because$ $*$ is associative)

$$= a * (a * b) * b * \ldots * (a * b) * b \quad \text{n times}$$

($\because$ $*$ is commutative)

$$= (a * a) * (b * b) * \ldots * (b * b) \quad \text{n times}$$

($\because$ $*$ is associative)

Repeating this n times we get

$$= a * a * \ldots * a(\text{n times}) * b * b * \ldots * b(\text{n times})$$

$$= a^n * b^n = \text{RHS}$$ **...Hence the proof.**

Ex. 5.5.7 : Show that in a group $< G, * >$, if for $a, b \in G$, $(a * b)^2 = a^2 * b^2$ then $< G, * >$ must be abelian.

☑ **Soln. :**

Given $< G, * >$ is a group and

$\forall\, a, b \in G, (a * b)^2 = a^2 * b^2$

$\therefore$ $*$ is associative.

Let e is be the identity element and a^{-1} is inverse of a.

Module V

Consider $(a * b)^2 = a^2 * b^2$

$\therefore \quad (a * b) * (a * b) = (a * a) * (b * b)$

$\therefore \quad a * (b * a) * b = a * (a * b) * b$

($\because$ * is associative)

$\therefore$ Include b^{-1} at the end on both sides.

$a * (b * a) * b * b^{-1} = a * (a * b) * b * b^{-1}$

$\therefore \quad a * (b * a) * e = a * (a * b) * e$

$\therefore \quad a * (b * a) = a * (a * b)$

Include a^{-1} at the beginning on both sides

$\therefore \quad a^{-1} * a * (b * a) = a^{-1} * a * (a * b)$

$\therefore \quad e * (b * a) = e * (a * b)$

$\therefore \quad b * a = a * b$

$\therefore$ * is commutative

$\therefore$ $< G, * >$ is an abelian group

UEx. 5.5.8 (MU - Dec. 2013, 4 Marks)

Show that if every element in a group is its own inverse, then the group must be abelian.

Soln. :

Given : an algebraic system $< G, * >$ is a group and $\forall\, a \in G,\ a^{-1} = a$.

$\therefore$ * is associative,

Let e be the identity element

$\forall\, a \in G$, we have $a^{-1} = a$

$\therefore \quad a * a^{-1} = e$

$\therefore \quad a * a = e$

Consider $a * b = a * b$

$\therefore \quad e * (a * b) = (a * b) * e$

$\therefore \quad (b * b) * (a * b) = (a * b) * (a * a)$

$\because \quad a * a = b * b = e.$

$\therefore \quad (b * b) * (a * b) = a * (b * a) * a$

$\because$ * is associative

Include a at the beginnings of both sides.

$\therefore \quad a * (b * b) * (a * b) = a * a * (b * a) * a$

$\therefore \quad a * (b * b) * (a * b) = e * (b * a) * a$

$a * (b * b) * (a * b) = (b * a) * a$

Include a at the end of both side

$\therefore \quad a * (b * b) * (a * b) * a = (b * a) * a * a$

$\therefore \quad a * (b * b) * (a * b) * a = (b * a) * e$

$\therefore \quad (a * b) * (b * a) * (b * a) = b * a \quad \because$ * is associative

Let $b * a = c$

$\therefore \quad (a * b) * c * c = b * a$

$\therefore \quad (a * b) * e = b * a$

$\therefore \quad a * b = b * a$

$\therefore$ * is Commutative

$\therefore$ $< G, * >$ is an abelian group

Hence the proof.

Ex. 5.5.9 : Let $< A, * >$ be an algebraic system where * is a binary operation such that, $\forall\, a, b \in A$, $a * b = a$. Check whether * is (i) associative (ii) commutative

Soln. :

Given : $< A, * >$ is an algebraic system

$\forall\, a, b \in A, \quad a * b = a$

(i) For * to be associative, we must prove

$a * (b * c) = (a * b) * c$

Consider

$LHS = a * (b * c) = a * b$

$= a$

$RHS = (a * b) * c = a * c$

$= a$

$\therefore \quad LHS = RHS$

$\therefore$ * is associative

(ii) For * to be commutative, we must prove that $a * b = b * a$

Consider $LHS = a * b = a$

$RHS = b * a = b$

$\therefore \quad LHS \neq RHS$

$\therefore$ * is not commutative

Ex. 5.5.10 : Let < A, * > be an algebraic system such that,

$\forall$ a, b, c, d $\in$ A,

a * a = a and (a * b) * (c * d) = (a * c) * (b * d)

Prove that

a * (b * c) = (a * b) * (a * c)

☑ Soln. :

Given : < A, * > is an algebraic system

$\forall$ a, b, c, d $\in$ A, we have

$$a * a = a \quad \ldots(I)$$

$$(a * b) * (c * d) = (a * c) * (b * d) \quad \ldots(II)$$

We have to prove

$$a * (b * c) = (a * b) * (a * c)$$

Consider LHS = a * (b * c)

= (a * a) * (b * c) ...By Equation (I)

= (a * b) * (a * c) ...By Equation (II)

= RHS

Hence the proof.

Ex. 5.5.11 : Let < A,* > be a semi group. Let 'a' be an element in A. Consider a binary operation □ on A, such that

$\forall$ x, y $\in$ A, x □ y = x * a * y

Show that □ is associative

☑ Soln. :

Given : < A,* > is semigroup

$\therefore$ * is associative

i.e. (a * b) * c = a * (b * c)

□ is a binary operation on A, such that

$\forall$ x, y $\in$ A, x □ y = x * a * y where a $\in$ A

We have to prove that □ is associative

i.e. (x □ y) □ z = x □ (y □ z)

Consider LHS

(x □ y) □ z

= (x * a * y) □ z = (x * a * y) * a * z

RHS = x □ (y □ z) = x □ (y * a * z)

= x * a * (y * a * z)

= (x * a * y) * a * z $\because$ * is associative

= LHS

$\therefore$ □ is associative.

Hence the proof.

Ex. 5.5.12 : Let < A,* > be a semigroup. Show that for a, b, c in A, if a * c = c * a and b * c = c * b, then (a * b) * c = c * (a * b)

☑ Soln. :

Given : < A,* > is a semigroup

$\therefore$ * is associative

i.e. $\forall$ a, b, c $\in$ A (a * b) * c = a * (b * c)

we have to prove that if a * c = c * a and b * c = c * b then (a * b) * c = c * (a * b).

Consider a * c = c * a

Include b on both sides at end

$\therefore$ (a * c) * b = (c * a) * b

$\therefore$ a * (c * b) = c * (a * b) $\because$ * is associative

$\therefore$ a * (b * c) = c * (a * b) $\because$ c * b = b * c

$\therefore$ (a * b)* c = c * (a * b) $\because$ * is associative

Hence the proof.

Ex. 5.5.13 : Let < {a, b}, * > be a semigroup where a * a = b show that a * b = b * a

☑ Soln. :

Given < {a, b}, * > is a semigroup

$\therefore$ * is associative

Given that a * a = b, we have to prove

$$a * b = b * a$$

Now consider a * a = b

Include a on both sides at end

$\therefore$ (a * a) * a = b * a

$\therefore$ a * (a * a) = b * a $\because$ * is associative

$\therefore$ a * b = b * a $\because$ a * a = b

Hence the proof.

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Ex. 5.5.14 : If < A, * > is a group, then show that $(a * b)^{-1} = b^{-1} * a^{-1}$.

Soln. :

Given : < A, * > is a group

$\therefore$ * is associative

Let e be the identity element. $\therefore$ $a * e = e * a = a$.

For an element a, a^{-1} is the inverse element

$$\therefore \quad a * a^{-1} = a^{-1} * a = e$$

We have to prove that $(a * b)^{-1} = b^{-1} * a^{-1}$

Consider

$$(a * b) * (a * b)^{-1} = e \qquad \because \text{ identity element}$$

Include a^{-1} on both sides at the beginning

$$\therefore \quad a^{-1} * (a * b) * (a * b)^{-1} = a^{-1} * e$$

$$(a^{-1} * a) * b * (a * b)^{-1} = a^{-1} \qquad \because \text{ * is associative}$$

$$\therefore \quad e * b * (a * b)^{-1} = a^{-1}$$

$$\therefore \quad b * (a * b)^{-1} = a^{-1}$$

Include b^{-1} on both sides at the beginning

$$\therefore \quad b^{-1} * b * (a * b)^{-1} = b^{-1} * a^{-1}$$

$$\therefore \quad e * (a * b)^{-1} = b^{-1} * a^{-1}$$

$$\therefore \quad (a * b)^{-1} = b^{-1} * a^{-1}$$

Hence the proof.

Ex. 5.5.15 : (MU - Dec. 2013, 6 Marks)

Let G be the set of rational numbers other than 1. Let define an operation on G by $a * b = a + b - ab$ for all $a, b \in G$. Prove that (G, *) is a group.

Soln. :

Given : G is set of rational numbers except 1. * is the operation on G, given by,

$$\forall a, b \in G, \quad a * b = a + b - ab$$

To prove : (G, *) is a group

(G, *) is a group if

(i) * is closed under G and associative.

(ii) G has an identity element w.r.t *

(iii) $\forall a \in G$, there exists an inverse, $a^{-1} \in G$,

(i) Consider, $\forall a, b \in G$.

$$a * b = a + b - ab$$

as a, b are rational, $a + b - ab$ is also rational

$\therefore$ * is closed under G

Now check for associativity – $(a * b) * c = a * (b * c)$

$$\begin{aligned}
\text{LHS} &= (a * b) * c \\
&= (a + b - ab) * c \\
&= (a + b - ab) + c - (a + b - ab)\, c \\
&= a + b - ab + c - ac - bc + abc \\
&= a + b + c - ab - ac - bc + abc
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= a * (b * c) \\
&= a + (b * c) - a\,(b * c) \\
&= a + b + c - bc - a\,(b + c - bc) \\
&= a + b + c - bc - ab - ac + abc \\
&= a + b + c - ab - ac - bc + abc
\end{aligned}$$

$\therefore$ LHS = RHS

* is associative.

(ii) Let e be the identity element.

i.e. $e \in G$ and $\forall a \in G$, $a * e = e * a = a$

$$\therefore \quad a * e = a + e - ae = a$$

$$\therefore \quad e - ae = 0$$

$$\therefore \quad e = ae$$

$$\therefore \quad e\,(1 - a) = 0$$

$\therefore$ Either $e = 0$ or $1 - a = 0$

$1 - a = 0$ if $a = 1$

But $1 \notin G$

$\therefore$ $e = 0 \in G$

$\therefore$ The identity element = 0

(iii) Let $a^{-1} \in G$ be the inverse element of $a \in G$

$$\therefore \quad a * a^{-1} = a^{-1} * a = e = 0$$

$$\therefore \quad a + a^{-1} - a \cdot a^{-1} = 0$$

$$\therefore \quad a^{-1} - aa^{-1} = -a$$

$\therefore \quad aa^{-1} - a^{-1} = a$

$\therefore \quad a^{-1}(a-1) = a$

$\therefore \quad a^{-1} = \dfrac{a}{a-1} \in G$

$\therefore$ For every element a in G, there exists an inverse element $a^{-1} = \dfrac{a}{a-1}$.

UEx. 5.5.16 (MU - Dec. 2013, 5 Marks)

If (G, .) is an abelian group, then for all a, b ∈ G, prove that by mathematical induction $(a.b)^n = a^n.b^n$.

Soln. :

Given : (G, *) is an abelian group.

To prove : $(a * b)^n = a^n * b^n$ using mathematical induction.

Proof : (G, *) is an abelian group.

$\therefore$ * is associative,

i.e. $\forall$ a, b, c ∈ G, $(a * b) * c = a * (b * c)$

There exist e ∈ G an identity element.

i.e. $a * e = e * a = a$

$\forall$ a ∈ G, there exists an inverse element a^{-1} such that, $a * a^{-1} = a^{-1} * a = e$

* is commutative.

i.e. $\forall$ a, b ∈ G. $a * b = b * a$

Proof by Mathematical Induction :

Let P(n) be the statement, $(a * b)^n = a^n * b^n$

Basis step : Let n = 1

$(a * b)^1 = a * b = a^1 * b^1$

$\therefore$ P (1) is true

Induction step : Assume that P (n) is true for n = k.

i.e. $(a * b)^k = a^k * b^k$

Induction hypothesis :

We know that $(a * b)^k = a^k * b^k$

$\therefore \quad (a * b)^k * (a * b) = (a^k * b^k) * (a * b)$

$\therefore \quad (a * b)^{k+1} = (a^k * b^k * a) * b$ {Associative}

$= (a^k * a * b^k) * b$ {Commutative}

$= a^{k+1} * (b^k * b)$ {Associative}

$= a^{k+1} * b^{k+1}$

$\therefore$ P (k + 1) is true.

$\therefore$ As per the principle of mathematical induction, if P(k) is true then P(k + 1) is also true.

$\therefore$ P(n) is true for all n ≥ 1

$\therefore \quad (a * b)^n = a^n * b^n$

UEx. 5.5.17 (MU - Dec. 2013, 4 Marks)

If f is a homorphism from a commutative group (S, .) to another group (T, .'), then prove that (T, .') is also commutative.

Soln. :

Given : (S, *) is commutative group. (T, *') is a group.

f is homomorphism from (S, *) to (T, *').

To prove : (T, *') is commutative

Proof : (S, *) is commutative group

$\therefore \quad \forall$ a, b ∈ S, $\quad a * b = b * a$

There exists a homomorphism f from (S, *) to (T, *').

$\therefore \quad \forall$ x, y ∈ T, we have a, b ∈ S, such that.

$f(a) = x \qquad f(b) = y$

Now consider

$x *' y = f(a) *' f(b)$

$= f(a * b)$

$= f(b * a)$ {Commutative}

$= f(b) *' f(a)$

$= y *' x$

$\therefore$ *' is commutative

$\therefore$ (T, *') is commutative.

UEx. 5.5.18 (MU - May 2014, 6 Marks)

Prove that the set G = {1, 2, 3, 4, 5, 6} is an abelian group under multiplication modulo 7.

Soln. :

To prove :

To set G = {1, 2, 3, 4, 5, 6} is a finite Abelian group of order 6, with respect to multiplication modulo 7.

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Proof

Let the operator X_7 denotes multiplication modulo 7.

$\therefore$ $\forall$ a, b $\in$ G, we have

$a\, x_7\, b = a \cdot b \bmod 7$

the following table represents x_7 on G.

x_7	1	2	3	4	5	6	
1	1	2	3	4	5	6	$2 \times 1 \bmod 7 = 2$
2	2	4	6	1	3	5	$2 \times 2 \bmod 7 = 4$
3	3	6	2	5	1	4	$2 \times 3 \bmod 7 = 6$
4	4	1	5	2	6	3	$2 \times 4 \bmod 7 = 1$
5	5	3	1	6	4	2	$2 \times 5 \bmod 7 = 3$
6	6	5	4	3	2	1	$2 \times 6 \bmod 7 = 5$

2nd row

And G is abelian group if G is closed under X_7

(i) X_7 is associative

(ii) There exists an identity element in G.

(iii) There exists an inverse element for all a $\in$ G.

(iv) X_7 is commutative.

All entries in the above table are from G.

$\therefore$ G is closed under x_7.

(i) X_7 is associative if $a\, X_7\, (b\, X_7\, C) = (a\, X_7\, b)\, X_7\, C$

$$\begin{aligned} LHS &= a\, X_7\, (b\, X_7\, C) \\ &= a\, X_7\, (b.c \bmod 7) \\ &= a.\, b.\, c \bmod 7 \\ RHS &= (a\, X_7\, b)\, X_7\, C \\ &= (a\, .\, b \bmod 7)\, X_7\, C \\ &= a.\, b.\, c \bmod 7 \end{aligned}$$

$\therefore$ RHS = LHS

$\therefore$ X_7 is associative

(ii) Let e $\in$ G be an identity element.

$\therefore$ $a *_7 e = e *_7 a = a$

$\therefore$ $a\, X_7\, e = a.e \bmod 7 = a$

From the given table it is clear that e = 1

(iii) $\forall$ a $\in$ G, if there exists as element a^{-1} then

$a\, X_7\, a^{-1} = a^{-1}\, X_7\, a = e = 1$

Search for 1 in the above table.

It exists exactly once in all the rows and columns.

$\therefore$ $1^{-1} = 1, 2^{-1} = 4, 3^{-1} = 5, 4^{-1} = 2, 5^{-1} = 3, 6^{-1} = 6$

(iv) $\forall$ a, b $\in$ G, we can write,

$$\begin{aligned} a\, X_7\, b &= a\, b \bmod 7 \\ &= b.a \bmod 7 \\ &= b\, X_7\, a \end{aligned}$$

$\therefore$ X_7 is commutative.

G is a finite set and has 6 elements in it Hence G is a finite abelian group of order 6, with respect to multiplication modulo 7.

UEx. 5.5.19 : (MU - May 2014, 6 Marks)

Let G be the set of real numbers and let G be the set of real numbers and let $a.\, b = \frac{ab}{2}$. Show that (G,) is a abelian group.

Soln. :

Given

G is set of all nonzero real numbers.

The operator $*$ is given by, $a * b = \frac{ab}{2}$

To prove

(G, $*$) is as abelian group

Proof

(G, $*$) is as abelian group, if , G is closed under $*$ and

(i) $*$ is associative

(ii) There exists and identity element in G.

(iii) There exists an inverse element for all a $\in$ G.

(iv) $*$ is commutative.

Consider $\forall$ a, b $\in$ G,

$$a * b = \frac{ab}{2} \in G$$

$\therefore$ G is closed under $*$.

(i) A, b, c ∈ G,

Consider (a * b) * c

$$= \left(\frac{ab}{2}\right) * c$$

$$= \frac{abc}{4}$$

Consider a * (b * c)

$$= a * \left(\frac{ab}{2}\right)$$

$$= \frac{abc}{2}$$

∴ (a * b) * c = a * (b * c)

∴ * is associative

(ii) Let e be the identity element.

∴ a * e = e * a = a

$$\therefore \frac{ae}{2} = a$$

$$\therefore e = 2$$

(iii) ∀ a ∈ G, let a^{-1} be the inverse element

$$\therefore a * a^{-1} = a^{-1} * a = e$$

$$\therefore \frac{aa^{-1}}{2} = 2$$

$$\therefore a^{-1} = \frac{4}{a}$$

(iv) Consider $a * b = \frac{ab}{2}$

$$\therefore a * b = b \times a$$

∴ * is commutative.

∴ (G, *) is an abelian group.

UEx. 5.5.20 : (MU - Q. 3(a), May 2018, 8 Marks)

Prove that set G = {1, 2, 3, 4, 5, 6} is a finite abelian group of order 6 with respect to multiplication module 7.

☑ Soln. :

Given set G = {1, 2, 3, 4, 5, 6}

The operation is multiplication module i.e. X_7.

The table is as follows :

X_7	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

$a X_7 b = (a \times b) \bmod 7$

Following are the characteristics of the abelian group.

1. **Associative :** Multiplication module 7 is associative.
2. **Identity element :** 1 is the identity element.
3. **Inverse element :** Every element in set G has inverse element as follows :

$1^{-1} = 1,\ 2^{-1} = 4,\ 3^{-1} = 5,\ 4^{-1} = 2,\ 5^{-1} = 3,\ 6^{-1} = 6$

4. Commutative-Multiplication module 7 is commutative.

∴ Given set G is an abelian group.

UEx. 5.5.21 : (MU - Q. 5(a), May 2018, 8 Marks)

Let G be a set of rational numbers other than 1. Let * be an operation on G defined by a * b = a + b – ab for all a, b ∈ G. Prove that (G, *) is a group.

☑ Soln. :

Given : G is set of rational numbers other than 1.

* is the operation given by,

a * b = a + b – ab for all a, b ∈ G.

To prove : (G, *) is a group.

☞ Proof

To be a group, (G, *) must be anociative, it must have as identity element and every element should have an inverse.

(i) Associative

Let a, b, c ∈ G

Consider (a * b) * c

= (a + b – ab) * c

= a + b – ab + c – (a + b – ab) · c

= a + b + c – ab – ac – bc + abc

Consider $a * (b * c)$

$$= a * (b + c - bc)$$

$$= a + b + c - bc - a(b + c - bc)$$

$$= a + b + c - ab - bc - ac + abc$$

$\therefore$ $(a * b) * c = a * (b * c)$

$\therefore$ $*$ is associative.

(ii) Identity element

Let e be the identity element.

$\therefore$ $a * e = e * a = a$

$\therefore$ $a + e - ae = a$

$\therefore$ $e(1 - a) = 0$

$\therefore$ $e = 0$

$\therefore$ Identity element is 0.

(iii) Inverse element

Let a^{-1} be inverse of a.

$\therefore$ $a * a^{-1} = a^{-1} * a = e$

$\therefore$ $a * a^{-1} = e = 0$

$\therefore$ $a + a^{-1} - a \cdot a^{-1} = 0$

$\therefore$ $a + a^{-1}(1 - a) = 0$

$$a^{-1} = \frac{-a}{1 - a}$$

$\therefore$ (G, *) is a group.

UEx. 5.5.22 : (MU - Q. 6(a), Dec. 2017, 6 Marks)

Prove that the set {1, 2, 3, 4, 5, 6} is group under multiplication modulo 7.

Soln. :

Given set G = {1, 2, 3, 4, 5, 6}

The operation is multiplication module i.e. X_7.

The table is as follows :

X_7	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

$a X_7 b = (a \times b) \bmod 7$

Following are the characteristics of the abelian group.

1. **Associative :** Multiplication module 7 is associative.
2. **Identity element :** 1 is the identity element.
3. **Inverse element :** Every element in set G has inverse element as follows :

$1^{-1} = 1,\ 2^{-1} = 4,\ 3^{-1} = 5,\ 4^{-1} = 2,\ 5^{-1} = 3,\ 6^{-1} = 6$

4. Commutative-Multiplication module 7 is commutative.

$\therefore$ Given set G is an abelian group.

5.6 HOMOMORPHISM

Given two groups (G,*) and (H,+), a function from (G,*) to (H,+) is given by h:G->H. the function h is called homomorphism if for all u and v in G we have

h(u*v)=h(u)+h(v).

5.7 ISOMORPHISM

If a homomorphism is bijective, (one-to-one and onto), then it is called isomorphism. In this case the reverse function is also homomorphism. Thus we have for two groups (G,*) and (H,+), the function i:G->H is isomorphism, if

for all u and v in G

$i(u*v) = i(u) + i(v)$

and for all x and y in H

$i^{-1}(x+y) = i^{-1}(x)*i^{-1}(y)$

5.8 AUTOMORPHISM

Automorphism is a function from a group to itself and it is isomorphism. For a group (G,*), a function a:G->G is called automorphism, if

for all u and v in G

a(u*v) = a(u)*a(v)

and

$a^{-1}(u*v) = a^{-1}(u)*a^{-1}(v)$

Syllabus Topic : Free and Cyclic Monoids and Groups

5.9 FREE AND CYCLIC MONOIDS AND GROUPS

There are special types of monoids and groups. We define them as follows.

5.9.1 Free Monoid

A free monoid is a special monoid that, involves strings i.e. It is represented as < S, · > where S is set of alphabets and '·' is concatenation operation '.' operation is associative, and the identity element is ∧, i.e. null or empty string.

w · ∧ = ∧ · w = w, where w is a string, made up of alphabets in S.

Example : Consider a monoid < {a, b, n}, · >,

This monoid consists of the sequences like,

∧, a, b, ab, ba, abb, babb, baabb etc.

A monoid is said to be free if it is isomorphic to the known free monoid.

Example

The monoid < {1}, + > is a free monoid. It consists of all sequences like "1", "1 + 1", "1 + 1 + 1 + 1" etc.

5.9.2 Cyclic Monoid

A moniod < S, * > is said to be cyclic, if there exists an elements a ∈ S, such that every element in S can be written as some powers of a.

i.e. ∀ x ∈ S, we have $x = a^k$ where k ∈ N, set of natural numbers.

The cyclic monoid is said to be generated by a.

Example : Consider the algebraic system < A, x > where

A = {1, 2, 4, 8...} and x is multiplication operation x is associative, and 1 is the identity element.

∴ < A, x > is a monoid.

Now consider the element 2 ∈ A,

We can write,

$1 = 2^0,\ 2 = 2^1,\ 4 = 2^2,\ 8 = 2^3 \ldots$

i.e. $1, 2, 4 = 2 \times 2,\ 8 = 2 \times 2 \times 2 \ldots$

∴ < A, x > is a cyclic monoid, generated by 2. This is an example of infinite cyclic monoid.

> **Note :** The algebraic system in Ex. 6.1.1 is an example of finite cyclic monoid. It is generated by 0.

5.9.3 Free Group

Free group is a free monoid, in which every element in the set has an inverse element.

5.9.4 Cyclic Group

Cyclic group is a cyclic monoid, in which every element in the set has an inverse element.

Syllabus Topic : Permutation Groups

5.10 PERMUTATION GROUPS

> **Definition :** A one-to-one function from a set A on to itself is called a permutation.

i.e. f : A → A is a permutation if f is one-to-one

Example : Let A = {a, b, c, d} and f be a function f : A → A given by,

f(a) = c, f(b) = d, f(c) = a, f(d) = b

∴ f is a permutation.

It is represented as

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$$\pi = \begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}$$

If a set has n elements, then there are n! permutations.

Example : Let A = {a, b, c} Here n = 3

∴ There are 3! i.e. 6 permutations as follows

$$\pi_1 = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} \quad \pi_2 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} \quad \pi_3 = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$$

$$\pi_4 = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \quad \pi_5 = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix} \quad \pi_6 = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$$

Consider on operator ∘, Composition of two functions, for a set A, let S be the set of all permutations. The algebraic system < S, ∘ > has the following properties. ∘ is associative.

The identity element is f(a) = a for all a ∈ A.

Every element in S has an inverse element.

∴ The algebraic system < S, ∘ > is a group.

This group is known as permutation group.

> **Definition :** A group < S, ∘ > is known as permutation group, if S = {$\pi_1, \pi_2, \ldots \pi_k$} where π_i is a permutation on a set S and O is composition operation. It is not necessary that the number of permutations must be always n!.

Example : Consider the algebraic system,

< {$\pi_1, \pi_2, \pi_3, \pi_4$}, ∘ > where

$$\pi_1 = \begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix} \pi_2 = \begin{pmatrix} a & b & c & d \\ b & a & c & d \end{pmatrix}.$$

$$\pi_3 = \begin{pmatrix} a & b & c & d \\ a & b & d & c \end{pmatrix} \pi_4 = \begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix}$$

The operator ∘ can be represented as,

∘	π_1	π_2	π_3	π_4
π_1	π_1	π_2	π_3	π_4
π_2	π_2	π_1	π_4	π_3
π_3	π_3	π_4	π_1	π_2
π_4	π_4	π_3	π_2	π_1

The identity element in π_1.

$$\pi_1^{-1} = \pi_1, \quad \pi_2^{-1} = \pi_2, \quad \pi_3^{-1} = \pi_3, \quad \pi_4^{-1} = \pi_4$$

∴ The given algebraic system is a permutation group.

Syllabus Topic : Substructures

5.11 SUB STRUCTURES

Till now we have seen different types of algebraic systems. Each of there have subsystems or substructures. They are defined as follows.

5.11.1 Sub Semigroup

> **Definition :** Let < S, * > be a semigroup. An algebraic system < T, *> is called subsemigroup of < S,*>, if T ⊆ S and the set T is closed under the operation *.

Example : < I^+, + > is a semigroup, where

I^+ is set of positive integens and + is addition operation.

The algebraic system, < E, + > is subsemigroup of < I^+, + > where

E = {2, 4, 6, 8, ...} set of even numbers

as (i) E is closed under operator +

i.e. sum of even numbers is even

(ii) E ⊆ I^+

5.11.2 Submonoid

> **Definition :** Let < S, * > be a monoid with identify element e. An algebraic system < T, *> is called submonoid of < S, * >, if T ⊆ S, set T is closed under the operation * and e ∈ T

Example < P, + > is a monoid where P = {0, 1, 2,...} and + is addition operation. The identity element is 0.

The algebraic system < P_1, + > is submonoid of < P, + >

Where, P_1 = {0, 2, 4, 6, ...}, as

(i) P_1 is closed under +.

(ii) $P_1 \subseteq P$

(iii) Identity element 0 ∈ P_1.

5.11.3 Subgroup

Definition : Let < S, * > be a group with identity element e and inverse of an element a is given by a^{-1}. An algebraic system < T, * > is called subgroup of <S, * > if

(i) $T \subseteq S$

(ii) $e \in T$

(iii) T is closed under operation *

(iv) $\forall$ $a \in T$ we have $a^{-1} \in T$.

Example : < I, + > is a group where I is set of integers and + is addition operation. The identity element is 0. Inverse of an element a = – a. The algebraic system < I_2, + > is subgroup of < I, + > where I_2 = {..., – 6, – 4, – 2, 0, 2, 4, 6, ...}

as i. $I_2 \subseteq I$.

ii. Identity element $0 \in I_2$

iii. I_2 is closed under +

iv. $\forall$ $a \in T$ we have $-a \in T$., i.e. inverse element

Syllabus Topic : Normal Subgroups

5.11.4 Normal Subgroup

Consider an algebraic system < A, * >. Let a be an element in A. Let $H \subseteq A$. We have a set known as left coset of H with respect to a that contains the result of a * h where $h \in H$.

Definition : The left coset of < H, * > with respect to a $\in$ A is given by

a * H = aH = {a * h | h $\in$ H}

The right coset of < H, *> with respect to a $\in$ A is given by

H * a = Ha = {h * a | h $\in$ H}

Example

Let <A, * > be an algebraic system where A = {0, 1, 2, 3, 4} and the operator * is given by, a * b = (a + b) mod 5.

Consider the algebraic system < H, * > where H = {0, 2, 3}. Now the left coset of H with respect to o, is given by,

$$0 * H = 0H = \{0 \times 0,\ 2 \times 0, 3 \times 0\} = \{0, 2, 3\}$$

Similarly we can have the following left cosets

1H = {1, 3, 4}

2H = {2, 4, 0}

3H = {3, 0, 1}

4H = {4, 1, 2}

Note : If the operator * is commutative, then the left and right cosets and equal.

Definition : A subgroup < H, * > of < G, * > is called a normal subgroup, if for any a $\in$ G, aH = Ha.

Note : Subgroup of an abelian group is always normal.

5.12 CODES AND GROUP CODES

- Codes are used in data compression, cryptography, error detection and correction while data transmission etc.
- The basic type of error detection and correction coding is parity check. In general one extra bit known as parity bit is added to the data stream in order to make the data even parity. It means that the number of 1's in the data must be even.

Original data	Encoded data (parity bit added to the left end (MSB))
1011101	11011101
0100101	10100101
1010101	01010101
1111111	11111111

Group code

Definition : The set $Z^n_2 = \{b_1, b_2, \ldots, b_n \mid b_i \in Z^n_2$, for $i=1,2,\ldots,n\}$ is a group under the binary operation modulo 2. This operation is denoted as $+_2$. This group $(Z^n_2, +_2)$ is called a group code.

Note that $0+_20=0$, $0+_21=1$, $1+_20=1$, $1+_21=0$

Ex. Following table gives group code for $(Z^2_2, +_2)$

$+_2$	00	01	10	11
00	00	01	10	11
01	01	00	11	10
10	10	11	00	01
11	11	10	01	00

5.13 SOLVED EXAMPLES ON CODES

UEx. 5.13.1 (MU - Dec. 2013, 4 Marks)

Consider the (3,5) group encoding function :

$e : B^3 \to B^5$ defined by

$e(000) = 00000$ $e(100) = 10011$

$e(001) = 00110$ $e(101) = 10101$

$e(010) = 01001$ $e(110) = 11010$

$e(011) = 01111$ $e(111) = 11100$

Decode the following words relative to a maximum likely hood decoding function.

(i) 11001 (ii) 01010 (iii) 00111.

Soln. :

Given :

Consider the (3,5) group encoding function :

$e : B^3 \to B^5$ defined by

$e(000) = 00000$ $e(100) = 10011$

$e(001) = 00110$ $e(101) = 10101$

$e(010) = 01001$ $e(110) = 11010$

$e(011) = 01111$ $e(111) = 11100$

Decode the following words relative to a maximum likely hood decoding function.

To find : (i) 11001 (ii) 01010 (iii) 00111.

The decoding table is prepared as follows :

The table is as shown below

000	001	010	011	100	101	110	110
00000	00110	01001	01111	10011	10101	11010	11100
00001	c	01000	01110	10010	10100	11011	11101
00010	00100	01011	01101	10001	10111	11000	11110
00100	00010	01101	01011	10111	10001	11110	11000
01000	01110	00001	00111	11011	11101	10010	10100
10000	10110	a	11111	00011	00101	b	01100

(a) 11001 is present in the column corresponding to 010.

$\therefore$ 11001 is decoded as 010.

(b) 01010 is present in the column corresponding to 110.

$\therefore$ 01010 is decoded as 110.

(c) 00111 is present in the column corresponding to 001.

$\therefore$ 00111 is decoded as 00111

UEx. 5.13.2 : (MU - May 2014, 8 Marks)

Let $H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ be parity check matrix.

Decode the following words relative to maximum likelyhood decoding function

(i) 011001 (ii) 101011

(iii) 111010 (iv) 110110.

Soln. :

Let the encoding function be,

$e_H : B^3 \to B^6$.

Where $B^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$

$B_6 = \{000000, 000001, \ldots, 111111\}$

$\forall$ (abc) $\in B^3$ the encoded word (abcdef) $\in B^6$ is calculated as follows, using the parity check matrix as.

$$d = a \cdot 1 + b \cdot \phi + c \cdot 0$$

$$e = a \cdot 0 + b \cdot 1 + c \cdot 1$$

$$f = a \cdot 0 + b \cdot 0 + c \cdot 1$$

i.e. d, e, f are calculated using columns 1, 2, 3 from the parity check matrix respectively.

Thus we get,

e (000) = 000000

e (001) = 001011

e (010) = 010110

e (011) = 011101

e (100) = 100100

e (101) = 101111

e (110) = 110010

e (111) = 111001

The decoding table is constructed as follows

The first row has the original data.

∴ Number of columns = 8

Each column represents encoding from 000 to 111.

The second row has the encoded data.

From 3rd row onwards, one bit from the encoded data is complemented from right to left.

There are 5 bits in the encoded data.

∴ Rows from 3 to 7 represent the encoded data with one bit complemented.

The table is as shown below.

000	001	010	011	100	101	110	111
000000	001011	010110	011101	100100	101111	110010	111001
000001	001010	010111	011100	100101	101110	110011	111000
000010	001001	010100	011111	100110	101101	110000	1111011
000100	001111	010010	i	100000	iv	v	111101
001000	000011	011110	010101	101100	100111	iii	110001
010000	011011	000110	001101	110100	111111	100010	ii

(i) 011001 is present in column corresponding to 011.

(ii) 101001 is present in column corresponding to 111.

(iii) 111010 is present in column corresponding to 110.

(iv) 101011 is present in column corresponding to 101.

(v) 110110 is present in column corresponding to 110.

∴ 011011 is decoded as 011.

101001 is decoded as 111.

111010 is decoded as 110.

101011 is decoded as 101.

110110 is decoded as 110.

Note : If we do not find the given words in the table, then construct another table in which 2 bits are complemented instead of 1. This can be continued by increasing the number of bits to be complemented.

This happens when the encoding function can correct the errors more than 1 bit.

UEx. 5.13.3 : (MU - Q. 4(c), Dec. 2018, 8 Marks)

Consider the (3, 6) encoding function $e : B^3 \rightarrow B^6$ defined by

e (000) = 000000 e (001) = 001100

e (010) = 010011 e (011) = 011111

e (100) = 100101 e (101) = 101001

e (110) = 110110 e (111) = 111010

Decode the following words relative to a maximum likelihood decoding function.

(i) 000101 (ii) 010101

Soln. :

(i) 000101 → 100

(ii) 010101 → This cannot be decoded as the number of bit changes is at least 2.

UEx. 5.13.4 : (MU - Q. 6(b), Dec. 2018, 6 Marks)

Let H = { $[0]_6$, $[3]_6$} find the left and right cosets in group Z_6. Is H a normal subgroup of group of Z_6.

Soln. :

Given : H = { $[0]_6$, $[3]_6$ }

Z_6 = {0, 1, 2, 3, 4, 5}

Left coset = {{0, 3}, {1, 4}, {2, 5}, {3, 0}, {4, 1}, {5, 2}}

Right coset = Left coset

H is normal subgroup of Z_6.

Module V

UEx. 5.13.5 : (MU - Q. 4(a), May 2018, 8 Marks)

Show that the (2, 5) encoding function $e : B^2 \to B^5$ defined by

$e(00) = 00000 \quad e(01) = 01110$

$e(10) = 10101 \quad e(11) = 11011$ is a group code.

How many errors will it detect and correct ?

Soln. :

Given encoding function $e : B^2 \to B^5$

$e(00) = 00000, \quad e(01) = 01110$

$e(10) = 10101, \quad e(11) = 11011$

Following is the table for $\oplus$

$\oplus$	00000	01110	10101	11011
00000	00000	01110	10101	11011
01110	01110	00000	11011	10101
10101	10101	11011	00000	01110
11011	11011	10101	01110	00000

All the elements of the table belong to the code $\oplus$ is anociative.

00000 is the identity element.

Inverse elements are given as follows :

$00000^{-1} = 00000$

$01110^{-1} = 01110$

$10101^{-1} = 10101$

$11011^{-1} = 11011$

$\therefore$ e is a group code.

$e : B^3 \to B^5$

$\therefore$ It can detect 3 errors and correct 1 error.

UEx. 5.13.6 : (MU - Q. 4(b), May 2018, 8 Marks)

Let $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Be a parity check matrix. Determine the group code $e_H : B^3 \to B^6$

Soln. :

Given parity check matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have to find group code $e_H : B^3 \to B^6$

Let $B_1, B_2\ B_3$ be the parity check $\in B^3$ then the next 3 bits are calculated as follows :

$B_4 = B_1 \oplus B_2, \quad B_5 = B_2 \oplus B_3, \quad B_6 = B_1 \oplus B_3$

$\therefore$ The group code is as follows :

B_1	B_2	B_3	B_4	B_5	B_6
1	0	0	1	0	1
0	1	1	1	0	1
1	1	1	0	0	0
1	0	0	1	0	1
0	1	0	1	1	0
0	0	1	0	1	1

UEx. 5.13.7 : (MU - Q. 4(d), Dec. 2017, 4 Marks)

Given the parity check matrix

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find the minimum distance of the code generated by H. How many errors it can detect and correct ?

☑ **Soln. :**

Minimum distance of H is calculated as follows :

$$110100 \oplus 011010 = 101110 = 4$$

$$110100 \oplus 101001 = 011101 = 4$$

$$011010 \oplus 101001 = 110011 = 4$$

$\therefore$ Minimum distance = 4

$\therefore$ It can detect 3 errors $(k + 1 \geq 4)$

and correct 1 error $(2k + 1 \geq 4)$

UEx. 5.13.8 : (MU - Q. 4(b), Dec. 2017, 6 Marks)

Consider the (2, 5) group encoding function

$e : B^2 \rightarrow B^5$ defined by

e (00) = 00000 e (01) = 01110

e (10) = 10101 e (11) = 11011

Decode the following works relative to a maximum likelihoods decoding function.

(i) 11110 (ii) 10011 (iii) 10100

☑ **Soln. :**

(i) decode ($\underline{1}$1110) = decode ($\underline{0}$1110) = 01

(ii) decode (1$\underline{0}$011) = decode (1$\underline{1}$011) = 11

(iii) decode (1010$\underline{0}$) = decode (1010$\underline{1}$) = 10

Chapter Ends...

❑❑❑

Module V

Note

Module VI

CHAPTER 6 Graph Theory

Syllabus

Types of graphs, Graph Representation, Sub graphs, Operations on Graphs, Walk, Path, Circuit, Connected Graphs, Disconnected Graph, Components, Homomorphism and Isomorphism of Graphs, Euler and Hamiltonian Graphs, Planar Graph, Cut Set, Cut Vertex, Applications.

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- **Chapter End** **6-19**

Syllabus Topic : Basic Terminology

6.1 BASIC TERMINOLOGY

- Many real life problems require the data to be represented in a two dimensional plane.
- **For example :** Road map of a city, Electrical connections in a housing colony, collection of small jobs which are part of a big project.
- Graph is used to represent such examples. In this chapter, we study graphs, their representations, their types, applications, few concepts and algorithms with respect to graphs.

> **Definition :** A graph is a non-linear data structure represented as an ordered pair as G = <V, E>, where V is set of vertices and E is set of edges.
> $V = \{V_1, V_2, \ldots\ldots V_n\}$ $E \subseteq V \times V$

- A graph is pictorially represented with n nodes and edges connecting nodes.

Example

- A graph G = <V, E> where V = {1, 2, 3, 4} and E = {<1, 2> <1, 3> <3, 3> <2, 4> <4, 1> <4, 3>} is pictorially represented as shown in Fig. 4.1.1.

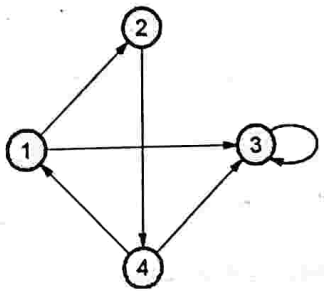


Fig. 6.1.1

6.2 TYPES OF GRAPHS

- The graphs can be classified into different types depending on the connectivity between nodes. When there is an edge from a node to itself, it is known as loop.

Fig. 6.2.1 : Loop

- If there are more than one edge between two nodes, then they are known as parallel edges.

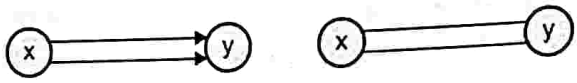


Fig. 6.2.2 : Parallel edges

Fig. 6.2.3 : Not parallel edges because <x, y> <y, x> are two different edges

6.2.1 Undirected Graph

> **Definition :** A graph in which the edges are not specified with direction or arrows, is known as undirected graph.

If there is an edge between two nodes x and y, then the graph is considered to have two edges <x, y> and <y, x>.

Example

V = {1, 2, 3, 4}

E = {<1, 2> <1, 3> <2, 1><2, 3> <2, 4> <3, 1> <3, 2> <3, 4> <4, 2> <4, 3> <4, 4>}

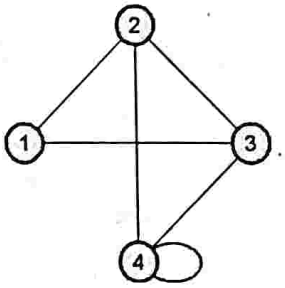


Fig. 6.2.4 : Undirected graph

- The number of edges that are connected to a vertex is known as degree of that vertex.

Module VI

6.2.2 Directed Graph or Digraph

Definition : A graph is known as directed graph, if the edges in the graph are directed.

Example

V = {1, 2, 3, 4}

E = {<1, 3>, <1, 4>, <2, 1>, <2, 2>, <3, 4>, <4, 2>, <4, 3>}

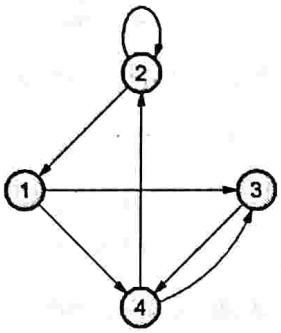


Fig. 6.2.5 : Digraph

- The number of edges incoming to a vertex is called indegree and number of edges outgoing from a vertex is called outdegree of the vertex.

6.2.3 Asymmetric Digraph

Definition : A digraph in which each pair of vertices have either one edge or no edges, is known as **asymmetric digraph.**

Example

V = {1, 2, 3, 4}

E = {<1, 2>, <2, 2>, <2, 3>, <3, 4>, <4, 2>, <1, 4>}

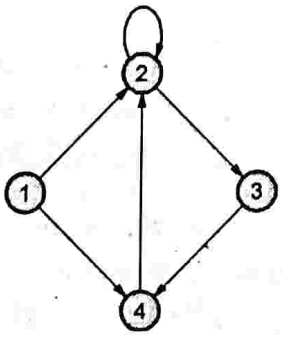


Fig. 6.2.6

6.2.4 Symmetric Digraph

Definition : A digraph in which each pair of vertices x, y have either no edges or two edges <x, y> and <y, x>.

Example

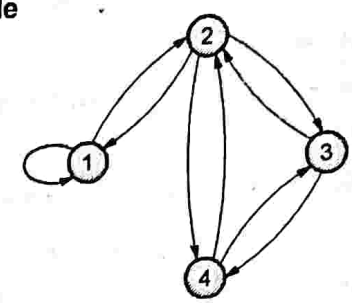


Fig. 6.2.7

V = {1, 2, 3, 4}

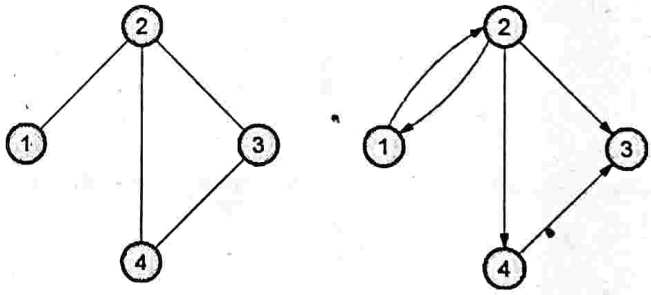
E = {<1, 1>, <1, 2>, <2, 1>, <2, 3>, <3, 2>, <2, 4> <4, 2>,<3, 4>, <4, 3>}

6.2.5 Simple Graph

Definition : A graph which has no self loop and no parallel edges is called simple graph.

A simple graph can be either directed or undirected.

Example



(a) Undirected simple graph **(b) Directed simple graph**

Fig. 6.2.8

6.2.6 Connected and Disconnected Graph

Definition : A graph is said to be connected, if every pair of vertices are connected i.e. one can move from one node to all other nodes by following the edges. If a graph is not connected, then it is called disconnected graph.

Example

- The examples of graphs that we have seen till now are all connected.
- Following is the example of disconnected graph.

$V = \{1, 2, 3, 4, 5\}$

$E = \{<1, 2>, <1, 3>, <2, 1>, <2, 3>, <3, 2>, <3, 1>, \{<4, 5>, <5, 4>\}$

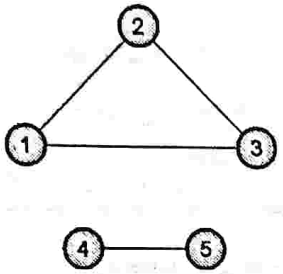


Fig. 6.2.9

6.2.7 Complete Graph

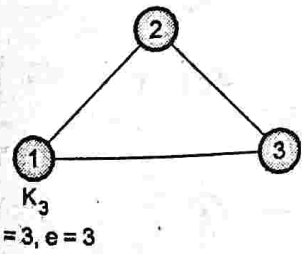
Definition : A simple graph is called complete graph, if it is undirected and there exists an edge between each pair of vertices.

- Let n be the number of vertices and e be the number of edges.
- If a graph has n number of nodes, then the number of edges is equal to $\frac{n(n-1)}{2}$.
- A complete graph is represented as K_n.

Examples

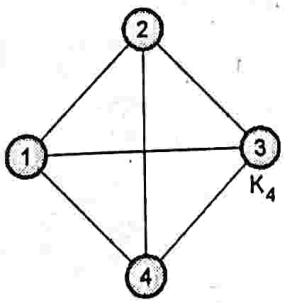
- Following graphs are examples of complete graphs.

(i)



n = 3, e = 3

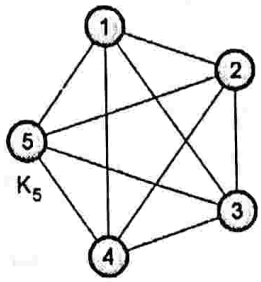
(ii)



n = 4, e = 6

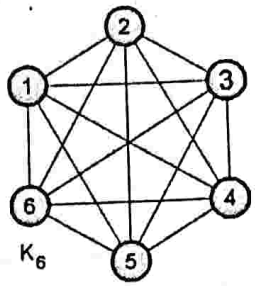
Fig. 6.2.10

(iii)



n = 5, e = 10

(iv)



n = 6, e = 15

Fig. 6.2.11

6.2.8 Bipartite Graph

Definition : An undirected simple graph is called bipartite graph, if the vertices of the graph can be divided into two sets, such that no two vertices in the same set have an edge connecting them.

- Let G = <V, E> be a bipartite graph, then there exists two sets V_1 and V_2 such that,

$V_1 \subseteq V,\ V_2 \subseteq V,\ V_1 \cap V_2 = \phi,\ V_1 \cup V_2 = V$

if $x, y \in V_1$ then $<x, y> \notin E$ and

if $x, y \in V_2$ then $<x, y> \notin E$

Example 1

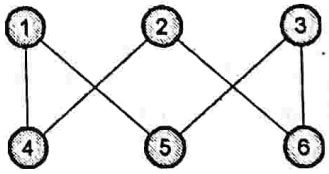


Fig. 6.2.12

$V = \{1, 2, 3, 4, 5, 6\}$

$E = \{<1, 4>, <1, 5>, <2, 4>, <2, 6>, <3, 5>, <3, 6>, \{<4, 1>, <5, 1>, <4, 2>, <6, 2>, <5, 3>, <6, 3>\}$

Example 2

Consider following graph

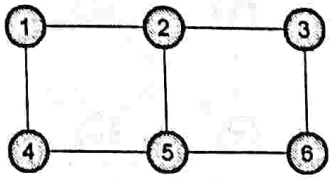


Fig. 6.2.13

The set of vertices

$V = \{1, 2, 3, 4, 5, 6\}$

$E = \{<1, 2>, <1, 4>, <2, 1>, <2, 3>, <2, 5>, <3, 2>, <3, 6>, <4, 1>, <4, 5>, <5, 2>, <5, 4>, <5, 6>, <6, 3>, <6, 5>\}$

It can be noted that, the nodes 1, 5, 3 do not have edges connecting them. Similarly nodes 2, 4, 6 also do not have edges connecting them. Thus the graph can be redrawn as,

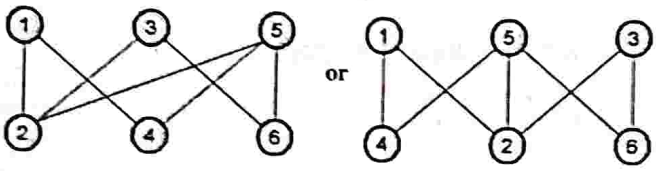


Fig. 6.2.14

∴ It is a bipartite graph.

6.2.9 Complete Bipartite Graph

Definition : A bipartite graph $G = <V, E>$ is called complete bipartite graph, if the set V is divided into two sets V1 and V2 and each vertex in V, has an edge with every vertex in V2. i.e. $G = <V, E>$ is complete bipartite graph, if then exist two sets V1 and V2 such that

$V1 \subseteq V,\ V2 \subseteq V,\ V1 \cap V2 = \phi,\ V1 \cup V2 = V$

If $x, y \in V_1$ then $<x, y> \notin E$

If $x, y, \in V_2$ then $<x, y> \notin E$

$\forall x \in V_1$ and $\forall y \in V_2$ $<x, y> \in E$

The complete bipartite graph is represented as $K_{m, n}$ where $m \leq n$ and m, n represent the number of elements in subsets V_1 and V_2.

Examples

Following are the examples of complete bipartite graphs

(i) $K_{2,2}$

$e = 4$

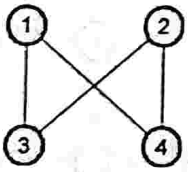


Fig. 6.2.15

(ii) $K_{2,3}$

$e = 6$

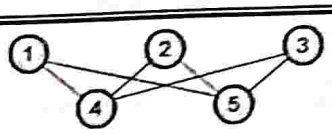


Fig. 6.2.16

(iii) $K_{3,4}$

$e = 12$

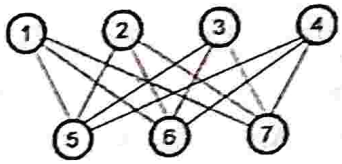


Fig. 6.2.16(a)

The number of edges in graph $K_{m,n}$ is $m \times n$.

Syllabus Topic : Weighted Graphs

6.2.10 Weighted Graph

A graph is said to be weighted graph, if the edges of the graph are associated with values, generally numbers.

Definition : A weighted graph is given by $G = <V, E>$ where we have a mapping $E \rightarrow N$ where N is set of numbers. A weighted graph may be directed or undirected.

Example

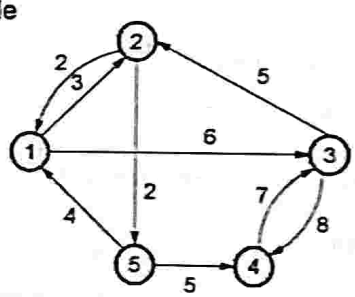


Fig. 6.2.17 : Directed weighted graph

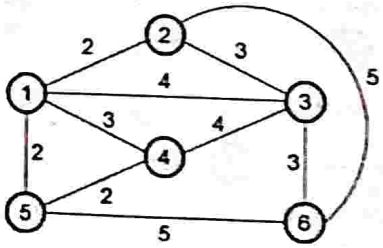


Fig. 6.2.18 : Undirected weighted graph

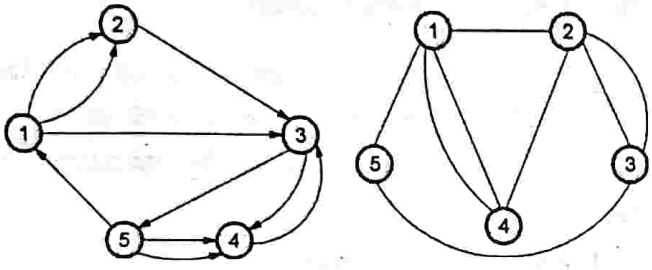
Syllabus Topic : Multi Graphs

6.2.11 Multi Graph

Definition : A graph is said to be multi graph, if there exist two or more edges (or parallel edges) between at least one pair of nodes.

A multi graph may be directed or undirected.

Example



(a) Directed multi graph (b) Undirected multi graph

Fig. 6.2.19

6.3 RELATED GRAPHS

Two graphs may be related in three different ways. In this section, we study these relations. It is not necessary that any two graphs are related.

Syllabus Topic : Isomorphic Graphs

6.3.1 Isomorphic Graphs

Definition : Two graphs $G_1 = \langle V_1, E_1 \rangle$ and $G_2 = \langle V_2, E_2 \rangle$ are isomorphic if $|V_1| = |V_2|$, $|E_1| = |E_2|$ and there exists one-to-one correspondence between vertices and edges of both graphs.

It means that, if we replace vertices of one graph with the others, the connection of vertices remains same.

Example

1.

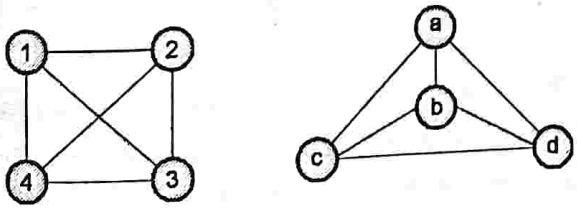


Fig. 6.3.1 : Isomorphic graphs

$1 \to a$, $2 \to b$, $3 \to c$, $4 \to d$

$\langle 1, 2 \rangle \to \langle a, b \rangle$, $\langle 1, 3 \rangle \to \langle a, c \rangle$,

$\langle 1, 4 \rangle \to \langle a, d \rangle$, $\langle 2, 3 \rangle \to \langle b, c \rangle$,

$\langle 2, 4 \rangle \to \langle b, d \rangle$ $\langle 3, 4 \rangle \to \langle c, d \rangle$,

2.

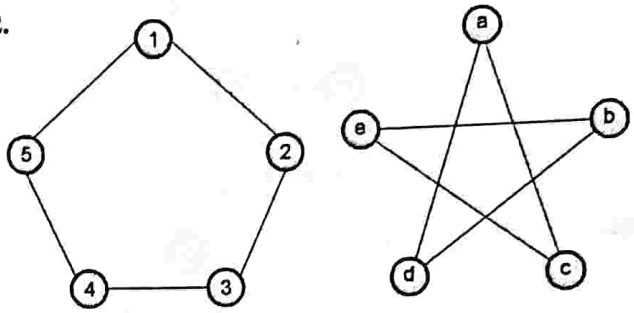


Fig. 6.3.2 : Isomorphic graphs

$1 \to a$, $2 \to c$, $3 \to e$, $4 \to b$, $5 \to d$

$\langle 1, 2 \rangle \to \langle a, c \rangle$, $\langle 2, 3 \rangle \to \langle c, e \rangle$, $\langle 3, 4 \rangle \to \langle e, b \rangle$,

$\langle 4, 5 \rangle \to \langle b, d \rangle$, $\langle 5, 1 \rangle \to \langle d, a \rangle$

6.3.2 Subgraph

Definition : A graph $G_1 = \langle V_1, E_1 \rangle$ is said to be subgraph of a graph $G = \langle V, E \rangle$ if $V_1 \subseteq V$ and $E_1 \subseteq E$ such that edges in E_1 are incident only with the vertices in V_1.

Example

Consider the following graph

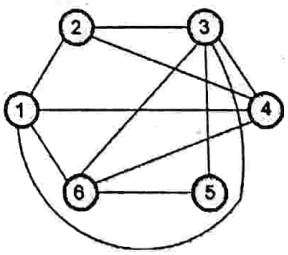


Fig. 6.3.3

Following are the subgraphs of above graph

(i)

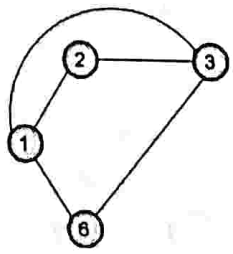


Fig. 6.3.4

(ii)

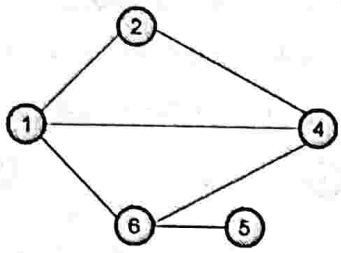


Fig. 6.3.5

(iii)

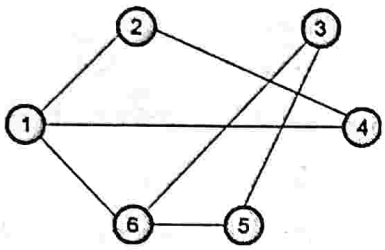


Fig. 6.3.6

6.3.3 Complement of a Graph

Definition : A graph $G_1 = <V_1, E_1>$ is said to be the complement of graph $G = <V, E>$ if $V_1 = V$ and $E_1 = V \times V - E$

Example

Consider the following graph

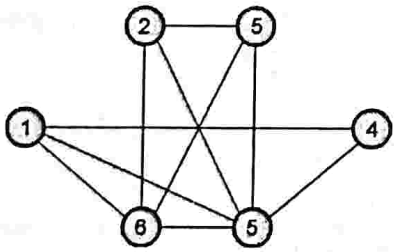


Fig. 6.3.7

The complement of the above graph is,

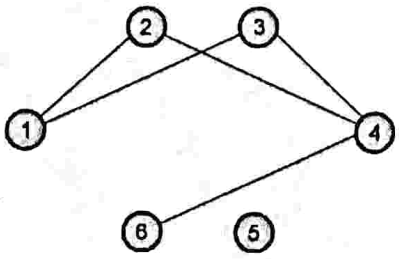


Fig. 6.3.8

Syllabus Topic : Representation of Graphs

6.4 REPRESENTATION OF GRAPHS

Representation of graphs indicates the way in which they are stored in the computer. There are three different types of data structures that are used to store a graph. Depending on the type of graph and its application, the suitable representation can be chosen.

6.4.1 Incidence Matrix

Incidence matrix is used to represent a graph which has no loops. The graph may contain parallel edges i.e. multipgrah. It is more advantageous when the number of edges are less.

It is used in electrical and switching networks. We can have two different incidence matrices for undirected and directed graph. Incidence matrix cannot be used to represent weighted graphs.

An incidence matrix $M(G) = [M_{ij}]$ represents a graph $G = <V, E>$. Let $|V| = v$ and $|E| = e$.

Then the matrix M has v rows and e columns. It is easier to represent vertices as $v_1, v_2, \ldots v_v$ and edge as $e_1, e_2, \ldots e_e$. For an undirected graph.

$$m_{ij} = 1, \text{ if } j^{th} \text{ edge is incident on } i^{th} \text{ vertex.}$$
$$= 0, \text{ otherwise.}$$

There will be exactly two 1's in each column.

For a directed graph,

$$m_{ij} = 1, \text{ if } j^{th} \text{ edge is outgoing from } i^{th} \text{ vertex.}$$
$$= -1, \text{ if } j^{th} \text{ edge is incoming to } i^{th} \text{ vertex.}$$
$$= 0, \text{ otherwise.}$$

There will be exactly one 1 and one – 1 in each column.

The number of non-zero elements in a row is equal to the degree of the corresponding vertex. If any two columns are same, then they represent parallel edges.

6.4.2 Solved Examples on Incidence Matrix

Example 1

Consider the following graph

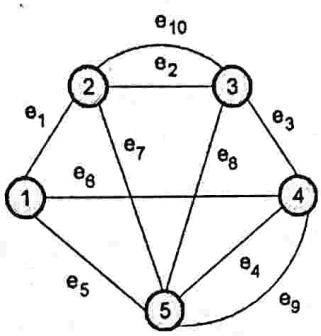


Fig. 6.4.1

Its incidence matrix is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Example 2

Consider the following digraph

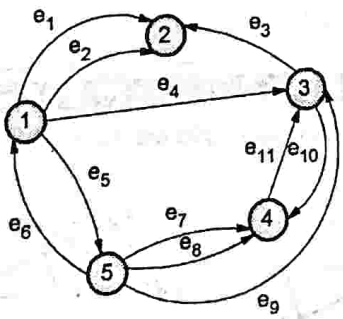


Fig. 6.4.2

Its incidence matrix is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

6.4.3 Adjacency Matrix

Adjacency matrix is used to represent a directed, undirected, weighted or graph with self loop. But it cannot be used to represent a multigraph.

An adjacency matrix $A(e_1) = [a_{ij}]$ of a graph $G = \langle V, E \rangle$ is an $n \times n$ matrix where $|V| = n$.

$$a_{ij} = \omega, \text{ if there is an edge from vertex i to j.}$$
$$= 0, \text{ otherwise.}$$

Where ω = 1, if graph is un-weighted

= Weight of the edge, otherwise

It is clear that for an undirected graph, the adjacency matrix is symmetric. If the graph has no loops, then the number of non-zero elements in a row of the adjacency matrix is equal to the degree of the corresponding vertex in the graph.

For a directed graph, the number of non-zero elements in a row represents out degree and number of non-zero elements in a column represents in degree of the corresponding vertex.

6.4.4 Solved Examples on Adjacency Matrix

Example 1

Consider the following graph.

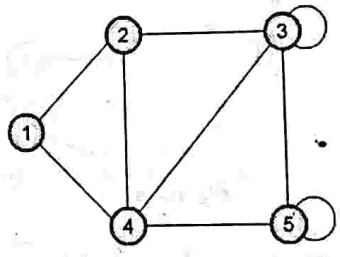


Fig. 6.4.3

The adjacency matrix representing the given graph in Example 1 is,

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Module VI

Example 2

Consider the following graph.

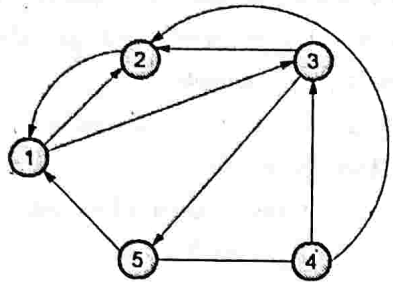


Fig. 6.4.4

The adjacency matrix representing the above graph is,

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Example 3

Consider the following graph

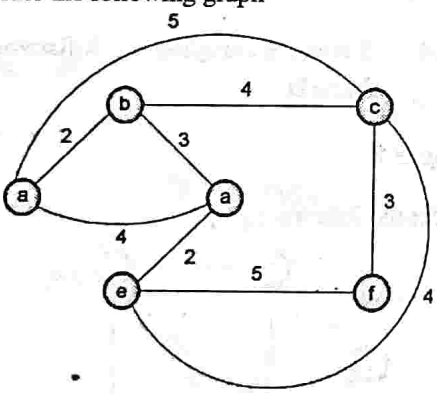


Fig. 6.4.5

The adjacency matrix of the given graph in Example 3 is,

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 2 & 5 & 4 & 0 & 0 \\ 2 & 0 & 4 & 3 & 0 & 0 \\ 5 & 4 & 0 & 0 & 0 & 3 \\ 4 & 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 5 \\ 0 & 0 & 3 & 0 & 5 & 0 \end{bmatrix} \end{array}$$

6.5 SOLVED EXAMPLES ON GRAPHS

UEx. 6.5.1 : MU - Dec. 2013, 4 Marks

How many vertices are necessary to construct a graph with exactly 6 edges in which each vertex is of degree 2.

Soln. :

Given : Number of edges = 6

Degree of each vertex = 2.

To find : Number of vertices

As per the handshaking lemma,

$$\sum_{i=1}^{n} d(v_i) = 2e. \quad \text{where } v_i \text{ is } i^{th} \text{ vertex}$$

$d(v_i)$ is degree of v_i.

n is number of vertices.

e is number of edges.

Substituting the given data we get,

$$\sum_{i=1}^{n} d(v_i) = 2 \times 6 = 12$$

$$d(v_i) = 2$$

$$\therefore \quad 2n = 12$$

$$\therefore \quad n = 6$$

$\therefore$ Number of vertices = 6

UEx. 6.5.2 : MU - Dec. 2013, Dec. 2014, 6 Marks

Show that the following graphs are isomorphic.

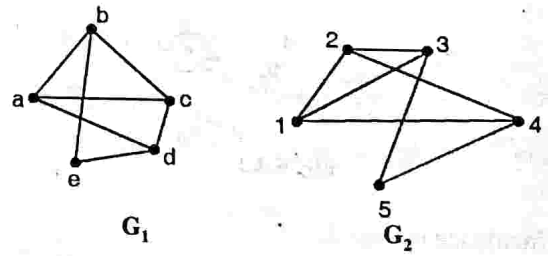


Fig. Ex. 6.5.2

Soln. :

For G_1 we have

$$V_1 = \{a, b, c, d, e\}$$

$$E_1 = \{(a, b), (b, c), (c, a), (d, a), (d, c), (e, b), (e, d)\}$$

For c_{12} we have,

$$V_2 = \{1, 2, 3, 4, 5\}$$

$$E_2 = \{(1, 3), (2, 1), (3, 2), (4, 1), (4, 2), (5, 3), (5, 4)\}$$

Comparing G_1 and G_2 we can write,

$$f : V_1 \rightarrow V_2$$

$$= \{(a, 1), (b, 3), (c, 3), (d, 4), (e, 5)\}$$

$\therefore$ G_1 and G_2 are isomorphic graphs

UEx. 6.5.3 : MU - May 2014, 8 Marks

Determine if following graphs g and G are isomorphic or not.

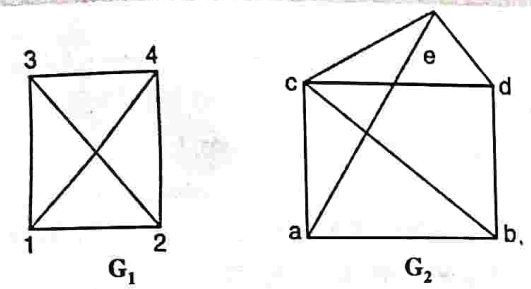


Fig. Ex. 6.5.3

Soln. :

In the given graphs, number of vertices are not equal.

$\therefore$ They are not isomorphic.

UEx. 6.5.4 : MU - May 2015, 8 Marks

Determine if following graphs g and G are isomorphic or not.

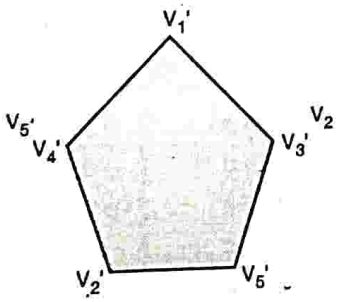


Fig. Ex. 6.5.4

Soln. :

For g we have,

$$V_1 = \{V_1, V_2, V_3, V_4, V_5\}$$

$$E_1 = \{(V_1, V_3), (V_1, V_4), (V_2, V_4), (V_2, V_5), (V_3, V_5)\}$$

For G we have,

$$V_2 = [V_1', V_2', V_3', V_4', V_5']$$

$$E_2 = \{(V_1', V_3'), (V_1', V_4'), (V_3', V_5') (V_5', V_2'), (V_1', V_4')\}$$

Comparing g and G we can write,

$$f : V_1 \rightarrow V_2$$

$$= \{(V_1, V_1'), (V_2, V_2'), (V_3, V_3') (V_4, V_4'), (V_5, V_5')\}$$

$\therefore$ g and G are isomorphic.

UEx. 6.5.5 : MU – Dec. 2015, 8 Marks

Determine the number of edges in a graph with 6 nodes, 2 nodes of degree 4 and 4 nodes of degree 2. Draw two such graphs.

Soln. :

Given : Number of nodes in a graph = 6, 2 nodes of degree 4 and 4 nodes of degree 2.

To find : Number of edges.

As per the handshaking lemma,

$$\sum_{i=1}^{n} d(V_i) = 2e \text{ where } V_i \text{ is the } i^{th} \text{ node or vertex.}$$

$d(V_i)$ is degree of V_i

n is number of vertices

e is number of edges.

Substituting the given data we get,

$$\sum_{i=1}^{n} d(V_i) = 2e$$

$$\therefore 2 \times 4 + 4 \times 2 = 2e$$

$$\therefore 16 = 2e$$

$$\therefore e = 8$$

$\therefore$ Number of edges = 8.

Following are examples of such graph.

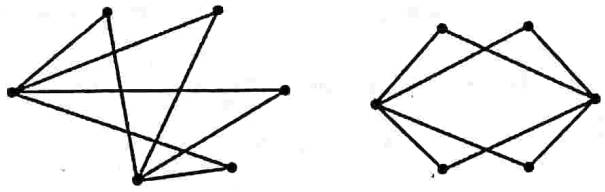


Fig. Ex. 6.5.5

UEx. 6.5.6 : MU - Dec. 2015, 8 Marks

Determine whether following graphs are isomorphic.

Soln. :

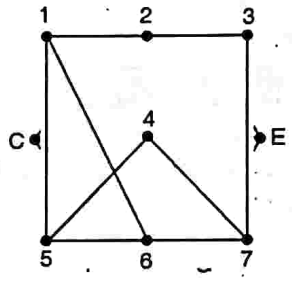


Fig. 1- Q. 6(a)

Soln. :

Let the graphs be G_1 and G_2.

For G_1 we have

$V_1 = \{1, 2, 3, 4, 5, 6, 7\}$

$E_1 = \{(1,2), (1,5), (1,6), (2,3), (3,7), (4,5), (4,7) (5,6), (6,7)\}$

For G_2 we have

$V_2 = \{A, B, C, D, E, F, G\}$

$E_2 = \{(A,B), (A,C), (A,D), (B,E), (C,F), (D,E), (D,G) (E,G), (F,G)\}$

Comparing G_1, G_2 we can write.

$F : V_1 \rightarrow V_2$

$= \{(1,A), (2,B), (3,E), (4,F), (5,C), (6,D), (7,G)\}$

$\therefore$ G_1, G_2 are isomorphic.

Ex. 6.5.7 : Show that following graphs are isomorphic.

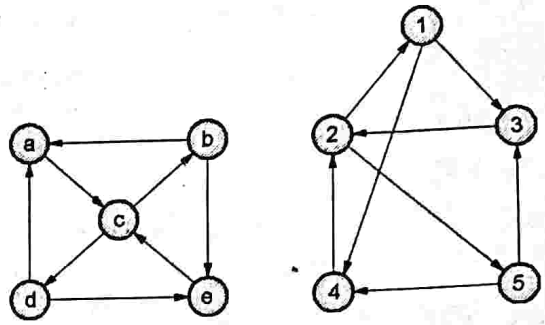


Fig. Ex. 6.5.7

Soln. :

- In the given graphs, vertex c and vertex 2 have two incoming and outgoing edges

  $\therefore$ vertex c corresponds to vertex 2
- The vertices a, e, 3, 4 have two incoming and one outgoing edge. The vertices b, d, 1, 5 have two outgoing and one incoming edge.
- By considering all there we can have the following correspondence.

a – 3
b – 1
c – 2
d – 5
e – 4

$a \rightarrow c$ – $3 \rightarrow 2$
$b \rightarrow a$ – $1 \rightarrow 3$
$b \rightarrow e$ – $1 \rightarrow 4$
$c \rightarrow b$ – $2 \rightarrow 1$
$c \rightarrow d$ – $2 \rightarrow 5$
$d \rightarrow a$ – $5 \rightarrow 3$
$d \rightarrow e$ – $5 \rightarrow 4$
$e \rightarrow c$ – $4 \rightarrow 2$

$\therefore$ The given graphs are isomorphic.

UEx. 6.5.8 : MU - Dec. 2016, 6 Marks

Prove that a connected graph with n vertices must have at least n-1 edges. Can a single undirected graph of 8 vertices have 40 edges excluding self loop.

Soln. :

To prove : A connected graph with n vertices must have at least n – 1 edges.

☞ **Proof**

Assume that a connected graph with n vertices has $n-2$ edges.

Let the vertices be $V_1, V_2, \ldots, V_n$. A graph is called connected, if there exists a path from every vertex to every other vertex. With $n-2$ edges, we can have a path from one vertex to $n-1$ vertices.

$\therefore$ There must be at least $n-1$ edges for a connected graph with n vertices

Given : Number of vertices = 8,

Number of edges = 40

To find : Whether such graph exist.

If the graph with a n vertices is a complete graph, then the number of edges is given by, $\frac{n(n-1)}{2}$.

In the given example, a complete graph with 8 vertices has, $\frac{(8 \times 7)}{2} = 28$ edges. i.e. maximum number of edges = 28.

$\therefore$ There cannot be a graph without self loop with 8 vertices and 40 edges.

UEx. 6.5.9 : MU - Q. 2(c), Dec. 2018, 8 Marks

Define isomorphic graphs. Show whether the following graphs are isomorphic or not.

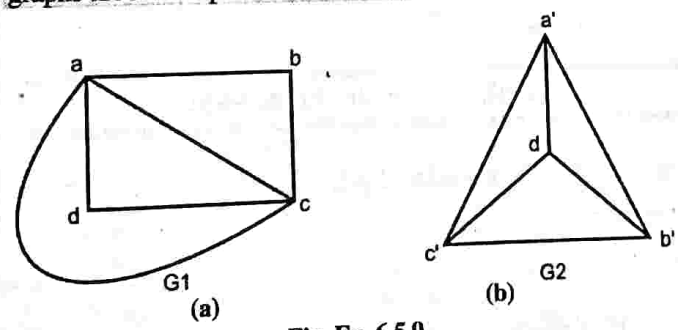


Fig. Ex. 6.5.9

☑ **Soln. :**

The graphs are not isomorphic.

Because nodes are not corresponding to one another.

Ex. Degrees of G1 are (4, 2, 4, 2)

Degrees of G2 are (3, 3, 3, 3)

UEx. 6.5.10 : MU - Q. 2(a), May 2018, 8 Marks

Define isomorphism of graphs. Find if the following two graphs are isomorphic. If yes, find the one-to-one correspondence between the vertices.

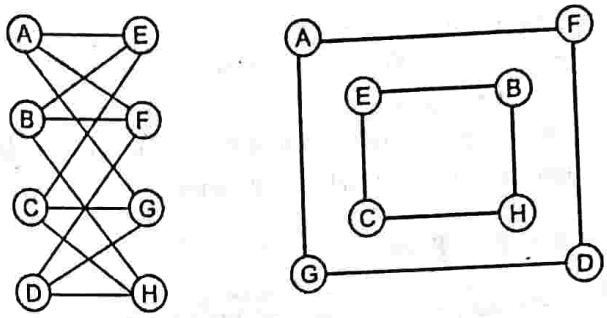


Fig. Ex. 6.5.10

☑ **Soln. :**

Given graphs are isomorphic.

$A \Leftrightarrow A,\ B \Leftrightarrow B,\ C \Leftrightarrow C,\ D \Leftrightarrow D,\ E \Leftrightarrow E,\ F \Leftrightarrow F,$
$G \Leftrightarrow G,\ H \Leftrightarrow H$

UEx. 6.5.11 MU - Q. 6(c), Dec. 2017, 4 Marks

Determine whether following graphs are isomorphic or not.

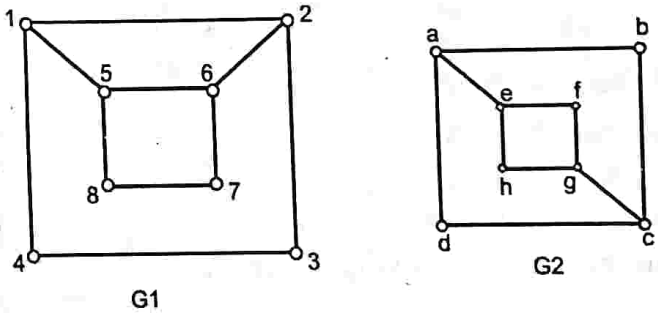


Fig. Ex. 6.5.11

☑ **Soln. :**

Given graphs are not isomorphic.

Vertices 7 and 8 have degree 2 and they are connected in graph 1.

Vertices h and f have degree 2 and they are not connected in graph 2.

$\therefore$ There is no one-to-one correspondence.

Syllabus Topic : Paths and Circuits

6.6 PATHS AND CIRCUITS

- In this section, we study the connectivity in graphs. To define paths and circuits we consider the edges, so we name the edge $e_1, e_2, \ldots e_n$ where n is the number of edges in the graph.
- If we have an edge $< V_x, V_y >$ then V_x is called initial vertex and V_y is called terminal vertex of the edge.

6.6.1 Path

Definition : A sequence of edge $(e_{i_1}, e_{i_2}, \ldots e_{i_k})$ is known as a path, if for every pair of edges e_{i_j} and $e_{i_{j+1}}$ where $1 \le j < k$, the terminal vertex of e_{i_j} is equal to initial vertex of $e_{i_{j+1}}$

Example

Consider the graph.

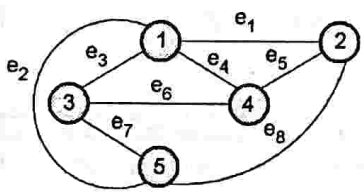


Fig. 6.6.1

Following are few paths

$(e_1, e_3, e_7, e_8, e_5, e_4)$

$(e_2, e_7, e_6, e_4, e_2, e_8)$

$(e_4, e_5, e_8, e_7, e_6, e_5, e_1, e_4, e_6)$

$(e_7, e_6, e_5, e_1, e_4, e_6, e_3, e_2, e_8, e_5, e_4, e_1)$

6.6.2 Simple Path

Definition : A path is said to be simple, if it does not include any edge two or more times. i.e. every edge will be present in a simple path either zero or one time.

Example

Consider the graph

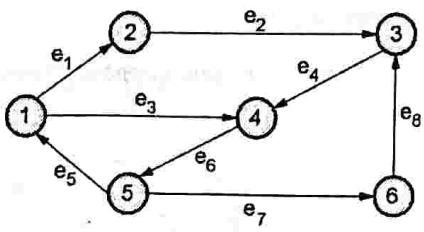


Fig. 6.6.2

Following are some of the simple paths

(1) $(e_1, e_2, e_4, e_6, e_7, e_8)$

(2) $(e_3, e_4, e_5, e_1, e_2)$

(3) (e_8, e_4, e_6, e_7)

(4) $(e_2, e_4, e_6, e_5, e_3)$

6.6.3 Elementary Path

Definition : A path is said to be elementary, if it does not include any vertex two or more times i.e. every vertex will be present in an elementary path either zero or one time.

Example

Consider the graph

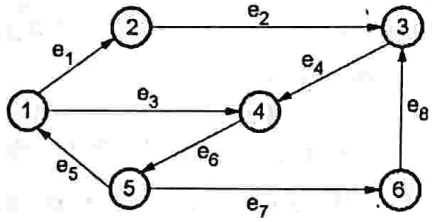


Fig. 6.6.3

Following are some of the elementary paths

1. $(e_1, e_2, e_4, e_6, e_7)$
2. (e_3, e_6, e_7, e_8)
3. (e_5, e_1, e_2, e_4)
4. (e_7, e_8, e_4)

Syllabus Topic : Euler Path

6.6.4 Eulerian Path

Definition : A path in a graph is said to be Eulerian path, if it has every edge in the graph exactly once i.e. it is a simple path with all edges.

It is not necessary that every graph should have an Eulerian path, A graph may have one or more Eulerian paths.

For example, the following graph does not have an Eulerian path.

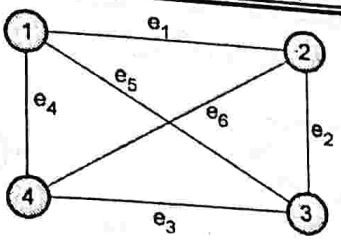


Fig. 6.6.4

☞ **Example**

Consider the following graph

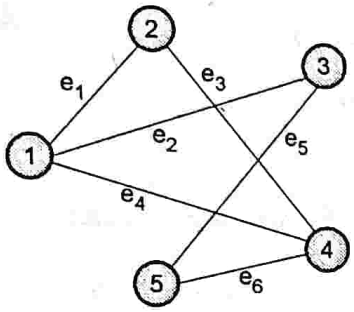


Fig. 6.6.5

Following are some of the Eulerian paths.

1. $(e_1, e_3, e_6, e_5, e_2, e_4)$
2. $(e_3, e_1, e_2, e_5, e_6, e_4)$
3. $(e_2, e_5, e_6, e_4, e_1, e_3)$

☞ **Theorem**

- An undirected graph has an Eulerian path if and only if it is connected, and either zero or two vertices are of odd degree.
- A complete graph K_n has an Eulerian path if $1 < n \leq 3$ or n is odd.
- A complete bipartite graph $K_{m,n}$ has as Eulerian path if
  - (i) Both m, n are even
  - (ii) m = 1, then n = 1 or 2
  - (iii) m = 2, n is either even or odd

Syllabus Topic : Hamiltonian Path

6.6.5 Hamiltonian Path

Definition : A path in a graph is said to be Hamiltonian Path, if it has every vertex in the graph exactly once.

i.e. it is an elementary path with all vertices.

- It is not necessary that every graph should have Hamiltonian path.
- A graph may have one or more Hamiltonian paths.
- For example, the following graph does not have a Hamiltonian path.

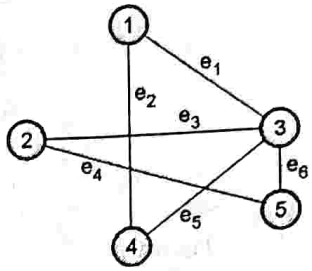


Fig. 6.6.6

But it has an Eulerian path, $(e_1, e_2, e_5, e_3, e_4, e_6)$

☞ **Example :** Consider the graph

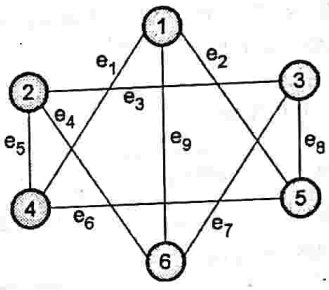


Fig. 6.6.7

Following are some of the Hamiltonian paths

(1) $(e_1, e_2, e_8, e_3, e_4)$

(2) $(e_5, e_4, e_7, e_8, e_2)$

(3) $(e_7, e_3, e_5, e_1, e_2)$

(4) $(e_5, e_1, e_9, e_7, e_8)$

- As in case of Eulerian path, there is no necessary condition, by satisfying which, a graph contains a Hamiltonian path.
- But still following theorem gives a sufficient condition.

☞ **Theorem**

- Let G be a linear graph with n vertices. Then it is sufficient that all pairs of vertices have their sum of degrees greater than or equal to n – 1, to have Hamiltonian path.

Module VI

- Remember that, this is sufficient condition, not necessary condition.
- For example, the following graph has Hamiltonian path, though the above theorem fails.

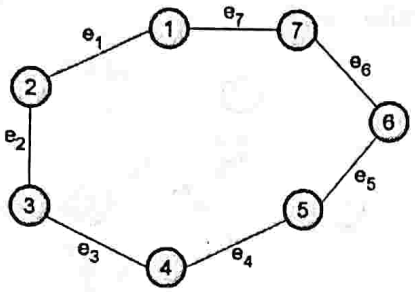


Fig. 6.6.8

- Here n = 7

  ∴ Sufficient condition is to have sum of degrees of each pair as at least 6.
- A complete graph K_n always has a Hamiltonian path.

  A complete bipartite graph $K_{m,n}$ has a Hamiltonian path if m = n or m = n + 1

6.6.6 Circuit

Definition : A sequence of edges $(e_{i_1}, e_{i_2}, \ldots e_{i_k})$ is known as a circuit, if it is a path and initial vertex of e_{i_1} is equal to terminal vertex of e_{i_k}.

Example : Consider the graph

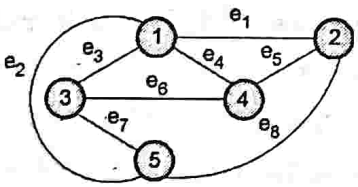


Fig. 6.6.9

Following are few circuits.

1. (e_1, e_5, e_4)
2. $(e_3, e_7, e_8, e_5, e_4)$
3. (e_5, e_8, e_2, e_4)
4. $(e_7, e_6, e_4, e_1, e_5, e_4, e_3, e_9)$
5. $(e_8, e_1, e_4, e_6, e_3, e_2)$

6.6.7 Simple Circuit

Definition : A circuit is said to be simple, if it does not include any edge two or more times. i.e. Every edge will be present in a simple either zero or one time.

Example : Consider the graph

Following are some of the simple circuits

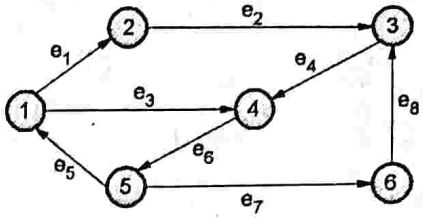


Fig. 6.6.10

(1) $(e_1, e_2, e_4, e_6, e_5)$

(2) (e_6, e_7, e_8, e_4)

(3) (e_3, e_6, e_5)

6.6.8 Elementary Circuits

Definition : A circuit is said to be elementary, if it does not include any vertex two or more times. i.e. Every vertex will be present in an elementary circuit either zero or one time.

- Remember that first and last vertices in a circuit must be same and thus this repetition is neglected.

Example

Consider the graph

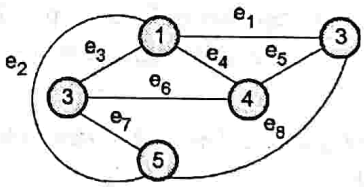


Fig. 6.6.11

Following are some of the elementary circuits.

(1) (e_3, e_6, e_4)

(2) (e_2, e_8, e_1)

(3) $(e_3, e_7, e_8, e_5, e_4)$

(4) $(e_2, e_7, e_6, e_5, e_1)$

6.6.9 Eulerian Circuit

Definition : A circuit in a graph is said to be Eulerian circuit, if it has every edge in the graph exactly once i.e. It is a simple circuit with all edges.

Example

Consider the following graph

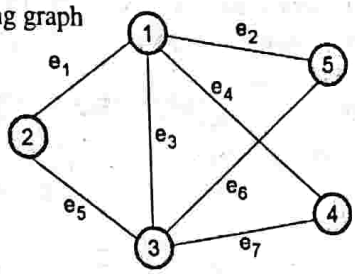


Fig. 6.6.12

Following are some of the Eulerian circuits.

1. $(e_1, e_5, e_3, e_4, e_7, e_6, e_2)$
2. $(e_1, e_5, e_7, e_4, e_3, e_6, e_2)$
3. $(e_5, e_1, e_3, e_7, e_4, e_2, e_6)$

Theorem

- An undirected graph has an Eulerian circuit, if and only if it is connected and the degree of every vertex is even.
- A complete graph K_n has an Eulerian circuit, if n is odd and $n \geq 3$.
- A complete bipartite graph $K_{m,n}$ has an Eulerian circuit, if both m and n are even.

6.6.10 Hamiltonian Circuit

Definition : A circuit in an graph is said to be Hamiltonian, if it has every vertex in the graph exactly once.
i.e. it is an elementary circuit with all vertices.

Example : Consider the following graph

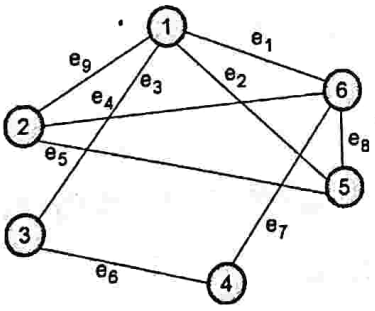


Fig. 6.6.13

- Following are the Hamiltanian circuit
  1. $(e_3, e_6, e_7, e_4, e_5, e_2)$
  2. $(e_8, e_5, e_9, e_3, e_6, e_7)$
- A complete graph K_n always has a Hamiltonian circuit
- A complete bipartite graph $K_{m,n}$ has a Hamiltonian circuit if m = n

6.7 SOLVED EXAMPLES

Ex. 6.7.1 : Show a graph that has both an Eulerian circuit and a Hamiltonian circuit.

Soln. :

- Every complete graph K_n has a Hamiltonian circuit, if $n \geq 3$ and n is odd, then it has Eulerian circuit.
- Let us consider K_3

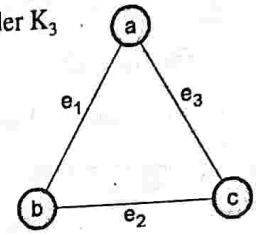


Fig. Ex. 6.7.1

- The graph has both Elerian and Hamiltonian circuits.
- Another example

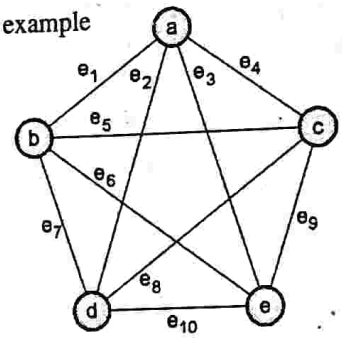


Fig. Ex. 6.7.1(a)

Ex. 6.7.2 : Show a graph that has an Eulerian circuit but has no Hamiltonian circuit.

Soln. :

- A complete bipartite graph $K_{m,n}$ has an Eulerian circuit. If m = n then it has Hamiltonian circuit.
- Now we need a graph that has Eulerian circuit but not Hamiltonian circuit.
- Consider complete bipartite graph $K_{2,4}$

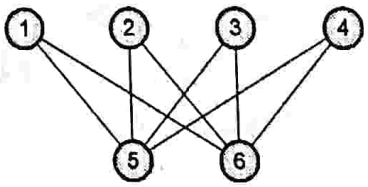


Fig. Ex. 6.7.2

- It has an Eulerian circuit, but not Hamiltonian circuit.

Ex. 6.7.3 : Show a graph that has no Eulerian circuit but has a Hamiltonian circuit.

Soln. : Consider the complete graph K_4

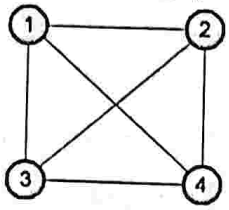


Fig. Ex. 6.6.3

It has Hamiltonian circuit, but not Eulerian circuit.

UEx. 6.7.4 : MU - May 2014, 6 Marks

Determine the Eulerian and Hamiltonian path, if exists, in the following graphs :

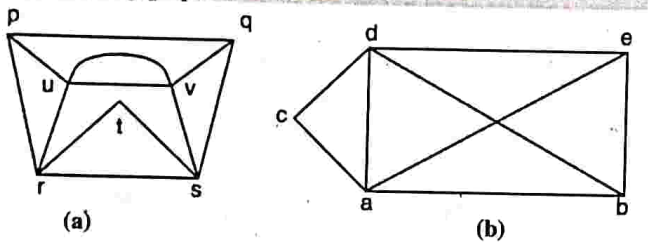


Fig. Ex. 6.7.4

Soln. :

a. Eulerian path ⇒ p, u, v, q, s, v, u, r, t, s, r, p, q

Hamiltonian path ⇒ p, u, v, q, s, t, r.

Hamiltonian circuit ⇒ p, u, v, q, s, t, r, p

b. Eulerian path ⇒ b, a, d, b, e, a, c, d, e.

Hamiltonian path ⇒ d, e, b, a, c.

Hamiltonian circuit ⇒ d, e, b, a, c, d.

UEx. 6.7.5 : MU - May 2015, 4 Marks

Determine Euler Cycle and path in graph shown below.

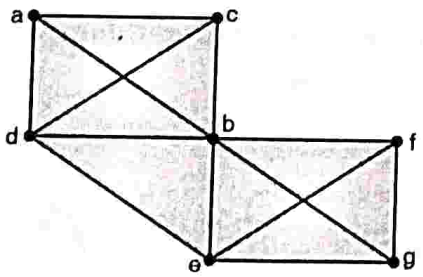


Fig. Ex. 6.7.5

Soln. :

In the given graph, number of vertices with odd degree are 4. They are a, c, f, g. We know that if a graph has at least one vertex with odd degree, then it cannot have a Euler circuit, and if it has atlest two vertices with odd degree, then it cannot have a Euler path. Thus the given graph has no Euler path or circuit.

UEx. 6.7.6 : MU - Q. 4(a), May 2016, 6 Marks

Define Euler path and Euler circuit, determine whether the given graph has Euler path and Euler circuit.

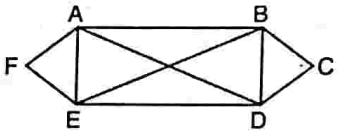


Fig. Ex. 6.7.6

Soln. :

For the given graph,

Euler path - A B D E B C D A F E.

Euler circuit - A B C D E F A D B E A.

UEx. 6.7.7 : MU - Q. 4(b), May 2016, 6 Marks

Define Hamiltonian path and Hamiltonian circuit, determine whether the given graph has Hamiltonian path and Hamiltonian circuit.

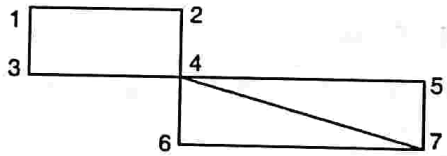


Fig. Ex. 6.7.7

Soln. :

For the given graph,

Hamiltonian path – 3 1 2 4 6 7 5

Hamiltonian circuit does not exist.

UEx. 6.7.8 : MU - Q. 4(c), May 2016, 8 Marks

Define isomorphic graphs. Show that the following two graphs are isomorphic.

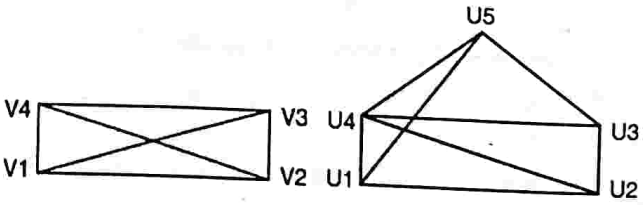


Fig. Ex. 6.7.8

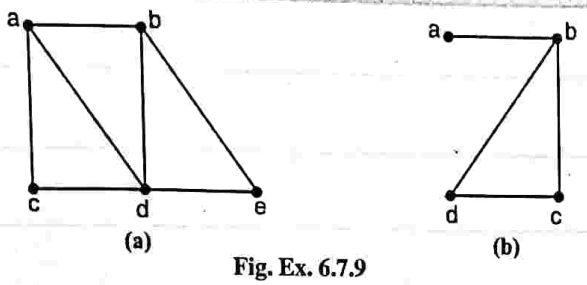
☑ **Soln. :** The given graphs are not isomorphic, as one has 4 vertices and other has 5.

UEx. 6.7.9 : MU - May 2017, 8 Marks

Define Euler Path ands Hamiltonian Path.

i) Determine Euler Cycle and path in graph shown in (a)

ii) Determine Hamiltonian Cycle and path in graph shown in (b).



(a) (b)

Fig. Ex. 6.7.9

☑ **Soln. :**

(i) Euler path = a c d a b a d e b

Euler circuit does not exist as the vertices a and b have odd degree.

(ii) Hamiltonian path = a b c d

Hamiltonian circuit does not exist as the vertices a, b have odd degree.

UEx. 6.7.10 : MU - May 2017, 8 Marks

Define Euler Path and Hamiltonian Path.

i) Determine Euler Cycle and path in graph shown in (a)

ii) Determine Hamiltonian Cycle and path in graph shown in (b).

☑ **Soln. :**

The given graphs G_1 and G_2 are not isomorphic, because, in G_1, two vertices E and F have degree 1.

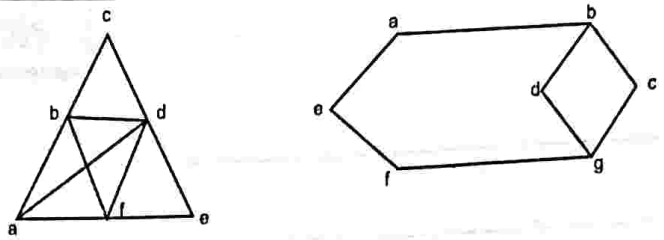
Where as in G_2, there is only one vertex Y, with degree 1.

∴ There dose not exist one-to-one correspondence between the vertices of to graphs.

∴ They are not isomorphic.

UEx. 6.7.11 : MU - Q. 3(a), Dec. 2018, 6 Marks

Determine which of the following graphic contains an Eulerian or Hamiltonian circuit.



(a) (b)

Fig. Ex. 6.7.11

☑ **Soln. :**

Fig. Ex.6.7.11(a) has Eulerian circuit

path d c b a f e d b f d a

But not Eulerian circuit.

It has Hamiltonian circuit - c b a f e d c

Fig. 6.7.11 has Eulerian path - b d g c b a e f g

But not Eulerian circuit.

It has Hamiltonian path - c b a e f g d

But not Hamiltonian circuit.

UEx. 6.7.12 : MU - Q. 6(c), May 2018, 4 Marks

Determine Euler Cycle and path in graph shown below :

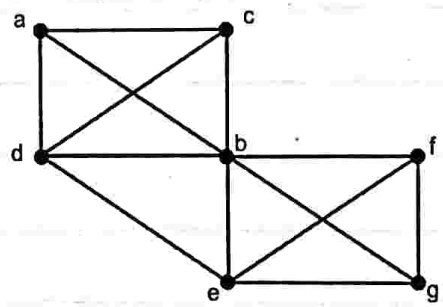


Fig. Ex. 6.7.12

☑ **Soln. :**

Euler path = a c d b f e g

Euler cycle = a d e g f b c a

UEx. 6.7.13 : MU- Q. 4(c)(i)(ii), Dec. 2017, 4 Marks

Is every Eulerian graph a Hamiltonian ?

(ii) Is every Hamiltonian graph a Eulerian ?

Explain with the necessary graph.

☑ **Soln. : Refer Ex. 6.7.1, 6.7.2 and 6.7.3.**

Chapter Ends...

Note

Q 128 bit

Q multiple edges.

Q *symmetric ✓

Q 24 12 vertices ✓

Q multiple edges

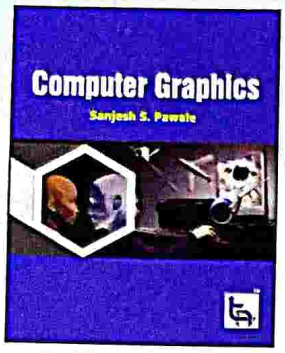
commutative.

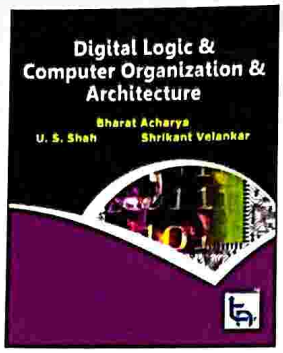
null

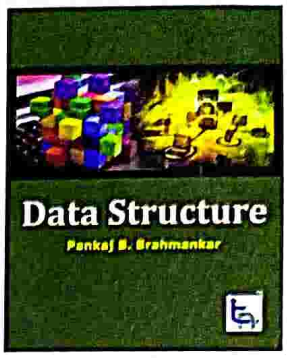
~~hours~~ $Hn = 2n$

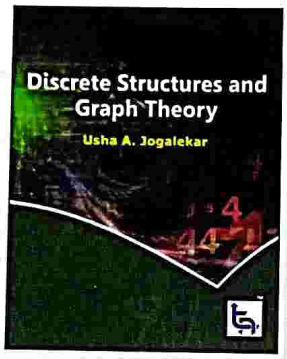
9

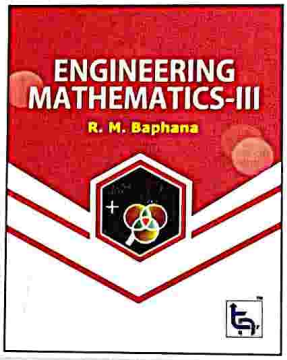
Semester 3 - Computer Engineering











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