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Tutorial -02 : Inverse Laplace Transformation

1] Find inverse Laplace of $\left[\frac{4s}{s^4+4}\right]$.

→ $L^{-1}\left[\frac{4s}{s^4+4}\right] = L^{-1}\left[\frac{4s}{(s^2+2)^2-4s^2}\right] = L^{-1}\left[\frac{4s}{(s^2+2-2s)(s^2+2+2s)}\right]$

$= L^{-1}\left[\frac{1}{(s^2+2-2s)} - \frac{1}{(s^2+2+2s)}\right]$

$= L^{-1}\left[\frac{1}{(s-1)^2+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$

$(s-1)\to s \qquad (s+1)\to s$

$= e^{t} L^{-1}\left[\frac{1}{s^2+1}\right] - e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right]$

$= e^{t}\sin t - e^{-t}\sin t = (e^{t}-e^{-t})\sin t$

$= 2\sinh t \sin t$.

$\therefore \boxed{L^{-1}\left[\frac{4s}{s^4+4}\right] = 2\sinh t.\sin t}$

2] Find Inverse Laplace Transform of $\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right]$.

→ Put $s^2+2s = u$

$L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right] = L^{-1}\left[\frac{u+3}{(u+2)(u+5)}\right]$

$= L^{-1}\left[\frac{A}{(u+2)} + \frac{B}{(u+5)}\right] = L^{-1}\left[\frac{1/3}{(u+2)} + \frac{2/3}{(u+5)}\right]$

$= \frac{1}{3} L^{-1}\left[\frac{1}{s^2+2s+2}\right] + \frac{2}{3} L^{-1}\left[\frac{1}{s^2+2s+5}\right]$

$$= \frac{1}{3} L^{-1}\left[\frac{1}{(s+1)^2+1}\right]_{(s+1)\to s} + \frac{2}{3} L^{-1}\left[\frac{1}{(s+1)^2+4}\right]_{(s+1)\to s} \quad \{FSP\}$$

$$= \frac{1}{3} e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{2}{3} e^{-t} L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{3} e^{-t} \sin t + \frac{\cancel{2}}{3} e^{-t} \frac{\sin(2t)}{\cancel{2}}$$

$$= \frac{1}{3} e^{-t} [\sin t + \sin 2t]$$

$$\therefore \boxed{L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right] = \frac{1}{3} e^{-t} [\sin t + \sin 2t]}$$

3] Find Inverse laplace Transform of $\tanh^{-1} s$.

→ w.k.t.,

$$\tanh^{-1}(s) = \frac{1}{2} \log\left(\frac{1+s}{1-s}\right)$$

$$L^{-1}[\tanh^{-1}(s)] = \frac{1}{2} L^{-1}\left[\log\left(\frac{1+s}{1-s}\right)\right]$$

By multiple by 't' property,

$$L^{-1}[f(s)] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds} f(s)\right]$$

$$\therefore \frac{1}{2} L^{-1}\left[\log\left(\frac{1+s}{1-s}\right)\right] = \frac{1}{2} \cdot \frac{-1}{t} L^{-1}\left[\frac{d}{ds} \log\left(\frac{1+s}{1-s}\right)\right]$$

$$= \frac{-1}{2t} L^{-1}\left[\frac{d}{ds}\left(\log(1+s) - \log(1-s)\right)\right]$$

$$= \frac{-1}{2t} L^{-1}\left[\frac{1}{(1+s)} + \frac{1}{(1-s)}\right]$$

$$= \frac{-1}{2t}\left(e^{-t} + e^{t}\right) = \frac{-1}{t} \sinh(t)$$

$$\therefore \boxed{L^{-1}[\tanh^{-1}(s)] = \frac{-\sinh(t)}{t}}$$

4] Find Inverse Laplace Transform of $L^{-1}\left\{\frac{s+29}{(s+4)(s^2+9)}\right\}$

$\rightarrow$ $L^{-1}\left\{\frac{s+29}{(s+4)(s^2+9)}\right\} = L^{-1}\left\{\frac{A}{(s+4)} + \frac{B}{(s^2+9)}\right\}$

$= L^{-1}\left\{\frac{1}{s+4} + \frac{-s+5}{s^2+9}\right\}$

$= L^{-1}\left[\frac{1}{s+4}\right] - L^{-1}\left[\frac{s}{s^2+9}\right] + 5L^{-1}\left[\frac{1}{s^2+9}\right]$

$= e^{-4t} - \cos(3t) + 5\sin(3t)$

$\therefore$ $\boxed{L^{-1}\left\{\frac{s+29}{(s+4)(s^2+9)}\right\} = e^{-4t} - \cos(3t) + 5\sin(3t)}$

5] Find Inverse Laplace Transform of $\tan^{-1}\left(\frac{s}{1}\right)$.

$\rightarrow$ $L^{-1}\left[\tan^{-1}(s)\right]$

By multiple by 't' property,

$L^{-1}[f(s)] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds} f(s)\right]$

$\therefore L^{-1}\left[\tan^{-1}(s)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left(\tan^{-1}(s)\right)\right]$

$= \frac{-1}{t} L^{-1}\left[\frac{1}{1+s^2}\right]$

$= \frac{-1}{t} \sin t$

$\therefore$ $\boxed{L^{-1}\left[\tan^{-1}(s)\right] = \frac{-\sin t}{t}}$

6] Find inverse laplace Transform of $\left[\frac{s-4}{(s-4)^2+2}\right]$

$\rightarrow$ $L^{-1}\left[\frac{s-4}{(s-4)^2+2}\right]_{(s-4)\to s} = e^{4t}\, L^{-1}\left[\frac{s}{s^2+2}\right]$ .... {FSP}

$= e^{4t}\cos(\sqrt{2}t)$

$\therefore \boxed{L^{-1}\left[\frac{s-4}{(s-4)^2+2}\right] = e^{4t}\cos(\sqrt{2}t)}$

7] Find Inverse laplace Transform of

i] $\left[\frac{2}{(s-2)(s+2)}\right]$

$\rightarrow$ $L^{-1}\left[\frac{2}{(s-2)(s+2)}\right] = \frac{1}{2} L^{-1}\left[\frac{1}{(s-2)} - \frac{1}{(s+2)}\right]$

$= \frac{1}{2}\left\{ L^{-1}\left[\frac{1}{s-2}\right] - L^{-1}\left[\frac{1}{s+2}\right]\right\}$

$= \frac{1}{2}\left\{ e^{2t} - e^{-2t} \right\}$

$= \sinh(2t)$.

$\therefore \boxed{L^{-1}\left[\frac{2}{(s-2)(s+2)}\right] = \sinh(2t)}$

ii] $\left[\frac{1}{s^2+4s+2}\right]$

$\rightarrow$ $L^{-1}\left[\frac{1}{s^2+4s+2}\right] = L^{-1}\left[\frac{1}{(s+2)^2-2}\right]_{(s+2)\to s}$ ... {FSP}

$= e^{-2t}\, L^{-1}\left[\frac{1}{s^2-2}\right]$

$$= e^{-2t}\,\frac{\sinh(\sqrt{2}t)}{\sqrt{2}}$$

$$\therefore \boxed{L^{-1}\left[\frac{1}{s^2+4s+2}\right] = \frac{e^{-2t}}{\sqrt{2}}\sinh(\sqrt{2}t)}$$

8] Find the inverse Laplace Transform of $\frac{s^2+5}{s^3+6s^2+11s-6}$

→ $L^{-1}\left[\frac{s^2+5}{(s+1)(s+2)(s+3)}\right] = L^{-1}\left[\frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}\right]$

$$= L^{-1}\left[\frac{3}{s+1} - \frac{9}{s+2} + \frac{7}{s+3}\right]$$

$$= 3L^{-1}\left[\frac{1}{s+1}\right] - 9 \cdot L^{-1}\left[\frac{1}{s+2}\right] + 7\,L^{-1}\left[\frac{1}{s+3}\right]$$

$$= 3e^{-t} - 9e^{-2t} + 7e^{-3t}$$

$$\therefore \boxed{L^{-1}\left[\frac{s^2+5}{s^3+6s^2+11s-6}\right] = 3e^{-t} - 9e^{-2t} + 7e^{-3t}}$$

9] Find the inverse Laplace ~~Theorem~~ Transform by convolution theorem of

i] $\frac{1}{(s-2)(s+2)^2}$

→ $L^{-1}\left[\frac{1}{(s-2)(s+2)^2}\right] = L^{-1}\left[\frac{1}{(s^2-4)(s+2)}\right]$ —— (I)

let $f(s) = \frac{1}{s^2-4}$, $g(s) = \frac{1}{s+2}$

$\therefore L^{-1}[f(s)] = L^{-1}\left[\frac{1}{s^2-4}\right] = \frac{\sinh(2t)}{2} = f(t)$ —— (II)

$L^{-1}[g(s)] = L^{-1}\left[\frac{1}{s+2}\right] = e^{-2t} = g(t)$ —— (III)

Put $t=u$ in $f(t)$,

$\therefore f(u) = \dfrac{\sinh(2u)}{2}$

Put $t=t-u$ in $g(t)$

$\therefore g(t-u) = e^{-2(t-u)}$

$\therefore$ By convolution theorem, we have,

$$L^{-1}[f(s)\cdot g(s)] = \int_{u=0}^{u=t} f(u)\cdot g(t-u)\, du.$$

$$\therefore L^{-1}\left[\frac{1}{(s^2-4)}\cdot\frac{1}{(s+2)}\right] = \int_{u=0}^{u=t} \frac{\sinh(2u)}{2}\cdot e^{-2(t-u)}\, du.$$

$$= \int_{u=0}^{t} \frac{1}{2}\cdot\frac{e^{2u}-e^{-2u}}{2}\cdot e^{-2(t-u)}\, du.$$

$$= \frac{1}{4}\int_{u=0}^{t} \left(e^{2u-2t+2u} - e^{-2u-2t+2u}\right) du$$

$$= \frac{1}{4}\int_{u=0}^{t} \left(e^{4u-2t} - e^{-2t}\right) du.$$

$$= \frac{1}{4}\left[\frac{e^{4u-2t}}{4} - e^{-2t}(u)\right]_{u=0}^{t}$$

$$= \frac{1}{4}\left[\frac{e^{2t}}{4} - e^{-2t}\cdot(t) - \frac{e^{-2t}}{4} + 0\right]$$

$$= \frac{1}{4}\left[\frac{\sinh(2t)}{2} - t\cdot e^{-2t}\right]$$

$$\therefore \boxed{L^{-1}\left[\frac{1}{(s-2)(s+2)^2}\right] = \frac{1}{4}\left[\frac{\sinh(2t)}{2} - t\cdot e^{-2t}\right]}$$

ii] $\dfrac{s}{(s^2+4)(s^2+1)}$

$\rightarrow$ $L^{-1}\left[\dfrac{s}{(s^2+4)(s^2+1)}\right] = L^{-1}\left[\dfrac{s}{(s^2+4)} \cdot \dfrac{1}{(s^2+1)}\right] = L^{-1}[f(s) \cdot g(s)]$

$\therefore f(s) = \dfrac{s}{s^2+4}$ and $g(s) = \dfrac{1}{s^2+1}$

$L^{-1}[f(s)] = L^{-1}\left[\dfrac{s}{s^2+4}\right] = \cos(2t) = f(t).$

$L^{-1}[g(s)] = L^{-1}\left[\dfrac{1}{s^2+1}\right] = \sin t = g(t).$

Put $t = u$ in $f(t)$,

$f(u) = \cos(2u).$

Put $t = t-u$ in $g(t)$,

$g(t-u) = \sin(t-u).$

By convolution theorem,

$L^{-1}[f(s) \cdot g(s)] = \int_{u=0}^{t} f(u) \cdot g(t-u)\, du$

$\therefore L^{-1}\left[\dfrac{s}{(s^2+4)(s^2+1)}\right] = \int_{u=0}^{t} \cos(2u) \cdot \sin(t-u)\, du$

$= \cancel{\dfrac{1}{2}\int_{u=0}^{t} \cos(2u+t-u)}$

$= \dfrac{1}{2}\int_{u=0}^{t} [\sin(2u+t-u) - \sin(2u-t+u)]\, du$

$= \dfrac{1}{2}\int_{u=0}^{t} [\sin(u+t) - \sin(3u-t)]\, du$

$= \dfrac{1}{2}\left[-\cos(u+t) - \dfrac{\cos(3u-t)}{3}\right]_{u=0}^{t}$

$= \dfrac{1}{2}\left[\left\{-\cos(2t) - \dfrac{\cos(2t)}{3}\right\} - \left\{-\cos t - \dfrac{\cos(-t)}{3}\right\}\right]$

$= \frac{1}{2}\left\{ \frac{-4}{3}\cos(2t) + \frac{4}{3}\cos t \right\}$

$= \frac{1}{3}\left(2\cos t - 2\cos(2t) \right)$

$$\therefore \boxed{L^{-1}\left[\frac{s}{(s^2+4)(s^2+1)}\right] = \frac{1}{3}\left[2\cos t - 2\cos(2t)\right]} \therefore$$

10] Find the inverse laplace transform by convolution theorem.

$\frac{s}{(s^2+a^2)^2}$

$\rightarrow L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+a^2)}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)} \cdot \frac{1}{(s^2+a^2)}\right] = L^{-1}\left[f(s) \cdot g(s)\right]$

$\therefore$ let $f(s) = \frac{s}{s^2+a^2}$ & $g(s) = \frac{1}{s^2+a^2}$

$L^{-1}[f(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos(at) = f(t)$

$L^{-1}[g(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \sin(at) = g(t)$.

Put $t = u$ in $f(t)$

$f(u) = \cos(au)$

Put $t = t-u$ in $g(t)$

$g(t-u) = \sin(a(t-u))$.

$\therefore$ By convolution theorem,

$L^{-1}[f(s) \cdot g(s)] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$

$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} \cos(au)\sin(at-au)\, du$.

$= \int_{0}^{t} [\sin(at) - \sin(2au - at)]\, du$

$= \cancel{\dfrac{-\cos(at)u + \cos(2au - at)}{a \qquad\qquad 2}}$

$= \left[\sin(at) \cdot u + \dfrac{\cos(2au - at)}{2a} \right]_0^t$

$= t \cdot \sin(at) + \dfrac{\cos(at)}{2a} - 0 - \dfrac{\cos(-at)}{2a}$

$= t \cdot \sin(at)$.

$\therefore \boxed{L^{-1}\left[\dfrac{s}{(s^2+a^2)^2} \right] = t\sin(at)}$

11. Find the inverse Laplace Transform of $\log\left(\dfrac{s^2+4}{s^2+25}\right)$.

→ ~~By multiple by 't' prop.~~

$\cancel{L^{-1}[f(s)] = \dfrac{-1}{t} L^{-1} \dfrac{d}{ds} \quad \dfrac{-1}{t} L^{-1}\left[\dfrac{d}{ds} \log\left(\dfrac{s^2+4}{} \right.\right.}$

By multiple by 't' property,

$L^{-1}[f(s)] = \dfrac{-1}{t} L^{-1}\left[\dfrac{d}{ds} \log\left(\dfrac{s^2+4}{s^2+25} \right) \right]$

$= \dfrac{-1}{t} L^{-1}\left[\dfrac{d}{ds} \left\{ \log(s^2+4) - \log(s^2+25) \right\} \right]$

$= \dfrac{-1}{t} L^{-1}\left[\dfrac{2s}{s^2+4} - \dfrac{2s}{s^2+25} \right]$

$= \dfrac{-2}{t} (\cos 2t - \cos 5t)$

$\therefore \boxed{L^{-1}\left[\log\left(\dfrac{s^2+4}{s^2+25} \right) \right] = \dfrac{-2}{t} (\cos 2t - \cos 5t)}$

12] Find inverse laplace Transform of

i] $\frac{1}{(s^2+4s+13)^2}$

→ $L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right] = L^{-1}\left[\frac{1}{((s+2)^2+9)^2}\right]_{s+2 \to s}$ ... {FSP}

$= e^{-2t} L^{-1}\left[\frac{1}{(s^2+9)^2}\right]$

$= e^{-2t} L^{-1}\left[\frac{1}{s} \cdot \frac{s}{(s^2+9)^2}\right]$

$= e^{-2t} \int_{t=0}^{t} L^{-1}\left[\frac{s}{(s^2+9)^2}\right] dt$

~~Put $s^2+9 = u \Rightarrow 2s\,ds = dt$~~

$= e^{-2t} \int_{t=0}^{t} t \cdot L^{-1}\left[\int_{s=s}^{\infty} \frac{s}{(s^2+9)^2} ds\right] dt$

... {Division by 't' property}

Put $s^2+9 = u \Rightarrow 2s\,ds = du$

s	s	∞
u	s^2+9	∞

$L^{-1}[f(s)] = e^{-2t} \int_{t=0}^{t} t \cdot L^{-1}\left[\int_{u=s^2+9}^{\infty} \frac{1}{u^2} \frac{du}{2}\right] dt$

$= e^{-2t} \int_{t=0}^{t} t \cdot L^{-1}\left[\frac{-1}{2u}\right]_{s^2+9}^{\infty} dt$

$= e^{-2t} \int_{t=0}^{t} t \cdot L^{-1}\left[\frac{1}{2(s^2+9)}\right] dt$

$= \frac{e^{-2t}}{2} \int_{t=0}^{t} \frac{t}{3} \sin(3t)\, dt$

$= \frac{e^{-2t}}{2} \left[t\left(\frac{-\cos(3t)}{3}\right) - (1)\left(-\frac{\sin 3t}{3}\right)\right]_0^t$

$= \frac{e^{-2t}}{2} \left[\frac{\sin(3t) - t\cos(3t)}{3}\right]$

$\therefore \boxed{L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right] = \frac{e^{-2t}}{6}\left[\sin(3t) - t.\cos(3t)\right]}$

ii] $f(s) = \dfrac{36\, e^{-5s}}{(s+2)^4}$

$\rightarrow \quad L^{-1}\left[\dfrac{36\, e^{-5s}}{(s+2)^4}\right]$

By second shifting property,

let $\dfrac{36}{(s+2)^4} = g(s)$

$\therefore \quad L^{-1}[g(s)] = L^{-1}\left[\dfrac{36}{(s+2)^4}\right]_{s+2 \to s}$

$= e^{-2t}\, L^{-1}\left[\dfrac{36}{s^4}\right]$

$= e^{-2t} \cdot t^3 = g(t).$

Now, put $t = t-5$ in $g(t)$.

$g(t-5) = e^{-2(t-5)} \cdot (t-5)^3.$

$$\therefore \boxed{f(t) = \begin{cases} 0, & 0 < t < 5 \\ e^{-2(t-5)} \cdot (t-5)^3 & 5 < t < \infty. \end{cases}}$$