Mod-3-Fourier series (20 Marks)

Dr. Uday Kashid at 26-09-2022 10:43

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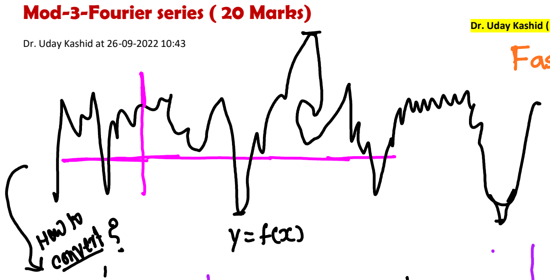

Fast Fouries -Transfor

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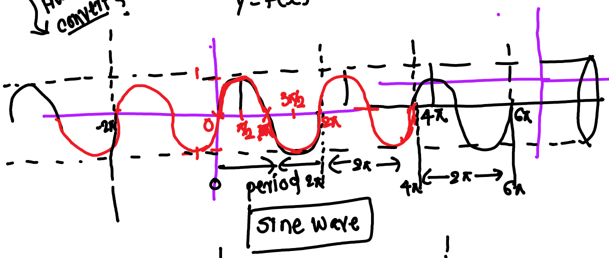
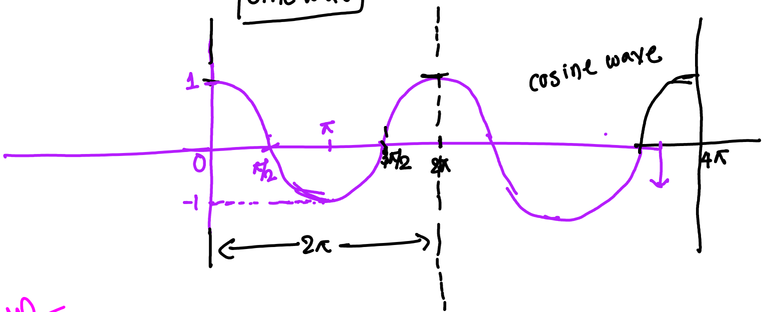


$\sin(2\pi+\theta) = \sin\theta$

period of fun

$\sin(P+\theta) = \sin\theta$



Defn) Period of the Function:-

IF $f(P+x) = f(x)$ Then 'p' is called period of $f(x)$

Ex. $f(x)=\cos(x)$ $\Rightarrow$ $p=2\pi$ $f(x+P)=\cos(2\pi+x)=\cos x$

$-1 \le f(x) \le 1$ sine cos

or period = $\boxed{\text{upper limit} - \text{Lower limit}}$

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ex② $f(x)=\tan x$ $f(x+P) = \tan(\pi+x) = \tan x$

period $=p=\pi$

$\tan x$ is discontineous fun

at $x=\pi/2 \Rightarrow \tan(\pi/2)=\infty$

$$f(x) = a_0\cos(0x) + a_1\cos(x) + a_2\cos(2x) + \cdots + a_n\cos(nx) + \cdots$$
$$b_0\sin(0x) + b_1\sin(x) + b_2\sin(2x) + \cdots + b_n\sin(nx)$$

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + 0 + \sum_{n=1}^{\infty} b_n\sin(nx)} \quad (1) \text{ Fourier series}$$

$0 \le x \le 2\pi$ or $-\pi \le x \le \pi$

where a_0, a_n, and b_n are called Fourier coefficients -

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} a_0\,dx + \sum_{n=1}^{\infty} a_n \int_0^{2\pi}\cos(nx)\,dx + \sum_{n=1}^{\infty} b_n \int_0^{2\pi}\sin(nx)\,dx$$

$$= a_0[x]_0^{2\pi} + \sum_{n=1}^{\infty} a_n\left[\frac{\sin(nx)}{n}\right]_0^{2\pi} + \sum_{n=1}^{\infty} b_n\left[\frac{-\cos nx}{n}\right]_0^{2\pi}$$

$$= a_0(2\pi-0) + \sum_{n=1}^{\infty} a_n\left[\frac{0-0}{n}\right] + \sum_{n=1}^{\infty} b_n\left[\frac{-1-(-1)}{n}\right]$$

$\sin(2n\pi)=0$ n is integer

$\sin(n\pi)=0$

$\cos(n\pi)=(-1)^n$

$\cos\pi = -1$

$\cos(2\pi) = 1$

$\cos(3\pi) = -1$

$\cos(2n\pi) = (-1)^{2n} = [(-1)^2]^n = [1]^n = 1$

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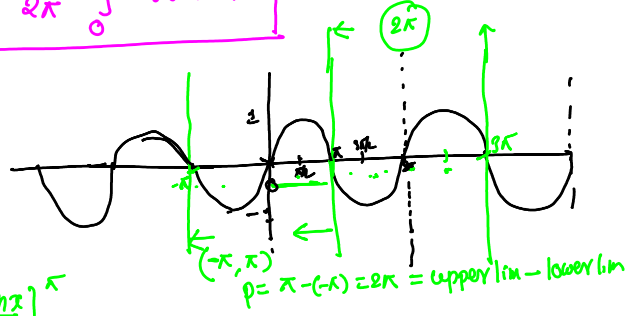
$$\int_0^{2\pi} f(x)\,dx = 2\pi a_0 + 0 + 0$$

$$a_0 = \frac{1}{\ \ } \int^{2\pi} f(x)\,dx$$

$$\int_0 f(x)\,dx = 2\pi a_0 + 0 + 0$$

$$\boxed{a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx}$$



$$\left[-\frac{\cos nx}{n}\right]_{-\pi}^{\pi}$$

$$-\frac{1}{n}\left[\cos(n\pi) - \underbrace{\cos(-n\pi)}_{\cos(n\pi)}\right] = -\frac{1}{n}\left[(-1)^n - (-1)^n\right] \to 0$$

Fourier series :- is decomposition of Random single into sine & cosine form. and expressed as

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right] \quad \text{--- (1)}$$

where period of sine and cosine functions is 2π.

Radian 2π ← period

$(0, 2\pi)$ $(-\pi, \pi)$

$(-\pi/2, \pi/2)$ π

$\frac{\pi}{2} - (-\pi/2) = \pi$

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case (I) If Range of $f(x)$ is $(0, 2\pi)$

$$\boxed{a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx}$$

$$\boxed{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx}$$

$$\boxed{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx}$$

case (II) Range $f(x) \to (-\pi, \pi)$

$$\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx}$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx}$$

$\pi \to L$ $(nx) \to \frac{n\pi x}{L}$

Note $\pi \Rightarrow L$

$(nx) \Rightarrow \frac{n\pi x}{L}$

2π

$(0, 2\pi)$

$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$

$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$

$(-\pi, \pi)$

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Neither even nor odd

$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

Even $f(-x) = f(x)$

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx)$

$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$

$b_n = 0$

odd $f(-x) = -f(x)$

$f(x) = \sum_{n=1}^{\infty} b_n\sin(nx)$

$a_0 = 0$

$a_n = 0$

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

Parseval's Identity:

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case ① $(0, 2\pi)$

$$\boxed{\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right]}$$

case ② $(-\pi, \pi)$

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right]}$$

Even

$$\boxed{\frac{1}{\pi}\int_0^{\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}a_n^2 + 0}$$

odd

$$\boxed{\frac{1}{\pi}\int_0^{\pi}[f(x)]^2\,dx = 0 + \frac{1}{2}\sum_{n=1}^{\infty}[0 + b_n^2] = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2}$$

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$f(x)$	E	E	O	O
$g(x)$	E	O	E	O
$\phi(x)=f(x)\cdot g(x)$	E	O	O	E

$\Rightarrow$ $f(x) \Rightarrow (-\pi, \pi)$

IF $n =$ Integer i.e. $n = \cdots -3, -2, -1, 0, 1, 2, 3, 4 \cdots$

(i) $\sin(n\pi) = 0$

(ii) $\sin(2n\pi) = 0$

(iii) $\sin(n\pm1)\pi = 0$

(iv) $\sin(2n\pm1)\pi = 0$

(i) $\cos(n\pi) = (-1)^n$

(ii) $\cos(2n\pi) = (-1)^{2n} = [(-1)^2]^n = 1$

(iii) $\cos(n\pm1)\pi = (-1)^{n\pm1}$
$= (-1)^n(-1)^{\pm1}$
$\cos(n\pm1)\pi = -(-1)^n$

(iv) $\cos(2n\pm1)\pi = (-1)^{2n\pm1} = (-1)^{2n}(-1)^{\pm1}$
$\cos(2n\pm1)\pi = -1$

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chain Rule of Integration:-

$$\int uv = [u][v_1] - [u'][v_2] + [u''][v_3] - \cdots$$

Ex ① Find Fourier series of $f(x) = x^2$ in $(0, 2\pi)$ and Hence deduce that

(Dec-09, 12, 17)

(i) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

(ii) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

$\rightarrow$ $f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$ — ①

$= a_0 + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$

$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{x=0}^{x=2\pi}$

$= \frac{1}{6\pi}[8\pi^3 - 0] = \boxed{\frac{4\pi^2}{3}}$ — (A)

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} \underset{u}{x^2}\,\underset{v}{\cos(nx)}\,dx$

$= \frac{1}{\pi}\left\{(x^2)\left[\frac{\sin nx}{n}\right] - (2x)\left[\frac{-\cos nx}{n^2}\right] + (2)\left[\frac{-\sin nx}{n^3}\right]\right\}_0^{2\pi}$

$$a_n = \frac{1}{\pi}\int_0 f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0 x \cos(nx)\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left[\frac{\sin nx}{n}\right] - (2x)\left[\frac{-\cos nx}{n\times n}\right] + (2)\left[\frac{-\sin nx}{n\times n\times n}\right]\right\}_{x=0}^{2\pi}$$

Note: If Range is $(0,2\pi)$ or $(-\pi,\pi)$ or $(0,\pi)$ Then All sine terms are becoming Zero

$$= \frac{1}{\pi}\left\{0 + \frac{2}{n^2}(2\pi)\cos(2n\pi) - \frac{2}{n^3}(0) - 0 - 0 - 0\right\}$$

$$a_n = \frac{1}{\pi}\left[\frac{4\pi}{n^2}(1)\right] = \boxed{\frac{4}{n^2}} \quad \text{--- (III)}$$

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$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left[\frac{-\cos nx}{n}\right] - (2x)\left[\frac{-\sin nx}{n\times n}\right] + (2)\left[\frac{\cos nx}{n\times n\times n}\right]\right\}_{x=0}^{x=2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(2\pi)^2}{n}\cos(2n\pi) + 0 + \frac{2}{n^3}\cos(2n\pi) + 0 + 0 - \frac{2}{n^3}\cos(0)\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n}(1) + \cancel{\frac{2}{n^3}(1)} - \cancel{\frac{2}{n^3}(1)}\right] = \boxed{\frac{-4\pi}{n}} \quad \text{--- (IV)}$$

$$\boxed{f(x) = x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(nx) + \sum_{n=1}^{\infty}\frac{-4\pi}{n}\sin(nx)} \quad \text{--- (V)}$$

put $x=0$ in eqn (V) But Range $(0,2\pi)$

$$0 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}(1) + 0$$

$$-\frac{\cancel{4}\pi^2}{3} = \cancel{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

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$$-\frac{\pi^2}{3} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\to -\frac{\pi^2}{3} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \text{--- (A)}$$

put $x=2\pi$

$$(2\pi)^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(2n\pi) + \sum_{n=1}^{\infty}\frac{-4\pi}{n}\sin(2n\pi)$$

$\sin 2n\pi = 0$
$\cos 2n\pi = 1$

$$\cancel{4}\pi^2 = \frac{\cancel{4}\pi^2}{3} + \cancel{4}\sum_{n=1}^{\infty}\frac{1}{n^2}(1) + 0$$

$$\pi^2 - \frac{\pi^2}{3} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{2\pi^2}{3} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \to \text{(B)}$$

let (A) + (B)

$$-\frac{\pi^2}{3} + \frac{2\pi^2}{3} = 2\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\boxed{\frac{\pi^2}{3\times 2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots}$$

$(0, 2\pi)$

$x=0$, $x=2\pi$ $x=\pi$ $x=\pi/2$, $x=\pi/4$

(II) put $x=\pi$ in eqn (V)

$$(\pi)^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(n\pi) + \sum_{n=1}^{\infty}\frac{-4\pi}{n}\sin(n\pi)$$

$\sin(n\pi) = 0$, $\cos(n\pi) = (-1)^n$

$$(\pi) = \frac{\pi^2}{3} + \sum_{n=1} \frac{}{n^2} \cos(n\pi) \ldots \sum_{n=1} \ldots n$$

$\sin(n\pi) = 0$, $\cos(n\pi) = (-1)^n$

$$\pi^2 - \frac{4\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n + 0$$

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$$\frac{-\pi^2}{3} = 4\left[\frac{1}{1^2}(-1)^1 + \frac{1}{2^2}(-1)^2 + \frac{1}{3^2}(-1)^3 + \frac{1}{4^2}(-1)^4 + \cdots\right]$$

$$-\frac{\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

Mech/civil

May-12,13,15,16

Dec-09,10

Ex. (2) Find F.S. of $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $\boxed{0 \le x \le 2\pi}$

Hence deduce that

$x=0$ (1) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

$x=\pi$ (2) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Addition

(3) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

(4) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$

→ We have $[0, 2\pi]$

$$f(x) = \frac{(\pi - x)^2}{4} = a_0 + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)] \quad \text{--- (1)}$$

(1) $a_0 = \frac{1}{2\pi}\int_{x=0}^{x=2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4}\,dx$

$\therefore$ let $\pi - x = y$, $\int y^2 = \frac{y^3}{3}$

$$= \frac{1}{8\pi}\left[\frac{(\pi - x)^3}{3(-1)}\right]_{x=0}^{2\pi} = \frac{1}{8\pi(-3)}\left[(-\pi)^3 - (\pi - 0)^3\right]$$

$$= \frac{1}{8\pi(-3)}\left[-\pi^3 - \pi^3\right] = \frac{-2\pi^3}{8\pi(-3)} = \boxed{\frac{\pi^2}{12}} = a_0$$

(2) $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{4\pi}\int_0^{2\pi}(\pi - x)^2 \cos(nx)\,dx$

$$= \frac{1}{4\pi}\left\{(\pi - x)^2\left[\frac{\sin(nx)}{n}\right] - \left[2(\pi - x)(-1)\right]\left[\frac{-\cos(nx)}{n \times n}\right] + \left[2(-1)(-1)\right]\left[\frac{-\sin nx}{n^2 \times n}\right]\right\}_{x=0}^{2\pi}$$

$$= \frac{1}{4\pi}\left[-\frac{2(\pi - x)\cos(nx)}{n^2}\right]_{x=0}^{2\pi} = -\frac{2}{4\pi n^2}\left[(-\pi)(1) - \pi(1)\right]$$

$$= \frac{-2(-2\pi)}{4\pi n^2} = \boxed{\frac{1}{n^2}} = a_n$$

put $n = 0$, $n = 1, -\infty$, $a_0 \to$ discon

(3) $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx = \frac{1}{4\pi}\int_0^{2\pi}(\pi - x)^2 \sin(nx)\,dx$

$$= \frac{1}{4\pi}\left\{(\pi - x)^2\left[\frac{-\cos(nx)}{n}\right] - \left[2(\pi - x)(-1)\right]\left[\frac{-\sin nx}{n \times n}\right] + \left[2(-1)(-1)\right]\left[\frac{\cos nx}{n^2 \times n}\right]\right\}_{x=0}^{2\pi}$$

$$= \frac{1}{4\pi}\left[-\frac{(\pi - x)^2\cos(nx)}{n} + \frac{2\cos nx}{n^3}\right]_{x=0}^{x=2\pi}$$

$$= \frac{1}{4\pi}\left[-\frac{(-\pi)^2}{n}(1) + \cancel{\frac{2}{n^3}(1)} - \left[-\frac{(\pi - 0)^2(1)}{n}\right] - \cancel{\frac{2}{n^3}(1)}\right] \quad \therefore \cos(2n\pi) = 1$$

$$= \frac{1}{4\pi}\left[-\frac{\pi^2}{n} + \frac{\pi^2}{n}\right] = 0 = b_n$$

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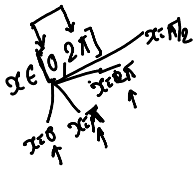
∞ $\quad \frac{1}{}(\pi - x)^2 \quad \pi^2 \quad \infty$



$$f(x)=\frac{(\pi-x)^2}{4} = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(nx)+\sum_{n=1} 0\sin(nx)$$

$$\boxed{\frac{(\pi-x)^2}{4} = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\boxed{\frac{1}{n^2}}\cos(nx)} \quad \text{—(II)} \rightarrow \text{cosine series Fourier series}$$

a_n

① put $x=0$ in eqⁿ (II)

$$\frac{(\pi-0)^2}{4}=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}(1)$$

$$\frac{\pi^2}{4}-\frac{\pi^2}{12}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{3\pi^2-\pi^2}{12}=\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots \quad \text{—(III)}$$

② We know $\cos(n\pi)=(-1)^n$ — odd: -1; even: 1

put $x=\pi$ in eqⁿ (II)

$$\frac{(\pi-\pi)^2}{4}=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(n\pi)$$

$$0-\frac{\pi^2}{12}=\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n$$

$$-\frac{\pi^2}{12}=\frac{1}{1^2}(-1)+\frac{1}{2^2}(1)+\frac{1}{3^2}(-1)+\frac{1}{4^2}(1)+\cdots$$

$$\boxed{\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots} \quad \text{—(IV)}$$

③ Add equⁿ (III) + (IV)

$$\frac{\pi^2}{6}+\frac{\pi^2}{12}=\left[\frac{1}{1^2}+\cancel{\frac{1}{2^2}}+\frac{1}{3^2}+\cancel{\frac{1}{4^2}}+\cdots\right]+\left[\frac{1}{1^2}-\cancel{\frac{1}{2^2}}+\frac{1}{3^2}-\cancel{\frac{1}{4^2}}+\cdots\right]$$

$$\frac{3\pi^2}{12}=2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right]$$

$$\boxed{\frac{\pi^2}{4\times 2}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots} \quad \text{—(V)}$$

④ By parseval's Identity in $[0,2\pi]$

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2dx = a_0^2+\frac{1}{2}\sum_{n=1}^{\infty}[a_n^2+b_n^2]$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{(\pi-x)^2}{4}\right]^2dx=\left(\frac{\pi^2}{12}\right)^2+\frac{1}{2}\sum_{n=1}^{\infty}\left[\left(\frac{1}{n^2}\right)^2+(0)^2\right]$$

$$\frac{1}{2\pi\times 16}\int_0^{2\pi}(\pi-x)^4dx=\frac{\pi^4}{144}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{32\pi}\left[\frac{(\pi-x)^5}{5(-1)}\right]_{x=0}^{2\pi}=\frac{\pi^4}{144}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$-\frac{1}{32\pi(5)}\left[(\pi-2\pi)^5-(\pi-0)^5\right]=\frac{\pi^4}{144}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{-32\pi(5)}\left[-\pi^5-\pi^5\right]=\frac{\pi^4}{144}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{-2\pi^5}{-32\pi(5)}=\frac{\pi^4}{144}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{80}-\frac{\pi^4}{144}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

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$$\frac{64\pi^4}{80\times 144}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{64\pi^4}{80\times 144} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{180}\times 2 = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\boxed{\frac{\pi^4}{90} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots}$$

Ex. (3) Find fourier series of $f(x) = x\sin(x)$ in $(0, 2\pi)$

$\rightarrow$ $f(x) = x\sin x = a_0 + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$ — (1)

$n = 1, \cdots \infty$
a_n
a_1
a_2
a_3
$\vdots$
a_n

(i) $a_0 = \frac{1}{2\pi}\int_{x=0}^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin(x)\,dx$

$= \frac{1}{2\pi}\left[x[-\cos x] - (1)[-\sin x]\right]_{x=0}^{2\pi}$

$= \frac{1}{2\pi}[-2\pi(1) - 0] = \boxed{-1} = a_0$

(ii) $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos(nx)\,dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x[2\cos(nx)\sin x]$ $\quad\because 2C_AS_B = S_{A+B} - S_{A-B}$

***

$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]\,dx$

$= \frac{1}{2\pi}\left\{x\left[\frac{-\cos(n+1)x}{(n+1)} - \frac{-\cos(n-1)x}{(n-1)}\right] - (1)\left[\frac{-\sin(n+1)x}{(n+1)(n+1)} + \frac{\sin(n-1)x}{(n-1)(n-1)}\right]\right\}_{x=0}^{x=2\pi}$

$= \frac{1}{\cancel{2\pi}}\left\{\cancel{2\pi}\left[\frac{-\cos 2(n+1)\pi}{(n+1)} + \frac{\cos 2(n-1)\pi}{(n-1)}\right] - 0\right\}$

$n = 1 \cdots \infty$
odd even

But $\cos 2(n\pm1)\pi = (-1)^{2(n\pm1)} = 1$

$= \frac{-1}{(n+1)} + \frac{1}{(n-1)} = \frac{-(n-1)+(n+1)}{(n-1)(n+1)} = \frac{2}{(n-1)(n+1)}$

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$\boxed{a_n = \frac{2}{(n-1)(n+1)}}$, $n \neq 1$, $n = 2, 3, 4, \cdots \infty$ $\quad a_1 = \infty$

at $n = 1$, we have discontinuity.

put $n = 1$ in Formula of a_n

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin(x)\cos(1x)\,dx$

$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\left[\frac{2\sin x\cos x}{2}\right]dx = \frac{1}{2\pi}\int_0^{2\pi} x[\sin(2x)]\,dx$

$= \frac{1}{2\pi}\left[x\left[-\frac{\cos 2x}{2}\right] - (1)\left[\frac{-\sin 2x}{2\times 2}\right]\right]_0^{2\pi}$

$= \frac{1}{2\pi}\left[-\frac{2\pi}{2}(1) - 0\right]$

$\boxed{a_1 = -\frac{1}{2}}$

$\because \cos 4\pi = 1$
$\cos n\pi = (-1)^n$

$\boxed{n = 1}$ $\quad n=1$
a_1 $\quad b_1$ $\quad b_n$

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(4) $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$

$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin(nx)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x[2\sin(nx)\sin(x)]\,dx$

$\because 2S_AS_B = C_{A-B} - C_{A+B}$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x \sin(nx)\,dx = \frac{1}{2\pi}\int_0 x[2\sin(nx)\sin(x)]\,dx$$

$\therefore\ 2S_AS_B = C_{A-B} - C_{A+B}$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n-1)x - \cos(n+1)x]\,dx$$

$$= \frac{1}{2\pi}\left\{ x\left[\frac{\cancel{\sin(n-1)x}}{(n-1)} - \frac{\cancel{\sin(n+1)x}}{(n+1)}\right] - (1)\left[\frac{-\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right]\right\}_{x=0}^{2\pi}$$

$$b_n = \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

But $\cos 2(n\pm1)\pi = 1$

$$b_n = \frac{1}{2\pi}\left[\cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{(n+1)^2}} - \cancel{\frac{1}{(n-1)^2}} + \cancel{\frac{1}{(n+1)^2}}\right]$$

$b_n = 0$ for $\forall n$ But $n \neq 1$

b_1 has discontinuity

Hence, put $n=1$ in formula of b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x \sin(1x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\left[\frac{1-\cos 2x}{2}\right]dx$$

$$= \frac{1}{2\pi}\left\{ x\left[x - \frac{\sin 2x}{2}\right] - (1)\left[\frac{x^2}{2} + \frac{\cos(2x)}{2\times 2}\right]\right\}_{x=0}^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi[2\pi - 0] - \frac{4\pi^2}{2} - \cancel{\frac{1}{4}} - 0 + 0 + \cancel{\frac{1}{4}}\right]$$

$$b_1 = \frac{1}{2\pi}\left[4\pi^2 - \frac{4\pi^2}{2} + 0\right] = \frac{1}{2\pi}\left[2\pi^2\right] = \pi \doteq \text{finite}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

$$x\sin x = a_0 + a_1\cos(1x) + \sum_{n=2}^{\infty} a_n\cos(nx) + b_1\sin(1x) + \sum_{n=2}^{\infty} b_n\sin(nx)$$

$$\boxed{x\sin x = -1 - \frac{1}{2}\cos(x) + \sum_{n=2}^{\infty}\frac{2}{(n-1)(n+1)}\cos(nx) + \pi\sin x + 0}$$

Type: II $[-\pi, \pi]$ Dr. Uday Kashid (PhD, Mathematics)

In this range, it is MUST to test either function is even or odd

IF $f(-x) = f(x) \Rightarrow$ Even Fu^n $\Rightarrow \boxed{b_n = 0}$

$f(-x) = -f(x) \Rightarrow$ odd Fu^n $\Rightarrow \boxed{a_0 = 0}\ \boxed{a_n = 0}$

Ex.① Find Fourier series of $f(x) = \frac{x(\pi - x)(\pi + x)}{12}$, $-\pi < x < \pi$

and Find sum of $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$ $\pm\pi$ 0 $\pi/2$

$\rightarrow$ $f(x) = \frac{x(\pi - x)(\pi + x)}{12} = \frac{x(\pi^2 - x^2)}{12}$

$$f(-x) = \frac{-x[\pi^2 - (-x)^2]}{12} = \frac{-x[\pi^2 - x^2]}{12} = -\left[\frac{x(\pi^2 - x^2)}{12}\right]$$

$\boxed{f(-x) = -f(x)} \Rightarrow f(x)$ is odd Fun.

$\Rightarrow \boxed{a_0 = 0}$ and $\boxed{a_n = 0}$

$\boxed{f(-x) = -f(x)} \Rightarrow$ f(x) is odd Fun

$\Rightarrow \boxed{a_0 = 0}$ and $\boxed{a_n = 0}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{x(\pi^2-x^2)}{12}\sin(nx)\,dx = \frac{1}{6\pi}\int_0^{\pi}(\pi^2 x - x^3)\sin(nx)\,dx$$

$$b_n = \frac{1}{6\pi}\left\{(\pi^2 x - x^3)\left[\frac{-\cos nx}{n}\right] - (\pi^2 - 3x^2)\left[\frac{-\sin nx}{n\times n}\right] + (0-6x)\left[\frac{\cos(nx)}{n^2\times n}\right] - (-6)\left[\frac{\sin(nx)}{n^3\times n}\right]\right\}_{x=0}^{x=\pi}$$

$$b_n = \frac{1}{6\pi}\left\{0 - \frac{6\pi(-1)^n}{n^3} - 0 - 0\right\} = \boxed{\frac{-(-1)^n}{n^3}}$$

$$\frac{x(\pi^2-x^2)}{12} = 0 + \sum_{n=1}^{\infty} 0\cos(nx) + \sum_{n=1}^{\infty}\frac{-(-1)^n}{n^3}\sin(nx)$$

$$\boxed{\frac{x(\pi^2-x^2)}{12} = \sum_{n=1}^{\infty}\frac{-(-1)^n}{n^3}\sin(nx)} \quad \text{— (I)}$$

× $x=0$ ×

× $x=\pi/-\pi$

$x=\pi/2$

$x=\pi/2$

$$\sin\left(\frac{n\pi}{2}\right) < \begin{matrix} 1, -1, & n=1,3 \\ 0 & n=2,4 \end{matrix}$$

put $x=\pi/2$ in eq (I)

$$\frac{\frac{\pi}{2}\left(\pi^2 - \frac{\pi^2}{4}\right)}{12} = \sum_{n=1}^{\infty}\frac{-(-1)^n}{n^3}\sin(n\pi/2)$$

$$\frac{\frac{\pi}{2}\left(\frac{3\pi^2}{4}\right)}{12} = \frac{-(-1)}{1^3}(1) + \frac{-(-1)^2}{2^3}(0) + \frac{-(-1)}{3^3}(-1) + 0 + \frac{-(-1)}{5^3}(1) + 0 + \cdots$$

$$\boxed{\frac{\pi^3}{32} = \frac{1}{1^3} + 0 - \frac{1}{3^3} + 0 + \frac{1}{5^3} + 0 - \frac{1}{7^3} + \cdots}$$

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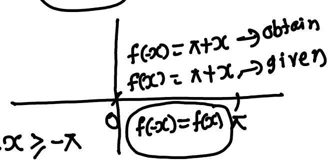
Ex. (2) $f(x) = \begin{cases} \pi - x, & -\pi \le x \le 0 \\ \pi + x, & 0 \le x \le \pi \end{cases}$ $[-\pi, \pi]$

$\rightarrow$ $f(-x) = \pi - (-x)$; $-\pi \le -x \le 0 \Rightarrow \pi \ge x \ge 0$

$f(-x) = \pi + x$; $0 \le x \le \pi$

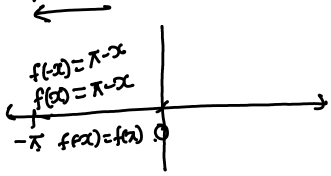
$\therefore \boxed{f(-x) = f(x),\ 0 \le x \le \pi}$ — (I)



$f(-x) = \pi + (-x)$; $0 \le -x \le \pi \Rightarrow 0 \ge x \ge -\pi$

$f(-x) = \pi - x$; $-\pi \le x \le 0$

$\Rightarrow \boxed{f(-x) = f(x)\ ; -\pi \le x \le 0}$ — (II)



From eq (I) & (II)

$f(-x) = f(x)$; $[-\pi, \pi]$ $\Rightarrow$ Even Function

$\boxed{b_n = 0}$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) \quad \text{— (III)}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi + x)\,dx$$

Even

$$\left[\ \ \pi^2\ \ \pi\ \ \right] = \frac{1}{\ }\left[\pi^2 + \frac{\pi^2}{\ }\ \ \right] = \frac{1}{\ }\left[\frac{3\pi^2}{\ }\right]$$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$ ... Even

$$a_0 = \frac{1}{\pi}\left[\pi x + \frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[\pi^2 + \frac{\pi^2}{2} - 0 - 0\right] = \frac{1}{\pi}\left[\frac{3\pi^2}{2}\right]$$

$$\boxed{a_0 = \frac{3\pi}{2}}$$

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

Even Even

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi + x)\cos(nx)\,dx = \frac{2}{\pi}\left\{(\pi+x)\left[\frac{\sin(nx)}{n}\right] - (1)\left[\frac{-\cos(nx)}{n^2}\right]\right\}_{x=0}^{x=\pi}$$

$$= \frac{2}{\pi}\left[0 + \frac{\cos(nx)}{n^2}\right]_{x=0}^{x=\pi} = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$\because \cos(n\pi) = (-1)^n$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} 0 & ; n = 2,4,6,\ldots \ (n \Rightarrow \text{Even}) \\ \frac{-4}{\pi n^2} & ; n = 1,3,5,\ldots \ (n = \text{odd}) \end{cases}$$

-2 $(-1-1)$

$\therefore [(-1)^n \pm 1]$ Even odd

Hence Fourier series of $f(x)$ is given as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) = \boxed{\frac{3\pi}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{-4}{\pi n^2}\cos(nx)}$$

Ex. 3 $f(x) = \cos x \quad ; -\pi \le x \le 0$
$\quad\quad = -\cos x \quad ; 0 \le x \le \pi$
Find F.S. of $f(x)$ $[-\pi, \pi]$

$\rightarrow$ $f(-x) = \cos(-x) \ ; \ -\pi \le -x \le 0 \Rightarrow \pi \ge x \ge 0$

$f(-x) = \cos(x) \quad ; \ 0 \le x \le \pi$ — (1)

$f(x) = -\cos x \quad ; \ 0 \le x \le \pi \rightarrow$ given (11)

$$\boxed{f(-x) = -f(x) \quad 0 \le x \le \pi} \text{ — (A)}$$

$f(-x) = -[\cos(-x)] \ ; \ 0 \le -x \le \pi \Rightarrow 0 \ge x \ge -\pi$

$f(-x) = -\cos(x) \quad ; -\pi \le x \le 0$

given $f(x) = \cos(x) \quad -\pi \le x \le 0$

$\}\ f(-x) = -f(x) \ ; -\pi \le x \le 0$ — (B)

$\Rightarrow f(-x) = -f(x)$, $[-\pi, \pi]$

$\Rightarrow$ Fun is odd Fun

$\boxed{a_0 = 0}$ $\boxed{a_n = 0}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

odd odd

$$b_n = \frac{2}{\pi}\int_{x=0}^{x=\pi} -\cos(x)\sin(nx)\,dx$$

$$= -\frac{1}{\pi}\int_0^{\pi} 2\sin(nx)\cos(1x)\,dx$$

$\because 2S_AC_B = S_{A+B} + S_{A-B}$

$$= -\frac{1}{\pi}\int_0^{\pi}\left[\sin(nx + x) + \sin(nx - x)\right]dx$$

$$= -\frac{1}{\pi}\left[\frac{-\cos(n+1)x}{(n+1)} - \frac{\cos(n-1)x}{(n-1)}\right]_{x=0}^{x=\pi}$$

$$= -\frac{1}{\pi}\left[\frac{-\cos(n+1)x}{(n+1)} - \frac{\cos(n-1)x}{(n-1)}\right]_{x=0}^{x=\pi}$$

$$\boxed{\begin{aligned}\cos(n+1)\pi &= (-1)^{n+1} = (-1)^n(-1)^1 = -(-1)^n \\ \cos(n-1)\pi &= (-1)^{n-1} = (-1)^n(-1)^{-1} = -(-1)^n\end{aligned}}$$

$$b_n = -\frac{1}{\pi}\left\{\frac{-[-(-1)^n]}{(n+1)} - \frac{[-(-1)^n]}{(n-1)}\right\} = -\frac{1}{\pi}\left[\frac{(-1)^n}{(n+1)} + \frac{(-1)^n}{(n-1)}\right]$$

$$= -\frac{1}{\pi}\left[\frac{(n-1)+(n+1)}{(n+1)(n-1)}\right](-1)^n = \boxed{\frac{-2n(-1)^n}{(n^2-1)\pi}}$$

Fourier series of

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

$$\boxed{= 0 + 0 + \sum_{n=1}^{\infty} \frac{-2n(-1)^n}{(n^2-1)\pi}\sin(nx)}$$

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Ex 5 Find F.S. of $f(x) = x + x^2$; $[-\pi, \pi]$

x → odd, x^2 → Even

Method-I

→ $f(-x) = (-x) + (-x)^2 = -x + x^2$

$f(-x) = -x + x^2$ $\quad$ $f(-x) = f(x)$ → Even ✗

$f(x) = x + x^2 = -(-x - x^2)$ $\quad$ $f(-x) = -f(x)$ → odd

⇒ $f(x)$ is neither even nor odd $(-\pi, \pi)$

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)$$

Method-II

$$f(x) = x + x^2 = f_1(x) + f_2(x)$$

F.S. of $f(x)$ = (F.S. of $f_1(x)$) + F.S. of $f_2(x)$

$f_1(x) = x$ → odd fun $\quad a_0 = 0,\ a_n = 0,$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$\text{F.S. of } f_1(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + \boxed{\sum_{n=1}^{\infty} b_n\sin(nx)} \quad \text{—(I)}$$

We have $f_2(x) = x^2$ → Even fun

$b_n = 0$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$\text{F.S. of } [f_2(x)] = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + 0 \quad \text{—(II)}$$

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General case $[\,\mathbf{0}, \mathbf{2L}\,]$ or $[\,-\mathbf{L}, \mathbf{L}\,]$

Radian $[0, 2\pi]$ or $[-\pi, \pi]$

$\pi \Leftrightarrow L$

$\rightarrow (nx) \Leftrightarrow \left(\frac{n\pi x}{L}\right)$

Ex ① Find Fourier series of $f(x) = e^{3x}$; $(0,3) = (0, 2L)$

$(3-0) = 3 = 2L$

$\rightarrow$ $3 = 2L \Rightarrow \boxed{L = 3/2}$

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)} \quad (0, 2L) \quad -①$$

$$a_0 = \frac{1}{2L}\int_0^{2L} f(x)\,dx = \frac{1}{2(3/2)}\int_0^3 e^{3x}\,dx = \frac{1}{3}\left[\frac{e^{3x}}{3}\right]_{x=0}^{3}$$

$$= \frac{1}{9}\left[e^9 - e^0\right] = \frac{1}{9}\left[e^9 - 1\right] \quad -⑪$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{(3/2)}\int_0^3 e^{3x}\cos\left(\frac{n\pi x}{3/2}\right)dx = \frac{2}{3}\int_0^3 e^{3x}\cos\left(\frac{2n\pi}{3}x\right)dx$$

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

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$$= \frac{2}{3}\left\{\frac{e^{3x}}{3^2 + \left(\frac{2n\pi}{3}\right)^2}\left[3\cos\left(\frac{2n\pi x}{3}\right) + \frac{2n\pi}{3}\sin\left(\frac{2n\pi x}{3}\right)\right]\right\}_{x=0}^{x=3}$$

$$= \frac{2}{3}\left\{\frac{e^9}{9 + \left(\frac{2n\pi}{3}\right)^2}\left[3\cos(2n\pi) + \frac{2n\pi}{3}\sin(2n\pi)\right]\right\}$$

$$- \frac{2}{3}\left\{\frac{e^0}{3^2 + \left(\frac{2n\pi}{3}\right)^2}\left[3(1) + \frac{2n\pi}{3}(0)\right]\right\}$$

$$\boxed{\begin{aligned}\sin(2n\pi) &= 0\\ \cos(2n\pi) &= 1\end{aligned}}$$

$$= \frac{2}{\cancel{3}\left[9 + \left(\frac{2n\pi}{3}\right)^2\right]}\left\{e^9[\cancel{3}] - 1(\cancel{3})\right\} = \frac{2}{9 + \left(\frac{2n\pi}{3}\right)^2}\left[e^9 - 1\right]$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \rightarrow \text{Home Work.}$$

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Ex.② Find F.S. of $f(x) = \begin{cases} \pi x & ; \ 0 \le x < 1 \\ 0 & \ x = 1 \Rightarrow 1 \le x \le 1 \\ \pi(x-2) & \ 1 < x \le 2 \end{cases}$ $[0, 2]$

$\rightarrow$ $[0, 2] = [0, 2L]$ $\Rightarrow \boxed{L = 1}$

$\pi \rightarrow L$

$nx \rightarrow \left(\frac{n\pi x}{1}\right)$

$\rightarrow \quad [0,2] = [0, 2L] \quad \Rightarrow \boxed{L=1}$ $\qquad \pi \rightarrow L$, $\quad n x \rightarrow \left(\frac{n\pi x}{L}\right)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{---} (1)$$

$$a_0 = \frac{1}{2L}\int_0^{2L} f(x)\,dx = \frac{1}{2(1)}\int_0^2 f(x)\,dx$$

$$= \frac{1}{2}\left\{ \int_0^1 \pi x\,dx + \int_{x=1}^{x=1} 0\,dx + \int_1^2 \pi(x-2)\,dx \right\}$$

Note: $x=1$, $1 \le x \le 1$

$$= \frac{1}{2}\left\{ \pi\left[\frac{x^2}{2}\right]_0^1 + 0 + \pi\left[\frac{x^2}{2} - 2x\right]_1^2 \right\}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0 + \pi\left[(2-4) - \left(\frac{1}{2} - 2\right)\right] \right\} = \frac{1}{2}\left[\frac{\pi}{2} + \pi\left[-2 + \frac{3}{2}\right]\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\pi}{2}\right] = 0$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{1}\int_0^2 f(x)\cos(n\pi x)\,dx$$

$$= \int_0^1 \pi x\cos(n\pi x)\,dx + \int_{x=1}^{x=1} 0\cdot\cos(n\pi x)\,dx + \int_1^2 \pi(x-2)\cos(n\pi x)\,dx$$

$$= \pi\left\{ x\left[\frac{\sin(n\pi x)}{(n\pi)}\right] - (1)\left[\frac{-\cos(n\pi x)}{(n\pi)(n\pi)}\right] \right\}_{x=0}^{x=1} + 0 + \pi\left[(x-2)\left[\frac{\sin(n\pi x)}{(n\pi)}\right] - (1)\left[\frac{-\cos(n\pi x)}{(n\pi)(n\pi)}\right] \right\}_{x=1}^{x=2}$$

$$= \pi\left\{ (0-0) + \frac{\cos(n\pi)}{(n^2\pi^2)} - \frac{1}{n^2\pi^2} \right] + \pi\left[(0-0) + \frac{\cos(2n\pi)}{n^2\pi^2} - \frac{\cos(n\pi)}{n^2\pi^2} \right]$$

$(-1)^n$

$\cos(n\pi) = (-1)^n \qquad \sin(n\pi) = 0$

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$\cos(2n\pi) = 1 \qquad \sin(2n\pi) = 0$

$$= \pi\left\{ \frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{1}{n^2\pi^2} - \frac{(-1)^n}{n^2\pi^2} \right] = 0$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{1}\int_0^2 f(x)\sin(n\pi x)\,dx$$

$$= \int_0^1 \pi x\sin(n\pi x)\,dx + \int_1^1 0\,\sin(n\pi x)\,dx + \int_1^2 \pi(x-2)\sin(n\pi x)\,dx$$

$$= \pi\left\{ x\left[\frac{-\cos(n\pi x)}{n\pi}\right] - (1)\left[\frac{-\sin(n\pi x)}{(n\pi)(n\pi)}\right] \right\}_{x=0}^{1} + 0$$

$$+\pi\left\{ (x-2)\left[\frac{-\cos(n\pi x)}{(n\pi)}\right] - (1)\left[\frac{-\sin(n\pi x)}{(n\pi)(n\pi)}\right] \right\}_{x=1}^{2}$$

$$b_n = \pi\left[\qquad \right] + \pi\left[0 - \frac{(-1)^n}{n\pi} - 0\right]$$

$$b_n = \pi\left[-\frac{(1)(-1)^n}{n\pi} - 0 + 0 + 0\right] + \pi\left[0 - \frac{(-1)^n}{n\pi} - 0\right]$$

$$= \pi\left[-\frac{(-1)^n}{n\pi}\right] = \frac{\pi}{n\pi}\left[-2(-1)^n\right] = \boxed{-\frac{2(-1)^n}{n}}$$

Replace $a_0 = 0$, $a_n = 0$ and $b_n = -\frac{2(-1)^n}{n}$ also $L=1$ in eq (1)

Hence F.S of given function

$$f(x) = 0 + \sum_{n=1}^{\infty} 0 + \sum \sin(n\pi x)$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin(n\pi x)} \longrightarrow \text{Ans}$$

Ex. (3) Find Fourier series of $f(x) = x^2$, $0 < x < 4$

$\rightarrow$ $0 < x < 4 \Rightarrow (0,4) \equiv (0, 2L) \Rightarrow \boxed{L = 2}$

$$f(x) = x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{— (1)}$$

(1) $$a_0 = \frac{1}{2L}\int_0^{2L} f(x)\,dx = \frac{1}{4}\int_0^4 x^2\,dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_{x=0}^{4} = \frac{1}{4\times3}\left[4\times4\times4 - 0\right]$$

$$\boxed{a_0 = \frac{16}{3}}$$

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(2) $$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_0^4 x^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left\{x^2\left[\frac{\sin(n\pi x/2)}{(n\pi/2)}\right] - (2x)\left[\frac{-\cos(n\pi x/2)}{(n\pi/2)(n\pi/2)}\right] + (2)\left[\frac{-\sin(n\pi x/2)}{(n\pi/2)^2(n\pi/2)}\right]\right\}_{x=0}^{x=4}$$

$$a_n = \frac{1}{2}\left\{(0-0) + 8\frac{(1)}{(n\pi/2)^2} - 0 + 0 - 0\right]$$

$\sin(2n\pi) = 0$ $\quad$ $\cos(2n\pi) = 1$

$$a_n = \frac{4}{n^2\pi^2/4} = \boxed{\frac{16}{n^2\pi^2}}$$

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$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_0^4 x^2 \sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left\{x^2\left[\frac{-\cos(n\pi x/2)}{(n\pi/2)}\right] - (2x)\left[\frac{-\sin(n\pi x/2)}{(n\pi/2)(n\pi/2)}\right] + (2)\left[\frac{\cos(n\pi x/2)}{(n\pi/2)^2(n\pi/2)}\right]\right\}_{x=0}^{x=4}$$

$$= \frac{1}{2}\left[\frac{-16(1)}{(n\pi/2)} - 0 + 0 - 0 + \frac{2(1)}{(n\pi/2)^3} - \frac{2(1)}{(n\pi/2)^3}\right]$$

$$= \frac{1}{2}\left[\frac{-16(1)}{(n\pi/2)} - 0 + 0 - 0 + \cancel{\frac{2(1)}{(n\pi/2)^3}} - \cancel{\frac{2(1)}{(n\pi/2)^3}}\right]$$

$$\boxed{b_n = \frac{-16}{n\pi}}$$

$$\boxed{f(x) = x^2 = \frac{16}{3} + \sum_{n=1}^{\infty} \frac{16}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} \frac{-16}{n\pi}\sin\left(\frac{n\pi x}{2}\right)}$$

Ex. 4 Find Fourier series of $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi} & 0 \le x \le \pi \end{cases}$

$\rightarrow$ (i) $f(-x) = 1 - \frac{2x}{\pi}$; $-\pi \le -x \le 0 \Rightarrow \pi \ge x \ge 0$

$$\boxed{f(-x) = 1 - \frac{2x}{\pi}, \quad 0 \le x \le \pi}$$

$\Rightarrow f(-x) = f(x)$, $0 \le x \le \pi$ — (i)

(ii) $f(-x) = 1 - \frac{2(-x)}{\pi}$, $0 \le -x \le \pi \Rightarrow 0 \ge x \ge -\pi$

$$\boxed{f(-x) = 1 + \frac{2x}{\pi} \quad ; -\pi \le x \le 0}$$

$$\boxed{f(-x) = f(x) \quad ; \quad -\pi \le x \le 0} \text{ — (ii)}$$

By eq (i) & (ii)

$f(-x) = f(x)$ $[-\pi, \pi]$ $\rightarrow$ $\boxed{\text{Even}}$ $\Rightarrow b_n = 0$

Homework

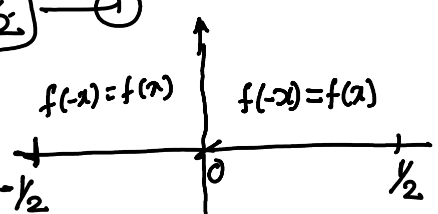
Ex. 5 Find F.S. of $f(x) = \begin{cases} \frac{1}{2} + x & ; -1/2 \le x \le 0 \\ \frac{1}{2} - x & ; 0 \le x \le 1/2 \end{cases}$

$\rightarrow$ $\left(-\frac{1}{2}, \frac{1}{2}\right) \equiv (-L, L)$ $\Rightarrow$ $\boxed{L = 1/2}$

(i) $f(-x) = \frac{1}{2} - x$; $-\frac{1}{2} \le -x \le 0 \Rightarrow \frac{1}{2} \ge x \ge 0$

$$\boxed{f(-x) = \frac{1}{2} - x \quad ; \quad 0 \le x \le 1/2}$$

$\Rightarrow \boxed{f(-x) = f(x) \quad 0 \le x \le 1/2}$ — (i)



(ii) $f(-x) = \frac{1}{2} - (-x)$; $0 \le -x \le 1/2$ $\Rightarrow 0 \ge x \ge -1/2$

(ii) $f(-x) = \frac{1}{2} - (-x) \;;\; 0 \le -x \le \frac{1}{2} \Rightarrow 0 \ge x \ge -\frac{1}{2}$

$$\boxed{f(-x) = \frac{1}{2} + x \;;\; -\frac{1}{2} \le x \le 0}$$

$$\Rightarrow \boxed{f(-x) = f(x) \;,\; -\frac{1}{2} \le x \le 0}$$

$\Rightarrow f(-x) = f(x) \quad [-\frac{1}{2}, \frac{1}{2}] = [-L, L]$

$\Rightarrow$ Even Fun $[-\frac{1}{2}, \frac{1}{2}]$

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$\Rightarrow \boxed{b_n = 0}$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{\;}\int_0^{1/2} f(x)\,dx$$

$$= 2\int_0^{1/2}\left(\frac{1}{2} - x\right)dx = 2\left[\frac{1}{2}x - \frac{x^2}{2}\right]_{x=0}^{x=1/2}$$

$$= 2\left[\frac{1}{4} - \frac{1}{8} - 0 + 0\right] = 2\left[\frac{1}{8}\right] = \boxed{\frac{1}{4}}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{E}\underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{E}dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{(1/2)}\int_0^{1/2}\left(\frac{1}{2} - x\right)\cos\left(\frac{n\pi x}{1/2}\right)dx = 4\int_0^{1/2}\left(\frac{1}{2} - x\right)\cos\left(\frac{2n\pi x}{1}\right)dx$$

$$= 4\left\{\left(\frac{1}{2} - x\right)\left[\frac{\sin(2n\pi x)}{(2n\pi)}\right] - (-1)\left[\frac{-\cos(2n\pi x)}{(2n\pi)(2n\pi)}\right]\right\}_{x=0}^{x=1/2}$$

$$= 4\left\{(0-0) - \frac{(-1)^n}{(2n\pi)^2} + \frac{1}{(2n\pi)^2}\right\}$$

$$a_n = \frac{4}{4n^2\pi^2}\left[1 - (-1)^n\right] = \begin{cases} 0 & n = \text{Even} \quad n = 2, 4, 6, \ldots \\ \dfrac{2}{n^2\pi^2} & n = \text{odd} \quad n = 1, 3, 5, 7, \ldots \end{cases}$$

Hence Fourier series of $f(x)$ is given as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{f(x) = \frac{1}{4} + \sum_{n=1,3,5\ldots}^{\infty} \frac{2}{n^2\pi^2}\cos(2n\pi x)} + 0$$

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Ex(5) Find F.S. of $f(x) = \begin{cases} -\sin\left(\frac{\pi x}{p}\right) & , -p \le x \le 0 \end{cases}$

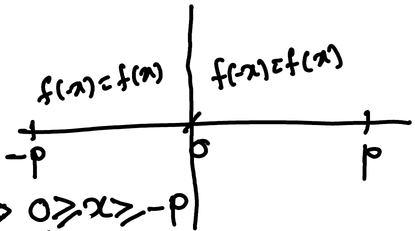
Ex (5) Find F.S. of $f(x) = \begin{cases} -\sin\left(\frac{\pi x}{p}\right) & , -p \le x \le 0 \\ \sin\left(\frac{\pi x}{p}\right) & ; 0 \le x \le p \end{cases}$

$\rightarrow \quad [-p, p] = [-L, L] \quad \Rightarrow \boxed{L = p}$

(i) $f(-x) = -\sin\left(\frac{-\pi x}{p}\right) \quad ; -p \le -x \le 0 \quad \Rightarrow p \ge x \ge 0$

$$\boxed{f(-x) = \sin\left(\frac{\pi x}{p}\right) \quad ; \quad 0 \le x \le p}$$

$$\boxed{f(-x) = f(x) \quad ; \quad 0 \le x \le p}$$



(ii) $f(-x) = \sin\left(-\frac{\pi x}{p}\right) \quad ; 0 \le -x \le p \quad \Rightarrow 0 \ge x \ge -p$

$$\boxed{f(-x) = -\sin\left(\frac{\pi x}{p}\right) \quad ; \quad -p \le x \le 0}$$

$$\boxed{f(-x) = f(x) \qquad -p \le x \le 0}$$

$\Rightarrow \quad f(-x) = f(x) \quad , [-p, p]$

$\Rightarrow \quad f(x)$ is Even Fu^{n} then $\boxed{b_n = 0}$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{p}\int_0^p \sin\left(\frac{\pi x}{p}\right) dx$$

$$a_0 = \frac{1}{p}\left[\frac{-\cos(\pi x/p)}{(\pi/p)}\right]_{x=0}^{x=p} = \frac{1}{p}\left(\frac{p}{\pi}\right)\left[-\cos(\pi) + \cos(0)\right]$$

$$a_0 = \frac{1}{\pi}[1+1] = \boxed{2/\pi}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{\text{Even}} \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\text{Even}} dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{p}\int_0^p \sin\left(\frac{\pi x}{p}\right)\cos\left(\frac{n\pi x}{p}\right) dx$$

$$= \frac{1}{p}\int_0^p 2\cos\left(n\frac{\pi x}{p}\right)\sin\left(\frac{\pi x}{p}\right) dx \qquad \because 2C_A S_B = S_{A+B} - S_{A-B}$$

$$= \frac{1}{p}\int_0^p \left\{\sin\left[(n+1)\frac{\pi x}{p}\right] - \sin\left[(n-1)\frac{\pi x}{p}\right]\right\} dx$$

$$= \frac{1}{p}\left\{\frac{-\cos(n+1)\frac{\pi x}{p}}{(n+1)\frac{\pi}{p}} + \frac{\cos(n-1)\frac{\pi x}{p}}{(n-1)\frac{\pi}{p}}\right\}_{x=0}^{x=p}$$

$$= \frac{1}{p} \times \frac{1}{(\pi/p)}\left[-\frac{\cos(n+1)\pi}{(n+1)} + \frac{\cos(n-1)\pi}{(n-1)} + \frac{1}{(n+1)} - \frac{1}{(n-1)}\right]$$

$$= \frac{1}{p} \times \frac{1}{(\pi/p)} \left[-\frac{\cos(n+1)\pi}{(n+1)} + \frac{\cos(n-1)\pi}{(n-1)} + \frac{1}{(n+1)} - \frac{1}{(n-1)} \right]$$

$$\therefore \cos(n\pm1)\pi = (-1)^{n\pm1} = (-1)^n (-1)^{\pm1} = -(-1)^n$$

$$a_n = \frac{1}{\pi} \left\{ \frac{(-1)^n}{(n+1)} - \frac{(-1)^n}{(n-1)} + \frac{1}{(n+1)} - \frac{1}{(n-1)} \right]$$

$$= \frac{1}{\pi} \left[\frac{[1+(-1)^n]}{(n+1)} - \frac{[1+(-1)^n]}{(n-1)} \right] = \frac{[1+(-1)^n]}{\pi} \left[\frac{1}{(n+1)} - \frac{1}{(n-1)} \right]$$

$$a_n = \frac{[1+(-1)^n][n-1-n-1]}{\pi(n^2-1)} = \frac{-2}{\pi(n^2-1)} [1+(-1)^n] \quad ; \; n \neq 1$$

$$a_n = \frac{-2}{\pi(n^2-1)} [1+(-1)^n] = \begin{cases} 0 & n=\text{odd}, \; n = 3,5,7,\dots \; n\neq 1 \\ \boxed{\frac{-4}{\pi(n^2-1)}} & n=\text{Even}, \; n = 2,4,6,\dots \end{cases}$$

At $n=1$, we have discontinuity —
put $n=1$ in Formula of a_n to find a_1 separately

$$a_1 = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{1\pi x}{L}\right) dx = \frac{2}{p} \int_0^p \sin\left(\frac{\pi x}{p}\right) \cos\left(\frac{\pi x}{p}\right) dx$$

$$= \frac{1}{p} \int_0^p 2 \sin\left(\frac{\pi x}{p}\right) \cos\left(\frac{\pi x}{p}\right) dx \qquad \because 2 S_A C_A = \sin(2A)$$

$$= \frac{1}{p} \int_0^p \sin\left(\frac{2\pi x}{p}\right) dx = \frac{1}{p} \left[\frac{-\cos\left(\frac{2\pi x}{p}\right)}{(2\pi/p)} \right]_{x=0}^{x=p}$$

$$= \frac{1}{2\pi} \left[-\cos(2\pi) + \cos(0) \right] = \frac{1}{2\pi} [-1+1] = \boxed{0 = a_1}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= a_0 + a_1 \cos\left(\frac{\pi x}{L}\right) + \underbrace{\sum_{n=2}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)}_{} + 0$$

$$\boxed{= \frac{2}{\pi} + 0 + \sum_{n=2,4,6,\dots}^{\infty} \frac{-4}{\pi(n^2-1)} \cos\left(\frac{n\pi x}{p}\right)}$$

Ex Find F.S. of $f(x) = \begin{cases} a(x-p) & -p \le x \le 0 \\ a(x+p) & 0 \le x \le p \end{cases}$

$\rightarrow$ $[-p, p] = [-L, L] \Rightarrow \boxed{L = p}$

① $f(-x) = a(-x-p)$; $-p \le -x \le 0 \Rightarrow p \ge x \ge 0$

(i) $f(-x) = a(-x-p)$; $-p \le -x \le 0 \Rightarrow p \ge x \ge 0$

$$\boxed{f(-x) = -a(x+p) \; ; \; 0 \le x \le p}$$

$$\Rightarrow \boxed{f(-x) = -f(x) \; ; \; 0 \le x \le p}$$

(ii) $f(-x) = a(-x+p)$; $0 \le -x \le p \Rightarrow 0 \ge x \ge -p$

$$\boxed{f(-x) = -a(x-p) \; ; \; -p \le x \le 0}$$

$$\boxed{f(-x) = -f(x) \; , \; -p \le x \le 0}$$

$\Rightarrow$ $f(-x) = -f(x)$ $[-p, p]$

$\Rightarrow$ $f(x)$ is odd Fun^n.

$\boxed{a_0 = 0}$ and $\boxed{a_n = 0}$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{\text{odd}} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}} dx = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

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$$b_n = \frac{2}{p}\int_0^p a(x+p)\sin\left(\frac{n\pi x}{p}\right) dx$$

$$= \frac{2a}{p}\left\{(x+p)\left[\frac{-\cos(n\pi x/p)}{(n\pi/p)}\right] - (1)\left[\frac{-\sin(n\pi x/p)}{(n\pi/p)(n\pi/p)}\right]\right\}_{x=0}^{p}$$

$$= \frac{2a}{p}\frac{1}{\left(\frac{n\pi}{p}\right)}\left[-2p(-1)^n + p(1)\right]$$

$$\boxed{b_n = \frac{2a}{n\pi}p\left[1 - 2(-1)^n\right]}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{= 0 + 0 + \sum_{n=1}^{\infty}\frac{2a}{n\pi}p\left[1-2(-1)^n\right]\sin\left(\frac{n\pi x}{p}\right)}$$ — Ans

Half range Cosine and Sine series in $(0,\pi)$, or $(0,L)$ Dr. Uday Kashid (PhD, Mathematics)

$(-\pi,\pi)$ $(-L,L)$

$(-\pi,\pi)$ To test Even/odd

Even $\Rightarrow b_n = 0 \Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$

$\boxed{\int_0}$

Fourier Cosine series

$(-\pi,\pi)$ odd $\Rightarrow a_0 = 0$ and $a_n = 0 \Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$

Fourier sine series

$g(-\pi,\pi)$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \boxed{\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx}$

odd odd

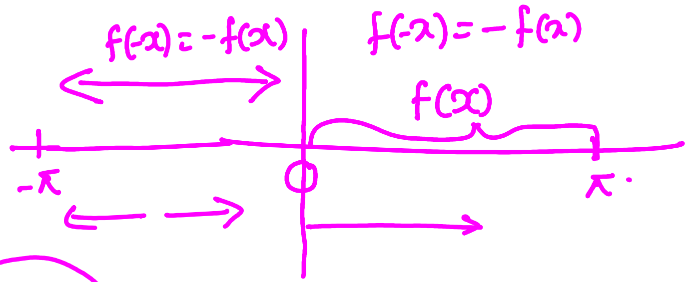
Assume even or odd

Half range

$(0,\pi) \to f(x)$

even / odd

cosine sine



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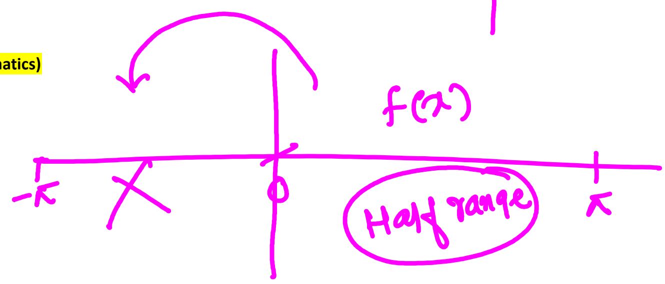

Half Range Fourier cosine series: $(-\pi,\pi)$ or $(-L,L)$

$\boxed{(0,\pi)}$ or $\boxed{(0,L)}$ → Half range

Assume $f(x)$ is Even fun in $(-\pi,\pi)$ or $(-L,L)$

Then find a_0 and a_n as like Even concept formulae Half R.

where $b_n = 0$. and $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$ ... $(0,\pi)$ Half Range

or $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ ... $(0,L)$

$\boxed{\text{In } (-\pi,\pi)}$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \boxed{\frac{1}{\pi}\int_0^{\pi} f(x)\,dx}$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{Even}}\,dx = \boxed{\frac{1}{\pi}\int_0^{\pi} f(x)\,dx}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{Even}} \underbrace{\cos(nx)}_{\text{Even}}\,dx = \boxed{\frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx}$$

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$\boxed{b_n = 0}$ → Even

In $(-L, L)$

For Half range F. cosine series : $(0, L)$ → Half range

Assume $f(x)$ is Even in $(-L, L)$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \boxed{\frac{1}{L}\int_0^{L} f(x)\,dx}$$

$$\boxed{a_n = \frac{2}{L}\int_0^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

$(0, L)$ → Half range

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(II) Half Range Fourier sine series $(0, \pi)$ or $(0, L)$ ← Half range

→ Assume $f(x)$ is odd fun in $(-\pi, \pi)$ or $(-L, L)$

$a_0 = 0$, and $a_n = 0$ and find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{odd}} \underbrace{\sin(nx)}_{\text{odd}}\,dx = \boxed{\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx} \quad \text{--}(0, \pi)$$

or

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{\text{odd}} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}}\,dx = \boxed{\frac{2}{L}\int_0^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx} \quad \text{--}(0, L)$$

Ex. (1) Find Half range Fourier cosine and sine series of

$$f(x) = \begin{cases} x & 0 < x < \pi/2 \\ (\pi - x), & \pi/2 < x < \pi \end{cases}$$

$(0, \pi)$ → Half range

$(0 | 2\pi)$ or $(-\pi, \pi)$

$(0 | 2L)$ or $(-L, L)$

→ 1) For Half range Fourier cosine series

$(0, 2\pi)$ ... $(0, 2L)$ or $(-L, L)$

→ I) For Half range Fourier cosine series

Assume $f(x)$ is Even Fu^n in $(-\pi, \pi)$

$$\boxed{b_n = 0}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \longrightarrow \text{H.R.C.S}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right\} = \frac{1}{\pi}\left\{\left(\frac{x^2}{2}\right)_{x=0}^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)_{x=\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{8} - 0 + \frac{\pi^2}{2} - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right\} = \frac{1}{\pi}\left\{\frac{\pi^2}{8} + \frac{\pi^2}{2} - \frac{3\pi^2}{8}\right\}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2 + 4\pi^2 - 3\pi^2}{8}\right] = \boxed{\frac{\pi}{4}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{Even}}\underbrace{\cos(nx)}_{\text{Even}}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} x\cos(nx)\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos(nx)\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[x\left[\frac{\sin(nx)}{n}\right] - (1)\left[\frac{-\cos(nx)}{n^2}\right]\right]_{x=0}^{\pi/2} + \left[(\pi - x)\left[\frac{\sin(nx)}{n}\right] - (-1)\left[\frac{-\cos(nx)}{n^2}\right]\right]_{x=\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\cancel{\frac{\pi}{2n}\sin(n\pi/2)} + \frac{\cos(n\pi/2)}{n^2} - 0 - \frac{1}{n^2} + 0 - \frac{(-1)^n}{n^2} - \cancel{\frac{\pi}{2n}\sin(n\pi/2)} + \frac{\cos(n\pi/2)}{n^2}\right\}$$

$$= \frac{2}{\pi}\left[\frac{2\cos(n\pi/2)}{n^2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2}\right] = \boxed{\frac{2\left[2\cos(n\pi/2) - 1 - (-1)^n\right]}{\pi n^2}}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{2\left[2\cos(n\pi/2) - 1 - (-1)^n\right]}{\pi n^2}\cos(nx) \rightarrow \boxed{\text{H.R.C.S}}$$

Dr. Uday Kashid (PhD, Mathematics)

II] Find Half range Fourier sine series

→ Assume $f(x)$ is odd Function in $(-\pi, \pi)$

$a_0 = 0$ and $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} x\sin(nx)\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin(nx)\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[x\left[-\frac{\cos nx}{n}\right]-(1)\left[-\frac{\sin(nx)}{n^2}\right]\right]_{x=0}^{\pi/2} + \left[(\pi-x)\left[-\frac{\cos nx}{n}\right]-(-1)\left[-\frac{\sin nx}{n^2}\right]\right]_{x=\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{\pi\cos(n\pi/2)}{2n} + \frac{\sin(n\pi/2)}{n^2} - 0 - 0 + 0 - 0 + \frac{\sin(n\pi/2)}{n^2}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{2\sin(n\pi/2)}{n^2} - \frac{\pi\cos(n\pi/2)}{2n}\right\} //$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

$$\boxed{= \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{2\sin(n\pi/2)}{n^2} - \frac{\pi\cos(n\pi/2)}{2n}\right]\sin(nx)} \rightarrow \text{H.R.S.S}$$

Dr. Uday Kashid (PhD, Mathematics)

Ex. ② Find Half range Fourier sine series for $f(x) = (lx - x^2)$, $0 < x < l$ — Dec-19

→ Assume $f(x)$ is odd fun in $(-l, l)$

$$\boxed{a_0 = a_n = 0}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underset{\text{odd}}{f(x)}\,\underset{\text{odd}}{\sin\left(\frac{n\pi x}{L}\right)}\,dx = \frac{2}{L}\int_0^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$(0, L) = (0, l) \Rightarrow L = l$

$$b_n = \frac{2}{l}\int_0^{l}(lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{2}{l}\left\{(lx - x^2)\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{(n\pi/l)}\right] - (l - 2x)\left[\frac{-\sin(n\pi x/l)}{(n\pi/l)^2}\right] + (-2)\left[\frac{\cos(n\pi x/l)}{(n\pi/l)^3}\right]\right\}_{x=0}^{x=l}$$

$$= \frac{2}{\ell}\left\{(0-0)+(0-0)-\frac{2\,(-1)^n}{(n\pi/\ell)^3}+\frac{2\,(1)}{(n\pi/\ell)^3}\right\}$$

$$b_n = \frac{2}{\ell}\,\frac{2}{(n\pi/\ell)^3}\left[1-(-1)^n\right]$$

$$b_n = \begin{cases} 0 & \text{Even} \quad n=2,4,6, \\ \dfrac{8\ell^2}{n^3\pi^3} & \text{odd} \quad n=1,3,5,\ldots \end{cases}$$

$$\boxed{f(x) = \ell x - x^2 = \sum_{n=1,3,5,\ldots}^{\infty} \frac{8\ell^2}{\pi^3 n^3}\sin\left(\frac{n\pi x}{\ell}\right)} \rightarrow \text{HRSS}$$

Ex. 3 Prove that in $(0,\pi)$ Dec-05, 08,

$$\frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}} = \frac{2}{\pi}\left[\frac{\sin(x)}{a^2+1^2}-\frac{2\sin(2x)}{a^2+2^2}+\frac{3\sin(3x)}{a^2+3^2}-\frac{4\sin(4x)}{a^2+4^2}+\cdots\right]$$

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F.S

Radian — period 2π

General — $2L$ period

$\pi \xrightarrow{\text{Replace}} L$

$(nx) \xrightarrow{\text{Replace}} \left(\frac{n\pi x}{L}\right)$

$[0, 2\pi]$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nx)+b_n\sin(nx)\right]$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

$[-\pi, \pi]$

Neither even nor odd

Even

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$\boxed{b_n = 0}$$

odd

$$\boxed{a_0 = 0}$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$