7. Functions

Let A and B be two non-empty sets then a subset F of $A \times B$ is said to be a function from A to B if

$\forall x \in A$,

$\exists$ a unique $y \in B$

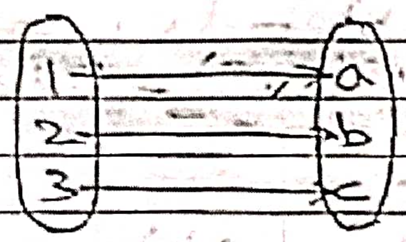
i.e. $F = \{(x,y) \mid x \in A$ and $y \in B$
$\forall x \in A, \exists$ a unique $y \in B\}$

eg. $A = \{1,2,3\}$, $B = \{a,b,c\}$

$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c)\}$

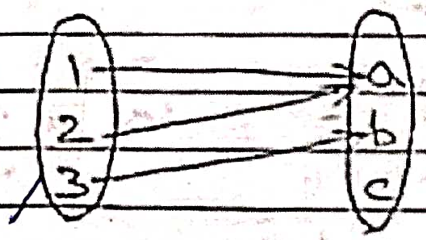
Consider,

$F = \{(1,a), (2,b), (3,c)\}$



$\therefore$ F is a function

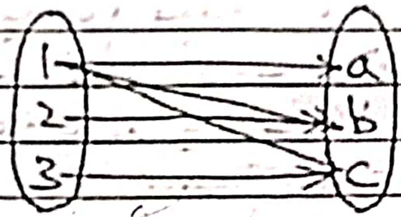
$F = \{(1,a), (2,a), (3,b)\}$



The function given has a unique mapping $\forall x \in A$. Also, F is a subset of $A \times B$.

$\therefore$ It is a function.

$F = \{(1,a),(1,b),(1,c),(2,b),(3,c)\}$



F is a subset of $A \times B$ for $1 \in A$ there exists multiple values of y which belongs to B which are a, b, c.
$\therefore$ As y is not unique, the given F is not a function.

* Domain, Codomain, Image and Range of a Function

Let $f: A \rightarrow B$ then

→ A is called the domain

→ B is called the codomain

→ Since f is a function $\forall x \in A$ $\exists$ a unique $y \in B$, then y is called Image of x.

→ Range
A subset of the codomain is said to be the range, if it contains all images $\forall x \in A$.

ex.1 $A = \{a,b,c\}$, $B = \{x,y,z\}$

1) $F = \{(a,y),(c,x)\}$
2) $F = \{(a,y),(c,x),(c,z),(b,z)\}$
3) $F = \{(a,x),(b,z),(c,x)\}$

Determine which are functions, write the domain, codomain and range.

Sol^n 1) F is a subset of $A \times B$ but $b \in A$ is not mapped to any element of B

$\therefore$ It is not a function.

2) F is a subset of $A \times B$ but $c \in A$ has been mapped to multiple elements of B.

$\therefore$ It is not a function.

3) F is a subset of $A \times B$ and every element in A has a unique mapping to elements in B.

$\therefore$ It is a function.

$\therefore$ Domain $= \{a, b, c\}$, Codomain $= \{x, y, z\}$, Range $= \{x, z\}$

ex. 2 Consider $A = \{1, 2, 3\}$, $B = \{5, 9, 13, 15, 17, 22\}$ and $F = \{(1,5), (2,9), (3,13)\}$. Find whether the given relation is a function. Find domain, codomain and range.

Sol^n F is a subset of $A \times B$ and every element of A has been uniquely mapped to elements of B.

$\therefore$ It is a function.

Domain $= A = \{1, 2, 3\}$

Codomain $= B = \{5, 9, 13, 15, 17, 22\}$.

Range $= \{5, 9, 13\}$

→ ** Injective Function (One-one Function)

Let $f: A \to B$ then f is said to be one-one function if $f(x_1) = f(x_2)$
$\Rightarrow x_1 = x_2$

→ Surjective Function (Onto Function)

Let $f: A \to B$ then f is said to be onto, if $\forall\ y \in B$ (codomain) $\exists$ a pre-image $x \in A$ (domain).

→ Bijective Function

Let $f: A \to B$ then f is said to be bijective if it is both injective and surjective.

→ Inverse Function

Let $f: A \to B$ then f is said to be invertible, if $f^{-1}: B \to A$

Note that, a function from A to B is invertible if f is bijective.

ex.) Let $f: R - \{2\} \to R$ which is defined by $f(x) = \dfrac{1}{x-2}$

Check if it is one-one, onto and bijective.

Sol^n 1) For injective

$$f(x_1) = f(x_2)$$

$$\frac{1}{x_1 - 2} = \frac{1}{x_2 - 2}$$

$$x_1 - 2 = x_2 - 2$$

$$x_1 = x_2$$

$\therefore$ It is injective

2) For surjective,

$$f(x) = y = \frac{1}{x-2}$$

$$y.(x-2) = 1$$
$$xy - 2y = 1$$
$$xy = 1 + 2y$$
$$x = \frac{1+2y}{y}$$

Since for all $y \in$ Codomain, some $x \in$ Domain cannot be defined. The given function is not surjective.
For $y = 0 \in$ Codomain, x is not defined in $R - \{2\}$

$\therefore$ The given function is not surjective.
Also, the function is not bijective as it is only injective.

ex.2 Let $f: R \to R$ defined by $f(x) = 2x - 3$. Prove that it is bijective and find its inverse.

Sol^n 1) For injective,

$$f(x_1) = f(x_2)$$
$$2x_1 - 3 = 2x_2 - 3$$
$$2x_1 = 2x_2$$
$$x_1 = x_2$$

$\therefore$ The given function is injective.

2) For surjective,

$$f(x) = y = 2x - 3$$
$$y + 3 = 2x$$
$$x = \frac{y+3}{2}$$

Since $\forall\ y \in \mathbb{R}\ \exists\ x \in \mathbb{R}$

$\therefore$ Range of f = Codomain

$\therefore$ f is surjective.

As the given function is both injective and surjective, it is bijective.

$\therefore$ f^{-1} exists.

3) For Invers,

$f(x) = y$

$\Rightarrow f^{-1}(y) = x = \frac{y+3}{2}$

$y = 2x - 3$

$x = 2y - 3$

$2y = x + 3$

$y = \frac{x+3}{2}$

$f^{-1}(x) = \frac{x+3}{2}$

$\therefore$ By rule of f^{-1},

$f^{-1}(x) = \frac{x+3}{2}$

ex.3 Let $f: \boxed{\mathbb{R} \to \mathbb{R}}$ defined by $f(x) = x^2$. ($N \to N$)

Check whether it is bijective.

Sol^n 1) For injective,

$f(x_1) = f(x_2)$

$x_1^2 = x_2^2$

$\therefore$ $x_1 = \pm x_2$

x	y
1	1
2	4
3	9
4	16

$\therefore$ The given function is not injective.

2) For surjective,

$f(x) = y = x^2$

$x = \sqrt{y}$

Since $\forall\ y \in$ Codomain, some $x \in$ Domain cannot be defined, The given function is not surjective.

Also, the function is not bijective, as it is neither injective nor surjective.

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ex.4 Let function $f : R - \left\{\frac{7}{3}\right\} \rightarrow R - \left\{\frac{4}{3}\right\}$

where $f(x) = \frac{4x-5}{3x-7}$

Prove that f is bijective and find f^{-1}.

Sol^n 1) For injective,

$$f(x_1) = f(x_2)$$

$$\frac{4x_1 - \cancel{7}5}{3x_1 - 7} = \frac{4x_2 - 5}{3x_2 - 7}$$

$$(4x_1 - 5)(3x_2 - 7) = (4x_2 - 5)(3x_1 - 7)$$

$$\cancel{12x_1x_2} - 28x_1 - 15x_2 + \cancel{35}$$

$$= \cancel{12x_1x_2} - 28x_2 - 15x_1 + \cancel{35}$$

$\therefore$ $7x_1 = 7x_2$

$\therefore$ $x_1 = x_2$

$\Rightarrow$ It is injective.

2) For surjective,

$$f(x) = y = \frac{4x-5}{3x-7}$$

$$y(3x-7) = 4x-5$$

$$3xy - 7y = 4x - 5$$

$$3xy - 4x = 7y - 5$$

$$x(3y-4) = 7y - 5$$

$\therefore$ $x = \frac{7y-5}{3y-4}$ $\qquad \frac{7y-5}{3\times\frac{4}{3}-4}$

Apart from $y = \frac{4}{3}$, x will exist.

$\Rightarrow$ It is surjective

$\Rightarrow$ It is bijective

$\therefore$ $f^{-1}(x) = \frac{7x-5}{3x-4}$

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* Composite Function :-

Let $f: A \to B$ and $g: B \to C$ then gof is said to be a composite function of f and g if $gof: A \to C$ and is defined as,

$$gof(x) = g\{f(x)\}$$

ex.1 $f: R \to R$, $f(x) = x+2$, $g: R \to R$, $g(x) = x-2$ and $h: R \to R$ is defined as $h(x) = 3x$. Find (1) $fogoh$ (2) $hogof$ (3) $fofof$.

Sol^n 1) $fogoh(x) = fog\{h(x)\}$
$= fog\{3x\}$
$= f\{g(3x)\}$
$= f(3x-2)$
$= 3x-2+2$
$= 3x$

2) $hogof(x) = hog\{f(x)\}$
$= hog\{x+2\}$
$= \cancel{hog}\ h\{g(x+2)\}$
$= h\{x+2-2\}$
$= h(x)$
$= 3x$

3) $fofof(x) = fof\{f(x)\}$
$= fof\{x+2\}$
$= f\{f(x+2)\}$
$= f\{x+2+2\}$
$= f(x+4)$
$= x+4+2$
$= x+6$

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ex. 2 Let $f: R \to R$, $f(x) = x^3$, $g: R \to R$, $g(x) = 4x^2+1$, $h: R \to R$, $h(x) = 7x-2$
Find $(goh) of$ and $go(hof)$.

Sol^n 1) $(goh) of = (g(h(x)) of$
$= \{g(7x-2)\} of$
$= \{4(7x-2)^2+1\} of$
$= \{4(49x^2-28x+4)+1\} of$
$= \{196x^2-112x+17\} \cdot \{f(x)\}$
$= \{196x^2-112x+17\} \{x^3\}$
$= 196x^6-112x^3+17$

2) $go(hof) = goh\{f(x)\}$
$= goh\{x^3\}$
$= g\{7x^3-2\}$
$= 4(7x^3-2)^2+1$
$= 4(49x^6-28x^3+4)+1$
$= 196x^6-112x^3+17$
$\Rightarrow (goh) of = go(hof)$

* Pigeon Hole Principle

If n pigeons are assigned to m pigeon holes where $m<n$ then at least one pigeon hole must have more than one pigeon.

* Extended Pigeon Hole Principle

If n pigeons are assigned to m pigeon holes where $m<n$ then at least one pigeon hole must have $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons.

ex.1 If 62 students are selected from a class then at least six of them must have their birthday on the same month of a year. Verify the statement.

Sol^n $n = 62$, $m = 12$

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{62-1}{12} \right\rfloor + 1$$

$$= \left\lfloor \frac{61}{12} \right\rfloor + 1$$

$$= \lfloor 5.08 \rfloor + 1$$

$$= 5 + 1$$

$$= 6$$

Hence verified.

ex.2 How many friends must have their birthday to guarantee that at least five of them will have their birthday in the same month of the year?

Sol^n $m = 12$

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = 5$$

$$\left\lfloor \frac{n-1}{12} \right\rfloor = 4$$

$$\lfloor n-1 \rfloor = 48$$

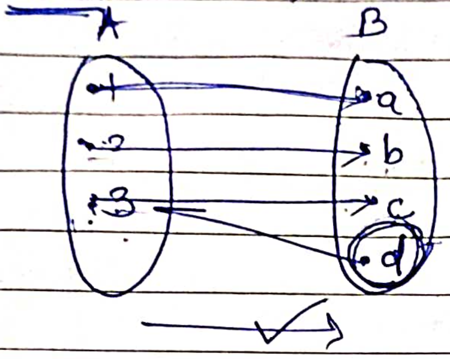
$$\lfloor n \rfloor = 49$$

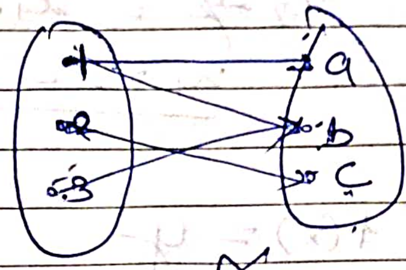
$\therefore$ $n \in \{49, 50, 51, \ldots, 60\}$

$F \subseteq R$

Functions

R $f(x)$





$\propto$ function

all elements of A has to be mapped to elements of B

$f(1) = a$

$f(1) = a, b \propto$

Domain, co-domain, range, image, preimage

Domain $f(x) = \{1, 2, 3\}$

co-domain $f(x) = \{a, b, c, d\}$

rang $f(x) = \{a, b, c\}$

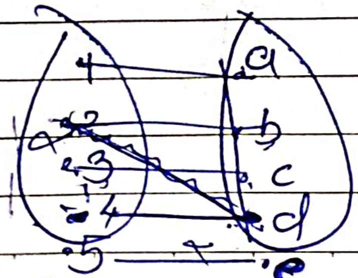
range $\subseteq$ co-domain

image

$f(1) = a$ $\quad f(2) = b$ $\quad f(3) = c$

image

pre-image



onto

one-one

Injective

$f(x_1) = f(x_2)$

$$\frac{1}{x_1-2} = \frac{1}{x_2-2}$$

$$x_2 - \cancel{2} = x_1 - \cancel{2}$$

$$x_1 = x_2$$

✓

$f(x) = y$

$$y = \frac{1}{x-2}$$

$$yx - 2y = 1$$

$$\cancel{x\ell = 1+2y}$$

$$\cancel{y(x-2) \in b}$$

$$xy = 1+2y$$

cod

$$x = \frac{1+2y}{y}$$

$$y = 0$$

$\times$ surjective

∴ Bijective not

$y = 2x - 3$

Switch value of x & y

$x = 2y - 3$

compute value of y

$2y = x + 3$

$y = \dfrac{x+3}{2}$

$f^{-1}(x) = \dfrac{x+3}{2}.$ — Bijective

$f: R \to R$ $\quad$ $g: R \to R$ $\quad$ $h: R \to R$

$f(x) = x + 2$ $\quad$ $g(x) = x - 2$ $\quad$ $h(x) = 3x$

(1) $f \circ g \circ h$ $\quad$ (2) $h \circ g \circ f$ $\quad$ (3) $f \circ f \circ f$

$f[g[h(x)]]$

$f[g(3x)]$

$f[3x - 2]$

$3x - \cancel{2} + \cancel{2}$

$3x$

n pigeons $\uparrow$ ~~7~~ 6

m holes ~~12~~ 5