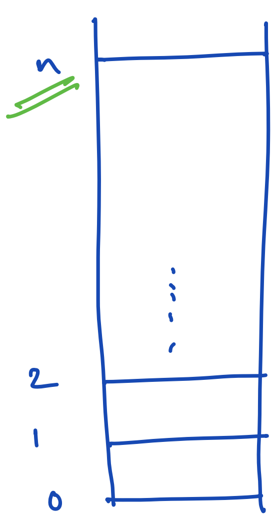
- HW 1 due on Friday before class time.

- Office hours by TAs will be announced on Piazza.

Mathematical Induction

Suppose we want to prove that a penguin can climb a ladder with n rungs, $\forall$ int $n \geq 0$.



Base case

(*) - Penguin can climb ladder of ht 0.

Induction step

Induction hypothesis

(*) - If [the penguin can climb a ladder with k rungs,] then it can climb a ladder with $k+1$ rungs. $(k \geq 0)$

$p \Rightarrow q$

Ex: Prove that for all int $n \geq 1$,

(*) $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Soln: We will prove the claim using induction on n.

Induction hypothesis: Let $\cancel{k \geq 1}$ $\cancel{k \geq 2}$ $\boxed{k \geq 3}$ be an integer. Assume that the claim holds when $n=k$. In other words assume that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

Base Case: $\cancel{n=1}$ $\quad$ $n=2$. LHS $=3$, RHS $= \frac{2(3)}{2} = 3$.

LHS $= \sum_{i=1}^{1} i = 1$, RHS $= \frac{1(1+1)}{2} = 1$.

Since LHS = RHS, the claim holds when $n=1$.

<u>Induction Step</u> : We want to prove the claim when $n=k+1$. In other words, we want to prove that

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

<u>LHS</u> $= \displaystyle\sum_{i=1}^{k+1} i$

$$= \sum_{i=1}^{k} i + k+1$$

$$\cancel{= \sum_{i=1}^{k/2} i + \sum_{i=\frac{k}{2}+1}^{k} i}$$

$$= \frac{k(k+1)}{2} + k+1 \quad \text{(By IH)}$$

$$= (k+1)\left(\frac{k}{2}+1\right)$$

$$= \frac{(k+1)(k+2)}{2}$$

Ex: Prove that the sum of the first n positive odd int is n^2.

Soln: We want to prove that

$$\sum_{i=0}^{n-1} 2i+1 = n^2, \quad \forall \text{ int } n \geq 1$$

Suppose $n = 3$

$$\sum_{i=0}^{2} 2i+1 = 1+3+5 = 9 = 3^2.$$

We will prove the claim using induction on n.

IH: Let $k \geq 1$ be an arb. but particular integer. Assume that the claim holds when $n = k$. That is, assume that

$$\boxed{\sum_{i=0}^{k-1} 2i+1 = k^2} \checkmark$$

<u>BC</u> : $n=1$

$$LHS = \sum_{i=0}^{0} 2i+1 = 2\cdot 0+1$$

$$= 1$$

$$RHS = 1^2 = 1. \checkmark$$

Induction Step: We want to prove that the claim holds when $n = k+1$.

In other words, we want to prove that

$$\sum_{i=0}^{k} 2i+1 = (k+1)^2.$$

$$\text{LHS} = \sum_{i=0}^{k-1} 2i+1 + 2k+1$$

$$= k^2 + 2k+1$$

$$= (k+1)^2.$$

Ex: Prove that for all non-negative integers n,

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

Proof: We will prove the claim by doing induction on n.

IH: Let $k \geq \cancel{1}\ 0$ be an arbitrary but particular integer. Assume that the claim holds when

$n = k$. In other words, assume that

$$\boxed{\sum_{i=0}^{k} 2^i = 2^{k+1} - 1}$$

$n = 0$

$LHS = 2^0 = 1$

$RHS = 2^{0+1} - 1 = 1$ ✓

BC : $n = 1$

$$LHS = \sum_{i=0}^{1} 2^i = 2^0 + 2^1$$

$$= 3$$

RHS: $2^{1+1} - 1 = \underline{\underline{3}}$

Thus the claim holds when $n=1$.

<u>IS</u>: We want to prove that the claim holds when $n=k+1$. That is, we want to prove that

$$\sum^{k+1} 2^i = 2^{k+2} - 1$$

$i = 0$

$$LHS = \sum_{i=0}^{k+1} 2^{i}$$

$$= \sum_{i=0}^{k} 2^{k} + 2^{k+1}.$$

$$= 2^{k+1} - 1 + 2^{k+1} \quad (\text{By IH})$$

$$= 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+2} - 1$$

Ex: Prove that for all non-negative int n, $2^{2n}-1$ is a multiple of 3.

Soln: We will prove the claim using induction on n.

IH: Let $k \geq 0$ be an arbitrary, but particular integer. Assume that the claim holds when $n=k$.

In other words, assume that

$$\boxed{2^{2k}-1 = 3 \cdot m}, \text{ for}$$

some int m.

<u>BC</u>: $n = 0$

$$2^{2n} - 1 = 2^{0} - 1 = 0 = 3 \cdot 0$$

<u>IS</u>: We want to prove that the claim holds when $n = k+1$. That is, we want to prove that

$$2^{2(k+1)} - 1 = 3 \cdot m',$$

where m' is an int.

LHS: $2^{2(k+1)} - 1$

$$= 2^{2} \cdot 2^{2k} - 1 - 3 + 3$$

$$= 2^{2}\left(2^{2k} - 1\right) + 3$$

$$= 2^{2}(3m) + 3$$

$$\left(2^{2} \right.$$

$= 3(2m^2+1)$

$= 3 \cdot m'$, where $m' = 2m^2+1$ is an int.

Ex: Prove that for all positive int $n>1$, $n! < n^n$.

Soln: We will prove that the claim holds by doing induction on n.

IH: Let $k \geq 2$ be an arb. but particular int. Assume that the claim holds when

$n=k$. That is assume that

$$k! < k^k.$$

BC: $n=2$

$2! = 2$

$2^2 = 4$

$2 < 4$ ✓.

IS: We want to prove the claim when $n=k+1$. That is, we want to prove that

$$(k+1)! < (k+1)^{k+1}.$$

$$LHS = (k+1)!$$

$$= k! \times (k+1)$$

$$< k^{k} \cdot k+1$$

k

$$< (k+1) \cdot (k+1)$$

$$= (k+1)^{k+1}$$