Curriculum Scheme: Rev2019

Examination: FE Semester I

Course Code: FEC101 and Course Name: Engineering Mathematics I

Time: 2 Hours 30 Minutes Max. Marks: 80

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Q No 1	Answers
1	B
2	D
3	A
4	D
5	B
6	C
7	D
8	B
9	B
10	D

EM-I-prelim-paper- Sol^n. 2.

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Q 2 (A) Expand $\sin^7\theta$ in the series of multiple of θ (05)

let $z = \cos\theta + i\sin\theta$ $\quad \frac{1}{z} = \cos\theta - i\sin\theta$

and $z^n = \cos(n\theta) + i\sin(n\theta)$, $\left(\frac{1}{z}\right)^n = \cos(n\theta) - i\sin(n\theta)$

$z^n + \frac{1}{z^n} = 2\cos(n\theta)$ — (I)

$z^n - \frac{1}{z^n} = 2i\sin(n\theta)$ — (II) i.e. $z - \frac{1}{z} = 2i\sin\theta$

$$2^7 i^7 \sin^7\theta = \left(z - \frac{1}{z}\right)^7 = z^7 - 7z^6 \cdot \frac{1}{z} + 21 z^5 \frac{1}{z^2} - 35 z^4 \frac{1}{z^3}$$
$$+ 35 z^3 \frac{1}{z^4} - 21 z^2 \frac{1}{z^5} + 7z\frac{1}{z^6} - \frac{1}{z^7}$$

$$2^7(-i)\sin^7\theta = \left(z^7 - \frac{1}{z^7}\right) - 7\left(z^5 - \frac{1}{z^5}\right) + 21\left(z^3 - \frac{1}{z^3}\right) - 35\left(z - \frac{1}{z}\right)$$
$$= (2i\sin 7\theta) - 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta)$$

$$\sin^7\theta = -\frac{1}{64}\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right].$$

$\boxed{Q\,2B}$. If $\operatorname{cosec}\left(\frac{\pi}{4} + ix\right) = u + iv$, where x, u, v are real

Then show that $(u^2+v^2)^2 = 2(u^2 - v^2)$

→ $\operatorname{cosec}\left(\frac{\pi}{4} + ix\right) = u + iv$

$$\frac{1}{\operatorname{cosec}\left(\frac{\pi}{4} + ix\right)} = \frac{1}{u+iv}$$

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$$\sin\left(\frac{\pi}{4} + ix\right) = \frac{u - iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$\sin\frac{\pi}{4}\cos(ix) + \cos\frac{\pi}{4}\sin(ix) = \frac{u-iv}{u^2+v^2}$$

$$\frac{1}{\sqrt{2}}\cosh(x) + i\frac{1}{\sqrt{2}}\sinh(x) = \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right)$$

$$\Rightarrow \frac{1}{\sqrt{2}}\cosh x = \frac{u}{u^2+v^2} \quad \& \quad \frac{1}{\sqrt{2}}\sinh x = -\frac{v}{u^2+v^2}$$

squaring & substracting

$$\frac{1}{2}\cosh^2 x - \frac{1}{2}\sinh^2 x = \frac{u^2}{(u^2+v^2)^2} - \frac{v^2}{(u^2+v^2)^2}$$

$$\frac{1}{2}\left[\cosh^2 x - \sinh^2 x\right] = \frac{(u^2-v^2)}{(u^2+v^2)^2}$$

$$\Rightarrow \boxed{(u^2+v^2)^2 = 2(u^2-v^2)}$$

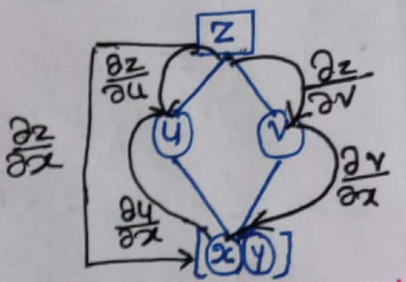
Q 2 (C) If $u = x^2 - y^2$, $v = 2xy$ and $z = f(x, y)$, Then P.T

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4\sqrt{u^2+v^2}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$\rightarrow$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}(2x) + \frac{\partial z}{\partial v}(2y)$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(-2y) + \frac{\partial z}{\partial v}(2x)$$

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$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left[2\left(\frac{\partial z}{\partial u}\cdot x + \frac{\partial z}{\partial v}\cdot y\right)\right]^2 + \left[2\left(-y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right)\right]^2$$

$$= 4\left\{x^2\left(\frac{\partial z}{\partial u}\right)^2 + \cancel{2xy\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}} + y^2\left(\frac{\partial z}{\partial v}\right)^2 + y^2\left(\frac{\partial z}{\partial u}\right)^2 - \cancel{2xy\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}} + x^2\left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4\left[(x^2+y^2)\left(\frac{\partial z}{\partial u}\right)^2 + (x^2+y^2)\left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4(x^2+y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] \quad \text{--- (1)}$$

But $u^2 + v^2 = (x^2-y^2)^2 + (2xy)^2 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2$

$u^2+v^2 = x^4 + 2x^2y^2 + y^4 = (x^2+y^2)^2$

$$\boxed{\sqrt{u^2+v^2} = (x^2+y^2)} \rightarrow \text{put in (1)}$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4\sqrt{u^2+v^2}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

Q 2 (D) If $y = \frac{1}{(3x-2)(x-3)^2}$, then find y_n

$\rightarrow$ $y = \frac{A}{(3x-2)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2}$

$1 = A(x-3)^2 + B(x-3)(3x-2) + C(3x-2)$

put $x = 2/3$

$1 = A\left(\frac{49}{9}\right) + 0 + 0 \Rightarrow \boxed{A = \frac{9}{49}}$

put $x = 3$

$1 = C(7) \Rightarrow \boxed{C = 1/7}$

put $x = 0$

$1 = 9A + 6B - 2C \Rightarrow \boxed{B = -3/49}$

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⑤

$$y = \frac{9}{49(3x-2)} - \frac{3}{49}\frac{1}{(x-3)} + \frac{1}{7}\frac{1}{(x-3)^2} \quad \text{(1)}$$

But we know that if $y = \frac{1}{(ax+b)^m}$ Then $y_n = \frac{(-1)^n a^n (m+n-1)!}{(m-1)!\,(ax+b)^{m+n}}$

(i) For $\frac{d^n}{dx^n}\frac{1}{(3x+2)^1} = \frac{(-1)^n (3)^n (1+n-1)!}{(1-1)!\,(3x+2)^{1+n}} = \frac{(-1)^n 3^n n!}{(3x+2)^{n+1}}$

(ii) $\frac{d^n}{dx^n}\frac{1}{(x-3)^1} = \frac{(-1)^n n!}{(x-3)^{n+1}}$ Dr. Uday Kashid

(iii) $\frac{d^n}{dx^n}\frac{1}{(x-3)^2} = \frac{(-1)^n (1)^n (2+n-1)!}{(2-1)!\,(x-3)^{n+2}} = \frac{(-1)^n (n+1)!}{(x-3)^{n+2}}$

Take nth derivative of eq (1)

$$y_n = \frac{9}{49}\frac{(-1)^n 3^n n!}{(3x+2)^{n+1}} - \frac{3}{49}\frac{(-1)^n n!}{(x-3)^{n+1}} + \frac{1}{7}\frac{(-1)^n (n+1)!}{(x-3)^{n+2}}.$$

$\boxed{\text{Q2 (E)}}$ If $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, Then S.T $(I-N)(I+N)^{-1}$ is a unitary matrix.

$\rightarrow$ $I-N = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

$I+N = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$

$|I+N| = 1-(4i^2-1) = 6$

$\text{adj}(I+N) = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$ Dr. Uday Kashid

$(I+N)^{-1} = \frac{\text{adj}(I+N)}{|I+N|} = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$

$(I-N)(I+N)^{-1} = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = A \text{ (say)}$

$A^{\theta} = \bar{A}' = [(I-N)(I+N)^{-1}]' = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

$AA^{\theta} = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}\frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = I$

similary $A^{\theta}A = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}\frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} = I$

Hence $A = [(I-N)(I+N)^{-1}]$ is unitary matrix.

$\boxed{Q2 \text{ (F)}}$ Determine the value of K for which the following system of eqns has non-trivial soln & find them in each case.

$(K-1)x+(4K-2)y+(K+3)z=0$, $(K-1)x+(3K+1)y+2Kz=0$

$2x+(3K+1)y+(3K-3)z=0$

$\rightarrow$ $AX=O$

$$\begin{bmatrix} K-1 & 4K-2 & K+3 \\ K-1 & 3K+1 & 2K \\ 2 & 3K+1 & 3K-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

$R_1 \to R_1-R_2$ & $R_3 \to R_3-R_2$

$$\begin{bmatrix} 0 & K-3 & -K+3 \\ K-1 & 3K+1 & 2K \\ -K+3 & 0 & K-3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$|A| = \begin{vmatrix} 0 & K-3 & -K+3 \\ K-1 & 3K+1 & 2K \\ -K+3 & 0 & K-3 \end{vmatrix} = 0-(K-3)\left[(K-3)(K-1)+2K(K-3)\right]-(K-3)\left[0+(K-3)(3K+1)\right]$$

$|A| = -(K-3)^2\left[3K-1\right]-(K-3)^2\left[3K+1\right] = -(K-3)^2(6K)$

But for Non-trivial soln, $|A|=0 = -(K-3)^2(6K)$

$\Rightarrow K=3$ or $K=0$

case (1) For $K=3$, put in (1)
$$\begin{bmatrix} 2 & 10 & 6 \\ 2 & 10 & 6 \\ 2 & 10 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2-R_1$, $R_3 \to R_3-R_1$

$$\begin{bmatrix} 2 & 10 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x+5y+3z=0$$

$\rho(A)=1$ LIV $=3-1=2$. Hence let $\boxed{y=\alpha}$ & $\boxed{z=\beta}$

$\boxed{x=-5\alpha-3\beta}$ $\rightarrow$ Infinite no. of soln.

case (2) For $K=0$ put in (1)
$$\begin{bmatrix} -1 & -2 & 3 \\ -1 & 1 & 0 \\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2-R_1$, $R_3 \to R_3+2R_1$

$$\begin{bmatrix} -1 & -2 & +3 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow R_3 \to R_3+R_2 \begin{bmatrix} -1 & -2 & +3 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A)=2$ LIV $=3-2=1$ $\quad -x-2y+3z=0$ & $3y-3z=0$

Let $z=\alpha$ Then $y-z=0$ $\Rightarrow \boxed{y=z=\alpha}$ and. $\cancel{x+2\alpha+3\alpha=0}$

$-x-2\alpha+3\alpha=0 \Rightarrow \boxed{x=\alpha}$

Thus For $K=3$
$x=-5\alpha-3\beta$
$y=\alpha$
$z=\beta$

and for
$K=0$
$x=\alpha$
$y=\alpha$
$z=\alpha$

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EM-1-prelim Exam. Soln.
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(05)

Q.3.(A) S.T. the roots of $z^7-1=0$ are $1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$, Hence S.T.
$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)(1-\alpha^5)(1-\alpha^6)=7$

$\rightarrow$ $z^7-1=0 \Rightarrow z^7=1=\cos(2k\pi+0)+i\sin(2k\pi+0)$ k=0,1,2---

$z^7 = e^{i(2k\pi)}$, $k=0,1,2,3,\ldots$

$z = e^{i\left(\frac{2k\pi}{7}\right)}$, $k=0,1,2,3,4,5,6$

For $k=0$, $z=\alpha_0=1$

For $k=1$, $z=\alpha=e^{i\left(\frac{2\pi}{7}\right)}$

$k=2$, $z=\alpha_1=e^{i(4\pi/7)}=\left[e^{i(2\pi/7)}\right]^2=\alpha^2$

$k=3$, $z=\alpha_2=e^{\frac{i6\pi}{7}}=\left[e^{i(2\pi/7)}\right]^3=\alpha^3$

$k=4$, $z=\alpha_3=e^{i8\pi/7}=\alpha^4$

$k=5$, $z=\alpha_4=e^{i10\pi/7}=\alpha^5$

$k=6$, $z=\alpha_5=e^{i(12\pi/7)}=\alpha^6$ Dr. Uday Kashid

Hence $1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$ are the roots of $z^7-1=0$.

But $z^7-1=(z-1)(z^6+z^5+z^4+z^3+z^2+z+1)$

$\therefore$ $(z-1)(z-\alpha)(z-\alpha^2)(z-\alpha^3)(z-\alpha^4)(z-\alpha^5)(z-\alpha^6)=$
$(z-1)(z^6+z^5+z^4+z^3+z^2+z+1)$

Now:
$(z-\alpha)(z-\alpha^2)(z-\alpha^3)(z-\alpha^4)(z-\alpha^5)(z-\alpha^6)=z^6+z^5+z^4+z^3+z^2+z+1$

Now put $z=1$
$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)(1-\alpha^5)(1-\alpha^6)=1+1+1+1+1+1+1$
$=7$

Q3(B) S.T. $\tan^{-1}[\cos\theta+i\sin\theta]=\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)-\frac{i}{2}\log\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$

$\rightarrow$ Let $\tan^{-1}[\cos\theta+i\sin\theta]=x+iy$ (1)

$\Rightarrow$ $\tan^{-1}[e^{i\theta}]=x+iy$

$\Rightarrow$ $e^{i\theta}=\tan(x+iy)$ Dr. Uday Kashid

$e^{-i\theta}=\tan(x-iy)$

But $\tan[(x+iy)+(x-iy)]=\dfrac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$

$$\tan(2x) = \frac{e^{i\theta}+e^{-i\theta}}{1-e^{i\theta}e^{-i\theta}} = \infty$$

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$$\Rightarrow \quad 2x = \tan^{-1}(\infty) = n\pi + \pi/2 \quad \text{[general Form]}$$

$$x = \frac{n\pi}{2} + \frac{\pi}{4} = \text{Real part of } \tan^{-1}[\cos\theta + i\sin\theta] \quad \text{—(II)}$$

Similarly

$$\tan[(x+iy)-(x-iy)] = \frac{\tan(x+iy) - \tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$\tan(2iy) = \frac{e^{i\theta}-e^{-i\theta}}{1+e^{i\theta}e^{-i\theta}} = \frac{e^{i\theta}-e^{-i\theta}}{2} = i\sin\theta$$

But $\tan(2iy) = i\tanh(2y) = i\sin\theta$

$$\tanh(2y) = \sin\theta \quad \Rightarrow \quad y = \frac{1}{2}\tanh^{-1}[\sin\theta]$$

$$y = \frac{1}{2}\cdot\frac{1}{2}\log\left[\frac{1+\sin\theta}{1-\sin\theta}\right] = \frac{1}{4}\log\left[\frac{1+\cos(\pi/2-\theta)}{1-\cos(\pi/2-\theta)}\right]$$

$$y = \frac{1}{4}\log\left[\frac{2\cos^2(\pi/4-\theta/2)}{2\sin^2(\pi/4-\theta/2)}\right] = \frac{1}{4}\log\left[\cot^2\left(\frac{\pi}{4}-\theta/2\right)\right]$$

$$= \frac{2}{4}\log[\cot(\pi/4-\theta/2)] = -\frac{1}{2}\log\left[\tan\left(\frac{\pi}{4}-\theta/2\right)\right] \quad \text{—(III)}$$

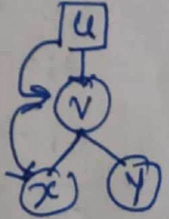
$= \text{Imaginary part of } \tan^{-1}[\cos\theta + i\sin\theta]$

put eq (II) & (III) in eq (I)

$$\tan^{-1}[\cos\theta + i\sin\theta] = \left(\frac{n\pi}{2}+\frac{\pi}{4}\right) - \frac{i}{2}\log\left[\tan\left(\frac{\pi}{4}-\theta/2\right)\right]$$

Q.3(c) If $u = \log[x^3+y^3-xy^2-yx^2]$, P.T. $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{-4}{(x+y)^2}$ (05)

→ let $u = \log v$ and $v = x^3+y^3-xy^2-yx^2$,



We know that

$$\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right]^2 u = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} = \frac{du}{dv}\frac{\partial v}{\partial x} = \frac{1}{v}[3x^2-y^2-2xy]$$

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$$\frac{\partial u}{\partial y} = \frac{du}{dv}\frac{\partial v}{\partial y} = \frac{1}{v}[3y^2-2xy-x^2]$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y} = \frac{1}{v}[2x^2+2y^2-4xy] = \frac{2[x^2+y^2-2xy]}{x^3+y^3-xy^2-yx^2}$$

$$= \frac{2[x^2+y^2-2xy]}{x^2(x-y)-y^2(x-y)} = \frac{2(x-y)^2}{(x-y)(x^2-y^2)} = \frac{2(x-y)^2}{(x-y)^2(x+y)}$$

$$= \frac{2}{(x+y)}$$

$$\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right]^2 u = \left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right]\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{2}{x+y}\right]+\frac{\partial}{\partial y}\left[\frac{2}{x+y}\right]$$

$$= \frac{-2}{(x+y)^2}-\frac{2}{(x+y)^2} = -\frac{4}{(x+y)^2} = \frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x \partial y}+\frac{\partial^2 u}{\partial y^2}$$

$\boxed{Q3\ (D)}$ If $y=(\sin^{-1}x)^2$, Then S.T $(1-x^2)y_{n+2}-(2n+1)y_{n+1}x-n^2y_n=0$

Hence find $y_n(0)$.

$\rightarrow$ $y=(\sin^{-1}x)^2$ — (I)

$y_1 = 2\sin^{-1}x \frac{1}{\sqrt{1-x^2}}$ — (II)

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squaring to B.S

$(1-x^2)y_1^2 = 4(\sin^{-1}x)^2 = 4y$

$(1-x^2)y_1^2-4y=0$ — (II)

$(1-x^2)2y_1y_2-(2x)y_1^2-4y_1=0$

$(1-x^2)y_2-xy_1-2=0$ — (III)

By Leibnitz Thm

$\left[(1-x^2)y_{n+2}+n(-2x)y_{n+1}+\frac{n(n-1)}{2}(-2)y_n\right]-\left[xy_{n+1}+n(1)y_n\right]=0$

$$\boxed{(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0}$$

put $x=0$ in (I), (II), (III) - -

$y(0)=0$, $y_1(0)=0$, $y_2(0)=2$ — $y_{n+2}(0)=n^2y_n(0)$

$y_{n+2}(0)=n^2y_n(0)$

for $n=2$, $y_4(0)=4y_2(0)=4(2)=2.2^2$

$n=4$, $y_6(0)=4^2y_4(0)=2.2^2.4^2$

$n=6$, $y_8(0)=6^2y_6(0)=2.2^2 4^2.6^2$

$y_n(0)=2.2^2.4^2.6^2.....(n-2)^2$ for n is Even Integer

For n is odd

$n=1$, $y_3(0)=1^2y_1(0)=0$

$n=3$, $y_5(0)=3^2y_3(0)=0$

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$y_n(0)=0$ for n is odd Integer

$$y_n(0) = \begin{cases} n \text{ is even} & 2.2^2.4^2.6^2.8^2...(n-2)^2, \quad n=2,4,6--- \\ n \text{ is odd} & 0, \quad n=1,3,5,7,-- \end{cases}$$

Q.3 (E) Find two non-singular matrices P & Q such that PAQ is in Normal Form, Hence find Rank of $A=\begin{bmatrix}1&2&3&2\\2&3&5&1\\1&3&4&5\end{bmatrix}$

$\rightarrow$

$$A_{3\times4} = I_{3\times3}\, A\, I_{4\times4}$$

$$\begin{bmatrix}1&2&3&2\\2&3&5&1\\1&3&4&5\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$R_2\rightarrow R_2-2R_1,\ R_3\rightarrow R_3-R_1$

$$\begin{bmatrix}1&2&3&2\\0&-1&-1&-3\\0&1&1&3\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-1&0&1\end{bmatrix}A\, I_4$$

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$R_3\rightarrow R_3+R_2$

$$\begin{bmatrix}1&2&3&2\\0&-1&-1&-3\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-3&1&1\end{bmatrix}A\, I_4$$

$C_2\rightarrow C_2-2C_1,\ C_3\rightarrow C_3-3C_1,\ C_4\rightarrow C_4-2C_1,$

$$\begin{bmatrix}1&0&0&0\\0&-1&-1&-3\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-3&1&1\end{bmatrix}A\begin{bmatrix}1&-2&-3&-2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$R_2\rightarrow(-1)R_2$

$$\begin{bmatrix}1&0&0&0\\0&1&1&3\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\2&-1&0\\-3&1&1\end{bmatrix}A\begin{bmatrix}1&-2&-3&-2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$C_3\rightarrow C_3-C_2\quad C_4\rightarrow C_4-3C_2$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\2&-1&0\\-3&1&1\end{bmatrix}A\begin{bmatrix}1&-2&-1&4\\0&1&-1&-3\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}I_2&0\\0&0\end{bmatrix}=PAQ$$

$\boxed{\rho(A)=2}$

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Q 3 (F) prove that $e^{2ai\cot^{-1}(b)}\left[\dfrac{bi-1}{bi+1}\right]^{-a}=1$

$\rightarrow$ $\dfrac{bi-1}{bi+1}=\dfrac{i(b+i)}{i(b-i)}=\dfrac{b+i}{b-i}$

let $b+i=re^{i\theta}$, $r=\sqrt{1+b^2}$, $\theta=\tan^{-1}[1/b]=\cot^{-1}(b)$

$b-i=re^{-i\theta}$

$$\frac{bi-1}{bi+1}=\frac{b+i}{b-i}=\frac{re^{i\theta}}{re^{-i\theta}}=e^{2i\theta}=e^{2i\cot^{-1}(b)}$$

We have $LHS = e^{2ai\cot^{-1}(b)}\left[\frac{bi-1}{bi+1}\right]^{-a} = e^{2ai\cot^{-1}(b)}\left[e^{2i\cot^{-1}(b)}\right]^{-a}$

$$= e^{2ai\cot^{-1}(b)}\, e^{-2ai\cot^{-1}(b)} = 1 = RHS$$

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$\boxed{Q4 \text{ (A)}}$ If $\cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$

Then P.T $\cos(3\alpha) + 8\cos(3\beta) + 27\cos(3\gamma) = 18\cos(\alpha+\beta+\gamma)$

$\rightarrow$ Let $a = \cos\alpha + i\sin\alpha = e^{i\alpha}$

$b = 2\cos\beta + 2i\sin\beta = 2e^{i\beta}$

$c = 3\cos\gamma + 3i\sin\gamma = 3e^{i\gamma}$

Now $a+b+c = (\cos\alpha + 2\cos\beta + 3\cos\gamma) + i(\sin\alpha + 2\sin\beta + 3\sin\gamma)$

$= 0 \quad \text{---(1)}$

We know that $a^3+b^3+c^3-3abc = (a+b+c)[a^2+b^2+c^2-ab-bc-ac]$

But $a+b+c=0$ By (1)

$a^3+b^3+c^3 = 3abc$

$e^{3i\alpha} + (2e^{i\beta})^3 + (3e^{i\gamma})^3 = 3\, e^{i\alpha}.(2e^{i\beta})(3e^{i\gamma})$

$e^{3i\alpha} + 8e^{3i\beta} + 27e^{3i\gamma} = 18\, e^{i(\alpha+\beta+\gamma)}$

$[\cos3\alpha + 8\cos3\beta + 27\cos3\gamma] + i[\sin3\alpha + 8\sin3\beta + 27\sin3\gamma]$

$= 18[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$

By comparing Real part

$\cos3\alpha + 8\cos3\beta + 27\cos3\gamma = 18\cos(\alpha+\beta+\gamma)$

$\boxed{Q4 \text{ (B)}}$ If $\cosh(x) = \sec\theta$, then P.T. (a) $x = \log(\sec\theta + \tan\theta)$

(b) $\tanh(x/2) = \tan(\theta/2)$

$\rightarrow$ (a) $\cosh(x) = \sec\theta \Rightarrow x = \cosh^{-1}[\sec\theta]$

$x = \log\left[\sec\theta + \sqrt{\sec^2\theta - 1}\right] = \log\left[\sec\theta + \sqrt{\tan^2\theta}\right]$

$x = \log[\sec\theta + \tan\theta]$ Dr. Uday Kashid

(b) $e^x = \sec\theta + \tan\theta = \frac{1+\sin\theta}{\cos\theta} = \frac{1+2\sin\theta/2\cos\theta/2}{\cos^2\theta/2 - \sin^2\theta/2}$

$= \frac{(\cos\theta/2 + \sin\theta/2)^2}{(\cos\theta/2 + \sin\theta/2)(\cos\theta/2 - \sin\theta/2)}$

$e^x = \frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2 - \sin\theta/2} = \frac{1+\tan\theta/2}{1-\tan\theta/2}$

$$\tanh(x/2) = \left[\frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}}\right] \times \frac{e^{x/2}}{e^{x/2}}$$

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$$= \frac{e^{x}-1}{e^{x}+1} = \frac{\frac{1+\tan\theta/2}{1-\tan\theta/2} - 1}{\frac{1+\tan\theta/2}{1-\tan\theta/2} + 1}$$

$$\tanh(x/2) = \frac{2\tan\theta/2}{2} = \tan(\theta/2)$$

$\boxed{Q.4\ (C)}$ If $u = \sin^{-1}\left[\frac{x^3-y^3}{7x-9y}\right]$, Then find the value of

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}.$$

→ put $x = \lambda x$, $y = \lambda y$

$$u(x,y) = \sin^{-1}\left[\frac{\lambda^3(x^3-y^3)}{\lambda(7x-9y)}\right] = \sin^{-1}\left[\lambda^2\frac{(x^3-y^3)}{7x-9y}\right] \neq \lambda^n u(x,y)$$

But if $z = \frac{x^3-y^3}{7x-9\lambda}$ is homogeneous fn of deg $= 2 = n$.

and $z = f(u) = \sin u$, Then By Euler's thm

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = G(u) = 2\frac{\sin u}{\cos u} = 2\tan u \quad \text{—(I)}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = G(u)[G'(u)-1] = 2\tan u[2\sec^2 u - 1] \quad \text{(II)}$$

Add (I) & (II)

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4\tan u\sec^2 u - 2\tan u + 2\tan u$$

$$= 4\tan u\sec^2 u = 4\frac{\sin u}{\cos u} \times \frac{1}{\cos^2 u} = \frac{4\sin u}{\cos^3 u}$$

$\boxed{Q4\ (D)}$ Find the extreme value of $x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$

→ let $u(x,y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$

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$$\frac{\partial u}{\partial x} = 3x^2 + y^2 - 24x + 21 = 0$$

$$\frac{\partial u}{\partial y} = 2xy - 4y = 0 \Rightarrow 2y(x-2) = 0$$

$y = 0$ or $x = 2$

For $y = 0$, $3x^2 + 0 - 24x + 21 = 0 \Rightarrow (x-7)(x-1) = 0$

$x = 7, 1$

$(7,0)$ & $(1,0)$ are stationary pts

For $x = 2$, $12 + y^2 - 48 + 21 = 0 \Rightarrow y^2 = 15 \Rightarrow y = \pm\sqrt{15}$

$(2, \sqrt{15})$ & $(2, -\sqrt{15})$ are stationary pts.

$r = \frac{\partial^2 u}{\partial x^2} = 6x-24$, $s = \frac{\partial^2 u}{\partial x \partial y} = 2y$, $t = \frac{\partial^2 u}{\partial y^2} = 2x-4$

Pts	$r = 6x-24$	$t = 2x-4$	$s = 2y$	$(rt-s^2)$	Remarks
$(7,0)$	$18>0$	$10>0$	0	$180>0$	fun^n is minimum
$(1,0)$	$-18<0$	$-2<0$	0	$36>0$	maximum
$(2,\sqrt{15})$	$-12<0$	0	$2\sqrt{15}$	$-60<0$	saddle pt
$(2,-\sqrt{15})$	$-12<0$	0	$-2\sqrt{15}$	$-60<0$	saddle pt

$u_{min}(7,0) = -98$

$u_{max}(1,0) = 10$

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Q4 (B) (E) Test the system of equations for consistency and solve if consistent

$$x_1 - 2x_2 + x_3 - x_4 = 2$$
$$x_1 + 2x_2 + 0x_3 + 2x_4 = 1$$
$$0x_1 + 4x_2 - x_3 + 3x_4 = -1$$

→ $AX = B$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 2 & 0 & 2 \\ 0 & 4 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 2 \\ 1 & 2 & 0 & 2 & 1 \\ 0 & 4 & -1 & 3 & -1 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 2 \\ 0 & 4 & -1 & 3 & -1 \\ 0 & 4 & -1 & 3 & -1 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 2 \\ 0 & 4 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A:B) = 2 = \rho(A) <$ no. of variables (4)

Hence equ^ns are consistent and have infinite no of sol^n

I.F.V = Total variables − Rank = 4 − 2 = 2

$x_1 - 2x_2 + x_3 - x_4 = 2$

$4x_2 - x_3 + x_4 = -1$

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let $x_3 = k_1$ and $x_4 = k_2$ Then $x_2 = \frac{k_1 - k_2 - 1}{4}$

and $x_1 = \left[\frac{k_1 - k_2 - 1}{2}\right] - k_1 + k_2 = -\frac{k_1}{2} + \frac{k_2}{2} - \frac{1}{2}$

Q4 (F) Using Lagrange's multipliers method find minimum distance of pt lying on the plane $x+2y+3z=14$ from origin $O(0,0,0)$.

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$\rightarrow$ Let $A(x,y,z)$ be a pt on plane $x+2y+3z=14$ and origin is $O(0,0,0)$.

Hence distance betn pt A and origin is

$$d(OA) = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = \sqrt{x^2+y^2+z^2} \quad \text{(I)}$$

Let $u = f(x,y,z) = x^2+y^2+z^2$ — (II)

and $\phi(x,y,z) = x+2y+3z-14 = 0$ — (III)

We can form Lagrange's fun as

$$L(x,y,z,\lambda) = f(x,y,z) + \lambda\phi(x,y,z)$$

$$L = x^2+y^2+z^2+\lambda(x+2y+3z-14) \quad \text{(IV)}$$

(i) $\frac{\partial L}{\partial x} = 2x+\lambda = 0 \Rightarrow x = -\lambda/2$ — (V)

(ii) $\frac{\partial L}{\partial y} = 2y+2\lambda = 0 \Rightarrow y = -\lambda$ — (VI)

(iii) $\frac{\partial L}{\partial z} = 2z+3\lambda = 0 \Rightarrow z = -3\lambda/2$ — (VII)

put eqn (V), (VI) & (VII) in (III)

$$-\lambda/2 + (-2\lambda) + \left(-\frac{9\lambda}{2}\right) - 14 = 0$$

$\Rightarrow \boxed{\lambda = -2}$

$\boxed{x = -\lambda/2 = 1}$, $\boxed{y = -\lambda = 2}$, $\boxed{z = -3\lambda/2 = 3}$

Thus, pt $A(x,y,z) = (1,2,3)$ lying on plane whose distance is minimum.

$$f_{min}(1,2,3) = 1^2+2^2+3^2 = 14$$

Thus, minimum distance of pt $A(1,2,3)$ from origin is $d(OA) = \sqrt{14}$

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