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Matrix: A matrix is a system of mn numbered arranged in m rows and n columns it is called and $m \times n$ matrices

e.g. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \vdots & a_{ij} & \vdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$

Types of matrices

1) Matrix which contain only one rows is called row matrix

e.g. $A = \begin{bmatrix} 2 & 3 & 21 & 63 & 21 \end{bmatrix}$

2) Matrix which contain only one column is called column matrix

e.g. $A = \begin{bmatrix} 5 \\ 23 \\ 15 \\ 8 \\ 10 \end{bmatrix}$

3) Matrix in which number of rows and columns are same is called square matrix

e.g. $A = \begin{bmatrix} 4 & 2 & 6 & 8 \\ 8 & 5 & 5 & 9 \\ 4 & 2 & 7 & 8 \\ 5 & 3 & 9 & 8 \end{bmatrix}$

4) Diagonal elements: In square matrix the elements lying along diagonal of a matrix are called diagonal elements

e.g. $A = \begin{bmatrix} \mathbf{4} & 2 & 6 & 8 \\ 8 & \mathbf{5} & 5 & 9 \\ 4 & 2 & \mathbf{7} & 8 \\ 5 & 3 & 9 & \mathbf{8} \end{bmatrix}$ Here **4, 5, 7, 8** are the diagonal elements

5) Diagonal Matrix: a square matrix in which all the non-diagonal elements are equal to zero are called diagonal matrix.

e.g. $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ Here 4, 5, 7, 8 are the diagonal elements

6) Scalar matrix: a diagonal matrix whose all diagonal elements are equal is called scalar matrix. e.g.
$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

7) Unit matrix: a diagonal matrix whose all diagonal elements equal to one is called unit matrix.

e.g. A= $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

8) Trace of a matrix: The sum of all the diagonal elements of a matrix is called trace of a matrix

If A= $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ then Trace of matrix A=$\sum_{i=1}^{n} a_{ii}$=$a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$

In given matrix Trace of A=4+5+7+8=24

9) Determinant of a square matrix A is denoted by $|A|$

10) Singular matrix: A square matrix A is said to be singular matrix if $|A| = 0$

e.g. , $C = [0], |C| = 0$,A=$\begin{bmatrix} 3 & 2 \\ 18 & 12 \end{bmatrix}$ then $|A| = 0$, $B = \begin{bmatrix} 1 & 3 & 5 \\ 8 & 4 & 3 \\ 2 & 6 & 10 \end{bmatrix}$ then $|B| = 0$

So matrices C, A, and B are singular matrices.

11) Non-singular matrix: A square matrix A is said to be non- singular matrix if $|A| \neq 0$

e.g. $A = [5], |A| = 5 \neq 0, B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, |B| = -1 \neq 0$ $C = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}, |C| = 27 \neq 0$

So matrices C, A, and B are singular matrices.

12) Upper triangular matrix: A matrix $A = [a_{ij}]$ *is* upper triangular matrix if $a_{ij} = 0 \; for \; i > j$

e.g. . A= $\begin{bmatrix} 4 & 2 & 6 & 8 \\ 0 & 5 & 5 & 9 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ i.e. all the elements below the diagonal elements are zero.

13) Lower triangular matrix: A matrix $A = [a_{ij}]$ *is* lower triangular matrix if $a_{ij} = 0 \; for \; i < j$

. A= $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 8 & 5 & 0 & 0 \\ 4 & 2 & 7 & 0 \\ 5 & 3 & 9 & 8 \end{bmatrix}$ i.e. all the elements above the diagonal elements are zero.

14) Transpose of a matrix: Matrix obtained by interchange of rows and column of a matrix A is called Transpose of a matrix and is denoted by A^T

If . A= $\begin{bmatrix} 4 & 2 & 6 & 8 \\ 8 & 5 & 5 & 9 \\ 4 & 2 & 7 & 8 \\ 5 & 3 & 9 & 8 \end{bmatrix}$ $then \; A^T = \begin{bmatrix} 4 & 8 & 4 & 5 \\ 2 & 5 & 2 & 3 \\ 6 & 5 & 7 & 9 \\ 8 & 9 & 8 & 8 \end{bmatrix}$

15) Conjugate of a matrix: The matrix obtained from a given matrix by replacing each element by it's complex conjugate is called the conjugate of the given matrix and is denoted by $\bar{A}$

Thus if $A = \begin{bmatrix} -2+3i & 3-2i & -6i \\ 4i & 1-2i & -5+2i \\ 3 & -4-6i & 8+2i \end{bmatrix}$ then $\bar{A} = \begin{bmatrix} -2-3i & 3+2i & 6i \\ -4i & 1+2i & -5-2i \\ 3 & -4+6i & 8-2i \end{bmatrix}$

16) Transposed conjugate of a matrix: The transpose of a complex conjugate of a given is called the transposed conjugate of a matrix A and is denoted by A^{θ}

i.e. $A^{\theta} = \bar{A}' = \bar{A'}$

e.g. if if $A = \begin{bmatrix} -2+3i & 3-2i & -6i \\ 4i & 1-2i & -5+2i \\ 3 & -4-6i & 8+2i \end{bmatrix}$ then $\bar{A} = \begin{bmatrix} -2-3i & 3+2i & 6i \\ -4i & 1+2i & -5-2i \\ 3 & -4+6i & 8-2i \end{bmatrix}$

and $A^{\theta} = \bar{A}' = \begin{bmatrix} -2-3i & -4i & 3 \\ 3+2i & 1+2i & -4+6i \\ 6i & -5-2i & 8-2i \end{bmatrix}$

17) Symmetric matrix: A matrix $A = [a_{ij}]$ is said to be symmetric matrix if $a_{ij} = a_{ji}\ \forall\ i, j$

Or $A^T = \mathrm{A}$

A=$\begin{bmatrix} 4 & -2 & 6 & 8 \\ -2 & 5 & -1 & -3 \\ 6 & -1 & 7 & 9 \\ 8 & -3 & 9 & 8 \end{bmatrix}$,A=$\begin{bmatrix} 1 & -2 & 3 \\ -2 & 5 & -7 \\ 3 & -7 & 8 \end{bmatrix}$,A=$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

18)Skew- symmetric matrix: A matrix $A = [a_{ij}]$ is said to be skew-symmetric matrix if $a_{ij} = -a_{ji}\ \forall\ i, j$
Or $A^T = -\mathrm{A}$

A=$\begin{bmatrix} 0 & 2 & -6 & -8 \\ -2 & 0 & 1 & 3 \\ 6 & -1 & 0 & -9 \\ 8 & -3 & 9 & 0 \end{bmatrix}$ A=$\begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 7 \\ 3 & -7 & 0 \end{bmatrix}$,A=$\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$

Note: In skew-symmetric matrix all the diagonal elements equal to zero

19) Hermitian matrix: Matrix $A = [a_{ij}]$ is said to be Hermitian matrix if $\overline{a_{ij}} = a_{ji}\ \forall\ i, j$

Or $A^{\theta} = A$

A=$\begin{bmatrix} 1 & 2-3i & 5+6i \\ 2-3i & 3 & 8-2i \\ 5-6i & 8+2i & 6 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 3-5i \\ 3+5i & 7 \end{bmatrix}$

Note: In Hermitian matrix all the diagonal elements are purely real.

20) Skew-Hermitian matrix: Matrix $A = [a_{ij}]$ is said to be Skew-Hermitian matrix if

$\overline{a_{ij}} = -a_{ji}\ \forall\ i, j$ Or $A^{\theta} = -A$

A=$\begin{bmatrix} 0 & 2-3i & 5+6i \\ -2-3i & 3i & 8-2i \\ -5+6i & -8-2i & -6i \end{bmatrix}$, $A = \begin{bmatrix} 2i & 3-5i \\ -3-5i & 0 \end{bmatrix}$

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Note: In Skew-Hermitian matrix all the diagonal elements are either zero or purely imaginary

Note:

1) $If\ a\ Matrix\ A\ is\ Hermition\ then\ iA\ is\ skew - Hermitian\ Matrix$

2) $If\ a\ matrix\ A\ \ is\ Skew - Hermitian\ then\ iA\ is\ Hermition\ Matrix$

Theorem: Every matrix A can be expressed as sum of symmetric and skew-symmetric matrices. i.e. $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

Where $\frac{1}{2}(A + A^T)\ is\ symmetric\ matrix, and\ \frac{1}{2}(A - A^T)\ is\ Skew - symmetric\ matrix$

e.g. Let $A = \begin{bmatrix} 6 & 10 & 16 \\ 20 & 26 & 30 \\ 40 & 50 & 60 \end{bmatrix}$ $\therefore\ A^T = \begin{bmatrix} 6 & 20 & 40 \\ 10 & 26 & 50 \\ 16 & 30 & 60 \end{bmatrix}$

Now $A + A^T = \begin{bmatrix} 12 & 30 & 56 \\ 30 & 52 & 80 \\ 56 & 80 & 120 \end{bmatrix}$, and $A - A^T = \begin{bmatrix} 0 & -10 & -24 \\ 10 & 0 & -20 \\ 24 & 20 & 0 \end{bmatrix}$

$\therefore \frac{1}{2}(A + A^T) = \begin{bmatrix} 6 & 15 & 28 \\ 15 & 26 & 40 \\ 28 & 40 & 60 \end{bmatrix}\ is\ symmetric, and$

$\frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & -5 & -12 \\ 5 & 0 & -10 \\ 12 & 10 & 0 \end{bmatrix}\ is\ skew - symmetric$

And $\begin{bmatrix} 6 & 10 & 16 \\ 20 & 26 & 30 \\ 40 & 50 & 60 \end{bmatrix} = \begin{bmatrix} 6 & 15 & 28 \\ 15 & 26 & 40 \\ 28 & 40 & 60 \end{bmatrix} + \begin{bmatrix} 0 & -5 & -12 \\ 5 & 0 & -10 \\ 12 & 10 & 0 \end{bmatrix}$ i.e. $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

Example: Express the following matrices as sum of symmetric and skew-symmetric matrices

1)$A = \begin{bmatrix} 5 & 7 \\ -3 & 4 \end{bmatrix}$,2)$A = \begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 3 & 5 & 11 \end{bmatrix}$, 3) $A = \begin{bmatrix} 0 & 5 & -3 \\ 1 & 1 & 1 \\ 4 & 5 & 9 \end{bmatrix}$,4)$A = \begin{bmatrix} 1 & 0 & 5 & 3 \\ -2 & 1 & 6 & 1 \\ 3 & 2 & 7 & 1 \\ 4 & -4 & 2 & 0 \end{bmatrix}$, 5) $A = \begin{bmatrix} 1 & 5 & 7 \\ -1 & -2 & -4 \\ 8 & 2 & 13 \end{bmatrix}$

Theorem: Every matrix A can be expressed as sum of Hermitian and skew-Hermitian matrices. i.e. $A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$

Where $\frac{1}{2}(A + A^{\theta})$ is Hermitian $matrix, and$ $\frac{1}{2}(A - A^{\theta})$ $is\ Skew -$ Hermitian $matrix$

e.g. Let $A = \begin{bmatrix} 2+4i & 4-6i & 6+8i \\ 8-10i & 10+12i & 12-14i \\ 14+16i & 16-18i & 18+20i \end{bmatrix} \therefore \bar{A} = \begin{bmatrix} 2-4i & 4+6i & 6-8i \\ 8+10i & 10-12i & 12+14i \\ 14-16i & 16+18i & 18-20i \end{bmatrix}$

$$A^{\theta} = \bar{A}^{T} = \begin{bmatrix} 2-4i & 8+10i & 14-16i \\ 4+6i & 10-12i & 16+18i \\ 6-8i & 12+14i & 18-20i \end{bmatrix}$$

Now

$A + A^{\theta} = \begin{bmatrix} 4 & 12+4i & 20-8i \\ 12-4i & 20 & 28+4i \\ 20+8i & 28-4i & 36 \end{bmatrix}$, and $A - A^{\theta} = \begin{bmatrix} 8i & -4-16i & -8+24i \\ 4-16i & 24i & -4-32i \\ 8+24i & 4-32i & 40i \end{bmatrix}$

$\therefore \frac{1}{2}(A + A^{\theta}) = \begin{bmatrix} 2 & 6+2i & 10-4i \\ 6-2i & 10 & 14+2i \\ 10+4i & 14-2i & 18 \end{bmatrix}$ $is\ Hermitian\ matrix\ , and$

$\frac{1}{2}(A - A^{\theta}) = \begin{bmatrix} 4i & -2-8i & -4+12i \\ 2-8i & 12i & -2-16i \\ 4+12i & 2-16i & 20i \end{bmatrix}$ $is\ skew - Hermitain\ matrix$

And $\begin{bmatrix} 2+4i & 4-6i & 6+8i \\ 8-10i & 10+12i & 12-14i \\ 14+16i & 16-18i & 18+20i \end{bmatrix} = \begin{bmatrix} 2 & 6+2i & 10-4i \\ 6-2i & 10 & 14+2i \\ 10+4i & 14-2i & 18 \end{bmatrix} + \begin{bmatrix} 4i & -2-8i & -4+12i \\ 2-8i & 12i & -2-16i \\ 4+12i & 2-16i & 20i \end{bmatrix}$

i.e. $A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$

Example: Express the following matrices as sum of symmetric and skew-symmetric matrices

1)$A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$, 2) $A = \begin{bmatrix} 2 & 2+i & 3 \\ -2+i & 0 & 4i \\ -i & 3-i & 1-i \end{bmatrix}$,3)$A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & i & 3i \end{bmatrix}$, 4) $A = \begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 0 \end{bmatrix}$, 5) $A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$, 6) $A = \begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}$

7) $A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3+2i & -1+2i & 0 \end{bmatrix}$, 8) $A = \begin{bmatrix} 2 & 4+i & 4i \\ 3i & 6-i & 2 \\ 6 & 4-2i & 1-i \end{bmatrix}$, 9) $A = \begin{bmatrix} 2 & 4+i & 6i \\ 6 & 5-i & 4 \\ 0 & 1-i & 8i \end{bmatrix}$

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Theorem: Every square matrix A can be expressed as $P + iQ$ where $P, and\ Q\ are\ Hermitian\ matrices$

Note:$(i)^\theta = -i$, $(-i)^\theta = i$

Proof: $A = \frac{1}{2}[A + A] = \frac{1}{2}[A + A^\theta + A - A^\theta] = \frac{1}{2}\left[(A + A^\theta) + i\frac{1}{i}(A - A^\theta)\right] = \left[\frac{1}{2}(A + A^\theta) + i\frac{1}{2i}(A - A^\theta)\right]$

$$A = P + iQ\ where\ P = \frac{1}{2}(A + A^\theta), and\ Q = \frac{1}{2i}(A - A^\theta)$$

Now $P^\theta = \left(\frac{1}{2}(A + A^\theta)\right)^\theta = \frac{1}{2}\left(A^\theta + (A^\theta)^\theta\right) = \frac{1}{2}(A^\theta + A) = P\ so\ P\ is\ hermition\ Matrix$

$Q = \left(\frac{1}{2i}(A - A^\theta)\right)^\theta = \frac{1}{2i^\theta}\left(A^\theta - (A^\theta)^\theta\right) = \frac{1}{2(-i)}(A^\theta - A) = \frac{-1}{2i}(A^\theta - A) = \frac{1}{2i}(A - A^\theta) = Q$

$So\ Q\ is\ hermitian\ matrix$

Thus
$A = P + iQ\ where\ P = \frac{1}{2}(A + A^\theta)\ , and\ Q = \frac{1}{2i}(A - A^\theta)$ then P and Q are hermitian matrices

Example: Express the matrix $A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix}$ as $P + iQ$, where P and Q are hermitian matrices

Solution: By given $A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix}, So\ \bar{A} = \begin{bmatrix} 2 & 3+i & 2-i \\ -i & 0 & 1+i \\ 1-2i & 1 & -3i \end{bmatrix}$

$$A^\theta = (\bar{A})^T = \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & 1 \\ 2-i & 1+i & -3i \end{bmatrix}$$

Now $(A + A^\theta) = \begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & 2-i \\ 3+i & 2+i & 0 \end{bmatrix}$, and $(A - A^\theta) = \begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & -i \\ -1+3i & -i & 6i \end{bmatrix}$

$\therefore \frac{1}{2}(A + A^\theta) = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & 2-i \\ 3+i & 2+i & 0 \end{bmatrix}$, and $\frac{1}{2i}(A - A^\theta) = \frac{1}{2i}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & -i \\ -1+3i & -i & 6i \end{bmatrix}$

Let $P = \frac{1}{2}(A + A^\theta) = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & 2-i \\ 3+i & 2+i & 0 \end{bmatrix}, Q = \frac{1}{2i}(A - A^\theta) = \frac{1}{2i}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & -i \\ -1+3i & -i & 6i \end{bmatrix}$

Then P and Q are Hermition Matrices

$Thus\ A = P + iQ$

i.e. $\begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & 2-i \\ 3+i & 2+i & 0 \end{bmatrix} + i\left(\frac{1}{2i}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & -i \\ -1+3i & -i & 6i \end{bmatrix}\right)$

Example: Express the matrix 1)$A=\begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}$,2)$A=\begin{bmatrix} 1+2i & 2 & 3-i \\ 2+3i & 2i & 1-2i \\ 1+i & 0 & 3+2i \end{bmatrix}$

Theorem: Every Hermition matrix A can be written as $A = B + iC, where\ B\ is\ real\ symmetric\ matrix\ and\ C\ is\ real\ skew-symmetric\ matrix$

Solution: By given A is hermition matrix $\therefore A^{\theta} = A\ i.e.(\bar{A})^T = A, or\ \overline{(A^T)} = A \cdots (*)$

Now $A = \frac{1}{2}(A + A) = \frac{1}{2}(A + \bar{A} + A - \bar{A}) = \frac{1}{2}(A + \bar{A}) + i\frac{1}{2i}(A - \bar{A}) = P + iQ \cdots (1)$

Where $P = \frac{1}{2}(A + \bar{A}), Q = \frac{1}{2i}(A - \bar{A}) \cdots (2)$

We know if $z = x + iy, \bar{z} = x - iy$, where x and y are real numbers ,then

$z + \bar{z} = 2x, z - \bar{z} = i2y \Rightarrow \frac{1}{2}(z + \bar{z}) = x, and\ \frac{1}{2i}(z - \bar{z}) = y \Rightarrow \frac{1}{2}(z + \bar{z}), and\ \frac{1}{2i}(z - \bar{z})\ are\ real\ numbers$

From equation (2) $P = \frac{1}{2}(A + \bar{A}), Q = \frac{1}{2i}(A - \bar{A})\ are\ real\ numbers$ i.e. P and Q are real numbers

Now From equation (2) $P^T = \left(\frac{1}{2}(A + \bar{A})\right)^T = \frac{1}{2}(A^T + \bar{A}^T) = \frac{1}{2}\left(A^T + A^{\theta}\right) = \frac{1}{2}(A^T + A)$ $\{using\ (*)\}$

$P^T = \frac{1}{2}(A + A^T) = P$ $\{Using\ (2)\}$ ⇒P is symmetric

Now From equation (2) $Q^T = \left(\frac{1}{2i}(A - \bar{A})\right)^T = \frac{1}{2i}(A^T - \bar{A}^T) = \frac{1}{2i}\left(A^T - A^{\theta}\right) = \frac{1}{2i}(A^T - A)$ $\{using\ (*)\}$

$Q^T = \frac{-1}{2i}(A - A^T) = -Q$ $\{Using\ (2)\}$ ⇒Q is skew-symmetric

Matrix A can be expressed as $A = P + iQ$where

$$P = \frac{1}{2}(A + \bar{A})is\ real\ and\ symmetric\ and\ Q = \frac{1}{2i}(A - \bar{A})is\ real\ and\ skew - symmetric$$

Example: Express the matrix $A = \begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix}$ $as\ A = P + iQ$, where $P\ is\ real\ and\ symmetric\ and\ Q\ is\ real\ and\ skew - symmetric$

Solution: $A = \begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix}, \bar{A} = \begin{bmatrix} 2 & 1-i & i \\ 1+i & 0 & -3+i \\ -i & -3-i & -1 \end{bmatrix}$

$A + \bar{A} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 0 & -6 \\ 0 & -6 & -2 \end{bmatrix} \Rightarrow \frac{1}{2}(A + \bar{A}) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix}$is symmetric

And $A - \bar{A} = \begin{bmatrix} 0 & 2i & -2i \\ -2i & 0 & -2i \\ 2i & 2i & 0 \end{bmatrix} \Rightarrow \frac{1}{2i}(A - \bar{A}) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$is skew-symmetric

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Thus $A = P + iQ$, Where $P = \frac{1}{2}(A + \bar{A}) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix}$ *is real and symmetric matrix*

And $Q = \frac{1}{2i}(A - \bar{A}) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$ is real and skew-symmetric *matrix*

Example: Express the matrix 1) $A = \begin{bmatrix} 1 & 2+i & -1+i \\ 2-i & 1 & 2i \\ -1-i & -2i & 0 \end{bmatrix}$, 2) $A = \begin{bmatrix} 2 & 2+i & -2i \\ 2-i & 3 & i \\ 2i & -i & 1 \end{bmatrix}$

Theorem: Every Skew-Hermitian matrix A can be expressed as $A = P + iQ$ where P is real and skew-symmetric and Q is real and symmetric.

Solution: By given A is skew-hermitian matrix $\therefore A^{\theta} = -A, or\ (\bar{A})^T = -A, or\ \overline{(A^T)} = -A$

Now $A = \frac{1}{2}(A + A) = \frac{1}{2}(A + \bar{A} + A - \bar{A}) = \frac{1}{2}(A + \bar{A}) + i\frac{1}{2i}(A - \bar{A}) = P + iQ \cdots (1)$

Where $P = \frac{1}{2}(A + \bar{A}), Q = \frac{1}{2i}(A - \bar{A}) \cdots (2)$

We know if $z = x + iy, \bar{z} = x - iy$, where x and y are real numbers ,then

$z + \bar{z} = 2x, z - \bar{z} = i2y \Rightarrow \frac{1}{2}(z + \bar{z}) = x, and\ \frac{1}{2i}(z - \bar{z}) = y \Rightarrow \frac{1}{2}(z + \bar{z}), and\ \frac{1}{2i}(z - \bar{z})\ are\ real\ numbers$

From equation (2) $P = \frac{1}{2}(A + \bar{A}), Q = \frac{1}{2i}(A - \bar{A})\ are\ real\ numbers$ i.e. P and Q are real numbers

Now From equation (2) $P^T = \left(\frac{1}{2}(A + \bar{A})\right)^T = \frac{1}{2}(A^T + \bar{A}^T) = \frac{1}{2}(A^T + A^{\theta}) = \frac{1}{2}(A^T - A)$ $\{using\ (*)\}$

$P^T = \frac{-1}{2}(A - A^T) = -P$ $\{Using\ (2)\}$ ⇒P is skew-symmetric

Now From equation (2) $Q^T = \left(\frac{1}{2i}(A - \bar{A})\right)^T = \frac{1}{2i}(A^T - \bar{A}^T) = \frac{1}{2i}(A^T - A^{\theta}) = \frac{1}{2i}(A^T + A)$ $\{using\ (*)\}$

$Q^T = \frac{1}{2i}(A + A^T) = Q$ $\{Using\ (2)\}$ ⇒Q is symmetric

Matrix A can be expressed as $A = P + iQ$ where

$$P = \frac{1}{2}(A + \bar{A})is\ real\ and\ symmetric\ and\ Q = \frac{1}{2i}(A - \bar{A})is\ real\ and\ skew - symmetric$$

1) Example: Express the matrix $A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$ $as\ A = P + iQ$, where $P\ is\ real\ and\ skew - symmetric\ and\ Q\ is\ real\ and\ symmetric$

Solution: $A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}, \bar{A} = \begin{bmatrix} -2i & 2-i & 1+i \\ -2-i & i & -3i \\ -1+i & -3i & 0 \end{bmatrix}$

$A + \bar{A} = \begin{bmatrix} 0 & 4 & 2 \\ -4 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \Rightarrow \frac{1}{2}(A + \bar{A}) = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ is skew-symmetric

And $A - \bar{A} = \begin{bmatrix} 4i & 2i & -2i \\ 2i & -2i & 6i \\ -2i & 6i & 0 \end{bmatrix} \Rightarrow \frac{1}{2i}(A - \bar{A}) = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ -1 & 3 & 0 \end{bmatrix}$ is symmetric

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Thus $A = P + iQ$, Where $P = \frac{1}{2}(A + \bar{A}) = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ $is\ real\ and\ skew - symmetric\ matrix$

And $Q = \frac{1}{2i}(A - \bar{A}) = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ -1 & 3 & 0 \end{bmatrix}$ is real and symmetric $matrix$

Example: Express the matrix 1) $A = \begin{bmatrix} i & 1-i & 2+3i \\ -1-i & 2i & -3i \\ -2+3i & -3i & -i \end{bmatrix}$, 2) $A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$