Maximum and Minimum for functions of two variables Dr. Uday Kashid, PhD (Mathematics)

Dr.Uday Kashid at 01-02-2022 13:22

Definition :

Let $f(x, y)$ be a function of two variables x and y. At $x = a$; $y = b$ i.e. (a, b)

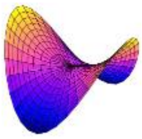
The $f(x, y)$ is said to have **maximum** If.. $f(a, b) > f(a + h, b + k)$ where h and k are small values.

or

$f(x, y)$ is have **minimum** value If $f(a, b) < f(a + h, b + k)$, where h and k are small values.

- **Extremum :** A function which have a **maximum** or **minimum** or **both** is called 'extremum' .
- **Extreme value** :- The maximum value or minimum value or both of a function is Extreme value.
- **Stationary points:** - To get stationary points(i.e. extreme or turning values) we solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

  i.e. the pairs $(a_1, b_1), (a_2, b_2)$ ……….. are called Stationary points of $f(x, y)$.
- **Saddle point :** A point at which the function $f(x, y)$ is **neither maximum nor minimum** is called a **saddle** point.
  i.e. At the saddle point, the function will be maximum in one direction while minimum in another direction.
  e.g. The hyperbolic paraboloid $z = f(x, y) = xy$ has a saddle point at the origin. The surface is called a hyperbolic paraboloid due to cross sections as parabolas/hyperbolas as shown in below. Dr. Uday Kashid, PhD (Mathematics)



Working Rules :

If $z = f(x, y)$ then to find Maximum or Minimum values of z , following rules are used…

step1 1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate them to zero. i.e. $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ and solve for x and y.

step 2. Calculate the values of $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x\, \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

step3 3. At any pairs of value of 'x' and 'y' for ex. Say (a, b) x, y Dr. Uday Kashid, PhD (Mathematics)

- If $(rt - s^2) > 0$ and $r < 0$ or $t < 0$ at (a, b) Then,
  $f(x, y)$ is Maximum at (a, b) and $f(a, b)$ is Maximum value of $z = f(x, y)$
- If $(rt - s^2) > 0$ and $r > 0$ or $t > 0$ at (a, b) Then,
- $f(x, y)$ is Minimum at (a, b) and $f(a, b)$ is Minimum value of $z = f(x, y)$
- If $(rt - s^2) < 0$ at (a, b) Then $z = f(x, y)$ is **neither maximum nor minimum** and in this case (a, b) is called a **saddle** point.
- If $(rt - s^2) = 0$ at (a, b) Then $z = f(x, y)$ is undecided. i.e. No conclusion can be drawn about maximum or minimum and needs further investigation.

Similarly we do this for other stationary points.

A) Find the stationary values and also find maximum and/or minimum values of following functions.

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1. If $z = f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ [MU -Dec-12,17] (6 marks)
2. If $z = f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ [MU -Dec-14, 15 May-17] (6 marks)
3. If $z = f(x, y) = x^3 + y^3 - 3axy,\ a > 0$ [MU -May-13] (6 marks)
4. If $z = f(x, y) = \sin x \sin y \sin(x + y)$ [MU -Dec-13] (6 marks)
5. If $z = f(x, y) = x^3y^2(1 - x - y)$ [MU -Dec-13] (6 marks) Ans: $f_{max}\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{432}$
6. Show that the minimum value of $z = f(x, y) = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ is $3a^3$. [MU -May-02 ,09] (6 marks)
7. Find three positive numbers whose sum is 100 and whose product is maximum.
8. Divide 24 into three parts such that the product of the first, square of the second and cube of the third is the maximum. [MU -May-14] (6 marks)

1. If $z = f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ [MU -Dec-12,17] (6 marks)

→ We have $z = f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ —(I)

I] To find stationary pts let $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$\therefore \frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72 = 0 \Rightarrow \frac{\partial f}{\partial x} = x^2 + y^2 - 10x + 24 = 0$ —(II)

$\frac{\partial f}{\partial y} = 6xy - 30y = 0 \Rightarrow 6y(x - 5) = 0$

$\Rightarrow y = 0$ or $x = 5$

For $y = 0$, put in (II) , Then

$\therefore x^2 - 10x + 24 = 0$

$\Rightarrow (x - 6)(x - 4) = 0$ Dr. Uday Kashid , Ph.D.(Mathematics)

$\Rightarrow \boxed{x = 6}$ or $\boxed{x = 4}$

Thus at $y = 0$ we have $x = 6$ & $x = 4$ i.e. we have $(6, 0)$ & $(4, 0)$ are stationary pts

Also for $x = 5$, put in (II) $25 + y^2 - 50 + 24 = 0 \Rightarrow y^2 = 1$

$\Rightarrow \boxed{y = \pm 1}$

Thus we have $(5, 1)$ and $(5, -1)$ are also stationary pts.

II] We have $r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = 6x - 30$

$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = 6y$

$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = 6x - 30$

Points $f(x, y)$	$r = \frac{\partial^2 f}{\partial x^2} = 6x - 30$	$t = \frac{\partial^2 f}{\partial y^2} = 6x - 30$	$s = \frac{\partial^2 f}{\partial x\, \partial y} = 6y$	$(rt - s^2)$	Remarks for $f(x, y)$ is	*Ans* $f(x, y)$
$(4, 0)$	$-6 < 0$	$-6 < 0$	0	$36 > 0$	Maximum	$f_{max}(4, 0) = 112$
$(6, 0)$	$6 > 0$	$6 > 0$	0	$36 > 0$	Minimum	$f_{min}(6, 0) = 108$
$(5, 1)$	0	0	6	$-36 < 0$	Neither max.nor min.	—
$(5, -1)$	0	0	-6	$-36 < 0$	Neither Max.nor min	—

112

108

Ans: $f_{max}(4, 0) = 112$ and $f_{min}(6, 0) = 108$

2. If $z = f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ [MU -Dec-14, 15 May-17] (6 marks)

→ We have $z = f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ —(I)

$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x = 0$ —(II) Also $\frac{\partial^2 f}{\partial x^2} = r = 6x - 6$

$\frac{\partial f}{\partial y} = 6xy - 6y = 0$ —(III) Also $\frac{\partial^2 f}{\partial y^2} = t = 6x - 6$

Dr. Uday Kashid , Ph.D.(Mathematics) Also $\frac{\partial^2 f}{\partial x \partial y} = s = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[6xy - 6y\right] = 6y$

From eqn (III) $\frac{\partial f}{\partial y} = 6xy - 6y = 0 \Rightarrow 6y(x - 1) = 0 \Rightarrow \boxed{y = 0}$ or $\boxed{x = 1}$

For $y = 0$, put in (II) $3x^2 + 0 - 6x = 0 \Rightarrow 3x(x - 2) = 0$

$\Rightarrow x = 0$ or $x = 2$

Hence (0, 0) & (2, 0) are stationary pts.

From eq (iii) $\frac{\partial f}{\partial y} = 6xy - 6y = 0 \Rightarrow 6y(x-1) = 0 \Rightarrow \boxed{y=0}$ or $\boxed{x=1}$

For $y=0$, put in (ii) $3x^2 + 0 - 6x = 0 \Rightarrow 3x(x-2) = 0$
$\Rightarrow x = 0$ or $x = 2$

Hence $(0,0)$ & $(2,0)$ are stationary pts.
For $x=1$, put in (ii) $3 + 3y^2 - 6 = 0 \Rightarrow y^2 = 1 \Rightarrow \boxed{y = \pm 1}$

Hence $(1,1)$ and $(1,-1)$ are also stationary pts for $z = f(x,y)$

Points $f(x,y)$	$r = \frac{\partial^2 f}{\partial x^2} = 6x-6$	$t = \frac{\partial^2 f}{\partial y^2} = 6x-6$	$s = \frac{\partial^2 f}{\partial x\,\partial y} = 6y$	$(rt - s^2)$	Remarks for $f(x,y)$ is
$(0,0)$	$-6<0$	$-6<0$	0	$36>0$	Maximum
$(2,0)$	$6>0$	$6>0$	0	$36>0$	Minimum
$(1,1)$	0	0	6	$-36<0$	Neither Maximum
$(1,-1)$	0	0	-6	$-36<0$	Nor minimum

Ans:- $f_{Max}(0,0) = 4$, and $f_{Min}(2,0) = 0$

3. If $z = f(x,y) = x^3 + y^3 - 3axy,\ a>0$ [MU -May-13] (6 marks) Dr. Uday Kashid , Ph.D.(Mathematics)

$\rightarrow$ We have $z = f(x,y) = x^3 + y^3 - 3axy,\ a>0$ —— (i)

$\therefore \frac{\partial f}{\partial x} = 3x^2 - 3ay \Rightarrow \frac{\partial^2 f}{\partial x^2} = r = 6x$

$\frac{\partial f}{\partial y} = 3y^2 - 3ax \Rightarrow \frac{\partial^2 f}{\partial y^2} = t = 6y$

And $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[3y^2 - 3ax\right] = -3a$

Now for stationary pts, $\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 3ay = 0$
$\Rightarrow y = \frac{x^2}{a}$ —— (ii)

Also $\frac{\partial f}{\partial y} = 0 \Rightarrow 3y^2 - 3ax = 0 \Rightarrow y^2 - ax = 0$ —— (iii)

Now put eq^n (ii) in equ^n (iii)

$\frac{x^4}{a^2} - ax = 0 \Rightarrow x^4 - a^3 x = 0$

$\Rightarrow x(x^3 - a^3) = 0 \Rightarrow \boxed{x=0}$ or $x^3 = a^3$ i.e. $\boxed{x=a}$

For $x=0$, put in (ii) Then $\boxed{y=0} \Rightarrow (0,0)$ is stationary pt.
For $x=a$, put in (ii) Then $y = \frac{a^2}{a} = a \Rightarrow (a,a)$ is stationary pt

Points $f(x,y)$	$r = \frac{\partial^2 f}{\partial x^2} = 6x$	$t = \frac{\partial^2 f}{\partial y^2} = 6y$	$s = \frac{\partial^2 f}{\partial x \partial y} = -3a$	$(rt - s^2)$	Remarks for $f(x,y)$ is
$(0,0)$	0	0	$-3a$	$-9a^2<0$	Neither Max. nor Min
(a,a)	$6a>0$	$6a>0$	$-3a$	$27a^2>0$	Minimum

ANS: $f_{min}(a,a) = a^3 + a^3 - 3a^3 = \boxed{-a^3}$

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4. If $z = f(x,y) = \sin x \sin y \sin(x+y)$ [MU -Dec-13] (6 marks)

$\rightarrow$ We have $z = f(x,y) = \sin(x)\sin(y)\sin(x+y)$ —— (i)

$\frac{\partial f}{\partial x} = \sin(y) \cdot \frac{\partial}{\partial x}\left[\sin(x)\sin(x+y)\right]$

$= \sin(y)\left\{\cos(x)\sin(x+y) + \sin(x)\cos(x+y)\right\}$

$\frac{\partial f}{\partial x} = \sin(y)\ \sin(2x+y)$ $\quad \therefore \sin A\cos B + \cos A \sin(B) = \sin(A+B)$

$\Rightarrow \frac{\partial^2 f}{\partial x^2} = r = \sin(y)\cos(2x+y)\cdot(2) = 2\sin(y)\cos(2x+y)$

Also $\frac{\partial f}{\partial y} = \sin(x)\frac{\partial}{\partial y}\left[\sin(y)\sin(x+y)\right] = \sin(x)\left[\cos(y)\sin(x+y) + \sin(y)\cos(x+y)\right]$

$\frac{\partial f}{\partial y} = \sin(x)\sin(x+2y)$ $\quad \therefore \sin A\cos B + \cos A\sin B = \sin(A+B)$

$\Rightarrow \frac{\partial^2 f}{\partial y^2} = t = 2\sin(x)\cos(x+2y)$ Dr. Uday Kashid , Ph.D.(Mathematics)

And $\frac{\partial^2 f}{\partial x \partial y} = s = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[\sin(x)\sin(x+2y)\right] = \cos(x)\sin(x+2y) + \sin(x)\cos(x+2y)$

$s = \frac{\partial^2 f}{\partial x \partial y} = \sin(2x+2y)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

Now For stationary pts Let $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\therefore \frac{\partial f}{\partial x} = \sin(y)\sin(2x+y) = 0 \Rightarrow \frac{1}{2}\left[2\sin(2x+y)\sin y\right] = 0$, $\Rightarrow \frac{1}{2}\left[\cos(2x) - \cos(2x+2y)\right] = 0$, $\Rightarrow \cos 2x - \cos(2x+2y) = 0$, $\Rightarrow \cos(2x) = \cos(2x+2y)$ —— (ii)

$\therefore \frac{\partial f}{\partial y} = \sin x \sin(x+2y) = 0 \Rightarrow \frac{1}{2}\left[2\sin(x+2y)\sin x\right] = 0$, $\Rightarrow \frac{1}{2}\left[\cos(2y) - \cos(2x+2y)\right] = 0$, $\Rightarrow \cos(2y) = \cos(2x+2y)$ —— (iii)

$\Rightarrow \cos(2x) = \cos(2y)$
$\Rightarrow \boxed{x=y}$ —— (iv)

Now, put $x=y$ in eq^n (ii), $\Rightarrow \cos(2x) = \cos(2x+2x) = \cos 4x$ $\therefore$ But $\boxed{\cos 4x = 2\cos^2(2x) - 1}$, Hence $\cos(2x) = 2\cos^2(2x) - 1$

$\Rightarrow 2\cos^2(2x) - \cos(2x) - 1 = 0 \Rightarrow$ By solving this quadratic equ^n

$\Rightarrow \cos(2x) = \frac{1 \pm \sqrt{1 - 4(2)(-1)}}{2(2)} = \frac{1 \pm \sqrt{9}}{4} = \frac{1 \pm 3}{4} = \boxed{1, -\tfrac{1}{2}}$

$\Rightarrow \cos(2x) = 1 = \cos(0)$ or $\cos(2x) = -\frac{1}{2} = \cos(\pi - \pi/3) = \cos(2\pi/3)$

$\Rightarrow \boxed{x=0}$ $\quad \Rightarrow 2x = 2\pi/3 \Rightarrow \boxed{x = \pi/3}$

$\Rightarrow$ But $x=y$, Hence
$(0,0)$ & $(\pi/3, \pi/3)$ are stationary pts.

$\therefore$

Points $f(x,y)$	$r = \frac{\partial^2 f}{\partial x^2} = 2\sin y\cos(2x+y)$	$t = \frac{\partial^2 f}{\partial y^2} = 2\sin x\cos(x+2y)$	$s = \frac{\partial^2 f}{\partial x\,\partial y} = \sin(2x+2y)$	$(rt - s^2)$	Remarks for $f(x,y)$ is
$(0,0)$	0	0	0	0	No conclusion
$(\frac{\pi}{3}, \frac{\pi}{3})$	$-\sqrt{3}<0$	$-\sqrt{3}<0$	$-\frac{\sqrt{3}}{2}$	$\frac{9}{4}>0$	maximum

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$\Rightarrow f_{max}(\frac{\pi}{3}, \frac{\pi}{3}) = \sin x \sin y \sin(x+y) = \sin \pi/3 \cdot \sin \pi/3 \sin(2\pi/3) = \boxed{\frac{3\sqrt{3}}{8}}$

5. If $z = f(x,y) = x^3y^2(1-x-y)$ [MU -Dec-13] (6 marks)

$\rightarrow$ We have $z = f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$ —— (i)

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ $\quad$ Also $\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$

$\frac{\partial^2 f}{\partial x^2} = r = 6xy^2 - 12x^2y^2 - 6xy^3$ $\quad \therefore \frac{\partial^2 f}{\partial y^2} = t = 2x^3 - 2x^4 - 6x^3y$

Now For stationary values Let $\frac{\partial f}{\partial x} = 0$ i.e. $3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$\Rightarrow x^2y^2(3 - 4x - 3y) = 0$ —— (ii) $\quad \Rightarrow x^2y^2 = 0$ or $3-4x-3y=0$
$\Rightarrow x=0$, or $y=0$ or $3-4x-3y=0$

And For $\frac{\partial f}{\partial y} = 0 \Rightarrow 2x^3y - 2x^4y - 3x^3y^2 = 0$
$\Rightarrow x^3y(2 - 2x - 3y) = 0$ —— (iii) $\quad \Rightarrow x^3y = 0$ or $2-2x-3y=0$
$\Rightarrow x=0$, or $y=0$, or $2-2x-3y=0$

Hence, When $x=0$, Then $y=0 \Rightarrow (0,0)$ is the stationary pt.
When $x=0$ and $2-2x-3y=0 \Rightarrow \boxed{y = 2/3}$ i.e. $(0, 2/3)$ is the stationary pt.
When $y=0$ and $2-2x-3y=0 \Rightarrow \boxed{x=1}$ i.e. $(1,0)$ is the stationary pt.

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Now From equ^n (iii)
When $x=0$ and $3-4x-3y=0 \Rightarrow \boxed{y=1}$ i.e. $(0,1)$ is the stationary pt.

When $y=0$ and $3-4x-3y=0 \Rightarrow x=3/4$, i.e. $(3/4,0)$ is the stationary pt.

And when $3-4x-3y=0$ and $2-2x-3y=0$, Then by solving them

$$
\begin{array}{l}
\quad 3-4x-3y=0 \\
- \quad \underline{-4+4x+6y=0} \\
\quad -1+0+3y=0 \\
\Rightarrow \boxed{y=1/3}
\end{array}
$$

Now put $y=1/3$ in $2-2x-3(1/3)=0 \Rightarrow x=1/2$

Thus, $(1/2,1/3)$ is also the stationary pt.

Thus, we have $(0,0),(0,2/3),(1,0),(0,1),(3/4,0)$ and $(1/2,1/3)$ are the stationary pts. of $f(x,y)$

Points $f(x,y)$	$r=\frac{\partial^2 f}{\partial x^2}$ $=6xy^2-12x^2y^2-6xy^3$	$t=\frac{\partial^2 f}{\partial y^2}$ $=2x^3-2x^4-6x^3y$	$s=\frac{\partial^2 f}{\partial x\,\partial y}$ $=6x^2y-8x^3y-9x^2y^2$	$(rt-s^2)$	Remarks for $f(x,y)$ is
$(0,0)$	0	0	0	0	No conclusion
$(0,2/3)$	0	0	0	0	No conclusion
$(1,0)$	0	0	0	0	No conclusion
$(0,1)$	0	0	0	0	No conclusion
$(3/4,0)$	0	$\frac{27}{128}$	0	0	No conclusion
$(1/2,1/3)$	$-\frac{1}{9}<0$	$-\frac{1}{8}<0$	$-\frac{1}{12}$	$\frac{1}{144}>0$	Maximum

Ans: $f_{max}(1/2,1/3) = x^3y^2(1-x-y) = (1/2)^3(1/3)^2\left[1-\frac{1}{2}-\frac{1}{3}\right] = \frac{1}{8}\times\frac{1}{9}\times\frac{1}{6} = \boxed{\frac{1}{432}}$

6. Show that the minimum value of $z=f(x,y)=xy+a^3\left(\frac{1}{x}+\frac{1}{y}\right)$ is $3a^3$. [MU -May-02 ,09] (6 marks)

$\Rightarrow$ We have $z=f(x,y)=xy+a^3\left(\frac{1}{x}+\frac{1}{y}\right)$ —— (I)

For the stationary pts, we have $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$

$\therefore \frac{\partial f}{\partial x} = y+a^3\left(-\frac{1}{x^2}+0\right) = y-\frac{a^3}{x^2}=0 \quad \Rightarrow \quad x^2y=a^3$ —— (II)

$\frac{\partial f}{\partial y} = x-\frac{a^3}{y^2}=0 \quad \Rightarrow \quad xy^2=a^3$ —— (III)

But from (II), $y=a^3/x^2$, put in equ^n (III)

$\Rightarrow x\frac{a^6}{x^4}=a^3 \quad \Rightarrow x^3=a^3 \quad \Rightarrow \boxed{x=a}$

$x=a$, put in (II) $\quad ay^2=a^3 \Rightarrow y^2=a^2 \Rightarrow \boxed{y=\pm a}$

$\Rightarrow (a,a)$ and $(a,-a)$ are the stationary pts. of $z=f(x,y)$

Now $\frac{\partial^2 f}{\partial x^2}=r=\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]=0+\frac{2a^3}{x^3}$ And $\frac{\partial^2 f}{\partial x\partial y}=s=\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]=1$

$\frac{\partial^2 f}{\partial y^2}=t=\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]=0+\frac{2a^3}{y^3}$

Points $f(x,y)$	$r=\frac{\partial^2 f}{\partial x^2}=\frac{2a^3}{x^3}$	$t=\frac{\partial^2 f}{\partial y^2}=\frac{2a^3}{y^3}$	$s=\frac{\partial^2 f}{\partial x\,\partial y}=1$	$(rt-s^2)$	Remarks for $f(x,y)$ is
(a,a)	$2>0$	$2>0$	1	$3>0$	Minimum
$(a,-a)$	$2>0$	$-2<0$	1	$-5<0$	Neither Max. nor Min.

Ans: $f_{min}(a,a) = xy+a^3\left(\frac{1}{x}+\frac{1}{y}\right) = a^2+a^3\left(\frac{1}{a}+\frac{1}{a}\right) = \boxed{3a^2}$

7. Find three positive numbers whose sum is 100 and whose product is maximum.

$\Rightarrow$ let x,y,z are three positive nos such that

(I) $x+y+z=100 \quad$ i.e. $z=[100-x-y]$

(II) $xyz = xy(100-x-y) = 100(xy)-x^2y-xy^2$

i.e. $f(x,y)=100(xy)-x^2y-xy^2$ —(I) [Note that we want product of three nos is maximum i.e. we want $f(x,y)$ is maximum]

For stationary values

$\frac{\partial f}{\partial x}=0 \quad$ i.e. $100(y)-2xy-y^2=0$ —— (II)

And $\frac{\partial f}{\partial y}=0 \quad$ i.e. $100(x)-x^2-2xy=0$ —— (III)

$\frac{\partial^2 f}{\partial x^2}=r=0-2y=-2y$

$\frac{\partial^2 f}{\partial y^2}=t=-2x \quad$ And $\frac{\partial^2 f}{\partial x\partial y}=s=\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]=100-2x-2y$

From eq^n (II) $\quad y[100-2x-y]=0 \quad \Rightarrow y=0$ or $100-2x-y=0$ —(IV)

From eq^n (III) $\quad x[100-x-2y]=0 \quad \Rightarrow x=0$ or $100-x-2y=0$ —(V)

From eq (IV) when $y=0$ and $x=0$, Then $(0,0)$ is the stationary pt

when $y=0$ and $100-x-2y=0$, Then $100-x-0=0 \Rightarrow x=100$

$\Rightarrow (100,0)$ is stationary pt

when $100-2x-y=0$ and $x=0$, Then $100-0-y=0 \Rightarrow y=100$

$(0,100)$ is stationary pt.

when $100-2x-y=0$ and $100-x-2y=0$, Then solving equ^s

$$
\begin{array}{l}
\quad 100-2x-y=0 \\
- \quad \underline{-200+2x+4y=0} \\
\quad -100+0+3y=0 \quad \Rightarrow \boxed{y=\frac{100}{3}}
\end{array}
$$

And $100-x-2y=0$ in put $y=100/3$

$100-2\left(\frac{100}{3}\right)=x \quad \Rightarrow \boxed{x=100/3}$

Thus $\left(\frac{100}{3},\frac{100}{3}\right)$ is also stationary pt.

Thus $(0,0),(100,0),(0,100)$ and $\left(\frac{100}{3},\frac{100}{3}\right)$ are the stationary pts.

Points $f(x,y)$	$r=\frac{\partial^2 f}{\partial x^2}=-2y$	$t=\frac{\partial^2 f}{\partial y^2}=-2x$	$s=\frac{\partial^2 f}{\partial x\,\partial y}$ $=100-2x-2y$	$(rt-s^2)$	Remarks for $f(x,y)$ is
$(0,0)$	0	0	100	$-10000<0$	Neither Maximum nor Minimum function
$(100,0)$	0	$-200<0$	-100	$-10000<0$	
$(0,100)$	$-200<0$	0	-100	$-10000<0$	
$\left(\frac{100}{3},\frac{100}{3}\right)$	$-\frac{200}{3}<0$	$-\frac{200}{3}<0$	$-\frac{100}{3}$	$\frac{10000}{3}>0$	Maximum

Ans:- Thus $f(x,y)=xyz$ has maximum value at $(x,y)=\left(\frac{100}{3},\frac{100}{3}\right)$

and $z = 100 - x - y = 100 - \frac{100}{3} - \frac{100}{3} = \frac{100}{3}$

Thus $x = \frac{100}{3}$, $y = \frac{100}{3}$ and $z = \frac{100}{3}$ are three positive nos whose sum is 100 and product is Max.

8. Divide 24 into three parts such that the product of the first, square of the second and cube of the third is the maximum. [MU -May-14] (6 marks)

→ let divide 24 into x, y, and $(24-x-y)$ three parts. Then

$$f(x,y) = (24-x-y)\,y^2 x^3 \quad \text{—— (I)}$$

To find x & y such that $x>0$ and $y>0$.

For stationary pts. let $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

But $f(x,y) = 24y^2x^3 - x^4y^2 - y^3x^3$ —— (II)

$$\frac{\partial f}{\partial x} = 72x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$ Dr. Uday Kashid , Ph.D.(Mathematics)

$\Rightarrow x^2y^2[72 - 4x - 3y] = 0$

$\Rightarrow x = 0$ or $y = 0$ or $72 - 4x - 3y = 0$ —— (III)

IIIy $\frac{\partial f}{\partial y} = 48x^3y - 2x^4y - 3x^3y^2 = 0$

$\Rightarrow x^3y[48 - 2x - 3y] = 0$

$\Rightarrow x = 0$, or $y = 0$, or $48 - 2x - 3y = 0$ —— (IV)

But Note that our $x>0$, $y>0$, Hence pt $(0,0)$ is not possible.

Hence solve (III) & (IV)

$$\begin{array}{r} 72 - 4x - 3y = 0 \\ -48 \mp 2x \mp 3y = 0 \\ \hline 24 - 2x + 0 = 0 \end{array} \Rightarrow \boxed{x = 12}$$

And $72 - 4(12) - 3y = 0 \Rightarrow 24 - 3y = 0 \Rightarrow \boxed{y = 8}$

And third part would be $(24-x-y) = 24-12-8 = 4$

Thus we can divide 24 into $(12, 8, 4)$ three parts.

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Lagrange's Method of Undetermined Multipliers

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Lagrange's method is useful to find the extreme values of a function of several variables. (These variables are not all independent and are connected by some relations).

Here we shall consider a function with one condition provided.

- **Note: Disadvantages of Lagrange's method**

The drawback of this method is that we cannot determine the nature of stationary values i.e. whether they are maximum or minimum.

* ***Working Rules :*** Dr. Uday Kashid, PhD (Mathematics)

To find stationary value of $f(x,y,z)$ under one condition $\phi(x,y,z)=0$

- **Step I :**

Let $u = f(x,y,z)$ $\quad\ldots(i)\quad$ **(be given function)**

Under the condition

$\emptyset(x,y,z) = 0$ $\quad\ldots(ii)$

Then construct new function called as Lagrange's function.

$L(x,y,z,\lambda) = f(x,y,z) + \lambda\,\phi(x,y,z)$ $\quad\ldots(iii)$

i.e. $L = u + \lambda\phi$

where λ is non-zero constant which is called as Lagrange's undetermined multiplier.

By differentiating equation (iii) w.r.t. x, y and z the stationary values occur when

$(i)\quad \frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \quad\ldots(iv)$

$(ii)\quad \frac{\partial L}{\partial y} = \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \quad\ldots(v)$ Dr. Uday Kashid, PhD (Mathematics)

$(iii)\quad \frac{\partial L}{\partial z} = \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \quad\ldots(vi)$

Step 2 :

By solving the four equations $[(ii),(iv),(v),(vi)]$ for four unknowns $(x,y,z$ and $\lambda)$ ⬧♏ ♑♏⧫ ⧫♒♏ ❑♏❑⧫♓❑♏♎ ⬧⧫♋⧫♓❑■♋❑⌂ ❑⧫⬧ ❑♐ $f(x,y,z)$ ⬧⧫♌♏♍⧫ ⧫❑ condition $\phi(x,y,z) = 0$

This method is called as Lagrange's method of undetermined multipliers.

1. **Using Lagrange's method, Find the maximum value of $f(x,y,z) = x^2y^3z^4$ subject to constraint $x+y+z=5$.**
2. **Using Lagrange's method, Find the minimum value of x^2+y^2, subject to condition $ax+by=c$.**
3. **Using Lagrange's method, Divide 24 into 3 parts such that the continued product of the first, square of second and cube of third is maximum.** [MU -April-21 online exam.]
4. **Using Lagrange's method, If $\frac{3}{x}+\frac{4}{y}+\frac{5}{z}=6$, find the values of x,y,z such that $x+y+z$ is minimum.** $f(x,y,z) = u = x+y+z$
5. **Using Lagrange's method, Find the minimum value of $x^2+y^2+z^2$ with the constraint $xy+yz+zx=3a^2$.** $\phi(x,y,z) = \frac{3}{x}+\frac{4}{y}+\frac{5}{z}-6=0$

1. **Using Lagrange's method, Find the maximum value of $f(x,y,z) = x^2y^3z^4$ subject to constraint $x+y+z=5$**

→ We have $u = f(x,y,z) = x^2y^3z^4$ —(I)

$\phi(x,y,z) = 0 \quad\Rightarrow\quad \phi(x,y,z) = x+y+z-5=0$ —(II)

$L(x,y,z,\lambda) = f(x,y,z) + \lambda\,\phi(x,y,z)$ → Lagrange's Fun

$L = x^2y^3z^4 + \lambda(x+y+z-5)$ —(III)

$\frac{\partial L}{\partial x} = 0 \Rightarrow 2xy^3z^4 + \lambda(1) = 0 \Rightarrow \lambda = -2xy^3z^4$ —(IV)

$\frac{\partial L}{\partial y} = 0 \Rightarrow 3x^2y^2z^4 + \lambda(1) = 0 \Rightarrow \lambda = -3x^2y^2z^4$ (V)

$\frac{\partial L}{\partial z} = 0 \Rightarrow 4x^2y^3z^3 + \lambda(1) = 0 \Rightarrow \lambda = -4x^2y^3z^3$ —(VI)

From eqn (IV) & (V) LHS = LHS $\Rightarrow -2x\cancel{y^3}\cancel{z^4} = -3x^2\cancel{y^2}\cancel{z^4}$

$\Rightarrow 2y = 3x \Rightarrow \boxed{y = \frac{3x}{2}}$ —(VII)

From eqn (IV) to (VI), LHS = LHS Dr. Uday Kashid, PhD (Mathematics)

$-2x\cancel{y^3}z^4 = -4x^2\cancel{y^3}z^3$

$2z = 4x \Rightarrow \boxed{z = 2x}$ —(VIII)

put eqn (VII) & (VIII) in eqn (II) $\quad x+y+z-5=0$

$x + \frac{3x}{2} + 2x = 5 \Rightarrow 3x + \frac{3x}{2} = 5$

$\Rightarrow 9x = 10$

$\Rightarrow \boxed{x = 10/9}$

$\Rightarrow y = \frac{3x}{2} = \frac{3}{2}\times\frac{10}{9} = \boxed{\frac{5}{3}}$

$\Rightarrow z = 2x = 2\times\frac{10}{9} = \boxed{\frac{20}{9}}$

so, stationary pt is $(x,y,z) = \left(\frac{10}{9}, \frac{5}{3}, \frac{20}{9}\right)$

$u = f(x,y,z) = x^2y^3z^4$ at stationary pt

$u = f\left(\frac{10}{9}, \frac{5}{3}, \frac{20}{9}\right) = \left(\frac{10}{9}\right)^2\left(\frac{5}{3}\right)^3\left(\frac{20}{9}\right)^4 = \frac{(5\times2)^2\, 5^3\, (2\times2\times5)^4}{(3\times3)^2\, 3^3\, (3\times3)^4} = \frac{2^{10}\times5^9}{3^{15}}$

max

$$\boxed{u = f_{max}(x,y,z) = \frac{2^{10}\times5^9}{3^{15}}}$$

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2. **Using Lagrange's method, Find the minimum value of x^2+y^2, subject to condition $ax+by=c$.**

→ $u = f(x,y) = x^2+y^2$ —(I)

$\phi(x,y) = 0 \Rightarrow ax+by-c=0 = \phi(x,y)$ —(II)

$L = u + \lambda\phi = (x^2+y^2) + \lambda(ax+by-c)$ —(III) Lagrange's Fun

$\frac{\partial L}{\partial x} = (2x) + \lambda(a) = 0 \Rightarrow 2x = -\lambda a \Rightarrow -\lambda = \frac{2x}{a}$ —(IV)

$\frac{\partial L}{\partial y} = (2y) + \lambda(b) = 0 \Rightarrow 2y = -\lambda b \Rightarrow -\lambda = \frac{2y}{b}$ —(V)

$\frac{\partial L}{\partial x} = (2x) + \lambda(a) = 0 \Rightarrow 2x = -\lambda a \Rightarrow -\lambda = \frac{2x}{a}$ —(IV)

$\frac{\partial L}{\partial y} = (2y) + \lambda(b) = 0 \Rightarrow 2y = -\lambda b \Rightarrow -\lambda = \frac{2y}{b}$ —(V)

From (IV) & (V) LHS = LHS $\Rightarrow \frac{2x}{a} = \frac{2y}{b}$

$\Rightarrow \boxed{y = \frac{bx}{a}}$ —(VI)

(x, y)

put eq^n (VI) in eq^n (II) $ax + by = c$

$ax + b\left[\frac{bx}{a}\right] = c \Rightarrow x\left[a + \frac{b^2}{a}\right] = c$

$x = \frac{c}{\left[\frac{a^2+b^2}{a}\right]} = \boxed{\frac{ac}{a^2+b^2}}$

Hence $y = \frac{bx}{a} = \frac{b}{a}\left[\frac{ac}{a^2+b^2}\right] = \frac{bc}{a^2+b^2}$

stationary pt is $(x,y) = \left(\frac{ac}{a^2+b^2}, \frac{bc}{a^2+b^2}\right)$

and $f_{min}(x,y) = x^2+y^2 = \left(\frac{ac}{a^2+b^2}\right)^2 + \left(\frac{bc}{a^2+b^2}\right)^2 = \frac{a^2c^2+b^2c^2}{(a^2+b^2)^2}$

$f_{min}(x,y) = \frac{c^2(a^2+b^2)}{(a^2+b^2)^2} = \boxed{\frac{c^2}{(a^2+b^2)}}$ → minimum value of $f(x,y)$

3. **Using Lagrange's method, Divide 24 into 3 parts such that the continued product of the first, square of second and cube of third is maximum.** [MU - April-21 online exam.]

→ Let divide 24 into x, y and z.

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$u = f(x,y,z) = xy^2z^3$ —(I)

$\phi(x,y,z) = x+y+z-24 = 0$ —(II) $x+y+z=24$

$L = f(x,y,z) + \lambda\phi(x,y,z) = xy^2z^3 + \lambda(x+y+z-24)$ —(III)

$L = u + \lambda_1\phi_1 + \lambda_2\phi_2 + \lambda_3\phi_3$

$\frac{\partial L}{\partial x} = 0 \Rightarrow y^2z^3 + \lambda(1) = 0 \Rightarrow -\lambda = y^2z^3$ —(IV)

$\frac{\partial L}{\partial y} = 0 \Rightarrow 2xyz^3 + \lambda(1) = 0 \Rightarrow -\lambda = 2xyz^3$ —(V)

$\frac{\partial L}{\partial z} = 0 \Rightarrow 3xy^2z^2 + \lambda(1) = 0 \Rightarrow -\lambda = 3xy^2z^2$ —(VI)

From eq^n (IV) & (V) LHS = LHS $\Rightarrow \cancel{y}^2 \cancel{z^3} = 2x\cancel{y}\cancel{z^3}$

$\boxed{y = 2x}$ —(VII)

From (IV) & (VI), LHS = LHS $\cancel{y^2}z^{\cancel{3}} = 3x\cancel{y^2}\cancel{z^2}$

$\boxed{z = 3x}$ —(VIII)

put $y = 2x$ & $z = 3x$ in eq^n (II)

$x+y+z-24 = 0 \Rightarrow x + 2x + 3x = 24$

$\boxed{6x = 24}$

$\Rightarrow \boxed{x = 4}$ → First part

$y = 2x = 8$ → second part

$z = 3x = 12$ → Third part

$(x, y, z) = (4, 8, 12)$ is Ans

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4. **Using Lagrange's method, If $\frac{3}{x} + \frac{4}{y} + \frac{5}{z} = 6$, find the values of x, y, z such that $x+y+z$ is minimum.**

→ $u = f(x,y,z) = x+y+z$ —(I)

$\phi(x,y,z) = \frac{3}{x} + \frac{4}{y} + \frac{5}{z} - 6 = 0$ —(II)

$L(x,y,z,\lambda) = f(x,y,z) + \lambda\phi(x,y,z)$

$L = (x+y+z) + \lambda\left(\frac{3}{x} + \frac{4}{y} + \frac{5}{z} - 6\right)$ —(III)

$\frac{\partial L}{\partial x} = 1 - \frac{3\lambda}{x^2} = 0 \Rightarrow 1 = \frac{3\lambda}{x^2} \Rightarrow x^2 = 3\lambda \Rightarrow x = \sqrt{3\lambda}$ —(IV)

$\frac{\partial L}{\partial y} = 1 - \frac{4\lambda}{y^2} = 0 \Rightarrow 1 = \frac{4\lambda}{y^2} \Rightarrow y^2 = 4\lambda \Rightarrow y = \sqrt{4\lambda}$ —(V)

$\frac{\partial L}{\partial z} = 1 - \frac{5\lambda}{z^2} = 0 \Rightarrow 1 = \frac{5\lambda}{z^2} \Rightarrow z^2 = 5\lambda \Rightarrow z = \sqrt{5\lambda}$ —(VI)

But $\frac{3}{x} + \frac{4}{y} + \frac{5}{z} = 6 \Rightarrow \frac{3}{\sqrt{3\lambda}} + \frac{4}{\sqrt{4\lambda}} + \frac{5}{\sqrt{5\lambda}} = 6$

$= \frac{1}{\sqrt{\lambda}}\left[\sqrt{3} + \sqrt{4} + \sqrt{5}\right] = 6$

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$\Rightarrow \sqrt{\lambda} = \left[\frac{\sqrt{3}+\sqrt{4}+\sqrt{5}}{6}\right]$ → (VII)

$\Rightarrow x = \sqrt{3}\sqrt{\lambda} = \frac{\sqrt{3}}{6}(\sqrt{3}+\sqrt{4}+\sqrt{5})$, $y = \sqrt{4}\sqrt{\lambda} = \frac{\sqrt{4}}{6}(\sqrt{3}+\sqrt{4}+\sqrt{5})$

and $z = \sqrt{5}\sqrt{\lambda} = \frac{\sqrt{5}}{6}(\sqrt{3}+\sqrt{4}+\sqrt{5})$

And $f_{min}(x,y,z) = x+y+z = \frac{(\sqrt{3}+\sqrt{4}+\sqrt{5})}{6}\left[\sqrt{3}+\sqrt{4}+\sqrt{5}\right]$

$= \frac{3+4+5+2\sqrt{3}\sqrt{4}+2\sqrt{3}\sqrt{5}+2\sqrt{4}\sqrt{5}}{6}$

And $f_{min}(x,y,z) = x+y+z = \frac{(\sqrt{3}+\sqrt{4}+\sqrt{5})}{6}(\sqrt{3}+\sqrt{4}+\sqrt{5})$

$$= \frac{3+4+5+2\sqrt{3}\sqrt{4}+2\sqrt{3}\sqrt{5}+2\sqrt{4}\sqrt{5}}{6}$$

$$f_{min}(x,y,z) = \frac{12+2(\sqrt{12}+\sqrt{15}+\sqrt{20})}{6} = \boxed{\frac{6+(\sqrt{12}+\sqrt{15}+\sqrt{20})}{3}}$$

5. Using Lagrange's method, Find the minimum value of $x^2 + y^2 + z^2$ with the constraint $xy + yz + zx = 3a^2$

→ We have $u = f(x,y,z) = x^2+y^2+z^2$ —(I)

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$\phi(x,y,z) = xy+yz+zx-3a^2 = 0$ —(II)

$L = f(x,y,z) + \lambda\phi(x,y,z)$

$= (x^2+y^2+z^2) + \lambda(xy+yz+zx-3a^2)$ —(III)

(I) $\frac{\partial L}{\partial x} = 2x + \lambda(y+z) = 0 \Rightarrow -\lambda = \frac{2x}{y+z}$ —(IV)

(II) $\frac{\partial L}{\partial y} = 2y + \lambda(x+z) = 0 \Rightarrow -\lambda = \frac{2y}{x+z}$ —(V)

(III) $\frac{\partial L}{\partial z} = 2z + \lambda(x+y) = 0 \Rightarrow -\lambda = \frac{2z}{x+y}$ —(VI)

From eq^n (IV), (V) and (VI)

Note This step ⇒ $\frac{2x}{y+z} = \frac{2y}{x+z} = \frac{2z}{x+y} = \frac{2x+2y+2z}{(y+z)+(x+z)+(x+y)} = \frac{2x+2y+2z}{2x+2y+2z} = 1$

$\frac{2x}{y+z} = 1 \Rightarrow 2x-y-z = 0$ —(A)

$\frac{2y}{x+z} = 1 \Rightarrow 2y-x-z = 0$ —(B)

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$\frac{2z}{x+y} = 1 \Rightarrow 2z-x-y = 0$ —(C)

Substract eq^n (A) − (B) —

$$\begin{aligned} 2x - y - z &= 0 \\ \mp x \pm 2y \mp z &= 0 \\ \hline 3x - 3y + 0 &= 0 \\ 3x - 3y &= 0 \Rightarrow \boxed{x=y} \end{aligned}$$

Substract eq^n (B) − (C)

$$\begin{aligned} -x + 2y - z &= 0 \\ - \quad \mp x \mp y \pm 2z &= 0 \\ \hline 0 + 3y - 3z &= 0 \end{aligned} \Rightarrow \boxed{y=z}$$

$\Rightarrow \boxed{x=y=z}$

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Hence, put $x=y=z$ in eq^n (II) $xy+yz+zx-3a^2=0$

$\Rightarrow x^2+x^2+x^2 = 3a^2$

$\Rightarrow 3x^2 = 3a^2 \Rightarrow x^2 = a^2$

$\Rightarrow \boxed{x = \pm a}$

$\Rightarrow \boxed{x = y = z = \pm a}$

$\Rightarrow (x,y,z) = (\pm a, \pm a, \pm a)$ is stationary pts and

$f_{min}(x,y,z) = x^2+y^2+z^2 = a^2+a^2+a^2 = \boxed{3a^2}$

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