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* Set theory :-

countable, uniqueness in quantity

1) Set is a collection of well-defined object.
2) Denoted by { }
3) Properties :-

i) Sets have unique element.

ii) Set is an unordered collection. eg. string in programming language is ordered, so string is not set.

4) Set representation :-

listing form (roster form)

$E = \{2, 4, 6, 8, \dots\}$

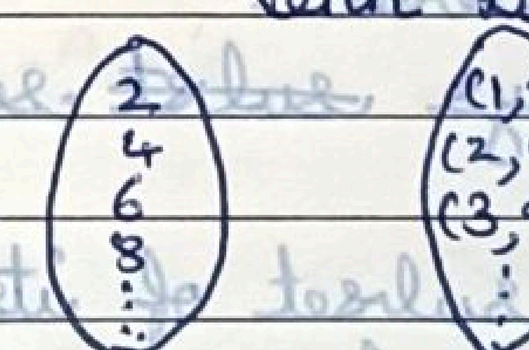
$A = \{(1,1), (2,4), (3,9), \dots\}$

set builder form

$E = \{x \mid x = 2n, n \in N\}$

$A = \{(x,y) \mid y = x^2, x \in N\}$

Venn Diagram



5) Terminology of set :-

i) Number / element of set :-

If a object 'a' belongs to set A then a is called member of set A.

It is written as $a \in A$.

eg. $A = \{1, 2, 3\}$ ∴ $2 \in A$ by $4 \notin A$.

ii) Cardinality of set :-

No. of elements in set. It is written as $|A|$.

eg. $A = \{1, 4, 6\}$ ∴ $|A| = 3$

6) Special sets :-

i) Empty set :-
A set which does not contain any element.
It is denoted by ϕ or { }.
$|\phi| = 0$.

ii) Universal set :-
A reference set which contain everything.
It is denoted by U.
$|U| = N$ ← (Total number of elements)

iii) Subset / Superset / Proper set :-
If every element set B is member of set A, then B is called subset of A, written as $B \subseteq A$ and A is called superset of B, written as $A \supseteq B$.
If $B \neq A$, then B is called proper subset of A, written as $B \subset A$.

a) Properties of subset :-
I) A set is always ~~subset~~ subset of universal set, i.e. $A \subseteq U$
II) Every set is subset of itself, i.e. $A \subseteq A$.
III) Empty set is always subset of any set, i.e. $\phi \subseteq A$.

iv) Power set :-
A set which is collection of all possible subset of A is called power set.
If set is A then its power set is written as $P(A)$ or 2^A

eg. $A = \{1, 2, 3\}$
$P(A) = 2^A = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

a) Properties of power set :-
I] If $|A| = n$, $|P(A)| = 2^n = 2^{|A|}$
II] Empty set is always member of power set.
III] A set is also member of its power set.

* Gate question :-
Q.1) Gate 2005
Which is true statement
I] $\phi \in A$ ×
II] $\phi \subseteq A$ ✓
Ans Only II

Q.2) Gate 2015

$A = \{5, \{6\}, \{7\}\}$
~~which~~ No of true statements
I] $\phi \in 2^A$ ✓
II] $\phi \subseteq 2^A$ ✓ // ϕ is always subset of every set
III] $\{5, \{6\}\} \subseteq 2^A$ ✓
IV] $\{5, \{6\}\} \in 2^A$ ×
Ans 3

Q.3) Gate 1993
Let $|A| = n$ then no. of elements in power set of $A \times A$ is
Ans 2^{n^2}

eg. $\{1, 2\} \times \{2, 3, 4\} = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4)\}$
$2 \times 3 = 6$
$\therefore A \times A = n \times n = n^2$
$\therefore$ ~~$|P(A)| = 2^{|P(A)|}$~~ $|P(A \times A)| = 2^{|A \times A|} = 2^{n^2}$

7) Set operation :-

i) Union :- (OR)

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

ii) Intersection :- (AND)

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

iii) Difference :-

~~$A - B = \{x \mid x \notin A$~~

$A - B = \{x \mid x \in A \text{ but } x \notin B\}$

iv) Symmetric difference :-

$A \Delta B = A \oplus B = \{x \mid x \in A \cup B \text{ but } x \notin A \cap B\}$

It is also modulus of sum of inputs by 2.

eg. $1 \oplus 1 = 2 \% 2 = 2$ (EX-OR) gate

eg. $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4, 5\}$

$\therefore A \Delta B = A \oplus B = \{1, 5\}$

v) Compliment :- (NOT)

$\bar{A} = \sim A = \{x \mid x \notin A\}$

$= U - A$

* Gate question :-

Q.1) Gate 2000

Which is true.

I] $P(P(S)) = P(S)$ ×

II] $P(S) \cap P(P(S)) = \phi$ × // operation of two set is always set. Here, ϕ is member.

III] $P(S) \cap S = P(S)$ ×

IV] $S \notin P(S)$ ×

Ans All wrong.

8) Set identifiers :-

i) Idempotent rule (self operation rule)

$A \cap A = A$

$A \cup A = A$

identity rule

$A \cup \phi = A$

$A \cap U = A$

domination rule

$A \cup U = U$

$A \cap \phi = \phi$

ii) Compliment rule :-

$\bar{\bar{A}} = A$

$A \cup \bar{A} = U$

$A \cap \bar{A} = \phi$

iii) Commutative rule :-

$A \cup B = B \cup A$

$A \cap B = B \cap A$

iv) Associative rule :-

$A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$

$A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$

v) Distribution rule :-

~~$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$~~

~~$A \cap (B \cup C) = (A \cap B) \cup (A \cup$~~

~~$A \cup (B \cup$~~

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

i) Partition of set

A ~~par portion~~ partition space π of a non-empty set A is a collection of non-empty subsets π_i such that

(i) $\forall \pi_i \in \pi,\ \pi_i \neq \phi$

(ii) $\forall \pi_i, \pi_j \in \pi,\ \pi_i \cap \pi_j = \phi$, where $i \neq j$

(iii) $\bigcup_{i=1}^{n} \pi_i = A$

eg. $A = \{1, 2, 3, 4\}$

$\pi' = \{\{1,2\}, \{3,4\}\}$ ✓

$\pi'' = \{\{1\}, \{2\}, \{3,4\}\}$ ✓

$B_1 = \{\{1,2\}, \{2,3,4\}\}$ ✗ // 2 is repeated

$B_2 = \{\{1,2\}, \{3\}\}$ ✗ // 4 is missing

$A = \{1, 2, 3, 4, 5\}$

$\pi = \{\{1,2\}, \{3\}, \{4\}, \{5\}\}$

partition space or partition ↑ ; Block π_1 ↑ ; π_2 ↑ ; π_3 ↑ ; π_4 ↑

* Gate question :-

Q1) No. of partition space

$A = \{1, 2\}$

Ans $\pi_1 = \{\{1\}, \{2\}\}$

$\pi_2 = \{\{1, 2\}\}$

$\therefore 2 = n$

$B = \{1, 2, 3\}$

$\pi_1 = \{\{1\}, \{2\}, \{3\}\}$

$\pi_2 = \{\{1\}, \{2,3\}\}$

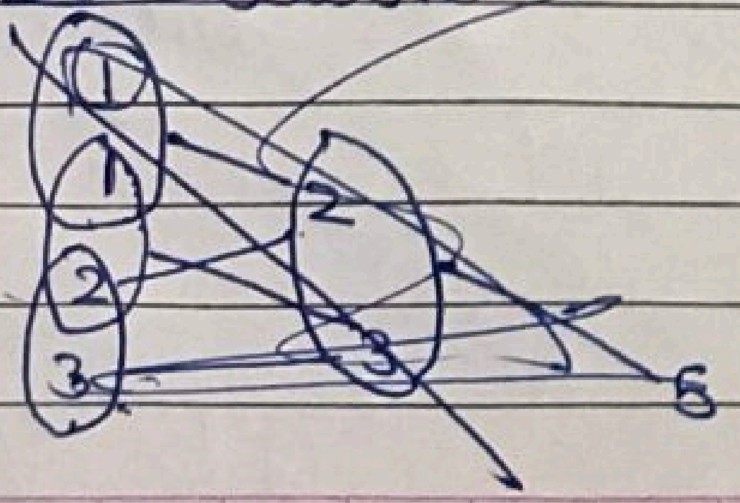
$\pi_3 = \{\{1,2\}, \{3\}\}$

$\pi_4 = \{\{2\}, \{1,3\}\}$

$\pi_5 = \{\{1,2,3\}\}$

$\therefore 5 = n$

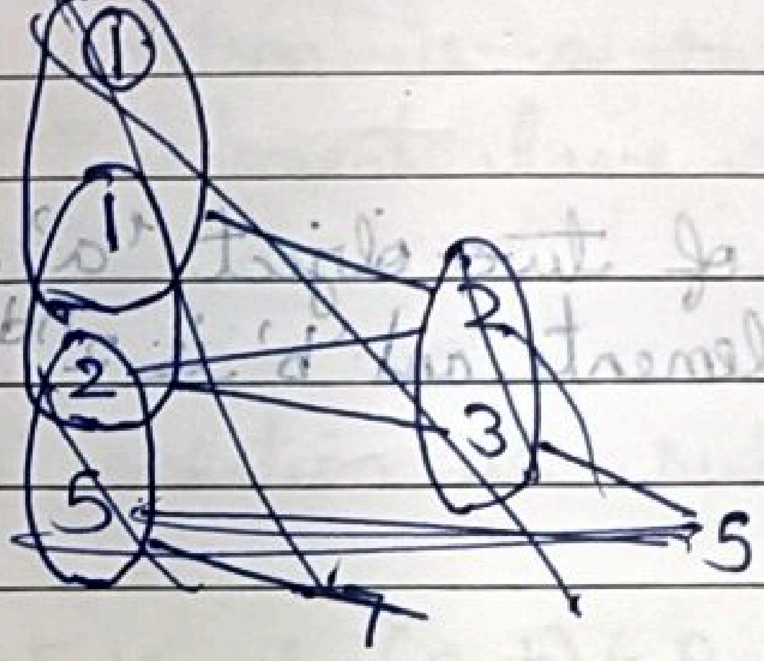
~~Bell curve~~

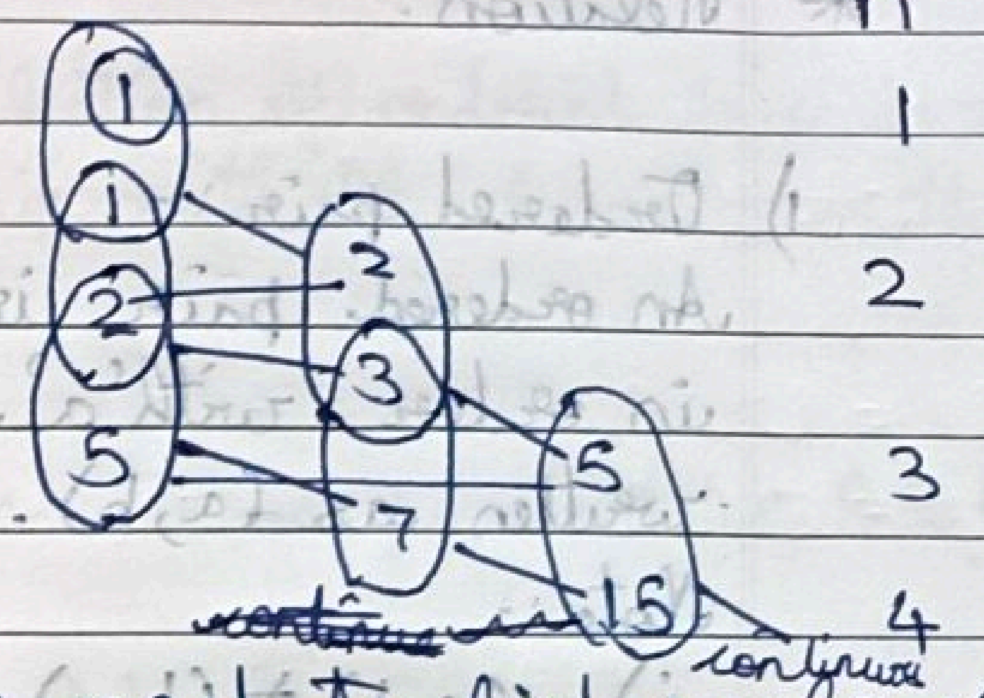


for 4, n = 15

for 5, n = 37

Bell curve :-





It is used to find number of partition space

* Gate question :-

Q.1) For $A = \{1, 2, 3, 4, 5\}$ which is / are correct partition on A.

$\pi_1 = \{\phi, \{1,2\}, \{3,4,5\}\}$ ✗ // ϕ is present

$\pi_2 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$ ✓

$\pi_3 = \{\{1\}, \{2,3\}, \{3,4,5\}\}$ ✗ // 3 is repeated

$\pi_4 = \{\{1\}, \{2,3\}, \{5\}\}$ ✗ // 4 is absent

* Relation:-

1) Ordered pair:-

An ordered pair is listing of two object 'a' and 'b' in order, with 'a' is 1st element and 'b' is 2nd element written as (a,b).

Note:-

i) $(a,b) \neq (b,a)$

ii) $(a,b) = (x,y)$ iff $a=x$ and $b=y$

2) Cartesian product:-

For sets A and B the cartesian product collection of ordered pair element (a,b) where $a \in A$, $b \in B$.

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

eg. $A = \{1,2,3\}$, $B = \{a,b\}$

~~$A \times B = \{(1,a)\}$~~

$$A \times B = \{1,2,3\} \times \{a,b\}$$
$$= \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$$

If $|A| = n$, $|B| = m$, then

$$|A \times B| = n \times m$$

eg. $A = \{1,2\}$

$$A \times A = \{1,2\} \times \{1,2\} = \{(1,1),(1,2),(2,1),(2,2)\}$$

$A = \{1,2,3,4\}$

$$A \times A = \{(1,1),(1,2),(1,3),(1,4),$$
$$(2,1),(2,2),(2,3),(2,4),$$
$$(3,1),(3,2),(3,3),(3,4),$$
$$(4,1),(4,2),(4,3),(4,4)\}$$

Relation
↓
$R = \{(a,b) \mid b = 2a,\ a,b \in A\}$ // All pairs must be from cartesian product

$R = \{(1,2),(2,4)\}$

✱ $|R|_{max} = m \times n$ for $|A| = m$, $|B| = n$; $|A \times B| = m \times n$

3) Relation:-

A relation is a rule or collection of ordered pair in which 1st element have relationship with 2nd element, written as aRb.

definition:-

A relation is subset of cartesian product i.e. $R \subseteq A \times B$

Note:- if $(a,b) \in R$, then aRb

if $(a,b) \notin R$, then $a \not R b$

5) Representation of relation:-

Let $A = \{1,2,3,4\}$, $B = \{1,2,3,4\}$, ~~where A has binary relation i.e. $A \times A$~~ in which $B = A$

1) Set builder form:-

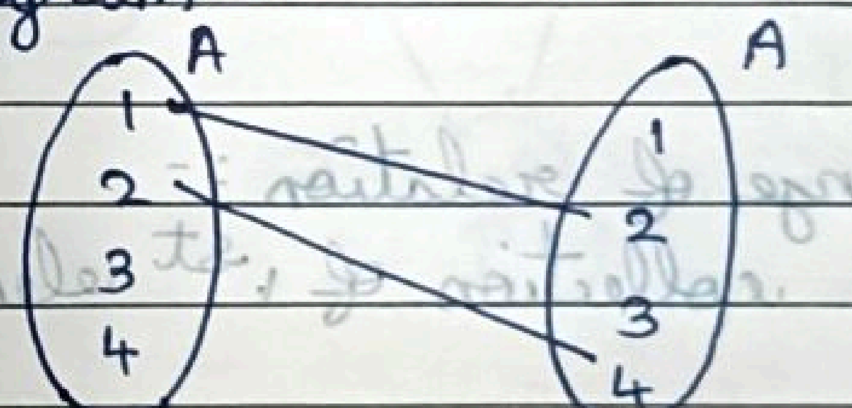
$$R = \{(a,b) \mid b = 2a\}$$

Here, A and B have binary relation that is $A = B$

2) Listing form:-

$$R = \{(1,2),(2,4)\}$$

3) Set diagram:-



4) Adjancey matrix:- // Used in Data structure (tree, graph)

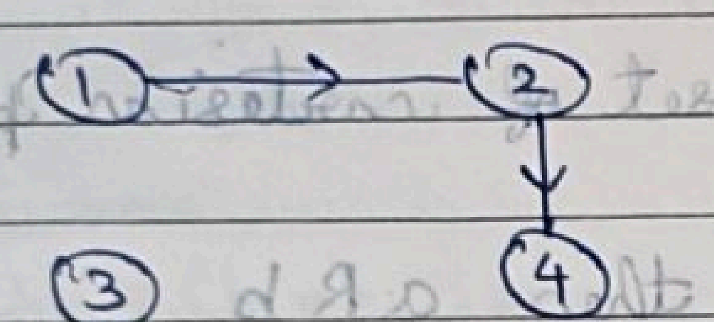
$$M_{i,j} = \begin{array}{c|cccc} A \backslash B & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

$$M_{i,j} = \begin{cases} 1 & a_i R b_j \\ 0 & a_i \not R b_j \end{cases}$$

5) Diagraph :- // Used in AI, networking

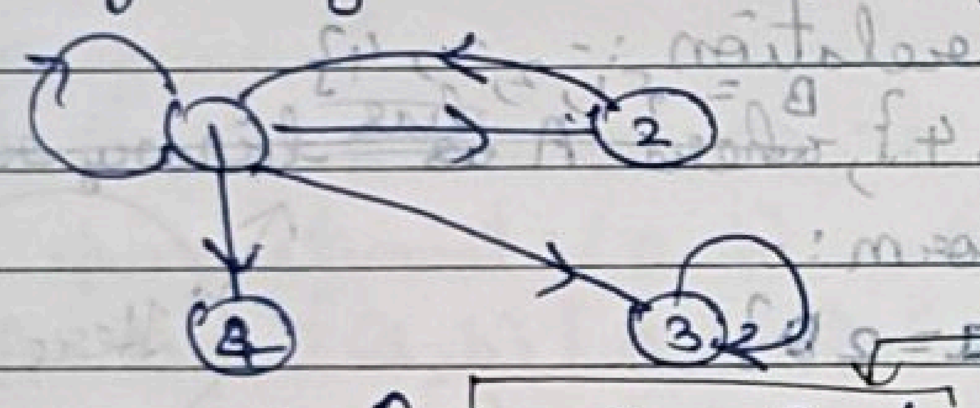
(Applicable only for binary relation i.e. $A = B$)

No. of vertices (#vertices) = $|A|$ or $|B|$ $\because A = B$



Q.1) Write Adjacency matrix for given relation :-

Ans



Range

$$\therefore M_{i,j} = \begin{array}{c|c} A \backslash B & 1 \quad 2 \quad 3 \quad 4 \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Domain → 3

Domain(R) = $\{1, 2, 3\}$

Range(R) = $\{1, 2, 3, 4\}$

6) Domain and range of relation :-

Domain of R is collection of 1st element of each element in R.

Range of R is collection of 2nd element of each element in R

Domain(R) = $\{a \mid (a,b) \in R\}$

Range(R) = $\{b \mid (a,b) \in R\}$

eg

$\therefore R = \{(1,1), (1,2), (3,4)\}$

Domain(R) = $\{1, 3\}$

Range(R) = $\{1, 2, 4\}$

7) Indegree and outdegree of vertices :-

Indegree (a) = number of $b \mid bRa$

Outdegree (a) = number of $a \mid aRb$

In AI

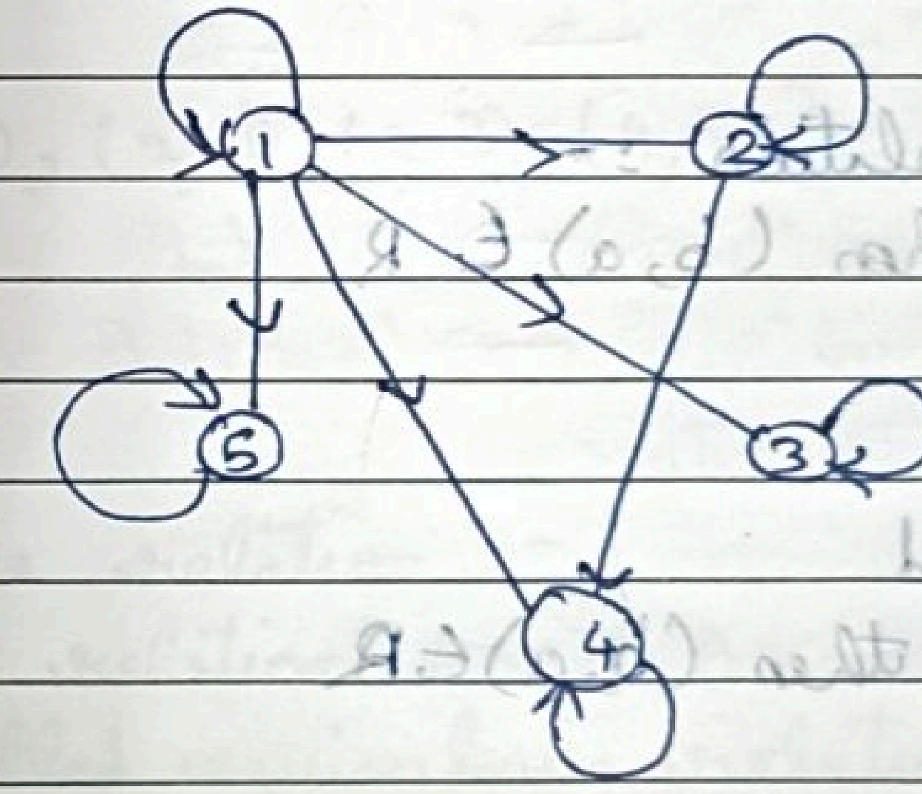
$a^+ = \text{In}$

$a^- = \text{out}$

Q.1) Find adjacency matrix R on A such that aRb iff b is multiple of a. Also find in and out degree of each vertex, where $A = \{1, 2, 3, 4, 5\}$

Ans

$$M_{i,j} = \begin{array}{c|c} A \backslash B & 1 \quad 2 \quad 3 \quad 4 \quad 5 \\ \hline \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



$1^+ = 1$, $1^- = 5$

$2^+ = 2$, $2^- = 2$

$3^+ = 2$, $3^- = 1$

$4^+ = 3$, $4^- = 1$

$5^+ = 2$, $5^- = 1$

8) Types of relation

i) Reflexive relation :-

if $\forall a \in A$, aRa, then the R is called reflexive relation.

ii) Irreflexive relation :-

If $\forall a \in A$, $a \not R a$, then R is called irreflexive relation.

iii) Equality relation / Diagonal relation :-

$\forall a \in A$, $(a,a) \in R$ but $(a,b) \notin R$ where $a \neq b$.

if relation is reflexive, transitive and symmetric then equal

eg. $A = \{1, 2, 3\}$

Reflexive relation $R = \{(1,1),(1,2),(2,2),(3,3)\}$

Irreflexive relation $R = \{(1,2),(3,3)\}$

Equality relation $R = \{(1,1),(2,2),(3,3)\}$

Q.1) $A = \{1, 2\}$

Reflexive relation $R = \{(1,1),(2,2),(1,2)\}$

Irreflexive relation $R = \{(1,2)\}$

Equality relation $R = \{(1,1),(2,2)\}$

iv) Symmetric relation :-

if $(a,b) \in R$ then $(b,a) \in R$

i.e.

$(a,a) \in A$

and

if $(a,b) \in R$ then $(b,a) \in R$

v) Antisymmetric relation :-

if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$

i.e.

$(a,a) \in R$

and

if $(a,b) \in R$ then $(b,a) \notin R$

vi) Asymmetric relation :-

if $(a,b) \in R$ then $(b,a) \notin R$

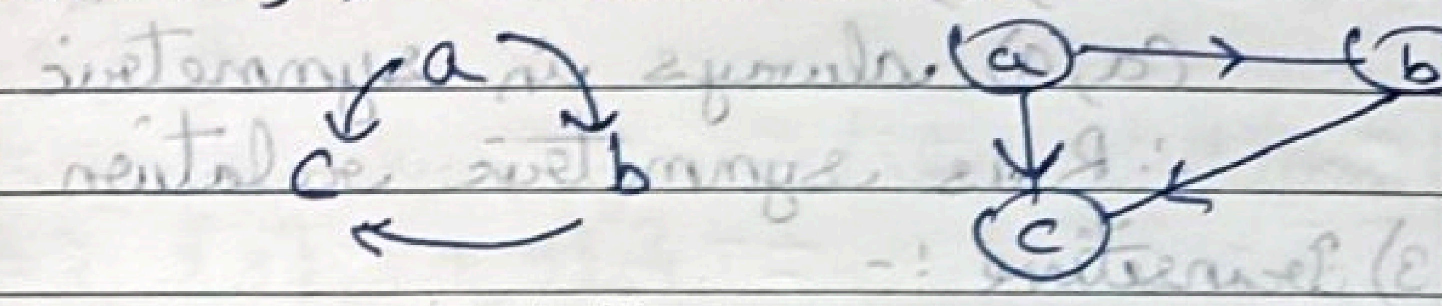
i.e.

$(a,a) \notin R$ and if $(a,b) \in R$ then $(b,a) \notin R$

Q.1) $R = \{(L_1, L_2) \mid L_1 \parallel L_2\}$

vii) Transitive relation :-

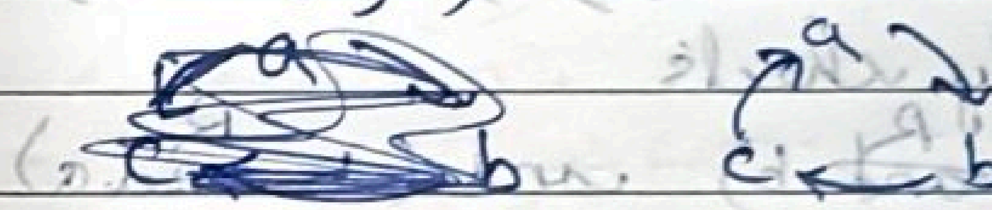
If $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$



// if if part is true and then part is false then not transitive

viii) Circular relation :-

If $(a,b) \in R$ and $(b,c) \in R$ then $(c,a) \in R$



Q.1) $A = \{1, 2, 3, 4\}$

$R = \{(1,1),(2,2),(3,3),(4,4)\} \Rightarrow$ equality and reflexive, transitive, circular

ix) Equivalence relation :-

Let R be relation on set A.

R is called equivalence relation iff

i) R is reflexive

ii) R is symmetric

iii) R is transitive

Q.1) Prove that :- $R = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4),(5,5)\}$ is equivalence relation on set $A = \{1,2,3,4,5\}$

Find equivalence class of each element. Also find A/R.

Ans 1) Reflexive? if $a \in A$, $(a,a) \in R$

eg. $(1,1),(2,2),(3,3),(4,4),(5,5)$ are in R

$\therefore$ R is reflexive

2) Symmetric :-

If $(a,b) \in R$ then $(b,a) \in R$

$(2,1), (1,2)$ in R

$(3,4), (4,3)$ in R

(a,a) always in symmetric relation

$\therefore$ R is symmetric relation

3) Transitive :-

If ~~a~~ aRb and bRc then aRc

① If (a,a) and (a,b) then (a,b)

If (a,b) and (b,b) then ~~(a,b)~~ (a,b)

//only checks

if $(a \overset{R}{} b)$ and $(b \overset{R}{} a)$

aRa, bRb

transitive

$(1,2)$ and $(2,1) \rightarrow (1,1) \in R$

$(2,1)$ and $(1,2) \rightarrow (2,2) \in R$

$(3,4)$ and $(4,3) \rightarrow (3,3) \in R$

$(4,3)$ and $(3,4) \rightarrow (4,4) \in R$

$\therefore$ R is transitive

$\therefore$ R is equivalence relation.

Equivalence class :-

If R is equivalence relation on set A, the every element of set of A have equivalence class.

$[a] = \{b \mid a R b, \text{ R is equivalence relation}\}$

$[1] = \{1, 2\}$

$[2] = \{1, 2\}$

$[3] = \{3, 4\}$

$[4] = \{4, 3\}$

$[5] = \{5\}$

A/R = collection of distinct equivalence classes.

$A/R = \{\{1,2\}, \{3,4\}, \{5\}\}$

Q.2) Let $A = \{1,2,3,4,5\}$ and $P_1 = \{1,2\}$, $P_2 = \{3\}$, $P_3 = \{4,5\}$ be partition set A.

Find equivalence relation formed by partition.

Ans $A/R = \{\{1,2\}, \{3\}, \{4,5\}\}$

$R = \{(1,1), (2,2), (1,2), (2,1), (3,3), (4,4), (5,5), (4,5), (5,4)\}$

Q.3) $R = \{(a,b) \mid a-b \text{ is odd positive integer}\}$ Prove that R is equivalence relation.

Ans Reflexive :- $\forall a \in A, (a,a) \in R$

$aRa \rightarrow a-a=0$

R is not equivalence

Q.4) R is relation on integers $R = \{(x,y) \mid \{x-y\} \text{ is divisible by } 3\}$

Prove that R is equivalence relation.

Ans 1) Reflexive :- $\forall a \in A, (a,a) \in R$

$aRa \Rightarrow a-a=0$

0 is divisible by 3

R is reflexive relation

2) Symmetric :- If aRb then bRa

Assume, $aRb \Rightarrow (a-b)\%3=0$

check :-

$bRa \Rightarrow (b-a) = -(a-b) = -$ divisible by 3

$\therefore$ R is symmetric

3) Transitive :-

If aRb and bRc the aRc

Assume; $aRb \Rightarrow (a-b)\%3=0 \Rightarrow a-b=3k_1$ — ①

$bRc \Rightarrow (b-c)\%3=0 \Rightarrow b-c=3k_2$ — ②

eq ① + eq ②

$a-\cancel{b}+\cancel{b}-c = 3k_1+3k_2$

$a-c = 3(k_1+k_2)$ is divisible by 3.

$\therefore (a-c)$ is transitive.

$\therefore aRc$

$\therefore R$ is equivalence relation.

Equivalence classes.

$[0] = \{0, \pm 3, \pm 6, \pm 9, \dots\} = \{3k_1 \mid k_1 = 1, 2, 3, \dots\}$

~~$[1] = \{1, \pm 4, \pm 7, \pm 10, \dots\} = \{3k+1 \mid k \in W\}$~~

~~$[2] = \{2, \pm 5, \pm 8, \pm 11, \dots\} = \{3k+2 \mid k \in W\}$~~

~~$[3] = [0]$~~

~~$[4] = [1]$~~

$[1] = \{1, 4, 7, 10, \dots\} = 3k_2 + 1$

$[2] = \{2, 5, 8, 11, \dots\} = 3k_3 + 2$

$[3] = [0] \qquad = 3k_1$

$[4] = [1]$

$Z/R = \{\{3k_1\}, \{3k_2+1\}, \{3k_3+2\}\}$

Q-5) $x, y \in Z$, xRy iff $2x+5y$ is divisible by 7.
If R equivalence relation?
Find equivalence ~~relation~~ classes.

If we want adjancey matrix of $\overline{R}$ just replace 0 by 1 and 1 by 0 that of R.
If we want adjancey matrix of R^{-1} just transpose adjancey matrix of A
If we want adjancey matrix of $R_1 \cup R_2$ is $M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$
- - - - - - - - - - - - - - - - - $R_1 \cap R_2$ is $M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2}$

* Operation on Relation :-

1) Compliment :-
$$\overline{R} = R^c = A \times A - R$$

2) Inverse :-
$$R^{-1} = \{ (b,a) \mid (a,b) \in R \}$$

3) Union :-
$$R_1 \cup R_2 = \{ (a,b) \mid (a,b) \in R_1 \text{ or } (a,b) \in R_2 \}$$

4) Intersection :-
$$R_1 \cap R_2 = \{ (a,b) \mid (a,b) \in R_1 \text{ and } (a,b) \in R_2 \}$$

5) Difference :-
$$R_1 - R_2 = \{ (a,b) \mid (a,b) \in R_1 \text{ but } (a,b) \notin R_2 \}$$

6) Symmetric difference :-
$$R_1 \oplus R_2 = \{ (a,b) \mid (a,b) \in R_1 \cup R_2 \text{ but } (a,b) \notin R_1 \cap R_2 \}$$

* Closure of relation :-

Reflexive closure :-
$$R^{\infty}_{ref} = R \cup \Delta$$

Symmetric closure :-
$$R^{\infty}_{sym} = R \cup R^{-1}$$

* Warshall's Algorithm (Transitive closure)

Find adjancey matrix M_{ij}

1) Warshall Matrix $W_0 = M_{ij}$
2) For $i = 1$ to n repeat step ① to ②
3) Consider i^{th} row and j^{th} column
   $\forall p,q$ for in $W_{qi} = 1$ and in rows W_{pi} $\cancel{W_{ip}} = 1$ column
   assign $W_{q,p} = 1$
4) Resulted matrix is transitive closure.

Q.1) Find transitive closure using warshall's algorithm.

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Ans

$$W_0 = M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R = \{(1,1), (1,4), (2,2), (2,3), (3,1), (4,3), (4,4)\}$

$R^{\infty} = \{(1,1), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,3), (3,4), (4,1), (4,3), (4,4), (1,3)\}$

$$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

eg. $R = \{(1,2),(2,3),(3,4),(2,1)\}$ on $A = \{1,2,3,4\}$

Find R^∞ using Warshall's algorithm

Ans

$$W_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Vertical column → Horizontal line

Horizonal column → vertical line

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$W_4 = W_3$

eg. $R = \{(4,3),(2,2),(2,1),(3,1),(1,2)\}$ ~~find~~ on $A = \{1,2,3,4\}$

Find R^∞ using Warshall's algorithm.

Ans

$$W_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$W_4 = W_3$

$R^\infty = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

Partial Order Relation :-

A ~~linear~~ binary relation 'R' on set 'A' is called partial order relation if

1) It is Reflexive.
2) It is Anti-Symmetric.
3) It is transitive.

Poset :-

If R is partial order relation on set A on relationship (i.e. operator). Then, the pair $(A, \subseteq)$ is together called poset.

eg. $(N, \subseteq)$ Is poset? $N = \{1, 2, 3, 4, \ldots\}$

Ans $\forall a \in N$ aRa and $a \subseteq a$ (True)

R is reflexive

$(a, b) \in R$ If aRb and bRa then $a = b$

$a \subseteq b$ and $b \subseteq a \Rightarrow a = b$ (True)

R is antisymmetric

If aRb and bRc then aRc

$a \subseteq b$ and $b \subseteq c \Rightarrow a \subseteq c$

∴ $\subseteq$ will always form ~~poset or~~ partial order relation

∴ $(A, \subseteq)$ is Poset.

eg. $(P(A), \subseteq)$ is Poset?

$A = \{1, 2, 3\}$

Ans $P(A) = \{\Phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

1) Reflexive :-

aRa and $a \subseteq a$ ∵ Set is subset of itself

It is true

$\forall a \subseteq$ ∴ It is reflexive

2) Antisymmetric :- If aRb and bRa then $a = b$

$a \subseteq b$ and $b \subseteq a$,

$a = b$

3) Transitive :-

If aRb, bRc then aRc

$a \subseteq b$ and $b \subseteq c$

$a \subseteq c$

This is true

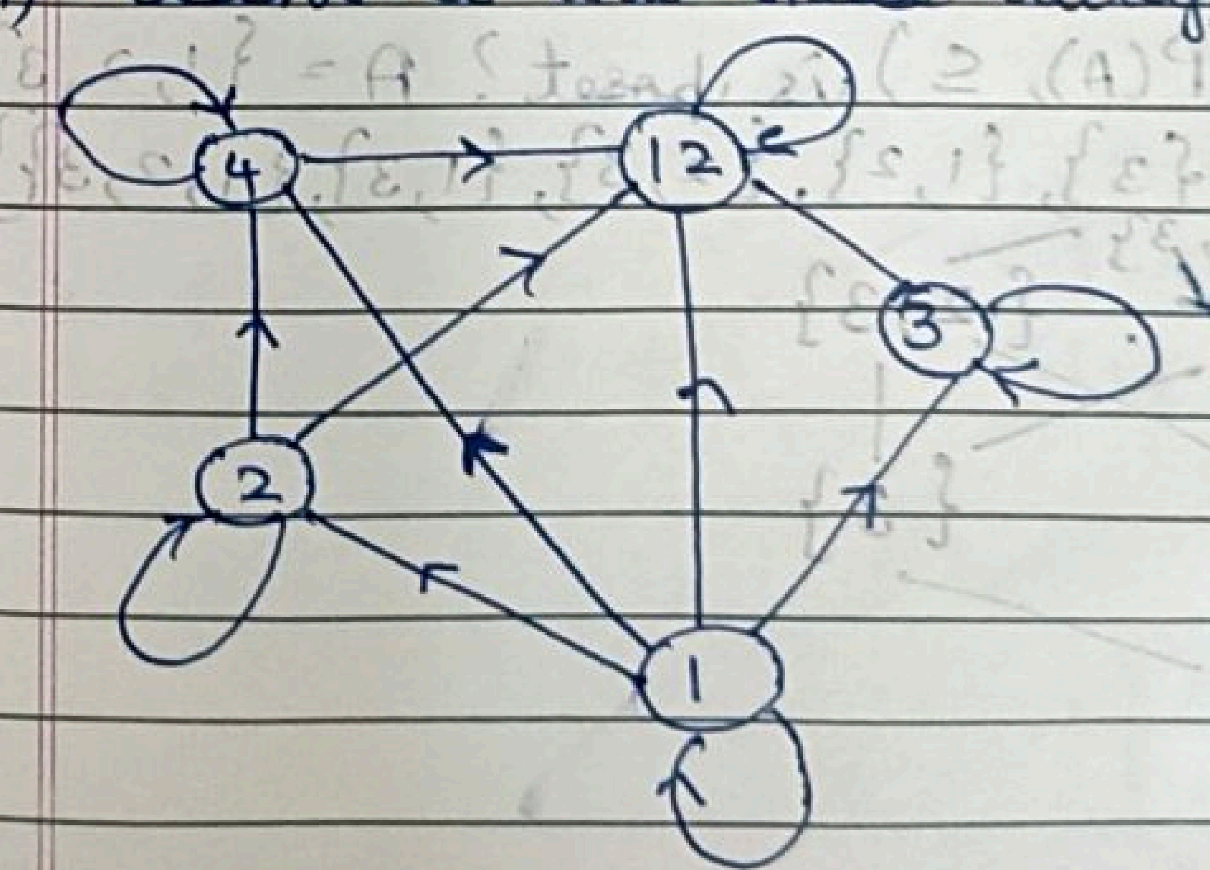
∴ $(P(A), \subseteq)$ is Poset.

* Representation of poset :-

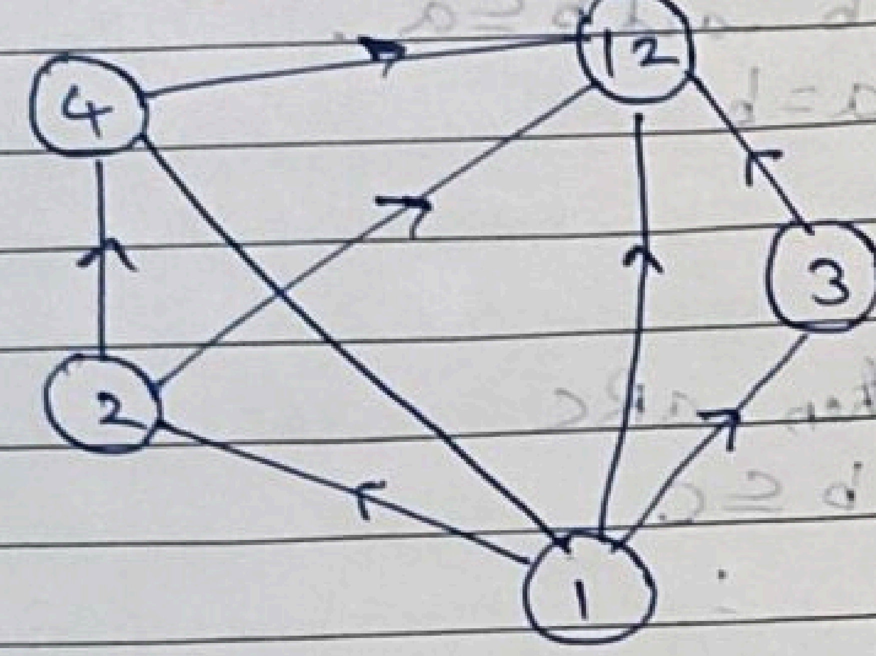
• Hasse diagram :-

1) Draw digraph of partial order relation
2) Remove self loop.
3) Remove transitive edges.
4) Remove arrow as
   if aRb $a \to b$. $\begin{matrix} b \\ | \\ a \end{matrix}$
   if cRb $c \to b$ $\begin{matrix} c \\ | \\ b \end{matrix}$
5) Resulted digraph is called Hasse diagram

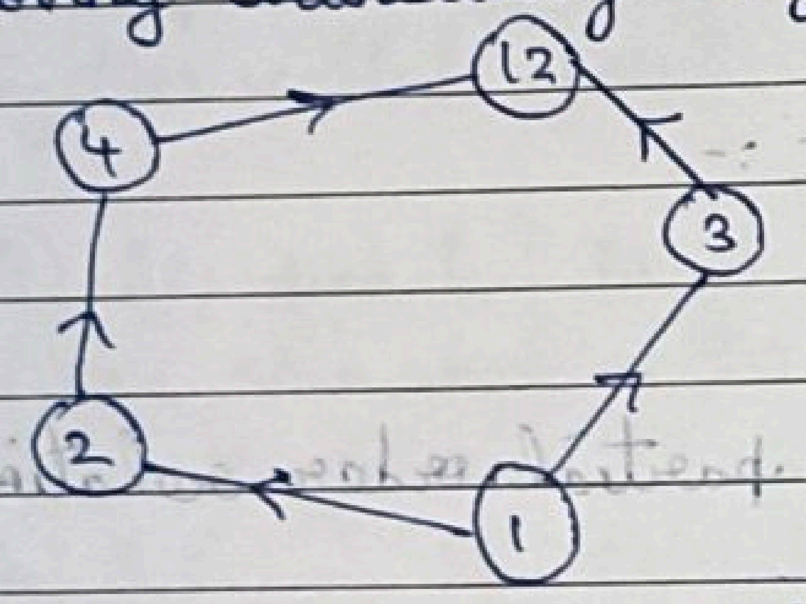
Q.1) Convert into Hasse diagram



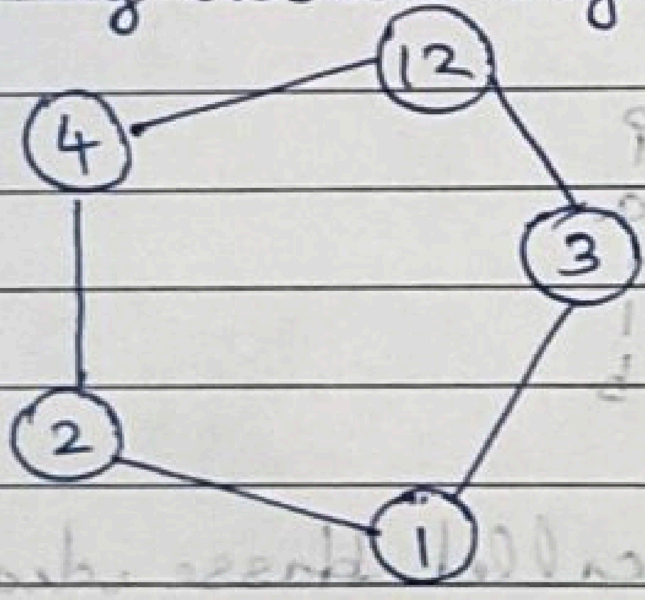
Ans Removing self loop



Removing transitivity edges



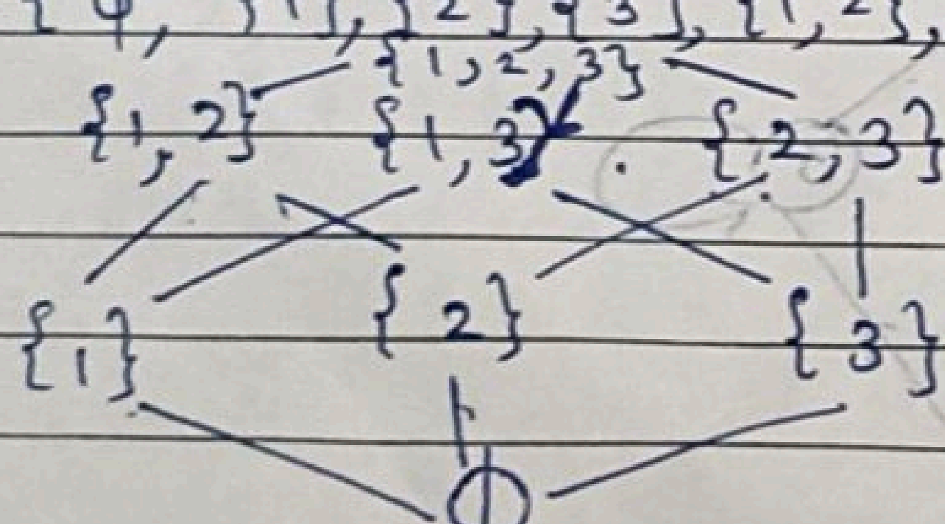
Removing arrow symbols



Hasse diagram of $(P(A), \subseteq)$ is poset? $A = \{1,2,3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

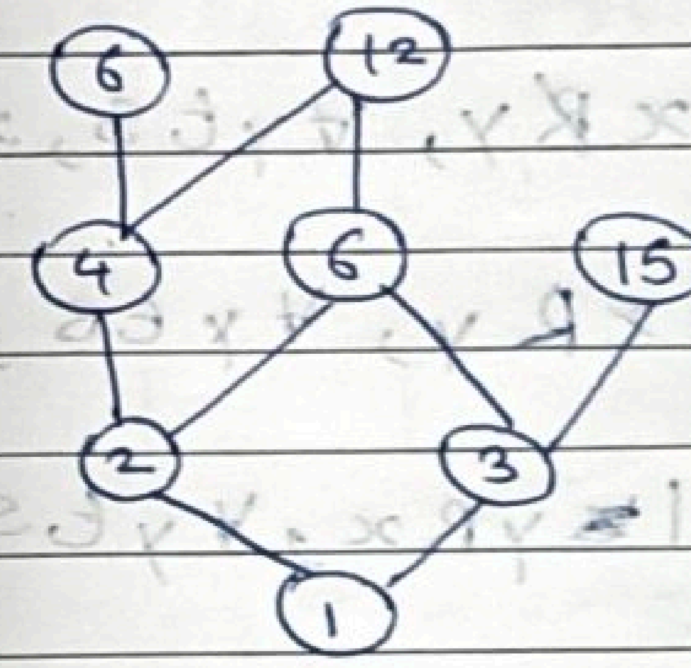
Ans



Q.2) Draw Hasse diagram:
$(\{1,2,3,4,6,8,12,15\}, |)$

$R = \{(a,b) \mid a/b\}$

Ans a divides b

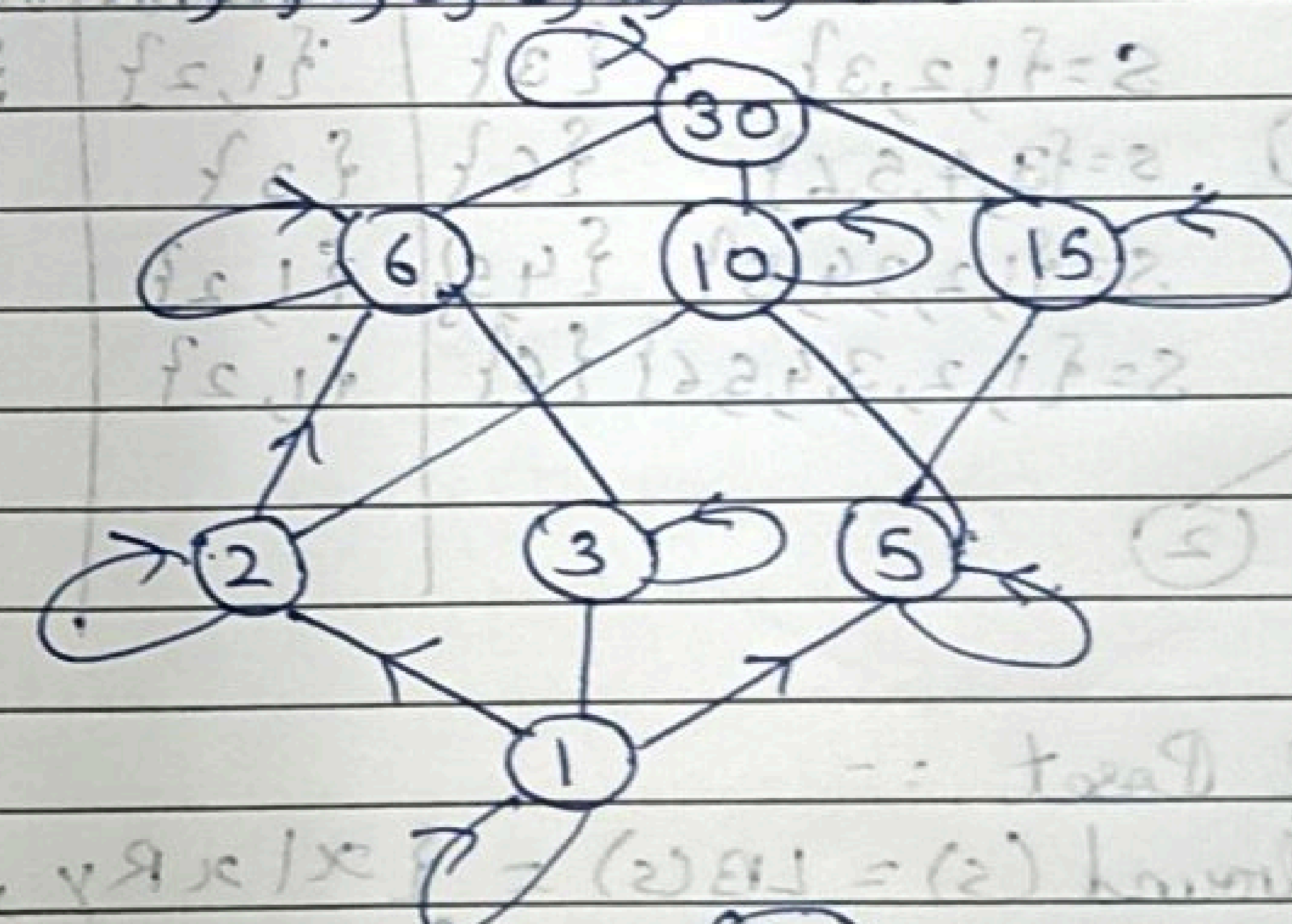


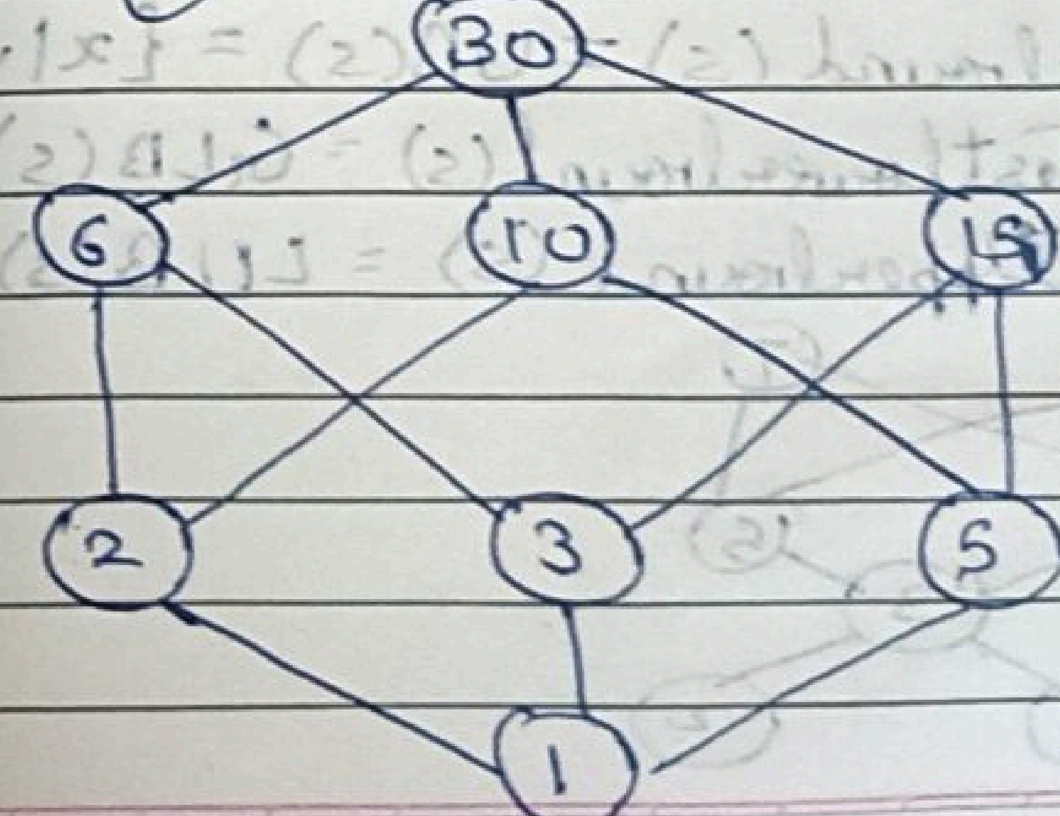
* Gate question:-

Q.1) Draw ~~the~~ Hasse diagram of D_{30}. ∵ D is divisor

Ans

1. $D_{30} = \{1,2,3,5,6,10,15,30\}$





*** Extremal element (A, R).**

if $S \subseteq P$, Then

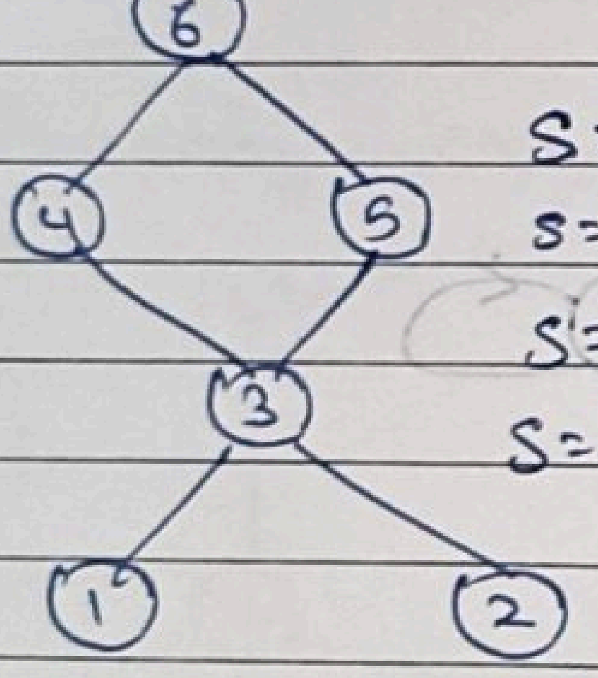
Minimal $(S) = \{x \mid y \not R x,$ ~~$\forall y$ is~~ $\forall y \in S, x \in S, x \neq y\}$

Maximal $(S) = \{x \mid x \not R y, \forall y \in S, x \in S, x \neq y\}$

Minimum $(S) = \{x \mid x R y, \forall y \in S, x \in S\}$

Maximum $(S) = \{x \mid y R x, \forall y \in S, x \in S\}$

Note:-
Maximum and minimum are unique and may or may not exists.



	Maximal	Minimal	Maximum	Minimum
$S=\{1,2,3\}$	$\{3\}$	$\{1,2\}$	$\{3\}$	ϕ
$S=\{3,4,5,6\}$	$\{6\}$	$\{3\}$	$\{6\}$	$\{3\}$
$S=\{1,2,3,4,5\}$	$\{4,5\}$	$\{1,2\}$	ϕ	ϕ
$S=\{1,2,3,4,5,6\}$	$\{6\}$	$\{1,2\}$	$\{6\}$	ϕ

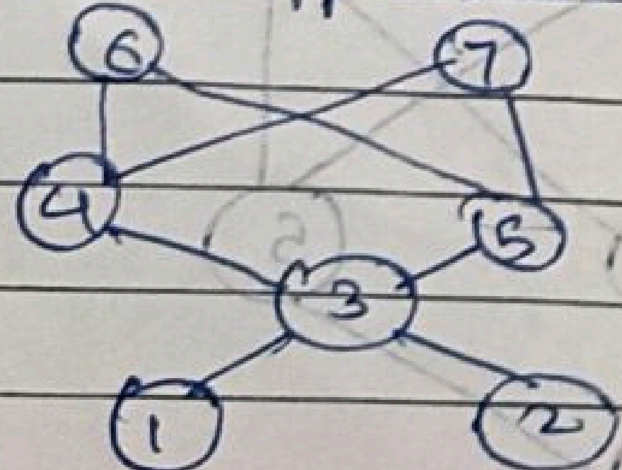
Q.1) Extremal of Poset :-

~~Let~~ Lowerbound $(S) = LB(S) = \{x \mid x R y, \forall y \in S, x \in P\}$

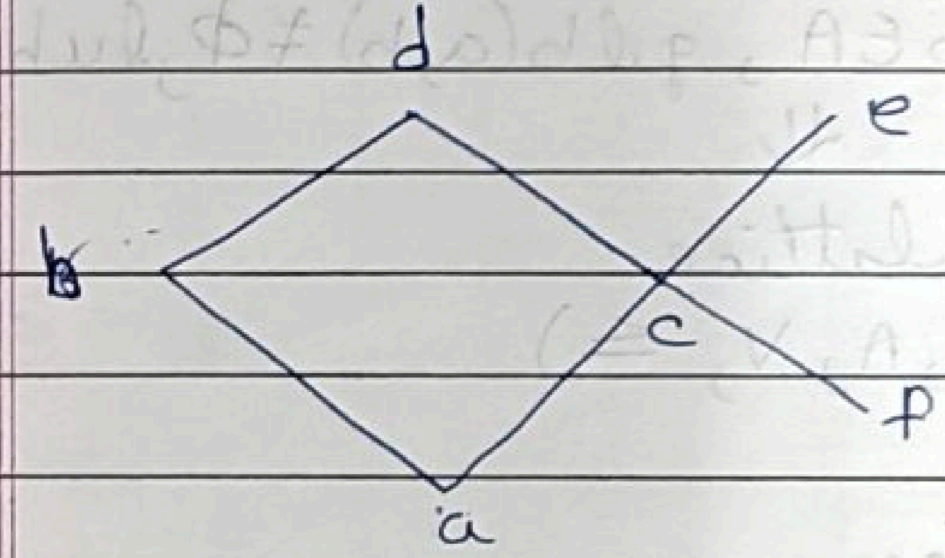
Upperbound $(S) = UB(S) = \{x \mid y R x, \forall y \in S, x \in P\}$

Greatestlowerbound $(S) = GLB(S) = \{x \mid y R x, \forall y \in LB(S), x \in P\}$

Lowestupperbound $(S) = LUB(S) = \{x \mid x R y, \forall y \in UB(S), x \in P\}$



S	LB	UB	GLB	LUB
$\{1,2,3\}$	ϕ	$\{3,4,5,6,7\}$	ϕ	3
$\{3,4,5\}$	$\{3,1,2\}$	$\{6,7\}$	$\{3\}$	ϕ
$\{1,3,7\}$	$\{1\}$	$\{7\}$	$\{1\}$	$\{7\}$
$\{4,5\}$	$\{3,1,2\}$	$\{6,7\}$	$\{3\}$	ϕ



S	LB	UB	GLB	LUB
$\{b,c\}$	$\{a\}$	$\{d\}$	$\{a\}$	$\{d\}$
$\{a,b,c,d\}$	$\{a\}$	$\{d\}$	$\{a\}$	$\{d\}$
$\{e,f\}$	$\{f\}$	$\{e\}$	$\{f\}$	$\{e\}$

Lattice

Poset $(A, \leq)$

$\forall a, b \in A$, glb$(a,b) \neq \phi$

OR

$\forall a, b \in A$, $a \wedge b \neq \phi$

$\Downarrow$

Meet Semi Lattice

$(L, \wedge, \leq)$

$\forall a, b \in A$, lub$(a,b) \neq \phi$

OR

$\forall a, b \in A$, $a \vee b \neq \phi$

$\Downarrow$

Join Semi Lattice

$(L, \vee, \leq)$

if $\forall a, b \in A$, glb$(a,b) \neq \phi$, lub$(a,b) \neq \phi$

$\Downarrow$

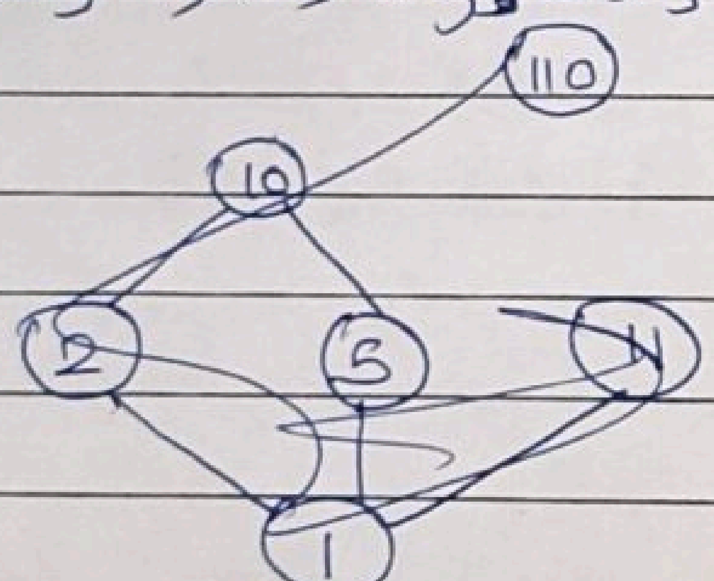
lattice

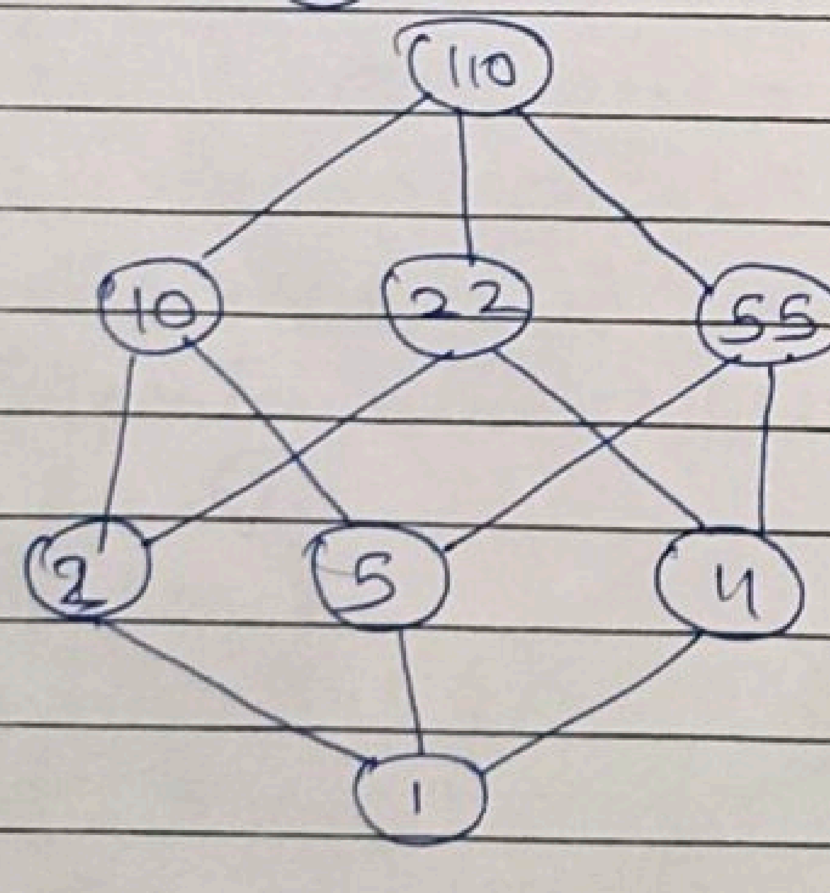
$(L, \wedge, \vee, \leq)$

Q.1) Prove that D_{110} is lattice

Ans

~~$D_{110} = \{1, 2, 5, 11, 10, 110, 55\}$~~ $D_{110} = \{1, 2, 5, 10, 11, 22, 55, 110\}$





// complement is either top or bottom respectively

Note :- HCF(a,b) = HCF(b,a) ← Find only upper or lower triangle and fill similarly in remaining

	$a \wedge b$	1	2	5	10	11	22	55	110
	1	1	1	1	1	1	1	1	1
	2	1	2	1	2	1	2	1	2
Find →	5	1	1	5	5	1	1	5	5
	10	1	2	5	10	1	2	5	10
HCF(a,b)	11	1	1	1	1	11	11	11	11
	22	1	2	1	2	11	22	11	22
	55	1	1	5	5	11	11	55	55
	110	1	2	5	10	11	22	55	110

Note :- LCM(a,b) = LCM(b,a)

	$a \vee b$	1	2	5	10	11	22	55	110
	1	1	2	5	10	11	22	55	110
	2	2	2	1			22		
Find →	5	5		5			110		
LCM(a,b)	10	10			10				
	11	11				11			
	22	22					22		
	55	55						55	
	110	110							110

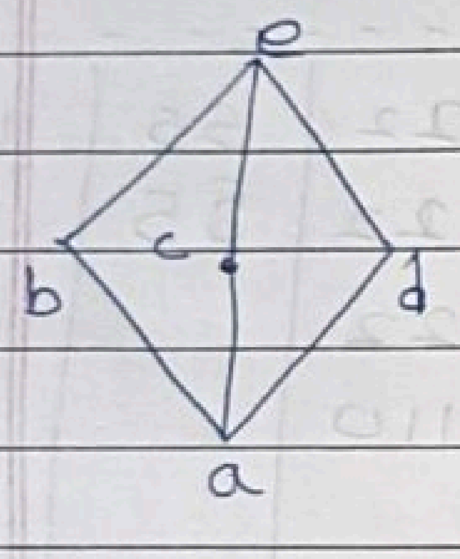
From above tables

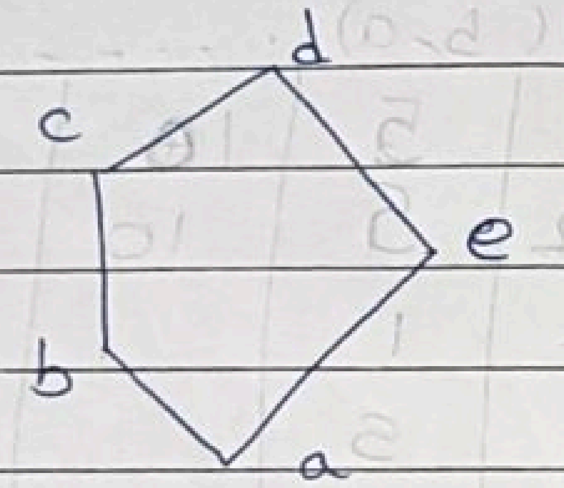
$\forall a, b \in A$, $a \wedge b \neq \phi$, $a \vee b \neq \phi$

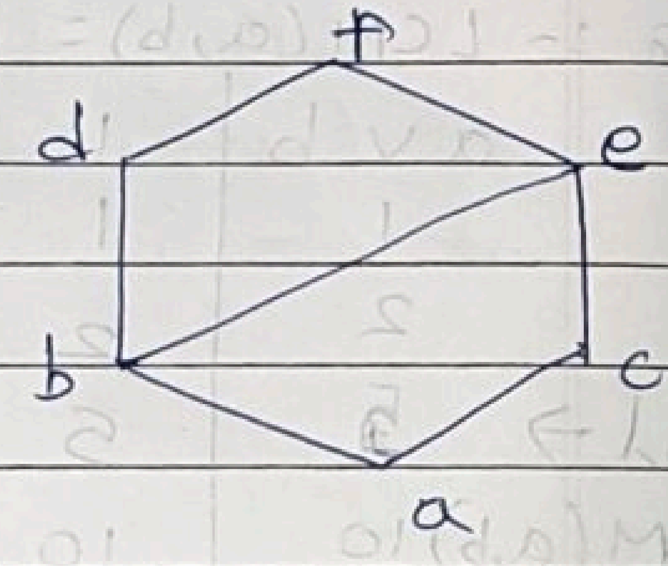
$\therefore D_{110}$ is lattice

Lattice $(L; \wedge, \vee, \leq)$

Complement lattice	Distributive lattice	Boolean lattice
$\forall a \in L,\ a^c \geq 1$	$\forall a \in L,\ a^c \leq 1$ OR $\forall a,b,c \in L$ $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$	$\forall a \in L,\ a^c = 1$



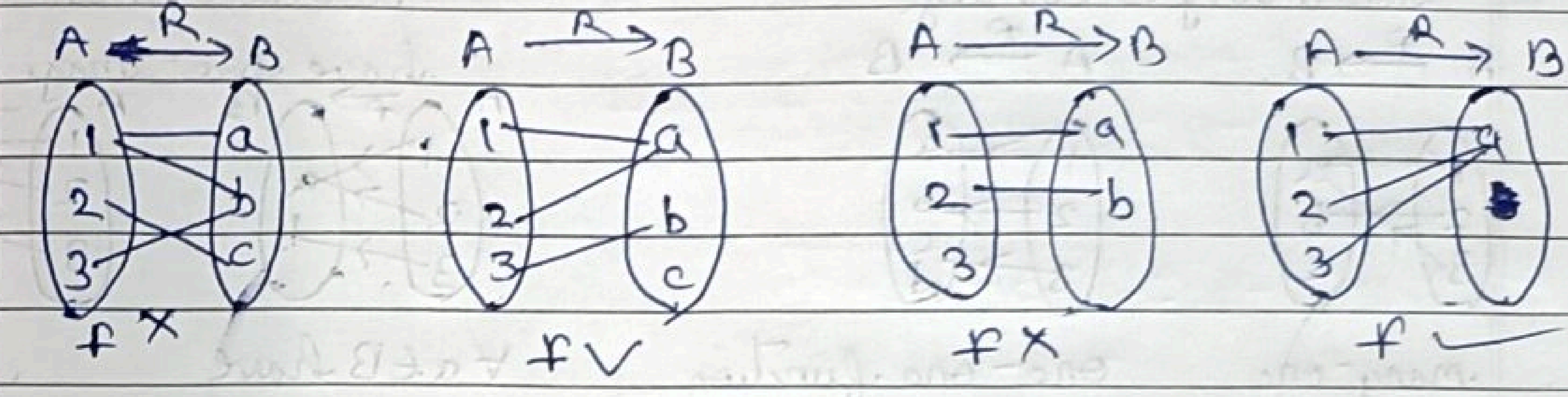




$a^c = e$	$a^c = d$	$a^c = f$
$e^c = a$	$d^c = a$	$f^c = a$
$b^c = \{c, d\}$	$b^c = e$	$b^c = \phi$
$c^c = \{b, d\}$	$c^c = e$	$d^c = c$
$d^c = \{b, c\}$	$e^c = \{b, c\}$	$c^c = d$
		$e^c = \phi$

Function

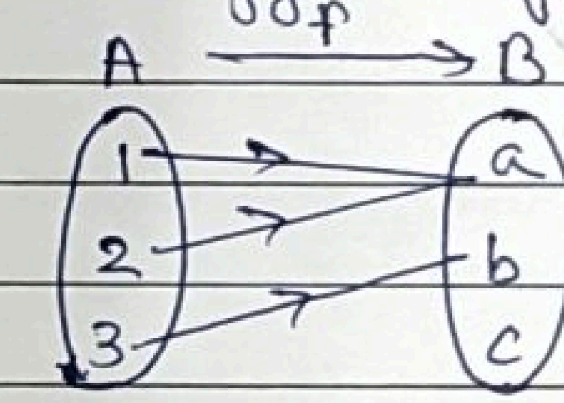
A function is rule or relation defined A to B such that every element of set A map to unique element in set B.



× 1 maped a
× 1 maped b

× 3 does not map any element

* Terminology in function :-



domain = set A = {1, 2, 3}
co-domain = set B = {a, b, c}
Range ⊆ co-domain = {a, b}

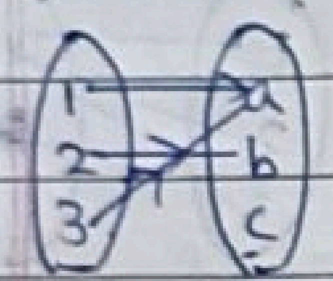
* image and Preimage

1 maped to a	
a is image of 1	1 is preimage of a
a is image of 2	2 is preimage of a
b is image of 3	3 is preimage of b

Types of function

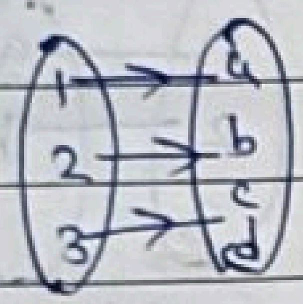
How many Pre-image | is all element of set B have pre-image in set A

$A \xrightarrow{f} B$



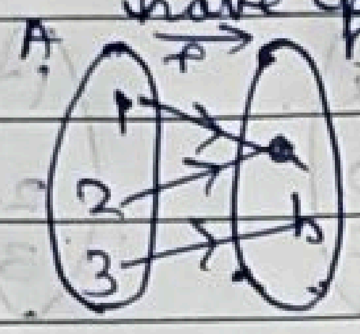
many-one function

$A \xrightarrow{f} B$



one-one function (injective)

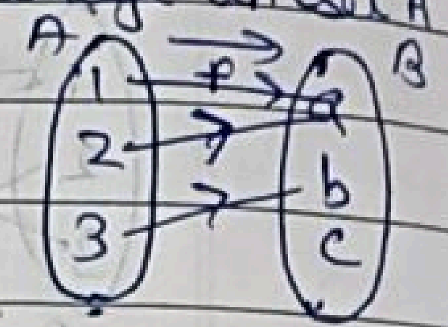
$A \xrightarrow{f} B$



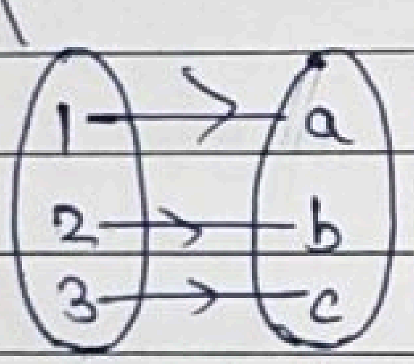
$\forall a \in B$ have preimage in A

onto function (surjective)

$A \xrightarrow{f} B$



into function



Bijective

* Injective function :-

$f: A \rightarrow B$ is said to be injective (one-one) function iff.

$f(x_1) = f(x_2) \longrightarrow x_1 = x_2$

or ~~f~~ $x_1 \neq x_2 \longrightarrow f(x_1) \neq f(x_2)$

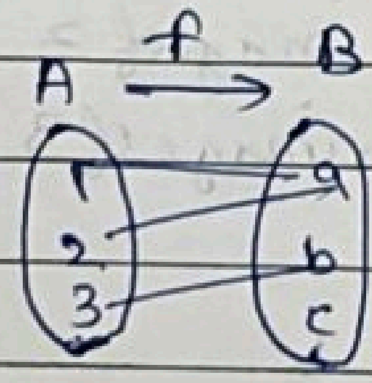
* Surjective :-

$f: A \rightarrow B \quad y = f(x)$

find $g: B \rightarrow A \quad x = g(y)$

if $\forall y \in B$ have images then f is surjective

$A \xrightarrow{f} B$



$A \xleftarrow{g} B$

* Bijective function :-

If $f: A \rightarrow B$ is injective and surjective, then f is called ~~Injective~~ bijective

* Invertable function :-

If $f: A \rightarrow B$ is bijective function then f is called invertable function defined as $g: B \rightarrow A$

g is called inverse of f

* Checking injective function :-

Let $f(x_1) = f(x_2)$

↓

Solve

↓

$x_1 = x_2$ injective | $x_1 \neq x_2$ — $x_1 = x_2$ or $x_1 \neq x_2$ not injective

* Check function injective or not :-

1) $f: R \rightarrow R \qquad f(x) = 2x^2 + 5x - 3$

Ans $f(x_1) = f(x_2)$

$2x_1^2 + 5x_1 \cancel{-3} = 2x_2^2 + 5x_2 \cancel{-3}$

$2x_1^2 + 5x_1 = 2x_2^2 + 5x_2$

$2x_1^2 - 2x_2^2 + 5x_1 - 5x_2 = 0$

$2(x_1^2 - x_2^2) + 5(x_1 - x_2) = 0$

$2(x_1 + x_2)(x_1 - x_2) + 5(x_1 - x_2) = 0$

$\therefore (x_1 - x_2) = 0$ or $2(x_1 + x_2) + 5 = 0$

$\therefore x_1 = x_2$ or $x_1 \neq x_2 \quad \because x_1 = \frac{-2x_2 - 5}{2}$

$\therefore$ Function is not injective

2) $f: R - \{2\} \rightarrow R \qquad f(x) = \frac{1}{x-2}$

Ans $f(x_1) = f(x_2)$

$\therefore \frac{1}{x_1 - 2} = \frac{1}{x_2 - 2}$

$\therefore x_1 - 2 = x_2 - 2$

$\therefore x_1 = x_2 \qquad \therefore$ function is injective

* Checking surjective function:-

eg. $f(x) = 2x^2+5x-3$

$y = 2x^2+5x-3$

$\therefore 2x^2+5x-3-y=0$

$\therefore x = \frac{-5 \pm \sqrt{25-4(2)(-3-y)}}{4}$

$\therefore x = \frac{-5 \pm \sqrt{25+24+8y}}{4}$

$\therefore x = \frac{-5 \pm \sqrt{49+8y}}{4}$

$\therefore$ For $y = -8$, $\sqrt{49+8y}$ is not defined

$\therefore$ Function is ~~sur~~ not surjective, since -8 does not have preimage.

eg. $f(x) = \frac{1}{x-2}$

$y = \frac{1}{x-2}$

$x = \frac{1}{y} + 2$

$g : R \to R - \{2\}$

for $y=0$, ~~it~~ does not exist x

$\therefore 0$ does not have pre-image

$\therefore$ Function not surjective

eg. $f : R \to R$ $f(x) = x^2 - 4x$

Ans $f(x) = x^2 - 4x$

$f(x_1) = f(x_2)$

$\therefore x_1^2 - 4x_1 = x_2^2 - 4x_2$

$\therefore x_1^2 - x_2^2 - 4x_1 + 4x_2 = 0$

$\therefore (x_1 - x_2)(x_1+x_2) - 4(x_1 - x_2) = 0$

$\therefore (x_1 - x_2)[(x_1+x_2) - 4] = 0$

$\therefore x_1 = x_2$ or $x_1 \neq x_2$

$\therefore$ Function is not injective

$\therefore y = x^2 - 4x$

$\therefore x^2 - 4x - y = 0$

$x = \frac{-(-4) \pm \sqrt{16-4(1)(-y)}}{2(1)}$

$x = \frac{4 \pm \sqrt{16+4y}}{2}$

$x = 2 \pm \sqrt{4+y}$

For $y = -5$, x does not exist

$\therefore$ It is not surjective

eg. $f : R - \{\frac{2}{5}\} \to R - \{\frac{4}{5}\}$ $f(x) = \frac{4x+3}{5x-2}$

Prove that its bijective function

Ans $f(x_1) = f(x_2)$

$\therefore \frac{4x_1+3}{5x_1-2} = \frac{4x_2+3}{5x_2-2}$

$\therefore (4x_1+3)(5x_2-2) = (4x_2+3)(5x_1-2)$

$\therefore \cancel{20x_1x_2} - 8x_1 + 15x_2 - \cancel{6} = \cancel{20x_1x_2} - 8x_2 + 15x_1 - \cancel{6}$

$\therefore -8(x_1 - x_2) - 15(x_1 - x_2) = 0$ or $15x_1 + 8x_1 = 15x_2 + 8x_2$

$\therefore (x_1 - x_2)(-8-15) = 0$

$\therefore x_1 = x_2$

$\therefore f(x)$ is injective function

$y = \frac{4x+3}{5x-2}$

$\therefore (5x-2)y = 4x+3$

$\therefore 5xy - 2y = 4x+3$

$\therefore 5xy - 4x - 2y - 3 = 0$

$\therefore x(5y-4) - 2y - 3 = 0$

$\therefore x = \frac{2y+3}{5y-4} = g(y)$

From given condition, $y \neq \frac{4}{5}$

$\therefore$ ~~The for $\forall x \in y$~~ for $\forall y \in R - \{\frac{4}{5}\}$

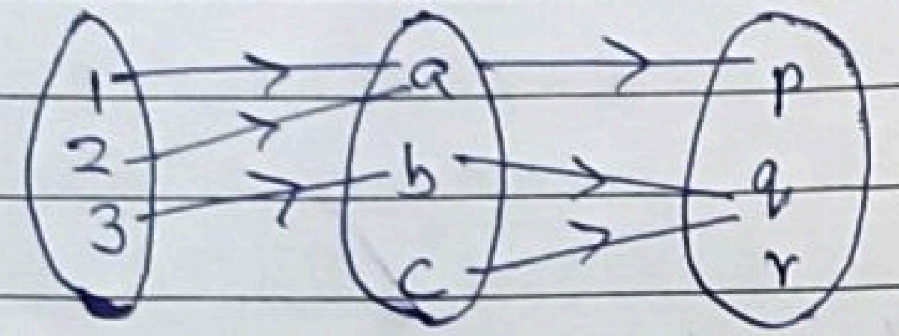
$\therefore$ function is surjective

$\therefore$ function is bijective

$\therefore f^{-1}(x) = \frac{2x+3}{5x-4}$

* Composition of function :-

$A \xrightarrow{f} B \xrightarrow{g} C$



Here

$gof(x) = g(f(x))$ ✓

but

$fog(x) = f(g(x))$ ✗

$\therefore gof(x) = A \to C$

$= \{(1,p), (2,p), (3,q)\}$

eg. $x = \{1, 2, 3\}$

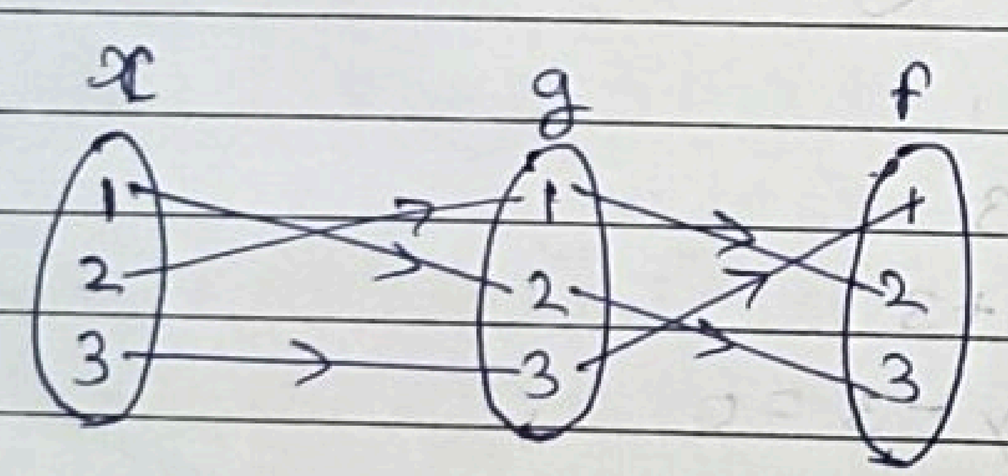
$f : x \to x$ $f = \{(1,2), (2,3), (3,1)\}$

$g : x \to x$ $g = \{(1,2), (2,1), (3,3)\}$

$h : x \to x$ $h = \{(1,1), (2,2), (3,1)\}$

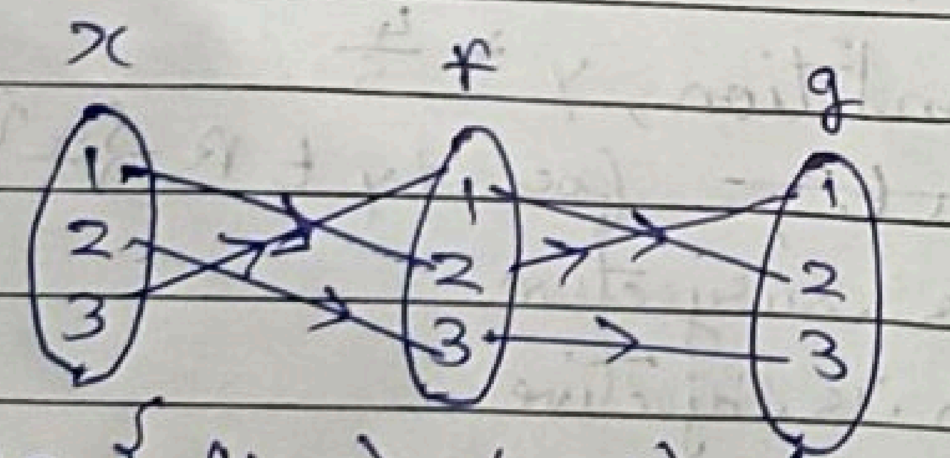
Find $fog(x)$, $gof(x)$

Ans $fog(x) =$



$\therefore fog(x) = \{(1,3), (2,2), (3,1)\}$

$gof(x) =$



$gof(x) = \{(1,1), (2,3), (3,2)\}$

Q.1) $f : R \to R$ $f(x) = x^3$

$g : R \to R$ $g(x) = 4x^2 + 1$

$h : R \to R$ $h(x) = 7x - 2$

Find gof, fog, $fofof$

Ans $gof = g[f(x)]$

$= g[x^3]$

$gof = 4x^6 + 1$

$fog = f[g(x)]$

$= f[4x^2 + 1]$

$= (4x^2 + 1)^3$

$= 64x^6 + 48x^2 + 12x^4 + 1$

$fofof = f[f[f(x)]]$

$= f[f(x^3)]$

$= f[(x^3)^3]$

$= f[x^9]$

$= (x^9)^3$

$fofof = x^{27}$

$gohof = g\{h[f(x)]\}$

$= g[h(x^3)]$

$= g(7x^3 - 2)$

$= 4(7x^3 - 2)^2 + 1$

$= 4[49x^6 - 28x^3 + 4] + 1$

$= 196x^6 - 112x^3 + 17$