Mathematical Foundations of Computer Science
Solutions to Homework Assignment 2
February 3, 2023

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1. Prove that the following propositions are true.

(a) The sum of any rational number and any irrational number is irrational.
(b) $\sqrt{13}$ is irrational.

Solution. (a) Let x and y be any particular but arbitrarily chosen rational number and irrational number respectively. Assume for contradiction that $x+y$ is rational. By definition, we have

$$\begin{aligned} x+y &= \frac{a}{b}, \text{ for some integers } a \text{ and } b \\ y &= \frac{a}{b} - x \\ &= \frac{a}{b} + (-x) \end{aligned}$$

The right hand side is a sum of two rational numbers which is rational. This is a contradiction since y is irrational.

(b) The proof can be obtained by replacing $\sqrt{2}$ by $\sqrt{13}$ and 2 by 13 in the proof using the unique factorization theorem for the statement "$\sqrt{2}$ is irrational".

Tim Frick (from my class in Spring'08) presented the following proof which is very nice as it not only proves that $\sqrt{13}$ is irrational but also proves that $\sqrt{n}$ is irrational where n is an integer that is not a perfect square. For his proof he first proved the following claim.

Claim: If p and q are relatively prime integers then so are p^2 and q^2.

Proof. By the prime factorization theorem we know that p, q, p^2, and q^2 can be represented as a unique product of primes.

$$\begin{aligned} p &= p_1^{e_1} p_2^{e_2} \dots p_k^{e_k} \\ q &= q_1^{e_1} q_2^{e_2} \dots q_\ell^{e_\ell} \\ p^2 &= p_1^{2e_1} p_2^{2e_2} \dots p_k^{2e_k} \\ q^2 &= q_1^{2e_1} q_2^{2e_2} \dots q_\ell^{2e_\ell} \end{aligned}$$

Since p and q are relatively prime $p_i \neq q_j$, for all i and j. This implies that p^2 and q^2 are relatively prime.

Let n be an arbitrary but specific integer that is not a perfect square. Assume, for the sake of contradiction, that $\sqrt{n}$ is a rational number. Then there are numbers a and b ($b \neq 0$) with no common factors such that

$$\sqrt{n} = \frac{a}{b}$$

Squaring both sides of the above equation gives

$$n = \frac{a^2}{b^2}$$

Since a and b are relatively prime we know that a^2 and b^2 are relatively prime integers. This is only possible if $b^2 = 1$ which means that $b = 1$, which in turn implies that $a = \sqrt{n}$, which contradicts the fact that a is an integer.

2. Let a, b, c be integers satisfying $a^2 + b^2 = c^2$. Prove that abc must be even.

Solution. We know that the product of odd numbers is an odd number. So for abc to be even, at least one of the three numbers must be even. We will prove the claim by proving the contrapositive, i.e., if a, b, and c all are odd then $a^2 + b^2 \neq c^2$. Let $a = 2k + 1$ and $b = 2k' + 1$, for some integers k and k'.

$$\begin{aligned}
\text{L.H.S} &= a^2 + b^2 \\
&= (2k+1)^2 + (2k'+1)^2 \\
&= 4k^2 + 4k + 1 + 4k'^2 + 4k' + 1 \\
&= 2(2k^2 + 2k'^2 + 2k + 2k' + 1)
\end{aligned}$$

This means that L.H.S. is even. But R.H.S. $= c^2$ is odd as it is a product of two odd numbers. Thus $a^2 + b^2 \neq c^2$. Hence at least one of the three numbers, a, b, or c must be even. Hence, abc is even.

3. Let $x_1, x_2, \ldots, x_n$ be n real numbers. Let $\overline{x} = (x_1 + x_2 + \ldots + x_n)/n$ be their average. Use a proof by contradiction to prove that at least one of $x_1, x_2, \ldots, x_n$ is greater than or equal to $\overline{x}$.

Solution. Assume otherwise, that is, assume that each x_i, $1 \leq i \leq n$, is less than $\overline{x}$. Then,

$$\begin{aligned}
\sum_{i=1}^{n} x_i &< n\overline{x_i} \\
\frac{\sum_{i=1}^{n} x_i}{n} &< \overline{x_i}
\end{aligned}$$

This is a contradiction because $\overline{x_i} = \sum_{i=1}^{n} x_i/n$.

4. For all $n \in \mathbb{N}$, prove that $3^{3n+1} + 2^{n+1}$ is divisible by 5.

Solution. Let $P(n)$ be the property that $3^{3n+1} + 2^{n+1}$ is divisible by 5.
Induction Hypothesis: Assume that $P(k)$ is true for some $k \geq 0$.
Base Case: $P(0)$ is true because L.H.S. $= 5$ and hence is divisible by 5.
Induction Step: We want to prove that $P(k+1)$ is true. In other words, we want to prove that $3^{3(k+1)+1} + 2^{(k+1)+1}$ is divisible by 5.

$$\begin{aligned} 3^{3(k+1)+1} + 2^{(k+1)+1} &= 3^3 \cdot 3^{3k+1} + 2 \cdot 2^{k+1} \\ &= 27(3^{3k+1} + 2^{k+1}) - 25 \cdot 2^{k+1} \end{aligned}$$

By the induction hypothesis, the first term in R.H.S. is divisible by 5. The second term is divisible by 5 since 25 is divisible by 5. Hence $P(k+1)$ is true.

5. Prove that for all $n > 1$,

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

Solution. Let $P(n)$ be the property that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

We will prove that $P(n)$ is true for all $n > 1$ using induction on n.
Base Case: $P(2)$ is true because L.H.S $= 1 + 1/4 = 5/4 < 1.5 =$ R.H.S.
Induction Hypothesis: Assume that $P(k)$ is true for some $k \geq 2$.
Induction Step:. To prove that $P(k+1)$ is true. In other words, we want to show that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$$

$$\begin{aligned} \text{L.H.S} &= 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \\ &< 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \qquad \text{(using induction hypothesis)} \\ &= 2 - \frac{(k+1)^2 - k}{k(k+1)^2} \\ &= 2 - \frac{k^2 + k + 1}{k(k+1)^2} \\ &< 2 - \frac{k^2 + k}{k(k+1)^2} \\ &< 2 - \frac{k(k+1)}{k(k+1)^2} \\ &= 2 - \frac{1}{k+1} \end{aligned}$$

6. The series

$$\sum_{k=1}^{n} \frac{1}{k}$$

is called the harmonic series. The sum of the first n numbers of the harmonic series is called the *nth harmonic number*, H_n. Thus,

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Using induction show that $H_{2^n} \geq 1 + \frac{n}{2}$. In other words, prove that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$$

Solution. Let $P(n)$ be the property that

$$H_{2^n} \geq 1 + \frac{n}{2}. \tag{1}$$

We will prove that $P(n)$ is true for all $n \geq 0$ using induction on n.
Base Case: $P(0)$ is true because when $n = 0$, L.H.S. of (**??**) is $H_{2^0} = H_1 = 1$ and so is the R.H.S.
Induction Hypothesis: Assume that $P(k)$ is true for some $k \geq 0$.
Induction Step: We want to prove $P(k+1)$. When $n = k+1$, L.H.S. of (1) is

$$\begin{aligned}
\text{L.H.S.} &= H_{2^{k+1}} \\
&= H_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots \frac{1}{2^{k+1}} \\
&\geq 1 + \frac{k}{2} + 2^k \times \frac{1}{2^{k+1}} \qquad \text{(using induction hypothesis).} \\
&= 1 + \frac{k}{2} + \frac{1}{2} \\
&= 1 + \frac{k+1}{2}
\end{aligned}$$

Thus we have proved that $P(k+1)$ is true and this completes the induction proof.

7. (a)Prove using induction that for all non-negative integers n and for all integers $x > 1$, $x^n - 1$ is divisible by $x - 1$.
(b) If n is a positive integer and $1 + x > 0$ then $(1+x)^n \geq 1 + nx$.

Solution. We will prove the claim using induction on n.
Base Case: $n = 1$. The claim is trivially true since $x^1 - 1$ is divisible by $x - 1$.
Induction Hypothesis: Assume that $x^k - 1$ is divisible by $x - 1$ for some $k > 0$.
Inductive Step: We want to prove that $x^{k+1} - 1$ is divisible by $x - 1$.

$$x^{k+1} - 1 = x(x^k - 1) + (x - 1)$$

We have expressed $x^{k+1} - 1$ as the sum of two terms. By induction hypothesis, the first term is divisible by $x - 1$. The second term is also divisible by $x - 1$. Thus, $x^{k+1} - 1$ is also divisible by $x - 1$.

(b) Let $P(n)$ be the following property: $\forall x$, such that $1 + x > 0$,

$$(1 + x)^n \geq 1 + nx \tag{2}$$

We will prove the claim by doing induction on n.
Base Case: The claim is clearly true for $n = 1$ since both sides of the expression are equal to $1 + x$.
Induction Hypothesis: Assume that $P(k)$ is true for some $k \geq 1$.
Induction Step: We want to prove that $P(k + 1)$ is true. When $n = k + 1$, the left side of (2) can be written as

$$\begin{aligned}
\text{L.H.S.} &= (1 + x)^{k+1} \\
&= (1 + x)(1 + x)^k \\
&\geq (1 + x)(1 + kx) \quad \text{(using induction).} \\
&= 1 + (k + 1)x + kx^2 \\
&\geq 1 + (k + 1)x \qquad \text{(since } kx^2 \text{ is always positive)}
\end{aligned}$$

This proves $P(k + 1)$ and hence completes the proof.