4.4.3. Continuous Probability Distribution

$\boxed{\text{4.4.3 : class work Problems}}$

Example-1: For a continuous random variable 'x' its Probability density function given by

$$f(x) = k(2-x) \qquad 0 < x < 2$$
$$= kx(x-2) \qquad 2 \le x \le 3$$
$$= 0 \qquad \text{o.w}$$

Find (i) k, (ii) mean and (iii) median for the distribution.

Solution: W.k.t total Probability $= 1$

$$\therefore \int_{x=-\infty}^{+\infty} f(x) = 1$$

$$\int_{x=-\infty}^{0} f(x)\,dx + \int_{x=0}^{2} f(x)\,dx + \int_{x=2}^{3} f(x)\,dx + \int_{x=3}^{\infty} f(x)\,dx = 1$$

$$\int_{x=-\infty}^{0} 0\,dx + \int_{x=0}^{2} k(2-x)\,dx + \int_{x=2}^{3} k\,x(x-2)\,dx + \int_{x=3}^{\infty} 0\,dx = 1$$

$$0 + k\left[2x - \frac{x^2}{2}\right]_0^2 + k\left[\frac{x^3}{3} - \frac{2x^2}{2}\right]_2^3 + 0 = 1$$

$$\therefore k\left[(4-2) - 0 + (9-9) - \left(\frac{8}{3} - 4\right)\right] = 1$$

$$k\left[2 - \left(\frac{-4}{3}\right)\right] = 1 \Rightarrow \frac{10k}{3} = 1 \therefore k = \left(\frac{3}{10}\right)$$

$$\therefore f(x) = \frac{3}{10}(2-x) \qquad 0 < x \le 2$$
$$= \frac{3}{10}\left[x^2 - 2x\right] \qquad 2 \le x \le 3$$
$$= 0 \qquad \text{o.w.}$$

$$\text{mean} = E(x) = \int_{x=-\infty}^{\infty} x f(x)\,dx = \int_{x=-\infty}^{0} x f(x)\,dx + \int_{x=0}^{2} x f(x)\,dx + \int_{x=2}^{3} x f(x)\,dx + \int_{x=3}^{\infty} x f(x)\,dx$$

$$= \int_{x=-\infty}^{0} x \cdot 0\,dx + \int_{x=0}^{2} x\,\frac{3}{10}(2-x)\,dx + \int_{x=2}^{3} x \cdot \frac{3}{10}(x^2 - 2x)\,dx + \int_{x=3}^{\infty} x \cdot 0\,dx$$

$$= 0 + \frac{3}{10}\left[x^2 - \frac{x^3}{3}\right]_0^2 + \frac{3}{10}\left[\frac{x^4}{4} - \frac{2x^3}{3}\right]_2^3 + 0$$

$$= \frac{3}{10}\left[\left(4 - \frac{8}{3}\right) - 0 + \left(\frac{81}{4} - 18\right) - \left(4 - \frac{16}{3}\right)\right] = \frac{59}{12} = 4.9167$$

Example-2: A continuous r.v. x has the p.d.f. $f(x) = \frac{4}{81} x(9-x^2)$ $0 \le x \le 3$
$= 0$ o.w.
Find the first four moment about the origin and about the mean.

Solution we know $E(x^r) = \int_{x=-\infty}^{+\infty} x^r f(x)\,dx = \int_{x=-\infty}^{0} x^r f(x)\,dx + \int_{x=0}^{3} x^r f(x)\,dx + \int_{x=3}^{\infty} x^r f(x)\,dx$

$\therefore E(x^r) = \int_{x=-\infty}^{0} x^r\, 0\,dx + \int_{x=0}^{3} x^r \frac{4}{81}(9x - x^3)\,dx + \int_{x=3}^{\infty} x^r . 0\,dx$

$E(x^r) = \frac{4}{81}\int_{x=0}^{3} (9x^{r+1} - x^{r+3})\,dx = \frac{4}{81}\left[9\frac{x^{r+2}}{r+2} - \frac{x^{r+4}}{r+4}\right]_{x=0}^{3}$

$\therefore \mu'_r = E(x^r) = \frac{4}{81}\left[\frac{3^2 . 3^{r+2}}{r+2} - \frac{3^{r+4}}{r+4} - (0-0)\right] = \frac{4}{81} . 3^{r+4}\left[\frac{1}{r+2} - \frac{1}{r+4}\right]$

$\mu'_r = 4 \times 3^r \frac{r+4-(r+2)}{(r+2)(r+4)} = \frac{8 \times 3^r}{(r+2)(r+4)}$

$\therefore \mu'_1 = \frac{8 \times 3}{3 \times 5} = \frac{8}{5} = 1.6$, $\mu'_2 = \frac{8 \times 9}{4 \times 6} = 3$, $\mu'_3 = \frac{8 \times 27}{5 \times 7} = 6.171429$, $\mu'_4 = \frac{8 \times 81}{6 \times 8} = 13.5$

$\therefore \mu_1 = 0$, $\mu_2 = \mu'_2 - (\mu'_1)^2 = 3 - (1.6)^2 = 0.44$, $\mu_3 = \mu'_3 - 3\mu'_2 \mu'_1 + 2(\mu'_1)^3 = \cancel{6.171429}$

$\therefore \mu_3 = 6.171429 - (3 \times 3 \times 1.6) + 2 \times (1.6)^3 = -0.036571$

~~Example-3~~:

$\mu_4 = \mu'_4 - 4\mu'_3 \mu'_1 + 6\mu'_2 (\mu'_1)^2 - 3(\mu'_1)^4 = 13.5 - (4 \times 6.171429 \times 1.6) + (6 \times \cancel{0.44}\, 3 \times (1.6)^2 - (3 \times (1.6)^4))$

$\therefore \mu_4 = 0.4201335$

Example-3: The daily consumption of electric power is a r.v. X with p.d.f. $f(x) = k x e^{-x/3}$ for $x > 0$
$= 0$ elsewhere

Find the value of k and the probability that on a given day the expected electric consumption electric consumption is more than the expected electric consumption

Solution $\because$ total probability $= 1$

$\therefore \int_{x=-\infty}^{+\infty} f(x)\,dx = 1 \Rightarrow \int_{x=-\infty}^{0} f(x)\,dx + \int_{x=0}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{x=-\infty}^{0} 0\,dx + \int_{x=0}^{\infty} k x e^{-\frac{x}{3}}\,dx = 1$

$0 + k\left[x \frac{e^{-x/3}}{-1/3} - 1\frac{e^{-x/3}}{(-1/3)^2}\right]_{x=0}^{\infty} = 1 \Rightarrow k\left[-\frac{1}{3} x e^{-x/3} + 9 e^{-x/3}\right]_0^{\infty} = 1$

$k\left[-\frac{1}{3}(0-0) + 9(0-1)\right] = 1 \Rightarrow 9k = 1 \Rightarrow k = \frac{1}{9}$

$\therefore f(x) = \frac{1}{9} x e^{-\frac{1}{3}x}$ for $x > 0$
$= 0$ elsewhere

$E(x) = \int_{x=0}^{\infty} x f(x)\,dx = \int_0^{\infty} \frac{1}{9} x^2 e^{-\frac{1}{3}x}\,dx = \frac{1}{9}\int_{x=0}^{\infty} e^{-\frac{1}{3}x} x^{3-1}\,dx = \frac{1}{9}\frac{\Gamma 3}{(1/3)^3} = \frac{27\,\Gamma 3}{9} = \frac{54}{9} = 6$

P(electric consumption more than expected electric consumption) $= P(6 \le x) = P(6 \le x < \infty) = \int_{x=6}^{\infty} f(x)\,dx = \int_{x=6}^{\infty} \frac{1}{9} x e^{-\frac{1}{3}x}\,dx$

$= P(x \ge 6)$

$= \frac{1}{9}\left[x \frac{e^{-\frac{1}{3}x}}{-1/3} - \frac{e^{-\frac{1}{3}x}}{(-1/3)^2}\right]_6^{\infty} = \frac{1}{9}\left[-3x e^{-x/3} - 9 e^{-x/3}\right]_6^{\infty}$

$= \frac{1}{9}\left[-3(0 - 6e^{-2}) - 9(0 - e^{-2})\right] = \frac{1}{9}\left[18 e^{-2} + 9 e^{-2}\right] = \frac{27}{9} e^{-2} = 3e^{-2}$

Example-4 The diameter say x of an electric cable is assumed to be a continuous variable with p.d.f $f(x) = 6x(1-x)$ $0 \le x \le 1$ (i) Is it Probability distribution function? (ii) obtain cumulative distribution function (iii) compute $P[x \le \frac{1}{2} / \frac{1}{3} \le x \le \frac{2}{3}]$ (iv) Determine k, so that $p(X < k) = p(X > k)$

Solution $f(x) = 6x(1-x)$ $0 \le x \le 1$

$$\int_{x=0}^{1} f(x)\,dx = \int_{x=0}^{1} 6x(1-x)\,dx = 6\int_{x=0}^{1} x^{2-1}(1-x)^{2-1}dx = 6\frac{\Gamma 2\,\Gamma 2}{\Gamma 4} = \frac{6 \times 1! \times 1!}{3!} = 1$$

$\therefore$ total probability $= 1$ $\therefore$ it is Probability distribution function

$$F_X(x) = P(X \le x) = \int_{x=0}^{x} f(x)\,dx = 6\int_{x=0}^{x}(x - x^2)\,dx = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^x = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right] = 3x^2 - 2x^3$$

$F_X(x) = 3x^2 - 2x^3$

(iii) $P(x \le \frac{1}{2} / \frac{1}{3} \le x \le \frac{2}{3}) = P[x < 0.5 / 0.33 \le x \le 0.67]$

$$= \frac{P[x < 0.5 \cap 0.33 \le x \le 0.67]}{P(0.33 \le x \le 0.67)} = \frac{P(0.33 \le x \le 0.5)}{P(0.33 \le x \le 0.67)} \quad \text{--- (1)}$$

Now $P(0.33 \le x \le 0.5) = P(\frac{1}{3} \le x \le \frac{1}{2}) = \int_{x=1/3}^{k} 6[x - x^2]\,dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/3}^{k}$

$$= 6\left[\frac{1}{2}\left(\frac{1}{4} - \frac{1}{9}\right) - \frac{1}{3}\left(\frac{1}{8} - \frac{1}{27}\right)\right] = \frac{13}{54}$$

$$P(0.3 \le x \le 0.67) = P(\tfrac{1}{3} \le x \le \tfrac{2}{3}) = \int_{1/3}^{2/3} 6(x - x^2)\,dx = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_{1/3}^{2/3}$$

$$= 6\left[\frac{1}{2}\left(\frac{4}{9} - \frac{1}{9}\right) - \frac{1}{3}\left(\frac{8}{27} - \frac{1}{27}\right)\right] = \frac{13}{27}$$

$\therefore$ From (1)

$$P\left(x \le \tfrac{1}{2} / \tfrac{1}{3} \le x \le \tfrac{2}{3}\right) = \frac{13/54}{13/27} = \frac{27}{54} = \frac{1}{2} = 0.5$$

(iv) Note take $P(X < k) = P(X > k)$ ie $P(-\infty < x < k) = P(k < x)$

ie $P(0 < x < k) = P(k < x < 1)$

$$\int_0^k 6(x - x^2)\,dx = \int_k^1 6(x - x^2)\,dx$$

$$\frac{x^2}{2} - \frac{x^3}{3}\Big|_0^k = \frac{x^2}{2} - \frac{x^3}{3}\Big|_k^1$$

$$\frac{k^2}{2} - \frac{k^3}{3} - 0 = \frac{1}{2} - \frac{1}{3} - \left(\frac{k^2}{2} - \frac{k^3}{3}\right)$$

$$\therefore 2\left(\frac{k^2}{2} - \frac{k^3}{3}\right) = \frac{1}{6}$$

$$\frac{2(3k^2 - 2k^3)}{6} = \frac{1}{6}$$

$\therefore 6k^2 - 4k^3 = 1$

$\Rightarrow 4k^3 - 6k^2 + 1 = 0$ $\therefore k = \frac{1-\sqrt{3}}{2}$ $k = \frac{1+\sqrt{3}}{2}$ $k = \frac{1}{2}$

$\therefore \boxed{k = \frac{1}{2}}$

Example-5 : Let 'x' be a continuous random variables with p.d.f
$f(x) = kx(1-x), 0 \le x \le 1$, Find k and determine a number b such that $p(x \le b) = p(x \ge b)$

Solution : $\because$ total probability = 1

$\therefore \int_{x=0}^{1} kx(1-x)\,dx = 1 \Rightarrow k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1 \Rightarrow k\left[\frac{1}{2} - \frac{1}{3}\right] = 1 \Rightarrow \frac{k}{6} = 1$ $\boxed{k=6}$

$\therefore f(x) = 6x(1-x) \quad 0 \le x \le 1$

Now $p(x \le b) = p(x \ge b) \Rightarrow p(x \le b) = p(b \le x) \Rightarrow p(0 \le x \le b) = p(b \le x \le 1)$

$\Rightarrow \int_{x=0}^{b} f(x)\,dx = \int_{x=b}^{1} f(x)\,dx \Rightarrow \int_{x=0}^{b} 6(x-x^2)\,dx = \int_{b}^{1} 6(x-x^2)\,dx$

$\Rightarrow \left.\frac{x^2}{2} - \frac{x^3}{3}\right|_0^b = \left.\frac{x^2}{2} - \frac{x^3}{3}\right|_b^1 \Rightarrow \frac{b^2}{2} - \frac{b^3}{3} = \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{b^2}{2} - \frac{b^3}{3}\right)$

$\Rightarrow 2\left(\frac{b^2}{2} - \frac{b^3}{3}\right) = \frac{1}{6} \Rightarrow \cancel{2b}\ 3b^2 - 2b^3 = 1 \Rightarrow 6b^2 - 4b^3 = 1$

$\Rightarrow 4b^3 - 6b^2 + 1 = 0 \quad b = \frac{1-\sqrt{3}}{2}, \ b = \frac{1+\sqrt{3}}{2}, \ b = \frac{1}{2}$

$\therefore$ Required answer is $b = \frac{1}{2}$

Example-6 Define Random variable with an example. Find k if the following is a p.d.f. $f(x) = kxe^{-4x^2} \quad 0 \le x \le \infty$, Also find mean

Solution:- $\because$ total Probability = 1

$\therefore \int_{x=0}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{x=0}^{\infty} kxe^{-4x^2}\,dx = 1 \Rightarrow \frac{k}{-8}\int_{x=0}^{\infty} e^{-4x^2}(-8x)\,dx = 1$

$\Rightarrow -\frac{k}{8} \left. e^{-4x^2} \right|_0^{\infty} = 1 \Rightarrow -\frac{k}{8}(0-1) = 1 \Rightarrow \frac{k}{8} = 1 \Rightarrow \boxed{k=8}$

$\therefore f(x) = 8xe^{-4x^2}, \ 0 \le x \le \infty$

$\text{mean} = E(x) = \int_{x=0}^{\infty} x f(x)\,dx = \int_{x=0}^{\infty} x\, 8x e^{-4x^2}\,dx = 8\int_{x=0}^{\infty} x^2 e^{-4x^2}\,dx$

Let $4x^2 = t$ i.e. $x = \frac{\sqrt{t}}{2} \quad dx = \frac{1}{2}\frac{1}{2}t^{\frac{1}{2}-1}\,dt$

$\therefore \text{mean} = 8\int_{t=0}^{\infty} e^{-t}\frac{t}{4}\frac{1}{4}t^{\frac{1}{2}-1}\,dt = \frac{1}{2}\int_{t=0}^{\infty} t^{\frac{3}{2}-1} e^{-t}\,dt = \frac{1}{2}\Gamma\frac{3}{2}$

$\therefore \text{Mean} = \frac{1}{2}\frac{1}{2}\Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{4}$

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Example-7 The p.d.f of random variable X is given by

$$f(x) = k x^2 (2-x) \quad 0 \le x \le 2$$
$$= 0 \quad \text{otherwise}$$

(i) Find k (ii) mean (iii) variance

Solution $\because$ total Probability $= 1$

i.e $\int_{x=-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{x=-\infty}^{0} f(x)dx + \int_{x=0}^{2} f(x)dx + \int_{x=2}^{\infty} f(x)dx = 1$

$$\int_{x=-\infty}^{0} 0\,dx + \int_{x=0}^{2} k(2x^2 - x^3)dx + \int_{x=2}^{\infty} 0\,dx = 1$$

$$0 + k\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = 1 \Rightarrow k\left[\frac{16}{3} - \frac{16}{4} - 0\right] = 1$$

$$16k\left(\frac{4-3}{12}\right) = 1 \Rightarrow \frac{16k}{12} = 1 \Rightarrow \frac{4k}{3} = 1 \Rightarrow k = \frac{3}{4}$$

$$\therefore f(x) = \frac{3}{4}x^2(2-x)dx$$

$$\mu_1' = \text{mean} = E(x) = \int_0^2 x f(x)dx = \int_{x=0}^{2} x\frac{3}{4}x^2(2-x)dx = \frac{3}{4}\int_{x=0}^{2}(2x^3 - x^4)dx$$

$$= \frac{3}{4}\left(\frac{2x^4}{4} - \frac{x^5}{5}\right)\Big|_0^2 = \frac{3}{4}\left[\frac{32}{4} - \frac{32}{5}\right] = \frac{3}{4}\times 32\left(\frac{1}{4} - \frac{1}{5}\right)$$

$$= 3\times 8\,\frac{1}{20} = \frac{6}{5} = \text{mean}$$

$$\mu_2' = E(x^2) = \int_{x=0}^{2} x^2 f(x)dx = \int_{x=0}^{2} x^2 \cdot \frac{3}{4}x^2(2-x)dx = \frac{3}{4}\int_{x=0}^{2}(2x^4 - x^5)dx$$

$$= \frac{3}{4}\left(\frac{2x^5}{5} - \frac{x^6}{6}\right)\Big|_0^2 = \frac{3}{4}\left(\frac{64}{5} - \frac{64}{6}\right) = \frac{3}{4}\times 64\left(\frac{6-5}{5\times 6}\right)$$

$$= \frac{8}{5}$$

$$\therefore \text{Variance} = \mu_2' - (\mu_1')^2 = \frac{8}{5} - \left(\frac{6}{5}\right)^2 = \frac{8}{5} - \frac{36}{25} = \frac{40-36}{25} = \frac{4}{25}$$

$\ldots = 0.3 + 0.1915 = 0.6915$

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Example-8 : A continuous random variable X has the following probability law $f(x) = kx^2 e^{-x}$ $x \geq 0$ Find k, mean & variance

Solution :: total probability = 1

$$\therefore \int_{x=0}^{\infty} k\, x^2 e^{-x} dx = 1$$

$$\therefore k \int_0^{\infty} e^{-x} x^{3-1} dx = 1 \Rightarrow k\, \Gamma 3 = 1 \Rightarrow 2k = 1 \quad k = \frac{1}{2}$$

$$\therefore f(x) = \frac{1}{2} e^{-x} \cdot x^2 \quad x \geq 0$$

$$\therefore \text{mean} = E(x) = \int_{x=0}^{\infty} x\, f(x)\, dx = \int_{x=0}^{\infty} x \frac{1}{2} e^{-x} \cdot x^2 dx = \frac{1}{2} \int_{x=0}^{\infty} e^{-x} x^{4-1} dx$$

$$\therefore \text{mean} = E(x) = \frac{1}{2} \Gamma 4 = \frac{3!}{2} = 3$$

$$E(x)^2 = \int_{x=0}^{\infty} x^2 f(x) dx = \int_{x=0}^{\infty} x^2 \frac{1}{2} e^{-x} x^2 dx = \frac{1}{2} \int_{x=0}^{\infty} e^{-x} \cdot x^{5-1} dx$$

$$\therefore E(x^2) = \frac{1}{2} \Gamma 5 = \frac{4!}{2} = 12$$

$$\therefore \text{variance} = E(x)^2 - (E(x))^2 = 12 - (3)^2 = 3$$

4.4.3 : Home work Problems

Example - 1: The length of time (in minutes), a day speaks on telephone is found to be a random variable with probability density function $f(x) = A e^{-x/3}$ for $x \geq 0$
$= 0$ elsewhere

find 'A' and the probability that she will speak for
(i) more than 10 minutes (ii) less than 5 minutes

Solution: $\because$ total Probability = 1

$$\therefore \int_{x=-\infty}^{+\infty} f(x)\,dx = 0 \Rightarrow \int_{x=-\infty}^{0} f(x)\,dx + \int_{x=0}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{x=-\infty}^{0} 0\,dx + \int_{x=0}^{\infty} A e^{-\frac{1}{3}x}\,dx = 1$$

$$\Rightarrow 0 + A \left.\frac{e^{-x/3}}{-1/3}\right|_0^{\infty} = 1 \Rightarrow -3A(0-1) = 1 \Rightarrow 3A = 1 \therefore A = \frac{1}{3}$$

$$\therefore f(x) = \frac{1}{3} e^{-\frac{1}{3}x}, \quad x \geq 0$$

$$p(X > 10) = p(10 < X) = p(10 < X < \infty) = \int_{x=10}^{\infty} f(x)\,dx = \int_{x=10}^{\infty} \frac{1}{3} e^{-\frac{1}{3}x}\,dx$$

$$= \frac{1}{3} \left.\frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}}\right|_{10}^{\infty} = -\left[0 - e^{-\frac{10}{3}}\right] = e^{\frac{10}{3}}$$

$$p(X < 5) = p(-\infty < X < 5) = \int_{x=-\infty}^{5} f(x)\,dx = \int_{x=0}^{5} \frac{1}{3} e^{-\frac{1}{3}x}\,dx$$

$$= \frac{1}{3} \left.\frac{e^{-\frac{1}{3}x}}{-1/3}\right|_0^5 = -\left[e^{-5/3} - 1\right] = \left(1 - e^{-5/3}\right)$$

Example - 2 A continuous random variable has p.d.f.

$f(x) = 1 - x \quad 0 < x < 1$
$= x - 1 \quad 1 < x < 2$ Find mean & variance
$= 0 \quad$ o.w.

Solution

$$E(X) = \int_{x=-\infty}^{\infty} f(x)\,dx = \int_{x=-\infty}^{0} x f(x)\,dx + \int_{x=0}^{1} x f(x)\,dx + \int_{x=1}^{2} x f(x)\,dx + \int_{x=2}^{\infty} x f(x)\,dx$$

$$\therefore E(X) = \int_{x=-\infty}^{0} x \cdot 0\,dx + \int_{x=0}^{1} x(1-x)\,dx + \int_{x=1}^{2} x(x-1)\,dx + \int_{x=2}^{\infty} x \cdot 0\,dx = 0 + \int_{x=0}^{1} (x - x^2)\,dx + \int_{x=1}^{2} (x^2 - x)\,dx + 0$$

$$\therefore E(X) = \left.\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 + \left.\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\right|_1^2 = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{8-1}{3}\right) - \frac{1}{2}(4-1) = \frac{1}{6} + \frac{7}{3} - \frac{3}{2} = 1 = \text{mean}$$

$$E(x^2) = 0 + \int_{x=0}^{1} x^2(1-x)\,dx + \int_{x=1}^{2} x^2(x-1)\,dx + 0 = \int_{x=0}^{1} (x^2 - x^3)\,dx + \int_{1}^{2} (x^3 - x^2)\,dx$$

$$E(x^2) = \left.\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\right|_0^1 + \left.\left(\frac{x^4}{4} - \frac{x^3}{3}\right)\right|_1^2 = \left(\frac{1}{3} - \frac{1}{4}\right) + \frac{1}{4}(16-1) - \frac{1}{3}(8-1) = \frac{3}{2}$$

$$\therefore \text{Variance} = E(x^2) - [E(X)]^2 = \frac{3}{2} - (1)^2 = \frac{3}{2} - 1 = \frac{1}{2}$$

Example-3-i If $f(x)$ is probability density function of a continuous random variable k, mean and variance

$f(x) = kx^2 \quad 0 \le x \le 1$

$= (2-x)^2 \quad 1 \le x \le 2$

Solution : total Probability $=1 \Rightarrow \int_{x=0}^{2} f(x)\,dx = 1 \Rightarrow \int_{x=0}^{1} f(x)\,dx + \int_{x=1}^{2} f(x)\,dx = 1$

$\int_{x=0}^{1} k x^2\,dx + \int_{x=1}^{2} (2-x)^2\,dx = 1 \Rightarrow k \frac{x^3}{3}\Big|_0^1 + \frac{(2-x)^3}{-3}\Big|_1^2 = 1$

$\therefore \frac{k}{3}(1-0) - \frac{1}{3}(0-1) = 1 \Rightarrow \frac{k}{3} + \frac{1}{3} = 1 \Rightarrow \frac{k}{3} = 1 - \frac{1}{3} = \frac{2}{3} \Rightarrow k = 2$

$\therefore f(x) = 2x^2 \quad 0 \le x \le 1$

$= (2-x)^2 \quad 1 \le x \le 2$

$E(x) = \int_{x=0}^{2} x f(x)\,dx = \int_{x=0}^{1} x f(x)\,dx + \int_{1}^{2} x f(x)\,dx = \int_{x=0}^{1} x\, 2x^2\,dx + \int_{x=1}^{2} x(4-4x+x^2)\,dx$

$E(x) = 2\int_{x=0}^{1} x^3\,dx + \int_{x=1}^{2} (4x-4x^2+x^3)\,dx = 2\frac{x^4}{4}\Big|_0^1 + \left[4\frac{x^2}{2} - 4\frac{x^3}{3} + \frac{x^4}{4}\right]_1^2$

$= \frac{1}{2}(1-0) + 2(4-1) - \frac{4}{3}(8-1) + \frac{1}{4}(16-1) = \frac{1}{2} + 6 - \frac{28}{3} + \frac{15}{4} = \frac{11}{12}$

$E(x) = \frac{11}{12} = \text{mean}$

$E(x^2) = \int_{x=0}^{2} x^2 f(x)\,dx = \int_{x=0}^{1} x^2 f(x)\,dx + \int_{x=1}^{2} x^2 f(x)\,dx$

$E(x^2) = \int_{x=0}^{1} x^2\, 2x^2\,dx + \int_{x=1}^{2} x^2(4-4x+x^2)\,dx = 2\int_{x=0}^{1} x^4\,dx + \int_{x=1}^{2} (4x^2-4x^3+x^4)\,dx$

$E(x^2) = 2\frac{x^5}{5}\Big|_0^1 + \left[4\frac{x^3}{3} - 4\frac{x^4}{4} + \frac{x^5}{5}\right]_1^2$

$= \frac{2}{5}(1-0) + \frac{4}{3}(8-1) - (16-1) + \frac{1}{5}(32-1)$

$= \frac{2}{5} + \frac{28}{3} - 15 + \frac{31}{5} = \frac{14}{15}$

$\therefore$ variance $= E(x)^2 - (E(x))^2 = \frac{14}{15} - \left(\frac{11}{12}\right)^2 = \frac{67}{720} = 0.0931$

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Example - 3 - ii A continuous random variable X has p.d.f. $f(x) = kx^2$, $0 \le x \le 2$. Determine k, $P(0.2 \le x \le 0.5)$ and $P(X \geq 0.75 / x > 0.5)$

Solution : total Probability = 1

$$\therefore \int_0^2 kx^2 dx = 1 \Rightarrow k \frac{x^3}{3}\Big|_0^2 = 1 \Rightarrow \frac{k}{3}(8-0) = 1 \Rightarrow k = \frac{3}{8}$$

$$\therefore f(x) = \frac{3}{8}x^2 \quad 0 \le x \le 2$$

$$P(0.2 \le x \le 0.5) = \int_{0.2}^{0.5} f(x) dx = \int_{x=0.2}^{0.5} \frac{3}{8}x^2 dx = \frac{3}{8}\frac{x^3}{3}\Big|_{0.2}^{0.5}$$

$$= \frac{1}{8}\left[(0.5)^3 - (0.2)^3\right] = 0.014625$$

$$P(x \geq 0.75 / x \geq 0.5) = P(0.75 < x / 0.5 \le x) = \cancel{P(0.75 < x < 2}$$

$$= P(0.75 < x < 2 / 0.5 \le x \le 2)$$

$$= \frac{P\left[(0.75 < x < 2) \cap (0.5 \le x \le 2)\right]}{P(0.5 \le x \le 2)}$$

$$= \frac{P(0.75 < x < 2)}{P(0.5 \le x \le 2)} \quad \text{— (1)}$$

Now

$$P(0.75 < x < 2) = \int_{0.75}^{2} f(x) dx = \int_{x=0.75}^{2} \frac{3}{8}x^2 dx = \frac{3}{8}\frac{x^3}{3}\Big|_{0.75}^{2}$$

$$= \frac{1}{8}(8 - (0.75)^3) = 0.9473 \quad \text{— (2)}$$

$$P(0.5 < x < 2) = \int_{0.5}^{2} f(x) dx = \int_{\cancel{0.75}\ 0.5}^{2} \frac{3}{8}x^2 dx = \frac{3}{8}\frac{x^3}{3}\Big|_{0.5}^{2}$$

$$= \frac{1}{8}(8 - (0.5)^3) = 0.984375 \quad \text{— (3)}$$

From (1), (2) & (3)

$$\therefore P(x \geq 0.75 / x \geq 0.75) = \frac{0.9473}{0.984375} = 0.962337$$

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Example-4:- The milage (in thousand of miles) which car owners get with a certain kind of types tyres is a random variable having p.d.f.

$$f(x) = \frac{1}{20} e^{-\frac{x}{20}} \quad x>0$$
$$= 0 \quad x \le 0$$

Find the probability that one of these tyres will last
(i) at most 10000 miles (ii) anywhere from 16000 to 24000 miles

Solution

$$\cancel{P(X<10000) = P(-\infty < x < 10000) = \int_{x=-\infty}^{10000} f(x)\,dx}$$
$$\cancel{= \int_{x=-\infty}^{0} f(x)\,dx + \int_{x=0}^{10000} f(x)\,dx = \int_{x=-\infty}^{0} 0\,dx + \int_{x=0}^{10000} \frac{1}{20} e^{-\frac{x}{20}}\,dx}$$
$$\cancel{= \frac{1}{20} \frac{e^{-\frac{x}{20}}}{-\frac{1}{20}} \Big|_0^{10000}} = -\left[e^{-\frac{10000}{20}} - 1\right] =$$

(i) $P(\text{at most } 10000 \text{ miles}) = P(x<10) = P(-\infty < x < 10) = \int_{-\infty}^{10} f(x)\,dx$

$$= \int_{x=-\infty}^{0} f(x)\,dx + \int_{x=0}^{10} f(x)\,dx = \int_{x=-\infty}^{0} 0\,dx + \int_{x=0}^{10} \frac{1}{20} e^{-\frac{x}{20}}\,dx$$
$$= 0 + \frac{1}{20} \frac{e^{-\frac{x}{20}}}{-\frac{1}{20}} \Big|_0^{10} = -\left[e^{-\frac{1}{2}} - 1\right] = \left(1 - e^{-0.5}\right)$$

(ii) $P(\text{anywhere from } 16000 \text{ to } 24000 \text{ miles})$

$$= P(16 < x < 24) = \int_{16}^{24} f(x)\,dx = \int_{16}^{24} \frac{1}{20} e^{-\frac{x}{20}}\,dx$$
$$= \frac{1}{20} \frac{e^{-\frac{x}{20}}}{-\frac{1}{20}} \Big|_{16}^{24} = -\left[e^{-\frac{24}{20}} - e^{-\frac{16}{20}}\right] = \left(e^{-\frac{4}{5}} - e^{-\frac{6}{5}}\right)$$
$$= \left[e^{-0.8} - e^{-1.2}\right]$$