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LAPLACE TRANSFORM

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ENGINEERING MATHEMATICS-III (CMPN/INFT)

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1. Laplace Transform (20 Marks)
2. Inverse Laplace Transform (20 Marks)
3. Fourier Series (20 Marks)
4. Complex Variables (20 Marks)
5. Statistical Techniques (20 Marks)
6. Probability (20 Marks)

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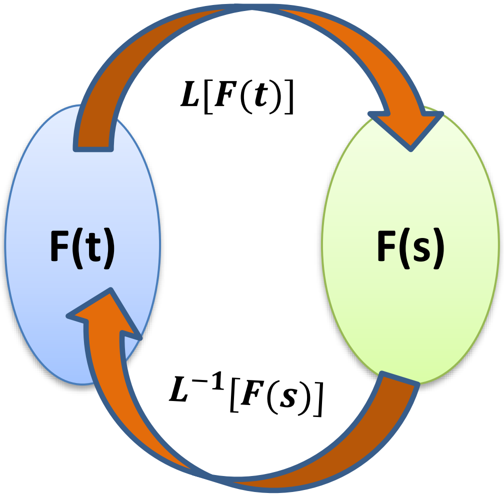
- Transformation:



- To evaluate Special type of Integration.
- To Solve Linear Ordinary and Partial Differential Equations with Const. Coefficients.

  Linear differential equation → Algebraic equation

  Under boundary conditions.

WHAT ARE THE APPLICATIONS OF L. TRANSFORM ?

- While sending signals over any two-way communication medium:
  (FM/AM stereo, 2-way radio sets, cellular phones).
  Cellular phones → Time-varying wave →Super-imposed on the medium→ propagate → At the receiving end, Medium wave's time functions → frequency functions
- Control systems: Analysis of HVAC (Heating, Ventilation and Air Conditioning)
- Modern Buildings and Constructions.
- Analysis of Electrical and Electronic Circuits.
- Digital Signal Processing.
- Nuclear Physics → True form of radioactive decay
- Determine structure of astronomical object from spectrum.
- Integrated circuit: To build required ICs and chips for systems.

WHAT IS LAPLACE TRANSFORM ?

- French Mathematician **Pierre-Simon Laplace** in 19th century
- **Definition LT** : If f(t)→Real valued function, *s* → Complex parameter

$$L[f(t)] = f(s)$$

Bilateral $(-\infty \le t \le \infty)$

$$\int_{t=-\infty}^{t=\infty} e^{-st} f(t)\, dt$$

Unilateral $(0 \le t \le \infty)$

$$\int_{t=0}^{t=\infty} e^{-st} f(t)\, dt$$

- **Definition Inverse LT:**

$$L^{-1}[f(s)] = f(t) = \frac{1}{2\pi i}\int_{s=\sigma-i\omega}^{s=\sigma+i\omega} f(s)\, e^{st} ds$$

CONDITION FOR EXISTENCE OF L. TRANSFORM

- If $f(t)$, $t \geq 0$ be Piecewise continuous on $[0, \infty)$ and of Exponential order a , then
  $L[f(t)] = f(s)$ exists for $s > a \geq 0$

- Piecewise continuous function: If $0 \leq \alpha \leq t \leq \beta$,there at most a finite number of points t_k , $\forall\ k = 1, 2 \ldots\ldots t_{k-1} < t_k$, at which $f(t)$ has finite discontinuities/jumps and is continuous on each interval $t_{k-1} < t < t_k$, then $f(t) \rightarrow$ *Piecewise cont. fun.*

- Exponential order a : If $f(t)$, $t \geq 0$ is said to be *Exponential order* a , if $\exists$ constants $a, k > 0,\ T > 0$ such that $|f(t)| \leq ke^{at}$, $\forall\ t > T$

- Hence , it means that $f(t)$ of Exponential order a , then

$$\lim_{t\to\infty} |f(t)e^{-at}| \leq k = \text{finite}$$

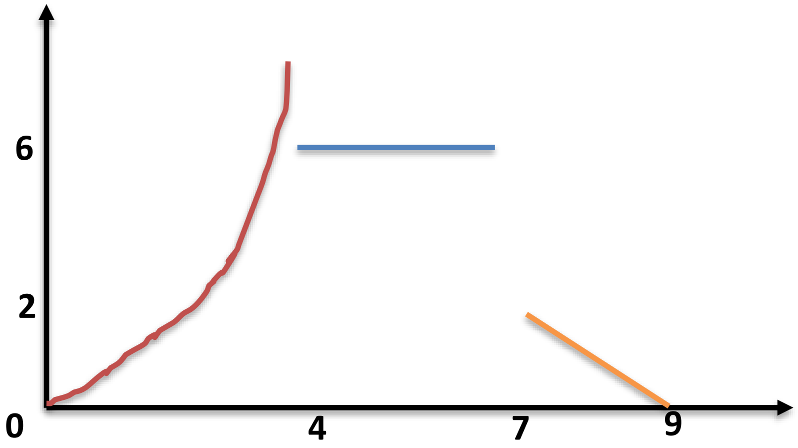
- Ex. $e^{\pm at}$, $Sin(at)$, $Cos(at)$, $Sinh(at)$, $Cosh(at)$ $\forall\ a \in \Re$ and $t^n\ \forall\ n > -1$

- **Continuous function :**

$$\lim_{t \to a} f(t) = f(a) \qquad \lim_{t \to a^+} f(t) = \lim_{t \to a^-} f(t) = f(a) = finite$$

- **Piecewise continuous function:** If $0 \le \alpha \le t \le \beta$,there at most a finite number of points t_k , $\forall\, k = 1, 2 \ldots \ldots t_{k-1} < t_k$, at which $f(t)$ has finite discontinuities/jumps and is continuous on each interval $t_{k-1} < t < t_k$, then $f(t)$→ ***Piecewise cont. fun.***

Ex. Let $f(t) = \begin{cases} t^2 & 0 \le t < 4 \\ 6 & 4 \le t < 7 \\ (9-t) & 7 \le t \le 9 \end{cases}$



EXPONENTIAL ORDER FUNCTIONS

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- **Exponential order a** : If $f(t)$, t ≥ 0 is said to be *Exponential order* a , if $\exists$ constants $a\,, k > 0,\ T > 0$ such that $|f(t)| \leq ke^{at}\,,\ \forall\ t > T$

➢ Hence , it means that $f(t)$ of Exponential order a , then

$$\lim_{t\to\infty} |f(t)e^{-at}| \leq\ k = \text{finite}$$

- **Ex.1.** If $f(t) = e^{2t}\, sin(2t)$ is of exponential order 2 for $t \geq 0$

➢ $|e^{2t}\, sin(2t)| = |e^{2t}|\ |sin(2t)| \ \leq\ e^{2t}$

$|sin(2t)| \leq 1$

Ex.2. If $f(t) = Cos(2t)$ is of exponential order 1, since $|Cos(2t)| \leq\ e^{t},\ t \geq 0$

Ex.3 If $f(t) = t^2$ is of exponential order 3, since $|t^2| \leq\ e^{3t},\ \ t \geq 0$

Ex.3 If $f(t) = t$ is of exponential order 1 , since $|t| \leq\ e^{t},\ \ t \geq 0$

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Example of a function not of ANY exponential order "a"

Let $f(t) = e^{t^2}$

$$\lim_{t\to\infty} |f(t)\, e^{-at}| \leq k = \text{finite}$$

$$\lim_{t\to\infty} \left|e^{t^2}\, e^{-at}\right| = \lim_{t\to\infty} \left| e^{\left(t-\frac{1}{2}a\right)^2 - \frac{a^2}{4}} \right|$$

$= \infty$, for any 'a'

Sufficient conditions for Existence of Laplace transform:

If $f(t)$, $t \geq 0$ be Piecewise continuous on $[0, \infty)$ and of Exponential order a , then

$L[f(t)] = f(s)$ exists for $s > a \geq 0$

- **Ex.** $e^{\pm at}$, $Sin(at)$, $Cos(at)$, $Sinh(at)$, $Cosh(at)$ $\forall\ a \in \Re$
  t^n $\forall\ n > -1$

PROOF : EXISTENCE THEOREM OF L. TRANSFORM

$$L[f(t)] = f(s) = \int_{t=0}^{t=\infty} e^{-st} f(t)\, dt$$

$$L[f(t)] = f(s) = \int_{t=0}^{t=T} e^{-st} f(t)\, dt + \int_{t=T}^{t=\infty} e^{-st} f(t)\, dt \;,\; T > t \geq 0$$

$$\text{Let } I = \left| \int_{t=T}^{t=\infty} e^{-st} f(t)\, dt \right| \leq \int_{t=T}^{t=\infty} |e^{-st} f(t)|\, dt \leq \int_{t=0}^{t=\infty} |e^{-st} f(t)|\, dt$$

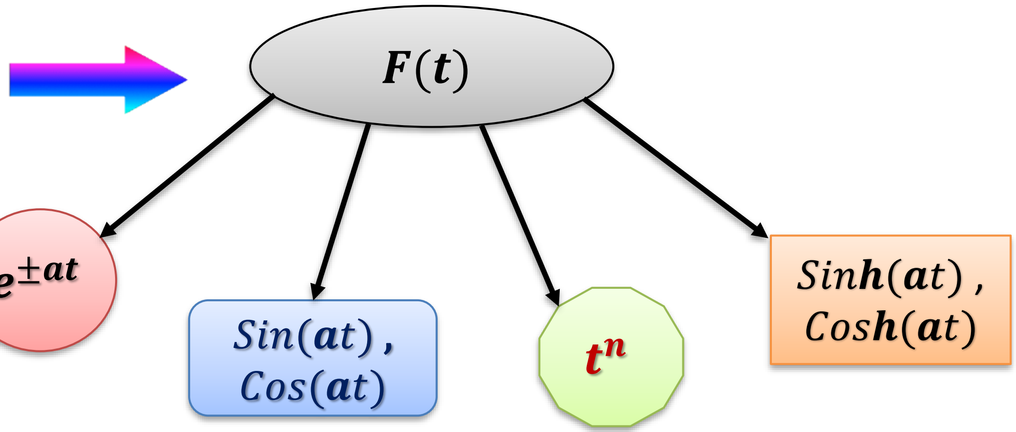
$$I \leq \int_{t=0}^{t=\infty} |e^{-st}|\; |f(t)|\, dt \quad \leq \int_{t=0}^{t=\infty} e^{-st}\, k e^{at} dt$$

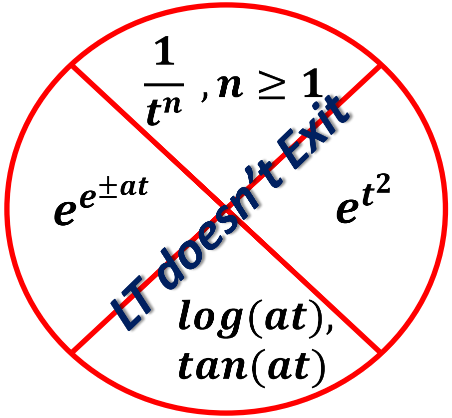
$$\boxed{|f(t)| \leq k e^{at}}$$

$$I \leq \int_{t=0}^{t=\infty} k\, e^{-(s-a)t}\, dt \;\leq k \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_{t=0}^{t=\infty} \leq \frac{k}{(s-a)} = finite$$

SUMMARY OF LECTURE

- ✓ Mathematical Operator and denoted by $L[f(t)]$
- ✓ Three main aspects → Transformation , Evaluation of integral, Differential Equations.
- ✓ Definitions of LT → Bilateral, Unilateral
- ✓ Existence conditions → Piecewise cont. , Exponential order
- ✓ Four types of functions:





Linear Property of LT:

$$L[k_1 f_1(t) \pm k_2 f_2(t)] = k_1\, L[f_1(t)] \pm k_2\, L[f_2(t)]$$

1. Let $f(t) = e^{at}$

$$L[f(t)] = L[e^{at}] = f(s) = \int_{t=0}^{t=\infty} e^{-st} e^{at}\, dt$$

$$L[e^{at}] = \int_{t=0}^{t=\infty} e^{-(s-a)t}\, dt$$

$$L[e^{at}] = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_{t=0}^{t=\infty} = \left[o - \frac{1}{-(s-a)}\right] = \frac{1}{(s-a)},\ s > a$$

$$L[e^{-at}] = \frac{1}{(s+a)}$$

$$L[e^{-0t}] = L[1] = \frac{1}{(s)},\ s > 0$$

LT OF STANDARD FUNCTIONS…

- Let $f(t) = t^n$, Then $L[t^n] = f(s) = \int_{t=0}^{t=\infty} e^{-st}\, t^n\, dt$
- Put st $= u \Rightarrow dt = \frac{du}{s}$, Then $L[t^n] = \int_{u=0}^{u=\infty} e^{-u}\left(\frac{u^n}{s^n}\right)\frac{du}{s}$
- $L[t^n] = \frac{1}{s^{n+1}}\int_{u=0}^{u=\infty} e^{-u}\, u^{(n+1)-1} du$

$$\boxed{L[t^n] = \frac{1}{s^{n+1}}\Gamma(n+1) = \frac{n!}{s^{n+1}},\quad n > -1}$$

Ex. $L\left[\, t^{\frac{-1}{2}}\,\right] =$ $\boxed{\frac{1}{s^{\frac{-1}{2}+1}}\Gamma\left(-\frac{1}{2}+1\right) = \frac{1}{s^{\frac{1}{2}}}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{s}}}$

- Let $f(t) = Sin(at)\ \ or\ \ Cos(at)$

LT OF STANDARD FUNCTIONS…

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- $Cos(at) + i\,Sin(at) = e^{i(at)}$
- $L[Cos(at) + i\,Sin(at)] = L[e^{(ia)t}]$
- $L[Cos(at)] + i\,L[\,Sin(at)] = \frac{1}{(s-ia)} = \frac{(s+ia)}{(s-ia)(s+ia)} = \frac{(s+ia)}{(s^2+a^2)}$
- $L[Cos(at)] = \frac{(s)}{(s^2+a^2)}$ and $L[\,Sin(at)] = \frac{(a)}{(s^2+a^2)}$
- $L[Cosh(at)] = L\left[\frac{e^{at}+e^{-at}}{2}\right] = \frac{1}{2}\left[\frac{1}{(s-a)} + \frac{1}{(s+a)}\right] = \frac{1}{2}\left[\frac{(s+a+s-a)}{(s-a)(s+a)}\right] = \frac{(s)}{(s^2-a^2)}$
- $L[Sinh(at)] = L\left[\frac{e^{at}-e^{-at}}{2}\right] = \frac{1}{2}\left[\frac{1}{(s-a)} - \frac{1}{(s+a)}\right] = \frac{(a)}{(s^2-a^2)}$

SUMMARY OF LECTURE

$f(t)$	$L[f(t)] = f(s)$	$f(t)$	$L[f(t)] = f(s)$
e^{at}	$\frac{1}{(s-a)}$	e^{-at}	$\frac{1}{(s+a)}$
1	$\frac{1}{(s)}$	t^n	$\frac{1}{s^{n+1}}\Gamma(n+1) = \frac{n!}{s^{n+1}}$
$Sin(at)$	$\frac{(a)}{(s^2+a^2)}$	$Cos(at)$	$\frac{(s)}{(s^2+a^2)}$
$Sinh(at)$	$\frac{(a)}{(s^2-a^2)}$	$Cosh(at)$	$\frac{(s)}{(s^2-a^2)}$

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1. First Shifting Property (Frequency Shift Prop.): If $L[f(t)] = f(s)$ Then,

$$L[e^{at}\, f(t)] = L[f(t)]_{s \to (s-a)} = f(s)_{s\to(s-a)} = f(s-a)$$

$$L[e^{-at}\, f(t)] = f(s)_{s\to(s+a)} = f(s+a)$$

Ex. $L[e^{5t}\, Sin(2t)] = \left[\frac{(2)}{(s^2+2^2)}\right]_{s\to(s-5)} = \frac{(2)}{[(s-5)^2+2^2]}$

2. Second Shifting Prop.: If $L[f(t)] = f(s)$ and $g(t) = \begin{cases} 0 & t<a \\ f(t-a) & t>a \end{cases}$ Then,

$$L[g(t)] = g(s) = e^{-as}\, L[f(t)]$$

- Ex. If $L[f(t)] = L[t^{100}] = \frac{100!}{s^{101}}$ and $g(t) = \begin{cases} 0 & t < 3 \\ f(t-3) & t > 3 \end{cases}$, then

$$L[g(t)] = g(s) = e^{-3s}\ \frac{100!}{s^{101}}$$

3. Change of Scale Prop.: If $L\,[f(t)] = f(s)$ Then,

$$L\,[f(at)] = \frac{1}{a} L[f(t)]_{s \to \frac{s}{a}} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

- Ex. If $L\,[f(t)] = L\,[erf(\sqrt{t})] = \frac{1}{S\sqrt{(S+1)}}$, Then Find $L\,[erf(5\sqrt{t})]$

- $f(25t) = erf(\sqrt{25\,t}) = erf(5\sqrt{t})$, Where $a = 25$

$$L\,[erf(5\sqrt{t})] = \frac{1}{25} \frac{1}{\frac{s}{25}\sqrt{\left(\frac{s}{25} + 1\right)}}$$

$$L\,[erf(5\sqrt{t})] = \frac{5}{s\sqrt{(s+25)}}$$

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4. Multiple by t^n Prop.: If $L\,[f(t)] = f(s)$ Then,

$$L\,[\,t^n\, f(t)] = (-1)^n \frac{d^n}{ds^n}\, L\,[f(t)]$$

Ex. $L[t\, Sinh(t)] = (-1)\frac{d}{ds}\left[\frac{(1)}{(s^2-1^2)}\right] = \frac{(2s)}{[(s)^2-1^2]^2}$

5. Division by t Prop.: If $L\,[f(t)] = f(s)$ Then,

$$L\left[\frac{f(t)}{t}\right] = \int_{s=s}^{s=\infty} L\,[f(t)]\;ds$$

$$L\left[\frac{f(t)}{t^3}\right] = \int_{s=s}^{s=\infty}\int_{s=s}^{s=\infty}\int_{s=s}^{s=\infty} L\,[f(t)]\,ds\,ds\,ds$$

6. Differentiation Prop.: If $L\,[f(t)] = f(s)$ Then,

$$L\left[\frac{d}{dt} f(t)\right] = s\,f(s) - f(0)$$

$$L\left[\frac{d^2}{dt^2} f(t)\right] = s^2\,f(s) - sf(0) - f'(0)$$

$$L\left[\frac{d^3}{dt^3} f(t)\right] = s^3\,f(s) - s^2\,f(0) - s\,f'(0) - f''(0)$$

7. Integration Prop.: If $L\,[f(t)] = f(s)$ Then,

$$L\left[\int_{t=0}^{t=t} f(t)\,dt\right] = \frac{1}{(s)}\,L\,[f(t)]$$

$$L\left[\int_{t=0}^{t=t}\int_{t=0}^{t=t}\int_{t=0}^{t=t} f(t)\,dt\,dt\,dt\right] = \frac{L\,[f(t)]}{s^3}$$

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Ex.1 Find $L\,[f(t)] = L\left[\frac{Cos(2t)Sin(t)}{e^{t}}\right]$ **[Extc-Nov-2019 (5M)]**

- $L\,[f(t)] = L\,[e^{-t}\,(Cos(2t)Sin(t))]$ **[FSP]**
- $= L\,[(Cos(2t)Sin(t))]_{s \to (s+1)}$ $\quad 2\,C_A\,S_B = S_{(A+B)} - S_{(A-B)}$
- $= \frac{1}{2}L\,[Sin(3t) - Sin(t)]_{s \to (s+1)}$
- $= \frac{1}{2}\,[L\{Sin(3t\} - L\{Sin(t)\}]_{s \to (s+1)}$ **{Linear prop}**
- $= \frac{1}{2}\left[\frac{(3)}{(s^2+3^2)} - \frac{(1)}{(s^2+1^2)}\right]_{s \to (s+1)} = \frac{1}{2}\left[\frac{(3)}{[(s+1)^2+3^2]} - \frac{(1)}{[(s+1)^2+1^2]}\right]$

Ex.2 Find $L\,[f(t)] = L\left[e^t\, t\,\sqrt{1+Sin(8t)}\right]$ **[Biom-Nov-2019 (5M)]**

- $= L\left[t\sqrt{1+Sin(8t)}\right]_{s\to(s-1)}$ **[FSP]**
- $= (-1)\left\{\frac{d}{ds}L\left[\sqrt{1+Sin(8t)}\right]\right\}_{s\to(s-1)}$

$\sqrt{1+Sin(8t)} = \sqrt{(Sin(4t)+Cos(4t))^2}$

- $= (-1)\left\{\frac{d}{ds}L[Sin(4t)+Cos(4t)]\right\}_{s\to(s-1)}$
- $= (-1)\left\{\frac{d}{ds}\left[\frac{(4)}{(s^2+4^2)}+\frac{(s)}{(s^2+4^2)}\right]\right\}_{s\to(s-1)} = (-1)\left\{\frac{d}{ds}\left[\frac{(4+S)}{(s^2+4^2)}\right]\right\}_{s\to(s-1)}$
- $= (-1)\left\{\frac{(s^2+4^2)1-(4+S)2s}{(s^2+4^2)^2}\right\}_{s\to(s-1)} = (-1)\left\{\frac{(-s^2-8S+16)}{(s^2+4^2)^2}\right\}_{s\to(s-1)} = \left\{\frac{\left((s-1)^2+8(s-1)-16\right)}{((s-1)^2+4^2)^2}\right\}$

Ex.3 Find $L\,[f(t)] = L\left[100^{4t}\right]$

$$100^{4t} = e^{log(100^{4t})} = e^{(4t)log(100)} \qquad x = e^{\log x}$$

$$L[100^{4t}] = L[e^{(4log100)t}] = \frac{1}{(S-4log100)}$$

Ex.4 Find $L\,[f(t)] = L\left[e^{-2t}\int_{t=0}^{t=t} t\,e^{3t}\,Cos(4t)\,dt\right]$ **[Extc-Nov-18(6M)]**

- $L\,[f(t)] = L\left[\int_{t=0}^{t=t} t\,e^{3t}\,Cos(4t)\,dt\right]_{s\to(s+2)}$ **[FSP]**
- $= \left\{\frac{1}{S}\,L\left[t\,e^{3t}\,Cos(4t)\right]\right\}_{s\to(s+2)}$ .....(1) **[Integral prop.]**
- Let $L\left[e^{3t}\,t\,Cos(4t)\right] = L[t\,Cos(4t)\,]_{s\to(s-3)} = \left\{(-1)\frac{d}{ds}L[\,Cos(4t)\,]\right\}_{s\to(s-3)}$

$$= \left\{(-1)\frac{d}{ds}\left[\frac{(s)}{(s^2+4^2)}\right]\right\}_{s\to(s-3)} = (-1)\left\{\frac{(-s^2+16)}{(s^2+4^2)^2}\right\}_{s\to(s-3)} = \left\{\frac{\left((s-3)^2-16\right)}{\left((s-3)^2+4^2\right)^2}\right\} \quad ....(2)$$

$$= \left\{\frac{1}{S}\left\{\frac{\left((s-3)^2-16\right)}{\left((s-3)^2+4^2\right)^2}\right\}\right\}_{s\to(s+2)} \qquad L\,[f(t)] = \left\{\frac{1}{(s+2)}\left\{\frac{\left((s-1)^2-16\right)}{\left((s-1)^2+4^2\right)^2}\right\}\right\}$$

Ex.5 Evaluate; $I = \int_{t=0}^{t=\infty} e^{-2t}\left[\frac{Cos(4t)-Cos(5t)}{t}\right] dt$ **[Biom-May-19(6M)]**

- $I = \int_{t=0}^{t=\infty} e^{-2t}\left[\frac{Cos(4t)-Cos(5t)}{t}\right] dt = \left[L\left[\frac{Cos(4t)-Cos(5t)}{t}\right]\right]_{S=2}$ (Defin of LT)

- $I = \left[\int_{s=s}^{s=\infty} L\{Cos(4t)-Cos(5t)\}\, ds\right]_{S=2} = \left\{\int_{s=s}^{s=\infty}\left[\frac{(s)}{(s^2+4^2)} - \frac{(s)}{(s^2+5^2)}\right] ds\right\}_{s=2}$

$$I = \left\{\frac{1}{2}\left[\log\left(s^2+16\right)-\log\left(s^2+25\right)\right]_{S=s}^{S=\infty}\right\}_{s=2} = \left\{\frac{1}{2}\left[\log\left(\frac{s^2+16}{s^2+25}\right)\right]_{S=s}^{S=\infty}\right\}_{s=2}$$

$$I = \left\{\frac{1}{2}\left[\log\left(\frac{1+\frac{16}{s^2}}{1+\frac{25}{s^2}}\right)\right]_{S=s}^{S=\infty}\right\}_{s=2} = \left\{\frac{1}{2}\left[\log 1-\log\left(\frac{1+\frac{16}{s^2}}{1+\frac{25}{s^2}}\right)\right]\right\}_{s=2}$$

$$I = \left\{\frac{1}{2}\left[0-\log\left(\frac{s^2+16}{s^2+25}\right)\right]\right\}_{s=2} = \left\{\frac{1}{2}\left[\log\left(\frac{s^2+25}{s^2+16}\right)\right]\right\}_{s=2}$$

$$I = \left\{\frac{1}{2}\left[\log\left(\frac{4+25}{4+16}\right)\right]\right\} = \left\{\frac{1}{2}\left[\log\left(\frac{29}{20}\right)\right]\right\}$$

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Ex.6 Find $L\left[e^{3t}Sin^3(t)\right]$

- $L[f(t)] = L\left[Sin^3(t)\right]_{s\to(s-3)}$

- $= \frac{1}{4}L\left[3Sin(t) - Sin(3t)\right]_{s\to(s-3)}$

$Sin(3t) = 3Sin(t) - 4[Sin^3(t)]$

- $= \frac{1}{4}\left[3\frac{(1)}{(s^2+1^2)} - \frac{(3)}{(s^2+3^2)}\right]_{s\to(s-3)}$

- $= \frac{1}{4}\left[3\frac{(1)}{((s-3)^2+1^2)} - \frac{(3)}{((s-3)^2+3^2)}\right]$

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Ex.7 Find $L\left[e^{-t}\int_{t=0}^{t=t}\frac{Sin(at)}{t}\,dt\right]$

- $L[f(t)] = L\left[\int_{t=0}^{t=t}\frac{Sin(at)}{t}\,dt\right]_{s\to(s+1)} = \left\{\frac{1}{s}L\left[\frac{Sin(at)}{t}\right]\right\}_{s\to(s+1)}$ .....(1)
- $L\left[\frac{Sin(at)}{t}\right] = \int_{s=s}^{s=\infty} L\{Sin(at)\}\,ds = \int_{s=s}^{s=\infty}\frac{(a)}{(s^2+a^2)}\,ds$
- $= \left[\tan^{-1}\left(\frac{S}{a}\right)\right]_{S=S}^{S=\infty} = \frac{\pi}{2} - \tan^{-1}\left(\frac{S}{a}\right) = \cot^{-1}\left(\frac{S}{a}\right)$ ....(2)
- $= \frac{1}{(s+1)}\cot^{-1}\left(\frac{S+1}{a}\right)$

EXAMPLES : LAPLACE TRANSFORMS

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Ex. 8 Find I) $L\left[Sin\left(\sqrt{t}\right)\right]$ **[ETRX -May-18 (5M)]** **II)** $L\left[\frac{Cos\left(\sqrt{t}\right)}{\sqrt{t}}\right]$ Ans$\left[\frac{\sqrt{\pi}}{\sqrt{S}}\, e^{-\frac{1}{4S}}\right]$

$$Sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$L[t^n] = \frac{1}{s^{n+1}} \left\lceil(n+1)\right.$$

$$\left\lceil(n+1) = n \left\lceil n\right.\right.$$

- Let $L[f(t)] = L\left[Sin\left(\sqrt{t}\right)\right]$
- $L\left[Sin\left(\sqrt{t}\right)\right] = L\left[t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots\right]$
- $= \frac{\left\lceil\frac{1}{2}+1\right.}{S^{3/2}} - \frac{\left\lceil\frac{3}{2}+1\right.}{3!\, S^{5/2}} + \frac{\left\lceil\frac{5}{2}+1\right.}{5!\, S^{7/2}} - \frac{\left\lceil\frac{7}{2}+1\right.}{7!\, S^{9/2}} + \cdots$
- $\frac{\frac{1}{2}\left\lceil\frac{1}{2}\right.}{S^{3/2}} - \frac{\frac{3}{2}\left\lceil\frac{3}{2}\right.}{(3\times 2)\, S^{3/2}\, s^1} + \frac{\frac{5}{2}\left\lceil\frac{5}{2}\right.}{(5\times4\times3\times2) S^{3/2}\, S^2} - \frac{\frac{7}{2}\left\lceil\frac{7}{2}\right.}{(7\times6\times5\times4\times3\times2) S^{3/2}\, S^3} + \cdots$

$$= \frac{\frac{1}{2}\left\lceil\frac{1}{2}\right.}{s^{3/2}} - \frac{\frac{3}{2}\,\frac{1}{2}\left\lceil\frac{1}{2}\right.}{(3\times 2)\, s^{3/2}\, s^{1}} + \frac{\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\left\lceil\frac{1}{2}\right.}{(5\times4\times3\times2) s^{3/2}\, s^{2}} - \frac{\frac{7}{2}\,\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\left\lceil\frac{1}{2}\right.}{(7\times6\times5\times4\times3\times2) s^{3/2}\, s^{3}} + \cdots$$

$$= \frac{\frac{1}{2}\left\lceil\frac{1}{2}\right.}{s^{3/2}}\left[1 - \frac{1}{4S} + \frac{1}{2!}\left(\frac{1}{4S}\right)^2 - \frac{1}{3!}\left(\frac{1}{4S}\right)^3 + \cdots\right]$$

$$e^{-x} = 1 - x + \frac{1}{2!}(x)^2 - \frac{1}{3!}(x)^3 + \cdots$$

$$= \frac{\sqrt{\pi}}{2\, s^{3/2}}\, e^{-\frac{1}{4S}}$$

$$\left\lceil\frac{1}{2}\right. = \sqrt{\pi}$$

Ex.9 Find $L[f(t)]$ **where** $f(t) = \begin{cases} Cos(t) & 0 < t < \pi \\ Sin(t) & \pi < t < \infty \end{cases}$ **[MU 02, 05,09,17]**

- $L[f(t)] = f(s) = \int_{t=0}^{t=\infty} e^{-st} f(t)\, dt$
- $L[f(t)] = \int_{t=0}^{t=\pi} e^{-st} Cos(t)\, dt + \int_{t=\pi}^{t=\infty} e^{-st} Sin(t) dt$
- **But,** $\int e^{-at} Cos(bt \pm c)\, dt = \frac{e^{-at}}{(a^2+b^2)} \{-a\, Cos(bt \pm c) + b\, Sin(bt \pm c)\}$
- $L[f(t)] = \left[\frac{e^{-St}}{(s^2+1^2)} \{-s\, Cos(t) + 1\, Sin(t)\}\right]_{t=0}^{t=\pi} + \left[\frac{e^{-St}}{(s^2+1^2)} \{-s\, Sin(t) - 1\, Cos(t)\}\right]_{t=\pi}^{t=\infty}$
- $L[f(t)] = \left[\frac{e^{-S\pi}}{(s^2+1^2)} \{-s\, Cos(\pi) + 1\, Sin(\pi)\} - \frac{e^{-0}}{(s^2+1^2)} \{-s\, Cos(0) + 1\, Sin(0)\}\right] +$
  $\left[\frac{e^{-\infty}}{(s^2+1^2)}\{-s\, Sin(t) - 1\, Cos(t)\} - \frac{e^{-s\pi}}{(s^2+1^2)}\{-s\, Sin(\pi) - 1\, Cos(\pi)\}\right]$
- $= \frac{1}{(s^2+1^2)} \{s\, e^{-S\pi} + s - e^{-S\pi}\}$

Ex.10 Evaluate; $I = \int_{t=0}^{t=\infty} e^{-\sqrt{2}t}\left[\frac{Sin(t)Sinh(t)}{t}\right] dt$ **[ETRX -May-19, 16,11,07,05,02]**

- $I = \int_{t=0}^{t=\infty} e^{-\sqrt{2}t}\left[\frac{Sin(t)Sinh(t)}{t}\right] dt = \left[L\left[\frac{Sin(t)Sinh(t)}{t}\right]\right]_{S=\sqrt{2}}$ (Defin of LT)

- $I = \left[\int_{S=s}^{S=\infty} L\{Sin(t)Sinh(t)\}\, ds\right]_{S=\sqrt{2}}$

- $I = \left[\int_{S=s}^{S=\infty} L\left\{Sin(t)\left[\frac{e^{t}-e^{-t}}{2}\right]\right\} ds\right]_{S=\sqrt{2}}$

- $I = \left[\int_{S=s}^{S=\infty} \frac{1}{2}\{L[e^{t}Sin(t)] - L[e^{-t}Sin(t)]\}\, ds\right]_{S=\sqrt{2}}$

- $I = \frac{1}{2}\left[\int_{S=s}^{S=\infty} \left\{\frac{(1)}{[(s-1)^2+1^2]} - \frac{(1)}{[(s+1)^2+1^2]}\right\} ds\right]_{S=\sqrt{2}}$

- $= \frac{1}{2}\left\{[\tan^{-1}(s-1) - \tan^{-1}(s+1)]_{S=S}^{S=\infty}\right\}_{s=\sqrt{2}}$
- $= \frac{1}{2}\left\{\left[\frac{\pi}{2} - \tan^{-1}(s-1) - \frac{\pi}{2} + \tan^{-1}(s+1)\right]\right\}_{s=\sqrt{2}}$
- $= \frac{1}{2}\{[\tan^{-1}(s+1) - \tan^{-1}(s-1)]\}_{s=\sqrt{2}}$

$\tan^{-1}(A) - \tan^{-1}(B) = \tan^{-1}\left(\frac{A-B}{1+AB}\right)$

- $= \frac{1}{2}\left\{\left[\tan^{-1}\left(\frac{(s+1)-(s-1)}{1+(s+1)(s-1)}\right)\right]\right\}_{s=\sqrt{2}}$
- $= \frac{1}{2}\left\{\left[\tan^{-1}\left(\frac{2}{1+s^2-1}\right)\right]\right\}_{s=\sqrt{2}} = \frac{1}{2}\left\{\left[\tan^{-1}\left(\frac{2}{2}\right)\right]\right\} = \frac{1}{2}\left\{\frac{\pi}{4}\right\}$
- $= \frac{\pi}{8}$

Ex.11 Find $L[f(t)]$ for $f(t) = |t-1| + |t+1|$ for $t \geq 0$ [NIT Kurukshetra 2003, IITD 2010]

- We have $f(t) = \begin{cases} 2 & 0 \leq t \leq 1 \\ 2t & 1 < t < \infty \end{cases}$

- $L[f(t)] = f(s) = \int_{t=0}^{t=\infty} e^{-st} f(t)\, dt$ (Def. of LT)

- $L[f(t)] = \int_{t=0}^{t=1} e^{-st}\, 2\, dt + \int_{t=1}^{t=\infty} e^{-st}\, 2t\, dt$

- $L[f(t)] = 2\left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=1} + 2\left[t\,\frac{e^{-st}}{-s} - 1\,\frac{e^{-st}}{(-s)(-s)}\right]_{t=1}^{t=\infty}$

- $L[f(t)] = 2\left[\frac{e^{-s}}{-s} - \frac{1}{-s}\right] + 2\left[0 - 0 - 1\,\frac{e^{-s}}{-s} - \frac{-\,e^{-s}}{(-s)(-s)}\right] = \frac{2}{s} + \frac{2\ e^{-s}}{(s)(s)} = \frac{2}{s}\left[1 + \frac{e^{-s}}{(s)}\right]$

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Exam.12 PT $L\left[erf(\sqrt{t})\right] = \frac{1}{S\sqrt{(S+1)}}$, Where $erf(x) = \frac{2}{\sqrt{\pi}}\int_{u=0}^{u=x} e^{-u^2}du$ [Dec-16, (6M)]

- We know $erf(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_{u=0}^{u=\sqrt{t}} e^{-u^2}du$

u	0	$\sqrt{t}$
y	0	t

- Let $u^2 = y \Rightarrow u = \sqrt{y} \Rightarrow du = \frac{1}{2\sqrt{y}}dy$
- $erf(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_{y=0}^{y=t} e^{-y}\ \frac{1}{2\sqrt{y}}\ dy$
- $L\left[erf(\sqrt{t})\right] = \frac{2}{\sqrt{\pi}} L\left[\int_{y=0}^{y=t} e^{-y}\ \frac{1}{2\sqrt{y}}\ dy\right]$
- $= \frac{1}{\sqrt{\pi}} L\left[\int_{y=0}^{y=t} e^{-y}\ y^{-\frac{1}{2}} dy\right]$

$$L\left[\int_{t=0}^{t=t} f(t)\,dt\right] = \frac{1}{(s)} L[f(t)]$$

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$$= \frac{1}{\sqrt{\pi}} \frac{1}{(s)} L\left[\, e^{-t} \; t^{-\frac{1}{2}} \,\right] = \frac{1}{\sqrt{\pi}} \frac{1}{(s)} L\left[\, t^{-\frac{1}{2}} \,\right]_{s \to (s+1)}$$

$$= \frac{1}{\sqrt{\pi}} \frac{1}{(s)} \left[\frac{1}{s^{-\frac{1}{2}+1}} \; \Gamma\!\left(-\frac{1}{2}+1\right) \right]_{s \to (s+1)}$$

$$= \frac{1}{\sqrt{\pi}} \frac{1}{(s)} \left[\frac{1}{s^{\frac{1}{2}}} \; \Gamma\frac{1}{2} \right]_{s \to (s+1)}$$

But, $\Gamma\frac{1}{2} = \sqrt{\pi}$

$$L\left[erf\left(\sqrt{t}\right)\right] = \frac{1}{S\sqrt{(S+1))}}$$

PRACTICE EXAMPLES : LAPLACE TRANSFORMS

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1. Find $L\left[\, e^{-3t}\, t\, Sin(4t)\right]$ [Biom- May-19,(5M)] Ans: $\frac{8(s+3)}{((s+3)^2+16)^2}$

2. Find $L\left[\, (Sin(2t) - Cos(2t))^2\right]$ [Extc-May-19,(5M)] Ans: $\frac{1}{s} - \frac{4}{s^2+16}$

3. Find $L\left[\frac{Sin(t)Sin(5t)}{t}\right]$ [Extc-May-19,(5M)] Ans: $\frac{1}{4}\left[\, Log\left(\frac{s^2+36}{s^2+16}\right)\right]$

4. Find $L[f(t)] = f(s)$ and $f(t) = \begin{cases} 2t & 0 < t < \pi \\ (\pi - t) & \pi < t < 2\pi \end{cases}$ [Biom-Nov-18,(6M)]

5. *Evaluate*; $I = \int_{t=0}^{t=\infty} \frac{3}{e^t}\left(\int_{u=0}^{u=t} u\, e^{3u}\, Cos(4u)du\right) dt$ [Biom-Nov-19,(6M)] Ans: $\frac{42}{2500}$

6. *Evaluate*; $I = \int_{t=0}^{t=\infty} e^{-2t} Cosh^5(t)\, dt$ [Extc-Dec-16, 19,(5M)] Ans: $\frac{2}{7}$