CSC 402
Exam 1 Solutions
February 20, 2023

---

[12] 1. Answer the following questions. No justification is necessary. No partial credit will be given for your answers.

a. State the negation of

If it is raining then the game is canceled.

Solution. It is raining and the game is not canceled.

b. State the contrapositive of

$$(\forall \epsilon \in \mathbb{R}^+(|a-b| < \epsilon)) \implies (a = b)$$

without using any negation symbols ($\neg$, !, $\sim$, or $\overline{p}$, for a proposition p).

Solution. $(a \neq b) \implies (\exists \epsilon \in \mathbb{R}^+(|a-b| \geq \epsilon))$

c. **True or False**: Let G be a graph (G may not be connected) with exactly two odd degree vertices u and v. Then there must be a path between u and v in G.

Solution. $\boxed{\text{False}}$

d. **True or False**: A graph with n vertices and $n-1$ edges is a tree.

Solution. $\boxed{\text{False}}$

e. **True or False**: There exists an undirected graph G whose vertices have the following degrees: 1, 2, 3, 4, 5.

Solution. $\boxed{\text{False}}$. A graoh must have an even number of odd degree vertices.

f. Consider a complete grapn on $2n$ vertices, i.e., a graph on $2n$ vertices in which there is an edge between every pair of vertices. What is the minimum number of edges that needed to be deleted to create two connected components of equal size?

Solution. $\boxed{n^2}$. There are n^2 edges that go between two sets of n vertices of a complete graph.

---

[12] 2. Prove using induction that for all integers $n \geq 1$,

$$1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n+1)! - 1$$

Solution. We will prove the claim using induction on n.

Induction Hypothesis: Assume that the claim holds when $n = k$, for some fixed integer $k \geq 1$. In other words, assume that for some fixed integer $k \geq 1$,

$$1(1!) + 2(2!) + 3(3!) + \cdots + k(k!) = (k+1)! - 1$$

Base Case: When $n = 1$ the left side equals $1(1!) = 1$ and the right hand side evaluates to $2! - 1 = 1$. Thus the claim holds when $n = 1$.
Induction Step: We want to prove that the claim is true when $n = k + 1$. That is, we want to prove that

$$1(1!) + 2(2!) + 3(3!) + \cdots + k(k!) + (k+1)((k+1)!) = (k+2)! - 1$$

Using the induction hypothesis we get

$$\begin{aligned} 1(1!) + 2(2!) + 3(3!) + \cdots + k(k!) + (k+1)((k+1)!) &= (k+1)! - 1 + (k+1)((k+1)!) \\ &= (k+1)!(1 + (k+1)) - 1 \\ &= (k+1)!(k+2) - 1 \\ &= (k+2)! - 1 \end{aligned}$$

Thus we have shown that the claim is true when $n = k+1$ and this completes the induction proof.

---

[14] 3. Prove or disprove each of the following propositions.

a. Let n be an integer. If $n^3 - 5$ is an odd integer, then n is even.

Solution. The statement is true. We can prove the claim by proving its contrapositive. That is, we will prove that if n is odd then $n^3 - 5$ is an even integer. Let $n = 2k + 1$, for some integer k. Thus=, we have

$$\begin{aligned} n^3 - 5 &= (2k+1)^3 - 5 \\ &= (8k^3 + 12k^2 + 6k + 1) - 5 \\ &= 8k^3 + 12k^2 + 6k - 4 \\ &= 2(4k^3 + 6k^2 + 3k - 2) \\ &= 2m, \qquad \text{where} m = 4k^3 + 6k^2 + 3k - 2 \text{is an integer} \end{aligned}$$

Hence, we proved that $n^3 - 5$ is an even integer.

b. For all rational numbers x and y, the number x^y is also rational.

Solution. The statement is false. A counterexample is $x = 2$ and $y = 1/2$. Thus $x^y = 2^{1/2} = \sqrt{2}$, which is irrational.

---

[12] 4. Prove that $\sqrt{8}$ is irrational.

Solution. Assume for contradiction that $\sqrt{8}$ is rational. By definition, for some integers p and q, where $q \neq 0$, we have

$$\begin{aligned} \sqrt{8} &= \frac{p}{q} \\ \therefore 2\sqrt{2} &= \frac{p}{q} \\ \therefore \sqrt{2} &= \frac{p}{2q} \end{aligned}$$

Observe that the right side is a rational number, since it is a ratio of two integers and since $q \neq 0$, the denominator $2q \neq 0$. The left side is $\sqrt{2}$, which is irrational. This is a contradiction! Hence $\sqrt{8}$ is irrational.

---

[12] 5. An acyclic graph is called a *forest*. Let G be a forest with n vertices and c components. Derive a formula for the number of edges in G.

Solution. Let n be the total number of vertices in the forest. Let $n_1, n_2, \cdots, n_c$ be the number of vertices in the connected components of the forest. Each connected component is a tree. Thus, there are $n_i - 1$ edges in each connected component i, $1 \leq i \leq c$. Hence, the total number of edges is given by $\sum_{i=1}^{c}(n_i - 1) = n - c$.

---

[13] 6. Consider a sequence S of *non-zero* real numbers $a_1, a_2, \ldots, a_n$ with the following constraints.

- $a_1 < 0$
- $a_n > 0$

Prove by induction on n that for any $n \geq 2$, the sequence S must have elements a_i and a_{i+1} such that $a_i < 0$ and $a_{i+1} > 0$.

Solution. **Base case:** $n = 2$. Clearly, any sequence a_1, a_2 where $a_1 < 0$ and $a_2 > 0$ satisfies the claim (set $a_i := a_1$), so the proposition holds for $n = 2$.

Induction hypothesis: For an arbitrary $k \in \mathbb{Z}_{\geq 2}$, suppose that any sequence of k nonzero real numbers $a_1, \ldots, a_k$ where $a_1 < 0$ and $a_k > 0$ satisfies the claim.

Inductive step: We must show that any sequence of $k + 1$ nonzero real numbers $b_1, \ldots, b_{k+1}$ where $b_1 < 0$ and $b_{k+1} > 0$ must have two consecutive elements b_i and b_{i+1} such that $b_i < 0$ and $b_{i+1} > 0$.

Consider the following two cases:

(i) $b_k < 0$. Then, since $b_{k+1} > 0$, we have found two consecutive elements that satisfy the claim.

(ii) $b_k > 0$. Then, we have a subsequence $b_1, \ldots, b_k$ of k non-zero real numbers where $b_1 < 0$ and $b_k > 0$. By the inductive hypothesis, the subsequence contains two consecutive elements b_j, b_{j+1} $(1 \leq j < k)$ such that $b_j < 0$ and $b_{j+1} > 0$. Since the sequence $b_1, \ldots, b_{k+1}$ contains these two consecutive elements, the claim holds true.

Since in both cases, two consecutive elements exist that satisfy the claim, the proposition is proven for $n = k + 1$.

Hence, the claim holds true by weak induction for all integers $n \geq 2$.

---

[12] 7. The *complement* of a graph G is a new graph formed by removing all the edges of G and replacing them by all possible edges that are not in G. Formally, consider a graph $G = (V, E)$. Then, the complement of the graph G is the graph $\overline{G} = (V, \overline{E})$, where

$$\overline{E} = \{\{x, y\} \,|\, x \neq y, \{x, y\} \notin E\}$$

Prove that for any graph G, G or $\overline{G}$ (or both) must be connected.

Solution. We want to show that if G is disconnected then $\overline{G}$ is connected. Since G is disconnected there must be two vertices u and v such that u and v belong to different components. We want to show that any two vertices x and y are connected by a path in $\overline{G}$. We consider the following cases.

Case 1: In G, x and y belong to different components.
Then there is an edge between x and y in $\overline{G}$.

Case 2: In G, x and y belong to the same component. Then in $\overline{G}$, there are edges from both x and y to the same vertex, either u or v.

---

[13] 8. Let G be a connected graph in which the average degree is less than 2. Prove that G is a tree.

Solution. The average degree is defined as the following:

$$\frac{1}{|V|} \sum_{v \in V} \deg(v)$$

Since we know that the average degree is less than 2, we have that:

$$\begin{aligned}
\frac{1}{|V|} \sum_{v \in V} \deg(v) &< 2 \\
\sum_{v \in V} \deg(v) &< 2|V| \\
2|E| &< 2|V| && \text{(Using the Handshaking Lemma)} \\
|E| &< |V|
\end{aligned}$$

However, we know that G is connected. Therefore, the only way G can have less edges than vertices than edges and still be connected is if it has exactly $|V| - 1$ edges. However, a connected graph with $|V| - 1$ edges is defined as a tree, and thus we have proved the claim.
