Successive Differentiation (10-12 Marks)

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If $y = f(x)$, then derivatives of 'y' with respect to 'x' are generally denoted as :

$y = f(x)$	First Order	Second	Third	....	n^{th} order
1	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	....	$\frac{d^ny}{dx^n}$
2	y'	y''	y'''	....	$y''''...(n\ times)$
3	$y^{(1)}$	$y^{(2)}$	$y^{(3)}$	....	$y^{(n)}$
4	Dy where $D = \frac{d}{dx}$	D^2y $D^2 = \frac{d^2}{dx^2}$	D^3y $D^3 = \frac{d^3}{dx^3}$	....	D^ny
5	$\dot{y}$	$\ddot{y}$	$\dddot{y}$	...	...
6	y_1	y_2	y_3	....	y_n

$y = f(x) \rightarrow$ (n is +ve Integer)

$\frac{dy}{dx} = y' = y^{(1)} = Dy = \dot{y} = y_1$

$\frac{d^2y}{dx^2} = y'' = y^{(2)} = D^2y = \ddot{y} = y_2$

$\frac{d^ny}{dx^n} = y^{'''...(n\ times)} = y^{(n)} = D^ny = y^{\cdots(n\ times)} = y_n$

y_{n+1}

n^{th} Derivative of some standard functions :

SN	$y = f(x)$	n^{th} Derivative	Remark
1	$y = (ax+b)^m$	$y_n = \frac{m!\ a^n(ax+b)^{m-n}}{(m-n)!}$	For a, b are constant, m & n positive integers, And $m > n$
		$y_n = m!\ a^n = n!\ a^n$	For $m = n$
		$y_n = 0$	For $m < n$
2	$y = \frac{1}{(ax+b)^m}$	$y_n = \frac{(-1)^n\ (m+n-1)!\ a^n}{(m-1)!\ (ax+b)^{m+n}}$	m & n positive integers
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3	$y = \frac{1}{(ax+b)}$	$y_n = \frac{(-1)^n\ (n)!\ a^n}{(ax+b)^{n+1}}$	$m = 1$ & n positive integers
4	$y = \log(ax+b)$	$y_n = \frac{(-1)^{n-1}\ (n-1)!\ a^n}{(ax+b)^n}$	
5	$y = e^{ax}$	$y_n = a^n e^{ax}$	
6	$y = a^x$	$y_n = a^x(\log a)^n$	
7	$y = Sin(ax+b)$	$y_n = a^n Sin\left(ax+b+n\frac{\pi}{2}\right)$	
8	$y = Cos(ax+b)$	$y_n = a^n Cos\left(ax+b+n\frac{\pi}{2}\right)$	
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9	$y = e^{ax}\ Sin(bx+c)$	$y_n = r^n e^{ax} Sin(bx+c+n\theta)$ Where $r = \sqrt{(a^2+b^2)}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$	a, b and c are constant
10	$y = e^{ax}\ Cos(bx+c)$	$y_n = r^n e^{ax}\ Cos(bx+c+n\theta)$ Where $r = \sqrt{(a^2+b^2)}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$	a, b and c are constant
11	Leibnitz's Thm $y = U\ V$	$y_n = U_n V + n\, U_{n-1} V_1 + \frac{n(n-1)}{2!} U_{n-2} V_2 + \frac{n(n-1)(n-2)}{3!} U_{n-3} V_3 + \cdots\cdots\cdots + UV_n$	

(I) $y = e^{ax}$

$y_1 = \frac{dy}{dx} = a e^{ax}$

$y_2 = a^2 e^{ax}$

$\boxed{y_n = a^n e^{ax}}$

(II) $y = a^x$

$y_1 = a^x(\log a)$

$y_2 = a^x(\log a)^2$

$y_3 = a^x(\log a)^3$

$\boxed{y_n = a^x(\log a)^n}$

(III) $y = (ax+b)^m$ where m is (+ve) Integer

$y_1 = m(ax+b)^{m-1} a$

$y_2 = m(m-1)a^2(ax+b)^{m-2}$

$y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}$

$y_4 = m(m-1)(m-2)(m-3)a^4(ax+b)^{m-4}$

$y_n = m(m-1)(m-2)(m-3)\cdots\cdots[m-(n-1)]\, a^n (ax+b)^{m-n}$

$= \frac{m(m-1)(m-2)(m-3)\cdots[m-(n-1)]\,[m-n]!}{(m-n)!}\, a^n (ax+b)^{m-n}$

$10 \cdot 9 \cdot 8 \cdots 3 \cdot 2 \cdot 1$

$10 \cdot 9 \cdot 8!$

$\boxed{y_n = \frac{m!\, a^n (ax+b)^{m-n}}{(m-n)!}}$; $m > n$

Ex. $y = x^4$ $m = 4$

$n=1$ $y_1 = 4x^3$ $m > n$

$n=2$ $y_2 = 4 \cdot 3 \cdot x^2$ $m > 2$

$y_3 = 4 \cdot 3 \cdot 2 \cdot x$ $4 > 3$

$n=4$ $y_4 = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$ $4=4$ $m=n$

$n=5$ $y_5 = 0$ $n > m$

$y_n = \frac{n!\, a^n (1)}{0!} = n!\, a^n$; $m = n$

$y_n = 0$ $n > m$

(IV) $y = (ax+b)^{-m} = \frac{1}{(ax+b)^m}$: where m is (+ve) Integer

$y_1 = -m(ax+b)^{-m-1} a$

$y_2 = (-m)(-m-1)a^2(ax+b)^{-m-2}$

$y_3 = (-m)(-m-1)(-m-2)a^3(ax+b)^{-m-3}$

$y_4 = (-m)(-m-1)(-m-2)(-m-3)a^4(ax+b)^{-m-4}$

$(n-1)$

$y_n = (-m)[(-m-1)(-m-2)(-m-3)\cdots\cdots[-m-(n-1)]]\, a^n (ax+b)^{-m-n}$

*** $= (-1)^n m(m+1)(m+2)(m+3)\cdots\cdots[m+(n-1)]\, a^n (ax+b)^{-m-n}$

$= (-1)^n [1 \cdot 2 \cdot 3 \cdot 4 \cdots (m-1)]\, m(m+1)(m+2)\cdots[m+(n-1)]\, a^n (ax+b)^{-m-n}$ ← Note This Step

$$= (-1)^{n} m(m+1)(m+2)(m+3) \cdots [m+(n-1)] a^n (ax+b)^{-m-n}$$

$$= \frac{(-1)^n [1.2.3.4 \cdots (m-1)] m(m+1)(m+2) \cdots [m+(n-1)] a^n (ax+b)^{-m-n}}{[1.2.3.4 \cdots (m-1)]}$$

Note This step

$$\boxed{y_n = \frac{(-1)^n a^n [m+(n-1)]!}{(m-1)!\,(ax+b)^{m+n}}} \Leftarrow \text{For } \boxed{y = \frac{1}{(ax+b)^m}}$$

(V) IF $m = 1$ Then $\boxed{y = \frac{1}{(ax+b)}}$

$$\boxed{y_n = \frac{(-1)^n a^n (n!)}{(ax+b)^{n+1}}}$$

(VI) IF $y = \frac{1}{(x+b)} \Rightarrow y_n = \frac{(-1)^n n!}{(x+b)^{n+1}}$

Ex. $y = \frac{1}{(x^2+5x+6)}$

$= \frac{1}{(x+3)(x+2)} = \frac{A}{(x+3)} + \frac{B}{(x+2)}$

$= \frac{1}{(x+2)} - \frac{1}{(x+3)} = \frac{2x+5}{(x+2)(x+3)}$

$y = \frac{1}{(ax+b)}$

$$y_n = \frac{(-1)^n n!}{(x+2)^{n+1}} - \frac{(-1)^n n!}{(x+3)^{n+1}}$$

(VII) $y = \log(ax+b)$

$$\frac{dy}{dx} = y_1 = \frac{1}{(ax+b)}\, a$$

Take (n-1) times derivative to B.S

$$\frac{d^{n-1}}{dx^{n-1}}\left[\frac{dy}{dx} = y_1\right] = a\,\frac{d^{n-1}}{dx^{n-1}}\left[\frac{1}{(ax+b)}\right]$$

$$\frac{d^n y}{dx^n} = y_n = a^1\,\frac{(-1)^{n-1} a^{n-1} (n-1)!}{(ax+b)^{n-1+1}} \longrightarrow \text{put } n \text{ by } n-1 \text{ in } y_n = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}$$

$$\boxed{y_n = \frac{(-1)^{n-1} (n-1)!\, a^n}{(ax+b)^n}} \Leftarrow \text{For } y = \log(ax+b)$$

(VIII) $y = \sin(ax+b)$

$y_1 = a\cos(ax+b)$

$\boxed{\sin(\frac{\pi}{2}+\theta) = \cos\theta}$

Note this step $\rightarrow$ $y_1 = a\sin\left[\frac{\pi}{2} + (ax+b)\right]$

$$y_2 = a^2\cos\left[\frac{\pi}{2} + (ax+b)\right] = a^2\sin\left[\frac{\pi}{2} + \left(\frac{\pi}{2} + ax+b\right)\right]$$

$$y_2 = a^2\sin\left[2\frac{\pi}{2} + (ax+b)\right]$$

$$y_3 = a^3\cos\left[2\frac{\pi}{2} + (ax+b)\right] = a^3\sin\left[\frac{\pi}{2} + 2\frac{\pi}{2} + ax+b\right]$$

$$= a^3\sin\left[3\frac{\pi}{2} + (ax+b)\right]$$

$$y_4 = a^4\sin\left[4\frac{\pi}{2} + (ax+b)\right]$$

$$\boxed{y_n = a^n\sin\left[n\frac{\pi}{2} + (ax+b)\right]} \Leftarrow \text{For } y = \sin(ax+b)$$

(IX) $y = \cos(ax+b) \Rightarrow y_n = a^n\cos\left[n\frac{\pi}{2} + (ax+b)\right]$

$y_1 = -a\sin(ax+b)$

$\boxed{\cos(\frac{\pi}{2}+\theta) = -\sin\theta}$

$$y_1 = -a\sin(ax+b)$$

$$\boxed{\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta}$$

$$= a\cos\left[\frac{\pi}{2}+(ax+b)\right]$$

$$y_2 = -a^2\sin\left[\frac{\pi}{2}+(ax+b)\right] = a^2\cos\left[\frac{\pi}{2}+\left(\frac{\pi}{2}+ax+b\right)\right]$$

$$y_2 = a^2\cos\left[2\,\frac{\pi}{2}+(ax+b)\right]$$

$$y_3 = \,?$$

$$y_4 = \,?$$

$$\boxed{y_n = a^n\cos\left[n\frac{\pi}{2}+(ax+b)\right]} \Leftarrow \text{For } y=\cos(ax+b)$$

(X) $y = e^{ax}\sin(bx+c)$ $\qquad d(uv) = u'v+uv'$

$$y_1 = a e^{ax}\sin(bx+c) + e^{ax}b\cos(bx+c)$$

$$= e^{ax}\left[a\sin(bx+c) + b\cos(bx+c)\right]$$

note this step $\rightarrow$ Let $a = r\cos\theta$ and $b = r\sin\theta$

$\Rightarrow r^2 = a^2+b^2 \Rightarrow r = \sqrt{a^2+b^2}$ $\rightarrow$ modulus

$\frac{b}{a} = \tan\theta \Rightarrow \theta = \tan^{-1}(b/a) \rightarrow$ argument/Amplitude

$$y_1 = e^{ax}\left[r\cos\theta\sin(bx+c) + r\sin\theta\cos(bx+c)\right]$$

$$= r e^{ax}\left[\sin(bx+c)\cos\theta + \cos(bx+c)\sin\theta\right]$$

$\sin A\cos B + \text{Co}$

$= \sin$

$$y_1 = r e^{ax}\sin(bx+c+\theta)$$

$$y_2 = r\left[a e^{ax}\sin(bx+c+\theta) + e^{ax}b\cos(bx+c+\theta)\right]$$

$$= r e^{ax}\left[r\cos\theta\sin(bx+c+\theta) + r\sin\theta\cos(bx+c+\theta)\right]$$

$$= r^2 e^{ax}\left[\sin(bx+c+\theta)\cos\theta + \cos(bx+c+\theta)\sin\theta\right]$$

$$y_2 = r^2 e^{ax}\sin(bx+c+\theta+\theta)$$

$$y_2 = r^2 e^{ax}\sin(bx+c+2\theta)$$

$$y_3 =$$

$$y_3 =$$

$s A \sin B$
$(A+B)$

$y_3 = r\, e \;\sin(bx + c + 3\theta)$

$$\boxed{y_n = r^n e^{ax} \sin(bx + c + n\theta)} \Leftarrow \text{For } y = e^{ax} \sin(bx + c)$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$

Ex- $y = e^{3x} \sin(4x)$ $\Rightarrow y_n = r^n e^{3x} \sin(4x + n\theta)$

$a = 3,\ b = 4,\ c = 0$

$r = \sqrt{a^2 + b^2} = \sqrt{9 + 16} = 5$

$\theta = \tan^{-1}(b/a) = \tan^{-1}(4/3)$

X1 $y = e^{ax} \cos(bx + c)$ $\Rightarrow y_n = r^n e^{ax} \cos(bx + c + n\theta)$

HW

$r = \sqrt{a^2 + b^2}$ & $\theta = \tan^{-1}(b/a)$

Examples: Successive Differentiation

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Examples based on algebraic function

1. If $y = (2x+3)^{25}$, Then find $y_{10},\ y_{25}$ *and* y_{30}
2. If $y = \frac{1}{(x+1)^{25}}$, Then find y_{10}
3. If $y = \frac{1}{(x^2-5x+6)}$, Then find y_n
4. If $y = \frac{8x}{(x^3-2x^2-4x+8)}$, Then find y_n
5. If $y = \frac{x^2+4}{(2x+3)(x-1)^2}$, Then find y_n [MU -Dec-2010]
6. If $y = \frac{x^3}{(x^2-1)}$, Then for $x = 0$ prove that $y_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -(n!) & \text{if } n \text{ is odd} \end{cases}$ [MU-Dec-2010]
7. If $y = \frac{x^n-1}{(x-1)}$ Then find y_n [Hint: $x^n - 1 = (x-1)\left(x^{n-1} + x^{n-2} + \cdots \cdots + x^3 + x^2 + x + 1\right)$]

   $x^3-1 = (x-1)(x^2+x+1)$
8. If $y = \frac{x^3}{(x+1)(x-2)}$, Then find y_n

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Improper

deg. of Nume < deg of demo

$\frac{R}{\text{Divisor}} = \frac{Ax+B}{(x^2+2)} + \frac{C}{(x-3)}$

$(x^2+2)(x-3)$

Examples based on Trigonometric function

9. If $y = \cos 3x \cos 2x \cos x$,Then find y_n [MU -May-2013]
10. If $y = \sin 3x \sin 2x \sin x$,Then find y_n [MU -May-2014] HW
11. If $y = \sin 3x \sin 2x \cos 4x$,Then find y_n [MU -May-2011] HW
12. ~~If~~ $y = \sin ax + \cos ax$,Then prove that $y_n = a^n\left[1 + (-1)^n \sin 2ax\right]^{1/2}$. Also find $y_8(\pi)$ *when* $a = \frac{1}{4}$ [MU -May-2014]
13. If $y = 2\tan\frac{x}{2}(1 + \cos x)$, Then find y_n $\Rightarrow y = 2\tan(x/2)\left[2\cos^2(x/2)\right] = 2\left[2\sin(x/2)\cos(x/2)\right] = 2\sin(x)$

    $y_n = 2(1)^n \sin(x + n\pi/2)$

Examples based on Exponential function

14. If $y = \cosh x \cos x$,Then find y_n
15. If $y = e^x \cos^2 x \sin x$,Then find y_n [MU -Dec-2018]
16. If $y = e^x \cos 3x \sin 2x$,Then find y_n [MU -May-2006]
17. If $y = 2^x \sin^2 x \cos^3 x$,Then find y_n [MU -May-2009,12]
18. If $y = e^{2x} \sin\frac{x}{2} \cos\frac{x}{2} \sin 3x$,Then find y_n [MU -Dec-2017]

Examples based on De'Moivre's theorem

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19. If $y = \tan^{-1} x$, Then find y_n
20. If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, Then find y_n [Hint: put $x = \tan\alpha$] /
21. If $y = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$, Then find y_n [Hint: put $x = \tan\alpha$, & $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$]
22. If $I_n = \frac{d^n}{dx^n}(x^n \log x)$, Then prove that $I_n = nI_{n-1} + (n-1)!$. Hence show that $I_n = n!\left[\log x + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{n}\right]$

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1. If $y = (2x+3)^{25}$, Then find $y_{10},\ y_{25}$ *and* y_{30}

$\rightarrow\ y = (ax+b)^m \Rightarrow y_n = \frac{m!\, a^n (ax+b)^{m-n}}{(m-n)!}$; $m > n$

$= m!\, a^n$; $m = n$

$= 0$; $n > m$

$y = (2x+3)^{25} \Rightarrow a = 2,\ b = 3,\ m = 25$

For $n = 10 \Rightarrow m > n$ $(25 > 10)$

$y_{10} = \frac{25!\, 2^{10} (2x+3)^{25-10}}{(25-10)!} = \frac{25!\, 2^{10} (2x+3)^{15}}{15!}$

For $n = 25 = m$ $y_{25} = n!\, a^n = 25!\,(2)^{25}$

For $n = 30 > m$ $\boxed{y_{30} = 0}$

2. If $y = \frac{1}{(x+1)^{25}}$, Then find y_{10}

$\rightarrow\ y = \frac{1}{(ax+b)^m} \Rightarrow y_n = \frac{(-1)^n a^n [m+n-1]!}{(m-1)!\,(ax+b)^{m+n}}$

For $n = 10$ $y_{10} = \frac{(-1)^{10} (1)^{10} [25+10-1]!}{(25-1)!\,(x+1)^{25+10}} = \frac{34!}{24!\,(x+1)^{35}}$

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3. If $y = \frac{1}{(x^2-5x+6)}$, Then find y_n

$\rightarrow\ y = \frac{1}{x^2-5x+6} = \frac{1}{(x-3)(x-2)} = \left[\frac{1}{(x-3)} - \frac{1}{(x-2)}\right]$ (1)

$\frac{1}{(x-3)(x-2)}$

IF $y = \frac{1}{(ax+b)} \Rightarrow y_n = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}$

① IF $y = \frac{1}{x-3} \Rightarrow y_n = \frac{(-1)^n (1)^n n!}{(x-3)^{n+1}}$

② If $y = \frac{1}{x-2} \Rightarrow y_n = \frac{(-1)^n (1)^n n!}{\ldots^{n+1}}$

$$x-3 \Rightarrow y_n = \frac{(-1)^n (1)^n n!}{(x-3)^{n+1}}$$

② If $y = \frac{1}{x-2} \Rightarrow y_n = \frac{(-1)^n (1)^n n!}{(x-2)^{n+1}}$

From ①

$$y_n = \frac{(-1)^n n!}{(x-3)^{n+1}} - \frac{(-1)^n n!}{(x-2)^{n+1}}$$

5. If $y = \frac{x^2+4}{(2x+3)(x-1)^2}$, Then find y_n [MU -Dec-2010]

$\rightarrow y = \frac{x^2+4}{(2x+3)(x-1)^2} = \frac{A}{(2x+3)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$ — (I)

$$\frac{x^2+4}{(2x+3)(x-1)^2} = \frac{A(x-1)^2}{(2x+3)(x-1)^2} + \frac{B(2x+3)(x-1)}{(2x+3)(x-1)^2} + \frac{C(2x+3)}{(2x+3)(x-1)^2}$$

$$\frac{x^2+4}{\cancel{(2x+3)(x-1)^2}} = \frac{A(x-1)^2 + B(2x+3)(x-1) + C(2x+3)}{\cancel{(2x+3)(x-1)^2}}$$

$x^2+4 = A(x-1)^2 + B(2x+3)(x-1) + C(2x+3)$ — (II)

put $2x+3=0 \Rightarrow x = -3/2$ in eqn (II)

$\frac{9}{4}+4 = A(-\frac{3}{2}-1)^2 + 0 + 0 \Rightarrow \frac{25}{4} = \frac{25}{4}A \Rightarrow \boxed{A=1}$

put $x-1=0$ or $x=1$ in eqn (II)

$5 = 0+0+C(5) \Rightarrow \boxed{C=1}$

put $x=0$, in eqn (II)

$4 = A - 3B + 3C \Rightarrow 4 = 1 - 3B + 3 \Rightarrow \boxed{B=0}$

$y = \frac{x^2+4}{(2x+3)(x-1)^2} = \frac{1}{(2x+3)} + 0 + \frac{1}{(x-1)^2}$

$\boxed{\text{If } y = \frac{1}{(ax+b)^m} \Rightarrow y_n = \frac{(-1)^n a^n (m+n-1)!}{(m-1)!(ax+b)^{m+n}}}$

$a=2$ $b=3$ $m=1$; $a=1$ $b=-1$ $m=2$

$$y_n = \frac{(-1)^n (2)^n (n!)}{(2x+3)^{n+1}} + \frac{(-1)^n (1)^n (n+1)!}{1!(x-1)^{n+2}}$$

6. If $y = \frac{x^3}{(x^2-1)}$, Then for $x = 0$ prove that $y_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -(n!) & \text{if } n \text{ is odd} \end{cases}$ [MU-Dec-2010]

$(x^2-1)\overline{)x^3}(x$
$-x^3 \mp x$
$0 + x$

$\boxed{\frac{\text{dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}}$

$$y = \frac{x^3}{(x^2-1)} = x + \frac{x}{(x^2-1)}$$

$$= x + \frac{x}{(x-1)(x+1)}$$

$$= x + \frac{1}{2}\left[\frac{1}{(x-1)} + \frac{1}{x+1}\right]$$

$$y = \frac{1}{(ax+b)} \Rightarrow y_n = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}$$

$$y_n = 0 + \frac{1}{2}\left[\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{(-1)^n n!}{(x+1)^{n+1}}\right] = \frac{(-1)^n n!}{2}\left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}}\right]$$

For $x=0$

$$y_n = \frac{(-1)^n n!}{2}\left[\frac{1}{(-1)^{n+1}} + \frac{1}{(1)^{n+1}}\right]$$

n is even, (n+1) is odd $= 0$, n is even (n+1) is odd

n is odd, (n+1) is even: $\frac{(-1)^{\text{odd}} n!}{\cancel{2}}\left[\frac{1}{1} + \frac{1}{1}\right] = -n!$, n is odd (n+1) is even

9. If $y = \cos 3x \cos 2x \cos x$,Then find y_n [MU -May-2013]

$\rightarrow$ $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$y = \cos(ax+b)$
$y_n = a^n \cos(ax+b+n\pi/2)$

$y = \frac{[2\cos 3x \cos 2x]}{2} \cos x$

$= \frac{1}{2}[\cos(5x) + \cos(x)]\cos x = \frac{1}{2}\{\cos 5x \cos x + \cos x \cos x\}$

$y = \frac{1}{2}\left[\frac{2\cos 5x \cos x}{2} + \frac{2\cos x \cos x}{2}\right] = \frac{1}{2\times 2}[\cos(6x) + \cos(4x) + \cos(2x) + \cos(0)]$

$y_n = \frac{1}{4}\left[6^n \cos(6x + n\frac{\pi}{2}) + 4^n \cos(4x + n\pi/2) + 2^n \cos(2x + n\pi/2) + 0\right]$

12. If $y = \sin ax + \cos ax$,Then prove that $y_n = a^n[1 + (-1)^n \sin 2ax]^{1/2}$ Also find $y_8(\pi)$ when $a = \frac{1}{4}$ [MU -May-2014]

$\rightarrow y = \sin(ax) + \cos(ax)$

$y_n = a^n \sin(ax + n\pi/2) + a^n \cos(ax + n\pi/2)$

Note this step $\Rightarrow y_n = a^n\{[\sin(ax+n\pi/2) + \cos(ax+n\pi/2)]^2\}^{1/2}$

$= a^n\{\sin^2(ax+n\pi/2) + \cos^2(ax+n\pi/2) + 2\sin(ax+n\pi/2)\cos(ax+n\pi/2)\}^{1/2}$

$= a^n[1 + \sin 2(ax+n\pi/2)]^{1/2}$ $\quad$ $2\sin\theta\cos\theta = \sin 2\theta$

$= a^n\left[\sin^2(ax+n\pi/2)+\cos^2(ax+n\pi/2)+2\sin(ax+n\pi/2)\cos(ax+n\pi/2)\right]$

$= a^n\left[1+\sin 2(ax+n\pi/2)\right]^{1/2}$ $\quad \because \boxed{2\sin\theta\cos\theta=\sin 2\theta}$

$= a^n\left[1+\sin(n\pi+2ax)\right]^{1/2}$ $\quad$ n: +ve Integer

$\sin(A+B)=\sin A\cos B+\cos A\sin B$ $\quad \sin(n\pi)=0$

$\cos(n\pi)=$ n odd (-1) 1, 3 ..., n even 1 2, 4 ...

$= a^n\left[1+\sin(n\pi)\cos(2ax)+\cos(n\pi)\sin(2ax)\right]^{1/2}$ $\quad \boxed{\cos(n\pi)=(-1)^n}$

$\sin(n\pi)=0$

$y_n(x)= a^n\left[1+0+(-1)^n\sin(2ax)\right]^{1/2}$

we have $y_8(\pi)$ means that $y_8(x)$ and $x=\pi$ also $a=1/4$

$y_8(\pi)=\left(\frac{1}{4}\right)^8\left[1+(-1)^8\sin(2\tfrac{1}{4}\pi)\right]^{1/2}$

$y_8(\pi)=\frac{1}{4^8}\left[1+\sin(\pi/2)\right]^{1/2} = \frac{2^{1/2}}{4^8}$

14. $y = \cosh x \cos x$,Then find y_n

15. If $y = e^x \cos^2 x \sin x$,Then find y_n [MU -Dec-2018]

HW 16. If $y = e^x \cos 3x \sin 2x$,Then find y_n [MU -May-2006]

17. If $y = 2^x \sin^2 x \cos^3 x$,Then find y_n [MU -May-2009,12]

HW 18. If $y = e^{2x} \sin\frac{x}{2}\cos\frac{x}{2}\sin 3x$,Then find y_n

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$\boxed{\text{Note:}}$ Hyperbolic Fun $\boxed{\cosh(x)=\frac{e^x+e^{-x}}{2}}$, $\boxed{\sinh(x)=\frac{e^x-e^{-x}}{2}}$

$\tanh(x)=\frac{\sinh x}{\cosh x}=\frac{e^x-e^{-x}}{e^x+e^{-x}}$

14. $y = \cosh x \cos x$,Then find y_n

$\rightarrow$ $y=\left(\frac{e^x+e^{-x}}{2}\right)\cos x=\frac{1}{2}\left[e^x\cos x+e^{-x}\cos x\right]$

IF $y=e^{ax}\cos(bx+c) \Rightarrow y_n=r^n e^{ax}\cos(bx+c+n\theta)$

$r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}(b/a)$

$y_n=\frac{1}{2}\left[r_1^n e^x\cos(x+n\theta_1)\right]+\frac{1}{2}\left[r_2^n e^{-x}\cos(x+n\theta_2)\right]$ $\quad a_2=-1, b_2=1$

But $r_1=\sqrt{a_1^2+b_1^2}=\sqrt{1+1}=\sqrt{2}$ $\quad r_2=\sqrt{a_2^2+b_2^2}=\sqrt{(-1)^2+1^2}=\sqrt{2}$

$\theta_1=\tan^{-1}[b_1/a_1]=\tan^{-1}(1)=\pi/4$ $\quad \theta_2=\tan^{-1}(b_2/a_2)=\tan^{-1}(-1)=3\pi/4$

$y_n=\frac{1}{2}\left\{2^{n/2}e^x\cos(x+n\tfrac{\pi}{4})+2^{n/2}e^{-x}\cos(x+n3\pi/4)\right\}$

15. If $y = e^x \cos^2 x \sin x$,Then find y_n [MU -Dec-2018]

$\rightarrow$ $y=e^x\left[\cos x\sin x\cos x\right]=e^x\frac{1}{2}\left[2\sin x\cos x\cdot\cos x\right]=\frac{e^x}{2}\left[\sin 2x\cdot\cos x\right]$

$=\frac{e^x}{2}\left[\frac{2\sin 2x\cos x}{2}\right]=\frac{e^x}{4}\left[\sin(3x)+\sin(x)\right]$ $\quad \therefore 2\sin A\cos B=\sin(A+B)+\sin(A-B)$

$y=\frac{1}{4}\left[e^{x}\sin(3x)+e^{1x}\sin(x)\right]$

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IF $y=e^{ax}\sin(bx+c) \Rightarrow y_n=r^n e^{ax}\sin(bx+c+n\theta)$

$r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}(b/a)$

$y_n=\frac{1}{4}\left[r_1^n e^x\sin(3x+n\theta_1)+r_2^n e^x\sin(x+n\theta_2)\right]$

$a_1=1,\ b_1=3$ $\quad a_2=1,\ b_2=1$

$r_1=\sqrt{1^2+3^2}=\sqrt{10}$ $\quad r_2=\sqrt{1^2+1^2}=\sqrt{2}$

$\theta_1=\tan^{-1}(b_1/a_1)=\tan^{-1}(3)=$ $\quad \theta_2=\tan^{-1}(b_2/a_2)=\tan^{-1}(1)=\pi/4$

$y=2\sin x\cos x$ $\quad (x^A)$

17. If $y = 2^x \sin^2 x \cos^3 x$,Then find y_n [MU -May-2009,12]

$\rightarrow$ We know $\boxed{A=e^{\log A}}$ $\quad$ u $\quad$ v $\quad$ un

Note this step

$(2^x)=e^{\log(2^x)}=e^{x\log 2}=e^{(\log 2)x}=e^{ax}$

$y=e^{(\log 2)x}\sin^2 x\cos^3 x=e^{(\log 2)x}\left[\frac{2\sin x\cos x}{2}\right]^2\cos x$

$=\frac{e^{(\log 2)x}}{4}\sin^2(2x)\cos x=\frac{e^{(\log 2)x}}{4}\left(\frac{1-\cos 4x}{2}\right)\cos x$

$=\frac{e^{(\log 2)x}}{8}\left[\cos x-\frac{2}{2}\cos 4x\cos x\right]$

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$=\frac{e^{(\log 2)x}}{8}\left[\cos x\right]-\frac{e^{(\log 2)x}}{16}\left[\cos(5x)+\cos(3x)\right]$

$=\frac{1}{8}\left[e^{(\log 2)x}\cos(x)\right]-\frac{1}{16}\left[e^{(\log 2)x}\cos(5x)\right]-\frac{1}{16}\left[e^{(\log 2)x}\cos(3x)\right]$

IF $y=e^{ax}\cos(bx+c) \Rightarrow y_n=r^n e^{ax}\cos(bx+c+n\theta)$

$r=\sqrt{a^2+b^2}$, $\theta=\tan^{-1}(b/a)$

$y_n=\frac{1}{8}\left[\sqrt{(\log 2)^2+1}\right]^n e^{(\log 2)x}\cos\left(x+n\tan^{-1}\left(\frac{1}{\log 2}\right)\right)$

If $y = e^{ax}\cos(bx+c) \Rightarrow y_n = r^n e^{ax}\cos(bx+c+n\theta)$, $r=\sqrt{a^2+b^2}$, $\theta = \tan^{-1}(b/a)$

$$y_n = \frac{1}{8}\left[\left[\sqrt{(\log 2)^2+1}\right]^n e^{(\log 2)x}\cos\left(x+n\tan^{-1}\left(\frac{1}{\log 2}\right)\right)\right]$$
$$-\frac{1}{16}\left[\left(\sqrt{(\log 2)^2+25}\right)^n e^{(\log 2)x}\cos\left(5x+n\tan^{-1}\left(\frac{5}{\log 2}\right)\right)\right]$$
$$-\frac{1}{16}\left[\left(\sqrt{(\log 2)^2+9}\right)^n e^{(\log 2)x}\cos\left(3x+n\tan^{-1}\left(\frac{3}{\log 2}\right)\right)\right]$$

Examples based on De'Moivre's theorem

1. If $y = \tan^{-1} x$, Then find y_n
2. If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, Then find y_n [Hint: put $x = \tan\alpha$] / $y = \tan^{-1}\left[\frac{\frac{1}{\cos\alpha}-1}{\frac{\sin\alpha}{\cos\alpha}}\right] = \tan^{-1}\left[\frac{1-\cos\alpha}{\sin\alpha}\right] = \tan^{-1}\left[\frac{2\sin^2\alpha/2}{2\sin\alpha/2\cos\alpha/2}\right] = \tan^{-1}\left[\tan\frac{\alpha}{2}\right] = \alpha/2 = \frac{1}{2}\tan^{-1}(x)$
3. If $y = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$, Then find y_n [Hint: put $x = \tan\alpha$, & $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$] $\Rightarrow y = \cos^{-1}\left[\frac{\frac{\sin\alpha}{\cos\alpha}-\frac{\cos\alpha}{\sin\alpha}}{\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}}\right] = \cos^{-1}\left[\frac{\sin^2\alpha-\cos^2\alpha}{1}\right] = \cos^{-1}\left[-\cos(2\alpha)\right] = \cos^{-1}\left[\cos(\pi-2\alpha)\right] = \pi-2\alpha = \pi - 2\tan^{-1}x$
4. If $I_n = \frac{d^n}{dx^n}(x^n \log x)$, Then prove that $I_n = nI_{n-1} + (n-1)!$. Hence show that $I_n = n!\left[\log x + \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n}\right]$

Note:- $z = x+iy = a+ib$, $x, y \in \mathbb{R}$

$\bar{z} = a - ib$

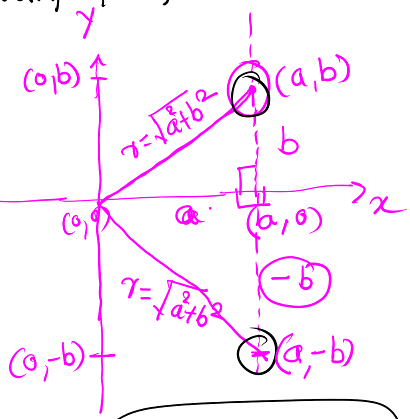
let $a = r\cos\theta$, $b = r\sin\theta$ $\quad \theta = \tan^{-1}(b/a)$

$r^2 = a^2+b^2$ $\quad r = \sqrt{a^2+b^2} \rightarrow$ modulus of z

$z = r\cos\theta + i r\sin\theta$

$z = r[\cos\theta + i\sin\theta] \longrightarrow$ polar Form

$z = r\left[e^{i\theta}\right] \longrightarrow$ exponential form



$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\boxed{x = i\theta}$

$e^{i\theta} = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \cdots$

$= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right]$

$e^{i\theta} = \cos\theta + i\sin\theta$

$z^n = r^n[\cos\theta + i\sin\theta]^n$

$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$

$\bar{z}^n = \bar{r}^n[\cos\theta + i\sin\theta]^{-n}$

$= \bar{r}^n[\cos(-n\theta) + i\sin(-n\theta)]$

$\bar{z}^n = r^{-n}[\cos(n\theta) - i\sin(n\theta)]$

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19. If $y = \tan^{-1} x$, Then find y_n

20. If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, Then find y_n [Hint: put $x = \tan\alpha$]

21. If $y = \cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$, Then find y_n [Hint: put $x = \tan\alpha$, & $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$]

22. If $I_n = \frac{d^n}{dx^n}(x^n \log x)$, Then prove that $I_n = nI_{n-1} + (n-1)!$. Hence show that $I_n = n!\left[\log x + \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n}\right]$

(19) $y = \tan^{-1}x$

$y_1 = \frac{1}{1+x^2} = \frac{1}{x^2-i^2} = \frac{1}{(x-i)(x+i)}$

$y_1 = \left[\frac{1}{(x-i)} - \frac{1}{(x+i)}\right]\frac{1}{2i}$

Take $(n-1)$ times derivative

$y_n = \frac{1}{2i}\left\{\frac{(-1)^{n-1}(1)^{n-1}(n-1)!}{(x-i)^{(n-1)+1}} - \frac{(-1)^{n-1}(1)^{n-1}(n-1)!}{(x+i)^{(n-1)+1}}\right\}$

$= \frac{1}{2i}(-1)^{n-1}(n-1)!\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right]$

$y = \frac{1}{(ax+b)^m}$

$y = \frac{1}{(ax+b)^1}$

$y_n = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}$

Note this step

let $(x+i) = re^{i\theta} = (r\cos\theta + ir\sin\theta)$

$\Rightarrow x = r\cos\theta$, $1 = r\sin\theta$

$r^2 = x^2+1^2 \Rightarrow \boxed{r = \sqrt{x^2+1}}$

$\boxed{\theta = \tan^{-1}\left[\frac{1}{x}\right]}$

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$r = x + 1 \Rightarrow$ [cut off]

$$\boxed{\theta = \tan^{-1}\left[\frac{1}{x}\right]}$$

$$x - i = r e^{-i\theta}$$

$$y_n = \frac{1}{(2i)}(-1)^{n-1}(n-1)! \left[\frac{1}{r^n e^{-in\theta}} - \frac{1}{r^n e^{in\theta}}\right]$$

$$\sin\theta = \boxed{\frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

$$= \frac{1}{(2i)}\frac{(-1)^{n-1}(n-1)!}{r^n}\left[e^{in\theta} - e^{-in\theta}\right]$$

$$\cos\theta = \boxed{\frac{e^{i\theta} + e^{-i\theta}}{2}}$$

$$= \frac{1}{2i}\frac{(-1)^{n-1}(n-1)!}{r^n}\left[\cos(n\theta) + i\sin(n\theta) - \left[\cos(n\theta) - i\sin n\theta\right]\right]$$

(19)

$$= \frac{1}{\cancel{2i}}\frac{(-1)^{n-1}(n-1)!}{r^n}\left[\cancel{2i}\sin(n\theta)\right]$$

But $1 = r\sin\theta$

$$\boxed{\frac{1}{r} = \sin\theta}$$

$$\frac{1}{r^n} = (\sin\theta)^n$$

$$\boxed{y_n = (-1)^{n-1}(n-1)!\,(\sin\theta)^n \sin(n\theta)}$$

$y = \tan^{-1} x$

1. If $I_n = \frac{d^n}{dx^n}(x^n \log x)$, Then prove that $I_n = nI_{n-1} + (n-1)!$. Hence show that $I_n = n!\left[\log x + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{n}\right]$

$(u \cdot v)^n = {}^nc_0 u^n v^0 + {}^nc_1 u^{n-1} v + {}^nc_2 u^{n-2} v^2 + \cdots + {}^nc_{n-1} u^1 v^{n-1} + {}^nc_n u^0 v^n$

If u and v are the two functions of x whose n^{th} derivatives are known or can *easily* be found out then, n^{th} derivative of the product of two functions i.e. $y = u \cdot v$ can be evaluated by Leibnitz's theorem as:

$$y_n = (u \cdot v)_n = {}^nc_0\,[u_n]\,v + {}^nc_1\,[u_{n-1}]\,v_1 + {}^nc_2\,[u_{n-2}]\,v_2 + {}^nc_3\,[u_{n-3}]\,v_3 + \cdots\cdots + {}^nc_{n-1}\,[u_1]\,v_{n-1} + {}^nc_n\,[u]\,v_n$$

But

$${}^nc_0 = 1, \quad {}^nc_1 = n, \quad {}^nc_2 = \frac{n(n-1)}{2!}, \quad {}^nc_3 = \frac{n(n-1)(n-2)}{3!}, \ldots\ldots {}^nc_{n-1} = n, \quad {}^nc_n = 1$$

OR

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$$\boxed{y_n = (u \cdot v)_n = [u_n]\,v + n\,[u_{n-1}]\,v_1 + \frac{n(n-1)}{2!}\,[u_{n-2}]\,v_2 + \frac{n(n-1)(n-2)}{3!}\,[u_{n-3}]\,v_3 + \cdots\cdots + n\,[u_1]\,v_{n-1} + [u]\,v_n}$$

Examples based on Leibnitz's theorem

1. If $y = x^3 \cos x$,Then find y_n
2. If $y = x^2 e^x \sin x$,Then find y_n [MU -Dec-98,08] (5 marks)
3. If $y = x\ \log(x+1)$, Then find y_n Dr. Uday Kashid; PhD(Mathematics)
4. If $y = a\cos(\log x) + b\sin(\log x)$, Then shown that $x^2\, y_{n+2} + (2n+1)x\, y_{n+1} + (n^2+1)y_n = 0$ [MU -May-2019] (8 marks)
5. If $y = x^n\ \log(x)$, Then prove that $y_{n+1} = \frac{n!}{x}$ [MU -03,04, Dec-08] (6 marks)
6. If $y = \sin x[\log(x^2+2x+1)]$, Then shown that $(x+1)^2\, y_{n+2} + (2n+1)(x+1)\, y_{n+1} + (n^2+4)y_n = 0$ [MU -Dec-12] (8 marks)
7. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, Then shown that $x^2\, y_{n+2} + (2n+1)x\, y_{n+1} + (2n^2)y_n = 0$ [MU -90, Dec-2011] (8 marks)
8. If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, Then shown that $(x^2-1)\, y_{n+2} + (2n+1)x\, y_{n+1} + (n^2-m^2)y_n = 0$ [MU -Dec-04, 07,10, May-13] (8 marks)
9. If $y = \cos(m\sin^{-1}x)$, Then shown that $(1-x^2)\, y_{n+2} - (2n+1)x\, y_{n+1} + (m^2-n^2)y_n = 0$ [MU -Dec-08, 13,17] (8 marks)
10. If $y = \sin(m\sin^{-1}x)$, Then shown that $(1-x^2)\, y_{n+2} - (2n+1)x\, y_{n+1} + (m^2-n^2)y_n = 0$ [MU -Dec-04, 15] (8 marks)
11. If $y = \sqrt{\frac{1+x}{1-x}}$, Then shown that $(1-x^2)\, y_n - [2(n-1)x+1]\, y_{n-1} - (n-1)(n-2)y_{n-2} = 0$ [MU -Dec-06] (8 marks)

1. If $y = x^3 \cos x$,Then find y_n

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$\to y = u \cdot v = \cos(x) \cdot x^3$

$u = \cos(ax+b) = \cos(x)$,
$u_n = a^n \cos(ax+b+n\pi/2)$

$v = x^3$, $v_1 = 3x^2$, $v_2 = 6x$, $v_3 = 6$, $v_4 = 0$, $v_n = 0$

By Leibnitz's thm

$y_n = u_n v + n\, u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \frac{n(n-1)(n-2)}{3!} u_{n-3} v_3 + 0 + 0 + \cdots + 0$

$y_n = \cos(x+n\pi/2)\, x^3 + n\cos[x+(n-1)\pi/2]\cdot[3x^2] + \frac{n(n-1)}{2}\cos[x+(n-2)\pi/2]\,[6x] + \frac{n(n-1)(n-2)}{6}\cos[x+(n-3)\pi/2]\,[6]$

2. $y = x^2 e^x \sin x$,Then find y_n

$\to y = u \cdot v = [e^x \sin x]\,[x^2]$

$u = e^{ax}\sin(bx+c) = e^x \sin(x)$
$u_n = r^n e^{ax} \sin(bx+c+n\theta)$
$r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}(b/a)$

$u = e^x \sin x$
$r = \sqrt{1+1} = \sqrt{2}$
$\theta = \tan^{-1}(1) = \pi/4$

$v = x^2$, $v_1 = 2x$, $v_2 = 2$, $v_3 = 0$, $v_4 = 0$, $v_n = 0$

$y_n = u_n v + n[u_{n-1}][v_1] + \frac{n(n-1)}{2!}[u_{n-2}][v_2] + 0 + 0 + \cdots + 0$

$y_n = (\sqrt{2})^n e^x \sin(x+n\pi/4) \cdot [x^2] + n[(\sqrt{2})^{n-1} e^x \sin(x+(n-1)\pi/4)][2x] + \frac{n(n-1)}{2}[(\sqrt{2})^{n-2} e^x \sin(x+(n-2)\pi/4)][2] + \frac{n(n-1)(n-2)}{3!}[(\sqrt{2})^{n-3} e^x \sin(x+(n-3)\frac{\pi}{4})][0] + 0 + \cdots$

3. If $y = x\ \log(x+1)$, Then find y_n Dr. Uday Kashid; PhD(Mathematics)

$y = \log(x+1) \cdot x = u \cdot v$

$u = \log(ax+b) = \log(x+1)$

$u_n = \frac{(-1)^{n-1}(n-1)!\, b\, a^n}{(ax+b)^n} = \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}$

$v = x$, $v_1 = 1$, $v_2 = 0$, $v_3 = 0$, $y_n = 0$

$\to y_n = u_n v + \boxed{n}\, u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} [v_2] + \cdots$

$= \left[\frac{(-1)^{n-1}(n-1)!}{(x+1)^n}\right][x] + n\left[\frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}}\right][1] + 0 + 0 + 0 + \cdots + 0$

4. If $y = a\cos(\log x) + b\sin(\log x)$, Then shown that $x^2\, y_{n+2} + (2n+1)x\, y_{n+1} + (n^2+1)y_n = 0$ [MU -May-2019] (8 marks)

$\to y_1 = \left[-a\sin(\log x)\cdot\frac{1}{x}\right] + \left[b\cos(\log x)\cdot\frac{1}{x}\right]$

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$x y_1 = -a\sin(\log x) + b\cos(\log x)$

again diff w.r.t. x

$y_1 + x y_2 = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$

multiple by x to B.S.

$x^2 y_2 + x y_1 = -[a\cos(\log x) + b\sin(\log x)]$

multiple by x to B.S.

$$x^2y_2 + xy_1 = -[a\cos(\log x) + b\sin(\log x)]$$
$$= -y$$

$$\boxed{x^2y_2 + xy_1 + y = 0} \quad \text{—— (A)}$$

Take nth derivative to B.S

$$\frac{d^n}{dx^n}[y_2x^2] + \frac{d^n}{dx^n}[y_1x] + \frac{d^n}{dx^n}[y] = 0 \quad \text{—— (B)}$$

By Leibnitz's Thm

(1) let $P = y_2x^2 = u.v$ $\quad u = y_2,\ u_n = y_{n+2}$ $\quad v = x^2,\ v_1 = 2x,\ v_2 = 2,\ v_3 = 0,\ v_n = 0$

$$\frac{d^n}{dx^n}[y_2x^2] = y_{n+2}[x^2] + n[y_{n+1}][2x] + \frac{n(n-1)}{2!}[y_n][2]$$

(2) $$\frac{d^n}{dx^n}[y_1x] = y_{n+1}[x] + n[y_n][1]$$

(3) $$\frac{d^n}{dx^n}[y] = y_n$$

$$\Rightarrow [x^2y_{n+2} + 2nxy_{n+1} + (n^2-n)y_n] + xy_{n+1} + ny_n + y_n = 0$$

$$\Rightarrow \boxed{x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0}$$

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5. If $y = sin[\log(x^2+2x+1)]$, Then shown that $(x+1)^2y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$ [MU -Dec-12] (8 marks)

$$\rightarrow y = \sin[\log(x+1)^2] = \sin[2\log(x+1)] \quad \text{—— (I)}$$

$$y_1 = \cos[2\log(x+1)] \cdot \left[2\frac{1}{(x+1)}\right]$$

$y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$

$$(x+1)y_1 = 2\cos[2\log(x+1)] \quad \text{—— (II)}$$

again diff w.r.t x

$$(x+1)y_2 + (1)y_1 = -2\sin[2\log(x+1)] \cdot \left[2\frac{1}{x+1}\right]$$

$$(x+1)^2y_2 + (x+1)y_1 = -4\sin[2\log(x+1)] = -4y$$

$$\boxed{(x+1)^2y_2 + (x+1)y_1 + 4y = 0} \quad \text{—— (A)}$$

By Leibnitz's Thm, take n^{th} derivative to B.S

$$\frac{d^n}{dx^n}[(x+1)^2y_2] + \frac{d^n}{dx^n}[(x+1)y_1] + 4\frac{d^n}{dx^n}(y) = 0 \quad \text{—— (B)}$$

(I) $$\frac{d^n}{dx^n}[y_2(x+1)^2] = \frac{d^n}{dx^n}[u.v] = u_n[v] + n[u_{n-1}][v_1] + \frac{n(n-1)}{2!}[u_{n-2}][v_2] + \cdots + [u][v_n]$$

$$= y_{n+2}(x+1)^2 + n[y_{n+1}][2(x+1)] + \frac{n(n-1)}{2!}[y_n][2] + 0 + 0 + 0 -$$

(II) $$\frac{d^n}{dx^n}[y_1(x+1)] = \frac{d^n}{dx^n}[u.v] = y_{n+1}(x+1) + n[y_n][1] + 0 + 0 + 0$$

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(III) $$\frac{d^n y}{dx^n} = y_n$$

put in (B)

$$(x+1)^2y_{n+2} + 2ny_{n+1}(x+1) + (n^2-n)y_n + (x+1)y_{n+1} + ny_n + 4y_n = 0$$

$$\boxed{(x+1)^2y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0}$$

Ans

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7. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, Then shown that $x^2y_{n+2} + (2n+1)xy_{n+1} + (2n^2)y_n = 0$ [MU -90, Dec-2011] (8 marks)
8. If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, Then shown that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$ [MU -Dec-04, 07,10, May-13] (8 marks)
9. If $y = cos(m\sin^{-1}x)$, Then shown that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$ [MU -Dec-08, 13,17] (8 marks)
10. If $y = sin(m\sin^{-1}x)$, Then shown that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$ [MU -Dec-04, 15] (8 marks) → HW
11. If $y = \sqrt{\frac{1+x}{1-x}}$, Then shown that $(1-x^2)y_n - [2(n-1)x+1]y_{n-1} - (n-1)(n-2)y_{n-2} = 0$ [MU -Dec-06] (8 marks)

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7. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, Then shown that $x^2y_{n+2} + (2n+1)xy_{n+1} + (2n^2)y_n = 0$ [MU -90, Dec-2011] (8 marks)

$$\rightarrow \frac{y}{b} = \cos[\log(x/n)^n]$$

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$$y = b\cos[n \cdot \log(x/n)] \quad \text{—— (I)}$$

$$y_1 = -b\sin\left[n\log\left(\frac{x}{n}\right)\right] \cdot \left[\cancel{n}\ \frac{1}{(x/n)} \cdot \frac{1}{\cancel{n}}\right]$$

$y = b\cos[n\cdot\log(x/n)]$ —— (I)

$y_1 = -b\sin\left[n\log\left(\frac{x}{n}\right)\right]\cdot\left[\cancel{n}\ \frac{1}{(x/n)}\cdot\frac{1}{\cancel{n}}\right]$

$y_1 = -\ b\sin\left[n\log\left(\frac{x}{n}\right)\right]\cdot\left(\frac{n}{x}\right)$

$xy_1 = -nb\sin\left[n\log\left(\frac{x}{n}\right)\right]$ —— (II)

Square to B.S

$x^2y_1^2 = n^2b^2\sin^2\left[n\log\left(\frac{x}{n}\right)\right] = n^2b^2\left[1-\cos^2\left[n\log\left(\frac{x}{n}\right)\right]\right]$

$x^2y_1^2 = n^2b^2 - n^2 b^2\cos^2[n\log(x/n)] = n^2b^2 - n^2y^2$

$x^2y_1^2 + n^2y^2 = n^2b^2$

again diff. w.r.t x

$x^2[2y_1y_2] + (2x)y_1^2 + n^2(2yy_1) = 0$

divide by $(2y_1)$

$$\boxed{x^2y_2 + xy_1 + n^2y = 0}$$ —— (A)

By Leibnitz's Thm, diff. w.r.t. x (n-times)

$\left[y_{n+2}(x^2) + ny_{n+1}(2x) + \frac{n(n-1)}{\cancel{2!}}y_n(\cancel{2})\right] + \left[y_{n+1}(x) + ny_n(1)\right] + n^2y_n = 0$

$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2 - \cancel{n} + \cancel{n} + n^2)y_n = 0$

$$\boxed{x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0}$$

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8. If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, Then shown that $(x^2-1)\ y_{n+2} + (2n+1)x\,y_{n+1} + (n^2-m^2)y_n = 0$

$\rightarrow$ $y^{\frac{1}{m}} + \frac{1}{y^{\frac{1}{m}}} = 2x \quad \Rightarrow \quad \frac{(y^{1/m})^2 + 1}{y^{1/m}} = 2x$

$$\boxed{(y^{1/m})^2 - 2xy^{1/m} + 1 = 0}$$

let $y^{1/m} = u$

$\boxed{u^2 - (2x)u + 1 = 0} \quad \Leftrightarrow \quad au^2 + bu + c = 0 \qquad a = 1,\ b = -2x,\ c = 1$

$u = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \Rightarrow u = \frac{2x \pm \sqrt{4x^2 - 4(1)(1)}}{2(1)}$

$y^{1/m} = u = \frac{2x \pm 2\sqrt{x^2-1}}{2} = \boxed{x \pm \sqrt{x^2-1}}$

$\Rightarrow \quad y = (x \pm \sqrt{x^2-1})^m$ —— (I)

** $y_1 = m(x \pm \sqrt{x^2-1})^{m-1}\cdot\left[1 \pm \frac{1}{\cancel{2}\sqrt{x^2-1}}(\cancel{2}x)\right]$

$= m(x \pm \sqrt{x^2-1})^{m-1}\left[\frac{\sqrt{x^2-1} \pm x}{\sqrt{x^2-1}}\right]$

$(\sqrt{x^2-1} \pm x) \begin{cases} (\sqrt{x^2-1} + x) = +(x + \sqrt{x^2-1}) \\ (\sqrt{x^2-1} - x) = -(x - \sqrt{x^2-1}) \end{cases} \pm(x \pm \sqrt{x^2-1})$

$(\sqrt{x^2-1})y_1 = m(x \pm \sqrt{x^2-1})^{m-1}\left[\pm(x \pm \sqrt{x^2-1})^1\right]$

$= \pm m(x \pm \sqrt{x^2-1})^{m-1+1}$

$(\sqrt{x^2-1}\,y_1) = \pm m(x \pm \sqrt{x^2-1})^m = \pm my$

Square to B.S.

$(x^2-1)y_1^2 = m^2y^2 \quad \Rightarrow \boxed{(x^2-1)y_1^2 - m^2y^2 = 0}$

again diff. w.r.t x

$(x^2-1)[2y_1y_2] + (2x)y_1^2 - m^2[2yy_1] = 0$

$2y_1\left[(x^2-1)y_2 + xy_1 - m^2y\right] = 0$

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$$\boxed{(x^2-1)y_2 + xy_1 - m^2y = 0}$$ —— (A)

By Leibnitz's Thm, diff. eq (A) w.r.t. x, (n-times)

$\left\{y_{n+2}(x^2-1) + ny_{n+1}(2x) + \frac{n(n-1)}{\cancel{2!}}y_n(\cancel{2})\right\} + \left[y_{n+1}(x) + ny_n(1)\right] - m^2y_n = 0$

$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - \cancel{n} + \cancel{n} - m^2)y_n = 0$

9. If $y = cos(m\sin^{-1}x)$, Then shown that $(1-x^2)\,y_{n+2} - (2n+1)x\,y_{n+1} + (m^2-n^2)y_n = 0$ **[MU -Dec-08, 13,17] (8 marks)**

$\rightarrow$ $y = \cos[m\sin^{-1}x]$ —— (I)

9. If $y = cos(m\sin^{-1}x)$, Then shown that $(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} + (m^2-n^2)y_n = 0$ [MU -Dec-08, 13,17] (8 marks)

$\rightarrow\ y = \cos[m\sin^{-1}x]$ —①

$y_1 = -\sin[m\sin^{-1}x]\cdot\left[m\frac{1}{\sqrt{1-x^2}}\right]$

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Multiple by $(\sqrt{1-x^2})$ to B.S

$\sqrt{1-x^2}\,y_1 = -m\sin[m\sin^{-1}x]$

square to B.S

$(1-x^2)y_1^2 = m^2\sin^2[m\sin^{-1}x] = m^2[1-\cos^2(m\sin^{-1}x)] = m^2 - m^2\cos^2(m\sin^{-1}x)$ y^2

$(1-x^2)y_1^2 = m^2 - m^2y^2$

$(1-x^2)y_1^2 + m^2y^2 = m^2$

again diff.

$(1-x^2)[2y_1y_2] + (-2x)y_1^2 + m^2[2yy_1] = 0$

divide by $2y_1$ to B.S

$\boxed{(1-x^2)y_2 - xy_1 + m^2y = 0}$ —Ⓐ

$\uparrow v \quad \uparrow u$

By Leibnitz's Thm $\Rightarrow \left[y_{n+2}(1-x^2) + n\,y_{n+1}(-2x) + \frac{n(n-1)}{2!}y_n(-2)\right] - \left[y_{n+1}(x) + n\,y_n(1)\right] + m^2y_n = 0$

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n)y_n - ny_n + m^2y_n = 0$

$\boxed{(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0}$

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11. If $y = \sqrt{\frac{1+x}{1-x}}$, Then shown that $(1-x^2)\,y_n - [2(n-1)x+1]\,y_{n-1} - (n-1)(n-2)y_{n-2} = 0$ [MU -Dec-06] (8 marks)

$\rightarrow\ y^2 = \frac{(1+x)}{(1-x)}$ —①

diff. eq ①

$2yy_1 = \frac{(1-x)(1) - (1+x)(-1)}{(1-x)^2} = \frac{(1-x)+(1+x)}{(1-x)^2} = 2\frac{1}{(1-x)^2}$

Note this step

let $(1-x^2)$ multiple to B.S

$\cancel{2}(1-x^2)yy_1 = \cancel{2}\cdot\frac{(1-x^2)}{(1-x)^2} = \frac{\cancel{(1-x)}(1+x)}{\cancel{(1-x)}(1-x)} = \frac{1+x}{1-x} = y^2$

divide by y to B.S

$(1-x^2)y_1 = y \Rightarrow \boxed{(1-x^2)y_1 - y = 0}$ —Ⓐ

Apply Leibnitz's Thm $(n-1)$ times to B.S

$\frac{d^{n-1}}{dx^{n-1}}[y_1(1-x^2)] - \frac{d^{n-1}}{dx^{n-1}}y = 0$

$\left[y_n(1-x^2) + (n-1)y_{n-1}(-2x) + \frac{(n-1)(n-2)}{\cancel{2!}}y_{n-2}(\cancel{-2})\right] - y_{n-1} = 0$

$(1-x^2)y_n - 2(n-1)xy_{n-1} - (n-1)(n-2)y_{n-2} - y_{n-1} = 0$

$\boxed{(1-x^2)y_n - [2(n-1)x+1]y_{n-1} - (n-1)(n-2)y_{n-2} = 0}$

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