MODULE-1- COMPLEX NUMBERS (14 Marks)

10 February 2022 10:00

Dr. Uday Kashid , Ph.D.(Mathematics)

1. $z = x + iy$: *Caspar Wessel in 1799*
   Caspar Wessel (1745-1818), a Norwegian, was the first one to obtain and publish a suitable presentation of complex numbers. On march 10, 1797, Wessel presented his paper "On the Analytic Representation of Direction: An Attempt" to the Royal Danish Academy of Sciences.
2. Operations: $+, -, \times, \div$
3. Polar and Exponetial form:
4. Properties of Modulus and Arguments(Amplitude)
5. Examples 1) Find Modulus and Arguments : $z = -1 + i$
   2) If $|z-1| = |z+1|$, Then prove that Re(z)=0
6. **De Moivre's theorem**

$\sqrt{4} = \pm 2$

$\sqrt{-4}$ $\quad i = \sqrt{-1}$ $\quad \boxed{i^2 = -1}$

$\sqrt{4i^2} = \pm 2i$

Note :- If $z = x + iy$, $x, y \in \mathbb{R}$ $\quad \bar{z} = x - iy$ $\quad Re(z) = x$ and $Im(z) = y$

$|z| = \text{modulus of } z = \sqrt{x^2+y^2} = \sqrt{(Re)^2 + (Im)^2} = r$

$z = x + iy = (x, y)$

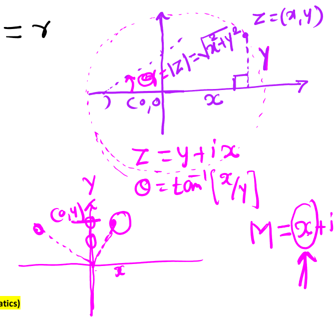
argument or Amplitude of z is denoted by 'θ'

$\arg(z) = amp(z) = \theta = \tan^{-1}\left[\frac{Im}{Re}\right] = \tan^{-1}\left[\frac{y}{x}\right]$

$\sin\theta = \frac{y}{r} \Rightarrow \boxed{y = r\sin\theta}$

$\cos\theta = \frac{x}{r} \Rightarrow \boxed{x = r\cos\theta}$

$\boxed{x^2 + y^2 = r^2}$



$M = x + iy$

If $z_1 = x_1 + iy_1$, $\quad z_2 = x_2 + iy_2$

$|z_1| = \sqrt{x_1^2 + y_1^2}$ $\quad |z_2| = \sqrt{x_2^2 + y_2^2}$

$\theta_1 = \tan^{-1}\left[\frac{y_1}{x_1}\right]$ $\quad \theta_2 = \tan^{-1}\left[\frac{y_2}{x_2}\right]$

① $z_1\bar{z_1} = (x_1 + iy_1)(x_1 - iy_1) = x_1^2 - i^2y_1^2 = x_1^2 + y_1^2$

$|z_1\bar{z_1}| = x_1^2 + y_1^2 = |z_1|^2 = |\bar{z_1}|^2$

② $|z_1 z_2| = |z_1| \cdot |z_2|$

③ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

④ $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

⑤ $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Examples

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1. **Find Modulus and Arguments : $z = -1 + i$**
2. **If $|z-1| = |z+1|$, Then prove that Re(z) = 0**
3. **Find the complex number z if $\arg(z+1) = \frac{\pi}{6}$ and $\arg(z-1) = \frac{2\pi}{3}$** [MU -Dec-11,May-08,12]
4. **Find the complex number z if $\arg(z+2i) = \frac{\pi}{4}$ and $\arg(z-2i) = \frac{3\pi}{4}$** [Ans: $z = x + iy = 2 + i0$]
5. **If $|z^2 - 1| = |z|^2 + 1$, Then prove that complex no z is purely imaginary.** [MU -Dec-07,16]
6. **Show that** $\left|\frac{z}{|z|} - 1\right| \le |\arg(z)|$, where $z = x + iy$ $\quad$ HW
7. **If $a^2 + b^2 + c^2 = 1$ and $b + ic = (a+1)z$, then prove that $\frac{1+iz}{1-iz} = \frac{a+ib}{1+c}$** [MU -Dec-11] $\quad$ HW.

1) $z = -1 + i = x + iy \Rightarrow x = -1, \; y = 1$

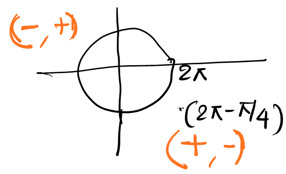
$|z| = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

$\theta = \arg(z) = \tan^{-1}\left[\frac{y}{x}\right] = \tan^{-1}\left[\frac{1}{-1}\right] = \tan^{-1}(-1)$

$\Rightarrow \tan\theta = -1 = \tan(\pi - \pi/4) = -\tan(\pi/4) = -1$

$\tan\theta = -1 = \tan(3\pi/4)$

$\boxed{\theta = 3\pi/4}$



2. If $|z-1| = |z+1|$, Then prove that Re(z) = 0

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→ Let $z = x + iy$

$z + 1 = (x+1) + iy$ $\quad z - 1 = (x-1) + iy$

$|z+1| = \sqrt{(x+1)^2 + y^2}$ —(I) $\quad |z-1| = \sqrt{(x-1)^2 + y^2}$ —(II)

We have $|z+1| = |z-1|$

$|z+1|^2 = |z-1|^2$

$(x+1)^2 + y^2 = (x-1)^2 + y^2$

$\Rightarrow \cancel{x^2} + 2x + \cancel{1} = \cancel{x^2} - 2x + \cancel{1}$

$4x = 0$

$x = 0$

$\boxed{Re(z) = x = 0}$

3. Find the complex number z if $\arg(z+1) = \frac{\pi}{6}$ and $\arg(z-1) = \frac{2\pi}{3}$ [MU -Dec-11,May-08,12]

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→ Let $z = x + iy$

$z + 1 = (x+1) + iy$

$\arg(z+1) = \theta_1 = \tan^{-1}\left[\frac{y}{x+1}\right] = \frac{\pi}{6}$

$\Rightarrow \frac{y}{x+1} = \tan(\pi/6) = \tan(30^\circ) = \frac{1}{\sqrt{3}}$

$\Rightarrow \sqrt{3}y = x + 1$

$\Rightarrow \boxed{x - \sqrt{3}y = -1}$ —(I)

$z - 1 = (x-1) + iy$

$\arg(z-1) = \theta_2 = \tan^{-1}\left[\frac{y}{x-1}\right] = \frac{2\pi}{3}$

$\frac{y}{x-1} = \tan(2\pi/3) = \tan(\pi - \pi/3)$

$= -\tan(\pi/3) = -\tan(60^\circ)$

$\frac{y}{x-1} = -\sqrt{3}$

$y = -\sqrt{3}x + \sqrt{3}$

$\boxed{\sqrt{3}x + y = \sqrt{3}}$ —(II)

$\Rightarrow$ solve eq (I) & (II)

$x - \sqrt{3}y = -1$

$+ \quad 3x + \sqrt{3}y = 3$ $\quad$ ←(eq (II) multiple by $\sqrt{3}$)

$4x + 0 = 2$

$\boxed{x = 1/2}$

$y = -\sqrt{3} \cdot \frac{1}{2} + \sqrt{3} = \sqrt{3}(1/2)$

$z = x + iy = \frac{1}{2} + \frac{\sqrt{3}}{2}i = \boxed{\frac{(1 + \sqrt{3}i)}{2}}$

1. If $|z^2 - 1| = |z|^2 + 1$, Then prove that complex no z is purely imaginary. [MU -Dec-07,16]

$\rightarrow$ let $z = x+iy$ $\qquad z = a+ib$

$z^2 = (x+iy)^2 = x^2+2xyi+y^2i^2 = (x^2-y^2)+i(2xy)$

$z^2-1 = (x^2-y^2-1)+i(2xy)$

$|z^2-1| = \sqrt{(x^2-y^2-1)^2+(2xy)^2}$ — (I)

$|z| = \sqrt{x^2+y^2} \Rightarrow |z|^2 = x^2+y^2$

$|z|^2+1 = x^2+y^2+1$ — (II)

sem-I – Complex Nos

III → Complex variable

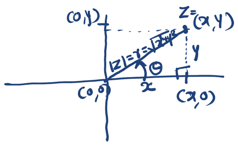
C.R. equ^n

IV Complex Integral

Residue Thm

Note:– Complex Nos.

1) Cartesian form :– $z = x+iy \qquad z = (x,y)$



2) Polar Form: -

$z = x+iy$ But $x = r\cos\theta$, $y = r\sin\theta$ $\Big\}\ x^2+y^2 = r^2$

$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \Rightarrow \theta = \tan^{-1}\left[\frac{y}{x}\right]$

$\boxed{z = r[\cos\theta + i\sin\theta]}$ → polar Form of complex No. z

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3) Exponential Form :-

We know $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Mod-6

Taylor series

$\rightarrow e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$ — (1)

$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^n}{n!} + \dots$

If $y = f(x)$

$f(-x) = \begin{cases} f(x) \rightarrow \text{Even Fu}^n \\ -f(x) \rightarrow \text{odd Fu}^n \end{cases}$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

Ex. If $y = \cos x = f(x)$

$f(-x) = \cos(-x) = \cos x = f(x)$ → Even

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

Ex. If $y = \sin x = f(x)$

$f(-x) = \sin(-x) = -\sin x = -f(x)$ → odd

put $x = i\theta$ in eq^n (1)

$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots$

$e^{i\theta} = \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right]$

$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$

$\rightarrow e^{-i\theta} = \frac{1}{e^{i\theta}} = \frac{1}{[\cos\theta + i\sin\theta]} \times \frac{[\cos\theta - i\sin\theta]}{[\cos\theta - i\sin\theta]} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta - i^2\sin^2\theta}$

$\boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$

(or) We know $e^{i\theta} = \cos\theta + i\sin\theta$

put θ by $(-\theta)$

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$\boxed{e^{-i\theta} = \cos\theta - i\sin(\theta)}$

3) Exponential Form

$z = r[\cos\theta + i\sin\theta]$

$\boxed{z = r\,e^{i\theta}}$

Exa. P.T $\left|\frac{z}{|z|} - 1\right| \leq |\arg z|$

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$\rightarrow z = x+iy = r[\cos\theta + i\sin\theta]$

$|z| = \sqrt{x^2+y^2} = r$

$\frac{z}{|z|} = \frac{r[\cos\theta + i\sin\theta]}{r} = \cos\theta + i\sin\theta$

$\left(\frac{z}{|z|} - 1\right) = (\cos\theta - 1) + i\sin\theta$ $\qquad 1-\cos\theta = 2\sin^2\theta/2$

$= -2\sin^2\theta/2 + i(2\sin\theta/2\cos\theta/2)$

$\dots\ [\cos\theta/2 + i\sin\theta/2]$

$\frac{z}{|z|}-1 = (\cos\theta - 1) + i\sin\theta$ $\qquad$ $1-\cos\theta = 2\sin^2\theta/2$

$= -2\sin^2\theta/2 + i(2\sin\theta/2\cos\theta/2)$

$= 2i^2\sin^2\theta/2 + 2i\sin\theta/2\cos\theta/2 = 2i\sin\theta/2[\cos\theta/2 + i\sin\theta/2]$

$\left|\frac{z}{|z|}-1\right| = \left|2i\sin\theta/2(\cos\theta/2 + i\sin\theta/2)\right|$ $\qquad$ $|z_1 . z_2\, z_3| = |z_1| . |z_2| . |z_3|$

$= 2|i|\;|\sin\theta/2|\;|\cos\theta/2 + i\sin\theta/2|$

But $|i| = |0+i| = \sqrt{0^2+1^2} = 1$ $\Rightarrow \boxed{|i| = |-i| = 1}$

$|\cos\theta/2 + i\sin\theta/2| = \sqrt{\cos^2\theta/2 + \sin^2\theta/2} = 1$

$\left|\frac{z}{|z|}-1\right| = 2|\sin\theta/2|$ $\qquad$ But $|\sin\theta| \leq |\theta|$

$|\sin\theta/2| \leq |\theta/2|$

$\leq 2|\theta/2|$

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$\leq |\theta|$

$\left|\frac{z}{|z|}-1\right| \leq |\arg z|$

De Moivre's theorem

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If $z = r(\cos\theta + i\sin\theta)$ then by De − Moivre's theorem $z^n = r^n(\cos n\theta + i\sin n\theta)$

$r = 1$

Note: If $z = \cos\theta + i\sin\theta$ Then

1. $z^n + \frac{1}{z^n} = 2\cos(n\theta)$
2. $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$

$\rightarrow z^n = r^n[\cos\theta + i\sin\theta]^n$

$\rightarrow \;= r^n[e^{i\theta}]^n = r^n e^{i(n\theta)}$

$\rightarrow z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$

$\boxed{z^n = r^n[\cos\theta + i\sin\theta]^n = r^n[\cos(n\theta) + i\sin(n\theta)]}$

$z = r[\cos\theta + i\sin\theta]$

$z^{-n} = r^{-n}[\cos\theta + i\sin\theta]^{-n}$

$z^{-n} = r^{-n}[\cos(-n\theta) + i\sin(-n\theta)]$

$\boxed{z^{-n} = r^{-n}[\cos(n\theta) - i\sin(n\theta)]}$

Examples:

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1. If $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$, and $\bar{z}$ is the conjugate of z, then prove that $(z)^{10} + (\bar{z})^{10} = 0$. [MU -Dec-06]
2. If $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $\bar{z}$ is the conjugate of z, then prove that $(z)^{15} + (\bar{z})^{15} = -2$.
3. Prove that $\left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n = \cos\left(\frac{n\pi}{2} - n\alpha\right) + i\sin\left(\frac{n\pi}{2} - n\alpha\right)$, [MU -Dec-01,04,May-15]
4. If $\alpha = 1+i$, $\beta = 1-i$, and $\cot(\varphi) = x+1$, then prove that $\frac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta} = \sin(n\varphi)\operatorname{cosec}^n(\varphi)$ [MU-Dec-02,04,11,14]
5. If $z = -1 + i\sqrt{3}$, Prove that $\left(\frac{z}{2}\right)^n + \left(\frac{2}{z}\right)^n = \begin{cases} -1, & \text{if 'n' is not multiple of 3} \\ 2 & \text{if 'n' is multiple of 3} \end{cases}$ [MU -Dec-15]
6. Prove that $\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^n + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^n = \begin{cases} -1, & \text{if } n = 3k \pm 1 \\ 2 & \text{if } n = 3k \end{cases}$, where k is an integer
7. Prove that $(1+i)^{100} + (1-i)^{100} = -2^{51}$
8. Prove that $\left(\frac{1+7i}{(2-i)^2}\right)^{4n} = (-4)^n$, where 'n' is positive integer [MU -Dec-04, 17]
9. If $x + iy = \sqrt[3]{a+ib}$, then $\frac{a}{x} + \frac{b}{y} = ?$

Thur − 2-3-phy
3-5-math

$\boxed{\text{IF } r = 1}$

- Note: If $z = \cos\theta + i\sin\theta$ Then
  - $z^n + \frac{1}{z^n} = 2\cos(n\theta)$
  - $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$

$z = r[\cos\theta + i\sin\theta] = 1[\cos\theta + i\sin\theta] = e^{i\theta}$

$\frac{1}{z} = \cos\theta - i\sin\theta = e^{-i\theta}$

(i) $z + \frac{1}{z} = [\cos\theta + i\sin\theta] + [\cos\theta - i\sin\theta]$

$\boxed{z + \frac{1}{z} = 2\cos\theta} \Rightarrow \cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right]$

$\Rightarrow \boxed{\cos\theta = \left[\frac{e^{i\theta} + e^{-i\theta}}{2}\right]}$

(ii) $\boxed{z - \frac{1}{z} = 2i\sin\theta}$

$\boxed{\sin\theta = \left[\frac{e^{i\theta} - e^{-i\theta}}{2i}\right]}$

$z^2 = [\cos\theta + i\sin\theta]^2 = \cos(2\theta) + i\sin(2\theta)$

$\frac{1}{z^2} = \cos(2\theta) - i\sin(2\theta)$

(i) $z^2 + \frac{1}{z^2} = 2\cos 2\theta \Rightarrow \cos(2\theta) = \left[\frac{e^{i2\theta} + e^{-i2\theta}}{2}\right]$

$z^2 - \frac{1}{z^2} = 2i\sin(2\theta) \Rightarrow \sin(2\theta) = \frac{e^{i2\theta} - e^{-i2\theta}}{2i}$

Similary

$z^n = [\cos\theta + i\sin\theta]^n = \cos(n\theta) + i\sin(n\theta)$

$\frac{1}{z^n} = \cos(n\theta) - i\sin(n\theta)$

(i) $z^n + \frac{1}{z^n} = 2\cos(n\theta)$

(ii) $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$

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1. If $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$, and $\bar{z}$ is the conjugate of z, then prove that $(z)^{10} + (\bar{z})^{10} = 0$. [MU -Dec-06]

$\rightarrow z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = x + iy \Rightarrow x = 1/\sqrt{2},\ y = 1/\sqrt{2}$

$z = r[\cos\theta + i\sin\theta] \Rightarrow r = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(1/\sqrt{2}\right)^2} = 1$, $\theta = \tan^{-1}\left[\frac{y}{x}\right] = \tan^{-1}\left[\frac{1/\sqrt{2}}{1/\sqrt{2}}\right] = \pi/4$

$z = 1\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$ — (i)

$\bar{z} = \cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)$ — (ii)

$z^{10} = \left[\cos(\pi/4) + i\sin(\pi/4)\right]^{10} = \cos\left(\frac{10\pi}{4}\right) + i\sin\left(\frac{10\pi}{4}\right)$

$(\bar{z})^{10} = \left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]^{10} = \cos\left(\frac{10\pi}{4}\right) - i\sin\left(\frac{10\pi}{4}\right)$

Addition

$(z)^{10} + (\bar{z})^{10} = 2\cos\left(\frac{10\pi}{4}\right) = 2\cos\left(5\frac{\pi}{2}\right)$ $\cos[n\pi/2] = 0$, $n = \text{odd}$

$(z^{10}) + (\bar{z})^{10} = 0$.

$[\cos\theta + i\sin\theta]^n$

3. Prove that $\left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n = \cos\left(\frac{n\pi}{2} - n\alpha\right) + i\sin\left(\frac{n\pi}{2} - n\alpha\right)$, [MU -Dec-01,04,15] Dr. Uday Kashid, PhD (Mathematics)

$\rightarrow$ Let $z^n = \left[\frac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha}\right]^n$

$\cos\left[\frac{\pi}{2} - \alpha\right] = \sin\alpha$

$\sin\left[\frac{\pi}{2} - \alpha\right] = \cos\alpha$

$z^n = \left[\frac{1+\cos\left(\frac{\pi}{2}-\alpha\right) + i\sin\left(\frac{\pi}{2}-\alpha\right)}{1+\cos\left(\frac{\pi}{2}-\alpha\right) - i\sin\left(\frac{\pi}{2}-\alpha\right)}\right]^n$

$\therefore \begin{cases} 1 + \cos\theta = 2\cos^2(\theta/2) \\ 1 + \cos\left(\frac{\pi}{2}-\alpha\right) = 2\cos^2\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \\ \sin\theta = 2\sin(\theta/2)\cos(\theta/2) \end{cases}$

$z^n = \left[\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) + i\,2\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) - i\,2\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right]^n$

$z = \cos\theta + i\sin\theta \Rightarrow z^n = [\cos(n\theta) + i\sin n\theta]$

$\Rightarrow \therefore \bar{z} = \frac{1}{z} = \cos\theta - i\sin\theta$ $(\bar{z})^n = \frac{1}{z^n} = [\cos(n\theta) - i\sin n\theta]$

$\frac{1}{z} = e^{-i\theta} \Rightarrow \frac{1}{z^n} = e^{-in\theta} = \cos n\theta - i\sin n\theta$

$z^n = \left[\frac{\cancel{2\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\left[\cos(\pi/4 - \alpha/2) + i\sin\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right]}{\cancel{2\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\left[\cos(\pi/4 - \alpha/2) - i\sin(\pi/4 - \alpha/2)\right]}\right]^n$

By De-moivre's Thm

$z^n = \frac{\left[\cos\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right) + i\sin\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right)\right]}{\left[\cos\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right) - i\sin\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right)\right]} = \frac{e^{i\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right)}}{e^{-i\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right)}} = e^{i\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right)} \cdot e^{i\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right)}$

$z^n = e^{i2\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right)} = e^{i\left(\frac{n\pi}{2} - n\alpha\right)}$

$\boxed{z^n = \cos\left(\frac{n\pi}{2} - n\alpha\right) + i\sin\left(\frac{n\pi}{2} - n\alpha\right)}$

4. If $\alpha = 1+i$, $\beta = 1-i$, and $\cot(\varphi) = x+1$, then prove that $\frac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta} = \sin(n\varphi)\operatorname{cosec}^n(\varphi)$ [MU-Dec-02,04,11,14]

$\rightarrow \alpha = 1 + i$ — (i) $\beta = 1 - i$ — (ii) $x = \cot\phi - 1$ — (iii)

(i) $x + \alpha = [\cot\phi - 1] + [1 + i] = \cot\phi + i = \frac{\cos\phi}{\sin\phi} + i = \frac{\cos\phi + i\sin\phi}{\sin\phi}$

$(x+\alpha)^n = \frac{[\cos\phi + i\sin\phi]^n}{\sin^n\phi} = \operatorname{cosec}^n\phi\,[\cos(n\phi) + i\sin(n\phi)]$ $\rightarrow$ By De-moivre's Thm — (iv)

(ii) $x + \beta = [\cot\phi - 1] + [1 - i] = \cot\phi - i = \frac{\cos\phi - i\sin\phi}{\sin\phi}$.

$\dfrac{}{\sin^n\phi} = \text{cosec}\,\phi\,[\cos(n\phi)+i\sin(n\phi)]$ — (IV)

(ii) $x+\beta = [\cot\phi - 1] + [1-i] = \cot\phi - i = \dfrac{\cos\phi - i\sin\phi}{\sin\phi}$

$(x+\beta)^n = \dfrac{[\cos\phi - i\sin\phi]^n}{\sin^n\phi} = [\cos(n\phi) - i\sin(n\phi)]\,\text{cosec}^n\phi$ — (V)

(iii) $\alpha - \beta = (1+i)-(1-i) = 2i$ — (VI)

$LHS = \dfrac{(x+\alpha)^n - (x+\beta)^n}{\alpha-\beta} = \dfrac{\text{cosec}^n\phi\,[\cos n\phi + i\sin n\phi] - \text{cosec}^n\phi\,[\cos n\phi - i\sin n\phi]}{2i}$

$LHS = \dfrac{\text{cosec}^n\phi}{\cancel{2i}}\left[\cancel{2i}\sin(n\phi)\right] = \boxed{\text{cosec}^n\phi\,\sin(n\phi)}$

5. If $z=-1+i\sqrt{3}$, Prove that $\left(\frac{z}{2}\right)^n+\left(\frac{2}{z}\right)^n = \begin{cases}-1, & \text{if 'n' is not multiple of 3}\\ 2 & \text{if 'n' is multiple of 3}\end{cases}$ [MU -Dec-15]

$\Rightarrow z = x+iy = -1+i\sqrt{3} = r(\cos\theta + i\sin\theta)$ $\Rightarrow x=-1,\ y=\sqrt{3}$, $z=(\overset{-}{-1},\overset{+}{\sqrt{3}})$ lies in IInd quadrant.

$r=\sqrt{x^2+y^2}=\sqrt{(-1)^2+(\sqrt{3})^2}=\sqrt{4}=2$

$\theta = \tan^{-1}\left[\frac{Im(z)}{Re(z)}\right] = \tan^{-1}\left[\frac{\sqrt{3}}{-1}\right] = \tan^{-1}[-\sqrt{3}]$ $\Rightarrow \tan\theta = -\sqrt{3}$

$\tan(\pi - \pi/3) = -\tan(\pi/3) = -\sqrt{3}$

$\tan(2\pi/3) = -\sqrt{3}$

$\Rightarrow \boxed{\theta = 2\pi/3}$

$z = 2[\cos 2\pi/3 + i\sin 2\pi/3]$ — (I)

$\bar{z} = 2[\cos 2\pi/3 - i\sin(2\pi/3)]$

$\rightarrow \frac{z}{2} = \cos(2\pi/3) + i\sin(2\pi/3)$

$\frac{2}{z} = \dfrac{1}{\cos(2\pi/3)+i\sin(2\pi/3)} = \cos(2\pi/3) - i\sin(2\pi/3) = (\bar{z}/2)$

$LHS = \left(\frac{z}{2}\right)^n + \left(\frac{2}{z}\right)^n = [\cos(2\pi/3)+i\sin(2\pi/3)]^n + [\cos(2\pi/3) - i\sin(2\pi/3)]^n$

By De-moivre's thm

$= \cos(2n\pi/3) + i\sin(2n\pi/3) + \cos(2n\pi/3) - i\sin(2n\pi/3)$

$\boxed{\left(\frac{z}{2}\right)^n + \left(\frac{2}{z}\right)^n = 2\cos\left(\frac{2n\pi}{3}\right)}$ — (A) $n \in \pm$ Integer, Fraction

$n = \frac{4}{5}$

$k=0$	± 1
$k=1$	$2, 4$
$k=2$	$5, 7$
$k=3$	$8, 10$
$k=4$	$11, 13$

case-(I) If n is multiple of 3 i.e. $n=3k$, $k=1,2,3,4,$

$\left(\frac{z}{2}\right)^n + \left(\frac{2}{z}\right)^n = 2\cos\left[\frac{2(3k)\pi}{3}\right] = 2\cos[2k\pi] = 2[1] = \boxed{2}$

For $k=1,2,3,\ldots$ $\cos(2k\pi) = 1$

case-(II) If n is not multiple of 3 $\Rightarrow$ i.e. $n = 3k\pm 1$, $k=0,1,2,3,\ldots$

$\left(\frac{z}{2}\right)^n + \left(\frac{2}{z}\right)^n = 2\cos\left[\frac{2(3k\pm1)\pi}{3}\right] = 2\cos\left[\frac{6\pi k \pm 2\pi}{3}\right] = 2\cos[2k\pi \pm 2\pi/3] = 2\cos(\pm 2\pi/3)$

$= 2\cos(2\pi/3) = 2\cos[\pi - \pi/3] = 2[-\cos(\pi/3)] = 2(-1/2) = \boxed{-1}$

7. Prove that $(1+i)^{100}+(1-i)^{100} = -2^{51}$

$\rightarrow z_1 = 1+i = x_1 + iy_1 = r_1[\cos\theta_1 + i\sin\theta_1]$ $\Rightarrow x_1 = 1,\ y_1 = 1$ (Ist quadrant)

$r_1 = \sqrt{1^2+1^2} = \sqrt{2}$, $\theta_1 = \tan^{-1}\left[\frac{1}{1}\right] = \pi/4$

$z_1 = \sqrt{2}[\cos\pi/4 + i\sin\pi/4]$

$z_1^{100} = (1+i)^{100} = (\sqrt{2})^{100}\left[\cos\left(\frac{100\pi}{4}\right) + i\sin\left(\frac{100\pi}{4}\right)\right]$ By De-moivre's Thm

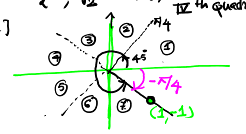
$= 2^{50}[\cos(25\pi) + i\sin(25\pi)]$ — (I)

$(1+3i)^{100} + (1-5i)^{100}$

let $z_2 = 1-i = x_2 + iy_2 = r_2[\cos\theta_2 + i\sin\theta_2]$ $x_2 = 1,\ y_2 = -1$ $\Rightarrow (1,-1)$ lies in IV th quadrant

$r_2 = \sqrt{1^2+(-1)^2} = \sqrt{2}$, $\theta_2 = \tan^{-1}\left[\frac{-1}{1}\right] = \tan^{-1}[-1]$

$\tan(\theta_2) = -1$



$\tan(2\pi - \pi/4) = -1$

$\tan(-\pi/4) = \tan[7\pi/4] = -1$

$\boxed{\theta_2 = -\pi/4}$

$z_2 = \sqrt{2}[\cos(-\pi/4) + i\sin(-\pi/4)] = \sqrt{2}[\cos(\pi/4) - i\sin(\pi/4)]$ — (II)

$z_2^{100} = (1-i)^{100} = (\sqrt{2})^{100}[\cos(\pi/4) - i\sin(\pi/4)]^{100} = 2^{50}[\cos(25\pi) - i\sin(25\pi)]$ → By De-moivre's Th

Add (I) & (II)

$(1+i)^{100} + (1-i)^{100} = 2^{50}\{\cos(25\pi) + i\sin(25\pi) + \cos(25\pi) - i\sin(25\pi)\} = 2^{50}\cdot 2\cos(25\pi)$

$(2i)^{50} + (-2i)^{50}$ $= 2^{51}(-1) = -\boxed{2^{51}}$

$+ 2^{50} i^{50}$

8. Prove that $\left(\frac{1+7i}{(2-i)^2}\right)^{4n} = (-4)^n$, where 'n' is positive integer [MU -Dec-04, 17]

$\rightarrow$ let $z = \dfrac{1+7i}{} = x+iy = \dfrac{(1+7i)}{} = \dfrac{(1+7i)}{}\times\dfrac{(3+4i)}{} = \dfrac{3+4i+21i-28}{} = \dfrac{-25+25i}{25}$

8. Prove that $\left(\frac{1+7i}{(2-i)^2}\right)^{4n} = (-4)^n$ **, where 'n' is positive integer** **[MU -Dec-04, 17]**

$\rightarrow$ let $z = \dfrac{1+7i}{(2-i)^2} = x+iy = \dfrac{(1+7i)}{4-4i-1} = \dfrac{(1+7i)}{(3-4i)} \times \dfrac{(3+4i)}{(3+4i)} = \dfrac{3+4i+21i-28}{(9+16)} = \dfrac{-25+25i}{25}$

$z = -1+i \Rightarrow x=-1,\ y=1 \rightarrow$ IInd quadrant

$r = \sqrt{2},\quad \theta = \tan^{-1}\left[\frac{1}{-1}\right] = \tan^{-1}(-1) = 3\pi/4$

$z = \sqrt{2}\left[\cos 3\pi/4 + i\sin 3\pi/4\right] \Rightarrow (z)^{4n} = \left[\dfrac{1+7i}{(2-i)^2}\right]^{4n} = (\sqrt{2})^{4n}\left[\cos(3n\pi) + i\sin(3n\pi)\right]$

$z = 2^{2n}\left[\cos(3n\pi) + i\sin(3n\pi)\right] = (2^2)^n\left[(-1)^n + 0\right]$

$= 4^n(-1)^n = [4(-1)]^n = [-4]^n$

$\because \sin(3n\pi) = 0,\ n = 0, 1, 2, 3 -$

$\cos(3n\pi) = (-1)^{3n},\ n = 0, 1, 2, 3 - = [(-1)^3]^n = (-1)^n$

$\cos[n\pi] = (-1)^n$

9. If $x+iy = \sqrt[3]{a+ib}$, then $\frac{a}{x}+\frac{b}{y} = ?$

$\rightarrow (x+iy)^3 = a+ib$

$x^3 + 3x^2(iy) + 3(x)(iy)^2 + (iy)^3 = a+ib$

$[x^3 - 3xy^2] + i[3x^2y - y^3] = a+ib$

$\Rightarrow x^3 - 3xy^2 = a$ & $3x^2y - y^3 = b$

divide by x — divide by y

$x^2 - 3y^2 = \frac{a}{x}$ — (I) $\qquad 3x^2 - y^2 = \frac{b}{y}$ — (II)

$\frac{a}{x} + \frac{b}{y} = (x^2 - 3y^2) + (3x^2 - y^2) = 4(x^2 - y^2)$

Examples based on expansion of $\cos^n(\theta), \sin^n(\theta)$ in terms of sine or cosine of multiple of θ

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Note: If $z = \cos\theta + i\sin\theta$, Then

1. $z + \frac{1}{z} = 2\cos(\theta) = (e^{i\theta} + e^{-i\theta})$
2. $z - \frac{1}{z} = 2i\sin(\theta) = (e^{i\theta} - e^{-i\theta})$

$\boxed{r=1}$

Note: If $z = \cos\theta + i\sin\theta$, Then

1. $z^n + \frac{1}{z^n} = 2\cos(n\theta) = e^{in\theta} + e^{-in\theta}$
2. $z^n - \frac{1}{z^n} = 2i\sin(n\theta) = e^{in\theta} - e^{-in\theta}$

Pascal's Triangle:-

- $(a+b)^0 \rightarrow$ 1
- $(a+b)^1 \rightarrow$ 1 1
- $(a+b)^2 \rightarrow$ 1 2 1
- $(a+b)^3 \rightarrow$ 1 3 3 1
- $(a+b)^4 \rightarrow$ 1 4 6 4 1
- $(a+b)^5 \rightarrow$ 1 5 10 10 5 1
- $(a+b)^6 \rightarrow$ 1 6 15 20 15 6 1
- $(a+b)^7 \rightarrow$ 1 7 21 35 35 21 7 1

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Examples:

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1. Show that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$ [MU -Dec-06, May-18] HW
2. Show that $\sin^7\theta = -\frac{1}{2^6}(\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta)$ [MU -May-14]
3. Show that $\cos^7\theta = \frac{1}{2^6}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$ [MU -Dec-12,14] HW
4. If $\sin^4\theta\cos^3\theta = a\cos\theta + b\cos 3\theta + c\cos 5\theta + d\cos 7\theta$, Then find a,b,c, d. [MU -2000,02,09,17]
5. Prove that $\cos^6\theta - \sin^6\theta = \frac{1}{16}(\cos 6\theta + 15\cos 2\theta)$ [MU -Dec-07,16]
6. ✓ Prove that $\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$ [MU -01,11,16]
7. ✓ If $\cos^6\theta + \sin^6\theta = (\alpha\cos 4\theta + \beta)$, Then Prove that $\alpha + \beta = 1$ [MU -Dec-15, May-16]
8. Prove that $\cos^8\theta + \sin^8\theta = \frac{1}{64}(\cos 8\theta + 28\cos 4\theta + 35)$ [MU -01,11,16]
9. If $\sin^3\theta\cos^5\theta = -\frac{1}{2^7}(\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta)$

2. Show that $\sin^7\theta = -\frac{1}{2^6}(\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta)$

$\rightarrow$ Let $z = \cos\theta + i\sin\theta$

$\frac{1}{z} = \bar{z} = \cos\theta - i\sin\theta$

$\Rightarrow z - \frac{1}{z} = 2i\sin\theta$

$\boxed{z + \frac{1}{z} = 2\cos\theta}$, $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$

$(A+B)^n = {}^nC_0 A^n B^0 + {}^nC_1 A^{n-1}B^1 + \cdots$

$= {}^7C_0 \quad {}^7C_1 \quad {}^7C_2 \quad {}^7C_3$

$1 \quad 7 \quad \frac{7\times 6}{2}, \quad \frac{7\times 6\times 5}{3\times 2}$

$\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$

$\sin^7\theta = \text{LHS} = \frac{1}{2^7 i^7}(z - \frac{1}{z})^7 = \frac{1}{2^7 (i^2)\cdot(i^2)(i^2)(i)}\left[z - \frac{1}{z}\right]^7$

$= \frac{1}{2^7(-i)}\left[z^7 - 7z^6(\frac{1}{z}) + 21z^5(\frac{1}{z^2}) - 35z^4(\frac{1}{z^3}) + 35z^3(\frac{1}{z^4}) - 21z^2(\frac{1}{z^5}) + 7z(\frac{1}{z^6}) - \frac{1}{z^7}\right]$

$= \frac{1}{2^7(-i)}\left[z^7 - 7z^5 + 21z^3 - 35z + 35\frac{1}{z} - 21\frac{1}{z^3} + 7\frac{1}{z^5} - \frac{1}{z^7}\right]$

$= \frac{-1}{2^7(i)}\left[(z^7 - \frac{1}{z^7}) - 7(z^5 - \frac{1}{z^5}) + 21(z^3 - \frac{1}{z^3}) - 35(z - \frac{1}{z})\right]$

But $z = \cos\theta + i\sin\theta \Rightarrow z^7 = \cos 7\theta + i\sin 7\theta$

$\frac{1}{z} = \cos\theta - i\sin\theta \Rightarrow \frac{1}{z^7} = \cos 7\theta - i\sin 7\theta$

$\Big\}\ z^7 - \frac{1}{z^7} = 2i\sin(7\theta)$

$z^n - \frac{1}{z^n} = 2i\sin(n\theta) \rightarrow$ Demoivre's Thm

$\sin^7\theta = \frac{-1}{2^7 i}\left[(2i)\sin 7\theta - 7(2i)\sin 5\theta + 21(2i)\sin 3\theta - 35(2i)\sin\theta\right]$

$= \frac{-1(2i)}{2^7(i)}\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right]$

$= -\frac{1}{2^6}\left[\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right]$

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4. If $\sin^4\theta\cos^3\theta = a\cos\theta + b\cos 3\theta + c\cos 5\theta + d\cos 7\theta$, Then find a,b,c, d. [MU -2000,02,09,17]

$\rightarrow$ let $z = \cos\theta + i\sin\theta \qquad \frac{1}{z} = \cos\theta - i\sin\theta$

$\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$ and $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$

$\text{LHS} = \sin^4\theta\cos^3\theta = \frac{1}{2^4 i^4}(z - \frac{1}{z})^4 \cdot \frac{1}{2^3}(z + \frac{1}{z})^3 = \frac{1}{2^7}\left[z - \frac{1}{z}\right]\left[(z - \frac{1}{z})(z + \frac{1}{z})\right]^3$

$= \frac{1}{2^7}(z - \frac{1}{z})\left[z^2 - \frac{1}{z^2}\right]^3 = \frac{1}{2^7}(z - \frac{1}{z})\left[(z^2)^3 - 3(z^2)^2(\frac{1}{z^2}) + 3(z^2)^1(\frac{1}{z^2})^2 - (\frac{1}{z^2})^3\right]$

$LHS = \sin\theta \cos\theta = \frac{1}{2^4 i^4}(z - \frac{1}{z}) \cdot \frac{1}{2^3}(z + \frac{1}{z}) = \frac{1}{2^7}[\ \ ][\ \ ]$

$$= \frac{1}{2^7}\left(z-\frac{1}{z}\right)\left[z^2-\frac{1}{z^2}\right]^3 = \frac{1}{2^7}\left(z-\frac{1}{z}\right)\left[(z^2)^3 - 3(z^2)^2\left(\frac{1}{z^2}\right) + 3(z^2)^1\left(\frac{1}{z^2}\right)^2 - \left(\frac{1}{z^2}\right)^3\right]$$

$$= \frac{1}{2^7}\left[\left(z^6 - \frac{1}{z^6}\right) - 3\left(z^2 - \frac{1}{z^2}\right)\right]\left(z - \frac{1}{z}\right)$$

$$= \frac{1}{2^7}\left\{z^7 - \frac{1}{z^5} - z^5 + \frac{1}{z^7} - 3(z^3) + 3\frac{1}{z} + 3z - 3\frac{1}{z^3}\right]$$

$$= \frac{1}{2^7}\left\{\left(z^7 + \frac{1}{z^7}\right) - \left(z^5 + \frac{1}{z^5}\right) - 3\left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)\right]$$

But $z^n + \frac{1}{z^n} = 2\cos(n\theta)$

$$LHS = \frac{1}{2^7}\left\{2\cos 7\theta - 2\cos 5\theta - 3(2)\cos 3\theta + 3(2)\cos(\theta)\right]$$

$$LHS = \frac{1}{2^6}\left[\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta\right] = a\cos\theta + b\cos 3\theta + c\cos 5\theta + d\cos 7\theta$$

$$a = \frac{3}{2^6}, \quad b = \frac{-3}{2^6}, \quad c = \frac{-1}{2^6}, \quad d = \frac{1}{2^6}$$

7. If $\cos^6\theta + \sin^6\theta = (\alpha\cos 4\theta + \beta)$, Then Prove that $\alpha + \beta = 1$

$\rightarrow$ Let $z = \cos\theta + i\sin\theta$ $\qquad \frac{1}{z} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

(I) $$\cos^6\theta = \frac{1}{2^6}\left(z+\frac{1}{z}\right)^6 = \frac{1}{2^6}\left[z^6 + 6z^5\frac{1}{z} + 15z^4\frac{1}{z^2} + 20z^3\frac{1}{z^3} + 15z^2\frac{1}{z^4} + 6z\frac{1}{z^5} + \frac{1}{z^6}\right]$$

$$= \frac{1}{2^6}\left[z^6 + 6z^4 + 15z^2 + 20 + 15\frac{1}{z^2} + 6\frac{1}{z^4} + \frac{1}{z^6}\right]$$

$$= \frac{1}{2^6}\left[\left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20\right]$$

$$\cos^6\theta = \frac{1}{2^6}\left[2\cos(6\theta) + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20\right] \quad \text{— (I)}$$

(II) $$\sin^6\theta = \frac{1}{2^6 i^6}\left(z - \frac{1}{z}\right)^6 = \frac{1}{2^6(-1)}\left[z^6 - 6z^5\frac{1}{z} + 15z^4\frac{1}{z^2} - 20z^3\frac{1}{z^3} + 15z^2\frac{1}{z^4} - 6z\frac{1}{z^5} + \frac{1}{z^6}\right]$$

$$= \frac{-1}{2^6}\left[z^6 - 6z^4 + 15z^2 - 20 + 15\frac{1}{z^2} - 6\frac{1}{z^4} + \frac{1}{z^6}\right]$$

$$= -\frac{1}{2^6}\left[\left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20\right]$$

$$\sin^6\theta = -\frac{1}{2^6}\left[2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20\right] \quad \text{— (II)}$$

Add (I) + (II)

$$\cos^6\theta + \sin^6\theta = \frac{1}{2^6}\left[\cancel{2\cos 6\theta} + 12\cos 4\theta + \cancel{30\cos 2\theta} + 20 - \cancel{2\cos 6\theta} + 12\cos 4\theta - \cancel{30\cos 2\theta} + 20\right]$$

$$= \frac{1}{2^6}\left[24\cos 4\theta + 40\right] = \frac{3\times2\times2\times2}{2^6}\cos 4\theta + \frac{5\times2\times2\times2}{2^6} = \frac{3}{8}\cos 4\theta + \frac{5}{8} = \alpha\cos 4\theta + \beta$$

Dr. Uday Kashid, PhD (Mathematics) $\Rightarrow \alpha = 3/8, \ \beta = 5/8 \Rightarrow \alpha + \beta = 1$

Expansion of Cos($n\theta$), sin($n\theta$) in power of cos (θ), sin (θ)

Dr. Uday Kashid, PhD (Mathematics)

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$z = \cos\theta + i\sin\theta$

Note: By De Moivre's thm:. $(\cos\theta + i\sin\theta)^n = \boxed{\cos n\theta + i\sin n\theta}$

But by binomial expansion to LHS

$^nC_0 \cos^n(\theta)(i\sin\theta)^0 + {}^nC_1 \cos^{n-1}(\theta)(i\sin\theta)^1 + {}^nC_2 \cos^{n-2}(\theta)(i\sin\theta)^2 + \cdots + {}^nC_n \cos^0(\theta)(i\sin\theta)^n = \cos n\theta + i\sin n\theta$

By equating Re & Im parts ...

- $\cos(n\theta) = {}^nC_0 \cos^n(\theta) - {}^nC_2 \cos^{n-2}(\theta)(\sin^2(\theta)) + {}^nC_4 \cos^{n-4}(\theta)(\sin^4(\theta)) - \cdots$ Dr. Uday Kashid, PhD (Mathematics)
- $\sin(n\theta) = {}^nC_1 \cos^{n-1}(\theta)(\sin\theta) - {}^nC_3 \cos^{n-3}(\theta)(\sin^3(\theta)) + {}^nC_5 \cos^{n-5}(\theta)\left(\sin^5(\theta)\right) - \cdots$

Examples: Dr. Uday Kashid, PhD (Mathematics)

1. **Prove that** i) $\cos 3\theta = \cos^3(\theta) - 3\cos^1(\theta)\sin^2(\theta)$, ii) $\sin 3\theta = 3\sin(\theta)\cos^2(\theta) - \sin^3(\theta)$

HW 2. **Show that** $\tan 5\theta = \frac{5\tan\theta - 10\tan^3(\theta) + \tan^5(\theta)}{1 - 10\tan^2(\theta) + 5\tan^4(\theta)}$ **[MU -Dec-03]**

HW 3. **Show that** $\tan 7\theta = \frac{7\tan\theta - 35\tan^3(\theta) + 21\tan^5(\theta) - \tan^7(\theta)}{1 - 21\tan^2(\theta) + 35\tan^4(\theta) - 7\tan^6(\theta)}$ **[MU -Dec-03,12]**

HW 4. **Using De Moivre's thm, Express** $\frac{\sin 7\theta}{\sin\theta}$ **in powers of** $\sin\theta$ **only.** **[MU -Dec-15]**

5. **If** $\cos 6\theta = \overset{{}^6C_0}{a}\cos^6(\theta) + \overset{-{}^6C_2}{b}\cos^4(\theta)\sin^2(\theta) + \overset{{}^6C_4}{c}\cos^2(\theta)\sin^4(\theta) + \overset{-{}^6C_6}{d}\sin^6(\theta)$, **Then find a, b, c, d.** **[MU -April-21 online exam]**

HW 6. **If** $\sin 6\theta = \overset{{}^6C_1}{a}\cos^5(\theta)(\sin\theta) + \overset{-{}^6C_3}{b}\cos^3(\theta)\sin^3(\theta) + \overset{{}^6C_5}{c}\cos(\theta)\sin^5(\theta)$, **Then find a, b, c.** **[MU -95,05]**

HW 7. **Using De Moivre's thm, Express** $\frac{\sin 6\theta}{\sin 2\theta} = 16\cos^4(\theta) - 16\cos^2(\theta) + 3$ **[MU -Dec-04,14]**

$n=6$ $\sin 6\theta = {}^6C_1\cos^5\theta\sin\theta - {}^6C_3\cos^3\theta\sin^3\theta + {}^6C_5\cos\theta\sin^5\theta$

$n=2$, $\sin 2\theta = {}^2C_1\cos\theta\sin\theta = 2\cos\theta\sin\theta$

1. $\cos 3\theta = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$, ii) $\sin 3\theta = 3\sin(\theta)\cos^2(\theta) - \sin^3(\theta)$

$\rightarrow$ let $z = \cos\theta + i\sin\theta$ $\Rightarrow (\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin 3\theta$

$\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 = \boxed{\cos 3\theta + i\sin 3\theta}$ Real Im

$[\cos^3\theta - 3\cos\theta\sin^2\theta] + i[3\cos^2\theta\sin\theta - \sin^3\theta] = \cos 3\theta + i\sin 3\theta$

By comparing Real & Im parts

$\boxed{\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta}$, $\boxed{\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta}$

$i^3 = -i$

Or

If $z = \cos\theta + i\sin\theta$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$\cos n\theta = {}^nC_0\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \cdots$

$\cos 3\theta = {}^3C_0\cos^3\theta - {}^3C_2\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \cdots$

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

3. **Show that** $\tan 7\theta = \frac{7\tan(\theta) - 35\tan^3(\theta) + 21\tan^5(\theta) - \tan^7(\theta)}{1 - 21\tan^2(\theta) + 35\tan^4(\theta) - 7\tan^6(\theta)}$ **[MU -Dec-03,12]** Dr. Uday Kashid, PhD (Mathematics)

$\rightarrow$ $\tan 7\theta = \frac{\sin 7\theta}{\cos 7\theta}$ —(I)

$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta + \cdots$

$n=7$

$\sin 7\theta = {}^7C_1\cos^6\theta\sin\theta - {}^7C_3\cos^4\theta\sin^3\theta + {}^7C_5\cos^2\theta\sin^5\theta - {}^7C_7\cos^0\theta\sin^7\theta$.

${}^7C_1 = 7$, ${}^7C_3 = \frac{7\times6\times5}{3\times2} = 35$, ${}^7C_5 = \frac{\cancel{7\times7\times6\times5\times}}{\cancel{5}\times2\times1} = 21$

$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$ —(II)

$\cos n\theta = {}^nC_0\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + -$

$\cos 7\theta = {}^7C_0\cos^7\theta - {}^7C_2\cos^5\theta\sin^2\theta + {}^7C_4\cos^3\theta\sin^4\theta - {}^7C_6\cos\theta\sin^6\theta$.

${}^7C_0 = 1$, ${}^7C_2 = {}^7C_5 = 21$, ${}^7C_4 = {}^7C_3 = 35$, ${}^7C_6 = 7$

$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$ —(III)

put eq (II) & (III) in (I)

$$\tan(7\theta) = \frac{7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta}{\boxed{\cos^7\theta} - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta}$$

divide by $\cos^7\theta$ to N & D.

$$\tan 7\theta = \frac{\frac{7\cos^6\theta\sin\theta}{\cos^7\theta} - \frac{35\cos^4\theta\sin^3\theta}{\cos^7\theta} + \frac{21\cos^2\theta\sin^5\theta}{\cos^7\theta} - \frac{\sin^7\theta}{\cos^7\theta}}{\frac{\cos^7\theta}{\cos^7\theta} - 21\frac{\cos^5\theta\sin^2\theta}{\cos^7\theta} + 35\frac{\cos^3\theta\sin^4\theta}{\cos^7\theta} - 7\frac{\cos\theta\sin^6\theta}{\cos^7\theta}}$$

$$\boxed{= \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}}$$

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Root of an equation

10 February 2022 11:56

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Note: De Moivre's thm can be used to find n^{th} root of a complex numbers.

- Let the equation be $z^n = cos\theta + i\,sin\theta$

$$z = (cos\theta + i\,sin\theta)^{\frac{1}{n}} = [cos(2k\pi+\theta) + i\,sin(2k\pi+\theta)]^{\frac{1}{n}}$$

Where $cos(2k\pi+\theta) = cos\theta$ and $sin(2k\pi+\theta) = sin\theta$, for $k = 0,1,2,3,\dots n \dots\dots$

By De Moivre's thm:-

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$$z = cos\left(\frac{2k\pi+\theta}{n}\right) + i\,sin\left(\frac{2k\pi+\theta}{n}\right) \quad where \ k = 0,1,2,3,\dots(n-1)$$

- *Note: Gate* Dr. Uday Kashid, PhD (Mathematics)

$2k\pi$

1. $z^n = 1 = cos(0) + i\,sin(0) = cos(2k\pi+0) + i\,sin(2k\pi+0)$, for $k = 0,1,2,3,\dots n \dots\dots$
2. $z^n = -1 = cos(\pi) + i\,sin(\pi) = cos(2k\pi+\pi) + i\,sin(2k\pi+\pi)$, for $k = 0,1,2,3,\dots n \dots\dots$
3. $z^n = i = cos\left(\frac{\pi}{2}\right) + i\,sin\left(\frac{\pi}{2}\right) = cos\left(2k\pi+\frac{\pi}{2}\right) + i\,sin\left(2k\pi+\frac{\pi}{2}\right)$, for $k = 0,1,2,3,\dots n \dots\dots$
4. $z^n = -i = cos\left(\frac{\pi}{2}\right) - i\,sin\left(\frac{\pi}{2}\right) = cos\left(2k\pi+\frac{\pi}{2}\right) - i\,sin\left(2k\pi+\frac{\pi}{2}\right)$, for $k = 0,1,2,3,\dots n \dots\dots$
5. $z^n = \beta = \beta(1) = \beta\{cos(0) + i\,sin(0)\} = \beta\{cos(2k\pi+0) + i\,sin(2k\pi+0)\}$
6. $z^n = -\beta = \beta(-1) = \beta\{cos(\pi) + i\,sin(\pi)\} = \beta\{cos(2k\pi+\pi) + i\,sin(2k\pi+\pi)\}$

Examples: Dr. Uday Kashid, PhD (Mathematics)

$cos\,n\theta \,/\, sin\,n\theta$

$cos^n\theta \,/\, sin^n\theta \to$ convert

1. Solve the equation $x^7 + x^4 + i(x^3+1) = 0$ [MU -Dec-15]
2. Solve the equation $x^6 - i = 0$ [MU -Dec-13]
3. Solve the equation $x^6 + 1 = 0$ [MU -Dec-14]
4. HW Show that all the roots of $(x+1)^6 + (x-1)^6 = 0$ are given by $-icot\left[\frac{(2k+1)\pi}{12}\right]$, $k = 0,1,2,3,4,5$ [MU -Dec-14,17]
5. Show that all the roots of $(x+1)^7 = (x-1)^7$ are given by $\pm icot\left[\frac{(k)\pi}{7}\right]$, $k = 1,2,3$ [MU -Dec-08]
6. Solve the equation $(x+1)^8 + (x)^8 = 0$
7. Solve the equation $x^{10} + 11x^5 + 10 = 0$ [MU -05]
8. Solve the equation $x^7 + 64x^4 + 64(x^3+64) = 0$
9. Solve the equation $z^3 = (z+1)^3$, then show that $z = \frac{-1}{2} + \frac{i}{2}cot\left[\frac{(k)\pi}{3}\right]$
10. Find the continued product of the roots of $x^4 = 1 + i$ [MU -April-21 online exam]

1. Solve the equation $x^7 + x^4 + i(x^3+1) = 0$ [MU -Dec-15]

$\to$ $x^4(x^3+1) + i(x^3+1) = 0$

$(x^3+1)(x^4+i) = 0$

$\Rightarrow \boxed{x^3+1=0}$ or $\boxed{x^4+i=0}$

(I) For $x^3+1=0 \Rightarrow x^3 = -1 = cos(\pi) + i\,sin(\pi) = cos(2k\pi+\pi) + i\,sin(2k\pi+\pi)$, $k=0,1,2,3,\dots$

$\to$ $x = [cos(2k\pi+\pi) + i\,sin(2k\pi+\pi)]^{1/3}$, $\boxed{k=0,1,2}$ $\boxed{3,4,5}$ $\boxed{6,7,8}$

$$\boxed{x = cos\left(\frac{2k\pi+\pi}{3}\right) + i\,sin\left(\frac{2k\pi+\pi}{3}\right)}$$

For $k=0$, $x_1 = cos(\pi/3) + i\,sin(\pi/3) = \boxed{\frac{1}{2} + i\frac{\sqrt{3}}{2}}$

$k=1$, $x_2 = cos(\pi) + i\,sin(\pi) = -1$

$k=2$, $x_3 = cos(5\pi/3) + i\,sin(5\pi/3) = cos(2\pi - \pi/3) + i\,sin(2\pi-\pi/3) = cos(\pi/3) - i\,sin(\pi/3)$

$x_3 = \boxed{\frac{1}{2} - i\frac{\sqrt{3}}{2}} = \boxed{\bar{x}_1}$

(II) For $x^4 + i = 0 \Rightarrow x^4 = -i = cos(\pi/2) - i\,sin(\pi/2) = cos(2k\pi+\pi/2) - i\,sin(2k\pi+\pi/2)$, $k=0,1,2,\dots$

$x = cos\left(2k\pi+\frac{\pi}{2}\right) - i\,sin\left(2k\pi+\pi/2\right)^{\frac{1}{4}} \Rightarrow \boxed{k=0,1,2,3}$

$$x = \left\{cos\left[\frac{4k\pi+\pi}{2}\right] - i\,sin\left[\frac{4k\pi+\pi}{2}\right]\right\}^{\frac{1}{4}} = cos\left[\frac{4k\pi+\pi}{8}\right] - i\,sin\left[\frac{4k\pi+\pi}{8}\right], \boxed{k=0,1,2,3}$$

For $k=0$, $x_4 = cos(\pi/8) - i\,sin(\pi/8)$

$k=1$, $x_5 = cos(5\pi/8) - i\,sin(5\pi/8)$

$k=2$, $x_6 = cos(9\pi/8) - i\,sin(9\pi/8)$

$k=3$, $x_7 = cos(13\pi/8) - i\,sin(13\pi/8)$

4. Show that all the roots of $(x+1)^6 + (x-1)^6 = 0$ are given by $-icot\left[\frac{(2k+1)\pi}{12}\right]$, $k = 0,1,2,3,4,5$

$x^n = 1, -1, i, -i$

$\to$ $(x+1)^6 = -(x-1)^6$

$\frac{(x+1)^6}{(x-1)^6} = -1 =$

$\left[\frac{x+1}{x-1}\right]^6 = -1 = cos(2k\pi+\pi) + i\,sin(2k\pi+\pi)$, $k=0,1,2,3,\dots$

$\frac{x+1}{x-1} = [cos(2k+1)\pi + i\,sin(2k+1)\pi]^{\frac{1}{6}}$, $\boxed{k=0,1,2,3,4,5.}$

$\frac{x+1}{x-1} = \boxed{cos(2k+1)\frac{\pi}{6} + i\,sin(2k+1)\frac{\pi}{6}}$, $k=0,1,2,3,4,5.$

Let $(2k+1)\frac{\pi}{6} = \theta$ (say)

$\frac{x+1}{x-1} = cos\theta + i\,sin\theta = \frac{e^{i\theta}}{1}$

By compodendo & dividendo Method $\left[\frac{N+D}{N-D}\right]$

Note this step

$\frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{e^{i\theta}+1}{e^{i\theta}-1} \Rightarrow \frac{2x}{2} = \frac{e^{i\theta}+1}{e^{i\theta}-1} \Rightarrow x = \left(\frac{e^{i\theta}+1}{e^{i\theta}-1}\right) \times \frac{e^{-i\theta/2}}{e^{-i\theta/2}} = \frac{e^{i\theta/2} + e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}}$

$$= \frac{\cancel{2}\cos(\theta/2)}{\cancel{2}\, i \sin(\theta/2)} = (-i)\cot(\theta/2)$$

By componendo & dividendo $[N-D]$

$$\frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{e^{i\theta}+1}{e^{i\theta}-1} \Rightarrow \frac{2x}{2} = \frac{e^{i\theta}+1}{e^{i\theta}-1} \Rightarrow x = \left[\frac{e^{i\theta}+1}{e^{i\theta}-1}\right] \times \left[\frac{e^{-i\theta/2}}{e^{-i\theta/2}}\right] = \frac{e^{i\theta/2}+e^{-i\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}}$$

$$\text{But } z = \cos\theta + i\sin\theta = e^{i\theta}, \quad \frac{1}{z} = \bar{z} = \cos\theta - i\sin\theta = e^{-i\theta}$$

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$$\boxed{\begin{aligned} e^{i\theta}+e^{-i\theta} &= 2\cos\theta \\ e^{i\theta}-e^{-i\theta} &= 2i\sin\theta \end{aligned}} \Rightarrow \boxed{\frac{1}{i} = -i}$$

$$x = -i\cot(\theta/2) = -i\cot\left[\frac{(2k+1)\pi}{12}\right], \quad k = 0, 1, 2, 3, 4, 5$$

5. Show that all the roots of $(x+1)^7 = (x-1)^7$ are given by $\pm i\cot\left[\frac{(k)\pi}{7}\right]$, $k = 1, 2, 3$

$\rightarrow \left[\frac{x+1}{x-1}\right]^7 = 1 = \cos(0) + i\sin 0 = \cos(2k\pi+0) + i\sin(2k\pi+0)$, $k = 0, 1, 2, 3 \ldots$

$\frac{x+1}{x-1} = [\cos(2k\pi) + i\sin(2k\pi)]^{\frac{1}{7}}$, $k = 0, 1, 2, 3, 4, 5, 6$.

$\frac{x+1}{x-1} = \cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right)$, $k = 0, 1, 2, 3 - 6$.

For $k=0$: $\frac{x+1}{x-1} = 1 \Rightarrow x+1 = x-1 \Rightarrow (1 = -1) \rightarrow$ Never possible

$\boxed{k=0 \text{ is discard}}$.

$\frac{x+1}{x-1} = \cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right)$, $k = 1, 2, 3, 4, 5, 6$.

$\frac{x+1}{x-1} = \frac{e^{i\left(\frac{2k\pi}{7}\right)}}{1}$

By componendo & dividendo $\left[\frac{N+D}{N-D}\right]$

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$$\frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{\left[e^{i(2k\pi/7)}+1\right]}{\left[e^{i(2k\pi/7)}-1\right]}$$

$$\frac{2x}{2} = \frac{\left[e^{i(2k\pi/7)}+1\right]}{\left[e^{i(2k\pi/7)}-1\right]} \times \frac{e^{-i(k\pi/7)}}{e^{-i(k\pi/7)}} = \frac{e^{i(k\pi/7)} + e^{-i(k\pi/7)}}{e^{i(k\pi/7)} - e^{-i(k\pi/7)}}$$

$$\frac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta, \quad \frac{e^{i\theta}-e^{-i\theta}}{2i} = \sin\theta$$

$$x = \frac{2\cos(k\pi/7)}{2i\sin(k\pi/7)} = -i\cot(k\pi/7), \quad k = 1, 2, 3, 4, 5, 6$$

For $k=1$, $x_1 = -i\cot(\pi/7)$

$k=2$, $x_2 = -i\cot(2\pi/7)$

$k=3$, $x_3 = -i\cot(3\pi/7)$

$k=4$, $x_4 = -i\cot(4\pi/7) = -i\cot\left(\pi - \frac{3\pi}{7}\right) =$

$x_4 = i\cot(3\pi/7) = \boxed{\overline{x_3}}$

$k=5$, $x_5 = -i\cot(5\pi/7) = -i\cot\left(\pi - 2\pi/7\right) = i\cot(2\pi/7) = \overline{x_2}$

$k=6$, $x_6 = -i\cot(6\pi/7) = -i\cot(\pi - \pi/7) = i\cot(\pi/7) = \overline{x_1}$

Hence Roots are $\boxed{x = \pm i\cot(k\pi/7), \ k = 1, 2, 3}$

9. Solve the equation $z^3 = (z+1)^3$, then show that $z = \frac{-1}{2} + \frac{i}{2}\cot\left[\frac{(k)\pi}{3}\right]$

$\rightarrow \left(\frac{z}{z+1}\right)^3 = 1 = \cos(2k\pi+0) + i\sin(2k\pi+0)$ $k = 0, 1, 2 \ldots$

$\frac{z}{z+1} = \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)$, $k = 0, 1, 2$

For $k=0$, $\frac{z}{z+1} = 1 \Rightarrow z = z+1 \Rightarrow \boxed{0=1} \rightarrow$ Never possible

$\boxed{k=0 \text{ is not possible}}$.

$\frac{z}{z+1} = \cos\theta + i\sin\theta$, $k = 1, 2$ and $\theta = \left(\frac{2k\pi}{3}\right)$

$\frac{z}{z+1} = \frac{e^{i\theta}}{1}$

$z^3 = (z+1)^3$

$\cancel{z^3} = \cancel{z^3} + 3z^2 + 3z + 1$

$\boxed{3z^2 + 3z + 1 = 0}$

By dividendo $\left[\frac{N}{D-N}\right]$

$$\frac{z}{(z+1)-z} = \left[\frac{e^{i\theta}}{1-e^{i\theta}}\right] \times \frac{e^{-i\theta/2}}{e^{-i\theta/2}}$$

$$z = \frac{e^{+i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} = -\frac{e^{i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} = \frac{-e^{i\theta/2}}{2i\sin(\theta/2)} = \frac{-1[\cos\theta/2 + i\sin\theta/2]}{2i\sin\theta/2}$$

$$= \frac{(i)}{2}\left[\cot(\theta/2) + i\right] = \left[\frac{i\cot(k\pi/3)}{2} - \frac{1}{2}\right] \qquad \theta = (2k\pi/3)$$

8. Solve the equation $x^7 + 64x^4 + 64(x^3 + 64) = 0$

$\rightarrow x^4(x^3+64) + 64(x^3+64) = 0 \Rightarrow (x^4+64)(x^3+64) = 0$

$\Rightarrow \boxed{x^4 = -64}$ or $\boxed{x^3 = -64}$

(I) For $x^4 = -64$

(II) For $x^3 = -64$

$x^4 = 64(-1) = 64[\cos(2k+1)\pi + i\sin(2k+1)\pi]$ $k = 0, 1, 2, 3, 4, \ldots$

$x = (64)^{1/4}\left[\cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}\right]$, $k = 0, 1, 2, 3$.

For $k=0$, $x_1 = \sqrt{8}\left[\cos\pi/4 + i\sin\pi/4\right] = \sqrt{8}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$

$k=1$, $x_2 = \sqrt{8}\left[\cos 3\pi/4 + i\sin 3\pi/4\right] = \sqrt{8}\left[\cos(\pi-\pi/4) + i\sin(\pi-\pi/4)\right] = \sqrt{8}\left[-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$

$k=2$, $x_3 = \sqrt{8}\left[\cos 7\pi/4 + i\sin 7\pi/4\right] = \sqrt{8}\left[\cos(2\pi-\pi/4) + i\sin(2\pi-\pi/4)\right] = \sqrt{8}\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = \overline{x_1}$

$$= \frac{\cancel{2}\cos(\theta/2)}{\cancel{2}i\sin(\theta/2)} = (-1)\cot(\theta/2)$$

For

$k=0,\ x_1 = \sqrt{8}\left[\cos \pi/4 + i \sin \pi/4\right] = \sqrt{8}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$

$k=1,\ x_2 = \sqrt{8}\left[\cos 3\pi/4 + i \sin 3\pi/4\right] = \sqrt{8}\left[\cos(\pi - \pi/4) + i \sin(\pi - \pi/4)\right] = \sqrt{8}\left[-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$

$k=3,\ x_3 = \sqrt{8}\left[\cos 7\pi/4 + i \sin \frac{7\pi}{4}\right] = \sqrt{8}\left[\cos(2\pi - \pi/4) + i \sin(2\pi - \pi/4)\right] = \sqrt{8}\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = \overline{x_1}$

$k=2,\ x_4 = \sqrt{8}\left[\cos 5\pi/4 + i \sin 5\pi/4\right] = \sqrt{8}\left[\cos(\pi + \pi/4) + i \sin(\pi + \pi/4)\right] = \sqrt{8}\left[-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = \overline{x_2}$

(11) For $x^3 = -64$ $\downarrow$ find $\boxed{x_5,\ x_6,\ \&\ x_7}$

Hyperbolic Functions

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Hyperbolic Functions

Definition

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\tanh x = \frac{\sinh x}{\cosh x}$

$\operatorname{csch} x = \frac{1}{\sinh x}$

$\operatorname{sech} x = \frac{1}{\cosh x}$

$\coth x = \frac{\cosh x}{\sinh x}$

Hyperbolic Identities

$\sinh(-x) = -\sinh x$

$\cosh(-x) = \cosh x$

$\cosh^2 x - \sinh^2 x = 1$

$1 - \tanh^2 x = \operatorname{sech}^2 x$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

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Derivatives of Hyperbolic Functions

$\frac{d}{dx}(\sinh x) = \cosh x$

$\frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Derivatives of Inverse Hyperbolic Functions

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, x > 1$

$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, |x| < 1$

$\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}, x \neq 0$

$\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}, 0 < x < 1$

$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}, |x| > 1$

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(i) $\sinh 2x = 2\sinh x \cosh x = \frac{2\tanh x}{1-\tanh^2 x}$

(ii) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1$

$= 1 + 2\sinh^2 x = \frac{1+\tanh^2 x}{1-\tanh^2 x}$

(iii) $2\cosh^2 x = \cosh 2x + 1$

(iv) $2\sinh^2 x = \cosh 2x - 1$

(v) $\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$

(vi) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

(vii) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

(viii) $\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$

(6) (i) $\cosh x + \sinh x = e^x$

(ii) $\cosh x - \sinh x = e^{-x}$

(iii) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

$\cosh(iz) = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos z$

$\sinh(iz) = \frac{1}{2}(e^{iz} - e^{-iz}) = i\sin z$

$\cosh(x+iy) = \cosh(x)\cos(y) + i\sinh(x)\sin(y)$

$\sinh(x+iy) = \sinh(x)\cos(y) + i\cosh(x)\sin(y)$

$\tanh(iz) = i\tan z$

$\cosh z = \cos(iz)$

$\sinh z = -i\sin(iz)$

$\tanh z = -i\tan(iz)$

$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$

$\sinh(2x) = \frac{2\tanh(x)}{1-\tanh^2 x}$

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\sinh x + \sinh y = 2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2}$

$\sinh x - \sinh y = 2\cosh\frac{x+y}{2}\sinh\frac{x-y}{2}$

$\cosh x + \cosh y = 2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$

$\cosh x - \cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$

$\sinh x \sinh y = \frac{\cosh(x+y) - \cosh(x-y)}{2}$

$\sinh x \cosh y = \frac{\sinh(x+y) + \sinh(x-y)}{2}$

$\cosh x \cosh y = \frac{\cosh(x+y) + \cosh(x-y)}{2}$

$\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$

$\sin(3x) = 3\sin(x) - 4\sin^3 x$

$\sinh(3x) = 3\sinh(x) + 4\sinh^3 x$

$\rightarrow \sin(ix) = -\frac{1}{i}\sinh(x) = i\sinh(x)$

$\tan(ix) = -\frac{1}{i}\tanh(x) = i\tanh(x)$

$\sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy)$

$\sinh(x+iy) = \sinh(x)\cosh(iy) + \cosh(x)\sinh(iy)$

$= \sinh(x)\cos(y) + i\cosh(x)\sin(y)$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B$

$\cosh(x+iy) = \cosh(x)\cosh(iy) + \sinh(x)\sinh(iy)$

$= \cosh(x)\cos(y) + i\sinh(x)\sin(y)$

Examples:

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1. If $x = \sqrt{3}$, then find the value of $\tanh(logx)$
2. If $\tanh(x) = \frac{2}{3}$, then find the value x and $cosh(2x)$ [MU -Dec-14]
3. If $7cosh(x) + 8\,sinh(x) = 1$, then find the real value x [MU -Dec-16]
4. Prove that $\left(\frac{1+\tanh(x)}{1-\tanh(x)}\right)^3 = cosh(6x) + sinh(6x)$ [MU -Dec-09,13]
5. If $\log(tanx) = y$, then prove that 1) $\cosh(ny) = \frac{1}{2}(tan^n x + cot^n x)$ 2) $\sinh(n+1)y + \sinh(n-1)y = 2\sinh(ny)\operatorname{cosec}(2x)$
6. If $u = \log\left(tan\left\{\frac{\pi}{4} + \frac{\theta}{2}\right\}\right)$, then prove that 1) $cosh(u) = sec(\theta)$ 2) $sinh(u) = tan(\theta)$ 3) $tanh(u) = sin(\theta)$ 4) $tanh\left(\frac{u}{2}\right) = tan\left(\frac{\theta}{2}\right)$
7. If $tanh\left(\frac{u}{2}\right) = tan\left(\frac{x}{2}\right)$, then prove that $u = \log\left(tan\left\{\frac{\pi}{4} + \frac{x}{2}\right\}\right)$ [MU -May-18]

Note:- Circular Fun:- $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

Hyperbolic Fun:- $\boxed{\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}}$, $\sinh\theta = \boxed{\frac{e^{\theta} - e^{-\theta}}{2}}$

$\boxed{\tanh\theta = \frac{\sinh\theta}{\cosh\theta} = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}}$, $\boxed{\operatorname{sech}\theta = \frac{1}{\cosh\theta} = \frac{2}{e^{\theta} + e^{-\theta}}}$

$\boxed{\operatorname{cosech}\theta = \frac{1}{\sinh\theta} = \frac{2}{e^{\theta} - e^{-\theta}}}$, $\boxed{\coth\theta = \frac{1}{\tanh\theta} = \frac{e^{\theta} + e^{-\theta}}{e^{\theta} - e^{-\theta}}}$

$i = \sqrt{-1}$ → complex NO

(i) $\cos(ix) = \cosh(x)$ (ii) $\sin(ix) = i\sinh(x)$ (iii) $\tan(ix) = i\tanh(x)$

→ $\cosh(ix) = \cos(x)$ $\sinh(ix) = i\sin(x)$ $\tanh(ix) = i\tan(x)$

→ (i) $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \Rightarrow \cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh(x)$

$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2(i)} = \frac{1}{2}(-i)\left[e^{-x} - e^{x}\right]$

$\sin(ix) = i\left[\frac{e^{x} - e^{-x}}{2}\right] = i\sinh(x)$

(iv) $\cot(ix) = -i\coth(x)$ (v) $\operatorname{cosec}(ix) = \frac{1}{\sin(ix)} = \frac{1}{i\sinh(x)} = -i\operatorname{cosech}(x)$

(vi) $\sec(ix) = \frac{1}{\cos(ix)} = \frac{1}{\cosh x} = \operatorname{sech} x$

(i) $\frac{d}{dx}[\cosh x] = \frac{d}{dx}\left[\frac{e^{x} + e^{-x}}{2}\right] = \frac{1}{2}\left[e^{x} - e^{-x}\right] = \sinh(x)$

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Hind:- $\sin(x)\sin(x)$ → change the sign in hyperbolic fun

$\boxed{\cos^2 x + \sin^2 x = 1}$ $\cos^2(ix) + \sin^2(ix) = 1$

$\boxed{\cosh^2 x - \sinh^2 x = 1}$ $\cosh^2(x) - \sinh^2 x = 1$

(ii) $1 + \tan^2 x = \sec^2 x$ $\Rightarrow \boxed{1 - \tanh^2 x = \operatorname{sech}^2 x}$

(iii) $1 + \cot^2 x = \operatorname{cosec}^2 x$

$\boxed{1 - \coth^2 x = -\operatorname{cosech}^2 x}$

1. If $x = \sqrt{3}$, then find the value of $\tanh(logx)$

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2. If $\tanh(x) = \frac{2}{3}$, then find the value x and $cosh(2x)$ [MU -Dec-14]

3. If $7cosh(x) + 8\,sinh(x) = 1$, then find the real value x [MU -Dec-16]

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4. Prove that $\left(\frac{1+\tanh(x)}{1-\tanh(x)}\right)^3 = cosh(6x) + sinh(6x)$ [MU -Dec-09,13]

5. If $\log(tanx) = y$, then prove that 1) $\cosh(ny) = \frac{1}{2}(tan^n x + cot^n x)$

2) $\sinh(n+1)y + \sinh(n-1)y = 2\sinh(ny)\,\mathrm{cosec}(2x)$

6. If $u = \log\left(tan\left\{\frac{\pi}{4} + \frac{\theta}{2}\right\}\right)$, then prove that 1) $cosh(u) = sec(\theta)$ 2) $sinh(u) = tan(\theta)$ 3) $tanh(u) = sin(\theta)$ 4) $tanh\left(\frac{u}{2}\right) = tan\left(\frac{\theta}{2}\right)$

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7. If $tanh\left(\frac{u}{2}\right)$ = $tan\left(\frac{x}{2}\right)$, then prove that $\mathrm{u} = \log\left(tan\left\{\frac{\pi}{4} + \frac{x}{2}\right\}\right)$ [MU -May-18]