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INVERSE LAPLACE TRANSFORM

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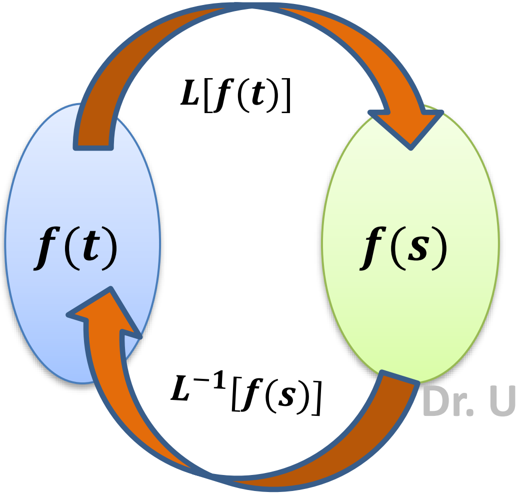
❑ Transformation:



$$L[f(t)] = f(s)$$

$$L^{-1}[f(s)] = f(t)$$

FORMULAE: INVERSE LAPLACE TRANSFORM

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$f(t)$	$L[f(t)] = f(s)$	$f(t)$	$L[f(t)] = f(s)$
e^{at}	$\frac{1}{(s-a)}$	e^{-at}	$\frac{1}{(s+a)}$
1	$\frac{1}{(s)}$	t^{n-1}	$\frac{1}{s^n}\Gamma(n) = \frac{(n-1)!}{s^n}$
$Sin(at)$	$\frac{(a)}{(s^2+a^2)}$	$Cos(at)$	$\frac{(s)}{(s^2+a^2)}$
$Sinh(at)$	$\frac{(a)}{(s^2-a^2)}$	$Cosh(at)$	$\frac{(s)}{(s^2-a^2)}$

Ex. 1 Find $i)\ L^{-1}\left[\frac{2s+3}{s^2+16}\right]$

➢ $L^{-1}\left[\frac{2s+3}{s^2+16}\right] = L^{-1}\left[2\left(\frac{s}{s^2+16}\right)+3\left(\frac{1}{s^2+16}\right)\right] = 2\,Cos(4t)+3\,\frac{Sin(4t)}{4}$

Ex. 2 Find $L^{-1}\left[\frac{2s+3}{s^2-6s-16}\right]$

➢ $L^{-1}\left[\frac{2s+3}{s^2-6s-16}\right] = L^{-1}\left[\frac{2s+3}{(s-8)(s+2)}\right] = L^{-1}\left[\frac{A}{(s-8)}+\frac{B}{(s+2)}\right] = L^{-1}\left[\frac{\frac{19}{10}}{(s-8)}+\frac{\frac{-1}{-10}}{(s+2)}\right]$

➢ $= \frac{19}{10}L^{-1}\left[\frac{1}{(s-8)}\right]+\frac{1}{10}L^{-1}\left[\frac{1}{(s+2)}\right] = \frac{19}{10}\,e^{8t}+\frac{1}{10}\,e^{-2t}$

Ex. 3 Find $L^{-1}\left[\frac{5s+15}{(25s^2-16)}\right] = L^{-1}\left[\frac{5s+15}{25\left(s^2-\frac{16}{25}\right)}\right] = L^{-1}\left[\frac{5\quad s}{25\left(s^2-\frac{16}{25}\right)}+\frac{15\ (1)}{25\left(s^2-\frac{16}{25}\right)}\right]$

➢ $= \frac{5}{25}L^{-1}\left[\frac{s}{\left(s^2-\left[\frac{4}{5}\right]^2\right)}\right]+\frac{15}{25}L^{-1}\left[\frac{1}{\left(s^2-\left[\frac{4}{5}\right]^2\right)}\right] = \frac{5}{25}\,\text{Cos}\left(\frac{4t}{5}\right)+\frac{15}{25}\,\frac{Sin\left(\frac{4t}{5}\right)}{\frac{4}{5}} = \frac{1}{5}\left[\text{Cos}\left(\frac{4t}{5}\right)+\frac{15}{4}Sin\left(\frac{4t}{5}\right)\right]$

FSP OF INVERSE LAPLACE TRANSFORMS

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1. **First Shifting Property (Frequency Shift Prop.):** If $L[f(t)] = f(s)$ Then,

$$L[e^{at} f(t)] = L[f(t)]_{s \to (s-a)} = f(s)_{s\to(s-a)} = f(s-a)$$

$$L[e^{-at} f(t)] = L[f(t)]_{s\to(s+a)} = f(s)_{s\to(s+a)} = f(s+a)$$

2. **For Inverse Laplace Transform:**

$$L^{-1}[f(s-a)]_{(s-a)\to s} = e^{at}\, L^{-1}[f(s)] = e^{at} f(t)$$

$$L^{-1}[f(s+a)]_{(s+a)\to s} = e^{-at}\, L^{-1}[f(s)] = e^{-at} f(t)$$

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Ex. 4 Find $L^{-1}\left[\frac{2s+3}{s^2-6s+16}\right]$

- $L^{-1}\left[\frac{2s+3}{s^2-6s+16}\right] = L^{-1}\left[\frac{2s+3}{(s-3)^2+7}\right] = L^{-1}\left[\frac{2(s-3)+9}{(s-3)^2+7}\right]_{(s-3)\to s}$
- $= e^{3t}\, L^{-1}\left[\frac{2\,s+9}{(s)^2+7}\right] = e^{3t}\, L^{-1}\left[\frac{2\;\;s}{(s)^2+7} + \frac{9\,(1)}{(s)^2+7}\right] = e^{3t}\left[2\,Cos(\sqrt{7}\;\,t) + 9\,\frac{Sin(\sqrt{7}\;\,t)}{\sqrt{7}}\right]$

Ex.5 Find $L^{-1}\left[\frac{s+4}{(s^2-1)(S+1)}\right]$

- $L^{-1}\left[\frac{s+4}{(s^2-1)(S+1)}\right] = L^{-1}\left[\frac{s+4}{(s-1)(S+1)(S+1))}\right] = L^{-1}\left[\frac{s+4}{(s-1)(s+1)^2}\right] = L^{-1}\left[\frac{A}{(s-1)} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2}\right]$
- $s + 4 = A(s\;+1)^2 + B(s-1)(s+1) + C(s-1)$
- For $s = 1$, $A = \frac{5}{4}$, For $s = -1$, $C = \frac{-3}{2}$ and for $s = 0$, $4 = A - B - C$, $B = \frac{5}{4} + \frac{3}{2} - 4 = \frac{-5}{4}$
- $= L^{-1}\left[\frac{5}{4}\,\frac{1}{(s-1)} - \frac{5}{4}\,\frac{1}{(s+1)} - \frac{3}{2}\,\frac{1}{(s+1)^2}\right] = \frac{5}{4}e^{t} - \frac{5}{4}e^{-t} - \frac{3}{2}\,e^{-t}\,L^{-1}\left[\frac{1}{s^2}\right] = \frac{5}{4}e^{t} - \frac{5}{4}e^{-t} - \frac{3}{2}\,e^{-t}\,\mathrm{t}$

Ex.6 Find $L^{-1}\left[\frac{4S}{(S^4+4)}\right]$ **[IIT B-17, MU-Dec 18]**

- $L^{-1}\left[\frac{4S}{(S^4+4)}\right] = L^{-1}\left[\frac{4S}{(S^2+2)^2-4S^2}\right] = L^{-1}\left[\frac{4S}{[(S^2+2)-2s][(S^2+2)+2s]}\right]$
- $= L^{-1}\left[\frac{1}{[(S^2+2)-2s]} - \frac{1}{[(S^2+2)+2s]}\right]$
- $= L^{-1}\left[\frac{1}{S^2-2s+2}\right] - L^{-1}\left[\frac{1}{S^2+2s+2}\right]$
- $= L^{-1}\left[\frac{1}{(S-1)^2+1}\right]_{(s-1)\to s} - L^{-1}\left[\frac{1}{(S+1)^2+1}\right]_{(s+1)\to s}$
- $= e^t L^{-1}\left[\frac{1}{(S)^2+1}\right] - e^{-t} L^{-1}\left[\frac{1}{(S)^2+1}\right] = e^t \sin(t) - e^{-t}\sin(t) = 2\,Sin(t)Sinh(t)$

Ex.7 Find $L^{-1}\left[\frac{S^2+2S+3}{(S^2+2S+2)(S^2+2S+5)}\right]$ **[IIT-BHU-18, MU-19]**

- Put $S^2 + 2S = u$, $L^{-1}\left[\frac{S^2+2S+3}{(S^2+2S+2)(S^2+2S+5)}\right] = L^{-1}\left[\frac{u+3}{(u+2)(u+5)}\right]$
- $= L^{-1}\left[\frac{A}{(u+2)} + \frac{B}{(u+5)}\right] = L^{-1}\left[\frac{\frac{1}{3}}{(u+2)} + \frac{\frac{2}{3}}{(u+5)}\right]$ Dr. Uday Kashid
- $= \frac{1}{3}L^{-1}\left[\frac{1}{(S^2+2S+2)}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(S^2+2S+5)}\right] = \frac{1}{3}L^{-1}\left[\frac{1}{(S+1)^2+1}\right]_{(s+1)\to s} + \frac{2}{3}L^{-1}\left[\frac{1}{(S+1)^2+4}\right]_{(s+1)\to s}$
- $= \frac{1}{3}e^{-t}L^{-1}\left[\frac{1}{S^2+1}\right] + \frac{2}{3}e^{-t}L^{-1}\left[\frac{1}{S^2+2^2}\right] = \frac{1}{3}e^{-t}\,Sin(t) + \frac{2}{3}e^{-t}\frac{Sin(2t)}{2}$
- $= \frac{1}{3}e^{-t}[Sin(t) + Sin(2t)]$

Ex.8 Find $L^{-1}\left[\frac{8S+20}{(S^2-12S+32)}\right]$ **[ETRX MU Dec-18,(6M)]**

- $L^{-1}\left[\frac{8S+20}{(S^2-12S+32)}\right] = L^{-1}\left[\frac{8S+20}{(S-4)(S-8)}\right]$
- $= L^{-1}\left[\frac{A}{(S-4)} + \frac{B}{(S-8)}\right]$
- $= L^{-1}\left[\frac{-\frac{52}{4}}{(S-4)} + \frac{\frac{84}{4}}{(S-8)}\right]$
- $= -13\, L^{-1}\left[\frac{1}{(s-4)}\right] + 21\, L^{-1}\left[\frac{1}{(s-8)}\right]$
- $= -13\ e^{4t} + 21\, e^{8t}$

Ex.9 Find $L^{-1}\left[\frac{S^2}{(S+1)^3}\right]$

- $L^{-1}\left[\frac{S^2}{(S+1)^3}\right] = L^{-1}\left[\frac{[(S+1)-1]^2}{(S+1)^3}\right]_{(s+1)\to s}$
- $= e^{-t}\ L^{-1}\left[\frac{[S-1]^2}{(S)^3}\right]$
- $= e^{-t}\ L^{-1}\left[\frac{[S]^2-2S+1}{(S)^3}\right] = e^{-t}\ L^{-1}\left[\frac{[S]^2}{(S)^3} - 2\frac{[S]}{(S)^3} + \frac{1}{S^3}\right]$
- $= e^{-t}\ L^{-1}\left[\frac{1}{S} - 2\frac{1}{(S)^2} + \frac{1}{S^3}\right]$
- $= e^{-t}\ \left(1 - 2t + \frac{t^2}{2!}\right)$

$$L^{-1}\left[\frac{1}{S^n}\right] = \frac{t^{(n-1)}}{(n-1)!}$$

2.1 HOMEWORK EXAMPLES: INVERSE LT

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Ex.1 Find $L^{-1}\left[\frac{1}{S^2(S+3)^2}\right]$ Ans $\left[\frac{1}{27}\left(3te^{-3t}+2e^{-3t}+3t-2\right)\right]$

Ex.2 Find $L^{-1}\left[\frac{S+29}{(S^2+9)(S+4)}\right]$ Ans $\left[e^{-4t}-Cos(3t)+\frac{5}{3}Sin(3t)\right]$

Ex.3 Find $L^{-1}\left[\frac{2S-1}{(S^4+S^2+1)}\right]$ Ans $\left[\frac{-1}{2}e^{-\frac{t}{2}}Cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{5}{2\sqrt{3}}e^{-\frac{t}{2}}Sin\left(\frac{\sqrt{3}}{2}t\right)\right]$

Ex.4 Find $L^{-1}\left[\frac{S+2}{(S+1)^3(S+3)}\right]$ Ans $\left[\frac{1}{8}e^{-3t}+\frac{1}{8}e^{-t}(2t^2+2t-1)\right]$

Ex.5 Find $L^{-1}\left[\frac{11S^2-2S+5}{2S^3-3S^2-3S+2}\right]$ Ans $\left[2e^{-t}+5e^{2t}-\frac{3}{2}e^{\frac{t}{2}}\right]$

MULTIPLE BY 't' PROP : INVERSE LAPLACE TRANSFORMS

- $L\,[\,t\;f(t)] = (-1)\dfrac{d}{ds}\,L\,[f(t)]$

- $t\;f(t) = L^{-1}\left[(-1)\dfrac{d}{ds}f(S)\right]$

- $t\;L^{-1}[f(s)] = (-1)\;L^{-1}\left[\dfrac{d}{ds}f(S)\right]$

- $L^{-1}[f(s)] = \dfrac{-1}{t}\;\;L^{-1}\left[\dfrac{d}{ds}f(S)\right]$

$f(s)$

log

tan^{-1}

cot^{-1}

$tanh^{-1}(s) = \frac{1}{2}\log\left(\frac{1+s}{1-s}\right)$

$$L\,[\,t^2\;f(t)] = \;\frac{1}{t^2}\;\;L^{-1}\left[\frac{d^2}{ds^2}f(S)\right]$$

Ex.10 Find $L^{-1}\left[\log\left(\frac{S^2+4}{S+4}\right)\right]$ **[MU-EXTC-NOV 18]**

- $L^{-1}[f(s)] = \frac{-1}{t}\; L^{-1}\left[\frac{d}{ds} f(S)\right]$ ...(1)

- Let $f(s) = \log\left(\frac{S^2+4}{S+4}\right) \quad \Rightarrow \frac{d}{ds} f(S) = \frac{d}{ds}\left[\log\left(\frac{S^2+4}{S+4}\right)\right]$

- $\frac{d}{ds} f(S) = \frac{d}{ds}\left[\log(S^2+4) - \log(S+4)\right] = \frac{2S}{S^2+4} - \frac{1}{S+4}$ ...(2)

- $L^{-1}[f(s)] = \frac{-1}{t}\; L^{-1}\left[\frac{2S}{S^2+2^2} - \frac{1}{S+4}\right]$

- $L^{-1}\left[\log\left(\frac{S^2+4}{S+4}\right)\right] = \frac{-1}{t}\left[2\,Cos(2t) - e^{-4t}\right]$

Ex.11 Find $L^{-1}\left[\log\left(1+\frac{a^2}{S^2}\right)\right]$ **[EXTC-NOV 18]**

- $L^{-1}[f(s)] = \frac{-1}{t}\ L^{-1}\left[\frac{d}{ds}f(S)\right]$ ...(1)
- Let $f(s) = \log\left(1+\frac{a^2}{S^2}\right) \Rightarrow \frac{d}{ds}f(S) = \frac{d}{ds}\left[\log\left(1+\frac{a^2}{S^2}\right)\right]$
- $\frac{d}{ds}f(S) = \frac{d}{ds}\left[\log\left(\frac{S^2+a^2}{S^2}\right)\right] = \frac{d}{ds}\left[\log(S^2+a^2) - \log(S^2)\right] = \frac{2S}{S^2+a^2} - \frac{S}{S^2}$ ...(2)
- $L^{-1}[f(s)] = \frac{-1}{t}\ L^{-1}\left[\frac{2\,S}{S^2+a^2} - \frac{1}{S}\right]$
- $L^{-1}\left[\log\left(1+\frac{a^2}{S^2}\right)\right] = \frac{-1}{t}\left[2\,Cos(at) - 1\right]$

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Ex.12 Find $L^{-1}\left[S\ \log\left(\frac{S+1}{S-1}\right)\right]$ **[NIT- 18]**

- $L^{-1}[f(s)] = \frac{-1}{t}\ L^{-1}\left[\frac{d}{ds}f(S)\right]$ ...(1)

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- **Let** $f(s) = S\ \log\left(\frac{S+1}{S-1}\right) \Rightarrow \frac{d}{ds}f(S) = \frac{d}{ds}\left[S\ \log\left(\frac{S+1}{S-1}\right)\right]$
- $\frac{d}{ds}f(S) = \frac{d}{ds}[S\{\log(S+1) - \log(S-1)\}]$
- $\frac{d}{ds}f(S) = \left[1\{\log(S+1) - \log(S-1)\} + S\left\{\frac{1}{S+1} - \frac{1}{S-1}\right\}\right]$
- $\frac{d}{ds}f(S) = \left[\{\log(S+1) - \log(S-1)\} + S\left\{\frac{-2}{(S^2-1)}\right\}\right]$ ...(2)

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- $L^{-1}[f(s)] = \frac{-1}{t}\ L^{-1}\left[\{\log(S+1) - \log(S-1)\} - 2\left\{\frac{S}{(S^2-1)}\right\}\right]$
- $= \frac{-1}{t}\ L^{-1}[\{\log(S+1) - \log(S-1)\}] + \frac{1}{t}\ L^{-1}\left[2\left\{\frac{S}{(S^2-1)}\right\}\right]$
- $= \frac{-1}{t}\left\{\frac{-1}{t}\ L^{-1}\left[\frac{d}{ds}[\{\log(S+1) - \log(S-1)\}]\right]\right\} + \frac{2}{t}Cosh(t)$
- $= \frac{1}{t^2}\left\{L^{-1}\left[\frac{1}{S+1} - \frac{1}{S-1}\right]\right\} + \frac{2}{t}Cosh(t)$

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- $= \frac{1}{t^2}\{e^{-t} - e^{t}\} + \frac{2}{t}Cosh(t) = \frac{-2}{t^2}\left\{\frac{e^t - e^{-t}}{2}\right\} + \frac{2}{t}Cosh(t)$
- $= \frac{-2}{t^2}Sinh(t) + \frac{2}{t}Cosh(t)$

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Ex.13 Find $L^{-1}\left[\tan^{-1}\left(\frac{S+a}{b}\right)\right]$ **[ETRX,BIOM-NOV 18, 15]**

- $L^{-1}[f(s)] = \frac{-1}{t}\ L^{-1}\left[\frac{d}{ds} f(S)\right]$ ...(1)
- Let $f(s) = \tan^{-1}\left(\frac{S+a}{b}\right) \Rightarrow \frac{d}{ds} f(S) = \frac{d}{ds}\left[\tan^{-1}\left(\frac{S+a}{b}\right)\right]$

$$\frac{d}{ds}[\tan^{-1} x] = \frac{1}{1+x^2}$$

- $\frac{d}{ds} f(S) = \frac{1}{1+\left(\frac{S+a}{b}\right)^2} \times \left[\frac{1}{b}\right] = \frac{b}{(S+a)^2 + b^2}$ ...(2)
- $L^{-1}[f(s)] = \frac{-1}{t}\ L^{-1}\left[\frac{b}{(S+a)^2 + b^2}\right]_{(s+a)\to s}$
- $= \frac{-1}{t}\ e^{-at} L^{-1}\left[\frac{b}{(S)^2 + b^2}\right] = \frac{-e^{-at}}{t}\ Sin(bt)$

Ex.14 Find $L^{-1}\left[\mathrm{Cot}^{-1}\left(\frac{1}{s}\right)\right]$

- $L^{-1}[f(s)] = \frac{-1}{t}\ L^{-1}\left[\frac{d}{ds}f(S)\right]$ ...(1)
- Let $f(s) = \mathrm{Cot}^{-1}\left(\frac{1}{s}\right) \Rightarrow \frac{d}{ds}f(S) = \frac{d}{ds}\left[\mathrm{Cot}^{-1}\left(\frac{1}{s}\right)\right]$
- But, $\frac{d}{ds}[\mathrm{Cot}^{-1}x] = \frac{-d}{ds}[\tan^{-1}x] = \frac{-1}{1+x^2}$

$$\mathrm{Cot}^{-1}x = \frac{\pi}{2} - \tan^{-1}x$$

- $\frac{d}{ds}\left[\mathrm{Cot}^{-1}\frac{1}{s}\right] = \frac{-d}{ds}\left[\tan^{-1}\frac{1}{s}\right] = \frac{-1}{1+\left(\frac{1}{s}\right)^2} \times \left(\frac{-1}{s^2}\right) = \frac{1}{s^2+1}$ ...(2)
- $L^{-1}[f(s)] = \frac{-1}{t}\ L^{-1}\left[\frac{1}{s^2+1}\right] = \frac{-1}{t}\ Sin(t)$

EXAMPLES: INVERSE LT

Ex.15 Find $L^{-1}[tanh^{-1}(S)]$ [IIT KGP-18, ETRX-18]

- We know $tanh^{-1}(s) = \frac{1}{2}\log\left(\frac{1+s}{1-s}\right) \Rightarrow L^{-1}[tanh^{-1}(S)] = \frac{1}{2}\ L^{-1}\left[\log\left(\frac{1+s}{1-s}\right)\right]$
- But $L^{-1}[f(s)] = \frac{-1}{t}\ L^{-1}\left[\frac{d}{ds}f(S)\right]$ ...(1)

- Let $f(s) = \log\left(\frac{1+S}{1-S}\right) \Rightarrow \frac{d}{ds}f(S) = \frac{d}{ds}\left[\log\left(\frac{1+S}{1-S}\right)\right]$
- $\frac{d}{ds}f(S) = \frac{d}{ds}[\{\log(1+S) - \log(1\ \ -S)\}\ ] = \frac{1}{S+1} - \frac{1}{1-S}(-1)$ ...(2)
- $L^{-1}[f(s)] = \frac{-1}{t}\ L^{-1}\left[\frac{1}{S+1} - \frac{1}{S-1}\right] = \frac{-1}{t}\ \{e^{-t} - e^{t}\}$
- $= \frac{2}{t}\left\{\frac{e^{t}-e^{-t}}{2}\right\} = \frac{2}{t}Sinh(t)$

2.2. HOMEWORK EXAMPLES: INVERSE LT

Ex.1 Find $L^{-1}\left[\log\left(\frac{S^2-4}{(S-3)^2}\right)\right]$ Ans $\left[\frac{2}{t}\left(e^{3t}-Cosh(3t)\right)\right]$

Ex.2 Find $L^{-1}\left[\log\left(\sqrt{\frac{S^2+1}{(S)^2}}\right)\right]$ Ans $\left[\frac{1}{t}(1-Cos(t))\right]$

Ex.3 Find $L^{-1}\left[\log\left(\frac{S^2+1}{S(S+1)}\right)\right]$ Ans $\left[\frac{1}{t}(-2Cos(t)+1+e^{-t})\right]$

Ex.4 Find $L^{-1}\left[\mathrm{Cot}^{-1}\left(\frac{2}{S^2}\right)\right]$ Ans $\left[-\frac{2}{t}Sint\ Sinh(t)\right]$

Ex.5 Find $L^{-1}[\mathrm{Cot}^{-1}(aS)]$ Ans $\left[\frac{1}{t}Sin\left(\frac{t}{2}\right)\right]$

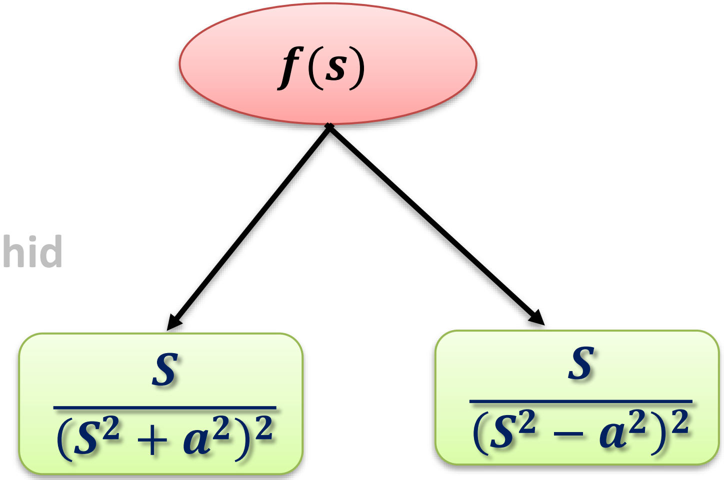
DIVISION BY 't' PROP: INVERSE LT

- ❖ $L\left[\frac{f(t)}{t}\right] = \int_{s=s}^{s=\infty} L\,[f(t)]\,ds$

- ➢ $\frac{f(t)}{t} = L^{-1}\left[\int_{s=s}^{s=\infty} f(s)\,ds\right]$

- ➢ $f(t) = t\ L^{-1}\left[\int_{s=s}^{s=\infty} f(s)\,ds\right]$

- ➢ $L^{-1}[f(S)] = t\ L^{-1}\left[\int_{s=s}^{s=\infty} f(s)\,ds\right]$



EXAMPLES: INVERSE LT

Ex.16 Find $L^{-1}\left[\frac{S}{(S^2+a^2)^2}\right]$

- $L^{-1}[f(S)] = t\ L^{-1}\left[\int_{s=S}^{s=\infty} f(s)\,ds\right]$ ...(1)

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- Let $I = \int_{s=S}^{s=\infty} f(s)\,ds = \int_{s=S}^{s=\infty} \frac{s}{(s^2+a^2)^2}\,ds$
- Put $s^2 + a^2 = u \Rightarrow 2s\,ds = du$

s	S	∞
u	$S^2 + a^2$	∞

- $I = \int_{u=S^2+a^2}^{u=\infty} \frac{1}{(u)^2}\frac{du}{2} = \frac{1}{2}\left[\frac{-1}{u}\right]_{u=S^2+a^2}^{\infty}$
- $I = \frac{1}{2}\left[0 + \frac{1}{S^2+a^2}\right]$ ...(2)
- $L^{-1}[f(S)] = t\ L^{-1}\left[\frac{1}{2}\left[\frac{1}{S^2+a^2}\right]\right]$
- $L^{-1}\left[\frac{S}{(S^2+a^2)^2}\right] = \frac{t}{2}\frac{Sin(at)}{a}$

INVERSE LT : PROP

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Derivative prop.

1) $L\left[\frac{d}{dt} f(t)\right] = s\,f(s) - f(0)$

If $f(0) = 0$, then

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$\frac{d}{dt} f(t) = L^{-1}[s\,f(s)]$

$L^{-1}[s\;f(s)] = \frac{d}{dt}\;L^{-1}[\,f(s)]$

2) If $f(0) = 0$ and $f'(0) = 0$, then

$L\left[\frac{d^2}{dt^2} f(t)\right] = s^2 f(s) - sf(0) - f'(0)$

$L^{-1}[s^2 f(s)] = \left[\frac{d^2}{dt^2} f(t)\right] = \frac{d^2}{dt^2} L^{-1}[\,f(s)]$

Integral prop.

$L\left[\int_{t=0}^{t=t} f(t)\,dt\right] = \frac{1}{(s)} L\,[f(t)]$

$\int_{t=0}^{t=t} f(t)\,dt = L^{-1}\left[\frac{1}{(s)} L\,[f(t)]\right]$

$L^{-1}\left[\frac{1}{(s)} f(s)\right] = \int_{t=0}^{t=t} L^{-1}[\,f(s)]\,dt$

Similarly,

$L^{-1}\left[\frac{1}{S^2} f(s)\right] = \int_{t=0}^{t=t} \int_{t=0}^{t=t} L^{-1}[\,f(s)]\,dt\,dt$

Ex.17 Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$

- $L^{-1}[s\ f(s)] = \frac{d}{dt}\ L^{-1}[\,f(s)]$

- $L^{-1}\left[s\ \frac{s}{(s^2+a^2)^2}\right] = \frac{d}{dt}\ L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

- $= \frac{d}{dt}\left[\frac{t\ \ Sin(at)}{2a}\right]$ where $f(0) = \lim_{t\to 0}\left(\frac{t\ \ Sin(at)}{2a}\right) = 0$

- $= \frac{1}{2a}[Sin(at) + \ a\,t\,Cos(at)]$

Ex.18 Find $L^{-1}\left[\frac{1}{(S^2+a^2)^2}\right]$

- $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{s}\ \frac{s}{(s^2+a^2)^2}\right]$
- But, $L^{-1}\left[\frac{1}{(s)}\ f(s)\right] = \int_{t=0}^{t=t} L^{-1}[f(s)]\, dt$
- $L^{-1}\left[\frac{1}{s}\ \frac{s}{(s^2+a^2)^2}\right] = \int_{t=0}^{t=t} L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] dt$
- $= \int_{t=0}^{t=t} \frac{t\ \ Sin(at)}{2a}\, dt = \frac{1}{2a}\left[t\left\{\frac{-Cos(at)}{a}\right\} - 1\left\{\frac{-Sin(at)}{a\times a}\right\}\right]_{t=0}^{t=t}$
- $= \frac{1}{2a}\left[t\left\{\frac{-Cos(at)}{a}\right\} + 1\left\{\frac{Sin(at)}{a\times a}\right\} - 0 - 0\right]$

Statement: If $L[f(t)] = f(s)$ and $L[g(t)] = g(s)$ then

$$L^{-1}[f(s) * g(s)] = \int_{u=0}^{u=t} f(u)\, g(t-u)\, du$$

Procedure :

Step 1: $\varphi(S) = f(s) * g(s)$

Step 2: Find $L^{-1}[f(s)] = f(t)$ and $L^{-1}[g(s)] = g(t)$

Step 3: Obtain $f(u)$ by replacing $'t'$ by $'u'$ in $f(t)$

Step 4: Obtain $g(t-u)$ by replacing $'t'$ by $(t-u)$ in $g(t)$

Step 5: Put $f(u)$ and $g(t-u)$ in convolution theorem and integrate.

EXAMPLES: INVERSE LT

Ex.19. Find $L^{-1}\left[\frac{1}{(S^2-S-6)}\right]$ **by using Convolution theorem.** [IIT K-Dec 17]

Step 1: $L^{-1}\left[\frac{1}{(S^2-S-6)}\right] = L^{-1}\left[\frac{1}{(S-3)\,(S+2)}\right] = L^{-1}\left[\frac{1}{(S-3)} * \frac{1}{(S+2)}\right] = L^{-1}[f(s) * g(s)]$

➢ where $f(s) = \frac{1}{(S-3)}$ and $g(s)=\frac{1}{(S+2)}$

Step 2: $L^{-1}[f(s)] = f(t) = L^{-1}\left[\frac{1}{(S-3)}\right] = e^{3t}$ | $L^{-1}[g(s)] = g(t) = L^{-1}\left[\frac{1}{(S+2)}\right] = e^{-2t}$

Step 3: $f(u) = e^{3u}$ | $g(t-u) = e^{-2(t-u)} = e^{-2t}\, e^{2u}$

Step 4: Convolution Thm.: $L^{-1}[f(s) * g(s)] = \int_{u=0}^{u=t} f(u)\, g(t-u)\, du$

➢ $L^{-1}\left[\frac{1}{(S^2-S-6)}\right] = \int_{u=0}^{u=t} e^{3u} e^{-2t}\, e^{2u} du = e^{-2t} \int_{u=0}^{u=t} e^{5u} du$

➢ $= e^{-2t}\left[\frac{e^{5u}}{5}\right]_{u=0}^{u=t} = \frac{e^{-2t}}{5}\left[e^{5t}-1\right]$

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Ex.20. Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ by using Convolution theorem. [EXTC-May 19,18 ,(6M)]

Step 1: $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)} * \frac{s}{(s^2+a^2)}\right] = L^{-1}[f(s) * g(s)]$

- where $f(s) = \frac{1}{(s^2+a^2)}$ and $g(s)=\frac{s}{(s^2+a^2)}$

Step 2: $L^{-1}[f(s)] = f(t) = L^{-1}\left[\frac{1}{(s^2+a^2)}\right]$

$$f(t) = \frac{Sin(at)}{a}$$

$L^{-1}[g(s)] = g(t) = L^{-1}\left[\frac{s}{(s^2+a^2)}\right]$

$$g(t) = Cos(at)$$

Step 3: $f(u) = \frac{Sin(au)}{a}$

$$g(t-u) = Cos\,[a(t-u)]$$

EXAMPLES: INVERSE LT

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Step 4: Convolution Thm.: $L^{-1}[\,f(s) * g(s)] = \int_{u=0}^{u=t} f(u)\, g(t-u)\, du$

- $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \int_{u=0}^{u=t} \frac{Sin(au)}{a}\, Cos\,[at-au]\, du$
- $= \frac{1}{2\,a}\int_{u=0}^{u=t} 2\;\; Sin(au)\, Cos\,[at-au]\, du$
- $= \frac{1}{2\,a}\int_{u=0}^{u=t} [Sin(at) + Sin\,[2au-at]]\, du = \frac{1}{2\,a}\left[u\;\, Sin(at) - \frac{Cos\,[2au-at\,]}{2a}\right]_{u=0}^{u=t}$
- $= \frac{1}{2\,a}\left[t\;\, Sin(at) - \frac{Cos(2at-at)}{2a} - 0 + \frac{Cos(0-at)}{2a}\right]$
- $= \frac{1}{2\,a}\left[t\;\, Sin(at) - \cancel{\frac{Cos\,[(at)\,]}{2a}} - 0 + \cancel{\frac{Cos\,[(at)\,]}{2a}}\right] = \frac{t\;\; Sin(at)}{2a}$

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Ex.21. Find $L^{-1}\left[\frac{s}{(s^2+4)(s^2+16)}\right]$ **by using Convolution theorem.** [BIOM-Nov 19,(6M)]

- **Step 1:** $L^{-1}\left[\frac{s}{(s^2+4)(s^2+16)}\right] = L^{-1}\left[\frac{1}{(s^2+4)} * \frac{s}{(s^2+16)}\right] = L^{-1}[f(s) * g(s)]$

- where $f(s) = \frac{1}{(s^2+2^2)}$ and $g(s) = \frac{s}{(s^2+4^2)}$

Step 2: $L^{-1}[f(s)] = f(t) = L^{-1}\left[\frac{1}{(s^2+2^2)}\right]$

$$f(t) = \frac{Sin(2t)}{2}$$

$L^{-1}[g(s)] = g(t) = L^{-1}\left[\frac{s}{(s^2+4^2)}\right]$

$$g(t) = Cos(4t)$$

Step 3: $f(u) = \frac{Sin(2u)}{2}$

$$g(t-u) = Cos\,[4(t-u)]$$

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Step 4: Convolution Thm.: $L^{-1}[f(s) * g(s)] = \int_{u=0}^{u=t} f(u)\, g(t-u)\, du$

- $L^{-1}\left[\frac{s}{(S^2+4)(S^2+16)}\right] = \int_{u=0}^{u=t} \frac{Sin(2u)}{2}\, Cos\,[4t-4u]\, du$
- $= \frac{1}{2\times 2}\int_{u=0}^{u=t} 2\; Sin(2u)\, Cos\,[4t-4u]\, du$

- $= \frac{1}{4}\int_{u=0}^{u=t} [Sin(4t-2u) + Sin\,[6u-4t]]\, du = \frac{1}{4}\left[-\frac{Cos\,[4t-2u]}{-2} - \frac{Cos\,[6u-4t]}{6}\right]_{u=0}^{u=t}$
- $= \frac{1}{4}\left[\frac{Cos\,[4t-2t]}{2} - \frac{Cos\,[6t-4t]}{6} - \frac{Cos(4t-0)}{2} + \frac{Cos(0-4t)}{6}\right]$
- $= \frac{1}{4}\left[\frac{Cos\,[2t]}{2} - \frac{Cos\,[2t]}{6} - \frac{Cos(4t)}{2} + \frac{Cos(4t)}{6}\right] = \frac{Cos(2t) - Cos(4t)}{12}$

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Ex.22. Find $L^{-1}\left[\frac{(S+2)^2}{(S^2+4s+8)^2}\right]$ **by using Convolution theorem.** **[EXTC-Nov 17,(6M)]**

- $L^{-1}\left[\frac{(S+2)^2}{(S^2+4s+8)^2}\right] = L^{-1}\left[\frac{(S+2)^2}{([S+2]^2+4)^2}\right]_{(s+2)\to s}$
- $= e^{-2t}\ L^{-1}\left[\frac{(S)^2}{(S^2+4)^2}\right] = e^{-2t}\ L^{-1}\left[\frac{S}{(S^2+4)} * \frac{S}{(S^2+4)}\right] = e^{-2t}\ L^{-1}\ [f(s) * g(s)]$ ...(1)

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- **Step 1:** where $f(s) = \frac{S}{(S^2+2^2)}$ and $g(s)=\frac{S}{(S^2+2^2)}$
- **Step 2:** $L^{-1}[\ f(s)] = f(t) = L^{-1}\left[\frac{S}{(S^2+2^2)}\right]$

  $f(t) = Cos(2t)$

  $L^{-1}[\ g(s)] = g(t) = L^{-1}\left[\frac{S}{(S^2+2^2)}\right]$

  $g(t) = Cos(2t)$

Step 3: $f(u) = Cos(2u)$ $g(t-u) = Cos\,[2(t-u)\,]$

Step 4: Convolution Thm.: $L^{-1}[\,f(s) * g(s)] = \int_{u=0}^{u=t} f(u)\,g(t-u)\,du$

- $L^{-1}\left[\frac{s}{(s^2+4)} * \frac{s}{(s^2+4)}\right] = \int_{u=0}^{u=t} Cos(2u)\,Cos\,[2t-2u\,]\,du$
- $= \frac{1}{2}\int_{u=0}^{u=t} 2\;Cos(2u)\,Cos\,[2t-2u\,]\,du = \frac{1}{2}\int_{u=0}^{u=t} [Cos(2t) + Cos\,[4u-2t\,]]\,du$

- $= \frac{1}{2}\left[u\;\;Cos(2t) + \frac{Sin\,[4u-2t\,]}{4}\right]_{u=0}^{u=t} = \frac{1}{2}\left[t\;\;Cos(2t) + \frac{Sin\,[4t-2t\,]}{4} - 0 - \frac{Sin\,[0-2t\,]}{4}\right]$
- $= \frac{1}{2}\left[t\;\;Cos(2t) + 2\,\frac{Sin\,[2t\,]}{4}\right]$ ...(2)
- **Hence** $L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right] = \frac{e^{-2t}}{2}\left[t\;\;Cos(2t) + \frac{Sin\,(2t)}{2}\right]$

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Ex.23. Find $L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right]$ **by using Convolution theorem.** **[ETRX-Nov 16,(6M)]**

- **Step 1:** $L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right] = L^{-1}\left[\frac{1}{(s-2)^4} * \frac{1}{(s+3)}\right] = L^{-1}[f(s) * g(s)]$

- where $f(s) = \frac{1}{(s-2)^4}$ and $g(s)=\frac{1}{(s+3)}$

Step 2: $L^{-1}[f(s)] = f(t) = L^{-1}\left[\frac{1}{(s-2)^4}\right]$

$$f(t) = e^{2t}\, L^{-1}\left[\frac{1}{(s)^4}\right] = e^{2t}\,\frac{t^3}{3\,!}$$

Step 3: $f(u) = e^{2u}\,\frac{u^3}{6}$

$L^{-1}[g(s)] = g(t) = L^{-1}\left[\frac{1}{(s+3)}\right]$

$$g(t) = e^{-3t}$$

$$g(t-u) = e^{-3(t-u)} = e^{(-3t)}\, e^{(3u)}$$

Step 4: Convolution Thm.: $L^{-1}[\,f(s) * g(s)] = \int_{u=0}^{u=t} f(u)\, g(t-u)\, du$

- $L^{-1}\left[\frac{1}{(s-2)^4} * \frac{1}{(s+3)}\right] = \int_{u=0}^{u=t} e^{2u}\, \frac{u^3}{6} e^{(-3t)}\, e^{(3u)} du$

- $= \frac{e^{(-3t)}}{6} \int_{u=0}^{u=t} u^3\, e^{(5u)}\, du$

- $= \frac{e^{(-3t)}}{6}\left[u^3\left(\frac{e^{(5u)}}{5}\right) - \{3u^2\}\left(\frac{e^{(5u)}}{5\times5}\right) + \{6u\}\left(\frac{e^{(5u)}}{5\times5\times5}\right) - \{6\}\left(\frac{e^{(5u)}}{5\times5\times5\times5}\right)\right]_{u=0}^{u=t}$

- $= \frac{e^{(-3t)}}{6}\left[t^3\left(\frac{e^{(5t)}}{5}\right) - \{3t^2\}\left(\frac{e^{(5t)}}{5^2}\right) + \{6t\}\left(\frac{e^{(5t)}}{5^3}\right) - \{6\}\left(\frac{e^{(5t)}}{5^4}\right) - 0 + 0 - 0 + \{6\}\left(\frac{1}{5^4}\right)\right]$

- $L^{-1}\left[\frac{1}{(s-2)^4} * \frac{1}{(s+3)}\right] = \frac{e^{(2t)}}{30}\left[t^3 - \left\{\frac{3t^2}{5}\right\} + \left\{\frac{6t}{25}\right\} - \left\{\frac{6}{125}\right\}\right] + \left\{\frac{e^{(-3t)}}{625}\right\}$

Ex.24. Find $L^{-1}\left[\frac{1}{(S^2+2s+2)(s+3)}\right]$ **by using Convolution theorem.** **[ETRX-Nov 16,(6M)]**

- **Step 1:** $L^{-1}\left[\frac{1}{(S^2+2s+2)(s+3)}\right] = L^{-1}\left[\frac{1}{(S^2+2s+2)} * \frac{1}{(s+3)}\right] = L^{-1}[f(s) * g(s)]$

- **where** $f(s) = \frac{1}{(S^2+2s+2)}$ **and** $g(s)=\frac{1}{(s+3)}$

Step 2: $L^{-1}[f(s)] = f(t) = L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$

$$f(t) = e^{-t}\, L^{-1}\left[\frac{1}{S^2+1}\right] = e^{-t}\, Sin(t)$$

$$L^{-1}[g(s)] = g(t) = L^{-1}\left[\frac{1}{(s+3)}\right]$$

$$g(t) = e^{-3t}$$

Step 3: $f(u) = e^{-u}\, Sin(u)$

$$g(t-u) = e^{-3(t-u)} = e^{(-3t)}\, e^{(3u)}$$

Step 4: Convolution Thm.: $L^{-1}[f(s) * g(s)] = \int_{u=0}^{u=t} f(u)\, g(t-u)\, du$

- $L^{-1}\left[\frac{1}{(S^2+2s+2)} * \frac{1}{(s+3)}\right] = \int_{u=0}^{u=t} e^{-u}\, Sin(u)\, e^{(-3t)}\, e^{(3u)} du$

- $= e^{(-3t)} \int_{u=0}^{u=t} e^{(2u)}\, Sin(u)\, du$

- $= e^{(-3t)} \left[\frac{e^{2u}}{[2^2+1^2]} [(2)\, Sin(u) - 1\, Cos(u)]\right]_{u=0}^{u=t}$

- $= \frac{e^{(-3t)}}{5}\left[e^{2t}\, [2\, Sin(t) - Cos(t)] - 1\, [2\, Sin(0) - Cos(0)]\right]$

- $L^{-1}\left[\frac{1}{(S^2+2s+2)} * \frac{1}{(s+3)}\right] = \frac{e^{(-3t)}}{5}\{e^{2t}\, [2\, Sin(t) - Cos(t)] + 1\}$

Ex.25. Find $L^{-1}\left[\frac{s^2+s}{(s^2+2s+2)(s^2+1)}\right]$ **by using Convolution theorem.** **[ETRX-Nov 16, (6M)]**

- **Step 1:** $L^{-1}\left[\frac{s^2+s}{(s^2+2s+2)(s^2+1)}\right] = L^{-1}\left[\frac{S(S+1)}{(s^2+2s+2)(s^2+1)}\right]$
- $= L^{-1}\left[\frac{(S+1)}{(s^2+2s+2)} * \frac{S}{(s^2+1)}\right] = L^{-1}[f(s) * g(s)]$
- where $f(s) = \frac{(S+1)}{(s^2+2s+2)}$ and $g(s)=\frac{S}{(s^2+1)}$

Step 2: $L^{-1}[f(s)] = f(t) = L^{-1}\left[\frac{(S+1)}{(s^2+2s+2)}\right]$

$$f(t) = L^{-1}\left[\frac{(S+1)}{(S+1)^2+1}\right]_{(s+1)\to s}$$

$$f(t) = e^{-t}\, L^{-1}\left[\frac{S}{(s^2+1)}\right] = e^{-t}\, Cos(t)$$

$L^{-1}[g(s)] = g(t) = L^{-1}\left[\frac{S}{(s^2+1)}\right]$

$$g(t) = Cos(t)$$

Step 3: $f(u) = e^{-u}\, Cos(u)$

$$g(t-u) = Cos(t-u)$$

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Step 4: Convolution Thm.: $L^{-1}[f(s) * g(s)] = \int_{u=0}^{u=t} f(u)\, g(t-u)\, du$

$$L^{-1}\left[\frac{s^2+s}{(s^2+2s+2)(s^2+1)}\right] = \int_{u=0}^{u=t} e^{-u}\, Cos(u)\, Cos(t-u)\, du$$

$$= \frac{1}{2}\int_{u=0}^{u=t} e^{-u}\,[2\, Cos(u)\, Cos(t-u)]du = \frac{1}{2}\int_{u=0}^{u=t} e^{-u}\,[Cos(t) + Cos(2u-t)]du$$

$$= \frac{1}{2}\int_{u=0}^{u=t} e^{-u}\,[Cos(t)]du + \frac{1}{2}\int_{u=0}^{u=t} e^{-u}\,[Cos(2u-t)]du$$

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$$= \frac{1}{2}\left[\frac{e^{-u}}{-1}\, Cos(t)\right]_{u=0}^{u=t} + \frac{1}{2}\left[\frac{e^{-u}}{[(-1)^2+2^2]}\,[(-1)\, Cos(2u-t) + 2\, Sin(2u-t)]\right]_{u=0}^{u=t}$$

$$= \frac{1}{2}\,[-e^{-t}\, Cos(t) + Cos(t)] + \frac{1}{2}\left[\frac{e^{-t}}{5}\,[-Cos(t) + 2\, Sin(t)] - \frac{1}{5}\,[-Cos(-t) + 2\, Sin(-t)]\right]$$

$$= \frac{1}{2}\,[-e^{-t}\, Cos(t) + Cos(t)] + \frac{1}{10}\,[e^{-t}\,[-Cos(t) + 2\, Sin(t)] + Cos(t) + 2\, Sin(t)]$$

$$= \frac{1}{10}\,[e^{-t}\,[-6\, Cos(t) + 2\, Sin(t)] + 6Cos(t) + 2\, Sin(t)]$$

2.3. HOMEWORK: CONVOLUTION THEOREM

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Ex.1 Find $L^{-1}\left[\frac{1}{S^2(S+2)^2}\right]$ Ans$\left[\frac{1}{8}\left(2\,t\,e^{-2t}+2\,e^{-2t}+2t-2\right)\right]$

Ex.2 Find $L^{-1}\left[\frac{1}{\left(S^2+4s+13\right)^2}\right]$ Ans$\left[\frac{e^{-2t}}{18}\left(\frac{Sin(3t)}{3}-t\,Cos(3t)\right)\right]$ [ETRX - Nov 18, (6M)]

Ex.3 Find $L^{-1}\left[\frac{1}{S\sqrt{S+4}}\right]$ Ans$\left[\frac{1}{2}\,erf\left(2\sqrt{t}\right)\right]$

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Ex.4 Find $L^{-1}\left[\frac{1}{S^3(S-1)}\right]$ Ans$\left[e^{-t}-1-t-\frac{t^2}{2}\right]$

Ex.5 Find $L^{-1}\left[\frac{(S+5)^2}{\left(S^2+10s+16\right)^2}\right]$ Ans$\left[\frac{e^{-5t}}{6}\left(Sinh(3t)+3t\,Cosh(3t)\right)\right]$