Exam2 Solutions are posted on Piazza.

Depth First Search.

DFS (G)

for each $u \in V$ do
    color [u] ← white
    π [u] ← NIL

$\Big\}$ $O(n)$

time ← 0

for each $u \in V$ do
    if color [u] is white then
        $\boxed{\text{DFS\_VISIT (u)}}$

DFS_VISIT (u)

color [u] ← Gray
time ← time + 1
d [u] ← time

$\Big\}$ $O(1)$

exploring the edge
e = (u, v)

for each $v \in N(u)$ do

"h" if color[v] is white then

$\pi[v] \leftarrow u$

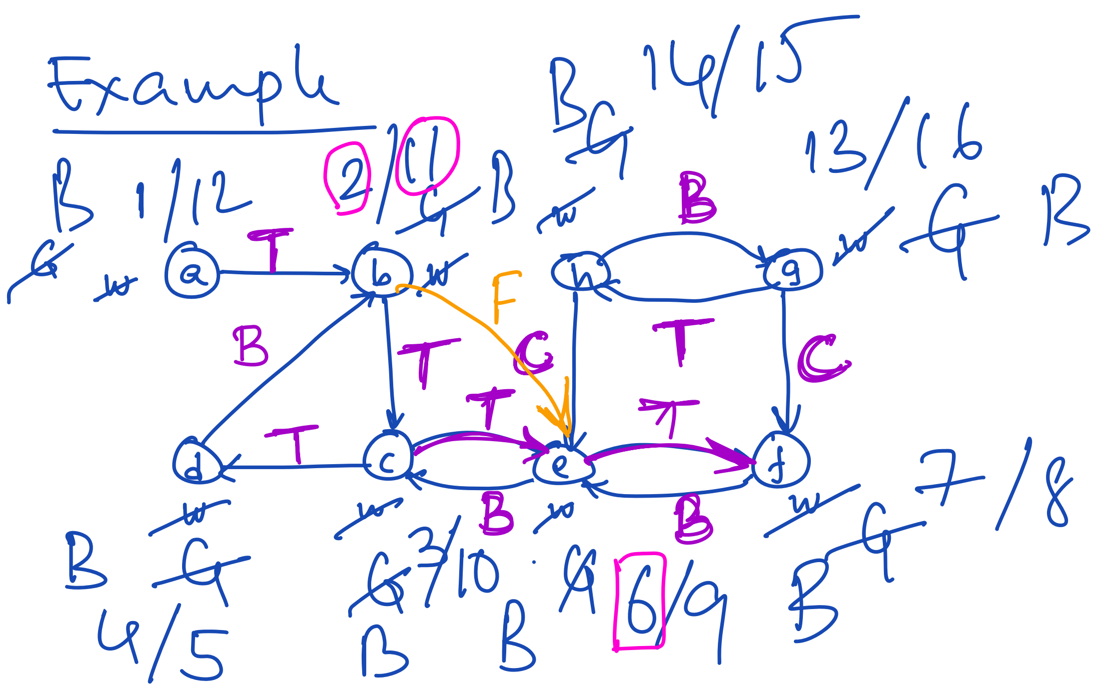
DFS_VISIT (v)

time ← time + 1

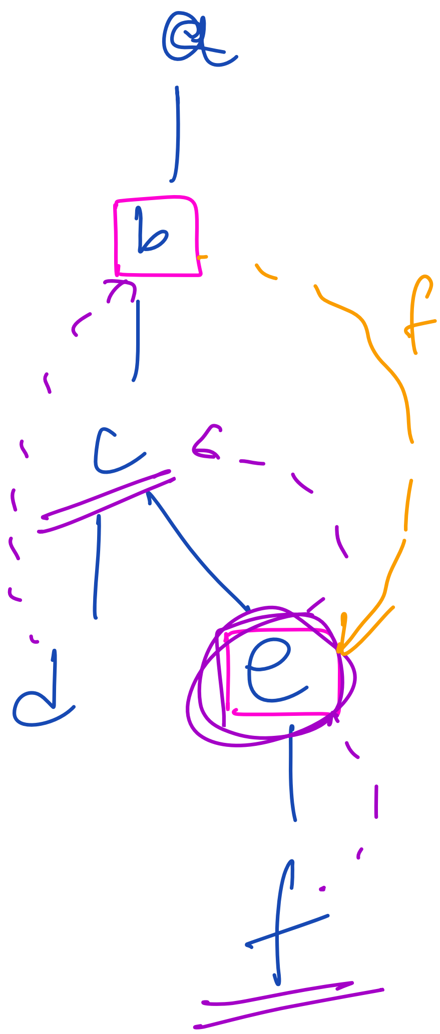
f[u] ← time

color[u] ← Black. } O(1)

Example



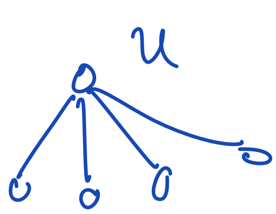
time ← ~~0~~ ~~1~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~ ~~6~~ ~~7~~ ~~8~~ ~~9~~ ~~10~~ ~~11~~ ~~12~~ ~~13~~ ~~14~~ ~~15~~ 16



Output of DFS :

DFS forest

Running time : $O(n+m)$



$$\sum_u \deg(u) = 2m = O(m)$$

Properties

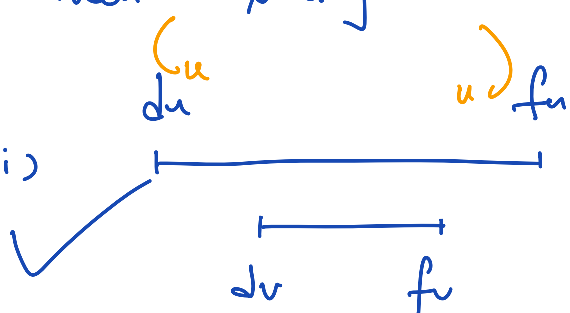
Q: When is a vertex v a descendant of vertex u in the DFS forest?

Property 1: v is a descendant of u iff v is discovered when u is gray.

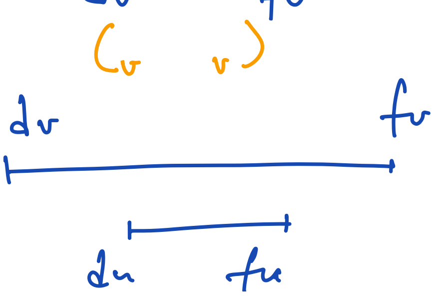
Property 2 (Parenthesis theorem)

Let u & v be any two vertices in G.

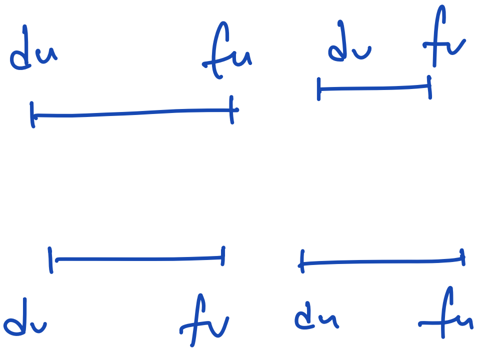
Then exactly one of the following is true.

(i) 

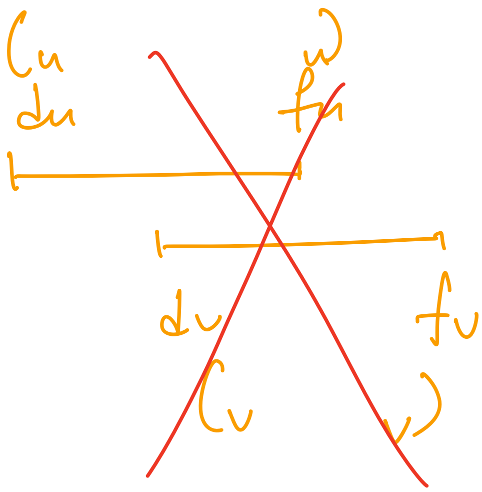
& v is a descendant of u in the DFS forest

(ii) 

& u is a descendant of v in the DFS forest

(iii) 

if neither u nor v is a descendant of the other. ✓

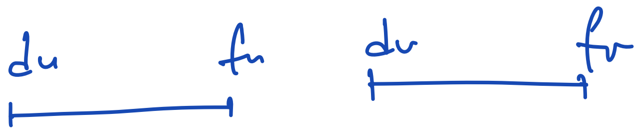


Proof: Without loss of generality, let

$d_u < d_v$.

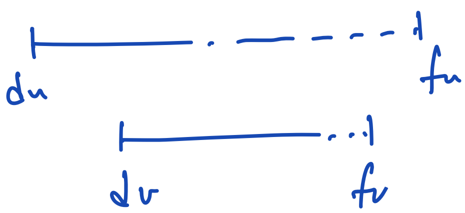
Case I: $f_u < d_v$

u finishes before v is discovered. Thus we

have  4

Since v is discovered when u is Black, by Property 1, v is not a descendant of u in the DFS forest.

<u>Case II</u> : $f_u > d_v$



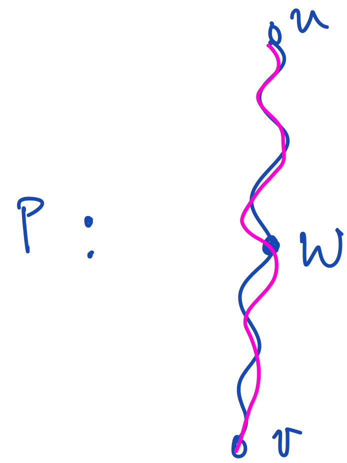
Since v is discovered after u, all neighbors of v are explored and then v finishes, before the search returns back to u. Thus $f_v < f_u$. Since v is discovered when u is gray, by property 1, v is a descendant of u in the DFS forest.

Corollary **: v is a descendant of u in the DFS forest iff $d_u < d_v < f_v < f_u$.

White Path Theorem.

Property 3 : Vertex v is a descendant of vertex u iff at time d(u) there is a white path (path consisting only of white vertices) from u to v in G.

Proof : ($\Rightarrow$) v is a descendant of u in the DFS forest $\Rightarrow$ at d(u) there is a WP from u to v in G.

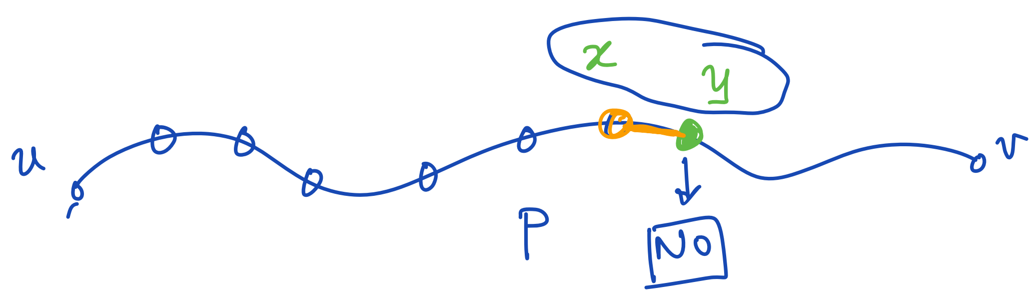


Let w be an arbitrary, but particular vertex in P.

By Parenthesis theorem, we have $d_u < d_w < f_w < f_u$ which means that w is not discovered at $d[u]$ & hence $color[w]$ is white at time $d[u]$.

($\Leftarrow$) At time $d[u]$ there is a white path, P from u to v in G
$\Rightarrow$ v is a descendant of u in the DFS forest.

<u>Proof</u>: Let P be the white path between u & v at time $d[u]$.

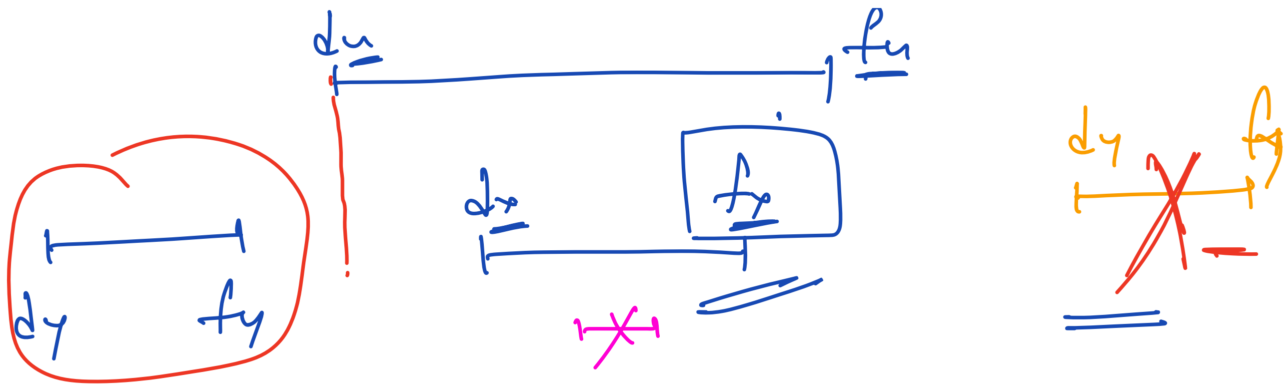


We want to prove that v is a descendant of u in the DFS forest.

Assume for contradiction that v is not a descendant of u in the DFS forest.

Walking from u towards v along P, let y be the first vertex that is not a descendant of u in the DFS forest.

Let x be the vertex just before y in P. Thus x is a descendant of u in the DFS forest. By the Parenthesis theorem, we have



By the Parenthesis thm, the interval (d_y, f_y) cannot be contained in the interval (d_u, f_u). Also, $d_y \not> f_u$. This is because, x cannot finish when it has a white neighbor.

Thus $d_y < d_u$ & $f_y < d_u$. But this is a contradiction as we know that at $d(u)$ there is a white path from x to y in G.

Edge Classification.

- label each edge in G w.r.t. a particular DFS traversal.

- an edge is given a label the very first time it is explored in DFS.

An edge $e = (u, v)$ is a

Tree edge if e is in the DFS forest.

Back edge if v is an ancestor of u in the DFS forest when e is explored

.

the first time.

color [v] = Gray.

forward edge if v is a descendant of u in the DFS forest when e is explored the first time.

color [v] = Black.

Cross edge, otherwise

color [v] = Black.

Theorem: DFS on an undirected graph G yields

Tree edges ✓

Back edges ✓

~~Forward edges~~ ✓ ✗

~~Cross edges~~ ✗

<u>Proof Sketch</u>: Let $e = (u,v)$ be an arbitrary but particular edge in G. WLOG, let $d_u < d_v$.

<u>Case I</u>: e is a tree edge.

✓

Case II : e is not a tree edge. It remains to show that e must be a back edge.

By White Path Thm., v is a descendant of u in the DFS forest. Thus, the edge (u,v) gets explored first when the search is at u. u is an ancestor of v at that time & hence e is a back edge.