* Addition Principle

Let A and B be finite sets which are disjoint then,

$$|A \cup B| = |A| + |B|$$

* Principle of Inclusion & Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

* ~~Principle~~ Mutual Inclusion and Exclusion for three sets

Let A, B, C be finite sets then

$$\boxed{|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|}$$

Proof:- Let D denote $B \cup C$, then

$A \cup B \cup C = A \cup D$

Hence,

$$|A \cup D| = |A| + |D| - |A \cap D| \quad \text{—— (1)}$$

$$|D| = |B \cup C| = |B| + |C| - |B \cap C| \quad \text{—— (2)}$$

$$\begin{aligned} |A \cap D| &= |A \cap (B \cup C)| \\ &= |(A \cap B) \cup (A \cap C)| \\ &= |A \cap B| + |A \cap C| - |A \cap B \cap C| \quad \text{—— (3)} \end{aligned}$$

Substitute eqⁿ ② & ③ in ①

$$|A \cup D| = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

$$\therefore |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Hence Proved.

ex.1 Find how many integers between 1 and 60 are not divisible by 2 nor by 3 nor by 5.

Solⁿ Let A denote numbers divisible by 2, B by 3, and C by 5.

$\therefore |A| = 30, |B| = 20, |C| = 12$

$$|A \cap B| = \frac{60}{2 \times 3} = 10$$

$$|B \cap C| = \frac{60}{3 \times 5} = 4$$

$$|A \cap C| = \frac{60}{2 \times 5} = 6$$

$$|A \cap B \cap C| = \frac{60}{2 \times 3 \times 5} = 2$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$= 30 + 20 + 12 - 10 - 4 - 6 + 2$$

$\therefore |A \cup B \cup C| = 44$

$\therefore$ Numbers not divisible by 2, 3, 5

$= 60 - 44$

$= 16$

ex.2 It is known that at the university 60% of students play tennis, 50% play cricket, 70% jog, 20% play tennis and cricket, 30% play tennis and jog and 40% play cricket and jog. If someone claims that 20% of students jog and play cricket and tennis, would this be true? Justify.

Solⁿ Let A be number of students which play tennis, B be cricket and C be jog.

$\therefore |A| = 60, |B| = 50, |C| = 70$

$\therefore |A \cup B \cup C| = 100$

$\therefore |A \cap B| = 20, |B \cap C| = 40, |A \cap C| = 30$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

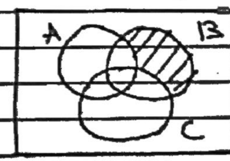
$$\therefore 100 = 60 + 50 + 70 - 20 - 40 - 30 + |A \cap B \cap C|$$

$\therefore |A \cap B \cap C| = 10\%$

$\therefore$ The claim is false

→ Also find the percent of students which play only cricket.

Solⁿ



$$|B - A - C| = |B| - |A \cap B| - |B \cap C| + |A \cap B \cap C|$$

$$= 50 - 20 - 40 + 10$$

$$= 0$$

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2. Graph Theory

A Graph consists of following parts :-

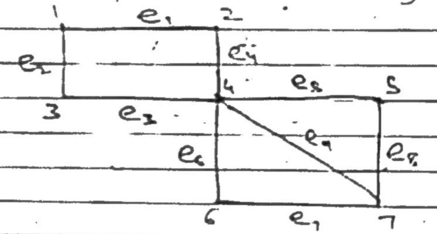
1) A Set $V = V(G)$ whose elements are called vertices or nodes.

2) A collection of edges $E = E(G)$ of unordered pairs of distinct vertices.

3) A function $\gamma(u,v)$ which maps any of the pair of vertices to edge name.

~~eg.~~

eg. Describe the following graph.



Ans $V = \{1,2,3,4,5,6,7\}$

$E = \{\{1,2\},\{1,3\},\{3,4\},\{2,4\},\{4,5\},\{4,6\},\{6,7\},\{7,5\},\{7,4\}\}$

$\gamma(v,w) = e_1 = \gamma(1,2)$, $\gamma(1,3) = e_2$,

$\gamma(3,4) = e_3$, $\gamma(2,4) = e_4$,

$\gamma(4,5) = e_5$, $\gamma(4,6) = e_6$,

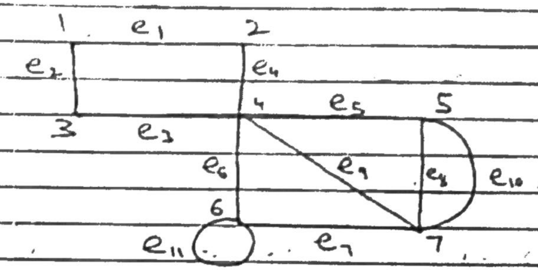
$\gamma(6,7) = e_7$, $\gamma(5,7) = e_8$,

$\gamma(4,7) = e_9$

* Multigraph

A multigraph $G = G(V,E)$ consists of set V of vertices and set E of edges except that E may contain multiple edges connecting same end points or may contain one or more loops as edges whose endpoints are same.

eg.



→ Adjacency Matrix

	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	0	1	1	0	0	0	0
V_2	1	0	0	1	0	0	0
V_3	1	0	0	1	0	0	0
V_4	0	1	1	0	1	1	1
V_5	0	0	0	1	0	0	1
V_6	0	0	0	1	0	0	1
V_7	0	0	0	1	1	1	0

→ Incidence Matrix.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}
V_1	1	1	0	0	0	0	0	0	0	0	0
V_2	1	0	0	1	0	0	0	0	0	0	0
V_3	0	1	1	0	0	0	0	0	0	0	0
V_4	0	0	1	1	1	1	0	0	1	0	0
V_5	0	0	0	0	1	0	0	1	0	1	0
V_6	0	0	0	0	0	1	1	0	0	0	1
V_7	0	0	0	0	0	0	1	1	1	1	0

* Trivial Graph

A graph with one vertex and no edges is called a trivial graph.

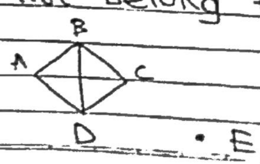
eg.

→ Trivial
$V = \{1\}$
$E = \phi$

* Isolated Vertex / Pendant Vertex

It is a vertex which is isolated and does not belong to any edge.

eg.



→ Pendant Vertex
$V = \{A, B, C, D, E\}$
$E = \{\{A,B\}, \{B,C\}, \{C,D\}, \{A,D\}, \{B,D\}\}$

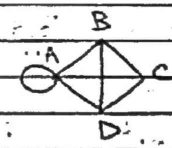
* Empty / Null Graph

A graph with no vertex or no edges is called Null Graph.

* Degree of a Vertex

It is equal to the number of edges which are incident on V or in other words, the number of edges which contain V as an end point.

eg.



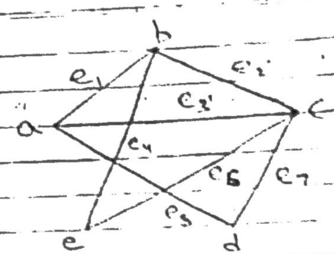
deg (A) = 4
deg (B) = 3
deg (C) = 2
deg (D) = 3

* Theorem

Sum of degree of vertices is equal to twice the number of edges in a graph.

→ In a graph, every edge is counted twice as it is incident on two vertices.

ex.1



Describe this graph.
Find the degree and parity of each vertex.
Verify if sum of degrees of vertices is equal to twice the number of edges.

Sol^n $V = \{a, b, c, d, e\}$

$E = \{\{a,b\}, \{a,d\}, \{a,c\}, \{e,b\}, \{e,c\}, \{d,c\}, \{c,b\}\}$

Odd parity $\longrightarrow$ a, b

Even parity $\longrightarrow$ c, d, e

Sum of degrees of vertices $= 3+3+4+2+2$
$= 14$

$2 \times$ no. of edges $= 2 \times 7 = 14$

$\therefore$ Sum of degrees of vertices is equal to twice the number of edges.

ex.2 A connected graph has 9 vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4, 5. How many edges are there in the said graph?

Sol^n $\Sigma d(V_i) = 2 \times E$

$28 = 2 \times E$

$\therefore E = \frac{28}{2} = 14$

ex.3 Find the no. of vertices of graph having 16 edges if degree of each vertex is 2.

Sol^n $\Sigma d(V_i) = 2 \times E$

$2 \times x = 2 \times 16$

$\therefore x = 16$

ex.4 How many vertices are necessary to construct a graph with exactly 6 edges in which each vertex is of degree 2?

Sol^n $\Sigma d(V_i) = 2 \times E$

$2 \times x = 2 \times 6$

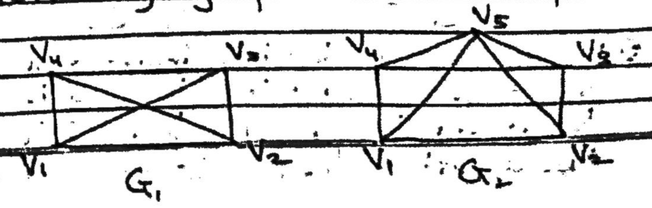
$\therefore x = 6$

* Isomorphic Graphs

Let G_1 and G_2 be any two given graphs then they are said to be isomorphic to each other, logically equivalent, if following conditions are satisfied.

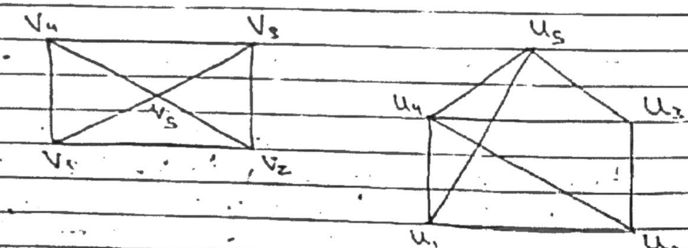
1) No. of vertices in G_1 = No. of vertices in G_2
2) No. of edges in G_1 = No. of edges in G_2
3) Both the graphs G_1 and G_2 must have equal no. of vertices having same degree

ex.1 Define isomorphic graph. Show that following graphs are isomorphic.



1) No. of vertices in $G_1 = 5$
No. of vertices in $G_2 = 5$
2) No. of edges in $G_1 = 8$
No. of edges in $G_2 = 8$
3) No. of vertices having degree 3 in $G_1 = 4$
No. of vertices having degree 4 in $G_1 = 1$
No. of vertices having degree 3 in $G_2 = 4$
No. of vertices having degree 4 in $G_2 = 1$
4) $V_1 \rightarrow V_1$
$V_2 \rightarrow V_2$
$V_3 \rightarrow V_3$
$V_4 \rightarrow V_4$
$V_5 \rightarrow V_5$

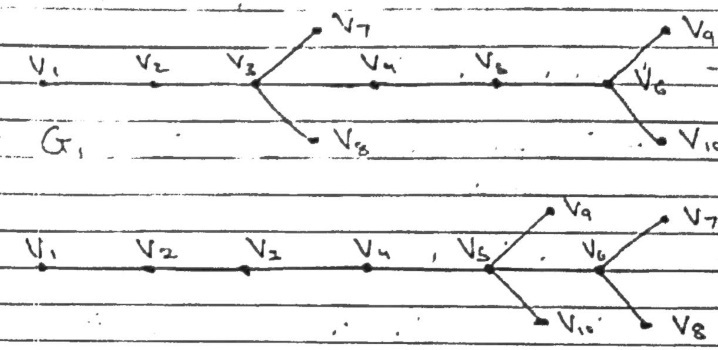
ex.2



Sol^n. 1) No. of vertices in $G_1 = 5$
No. of vertices in $G_2 = 5$
$\therefore$ No. of vertices in G_1 = No. of vertices in G_2.
2) No. of edges in $G_1 = 8$
No. of edges in $G_2 = 8$
$\therefore$ No. of edges in G_1 = No. of edges in G_2
3) No. of vertices having degree 3 in $G_1 = 4$
No. of vertices having degree 4 in $G_1 = 1$
No. of vertices having degree 3 in $G_2 = 4$
No. of vertices having degree 4 in $G_2 = 1$

$\therefore$ All the conditions of isomorphism are satisfied.
$\therefore$ G_1 and G_2 are isomorphic.

ex.3



Sol^n The Graphs G_1 and G_2 are not isomorphic because under any isomorphism, the vertex V_3 of degree 4 must correspond to vertex V_5 of degree 4 and vertex V_6 of degree 3 must correspond to V_6 of G_2 of degree 3. but V_3 and V_6 are not adjacent in G_1, whereas V_5 and V_6 are adjacent in G_2. Hence adjacency is not preserved

* Euler Graph

A graph G is said to be Euler if there exists a Euler circuit in it.

→ **Euler circuit**

A Euler circuit is a Euler path which is closed.

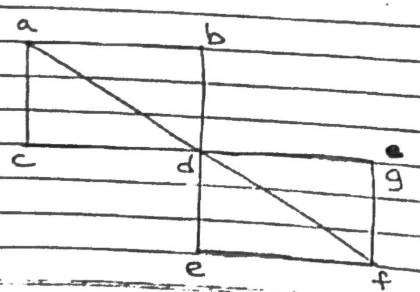
→ **Euler Path**

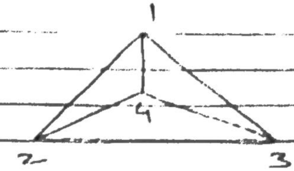
A Euler Path in a graph is a path in which every edge is travelled only once.

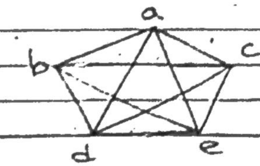
→ **Necessary and sufficient conditions for a Euler circuit**

1) Every vertex must have even degree.
2) If there are exactly two vertex of odd degree then Euler path exists which begins with one vertex of odd degree and ends with another vertex of odd degree.

ex. i







Which of the given graphs have Euler circuit and Euler path? State the Euler path if it is present.

Sol^n In Graph 1,

✓ deg (a) = 3
deg (b) = 2
deg (c) = 2
deg (d) = 6
deg (e) = 2
✓ deg (f) = 3
deg (g) = 2

There are exactly two vertices of odd degree.
Hence, a Euler path exists.
Euler path is,
a – c – d – e – f – g – d – b – a – d – f

In Graph 2,
deg (1) = 3
deg (2) = 3
deg (3) = 3
deg (4) = 3
There are more than two vertices of odd degree.
Hence, no Euler path exists.

In Graph 3,

deg(a) = 4
deg(b) = 4
deg(c) = 4
deg(d) = 4
deg(e) = 4

Every vertex has an even degree.
Hence Euler path exists.
Euler path is
a-b-d-e-c-a-d-c-b-e-a

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* Hamiltonian Graph

A graph is said to be hamiltonian if it has a hamiltonian circuit in it.

→ **Hamiltonian Path**

A Hamiltonian path in a graph is a path in which each vertex is visited only once.

→ **Hamiltonian Circuit**

Hamiltonian Circuit is a Hamiltonian path which is closed meaning the initial vertex is same as final vertex and only this vertex is visited twice.

→ **Conditions for Hamiltonian Circuit:**

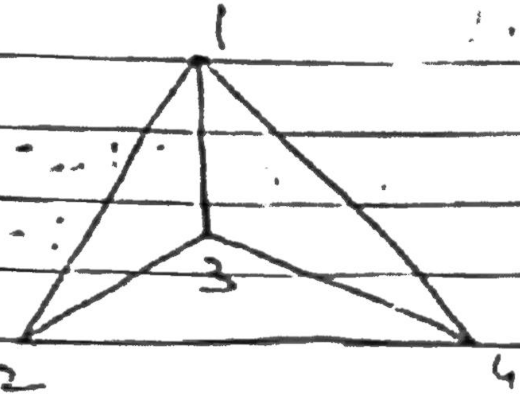
1) If the sum of degrees of any 2 adjacent vertices is greater than or equal to 'n' where n is the no. of vertices, the graph is said to be Hamiltonian.

OR

2) If the degree of each vertex is greater than or equal to (n/2) where n is no. of vertices then the graph is Hamiltonian.

However, these conditions are only sufficient conditions but not necessary to prove that a graph has Hamiltonian circuit in it.

ex.1



Solⁿ deg (1) = 3
deg (2) = 3
deg (3) = 3
deg (4) = 3
n = 4

1) Sum of degrees of adjacent vertices = 3+3
= 6 > n

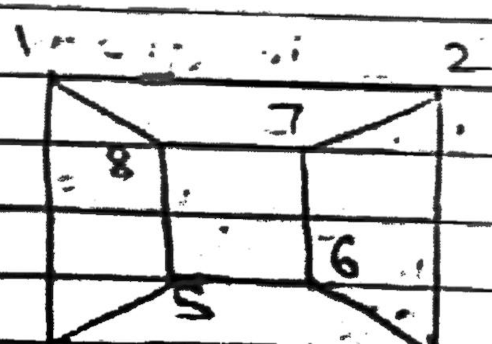
2) n/2 = 4/2 = 2 < 3

Hence, Hamiltonian Circuit and Path exists

∴ Hamiltonian Path = 1 - 3 - 4 - 2

Hamiltonian Circuit = 1 - 3 - 4 - 2 - 1

ex-2



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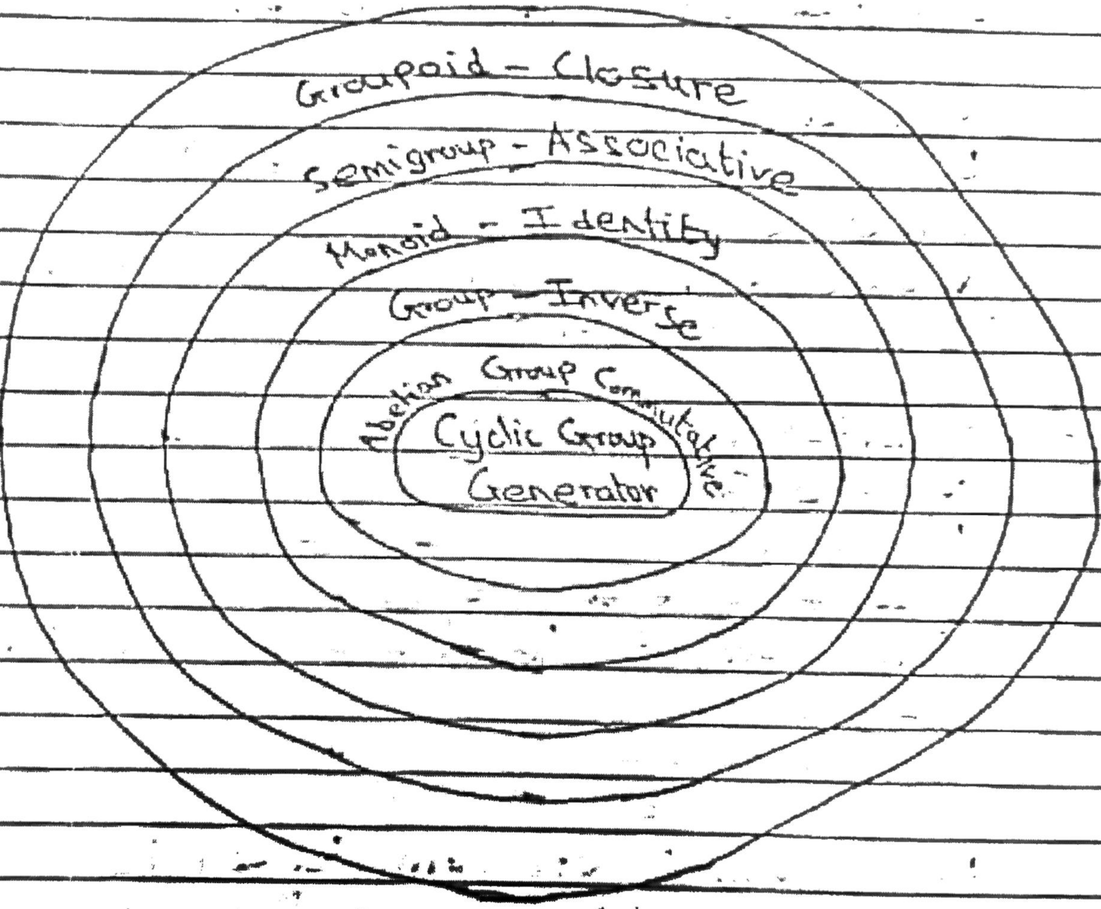
3. Algebraic Structures

An Algebraic Structure is an ordered pair of A, F, where, A is set of elements and F is a finite set of operators which is to be applied on A.



① Groupoid

Let G be a non-empty set and $*$ be the binary operation then the algebraic system $(G, *)$ is said to be a groupoid if it follows closure axiom, $\forall a, b \in G$

$\Rightarrow a * b \in G$

(2) Semigroup

An Algebraic system $(G, *)$ is said to be a semigroup if it satisfies
(1) closure, (2) associativity
$\forall a,b,c \in G$
$\Rightarrow a*(b*c) = (a*b)*c$

(3) Monoid

An algebraic system $(G,*)$ is said to be a monoid if it satisfies
(1) closure, (2) associativity, (3) it has an identity element

→ Identity element
$\forall a \in G$
there exists an element 'e' such that
$a*e = e*a = a$.
The element 'e' here is an identity element.

(4) Group

An algebraic system $(G,*)$ is said to be a group if it satisfies
(1) closure, (2) associativity, (3) existence of identity element, (4) existence of inverse

→ Inverse Element
Let $a \in G$
then an element 'α' in G such that
$a * \alpha = \alpha * a = e$
where, e is the identity element w.r.t. $*$.

(5) Abelian Group

A group $(G,*)$ is said to be Abelian or Commutative group if it follows commutativity.

→ Commutativity
$\forall a,b \in G$
$\Rightarrow a*b = b*a$

(6) Cyclic Group

A finite group $(G,*)$ is said to be cyclic if all elements of G are generated by a single element of it. This single element is called generator of G.

ex.1 Prove that a set $G = \{1,-1,i,-i\}$ is an Abelian group w.r.t. multiplication. Check whether it is cyclic.

Sol^n

$*$	1	−1	i	−i
1	1	−1	i	−i
−1	−1	1	−i	i
i	i	−i	−1	1
−i	−i	i	1	−1

i) closure
$\forall a,b \in G$
$a*b \in G$
$\Rightarrow$ It satisfies closure property

ii) associativity

$(1 * i) * (-i) = i * (-i) = 1$

$1 * (i * (-i)) = 1 * 1 = 1$

$\Rightarrow$ It is associative.

iii) identity

$1 * 1 = 1$

$-1 * 1 = -1$

$i * 1 = i$

$-i * 1 = -i$

$\Rightarrow$ '1' is the identity element.

iv) inverse

$1 * 1 = 1 \Rightarrow (1)^{-1} = 1$

$-1 * -1 = 1 \Rightarrow (-1)^{-1} = -1$

$i * -i = 1 \Rightarrow (i)^{-1} = -i$

$-i * i = 1 \Rightarrow (-i)^{-1} = i$

v) commutativity

$1 * i = i$

$i * 1 = i$

$\Rightarrow$ Commutative

vi) cyclic

$(i)^1 = i$

$(i)^2 = i * i = -1$

$(i)^3 = i * i * i = -i$

$(i)^4 = i * i * i * i = 1$

$\Rightarrow$ Cyclic

ex.2 Prove that a set $G = \{1, \omega, \omega^2\}$ forms an Abelian group w.r.t. multiplication. Check if it is cyclic.

Sol^n

$*$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

i) closure

$\forall\, a, b \in G$

$a * b \in G$

$\Rightarrow$ It is closed

ii) associativity

$1 * (\omega * \omega^2) = 1 * 1 = 1$

$(1 * \omega) * \omega^2 = \omega * \omega^2 = 1$

$\Rightarrow$ Associative

iii) identity

$1 * 1 = 1$

$\omega * 1 = \omega$

$\omega^2 * 1 = \omega^2$

$\Rightarrow$ '1' is identity element.

iv) inverse

$1 * 1 = 1 \Rightarrow (1)^{-1} = 1$

$\omega * \omega^2 = 1 \Rightarrow (\omega)^{-1} = \omega^2$

$\omega^2 * \omega = 1 \Rightarrow (\omega^2)^{-1} = \omega$.

v) commutativity

$1 * \omega = \omega$

$\omega * 1 = \omega$

$\Rightarrow$ Commutative / Abelian group

vi) cyclic

$(\omega)^1 = \omega$

$(\omega)^2 = \omega * \omega = \omega^2$

$(\omega)^3 = \omega * \omega * \omega = 1$

$\Rightarrow$ Cyclic

* Addition Modulo m

Let m be a positive integer then
$\forall a, b \in I$
$a +_m b = r$
where, r is the remainder when $a+b$ is divided by m.

eg. $3 +_4 7 = 2$

* Multiplication Modulo m

Let m be a positive integer then
$\forall a, b \in I$
$a \times_m b = r$
where, r is the remainder when $a \times b$ is divided by m.

eg. $3 \times_4 7 = 1$

* Order of a group

The least positive integer m, if it exists such that
$a^m = e$
where, e is the identity element in G is called order of element a.

It is denoted as $O(a) = m$.

ex.3 Let $G = \{0, 1, 2, 3, 4, 5\}$

1) Prepare composition table w.r.t. Addition Modulo 6 $(+_6)$.
2) Prove that G is an Abelian group w.r.t. addition modulo 6.
3) Find inverse of 2, 3, 5.
4) Check if it is cyclic
5) Find the order of 2, 3 and sub-group generated by these elements.

Sol^n 1)

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

2) i) closure

$\forall a, b \in G$

$a +_6 b \in G$

$\Rightarrow$ Closure

ii) associativity

$(0 +_6 1) +_6 5 = 1 +_6 5 = 0$

$0 +_6 (1 +_6 5) = 0 +_6 0 = 0$

$\Rightarrow$ associative

iii) identity

$0 +_6 0 = 0$

$1 +_6 0 = 1$

$2 +_6 0 = 2$

$3 +_6 0 = 3$

$4 +_6 0 = 4$

$5 +_6 0 = 5$

$\Rightarrow$ '0' is identity element

iv) inverse

$0 +_6 0 = 0 \Rightarrow (0)^{-1} = 0$

$1 +_6 5 = 0 \Rightarrow (1)^{-1} = 5$

$2 +_6 4 = 0 \Rightarrow (2)^{-1} = 4$

$3 +_6 3 = 0 \Rightarrow (3)^{-1} = 3$

$4 +_6 2 = 0 \Rightarrow (4)^{-1} = 2$

$5 +_6 1 = 0 \Rightarrow (5)^{-1} = 1$

v) commutativity

$0 +_6 1 = 1$

$1 +_6 0 = 1$

$\Rightarrow$ commutative / Abelian group

3) $(2)^{-1} = 4$

$(3)^{-1} = 3$

$(5)^{-1} = 1$

4) cyclic

$(1)^1 = 1$

$(1)^2 = 2$

$(1)^3 = 3$

$(1)^4 = 4$

$(1)^5 = 5$

$(1)^6 = 0$

$\Rightarrow$ ~~Subgroup of~~ Cyclic

5) $(2)^1 = 2$

$(2)^2 = 2 +_6 2 = 4$

$(2)^3 = 2 +_6 2 +_6 2 = 0$

$\Rightarrow$ Subgroup of $2 = \{0, 2, 4\}$

$\Rightarrow O(2) = 3$

$(3)^1 = 3$

$(3)^2 = 0$

$(3)^3 = 3$

$\Rightarrow$ Subgroup of $3 = \{0, 3\}$

$\Rightarrow O(3) = 2$

ex.4 Let $G = \{1, 2, 3, 4, 5, 6\}$. Check whether the given set is a cyclic group. Find order and subgroups generated by element 2 and 3, w.r.t. multiplication module 7 (X_7)

Sol^n. 1)

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

i) closure

$\forall\ a, b \in G$

$a \times_7 b \in G$

$\Rightarrow$ Closure

ii) associativity

$(1 \times_7 2) \times_7 3 = 2 \times_7 3 = 6$

$1 \times_7 (2 \times_7 3) = 1 \times_7 6 = 6$

$\Rightarrow$ Associative

iii) identity

$1 \times_7 1 = 1$

$2 \times_7 1 = 2$

$3 \times_7 1 = 3$

$4 \times_7 1 = 4$

$5 \times_7 1 = 5$

$6 \times_7 1 = 6$

$\therefore \Rightarrow$ '1' is identity element

iv) inverse

$1 \times_7 1 = 1 \Rightarrow (1)^{-1} = 1$

$2 \times_7 \cancel{3}4 = 1 \Rightarrow (2)^{-1} = 4$

$3 \times_7 \cancel{3}5 = 1 \Rightarrow (3)^{-1} = 5$

$4 \times_7 \cancel{4}2 = 1 \Rightarrow (4)^{-1} = 2$

$5 \times_7 \cancel{5}3 = 1 \Rightarrow (5)^{-1} = 3$

$6 \times_7 \cancel{6}6 = 1 \Rightarrow (6)^{-1} = 6$

v) commutativity

$1 \times_7 2 = 2$

$2 \times_7 1 = 2$

$\Rightarrow$ Commutative

vi) cyclic

$(3)^1 = 3$

$(3)^2 = 2$

$(3)^3 = 6$

$(3)^4 = 4$

$(3)^5 = 5$

$(3)^6 = 1$

$\Rightarrow$ Cyclic

2) $(2)^1 = 2$

$(2)^2 = 4$

$(2)^3 = 1$

$(2)^4 = 2$

Subgroup of $2 = \{2, 4, 1\}$

$O(2) = 3$

$(3)^1 = 3$

$(3)^2 = 2$

$(3)^3 = 6$

$(3)^4 = 4$

$(3)^5 = 5$

$(3)^6 = 1$

Subgroup of $3 = \{1, 2, 3, 4, 5, 6\}$

ex.5 Let $G = \{0, 3, 6, 9, 12\}$. Check whether it is a cyclic group under the operation multiplication modulo 15 ($\times_{15}$)

Sol^n

$\times_{15}$	0	3	6	9	12
0	0	0	0	0	0
3	0	9	3	12	6
6	0	3	6	9	12
9	0	12	9	6	3
12	0	6	12	3	9

i) closure

$\forall\, a, b \in G$

$a \times_{15} b \in G$

$\Rightarrow$ Closure

ii) associativity

$(0 \times_{15} 3) \times_{15} 6 = 0 \times_{15} 6 = 0$

$0 \times_{15} (3 \times_{15} 6) = 0 \times_{15} 3 = 0$

$\Rightarrow$ associative

iii) identity

$0 \times_{15} 6 = 0$

$3 \times_{15} 6 = 3$

$6 \times_{15} 6 = 6$

$9 \times_{15} 6 = 9$

$12 \times_{15} 6 = 12$

$\Rightarrow$ '6' is identity element.

iv) inverse

$0 \times_{15} \quad = 6$

$\Rightarrow$ Inverse of '0' does not exist

$\Rightarrow$ Not a Cyclic Group

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ex.6 Determine the set together with binary operation $*$ is a semigroup monoid or a group. It is a set of real numbers with $a * b = a + b + 2$

Sol^n i) closure

$a + b \in R$

$2 \in R$

$\therefore\ a + b + 2 \in R$

$\Rightarrow$ It is closed

ii) associativity

$(a * b) * c = (a + b + 2) * c$

$= (a + b + 2) + c + 2$

$= a + (b + c + 2) + 2$

$= a * (b + c + 2)$

$= a * (b * c)$

$\Rightarrow$ Associative

iii) identity

$a * e = a$

$\cancel{a} + e + 2 = \cancel{a}$

$\therefore\ e = -2 \in R \Rightarrow$ Identity exists

iv) inverse

$a * \alpha = e$

$a + \alpha + 2 = -2$

$\alpha = -a - 4 \in R$

$\Rightarrow$ Inverse exists

$\Rightarrow$ The set is a semigroup, monoid and a group.

ex.7 Prove that a set of non-zero numbers form an Abelian group w.r.t. binary operation $*$ where, $*$ is defined as,

$\forall\, a, b \in R$

$a * b = \frac{ab}{2}$

Solⁿ i) closure

$ab \in R$

$\frac{1}{2} \in R$

$\Rightarrow$ Closure

ii) associativity

$(a*b)*c = \left(\frac{ab}{2}\right) * c$

$= \frac{\left(\frac{ab}{2}\right) \cdot c}{2}$

$= \frac{a \cdot \left(\frac{bc}{2}\right)}{2}$

$= a * \left(\frac{bc}{2}\right)$

$= a * (b*c)$

$\Rightarrow$ ~~ac~~ associative

iii) identity

$a * e = a$

$\frac{ae}{2} = a$

$e = 2 \in R$

$\Rightarrow$ identity

iv) inverse

$a * \alpha = e$

$\frac{a\alpha}{2} = 2$

$\alpha = \frac{4}{a} \in R$

$\Rightarrow$ Inverse exists

v) commutativity

$a * b = \frac{ab}{2}$

$= \frac{ba}{2}$

$= b * a$

$\Rightarrow$ Commutative

$\Rightarrow$ Abelian group

ex.8 State and prove

i) Left Cancellation Law

In a group $(G, *)$ for all $a, b, c \in G$,

if $a * b = a * c$

$\Rightarrow b = c$

$\rightarrow$ Proof :-

$a \in G \Rightarrow a^{-1} \in G$ as G is a group and its inverse exists.

Here, $a * b = a * c$

Multiplying a^{-1} on both sides, we get,

$a^{-1} * a * b = a^{-1} * a * c$

$\therefore e * b = e * c$

$\therefore b = c$

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2) Right Cancellation Law

In a group $(G,*)$, $\forall\, a,b,c \in G$
$b * a = c * a \Rightarrow b = c$

3) In a group $(G,*)$. prove that identity element is unique.

→ Proof:-

Let e_1 and e_2 be two identity elements then by definition,

$a * e_1 = a = e_1 * a$ —— ①

$a * e_2 = a = e_2 * a$ —— ②

From ① and ②

$a * e_1 = a * e_2$

$e_1 = e_2$

Therefore, identity element is unique for any group.

4) Prove that inverse of each element is unique in a group $(G,*)$

→ Proof:-

Let $a \in G$ and its inverse is not unique. Suppose $\alpha_1, \alpha_2 \in G$ are two inverses of a.

$\therefore$ $a * \alpha_1 = e = \alpha_1 * a$ —— ①

$a * \alpha_2 = e = \alpha_2 * a$ —— ②

From ① and ②

$a * \alpha_1 = a * \alpha_2$

$\alpha_1 = \alpha_2$

Therefore, inverse of each element is unique in a group.

5) Prove that every cyclic group is an Abelian group.

→ Proof:-

Let $(G,*)$ be a cyclic group and 'a' be its generator element.

Let $b, c \in G$ such that $b = a^m$ & and $c = a^n$ for some $m, n \in N$

$\therefore$ $b * c = a^m * a^n$

$= a^{m+n}$

$= a^{n+m}$

$= a^n * a^m$

$= c * b$

$\Rightarrow$ Commutative

ex.9 Let $(A,*)$ be a semigroup. Let $a \in A$. Consider the binary operation $+$ such that $\forall\, x,y \in A$ $x + y = x * a * y$. Show that '+' is associative operation.

Sol^n $x + (y+z) = x + (y * a * z)$

$= x * a * (y * a * z)$

$= (x * a * y) * a * z$

$= (x+y) * a * z$

$= (x+y) + z$

$\Rightarrow$ Associative

ex.10 Show that in a group $\forall\, a,b \in G$ $(a*b)^2 = a^2 * b^2$, if and only if $(G,*)$ must be Abelian.

Sol^n $\forall\, a,b \in G$

$(a*b)^2 = a^2 * b^2$

$\Rightarrow (a*b)*(a*b) = (a*a)*(b*b)$

$\therefore \quad a*b*a*b = a*a*b*b$

$\therefore \quad a*(b*a)*b = a*(a*b)*b$

Using Left and Right Cancellation Law,

$\therefore \quad b*a = a*b$

* Group Codes

Coding Theory has developed techniques for introducing redundant information in the transmitted data over the network that help in detecting or at sometimes correcting errors. Some of these make use of group theory.

① Encoding Function

Let m, n be positive integers such that $m<n$ then a one to one function e which is equal to $e: B^m \to B^n$ is called encoding function.

② Weight

Let $x \in B^n$ then weight of x is denoted by $|x|$ and is defined as number of ones.

eg. $x = 001100 \Rightarrow |x| = 2$

③ Hamming Distance

Let $x, y \in B^n$ then the Hamming distance between x & y is denoted by $\delta(x,y)$ and defined as

$\delta(x,y) = |x \oplus y|$

eg. $x = 00000$

$y = 10110$

$\delta(x,y) = |x \oplus y| = 3$

④ Minimum Distance

Let $m<n$ and $e: B^m \to B^n$ be the encoding function then the minimum distance of encoding function e is the minimum of Hamming distance between all distinct pair of codewords.

$\delta(e_x, e_y)$ such that $x, y \in B^m$

If $m<n$ and $e: B^m \to B^n$ is the encoding function then it can detect k or fewer errors if the minimum distance of e is $k+1$.

If $m<n$ and $e: B^m \to B^n$ then the given encoding function can correct k or fewer errors if the minimum distance is $2k+1$.

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ex.1 Consider (2,5) encoding function e defined as $e: B^2 \to B^5$. Here $e(00) = 00000$, $e(01) = 01110$, $e(10) = 00111$, $e(11) = 11111$

(1) Find the minimum distance

(2) How many errors can e detect?

(3) How many errors will e correct?

Sol^n (1) $e(00) = 00000$ —— x_0

$e(01) = 01110$

$e(10) = 00111$

$e(11) = 11111$

$|x_0 \oplus x_1| = 3$

$|x_0 \oplus x_2| = 3$

$|x_0 \oplus x_3| = 5$

$|x_1 \oplus x_2| = 2$

$|x_1 \oplus x_3| = 2$

$|x_2 \oplus x_3| = 2$

$\therefore$ Minimum distance = 2

(2) $k+1 = 2$

$k = 1$

$\Rightarrow$ e can detect 1 error

(3) $2k+1 = 2$

$2k = 1$

$k = \frac{1}{2} \approx 0$

$\Rightarrow$ e can correct 0 errors

ex.2 Consider a group (3,5) where encoding function $e: B^3 \to B^5$

Sol^n $e(000) = 00000$

$e(001) = 00110$

$e(010) = 01001$

$e(011) = 01111$

$e(100) = 10011$

$e(101) = 10101$

$e(110) = 11010$

$e(111) = 11100$

Decode the following codewords

(1) 11001

(2) 01010

(3) 00111

Sol^n (1) $x_t = 11001$

$|x_t \oplus x_0| = |11001 \oplus 00000| = |11001| = 3$

$|x_t \oplus x_1| = |11001 \oplus 00110| = |11111| = 5$

$|x_t \oplus x_2| = |11001 \oplus 01001| = |10000| = 1$

$\therefore$ The encoding function e(010) will be used for decoding 11001

(2) $x_t = 01010$

$|x_t \oplus x_0| = |01010 \oplus 00000| = |01010| = 2$

$|x_t \oplus x_1| = |01010 \oplus 00110| = |01100| = 2$

$|x_t \oplus x_2| = |01010 \oplus 01001| = |00011| = 2$

$|x_t \oplus x_3| = |01010 \oplus 01111| = |00101| = 2$

$|x_t \oplus x_4| = |01010 \oplus 10011| = |11001| = 3$

$|x_t \oplus x_5| = |01010 \oplus 10101| = |11111| = 5$

$|x_t \oplus x_6| = |01010 \oplus 11010| = |10000| = 1$

$\therefore$ The encoding function e(110) will be used for decoding 01010.

(3) $x_t = 00111$

$|x_t \oplus x_0| = |00111 \oplus 00000| = |00111| = 3$

$|x_t \oplus x_1| = |00111 \oplus 00110| = |00001| = 1$

$\therefore$ The encoding function e(001) will be used for decoding 00111.

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ex.3 Show that encoding function $e: B^2 \to B^5$ defined as
$e(00) = 00000$
$e(01) = 01110$
$e(10) = 10101$
$e(11) = 11011$
Check whether it is a group code. Find the minimum distance.

Sol^n 1) Closure

		$\oplus$	x_0	x_1	x_2	x_3
No. of 1's = 0	00000	x_0	x_0	x_1	x_2	x_3
= 3	01110	x_1	x_1	x_0	x_3	x_2
= 3	10101	x_2	x_2	x_3	x_0	x_1
= 4	11011	x_3	x_3	x_2	x_1	x_0

$\Rightarrow$ It is closed.

2) Associativity
$(x_2 \oplus x_3) \oplus x_1 = x_0$
$x_2 \oplus (x_3 \oplus x_1) = x_0$
$\Rightarrow$ Associative

3) Identity
$x_0 \oplus x_0 = x_0$
$x_1 \oplus x_0 = x_1$
$x_2 \oplus x_0 = x_2$
$x_3 \oplus x_0 = x_3$
$\Rightarrow x_0$ is identity element.

4) Inverse
$x_0 \oplus x_0 = x_0 \Rightarrow (x_0)^{-1} = x_0$
$x_1 \oplus x_1 = x_0 \Rightarrow (x_1)^{-1} = x_1$
$x_2 \oplus x_2 = x_0 \Rightarrow (x_2)^{-1} = x_2$
$x_3 \oplus x_3 = x_0 \Rightarrow (x_3)^{-1} = x_3$
$\Rightarrow$ It is a group code.

$\therefore$ Minimum distance
= Minimum no. of 1's
= 3

* Parity Check Matrix

Let $m < n$ and $r = n - m$, a boolean matrix of order $n \times r$

$$H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1r} \\ h_{21} & h_{22} & \cdots & h_{2r} \\ h_{31} & h_{32} & \cdots & h_{3r} \\ 1 & 0 & \cdots & \cdots \\ 0 & 1 & \cdots & \cdots \\ 0 & 0 & 1 \cdots & \cdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n \times r}$$

whose last r rows is an identity matrix of order r is called parity check matrix. ~~to define group code. e^H:~~ We can use parity check matrix to define group code.
$e^H : B^m \to B^n$

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ex.1 Let $m=2, n=5$ and

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{5\times 3}$$

Determine the group code $e : B^2 \to B^5$

Sol^n $[0 \ 0]_{1\times 2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3} = [0 \ 0 \ 0]_{1\times 3}$

$\therefore$ $e(00) = 00000$

$[0 \ 1] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [0 \ 1 \ 1]$

$\therefore$ $e(01) = 01011$

$[1 \ 0] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [1 \ 1 \ 0]$

$\therefore$ $e(10) = 10110$

$[1 \ 1] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [1 \ 0 \ 1]$

$\therefore$ $e(11) = 11101$

ex.2 Let $H = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ be the parity check matrix. Find the encoding function and decode the following codewords

(1) 0101

(2) 1010

Sol^n $n = 4, \ r = 2$

$m = n - r = 4 - 2 = 2$

$\therefore$ $e : B^2 \to B^4$

$[0 \ 0] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = [0 \ 0]$

$\therefore$ $e(00) = 0000$

$[0 \ 1] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = [1 \ 0]$

$\therefore$ $e(01) = 0110$

$[1 \ 0] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = [1 \ 1]$

$\therefore$ $e(10) = 1011$

$[1 \ 1] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = [0 \ 1]$

$\therefore$ $e(11) = 1101$

1) $x_t = 0101$

$|x_t \oplus x_0| = |0101 \oplus 0000| = |0101| = 2$

$|x_t \oplus x_1| = |0101 \oplus 0110| = |0011| = 2$

$|x_t \oplus x_2| = |0101 \oplus 1011| = |1110| = 3$

$|x_t \oplus x_3| = |0101 \oplus 1101| = |1000| = 1$

$\therefore$ Minimum distance $= 1$

$\therefore$ $e(11) = 1101$ will be used for decoding.

2) $x_t = 1010$

$|x_t \oplus x_0| = |1010 \oplus 0000| = |1010| = 2$

$|x_t \oplus x_1| = |1010 \oplus 0110| = |1100| = 2$

$|x_t \oplus x_2| = |1010 \oplus 1011| = |1000| = 1$

$\therefore$ Minimum distance $= 1$

$\therefore$ $e(10)$ ~~wi~~ $= 1011$ will be used for decoding.

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4. Mathematical Induction

ex.1 Prove by induction

$$1+2+3+\dots+n = \frac{n(n+1)}{2} \quad \forall n \in N$$

Soln Step I:- Verification

put $n=1$

LHS $= n = 1$

RHS $= \frac{1(1+1)}{2} = \frac{2}{2} = 1$

$\therefore$ LHS = RHS

$\therefore$ The result is true for $n=1$.

Step II:- Inductive Hypothesis

let $n=k$.

$\therefore \; 1+2+3+\dots+k = \frac{k(k+1)}{2}$ —— (1)

Step III:- Conclusion

put $n=k+1$.

LHS $= 1+2+3+\dots+k+(k+1)$

$= \frac{k(k+1)}{2} + (k+1)$ (From (1))

$= (k+1)\left(\frac{k}{2}+1\right)$

$= \frac{(k+1)(k+2)}{2}$

RHS $= \frac{(k+1)(k+1+1)}{2}$

$= \frac{(k+1)(k+2)}{2}$

$\therefore$ LHS = RHS

$\therefore$ $P(k) \Rightarrow P(k+1)$

$\therefore$ The result is ~~tr~~ true for all $n=k+1$.

ex.2 $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}$

$\forall n \in N$

Soln 1) Verification

put $n=1$,

LHS $= (2(1)-1)^2 = (1)^2 = 1$

RHS $= \frac{1(2(1)+1)(2(1)-1)}{3} = \frac{3}{3} = 1$

$\therefore$ LHS = RHS

$\therefore$ The result is true for $n=1$.

2) Inductive Hypothesis

let $n=k$.

$1^2+3^2+5^2+\dots+(2k-1)^2 = \frac{k(2k+1)(2k-1)}{3}$ —— (1)

3) Conclusion

put $n=k+1$,

LHS $= 1^2+3^2+5^2+\dots+(2k-1)^2+(2(k+1)-1)^2$

$= \frac{k(2k+1)(2k-1)}{3} + (2k+1)^2$ (From (1))

$= (2k+1)\left[\frac{k(2k-1)}{3} + (2k+1)\right]$

$= (2k+1)\left[\frac{2k^2-k+6k+3}{3}\right]$

$= (2k+1)\left[\frac{2k^2+5k+3}{3}\right]$

$= (2k+1)\left[\frac{2k^2+2k+3k+3}{3}\right]$

$= (2k+1)\left[\frac{(2k+3)(k+1)}{3}\right]$

$\therefore$ LHS $= \frac{(k+1)(2k+3)(2k+1)}{3}$

RHS $= \frac{(k+1)(2(k+1)+1)(2(k+1)-1)}{3}$

$= \frac{(k+1)(2k+3)(2k+1)}{3}$

$\therefore$ LHS = RHS

$\therefore$ $P(k) \Rightarrow P(k+1)$

$\therefore$ The result is true for all $n=k+1$.

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Ex. 3 $5 + 10 + 15 + \ldots + 5n = \frac{5n(n+1)}{2} \;\forall\; n \geq 1$

Sol^n 1) Verification

put $n=1$

LHS $= 5(1) = 5$

RHS $= \frac{5(1)(1+1)}{2} = \frac{5\times 2}{2} = 5$

$\therefore$ LHS = RHS

$\therefore$ The result is true for $n=1$.

2) Inductive Hypothesis

let $n=k$,

$5+10+15+\ldots+5k = \frac{5k(k+1)}{2}$ —— (1)

3) Conclusion

put $n=k+1$,

LHS $= 5+10+15+\ldots+5k+5(k+1)$

$\therefore$ LHS $= \frac{5k(k+1)}{2} + 5(k+1)$ (From (1))

$= 5(k+1)\left(\frac{k}{2}+1\right)$

$= \frac{5(k+1)(k+2)}{2}$

RHS $= \frac{5(k+1)(k+1+1)}{2}$

$= \frac{5(k+1)(k+2)}{2}$

$\therefore$ LHS = RHS

$\therefore$ $P(k) \Rightarrow P(k+1)$

$\therefore$ The result is true for all $n=k+1$.

ex.4 $1^3+2^3+3^3+\dots+n^3 = \dfrac{n^2(n+1)^2}{4} \quad \forall\, n \in N$

Sol^n 1) Verification

put $n=1$.

$LHS = 1^3 = 1$

$RHS = \dfrac{1^2(1+1)^2}{4} = \dfrac{4}{4} = 1$

$\therefore$ $LHS = RHS$

$\therefore$ The result is true for $n=1$.

2) Inductive Hypothesis

let $n=k$,

$1^3+2^3+3^3+\dots+k^3 = \dfrac{k^2(k+1)^2}{4}$ —— ①

3) Conclusion

put $n=k+1$

$LHS = 1^3+2^3+3^3+\dots+k^3+(k+1)^3$

$= \dfrac{k^2(k+1)^2}{4} + (k+1)^3$ (From ①)

$= (k+1)^2\left(\dfrac{k^2}{4}+k+1\right)$

$= (k+1)^2\left(\dfrac{k^2+4k+4}{4}\right)$

$= \dfrac{(k+1)^2(k+2)^2}{4}$

$RHS = \dfrac{(k+1)^2(k+1+1)^2}{4}$

$= \dfrac{(k+1)^2(k+2)^2}{4}$

$\therefore$ $LHS = RHS$

$\therefore$ $P(k) \Rightarrow P(k+1)$

$\therefore$ The result is true for all $n=k+1$.

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ex.5 $1+a+a^2+\dots+a^{n-1} = \dfrac{a^n-1}{a-1} \quad \forall\, a \neq 1,\ n \geq 1$

Sol^n 1) Verification

put $n=1$.

$LHS = a^{1-1} = a^0 = 1$

$RHS = \dfrac{a^1-1}{a-1} = \dfrac{a-1}{a-1} = 1$

$\therefore$ $LHS = RHS$

$\therefore$ The result is true for $n=1$.

2) Inductive Hypothesis

let $n=k$,

~~LHS =~~ $1+a+a^2+\dots+a^{k-1} = \dfrac{a^k-1}{a-1}$ —— ①

3) Conclusion

put $n=k+1$.

$LHS = 1+a+a^2+\dots+a^{k-1}+a^{k+1-1}$

$= \dfrac{a^k-1}{a-1} + a^k$ (From ①)

$= \dfrac{a^k-1+a^k(a-1)}{a-1}$

$= \dfrac{a^k-1+a^{k+1}-a^k}{a-1}$

$= \dfrac{a^{k+1}-1}{a-1}$

$RHS = \dfrac{a^{k+1}-1}{a-1}$

$\therefore$ $LHS = RHS$

$\therefore$ $P(k) \Rightarrow P(k+1)$

$\therefore$ The result is true for all $n=k+1$.

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ex.6 $\frac{1}{1\times3}+\frac{1}{3\times5}+\dots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}, \forall n\in N$

Sol^n 1) Verification

put $n=1$

$LHS=\frac{1}{(2(1)-1)(2(1)+1)}=\frac{1}{1\times3}=\frac{1}{3}$

$RHS=\frac{1}{2(1)+1}=\frac{1}{3}$

$\therefore$ $LHS=RHS$

$\therefore$ The result is true for $n=1$.

2) Inductive Hypothesis

let $n=k$,

$\frac{1}{1\times3}+\frac{1}{3\times5}+\dots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$ —— (1)

3) Conclusion

put $n=k+1$,

$LHS=\frac{1}{1\times3}+\frac{1}{3\times5}+\dots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$

$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$ (From (1))

$=\frac{1}{2k+1}\left(k+\frac{1}{2k+3}\right)$

$=\frac{1}{2k+1}\left(\frac{2k^2+3k+1}{2k+3}\right)$

$=\frac{1}{2k+1}\left(\frac{2k^2+2k+k+1}{2k+3}\right)$

$=\frac{1}{\cancel{(2k+1)}}\cdot\frac{\cancel{(2k+1)}(k+1)}{(2k+3)}$

$=\frac{k+1}{2k+3}$

$RHS=\frac{k+1}{2(k+1)+1}$

$=\frac{k+1}{2k+3}$

$\therefore$ $LHS=RHS$

$\therefore$ $P(k)\Rightarrow P(k+1)$

$\therefore$ The result is true for all $n=k+1$.

ex.7 Show that n^3+2n is divisible by 3 $\forall n\geq1$.

Sol^n 1) Verification

put $n=1$,

$n^3+2n=1^3+2(1)=1+2=3$

It is divisible by 3.

$\therefore$ The result is true for $n=1$.

2) Inductive Hypothesis

let $n=k$

k^3+2k is divisible by 3. —— (1)

3) Conclusion

put $n=k+1$,

$(k+1)^3+2(k+1)$

$=k^3+3k^2+3k+1+2k+2$

$=(k^3+\cancel{3k^2}\,2k)+(3k^2+3k+3)$

$=(k^3+2k)+3(k^2+k+1)$

From (1), we can prove that k^3+2k is divisible by 3.

The entire remaining term is a multiple of 3. Hence, it is also divisible by 3.

$\therefore$ The entire expression is divisible by 3.

$\therefore$ $P(k) \Rightarrow P(k+1)$

$\therefore$ The result is true for all $n=k+1$.

ex.8 Show that 5^n-4n-1 is divisible by 16 $\forall\, n \geq 1$.

Sol^n 1) Verification

~~for~~ put $n=1$.

$5^n-4n-1 = 5^1-4(1)-1 = 5-4-1 = 0$

It is divisible by 16.

$\therefore$ The result is true for $n=1$.

2) Inductive Hypothesis

let $n=k$.

5^k-4k-1 is divisible by 16: —— ①

3) Conclusion

put $n=k+1$

~~LHS~~ $= 5^{k+1}-4(k+1)-1$

$= 5^k.5-4k-4-1$

$= 5^k.5-20k+16k-5$

$= 5(5^k-4k-1)+16k$

From ①, we can prove that $5(5^k-4k-1)$ is divisible by 16.

The remaining term is a multiple of 16.

Hence, it is also divisible by 16.

$\therefore$ The entire expression is divisible by 16.

$\therefore$ $P(k) \Rightarrow P(k+1)$

$\therefore$ The result is true for all $n=k+1$.

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ex.9 Prove that n^4-4n^2 is divisible by 3 $\forall\, n \geq 2$.

Sol^n 1) Verification

put $n=2$

$n^4-4n^2 = (2)^4-4(2)^2 = 16-16 = 0$

It is divisible by 3.

$\therefore$ The result is true for $n=1$.

2) Inductive Hypothesis

let $n=k$.

~~k^4-2k~~ k^4-4k^2 is divisible by 3 —— ①

3) Conclusion

put $n=k+1$

$(k+1)^4-4(k+1)^2$

$= k^4+4k^3+6k^2+4k+1-4k^2-8k-4$

$= (k^4-4k^2)+(4k^3+6k^2-4k-3)$

$= (k^4-4k^2)+4k(k^2-1)+3(2k^2-1)$

$= (k^4-4k^2)+4k(k-1)(k+1)+3(2k^2-1)$

From ①, we can prove that (k^4-4k^2) is divisible by 3.

$4k(k-1)(k+1)$ contains three consecutive terms

Hence, it is also divisible by 3.

The remaining term is ~~a~~ multiple of 3.

Hence, it is also divisible by 3.

$\therefore$ The entire expression ~~is~~ divisible by 3.

$\therefore$ $P(k) \Rightarrow P(k+1)$

$\therefore$ The result is true for all $n=k+1$.

ex.10 Prove that 8^n-3^n is a multiple of 5, $\forall\, n \geq 1$

Sol^n 1) Verification

put $n=1$

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$8^n - 3^n = 8^1 - 3^1 = 8 - 3 = 5$

It is ~~divisible~~ multiple ~~by~~ of 5

$\therefore$ The result is true for $n=1$.

2) Inductive Hypothesis

let $n=k$

$8^k - 3^k$ is ~~divisible~~ multiple ~~by~~ of 5 —— (1)

3) Conclusion

put $n=k+1$

$8^{k+1} - 3^{k+1}$

$= 8^k.8 - 3^k.3$

$= 8^k(5+3) - 3^k.3$

$= 8^k.5 + 8^k.3 - 3^k.3$

$= 5.8^k + 3(8^k - 3^k)$

From (1) we can prove that $3(8^k-3^k)$ is ~~di~~ multiple of 5.

The remaining term is a multiple of 5.

$\therefore$ The entire expression is a multiple of 5.

$\therefore$ $P(k) \Rightarrow P(k+1)$

$\therefore$ The result is true for all $n=k+1$.

ex 11. $6^{n+2} + 7^{2n+1}$ is divisible by 43, $\forall n \geq 1$

Soln 1) Verification

put $n=1$,

$6^{n+2} + 7^{2n+1} = 6^{1+2} + 7^{2(1)+1}$

$= 6^3 + 7^3$

$= 216 + 343$

$= 559$

It is divisible by 43. $(\because 559/43 = 13)$

$\therefore$ The result is true for $n=1$.

2) Inductive Hypothesis

let $n=k$

$6^{k+2} + 7^{2k+1}$ is divisible by 43 —— (1)

3) Conclusion

put $n=k+1$

$6^{k+1+2} + 7^{2(k+1)+1}$

$= 6^{k+3} + 7^{2k+3}$

$= 6^{k+2}.6 + 7^{2k+1}.49$

$= 6^{k+2}.6 + 7^{2k+1}(43+6)$

$= 6^{k+2}.6 + 7^{2k+1}.6 + 7^{2k+1}.43$

$= 6(6^{k+2} + 7^{2k+1}) + 43.7^{2k+1}$

From (1), we can prove that $6(6^{k+2}+7^{2k+1})$ is divisible by 43.

The remaining term is a multiple of 43

Hence, it is also divisible by 43.

$\therefore$ The entire expression is divisible by 43

$\therefore$ $P(k) \Rightarrow P(k+1)$

$\therefore$ The result is true for all $n=k+1$.

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Prove by induction.

University question
Opcode 40418
5mks

Q. $1+2+3+\dots+n = \frac{n(n+1)}{2} \quad \forall n = \varepsilon$

Verification:-
When $n=1$,

$LHS = 1$

$RHS = \frac{1(1+1)}{2} = 1$

$\therefore$ Result is true for all $n=1$

Using inductive hypothesis,
~~Assume~~ $n = K$,

$1+2+3+\dots K = \frac{K(K+1)}{2}$ (i)

← inductive hypothesis

$P(K+1)$ is true.

ie $1+2+3+\dots K+K+1 = \frac{(K+1)(K+2)}{2}$

$= \frac{K(K+1)}{2} + K+1$

$= \frac{K(K+1)+2K+2}{2}$

$= \frac{(K+1)(K+2)}{2} = RHS$

$\therefore P(K) \Rightarrow P(K+1)$

$\therefore$ the result is true for $n=k+1$.

Hence $1+2+3\dots+n = \frac{n(n+1)}{2}$ is true for

Q 1^2+3^2+5
$\Rightarrow$
For $n=1$
$LHS = 1^2$
$RHS = 1$
$\therefore LHS = R$
For $n=k$
1^2+3^2+5
Now to
Hence
$P(K+1) =$
$LHS =$
$1^2+3^2+5^2$
$= 1^2+3^2+$
$= 1^2+3^2+$
$= 2k+1)$

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Q. $1^2+3^2+5^2+ \dots + (2n-1)^2$

$$= \frac{n(2n+1)(2n-1)}{3} \quad \forall n \in N.$$

$\Rightarrow$

For $n=1$,

$LHS = (2n-1)^2 = (2-1)^2 = \underline{\underline{1}}$

$RHS = \frac{1(2+1)(2-1)}{3} = \frac{1(3)(1)}{3} = \frac{3}{3} = \underline{\underline{1}}$

$\therefore LHS = RHS.$

For $n=k$. Assuming it to be true.

$1^2+3^2+5^2 \dots + (2k-1)^2$

$$= \frac{k(2k+1)(2k-1)}{3} \quad \text{(i)}$$

For $n = k+1$.

$$LHS = \underbrace{1^2 + 3^2 + \cdots\cdots (2k-1)^2}_{\text{From (i)}} + (2k+1)^2$$

$$= \frac{k(2k+1)(2k-1)}{3} + (2k+1)^2$$

$$= (2k+1)\left[\frac{k(2k-1) + 3(2k+1)}{3}\right]$$

$$= \frac{(2k+1)}{3}\left[2k^2 - k + 6k + 3\right]$$

$$= \frac{(2k+1)}{3}\left[\cancel{2k(k+1)}\ 2k^2 + 5k + 3\right]$$

$$= \frac{(2k+1)}{3}\left[2k^2 + 2k + 3k + 3\right]$$

$$= \frac{(2k+1)}{3}\left[2k(k+1) + 3(k+1)\right]$$

$$\frac{(2k+1)(2k+3)(k+1)}{3} = \underline{\underline{RHS}}.$$

Q. $1^2+3^2+5^2+\dots+(2n-1)^2$

$$= \frac{n(2n+1)(2n-1)}{3} \quad \forall n \in N.$$

$\Rightarrow$

For $n=1$,

$LHS = (2n-1)^2 = (2-1)^2 = \underline{1}$

$RHS = \frac{1(2+1)(2-1)}{3} = \frac{1(3)(1)}{3} = \frac{3}{3} = 1$

$\therefore LHS = RHS.$

For $n=k$. Assuming it to be true

$1^2+3^2+5^2 \dots + (2k-1)^2$

$$= \frac{k(2k+1)(2k-1)}{3} \quad (i)$$

For $n=k+1$.

$\cancel{1^2+3^2}+\dots+\cancel{(2(k+1)-1)}^2.$

$1^2+3^2+\dots+(2k-1)^2+(2(k+1)-1)^2$

$$= \frac{k+1\,(2(k+1)+1)(2(k+1)-1)}{3}$$

$1^2+3^2 \dots + (2k-1)^2 + (2k+1)^2$

$$= \frac{(k+1)(2k+3)(2k+1)}{3}$$

$$LHS = \underbrace{1^2 + 3^2 + \cdots (2k-1)^2}_{\text{From (i)}} + (2k+1)^2$$

$$= \frac{k(2k+1)(2k-1)}{3} + (2k+1)^2$$

$$= (2k+1)\left[\frac{k(2k-1) + 3(2k+1)}{3}\right]$$

$$= \frac{(2k+1)}{3}\left[2k^2 - k + 6k + 3\right]$$

$$= \frac{(2k+1)}{3}\left[\cancel{2k(k+1)}\ 2k^2 + 5k + 3\right]$$

$$= \frac{(2k+1)}{3}\left[2k^2 + 2k + 3k + 3\right]$$

$$= \frac{(2k+1)}{3}\left[2k(k+1) + 3(k+1)\right]$$

$$\frac{(2k+1)(2k+3)(k+1)}{3} = \underline{\underline{RHS}}.$$

$\therefore$ $P(k) \Rightarrow \underline{\underline{P(k+1)}}$.

$\therefore$ result is true for $n = k+1$.

Assuming stmt to be true for $n = k$, LHS.

$$5 + 10 + 15 + \dots 5k = \frac{5k(k+1)}{2}$$

To prove stmt is true for $n = k+1$
$P(k) \Rightarrow P(k+1)$.

$$5 + 10 + 15 + \dots 5k + 5(k+1) = \frac{5(k+1)(k+1+1)}{2}$$

$$\underbrace{5 + 10 + 15 + \dots 5k}_{\downarrow} + 5(k+1) = \frac{5(k+1)(k+2)}{2}$$

$$\frac{5k(k+1)}{2} + 5(k+1) = \frac{5(k+1)(k+2)}{2}$$

Q. $1+2+2^2+2^3+\ldots+2^n = 2^{n+1}-1$

For $n=1$,

$LHS = 2^1$

$$1^3+2^3+3^3+\ldots n^3 = \frac{n^2(n+1)^2}{4}$$

For $n=1$,

$LHS = n^3 = \underline{\underline{1}}$ $\quad$ $RHS = \frac{(1)^2(1+1)^2}{4} = \frac{4}{4} = \underline{\underline{1}}$.

$\therefore LHS = RHS$.

Hence stmt is true for $n=1$.

Let $n=k$, assume true.

$$1^3+2^3+3^3+\ldots k^3 = \frac{k^2(k+1)^2}{4} \quad (-1)$$

To prove $P(k) \Rightarrow P(k+1)$

Assuming $n = k+1$.

$$1^3+2^3+3^3+\ldots k^3+(k+1)^3 = \frac{(k+1)^2(k+1+1)^2}{4}$$

$$\underbrace{1^3+2^3+3^3\ldots+k^3}_{\frac{k^2(k+1)^2}{4}}+(k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

$$\frac{k^2(k+1)^2}{4}+(k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

$$LHS = \frac{k^2(k+1)^2+4(k+1)^3}{4} = \frac{(k+1)^2}{4}\left[k^2+4(k+1)\right]$$

$$= \frac{(k+1)^2}{4}\left[k^2+4(k+1)\right]$$

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$$LHS = \cancel{k^2(k+1)^2 + (k+1)^3 \over 4}$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right]$$

$$= \frac{(k+1)^2}{4}\left(k^2+4k+4\right)$$

$$= \frac{(k+1)^2(k+2)^2}{4} = RHS$$

$\therefore$ Result is true for $n = k+1$.

Q. $1+5+9+\dots+(4n-3) = n(2n-1)$

$n=1$.

$LHS = (4(1)-3) = 1$

$RHS = 1(2(1)-1) = 1$

$LHS = RHS$.

For $n=k$,

$1+5+9+\dots+(4k-3) = k(2k-1)$ (i)

For $n = k+1$.

$1+5+9+\dots+\cancel{(4(k+1))}(4k-3) + [4(k+1)-3]$
$= (k+1)(2(k+1)-1)$

$1+5+9+\dots+(4k-3)+(4k+4-3)$
$= (k+1)(2k+1)$

$$1+5+9+\cdots(4k-3)+(4k+1) = (k+1)(2k+1)$$

$$k(2k-1)+(4k+1) = (k+1)(2k+1).$$

$$\begin{aligned}
LHS &= k(2k-1)+(4k+1)\\
&= 2k^2-k+4k+1\\
&= 2k^2+3k+1\\
&= 2k^2+2k+k+1\\
&= 2k(k+1)+1(k+1)\\
&= (2k+1)(k+1) = RHS.
\end{aligned}$$

Q $1+a+a^2+\cdots+a^{n-1} = \dfrac{a^n-1}{a-1}$, $a \neq 1$

For $n=1$,

$LHS = a^{(1-1)} = a^0 = 1$

$RHS = \dfrac{a^1-1}{a-1} = \dfrac{a-1}{a-1} = 1$

$\therefore LHS = RHS.$

For $n=k$.

$$1+a+a^2+\cdots+a^{k-1} = \frac{a^k-1}{a-1}$$

For $n=k+1$

$$1+a+a^2+\cdots+a^{k-1}+a^{k+1-1} = \frac{a^{k+1}-1}{a-1}$$

$$1+a+a^2\cdots+a^{k-1}+a^k = \frac{a^{k+1}-1}{a-1}$$

$$1+a+a^2+\cdots a^{k-1}+a^k = \frac{a^{k+1}-1}{a-1}$$

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$$\frac{a^k-1}{a-1}+a^k=\frac{a^{k+1}-1}{a-1}$$

$$LHS = \frac{a^k-1}{a-1}+a^k$$

$$= \frac{a^k-1+a^k(a-1)}{a-1}$$

$$= \frac{a^k-1+a^{k+1}-a^k}{a-1}$$

$$= \frac{a^{k+1}-1}{a-1} = \underline{\underline{RHS}}.$$

Q. $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\dots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$ for $n>1$

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$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+2-1)(2k+2+1)}$$

$$= \frac{k+1}{2k+3}$$

$$\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$$

$$LHS = \frac{k(2k+1)(2k+3) + (2k+1)}{(2k+1)(2k+3)}$$

$$= (2k+1)\left[\frac{2k^2+3k+1}{(2k+1)(2k+3)}\right]$$

$$= \frac{2k^2+2k+k+1}{2k+3}$$

$$= \frac{2k(k+1)+1(k+1)}{2k+3}$$

$$= \frac{(2k+1)(k+1)}{(2k+3)}$$

$$LHS = \frac{k}{(2k+1)} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{1}{(2k+1)}\left[k + \frac{1}{(2k+3)}\right]$$

$$= \frac{1}{2k+1}\left[\frac{2k^2+3k+1}{2k+3}\right] \rightarrow \begin{array}{l} 2k^2+2k+k+1 \\ 2k(k+1)+1(k+1) \end{array}$$

$$= \frac{1}{2k+1}\left[\frac{(2k+1)(k+1)}{2k+3}\right]$$

$$= \frac{k+1}{2k+3}$$

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ST. $n^3 + 2n$ is divisible by 3, $\forall n \in N$.

$P(n) = n^3 + 2n$

Verificatn step:-

For $n = 1$,

$1^3 + 2(1) = 3$.

Inductive hypothesis :-

For $n = k$

$P(k) = k^3 + 2(k)$ —(1)

Conclusion step:-

For $n = k+1$.

$(k+1)^3 + 2(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= k^3 + 3k^2 + 5k + 3$

$= (k^3 + 2k) + (3k^2 + 3k + 3)$

$= \quad 3(k^2 + k + 1)$

Divisible by 3 from (1)

Divisible by 3 because 3 is a multiple

Type 2

Q. $5^n - 4n - 1$ is divisible by 16 for $n \geq 1$

For $n=1$,

$5^1 - 4(1) - 1 = 5-4-1 = 0$ — divisible by 16.

For $n=k$

$5^k - 4k - 1$ — divisible by 16.

For $n = k+1$.

$$5^{k+1} - 4(k+1) - 1 = 5^{k+1} - 4k - 4 - 1$$
$$= 5^{k+1} - 4k - 5$$
$$= 5^k \cdot 5 - 4k - 1 - 4$$
$$= 5^k \cdot 5 - (5)(4)k + 16k - 5$$
$$= \underbrace{5(5^k - 4k - 1)}_{\text{divisible by 16}} + \underbrace{16k}_{\text{multiple of 16}}.$$

Q. $n(n^2-1)$ is divisible by 24, n is odd integer.

For $n=1$.

$1(1^2-1) = 1(0) = \underline{0}$ — divisible by 24

For $n=k$.

$k(k^2-1)$ — assume divisible by 24.

For $n = k+2$

$$(k+2)((k+2)^2 - 1) = (k+2)(k^2+4k+4-1)$$
$$= (k+2)(k^2+4k+3)$$
$$= (k+2)(k^2-1+4k+4)$$

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ST :- $n^4 - 4n^2$ is divisible by 3 for $n \geq 2$.

For $n = 2$

$2^4 - 4(2)^2 = 16 - 16 = 0$ – divisible by 3

For $n = k$,

$k^4 - 4(k)^2$ – assume divisible by 3.

(i)

For $n = k+1$

$(k+1)^4 - 4(k+1)^2$

$= k^4 + 4k^3 + 6k^2 + 4k + 1 - 4(k^2 + 2k + 1)$

$= k^4 + 4k^3 + 6k^2 + 4k + 1 - 4k^2 - 8k - 4$

$= k^4 + 4k^3 + 2k^2 - 4k - 3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(k+1)^4 - 4(k+1)^2$

$= k^4 + 4k^3 + 6k^2 + 4k + 1 - 4(k^2 + 2k + 1)$

$= k^4 + 4k^3 + 6k^2 + 4k + 1 - 4k^2 - 8k - 4$

$= (k^4 - 4k^2) + 4k^3 + 6k^2 - 4k - 3$

$= (k^4 - 4k^2) + 4k(k^2 - 1) + 3(2k^2 - 1)$

$4k(k+1)(k-1)$

$(k^4 - 4k^2)$ ↓ from (i)

$4k(k+1)(k-1)$ ↓ 3 consecutive divisible

$3(2k^2 - 1)$ ↓ multiple of 3.

2k6

6mk

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Q. $8^n - 3^n$ is multiple of 5 for $n \geq 1$.

$n=1$, $8^1 - 3^1 = 8-3 = 5$ – divisible by 5

$n=k$, $8^k - 3^k$ – assume divisible by 5 (1)

$n = k+1$

$8^{k+1} - 3^{k+1}$

$8^k \cdot 8 - 3^k \cdot 3$

$8 \cdot 8^k - 8 \cdot 3^k + 5 \cdot 3^k$

$8\underbrace{(8^k - 3^k)}_{\text{from 1}} + 5 \cdot 3^k$

divisible by 5 multiple of 5.

246628

Q.4b $6^{n+2} + 7^{2n+1}$ divisible by 43.

6mks

$n=1$, $6^{1+2} + 7^{2+1}$

$= 6^3 + 7^3 = 216 + 343 = \underline{\underline{559}}$

divisible by 43

$n=k$, $6^{k+2} + 7^{2k+1}$ – assume divisible by 43. (I)

$n = k+1$, $6^{k+1+2} + 7^{2k+2+1}$

$6^{k+3} + 7^{2k+3}$

$6 \cdot 6^{k+2} + 7^2 \cdot 7^{2k+1}$

$6 \cdot 6^{k+2} + 49 \cdot 7^{2k+1}$

$6 \cdot 6^{k+2} + 43 \cdot 7^{2k+1} + 6 \cdot 7^{2k+1}$

$6(6^{k+2} + 7^{2k+1}) + 43 \cdot 7^{2k+1}$ multiple of 43.

*

Q $n^3 + 2n$ is divisible by 3 $n \geq 1$.

For $n = 1$.

$1^3 + 2(1) = 1 + 2 = 3$ - ✓

For $n = k$

$k^3 + 2k$ - ✓ (i)

For $n = k+1$

$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= k^3 + 2k + 3k^2 + 3k + 3$

$= \underbrace{(k^3 + 2k)}_{(i)} + \underbrace{3(k^2 + k + 3)}_{3(x)}$

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$8^n - 3^n$ is multiple of 5 for $n \geq 1$.

for $n=1$,

$8-3=5$ ✓

For $n=k$

$8^k - 3^k$ ✓ (I)

For $n=k+1$

$8^{k+1} - 3^{k+1}$

$8^k \cdot 8 - 3^k \cdot 3$

$8 \cdot 8^k - 3 \cdot 3^k$

$8 \cdot 8^k - 8 \cdot 3^k + 5 \cdot 3^k$

$8(8^k - 3^k) + 5 \cdot 3^k$

$\downarrow$ From (I) $\quad \downarrow$ multiple of 5.

* LOGIC :- used to determine if a particular reasoning or argument is valid. Rules of logic are used to provide proofs of theorems

15/9/19

* Generating Functions

① Non-Homogenous Linear Recurrence Relation

ex.1 Determine the sequence a_n whose recurrence relation is,

$a_n = a_{n-1} + 3$

with the initial condition $a_1 = 2$.

Sol^n $a_n = a_{n-1} + 3$

$= (a_{n-2} + 3) + 3$

$= (a_{n-3} + 3) + 3 + 3$

$= a_{n-(n-1)} + 3 + 3 + \dots (n-1)$ times

$\therefore$ $a_n = a_1 + (n-1)3$

For $n=1$, $a_1 = a_1 + (1-1)3$

$= 2 + 0$

$a_1 = 2$

For $n=2$, $a_2 = a_1 + (2-1)3$

$= 2 + 3$

$a_2 = 5$

For $n=2$, $a_2 = a_1 + (3-1)3$

$= 2 + 6$

$\therefore$ $a_2 = 8$

② Homogenous Linear Recurrence Relation

→ If the characteristic eq^n $ax^2 + bx + c = 0$ has two roots S_1 & S_2 then, write

1) If $S_1 \neq S_2$, then the formula for the sequence is given by,

$a_n = uS_1^n + vS_2^n$

where, u & v are constants.

2) If it has equal roots i.e. $s_1 = s_2$, then the formula for the sequence is given by,

$$a_n = (u + vn)\, s^n$$

→ If the characteristic eq^n is of the form $ax^3 + bx^2 + cx + d = 0$ and it has three roots s_1, s_2, s_3 then,

1) If $s_1 = s_2 = s_3$ then the sequence is given by, ~~$a_n =$~~

$$a_n = (u + vn + wn^2)\, s^n$$

2) If $s_1 = s_2 = s$ and $s \neq s_3$, then the sequence is given by,

$$a_n = (u + vn)\, s^n + w\, s_3^n$$

3) If $s_1 \neq s_2 \neq s_3$, then the sequence is given by,

$$a_n = u s_1^n + v s_2^n + w s_3^n$$

where, u, v, w are constants.

ex. 1 Determine the sequence whose relation is given ~~by~~ by,

$$c_n = 3c_{n-1} - 2c_{n-2}$$

with initial condition $c_1 = 5$, $c_2 = 3$

Sol^n: $c_1 = a_1 = 5$, $c_2 = a_2 = 3$

$$c_n = 3c_{n-1} - 2c_{n-2}$$

$$x^2 = 3x - 2$$

$\therefore\ x^2 - 3x + 2 = 0$

$\therefore\ x = 1, 2 \longrightarrow S_1, S_2$

$a_n = uS_1^n + vS_2^n$

$\therefore\ a_n = u(1)^n + v(2)^n$

put ~~For~~ $n = 1$,

$a_1 = u + 2v$

$5 = u + 2v$ ——— ①

put $n = 2$

$a_2 = u + 4v$

$3 = u + 4v$ ——— ②

Solving ① & ②, we get,

$u = 7,\ v = -1$

$\therefore\ a_n = 7(1)^n - 1(2)^n$

$\therefore\ a_n = 7 - (2)^n$

For $n = 1$, $a_1 = 7 - (2)^1$

$= 7 - 2$

$\therefore\ a_1 = 5$

For $n = 2$; $a_2 = 7 - (2)^2$

$= 7 - 4$

$\therefore\ a_2 = 3$

For $n = 3$, $a_3 = 7 - (2)^3$

$= 7 - 8$

$\therefore\ a_3 = -1$

ex.2 Determine the sequence whose recurrence relation is,

$a_n = 2a_{n-1} - a_{n-2}$

with initial condition. $a_1 = 1.5$; $a_2 = 3$

Sol^n $a_n = 2a_{n-1} - a_{n-2}$

$\therefore\ x^2 = 2x - 1$

$\therefore\ x^2 - 2x + 1 = 0$

$\therefore\ x = 1, 1 \longrightarrow S_1, S_2$

Let $s_1 = s_2 = s = 1$

$a_n = (u + vn)s^n$

$= (u + vn)(1)^n$

$\therefore\ a_n = u + vn$

put $n = 1$,

$a_1 = u + v$

$1.5 = u + v$ ——— ①

put $n = 2$,

$a_2 = u + 2v$

$3 = u + 2v$ ——— ②

Solving ① & ②, we get,

$u = 0,\ v = 1.5$

$\therefore\ a_n = 0 + (1.5)n$

$\therefore\ a_n = 1.5n$

For $n = 1$, $a_1 = 1.5(1)$

$= 1.5$

For $n = 2$, $a_2 = 1.5(2)$

$= 3$

For $n = 3$, $a_3 = 1.5(3)$

$= 4.5$

ex.3 Determine the sequence whose recurrence relation is,

$a_n = 4a_{n-1} + 5a_{n-2}$

where, $a_1 = 2$, $a_2 = 6$.

Sol^n $a_n = 4a_{n-1} + 5a_{n-2}$

$x^2 = 4x + 5$

$\therefore\ x^2 - 4x - 5 = 0$

$\therefore\ x = 5, -1 \longrightarrow S_1, S_2$

$a_n = uS_1^n + vS_2^n$

$\therefore\ a_n = u(5)^n + v(-1)^n$

put $n = 1$

$a_1 = u(5)^1 + v(-1)^1$ —— (1)

$2 = 5u - v$

put $n = 2$

$a_2 = u(5)^2 + v(-1)^2$ —— (2)

$6 = 25u + v$

Solving (1) & (2), we get,

$u = \frac{8}{30}, \; v = -\frac{2}{3}$

$a_n = \frac{8}{30}(5)^n - \frac{2}{3}(-1)^n$

$a_n = \frac{4}{15}(5)^n - \frac{2}{3}(-1)^n$

For $n=1$, $a_1 = \frac{4}{15}(5)^1 - \frac{2}{3}(-1)^1$

$= \frac{4}{3} + \frac{2}{3}$

$a_1 = 2$

For $n=2$, $a_2 = \frac{4}{15}(5)^2 - \frac{2}{3}(-1)^2$

$= \frac{20}{3} - \frac{2}{3}$

$a_2 = 6$

For $n=3$, $a_3 = \frac{4}{15}(5)^3 - \frac{2}{3}(-1)^3$

$= \frac{100}{3} + \frac{2}{3}$

$a_3 = 34$

ex. Determine the function to be used to generate Fibonacci sequence

Sol. Fibonacci sequence is generated by,

$f_n = f_{n-1} + f_{n-2}$ and $a_1 = 1, a_2 = 1$

$x^2 = x + 1$

$\therefore x^2 - x - 1 = 0$

$\therefore x = \frac{1 \pm \sqrt{1+4}}{2}$

$\therefore x = \frac{1+\sqrt5}{2}, \frac{1-\sqrt5}{2} \longrightarrow s_1, s_2$

$a_n = u s_1^n + v s_2^n$

$\therefore a_n = u\left(\frac{1+\sqrt5}{2}\right)^n + v\left(\frac{1-\sqrt5}{2}\right)^n$

put $n=1$,

$a_1 = u\left(\frac{1+\sqrt5}{2}\right)^1 + v\left(\frac{1-\sqrt5}{2}\right)^1$

$\therefore 1 = u\left(\frac{1+\sqrt5}{2}\right) + v\left(\frac{1-\sqrt5}{2}\right)$ —— (1)

put $n=2$,

$a_2 = u\left(\frac{1+\sqrt5}{2}\right)^2 + v\left(\frac{1-\sqrt5}{2}\right)^2$

$\therefore 1 = u\left(\frac{1+2\sqrt5}{2}\right)^2 + v\left(\frac{1-2\sqrt5}{2}\right)^2$ —— (2)

$\therefore$ ~~$1 = u(3+\sqrt5) + v(3-\sqrt5)$ —— (2)~~

$\therefore 1 = u\left(\frac{6+2\sqrt5}{4}\right) + v\left(\frac{6-2\sqrt5}{4}\right)$

$\therefore 1 = u\left(\frac{3+\sqrt5}{2}\right) + v\left(\frac{3-\sqrt5}{2}\right)$

Solving (1) & (2),

(1) $\times \left(\frac{1+\sqrt5}{2}\right)$ − (2),

$$\frac{1+\sqrt{5}}{2} - 1 = v\left(\frac{1-5}{4}\right) - v\left(\frac{3-\sqrt{5}}{2}\right)$$

$$\therefore \frac{\sqrt{5}-1}{2} = v(-1) - v\left(\frac{3-\sqrt{5}}{2}\right)$$

$$\therefore \frac{\sqrt{5}-1}{2} = v\left(\frac{\sqrt{5}-5}{2}\right)$$

$$\therefore v = \frac{\sqrt{5}-1}{\sqrt{5}-5} \times \frac{(\sqrt{5}+5)}{(\sqrt{5}+5)}$$

$$= \frac{5+4\sqrt{5}-5}{5-25}$$

$$= \frac{4\sqrt{5}}{-20}$$

$$= \frac{-\sqrt{5}}{5}$$

$$\therefore v = \frac{-1}{\sqrt{5}}$$

$$① \times \left(\frac{1-\sqrt{5}}{2}\right) - ②$$

$$\frac{1-\sqrt{5}}{2} - 1 = \left(\frac{1-5}{4}\right)u - \left(\frac{3+\sqrt{5}}{2}\right)$$

$$\therefore u = \frac{\sqrt{5}+1}{\sqrt{5}+5} \times \frac{(\sqrt{5}-5)}{(\sqrt{5}-5)} = \frac{1}{\sqrt{5}}$$

$$\therefore a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

For $n=1$, $a_1 = 1$

For $n=2$, $a_2 = 1$

For $n=3$, $a_3 = 2$

* ## Procedure to find particular solution ~~for a certain function f_n~~

For a certain function f_n, check the right side of the eq^n to have polynomials in n, powers in n, and constants.

The forms of the corresponding particular solution would be as follows:-

	Forms of f_n	Forms of $(a_n)^{(p)}$ to be assumed
1)	constant	A, a constant
2)	n	$nA_0 + A_1$
3)	n^2	$A_0 n^2 + A_1 n + A_2$
4)	n^t	$A_0 n^t + A_1 n^{t-1} + \dots + A_n$
5)	r^n, $r \in R$	$A r^n$

The total solution to the recurrence relation is given by

$$a_n = a_n^{(h)} + a_n^{(p)}$$

ex.) Solve the recurrence ~~&~~ relation,

$a_r - a_{r-1} - 6a_{r-2} = -30$, $a_0 = 20$, $a_1 = -5$.

Sol^n $x^2 - x - 6 = \cancel{-30}\ 0$ $\qquad x^2 - x - 6 = 0$

$\therefore$ ~~$x^2 - x + 24 = 0$~~ $x = -2, 3 \rightarrow S_1, S_2$ $\qquad x^2$ $\qquad \cancel{2x3}$

$(x-2)(x+3)$

$x^2 - 2x + 3x - 6$

-6

$a_r^{(H)} = u(-2)^r + v(3)^r$

put $\left.\begin{array}{l} a_r = P \\ a_{r-1} = P \\ a_{r-2} = P \end{array}\right\}$ since RHS is constant

$P - P - 6P = -30$

$-6P = -30$

$P = 5 = a_r^{(P)}$

$\therefore a_r = a_r^{(H)} + a_r^{(P)}$

$\therefore a_r = u(-2)^r + v(3)^r + 5$

put $r = 0$,

$a_0 = u(-2)^0 + v(3)^0 + 5$

$20 = u + v + 5$

$\therefore u + v = 15$ —— (1)

put $r = 1$,

$a_1 = u(-2)^1 + v(3)^1 + 5$

$-5 = -2u + 3v + 5$

$\therefore 2u - 3v = 10$ —— (2)

Solving (1) & (2), we get,

$u = 11$, $v = 4$

$\therefore a_r = 11(-2)^r + 4(3)^r + 5$

For $r = 0$, $a_0 = 11(-2)^0 + 4(3)^0 + 5$

$= 11 + 4 + 5$

$= 20$

For $r = 1$, $a_1 = 11(-2)^1 + 4(3)^1 + 5$

$= -22 + 12 + 5$

$= -5$

For $r = 2$, $a_2 = 11(-2)^2 + 4(3)^2 + 5$

$= 44 + 36 + 5$

$= 85$

ex.2 Find the solution for recurrence relation, $a_0 = 1$, $a_1 = 2$

~~$a_{n-1}(n-1)+$~~

$a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n$

Sol^n $x^2 - 7x + 10 = 0$

$\therefore x = 2, 5 \longrightarrow s_1, s_2$

$a_n^{(H)} = u s_1^n + v s_2^n$

$= u(2)^n + v(5)^n$

put $a_n = P_0 n + P_1$

$a_{n-1} = P_0(n-1) + P_1$

$a_{n-2} = P_0(n-2) + P_1$

$\therefore (P_0 n + P_1) - 7[P_0(n-1) + P_1] + 10[P_0(n-2) + P_1]$

$\cancel{-7P_0n + 7P_0 - 7P_1 + 10P_0n - 20P_0 + 10P_1} = 6 + 8n$

$\therefore 4P_0 n - 13P_0 + 4P_1 = 6 + 8n$

Comparing LHS and RHS, we get,

$4P_0 = 8$, $-13P_0 + 4P_1 = 6$

$P_0 = 2$, $-13(2) + 4P_1 = 6$

$-26 + 4P_1 = 6$

$4P_1 = \cancel{20}\ 32$

$P_1 = \cancel{5}\ 8$

$\therefore a_r^{(P)} = P_0 n + P_1 = 2n - 5$

$a_r = a_r^{(H)} + a_r^{(P)}$

$\therefore a_r = u(2)^n + v(5)^n + 2n + 8$

put $r = 0$,

$a_0 = u(2)^0 + v(5)^0 + 2(0) + 8$

$1 = u + v + 8$

$\therefore u + v = -7$ —— (1)

$a_1 = u(2)^1 + v(5)^1 + 2(1) + 8$

$2 = 2u + 5v + 10$

$\therefore 2u + 5v = -8$ —— (2)

Solving (1) & (2), we get,

$u = -9$, $v = 2$

$\therefore \quad a_n = -9(2)^n + 2(5)^n + 2n + 8$

For $n=0$, $a_0 = -9(2)^0 + 2(5)^0 + 2(0) + 8$

$= -9 + 2 + 8$

$= 1$

For $n=1$, $a_1 = -9(2)^1 + 2(5)^1 + 2(1) + 8$

$= -18 + 10 + 2 + 8$

$= 2$

For $n=2$, $a_2 = -9(2)^2 + 2(5)^2 + 2(2) + 8$

$= -36 + 50 + 4 + 8$

$= 26$

18/9/19

ex.3 Find the general solution for

$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$

Sol^n $a_n = a_n^{(H)} + a_n^{(P)}$

For $a_n^{(H)}$,

$x^2 + 5x + 6 = 0$

$\therefore \quad x = -2, -3 \longrightarrow s_1, s_2$

$\therefore \quad a_n^{(H)} = u(-2)^n + v(-3)^n$

For $a_n^{(P)}$,

let $a_n = P_0 + P_1 n + P_2 n^2$

$a_{n-1} = P_0 + P_1(n-1) + P_2(n-1)^2$

$a_{n-2} = P_0 + P_1(n-2) + P_2(n-2)^2$

$\therefore \quad P_0 + P_1 n + P_2 n^2 + 5[P_0 + P_1(n-1) + P_2(n-1)^2]$

$+ 6[P_0 + P_1(n-2) + P_2(n-2)^2] = 3n^2$

$\therefore \quad (P_2 + 5P_2 + 6P_2)n^2 + (P_1 - \overset{10}{}P_2 + 5P_1 + 6P_1 \overset{-24}{} P_2)n$

$(P_0 + 5P_0 - 5P_1 + 5P_2 + 6P_0 - 12P_1 + 24P_2) = 3n^2$

$\therefore \quad 12P_2 n^2 + (8P_1 - 6P_2)n + (12P_0 - 17P_1 + 29P_2) = 3n^2$

Comparing LHS and RHS,

$12P_2 = 3 \Rightarrow P_2 = \frac{1}{4}$

$12P_1 - \overset{34}{}P_2 = 0 \Rightarrow P_1 = \frac{17}{24}$

$12P_0 - 17P_1 + 29P_2 = 0 \Rightarrow P_0 = \left(\frac{289}{24} - \frac{29}{4}\right)\frac{1}{12}$

$= \left(\frac{289 - 174}{24}\right)\frac{1}{12}$

$= \frac{115}{24 \times 12}$

$= \frac{115}{288}$

$\therefore \quad a_n^{(P)} = P_0 + P_1 n + P_2 n^2$

$\therefore \quad a_n = u(-2)^n + v(-3)^n + \frac{115}{288} + \frac{17}{24}n + \frac{1}{4}n^2$

⟹ ~~Solve~~ If the function on RHS, $f(r)$ is an exponential function db^r provided b is not a characteristic root then its particular solution is given by Pb^r.

⟹ ~~If f(r)~~

ex.5 If $f(r)$ is an exponential function which is equal to db^r and if b is the root with multiplicity $(m-1)$ then its particular solution is given by $P r^{m-1} \cdot b^r$ $\quad db^r \quad Pb^r r^2$

~~Find the soln~~

ex.1 Find the solution for $a_n - a_{n-1} = 3n2^n$

Sol^n $x - 1 = 0$

$x = 1 \to S \Rightarrow a_n^{(H)} = u(1)^n$

$\therefore$ Particular sol^n is,

$a_n^{(P)} = P2^n(P_0 + P_1 n)$

$a_{n-1} = P2^{n-1}(P_0 + P_1(n-1))$

$\therefore P2^n(P_0+P_1 n) + P2^{n-1}(P_0 + P_1(n-1)) = 3n2^n$

$\therefore P2^n\left(P_0+P_1 n + \frac{P_0 + P_1 n - P_1}{2}\right) = 3n2^n$

$\therefore P2^n\left(\frac{3P_0}{2} - \frac{P_1}{2} + \frac{3P_1 n}{2}\right) = 3n\cancel{2^n}$

$\therefore \frac{3P_0P - P_1P}{2} + \frac{3PP_1 n}{2} = 3n$

Comparing LHS and RHS,

$\frac{3P_0P - P_1P}{2} = 0, \quad \frac{\cancel{3}PP_1}{2} = \cancel{3}$ $\quad \frac{3P_0P - 2}{2} = 0$

$PP_0 = \frac{2}{3}, \quad PP_1 = 2$ $\quad 3PP_0 = 2$ $\quad PP_0 = \frac{2}{3}$

$\therefore a_n^{(P)} = P2^n(P_0 + P_1 n)$

$= 2^n(PP_0 + P_1P n)$

$= 2^n\left(\frac{2}{3} + 2n\right)$ $\quad 2^n \cdot 2\left(\frac{1}{3} + n\right)$

~~2^{n+1}~~ $\quad \frac{2^{n+1}}{3}(3n+1)$

$\therefore a_n^{(P)} = \frac{2^{n+1}}{3}(3n+1)$

$a_n = a_n^{(H)} + a_n^{(P)}$

$= u(1)^n + \frac{2^{n+1}}{3}(3n+1)$

ex.2 Find the solution ~~for~~ of $a_n - 3a_{n-1} - 4a_{n-2} = 4^n$

Sol^n ~~Find~~ $x^2 - 3x - 4 = 0$

$x = 4, -1 \to S, S_2$

$\therefore a_n^{(H)} = u(4)^n + v(-1)^n$

$\therefore$ Particular sol^n is,

$a_n^{(P)} = Pn4^n$

$a_{n-1} = P(n-1)4^{n-1}$

$a_{n-2} = P(n-2)4^{n-2}$

$\therefore Pn4^n - 3P(n-1)4^{n-1} - 4P(n-2)4^{n-2} = 4^n$

$\therefore P\cancel{4^n}\left(n - \frac{(3n-3)}{4} - \frac{(4n-8)}{16}\right) = \cancel{4^n}$

$\therefore P\left(\frac{4\cancel{n} - 3\cancel{n} + 3 - \cancel{n} + 2}{4}\right) = 1$

$\therefore P(\cancel{-3n+}5) = 4$

~~$-3Pn + 4P - 4 = 0$~~ $\quad P = \frac{4}{5}$

$\therefore a_n^{(P)} = \frac{4n4^n}{5}$

$\therefore a_n = u(4)^n + v(-1)^n + \frac{n \cdot 4^{n+1}}{5}$

ex.3 Find the solution of $a_n - 4a_{n-1} + 4a_{n-2} = 2^n$ $\quad P2^n \; n^2$

Sol^n $x^2 - 4x + 4 = 0$

$x = 2, 2 \to S$

$\therefore a_n^{(H)} = (u + vn)(2)^n$

Particular solution is,

$a_n^{(P)} = Pn^2 2^n$

$a_{n-1} = P(n-1)^2 2^{n-1}$

$a_{n-2} = P(n-2)^2 2^{n-2}$

$\therefore Pn^2 2^n - 4P(n-1)^2 2^{n-1} + 4P(n-2)^2 2^{n-2} = 2^n$

$\cancel{P 2^n \left[n^2 - \frac{(n-1)^2}{2} + \frac{(n-2)^2}{4} \right] = 2^n}$

$\cancel{P [2n^2 - 4n\ 2}$

$\therefore P 2^n \left[n^2 - \frac{4(n-1)^2}{2} + \frac{4(n-2)^2}{4} \right] = 2^n$

$\therefore P \left[n^2 - 2(n^2 - 2n + 1) + (n^2 - 4n + 4) \right] = 1$

$\therefore P(2) = 1$

$\therefore P = \frac{1}{2}$

$\therefore a_n = (u + vn) 2^n + n^2 2^{n-1}$

01/10/17

23/6/17

ex. 11 Find the generating function for the given sequence $\{0, 1, 2, 3, 4, 5, \ldots\}$ which is infinite.

Sol^n $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$= 0x^0 + 1x^1 + 2x^2 + 3x^3 + \ldots$

$= x + 2x^2 + 3x^3 + \ldots$

$= x(1 + 2x + 3x^2 + \ldots)$

$= x(1-x)^{-2}$

$\therefore G(x) = \frac{x}{(1-x)^2}$

$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

For infinite sequence which is,

$S_\infty = a + ar + ar^2 + ar^3 + \ldots$

$= \frac{a}{1-r}$

where, a is first term

r is common ratio

For finite sequence,

$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$

$= \frac{a(1-r^n)}{1-r}$

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ex.2 ~~{1, 2, 3, 4, 5}~~ {2, 2, 2, 2, 2}

Sol^n $G(x) = \sum_{n=0} a_n x^n$

$= 2x^0 + 2x^1 + 2x^2 + 2x^3 + 2x^4$

$= 2(1 + x + x^2 + x^3 + x^4)$

$\therefore\ G(x) = \dfrac{2(1-x^5)}{(1-x)}$

ex.3 {2, 2, 2, ----}

Sol^n $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$= 2x^0 + 2x^1 + 2x^2 + \text{--}$

$= 2(1 + x + x^2 + \text{---})$

$\therefore\ G(x) = \dfrac{2}{1-x}$

ex.4 {0, 0, 0, 1, 1, 1, 1}

Sol^n $G(x) = \sum_{n=0}^{6} a_n x^n$

$= 0x^0 + 0x^1 + 0x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6$

$= x^3 + x^4 + x^5 + x^6$

$\therefore\ G(x) = \dfrac{x^3(1-x^4)}{1-x}$

ex.5 $\{0, 0, 0, \underbrace{1, 1, 1, 1, \text{---}}_{\text{Repeating}}\}$

Sol^n $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$= 0x^0 + 0x^1 + 0x^2 + 1x^3 + 1x^4 + 1x^5 + \text{---}$

$= x^3 + x^4 + x^5 + \text{---}$

$\therefore\ G(x) = \dfrac{x^3}{1-x}$