Mathematical Foundations of Computer Science
Homework Assignment 2

Given: January 27, 2023 **Due:** February 3, 2023

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1. Prove that the following propositions are true.

(a) The sum of any rational number and any irrational number is irrational.
(b) $\sqrt{13}$ is irrational.

2. Let a, b, c be integers satisfying $a^2 + b^2 = c^2$. Prove that abc must be even.

3. Let $x_1, x_2, \ldots, x_n$ be n real numbers. Let $\overline{x} = (x_1 + x_2 + \ldots + x_n)/n$ be their average. Use a proof by contradiction to prove that at least one of $x_1, x_2, \ldots, x_n$ is greater than or equal to $\overline{x}$.

4. For all $n \in \mathbb{N}$, prove that $3^{3n+1} + 2^{n+1}$ is divisible by 5.

5. Prove that for all $n > 1$,

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

6. The series

$$\sum_{k=1}^{n} \frac{1}{k}$$

is called the harmonic series. The sum of the first n numbers of the harmonic series is called the *nth harmonic number*, H_n. Thus,

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Using induction show that $H_{2^n} \geq 1 + \frac{n}{2}$. In other words, prove that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$$

7. (a)Prove using induction that for all non-negative integers n and for all integers $x > 1$, $x^n - 1$ is divisible by $x - 1$.
(b) If n is a positive integer and $1 + x > 0$ then $(1 + x)^n \geq 1 + nx$.