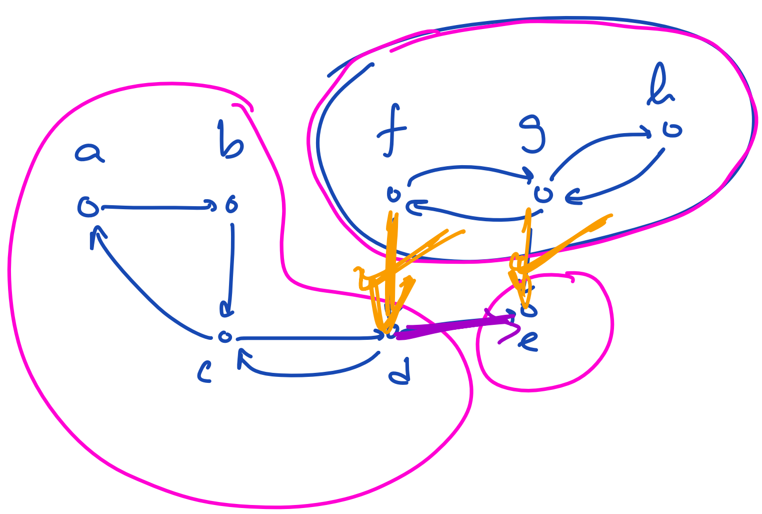
Strongly Connected Components (SCC)

$H = (V_H, E_H)$ is a SCC of $G = (V, E)$ if

- H is a subgraph of G
- $\forall\, u, v$ in H, $u \neq v$, there must be a $u \rightsquigarrow v$ path in G and a $v \rightsquigarrow u$ path in G.
- H is maximal

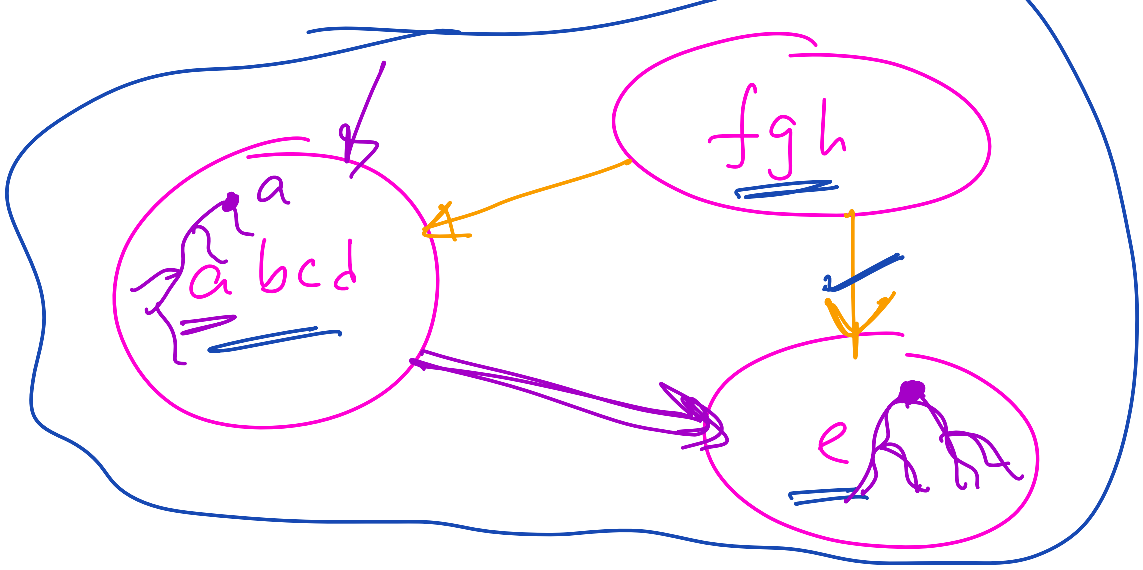
<u>Input:</u> Directed graph $G = (V, E)$

<u>Output:</u> To output all SCCs of G.



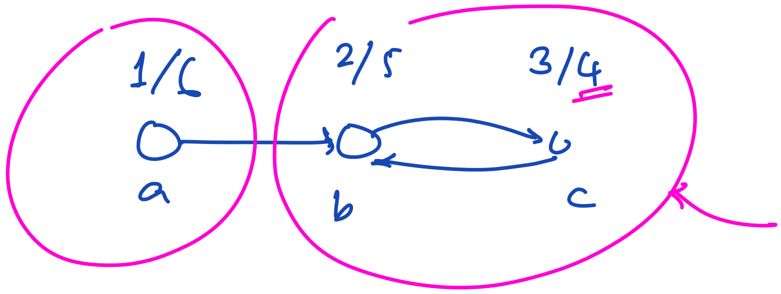
Obs 1 : Flipping the directions of every edge in G does not affect the SCCs of G.

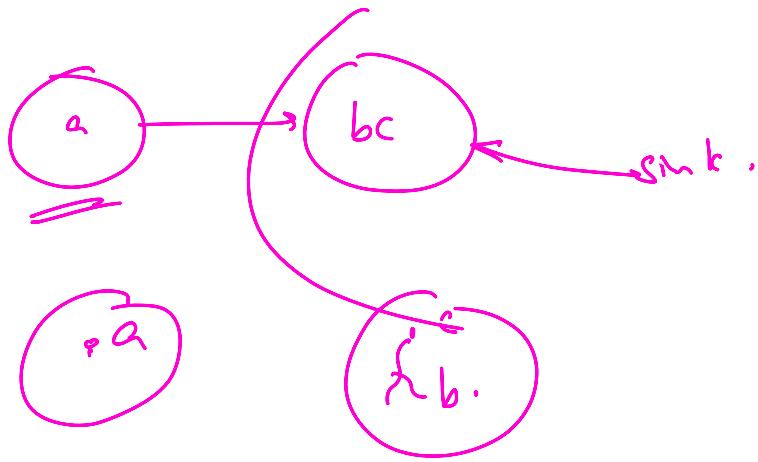
Obs 2 : $G^{SCC} = (V^{SCC}, E^{SCC})$



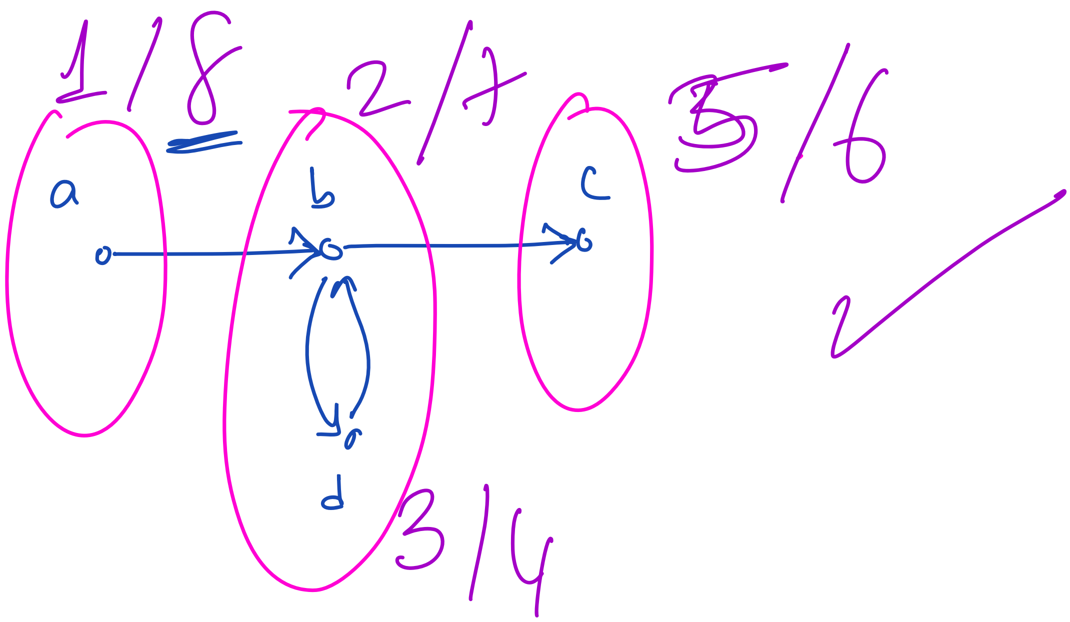
G^{SCC} is a DAG.

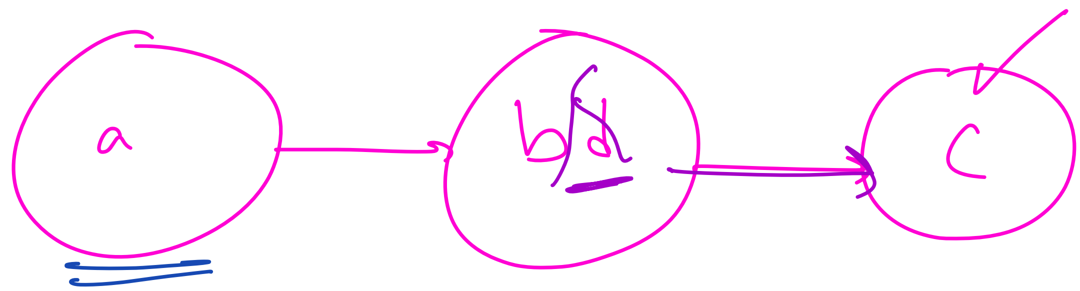
Conjecture : ~~If~~ we do DFS(G) then the vertex with the smallest f[·] belongs to the sink vertex of G^{scc}.





Ex :





Conjecture: Vertex with the largest $f[\cdot]$ in DFS(G) belongs to the source of G^{scc}.

Alg.

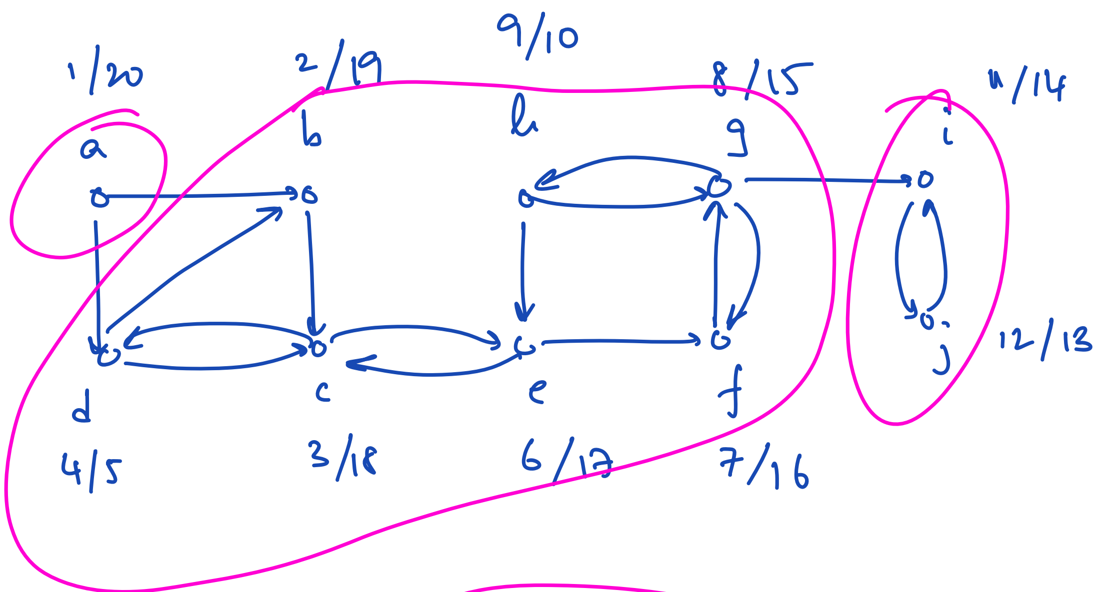
1. Do DFS(G) and note $f[\cdot]$. $O(n+m)$

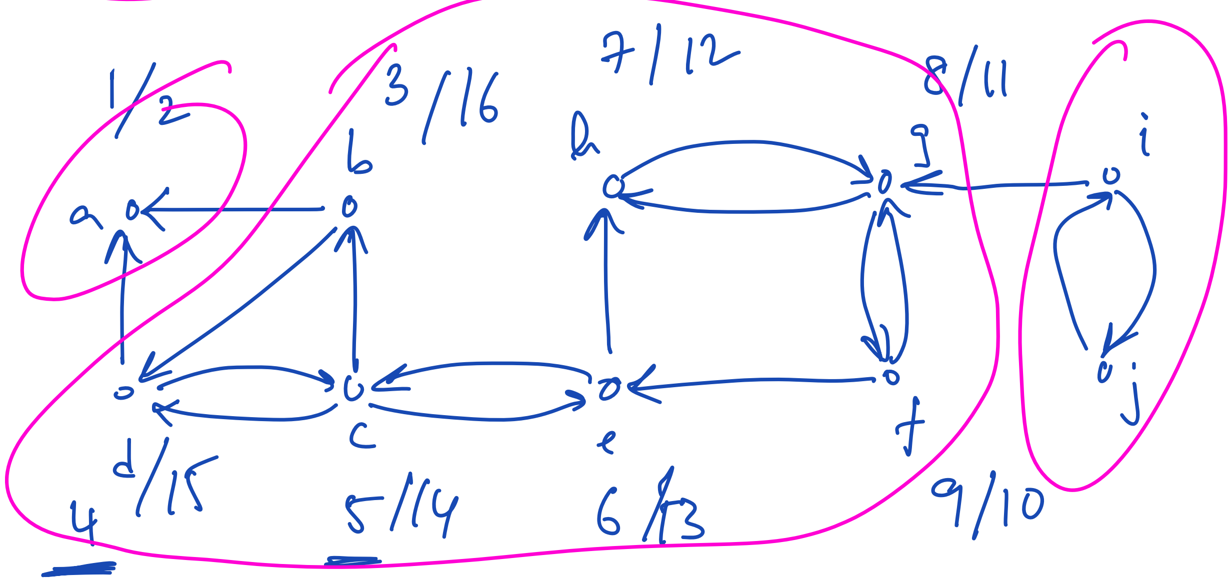
2. Compute G^T $O(n+m)$

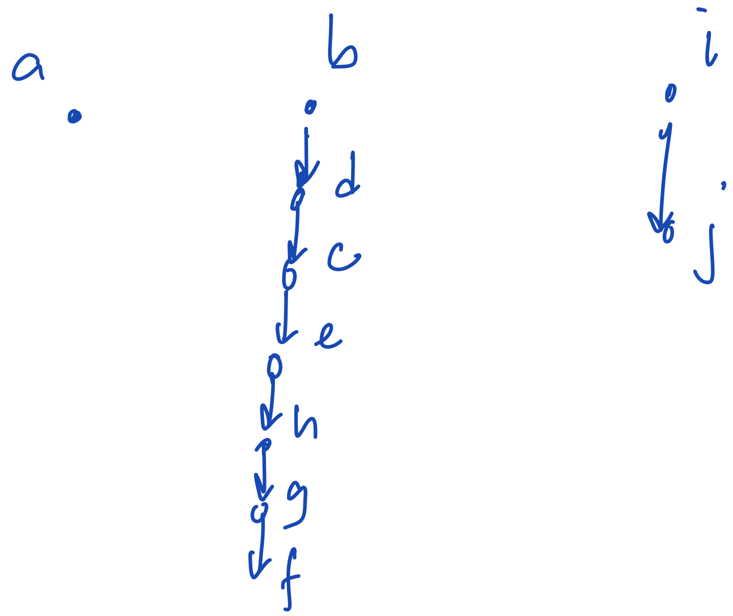
3. Do DFS(G^T), but whenever we want to choose a new root, we will always pick $O(n+m)$

the vertex with the largest f[·] in G.

4. Vertices in each DFS tree forms a SCC in G.







<u>Running time:</u> $O(n+m)$

<u>Correctness</u>

For any set $S \subseteq V$, let us define

$$d[S] = \min_{u \in S} \{ d[u] \}$$

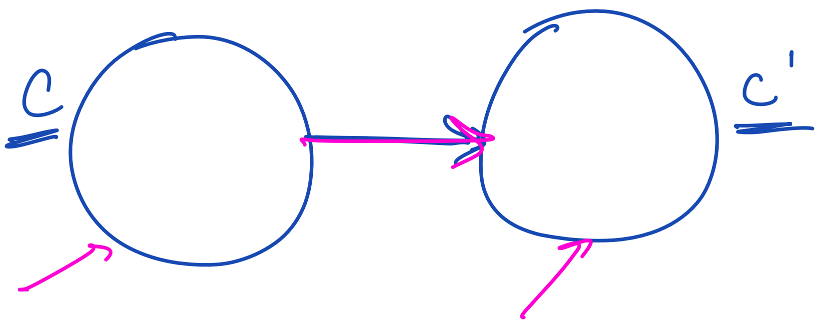
$$f[S] = \max_{u \in S} \{ f[u] \}$$

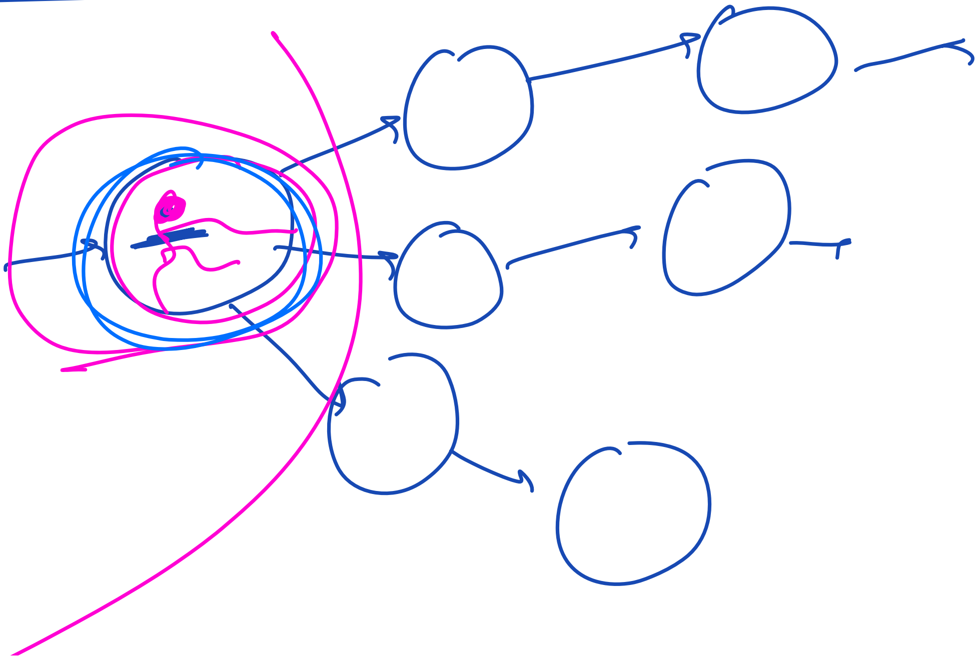
Lemma: Let C and C' be SCCs in G.

Thus C & C' are vertices in G^{SCC}.

Let $(C, C') \in E^{SCC}$. Then

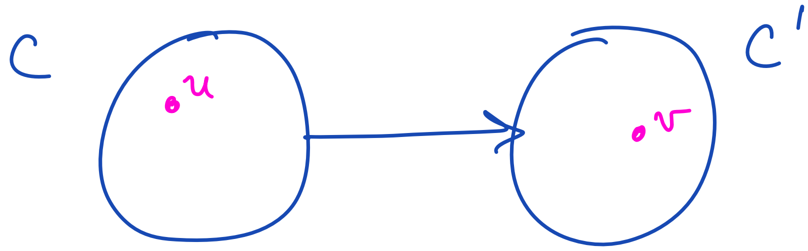
$$\boxed{f[C] > f[C']}$$





Proof of the lemma

Case I : $d[C] < d[C']$



Let $d[C] = d[u]$. To show that $f[C] > f[C']$, i.e., to show that some vertex in C finishes after all vertices in C'.

Let v be an arbitrary but

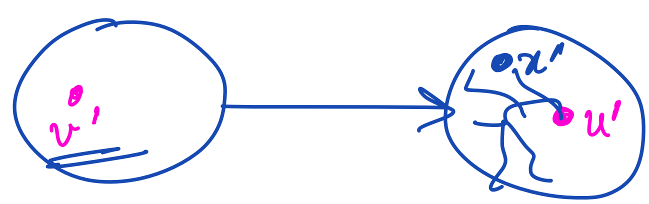
particular vertex in C'.

At time $\underline{d[u]}$, there is a $\underline{\text{white path}}$ from $\underline{u}$ to $\underline{v}$. Thus vertex v is a descendant of vertex u.

By the Parenthesis thm, we have

$$d_u < d_v < \boxed{f_v < f_u}$$ ✓

<u>Case II</u> : $d[C] > d[C']$

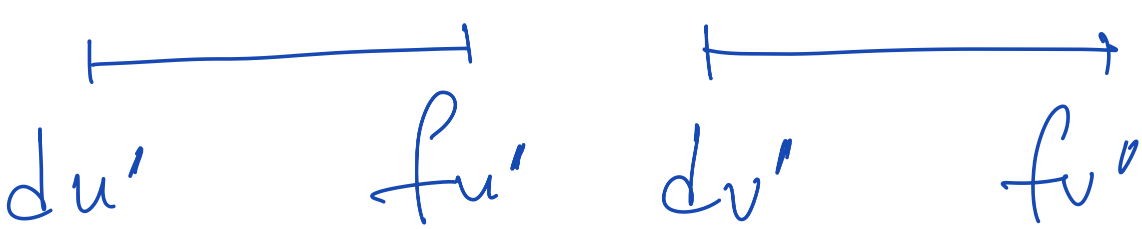
C  C'

Let $d[c'] = d[u']$.

At time $d[u']$ there is no WP from u' to v' in G.

By the WPT, v' is not a descendant of u'.

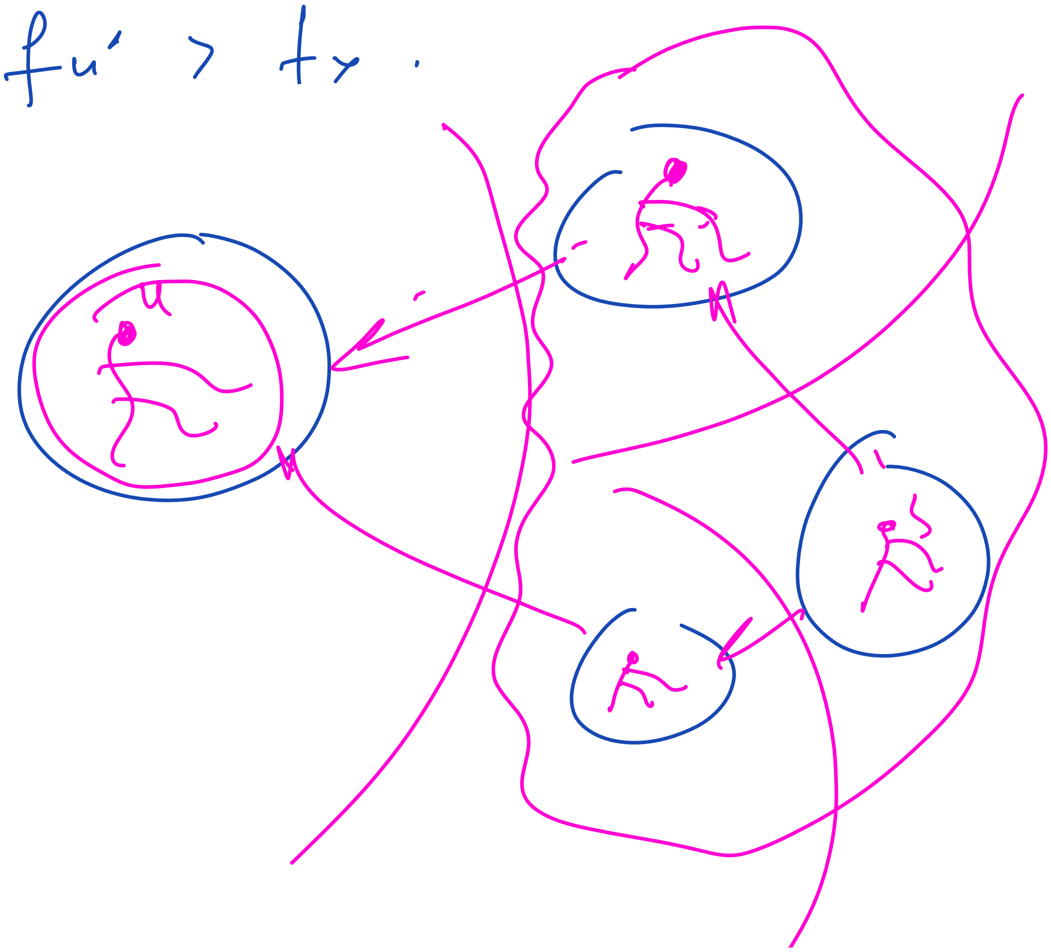
By the PT,



$f_{v'} > f_{u'}$.

At time $d_{u'}$ there is a WP from u' to x. Thus by WPT, x is a desc of u' in the DFS forest. Thus

$f_{u'} > f_x$.



Assume: We can compute G^{SCC} of a directed graph G in $O(n+m)$ time.

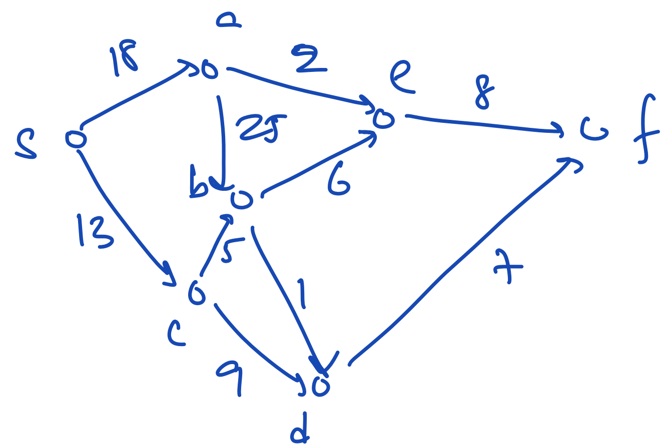
Shortest Paths

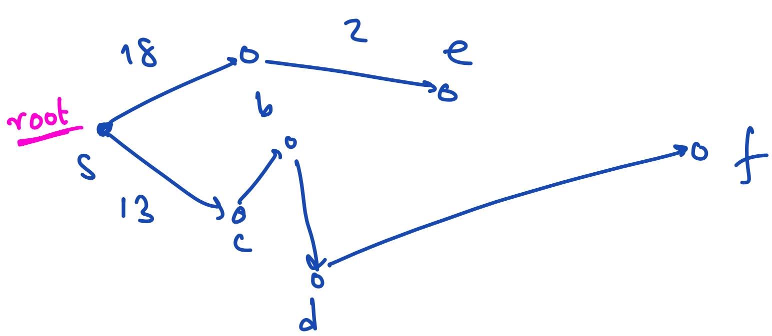
Input: Directed graph $G = (V, E)$

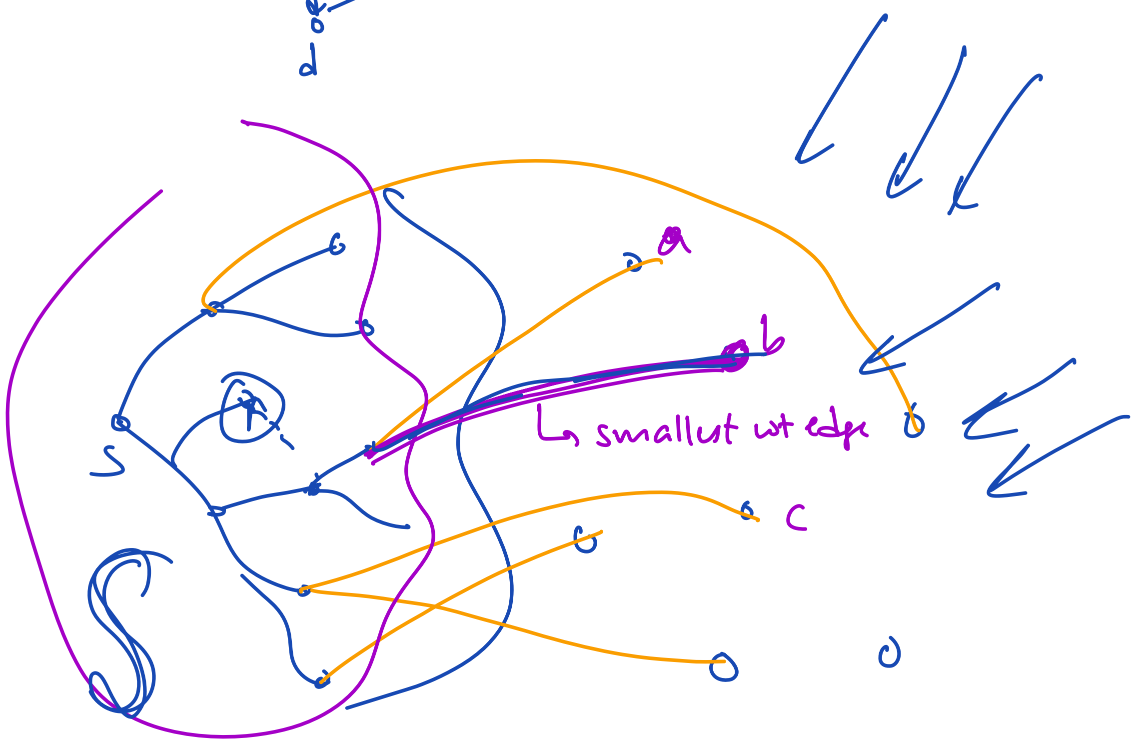
wts on edges (positive)

$s \in V$

Objective: To find shortest paths from s to every vertex in G.


a
18
2
e
8
s
25
f
b
6
13
5
1
7
c
9
d


18
2
e
root
b
s
f
13
c
d


a
b
s
smallest wt edge
c
S

for each $u \in V$ do

$d[u] \leftarrow \infty$ // $d[u]$ is the length of // the shortest path from // s to u in G.

$\pi[u] \leftarrow NIL$

$d[s] \leftarrow 0$

$S \leftarrow \phi$

while $S \neq V$ do // assume all vertices are reachable from s.

$u \leftarrow$ vertex with the smallest $d[\cdot]$ in $V \setminus S$.

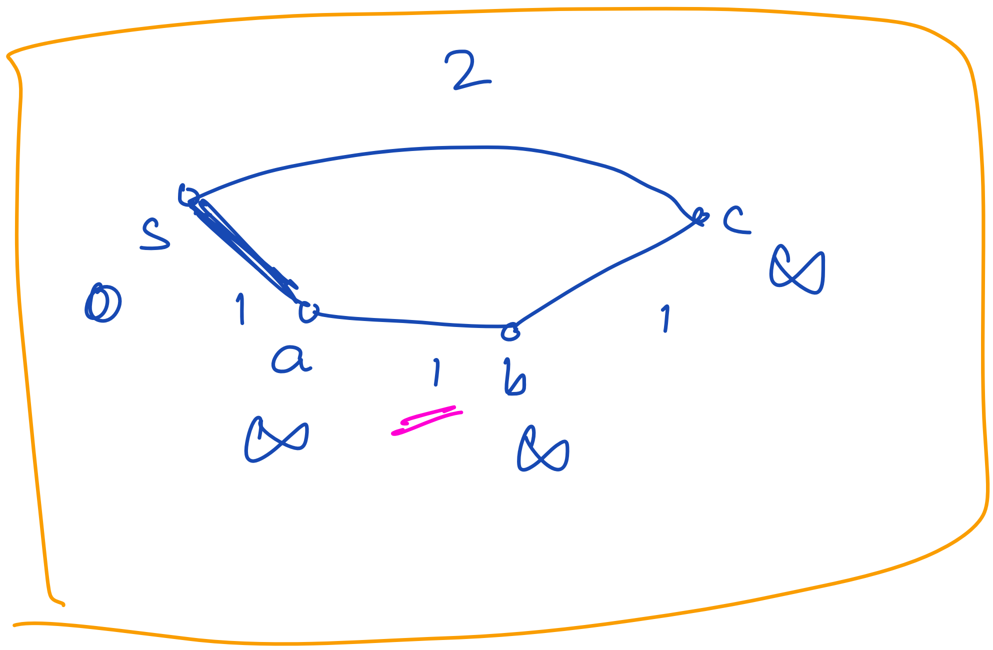
$S \leftarrow S \cup \{u\}$

for each $v \in \boxed{N(u) \cap (V \setminus S)}$ do

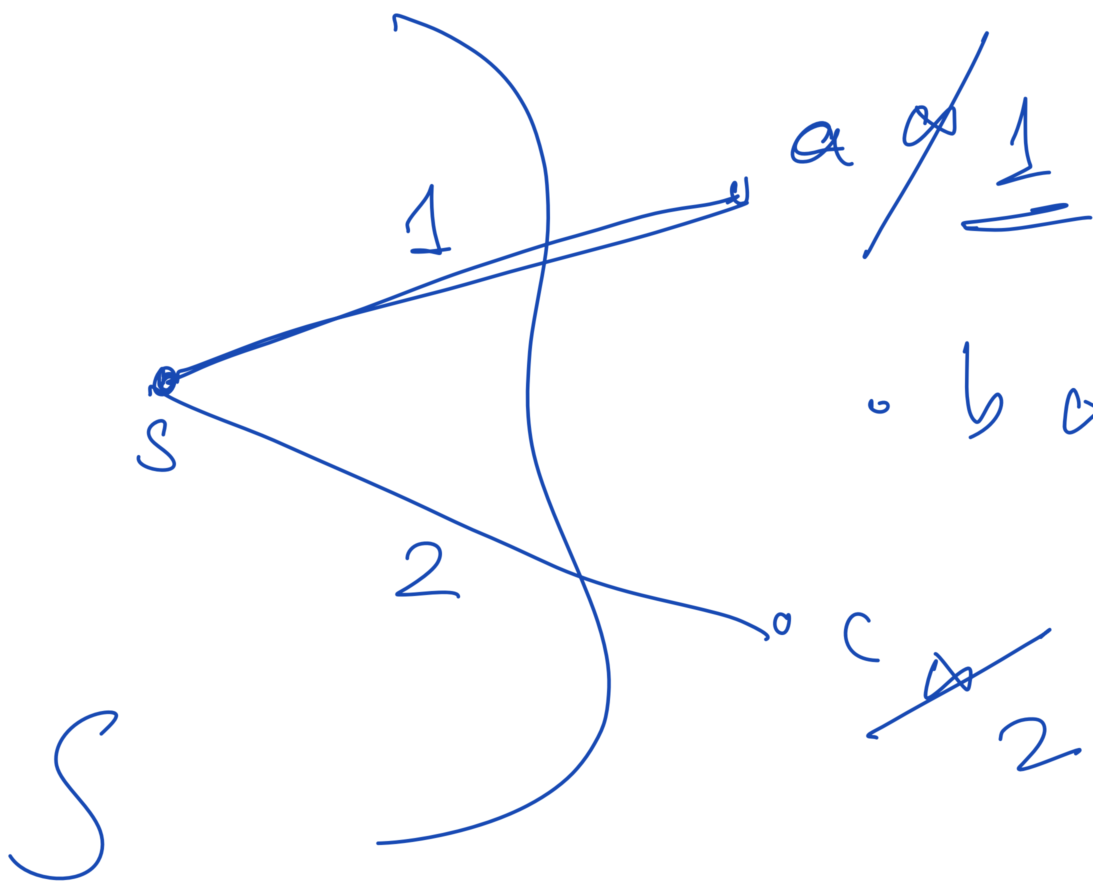
if $d[v] > w_{uv}$ then

$d[v] = w_{uv}$

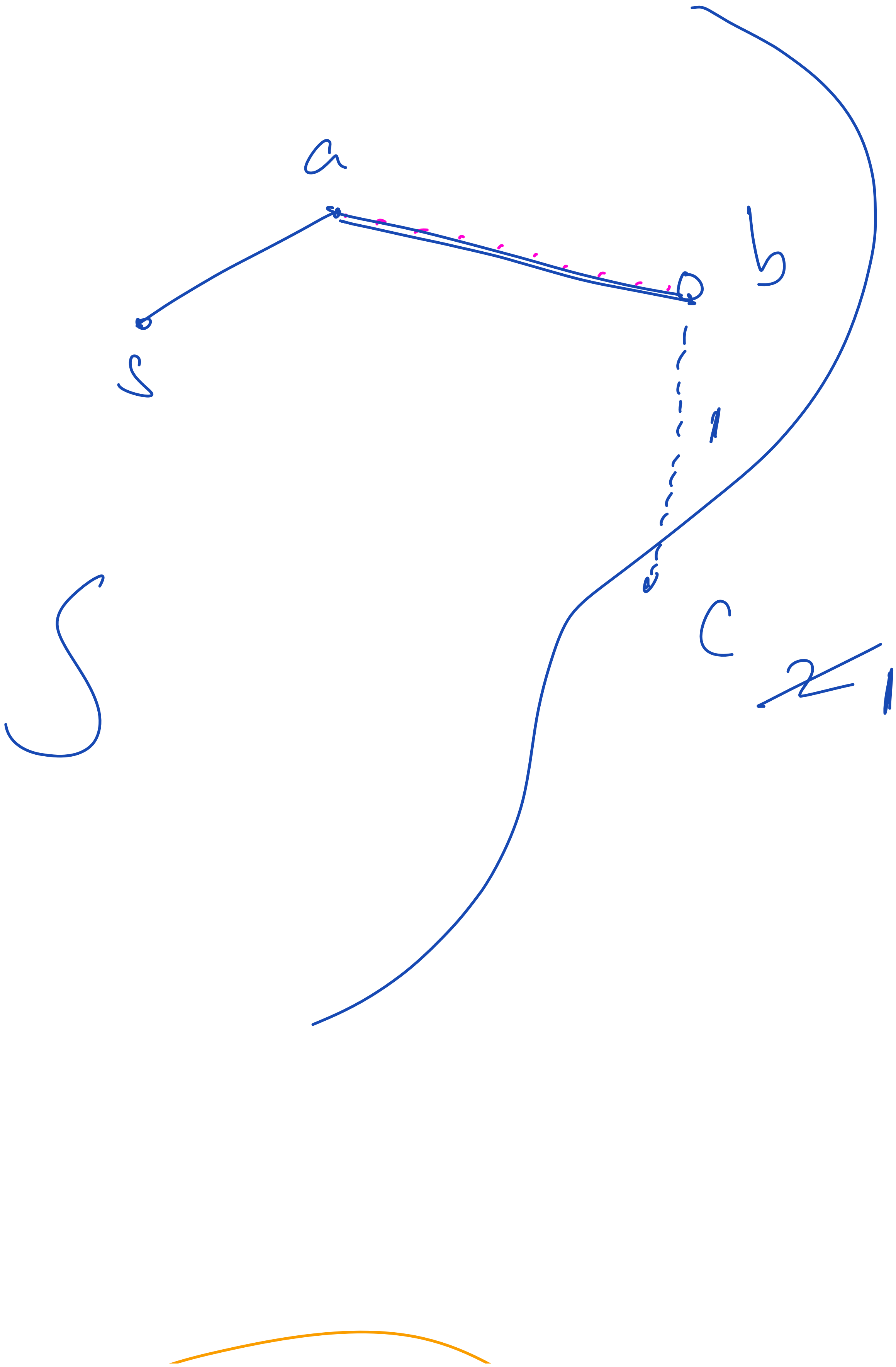
$\pi[v] \leftarrow u$

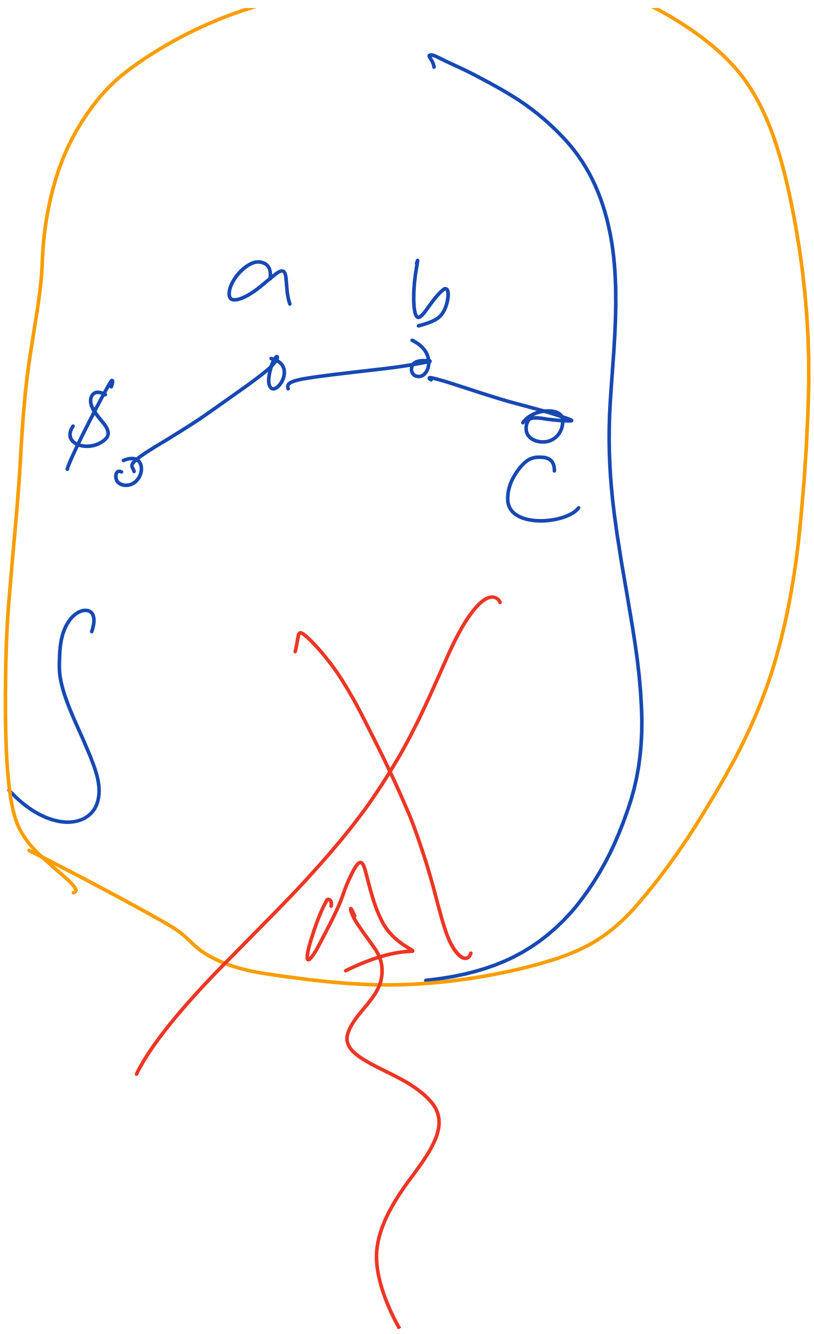


2
s
c
0
1
∞
1
a
1
b
∞
∞

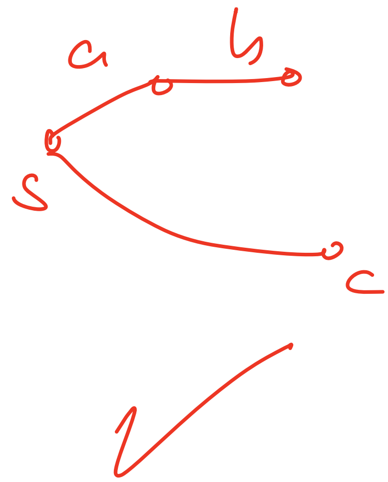


a
~~∞~~
1
1
S
b
∞
2
c
~~∞~~
2
S


a
b
s
1
c
~~2~~ 1
S



a
b
$
c



a
b
s
c