HW 1 deadline : 11:59 pm IST TODAY.

HW2 : will be posted shortly.

same zoom link as the lecture

My office hours : TODAY @ 7:30 pm IST

- bring any questions that you may have.
- we can also chat about various things.

Recitation on Saturday @ 9:30 pm IST

Ex: Let $A_1, A_2, \cdots, A_n$ $(n \geq 2)$ be

a set is an unordered collection of elements.

n sets s.t. for any two sets $A_i$ & $A_j$,

$A_i \subseteq A_j$ or $A_j \subseteq A_i$. Prove that

there is one set that is a subset of

all sets.

$A = \{1, 2\}$, $B = \{1, 2, 5\}$

$A \subseteq B$.

Proof: We will prove the claim using induction on $n$.

IH: Let $k \geq 2$ be an integer. Assume that the claim holds when $n = k$. That is if we have sets $A_1, A_2, \cdots, A_k$ with cond$^n$ that $A_i \subseteq A_j$ or $A_j \subseteq A_i$, $\forall i, j, i \neq j$ then there is a set that is a subset of $A_1, A_2, \cdots, A_k$.

BC : $n = 2$

$A_1$ , $A_2$

Case I : $A_1 \subseteq A_2$

Since $A_1 \subseteq A_1$ , $A_1 \subseteq A_2$ , $A_1$ is a subset of every set.

Case II : $A_2 \subseteq A_1$

Similarly, $A_2$ is a subset of every set.

Induction Step : We want to prove the claim when $n = k+1$. Let $A_1, A_2, \ldots, A_{k+1}$ be the sets s.t. for any two sets, $A_i$ & $A_j$, $i \neq j$, $A_i \subseteq A_j$ or $A_j \subseteq A_i$.

We want to show that there is a set that is a subset of every set.

Consider $A_1, A_2, \ldots, A_k$. The condition of the problem holds for $A_1, \ldots, A_k$. Thus by IH, there is a set, say $A_m$, $1 \le m \le k$ s.t.

$A_m \subseteq A_1, A_m \subseteq A_2, \ldots, A_m \subseteq A_k$.

<u>Case I</u>: $A_m \subseteq A_{k+1}$.

Clearly, $A_m$ is a subset of every set.

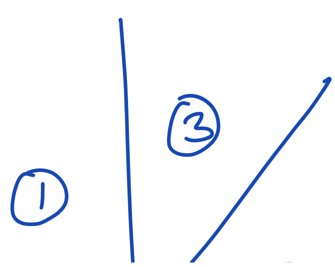
Case II : $A_{k+1} \subseteq A_m$.

$A_{k+1} \subseteq A_1,\ A_{k+1} \subseteq A_2,\ \dots,\ A_{k+1} \subseteq A_{k+1}$.

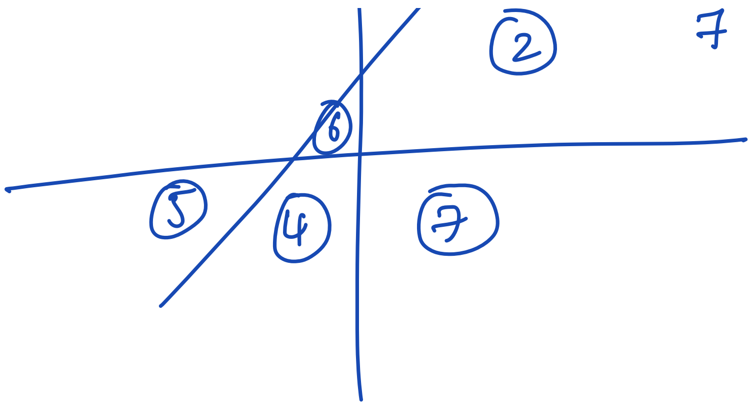
Ex : Prove that for all int $n \geq 1$, $n$ lines separate the plane into $\frac{n^2+n+2}{2}$ regions.

(no two lines are parallel & no three intersect at a common pt)

Proof :




$4 = \frac{2^2+2+2}{2}$




$$7 = \frac{3^2+3+2}{2}$$

<u>IH:</u> Let $k \geq 1$ be an int.

Assume that the claim holds when $n=k$. That is, $k$ lines separate the plane into $\frac{k^2+k+2}{2}$ regions.

BC: $n = 1$.

This creates two regions.

$$\frac{1^2+1+2}{2} = \frac{4}{2} = 2.$$

IS: We want to prove the claim when $n = k+1$.

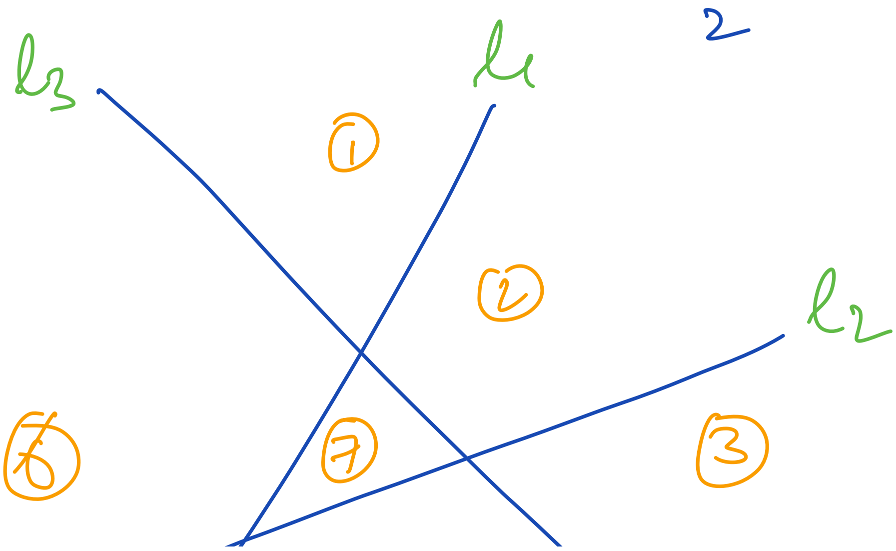
Suppose we have $k+1$ lines — $l_1, l_2, \cdots, l_k, l_{k+1}$.

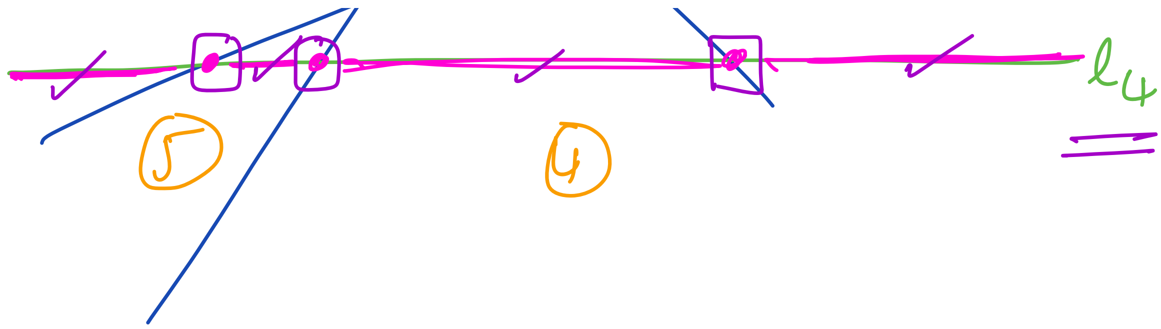
We want to show that these lines separate the plane into

$$\frac{(k+1)^2 + (k+1) + 2}{2} \text{ regions.}$$

Consider lines $l_1, l_2, \ldots, l_k$.

By IH these $k$ lines divide the region into $\frac{k^2+k+2}{2}$ regions.






Consider line $\ell_{k+1}$. $\ell_{k+1}$ must intersect $\ell_1, \ell_2, \cdots, \ell_k$ in $k$ pts. These $k$ points divide $\ell_{k+1}$ into $k+1$ line segments. Each line segment will divide an existing region into two regions, thus creating one new region. Thus $l_{k+1}$ introduces $k+1$ new regions.

Therefore, the total # regions

$$= \frac{k^2+k+2}{2} + (k+1)$$

$$= \frac{k^2+k+2+2k+2}{2}$$

$$= \frac{k^2+2k+1 \quad + \quad k+1 \;+ 2}{2}$$

$$= \frac{(k+1)^2+(k+1)+2}{2}$$

Ex: Let $n$ be a non-negative integer.

Show that any $2^n \times 2^n$ region with

any

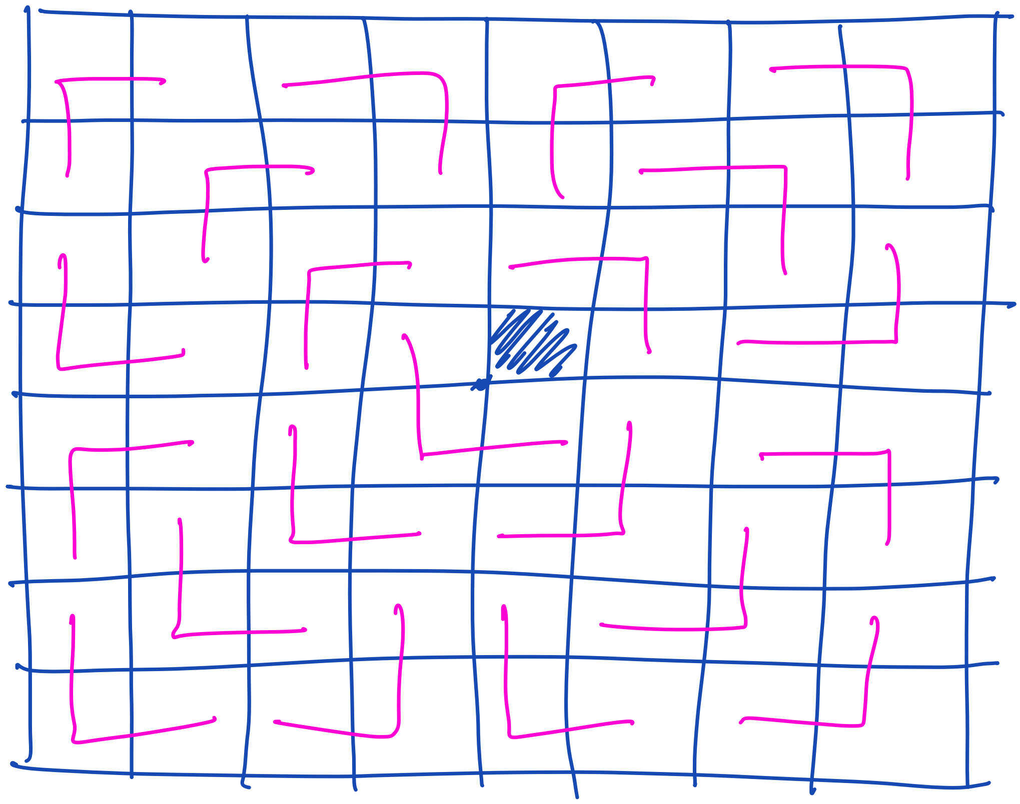
one ~~central~~ sq. removed can be tiled

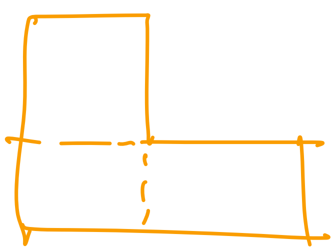
using L-shaped pieces, where each

piece covers 3 squares at a time.

2

$2^3$





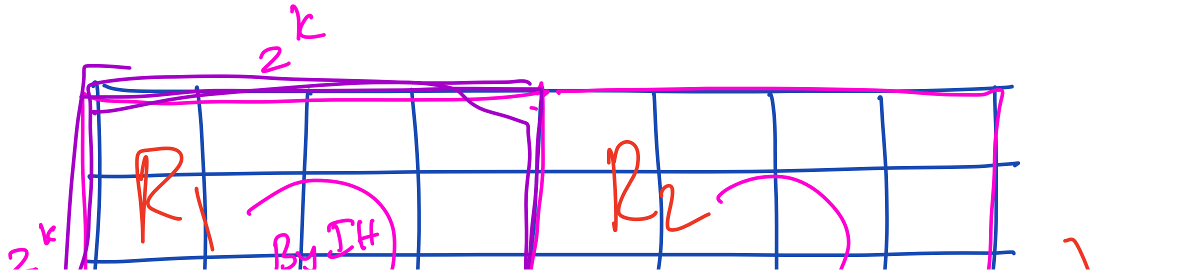
<u>IH:</u> Let $k \geq 1$ be an integer. Assume that $2^k \times 2^k$ region with ~~a central~~ any sq.

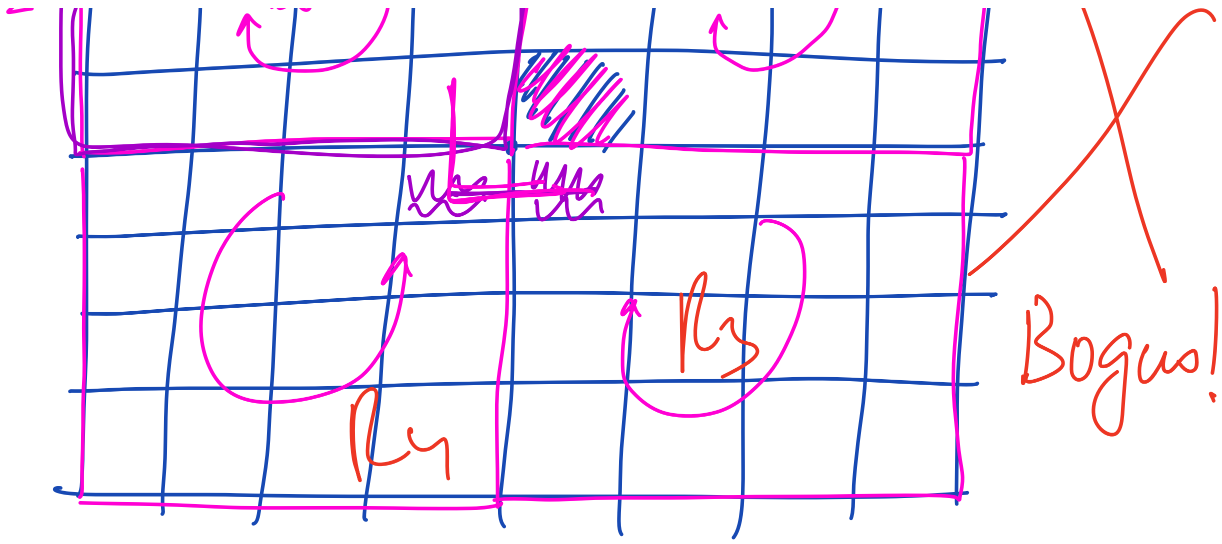
removed can be tiled using L-shaped pieces.

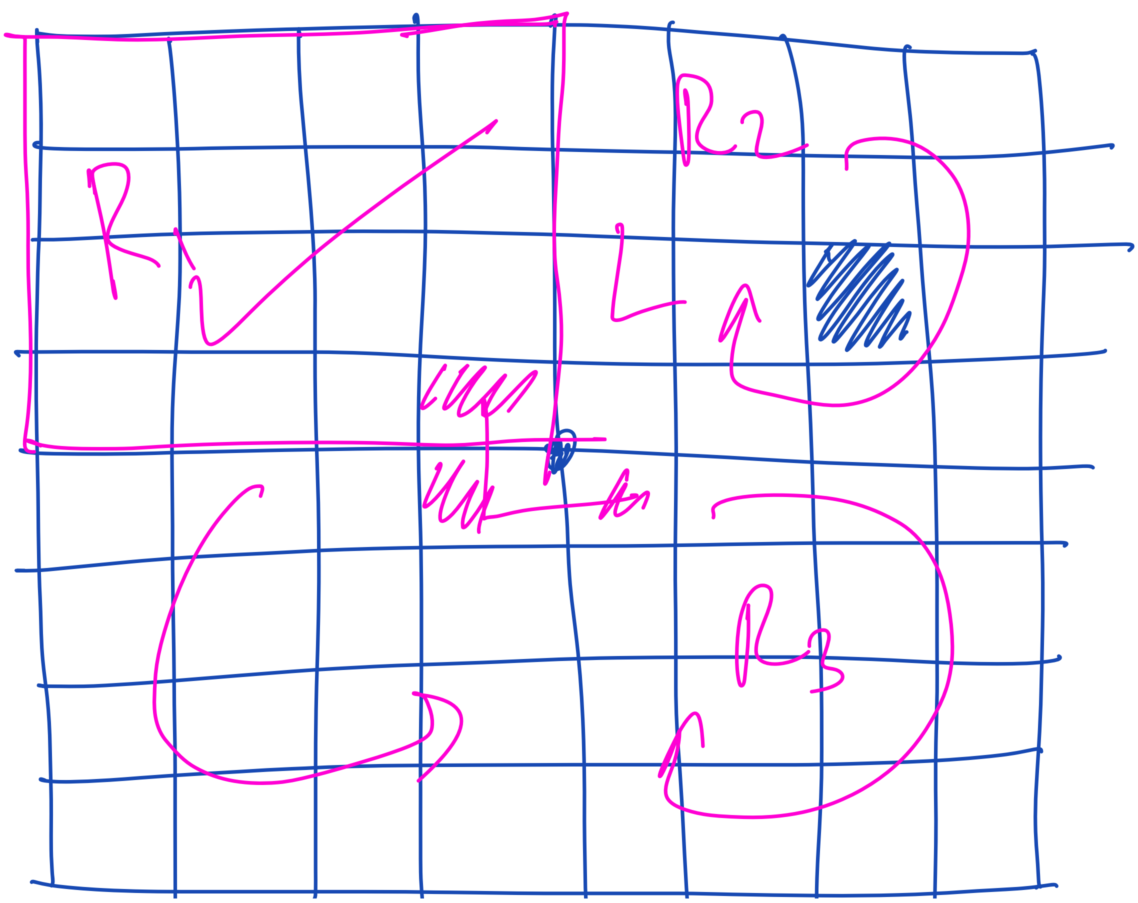
BC : $2^0 \times 2^0$ region ✓

IS : We want to prove the claim when $n = k+1$. Let $R$ be a $2^{k+1} \times 2^{k+1}$ region with a ~~central~~ any square removed.

We want to show that $R$ can be tiled using L-shaped pieces.

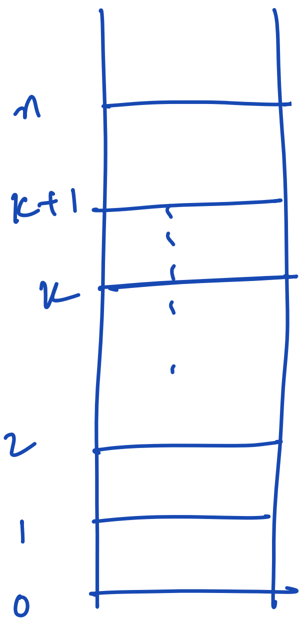



R4
R3
Bogus!


R1
R2
L
R3

## Strong induction.

T.P.T. $\forall$ int $n \geq 0$, Penguin can climb ladder of ht $n$.




- Penguin can climb ladder of height 0.
- $P_0 \wedge P_1 \wedge \dots \wedge P_k \Rightarrow P_{k+1}$

| Weak induction | Strong induction. |
| --- | --- |
| - $P_0$ | - $P_0$ |
| - $P_k \Rightarrow P_{k+1}$ | - $P_0 \wedge P_1 \wedge \dots \wedge P_k \Rightarrow P_{k+1}$. |

Ex: Prove that if $n$ is an integer greater than 1 then either $n$ is a prime or it can be written as a product of primes.

Proof: We will prove the claim using induction on $n$.

IH: Let $k \geq 2$ be an arb, but particular integer. Assume that the

$h \quad 2 \le h \le k$

Claim holds when $n = \cancel{k}$. That is, $\boxed{\cancel{k}}$ $h$ is either a prime or it can be written as a product of primes.

**BC**: $n = 2$

$2$ is a prime. ✓

**IS**: We want to prove the claim when $n = k+1$. That is, we want to prove that $k+1$

is a prime or it can be written as a product of primes.

Case I : $k+1$ is a prime.

✓

Case II : $k+1$ is composite.

By defn,

$2 \le a \le k$

$2 \le b \le k$.

$k+1 = a \times b$, where

$a$ & $b$ are int $\geq 2$.

strong induction.

By IH, $a$ & $b$ are either

primes or they can be

written as product of primes.

$$k+1 = \underset{2\le a\le k}{a} \times \underset{2\le b\le k}{b}$$

$\therefore$ $k+1$ is a product of primes.

Ex: Let $x_1, x_2, \ldots, x_n$ be

$n$ distinct real nos. Prove that

no matter how the parenthesis

are inserted into the product of

the $n$ nos, the # multiplication

$= n-1$.

$$\Big(\underbrace{\underbrace{\big(\underbrace{(x_1\;x_2)}_{3}\;x_3\big)}_{4}\;\underbrace{\big(x_4\;\underbrace{(x_5\;x_6)}_{1}\big)}_{2}}_{5}\Big)$$

<u>Proof</u>: Induction on $n$.

<u>IH</u>: Let $k \geq 1$ be an arbitrary,

but particular integer. Assume that the claim holds when $n = k$.

BC: $n = 1$

$(x_1)$ # mult $= 0$ ✓.

IS:

Bogus!

$$\underbrace{x_1 \times x_2 \times x_3 \times \cdots \times x_k}_{k-1 \text{ (By IH)}} \times x_{k+1}$$

$+1$

$k-1+1 = \boxed{k}$.