Random Variable

$\boxed{4\text{-}2\text{-}1: \text{class work Problem}}$

(1-a) If X_1 has mean 5 and variance 5, X_2 has mean -2 and variance 3 and X_1, X_2 are ~~dependent~~ independent

Find (i) $E(X_1+X_2)$, $var(X_1+X_2)$ (ii) $E(X_1-X_2)$ $var(X_1-X_2)$

(iii) $E(2X_1+3X_2-5)$, $var(2X_1+3X_2-5)$

Solution: By given $E(X_1)=5$, $var(X_1)=5$, $E(X_2)=-2$, $var(X_2)=3$

Now (i) $E(X_1+X_2) = E(X_1)+E(X_2) = 5+(-2) = 3$

$var(X_1+X_2) = var(1X_1+1X_2) = 1^2 var(X_1)+1^2 var(X_2) = (1\times5)+(1\times3) = 8$

(ii) $E(X_1-X_2) = E(X_1)-E(X_2) = \cancel{5-2=3} = 5-(-2) = 7$

$var(X_1-X_2) = var[(1X_1)+(-1)X_2] = 1^2 var(X_1) + (-1)^2 var(X_2)$

$= (1\times5)+(1\times3) = 5+3 = 8$

(iii) $E(2X_1+3X_2-5) = 2E(X_1)+3E(X_2)-5 = (2\times5)+(3\times(-2))-5$

$= 10-6-5 = -1$

$var(2X_1+3X_2-5) = 2^2 var(X_1)+3^2 var(X_2) = (4\times5)+(9\times3)$

$= 20+27 = 47$

(1-b) Let X & Y be ~~the~~ numbers obtained on two dice

Find $E(X)$, $E(X-Y)$ $E(2X-3Y)$

Sol:

X	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Y	1	2	3	4	5	6
$P(Y=y)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$\therefore E(X) = \Sigma x\, p(x)$

$= 1\cdot\frac{1}{6}+2\cdot\frac{1}{6}+3\cdot\frac{1}{6}+4\cdot\frac{1}{6}+5\cdot\frac{1}{6}+6\cdot\frac{1}{6}$

$E(X) = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2}$

$E(Y) = \Sigma y\, p(y)$

$= 1\cdot\frac{1}{6}+2\cdot\frac{1}{6}+3\cdot\frac{1}{6}+4\cdot\frac{1}{6}+5\cdot\frac{1}{6}+6\cdot\frac{1}{6}$

$= \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2}$

Now $E(X-Y) = E(X)-E(Y) = \frac{7}{2}-\frac{7}{2} = 0$

& $E(2X-3Y) = 2E(X)-3E(Y) = 2(\frac{7}{2})-3\frac{7}{2} = -3\cdot\frac{7}{2} = -\frac{21}{2}$

Example-2: The probability density function of a random variable X is zero except at $x=0,1,2$ and $P(0) = 3c^3$, $P(1) = 4c-10c^2$, $P(2) = 5c-1$, Find (i) c (ii) $P(0<x\leq 2)$

Solution: $\because$ total probability $=1$

ie $P(0)+P(1)+P(2)=1$

$\therefore 3c^3+4c-10c^2+5c-1=0$

$\therefore 3c^3-10c^2+9c-1=0$

$\Rightarrow c=\frac{1}{3}$

$\therefore P(0)=3\cdot\left(\frac{1}{3}\right)^3=\frac{1}{9}$, $P(1)=\frac{4}{3}-\frac{10}{9}=\frac{2}{9}$, $P(2)=\frac{5}{3}-1=\frac{2}{3}$

$\therefore P(0<x\leq 2)=P(X=1,2)=P(X=1)+P(X=2)=\frac{2}{9}+\frac{2}{3}=\frac{2+6}{9}=\frac{8}{9}$

Example-3 The probability density function of a random variable X is

X	0	1	2	3	4	5	6
$P(X=x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find (i) k, (ii) $P(x\leq 4)$ (iii) $P(3<x\leq 6)$

Solution: $\because$ total probability $=1$

$\therefore k+3k+5k+7k+9k+11k+13k=1 \Rightarrow 49k=1 \therefore \boxed{k=\frac{1}{49}}$

$P(x\leq 4)=P(X=0,1,2,3,4)=P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$\therefore P(x\leq 4)=k+3k+5k+7k+9k=15k=\frac{15}{49}$

$P(3<x\leq 6)=P(X=4,5,6)=P(X=4)+P(X=5)+P(X=6)$

$=9k+11k+13k=33k=\frac{33}{49}$

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Example-4 A fair coin is tossed till head appears. what is the ~~probability that~~ Expectation of number of tosses required ?

event : H, TH, TTH, TTTH, TTTTH, TTTTTH, -----

let random variable x denotes the number of tosses required to get head

x	1	2	3	4	5	--
$p(x=x)$	$\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}=\left(\frac{1}{2}\right)^2$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\left(\frac{1}{2}\right)^3$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\left(\frac{1}{2}\right)^4$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\left(\frac{1}{2}\right)^5$	----

$\therefore E(x) = \sum x\,p(x=x) = \left(1\cdot\frac{1}{2}\right)+\left(2\cdot\left(\frac{1}{2}\right)^2\right)+3\left(\frac{1}{2}\right)^3+4\left(\frac{1}{2}\right)^4+5\left(\frac{1}{2}\right)^5 ----$

$\therefore E(x) = \frac{1}{2}\left[1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+5\left(\frac{1}{2}\right)^4 + --\right] = \frac{1}{2}\,\frac{1}{\left(1-\frac{1}{2}\right)^2} = \frac{1}{2}\cdot\frac{1}{\left(\frac{1}{2}\right)^2} = \frac{1}{2}\cdot\frac{4}{1} = 2$

Example-5 : A pair of fair dice is rolled once. Let x be the random variable whose values for any outcome is the sum of two numbers on dice.

(i) Find probability function for x and construct the probability table.

(ii) Find the probability that x is an odd number

(iii) Find the probability that x is lies between 3 and 9

solution If we toss a pair of dice then

$S = \{$ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)
(6,1), (6,2), (6,3) (6,4), (6,5), (6,6) $\}$

(i)

x :	2	3	4	5	6	7	8	9	10	11	12
$p(x=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(ii) $p(x \text{ is odd number}) = p(x=1)+p(x=3)+p(x=5)+p(x=7)+p(x=9)+p(x=11)$
$= \frac{2}{36}+\frac{4}{36}+\frac{6}{36}+\frac{4}{36}+\frac{2}{36} = \frac{18}{36} = \frac{1}{2}$

(iii) $p(x \text{ lies between } 3 \text{ and } 9) = p(3<x<9) = p(x=4,5,6,7,8)$
$= p(x=4)+p(x=5)+p(x=6)+p(x=7)+p(x=8)$
$= \frac{3}{36}+\frac{4}{36}+\frac{5}{36}+\frac{6}{36}+\frac{5}{36} = \frac{23}{36}$

4·2·1 : Home work problems

① A random variable x has the following probability function

x :	1	2	3	4	5	6	7
$p(x=x_i)$:	k	$2k$	$3k$	k^2	k^2+k	$2k^2$	$4k^2$

Find (i) k (ii) $p(x<5)$ (iii) $p(x>5)$ (iv) $p(0 \le x \le 5)$

Solution : ∵ total probability = 1

$\therefore k+2k+3k+k^2+k^2+k+2k^2+4k^2=1 \Rightarrow 8k^2+7k-1=0$

$\Rightarrow 8k^2+8k-k-1=0 \Rightarrow 8k(k+1)-1(k+1)=0 \Rightarrow (k+1)(8k-1)=0$

$\Rightarrow k=-1$ or $k=\frac{1}{8}$

∵ probability is in between 0 & 1 $\therefore k=\frac{1}{8}$

∴ x	1	2	3	4	5	6	7
$p(x=x_i)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{1}{64}$	$\frac{9}{64}$	$\frac{2}{64}$	$\frac{4}{64}$

Now $p(x<5) = p(x=1,2,3,4) = p(x=1)+p(x=2)+p(x=3)+p(x=4)$

$\therefore p(x<5) = \frac{1}{8}+\frac{2}{8}+\frac{3}{8}+\frac{1}{64} = \frac{49}{64}$

$p(x>5) = p(x=6,7) = p(x=6)+p(x=7) = \frac{2}{64}+\frac{4}{64} = \frac{6}{64} = \frac{3}{32}$

$p(0 \le x \le 5) = p(x=1,2,3,4,5,6) = p(x=1)+p(x=2)+p(x=3)+p(x=4)+p(x=5)$

$= \frac{1}{8}+\frac{2}{8}+\frac{3}{8}+\frac{1}{64}+\frac{9}{64} = \frac{58}{64}$

② Find the value of k for the following data

x	0	10	15
$p(x=x_i)$	$\frac{k-6}{5}$	$\frac{2}{k}$	$\frac{14}{5k}$

Also Find the cumulative probability distribution function and mean of x

Solution : ∵ total probability = 1

$\therefore \frac{k-6}{5}+\frac{2}{k}+\frac{14}{5k}=1$

$k(k-6)+10+14=5k$

$k^2-6k-5k+24=0$

$k^2-11k+24=0$

$k^2-8k-3k+24=0$

$k(k-8)-3(k-8)=0$

$(k-3)(k-8)=0$

$k=3$, $\boxed{k=8}$

X

∴ x	0	10	15
$p(x=x_i)$	$\frac{2}{5}$	$\frac{2}{8}$	$\frac{7}{20}$
$F(x=x_i)$	$\frac{2}{5}$	$\frac{13}{20}$	1

mean $= E(x) = \sum_x x\, p(x=x_i)$

$= (0\times\frac{2}{5})+(10\times\frac{2}{8})+(15\times\frac{7}{20})$

$= 0+\frac{5}{2}+\frac{21}{4} = \frac{10+21}{4} = \frac{31}{4}$

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③ If the following distribution of a discrete random variable X has mean = 16 then find m, n and the variance of X

X	8	12	16	20	24
$p(x=x)$	$\frac{1}{8}$	m	n	$\frac{1}{4}$	$\frac{1}{12}$

Solution: ∵ total probability = 1

$\frac{1}{8}+m+n+\frac{1}{4}+\frac{1}{12}=1 \Rightarrow m+n+\frac{11}{24}=1 \Rightarrow m+n=\frac{13}{24}$ —①

mean $= E(x) = \sum_x x\,p(x=x) = (8\cdot\frac{1}{8})+(12\cdot m)+(16\cdot n)+(20\cdot\frac{1}{4})+(24\cdot\frac{1}{12})=16$

$\therefore 1+12m+16n+5+2=16 \Rightarrow 12m+16n=8 \Rightarrow 3m+4n=2$ —②

$\therefore m=\frac{1}{6},\ n=\frac{3}{8}$

∴ X	8	12	16	20	24
$p(x=x)$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{12}$

By given mean $= \bar{x} = E(x) = 16$

$E(x^2) = \sum_x x^2 p(x=x) = (8^2\cdot\frac{1}{8})+(12^2\cdot\frac{1}{6})+(16^2\cdot\frac{3}{8})+(20\cdot\frac{1}{4})+(24\cdot\frac{1}{12}) = \cancel{135} = 276$

$\therefore Var(x) = E(x^2)-(E(x))^2 = \cancel{135} = 276-(16)^2 = 20$

*4.2.2 Expectation, variance, Probability Density Function & Cumulative Density function

4.2.2 : Class work Problems

Example-1: A box contain 'a' white balls and 'b' black balls. Now c balls are drawn at random from the box. Find the expected value of number of white balls.

Solution:- Let x_i be the variable denoting the result of the ith draw

Let $x_i=1$ if ith ball drawn is white and

$x_i=0$ if ith ball drawn is black

∵ c balls are drawn the sum of the white ball will be

$S = x_1+x_2+x_3+\cdots+x_n = \sum_{i=1}^{c} x_i$

Now $p(x_i=1) = p(\text{drawing a white ball}) = \frac{a}{a+b}$

$p(x_i=0) = p(\text{drawing a black ball}) = \frac{b}{a+b}$

$E(x_i) = \sum_i x_i p(x_i) = 1\cdot p(x_i=1)+0\cdot p(x_i=0) = 1\cdot\frac{a}{a+b}+0\cdot\frac{b}{a+b} = \frac{a}{a+b}$

$\therefore E(S) = E(x_1)+E(x_2)+E(x_3)+\cdots+E(x_n) = \frac{a}{a+b}+\frac{a}{a+b},\cdots$ (c times) $= c\,\frac{a}{a+b}$

③ A coin is tossed until a head appears. whate is the exceptation of the number of tosses required ?

Solution: Answer of example 4 of $\boxed{\text{4.2.1: class work Problems}}$

Example 4: A and B toss a fair coin alternately one who gets a head first win Rs. 12. A start Find their Expectation.

Solution A can win the game in 1st, IIIrd, 5th, 7th ... throw

~~Event P(A) = P[(A) ∪ (ĀB̄A) ∪ (ĀB̄ĀB̄A) ∪ ...]~~

∴ ~~P(A) = p~~

P(A winning the game)

$= P(A \cup (\bar{A}\bar{B}A) \cup (\bar{A}\bar{B}\bar{A}\bar{B}A) \cup (\bar{A}\bar{B}\bar{A}\bar{B}\bar{A}\bar{B}A) \cup \dots)$

$= P(A \cup (\bar{A}\bar{B}A) \cup (\bar{A}\bar{B}\bar{A}\bar{B}A) \cup (\bar{A}\bar{B}\bar{A}\bar{B}\bar{A}\bar{B}A) + \dots$

$= P(A) + P(\bar{A}\bar{B}A) + P(\bar{A}\bar{B}\bar{A}\bar{B}A) + P(\bar{A}\bar{B}\bar{A}\bar{B}\bar{A}\bar{B}A) + \dots$

$= \frac{1}{6} + \left(\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}\right) + \left(\frac{5}{6}\cdot\frac{5}{6}\right)\left(\frac{5}{6}\cdot\frac{5}{6}\right)\cdot\frac{1}{6} + \left(\frac{5}{6}\cdot\frac{5}{6}\right)\left(\frac{5}{6}\cdot\frac{5}{6}\right)\left(\frac{5}{6}\cdot\frac{5}{6}\right)\frac{1}{6} + \dots$

$= \frac{1}{6}\left[1 + \left(\frac{25}{36}\right) + \left(\frac{25}{36}\right)^2 + \left(\frac{25}{36}\right)^3 + \dots\right]$

$= \frac{1}{6}\cdot\frac{1}{1-\frac{25}{36}} = \frac{1}{6}\cdot\frac{36}{36-25} = \frac{6}{11}$

Thus P(A winning the game) $= \frac{6}{11}$

∴ P(B winning the game) $= 1 - \frac{6}{11} = \frac{5}{11}$

∴ $E(A) = \sum A P(A) = 12\cdot\frac{6}{11} = \frac{72}{11}$

$E(B) = \sum B P(B) = 12\cdot\frac{5}{11} = \frac{60}{11}$

$= \frac{1}{2} + \left(\frac{1}{2}\cdot\frac{1}{2}\right)\frac{1}{2} + \left(\frac{1}{2}\cdot\frac{1}{2}\right)\left(\frac{1}{2}\cdot\frac{1}{2}\right)\frac{1}{2} + \left(\frac{1}{2}\cdot\frac{1}{2}\right)\left(\frac{1}{2}\cdot\frac{1}{2}\right)\left(\frac{1}{2}\cdot\frac{1}{2}\right)\frac{1}{2} + \dots$

$= \frac{1}{2}\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \dots\right] = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{4}} = \frac{1}{2}\cdot\frac{4}{3} = \frac{2}{3}$

∴ P(B winning the game) $= 1 - \frac{2}{3} = \frac{1}{3}$

∴ Expectation of A $= 12\cdot\frac{2}{3} = 8$

Expectation of B $= 12\cdot\frac{1}{3} = 4$

Example-5: Find the distribution function for the sum of numbers appearing on the toss of two unbiased dice. Also find cumulative p.d.f.

Solution:-

X	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
$F(X=x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	1

Example-6:- Determine the discrete probability distribution, expectation and variance of a d.r.v. X which denotes the minimum of the two numbers that appear when a pair of fair dice is thrown

Solution:- Let S be the sample if we throw a pair of dice

$\therefore S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$

$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$

$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$

$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$

$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$

$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$

Let $X = \min(x_1, x_2)$ x_1 be the number on first die

x_2 be the number on second die

$\therefore X$	1	2	3	4	5	6
$P(X=x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

$E(X) = \sum_x x\, p(X=x) = (1\cdot\frac{11}{36}) + (2\cdot\frac{9}{36}) + (3\cdot\frac{7}{36}) + (4\cdot\frac{5}{36}) + (5\cdot\frac{3}{36}) + (6\cdot\frac{1}{36}) = \frac{91}{36}$

$\therefore$ mean $= E(X) = \frac{91}{36}$

$E(X^2) = \sum_x x^2 p(X=x) = (1^2\cdot\frac{11}{36}) + (2^2\cdot\frac{9}{36}) + (3^2\cdot\frac{7}{36}) + (4^2\cdot\frac{5}{36}) + (5^2\cdot\frac{3}{36}) + (6^2\cdot\frac{1}{36})$

$\therefore E(X^2) = \frac{301}{36}$ $\therefore Var(X) = E(X^2) - (E(X))^2 = \frac{301}{36} - \left(\frac{91}{36}\right)^2 = 1.9715$

Example-7 A box contains 2^n tickets of which ${}^nC_r$ tickets bear the number r $(r = 0, 1, 2, 3, \ldots n)$ (i) m tickets are drawn from the box. Find the expectation of the sum of their numbers.
(ii) 2 tickets are drawn from the box find the expectation of the sum of their numbers.

Solution

By given there are ${}^nC_0$ tickets having number $x=0$
${}^nC_1$ tickets having number $x=1$
${}^nC_2$ tickets having number $x=2$
${}^nC_3$ tickets having number $x=3$
$\vdots$
${}^nC_n$ tickets having number $x=n$

let $S = x_1 + x_2 + x_3 + x_4 + \ldots + x_m$ — (1)

Let X represent the number on ticket drawn

$$\therefore \begin{array}{c|cccccccc} X: & 0 & 1 & 2 & 3 & 4 & \cdots & n \\ P(X=x) & \frac{{}^nC_0}{2^n} & \frac{{}^nC_1}{2^n} & \frac{{}^nC_2}{2^n} & \frac{{}^nC_3}{2^n} & \frac{{}^nC_4}{2^n} & \cdots & \frac{{}^nC_n}{2^n} \end{array}$$

$$\therefore E(X) = \sum_x x\,P(X=x) = 0\cdot P(X=0) + 1\cdot P(X=1) + 2\cdot P(X=2) + 3\cdot P(X=3) + 4\cdot P(X=4) + \ldots + n\,P(X=n)$$

$$= 0\cdot\frac{{}^nC_0}{2^n} + 1\cdot\frac{{}^nC_1}{2^n} + 2\,\frac{{}^nC_2}{2^n} + 3\cdot\frac{{}^nC_3}{2^n} + 4\cdot\frac{{}^nC_4}{2^n} + \ldots + n\cdot\frac{{}^nC_n}{2^n}$$

$$\therefore E(X) = \frac{1}{2^n}\left[1\cdot{}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + 4\,{}^nC_4 + \ldots + n\cdot{}^nC_n\right]$$

$$= \frac{1}{2^n}\left[1\,\frac{n!}{(n-1)!\,1!} + 2\,\frac{n!}{(n-2)!\,2!} + 3\,\frac{n!}{(n-3)!\,3!} + 4\,\frac{n!}{(n-4)!} + \ldots + n\,\frac{n!}{(n-n)!\,n!}\right]$$

$$= \frac{1}{2^n}\left[\cancel{1}\,\frac{n(n-1)!}{[(n-1)-0]!\,\cancel{1}\cdot 0!} + \frac{\cancel{2}\,n(n-1)!}{[(n-1)-1]!\,\cancel{2}\cdot 1!} + \frac{\cancel{3}\,n(n-1)!}{[(n-1)-2]!\cdot\cancel{3}\cdot 2!} + \frac{\cancel{4}\,n(n-1)!}{[(n-1)-3]!\,\cancel{4}\cdot 3!} + \ldots + \frac{\cancel{n}\cdot n(n-1)!}{[(n-1)-(n-1)]!\,\cancel{n}(n-1)!}\right]$$

$$= \frac{n}{2^n}\left[\frac{(n-1)!}{[(n-1)-0]!\,0!} + \frac{(n-1)!}{[(n-1)-1]!\,1!} + \frac{(n-1)!}{[(n-1)-2]!\,2!} + \frac{(n-1)!}{[(n-1)-3]!\,3!} + \ldots + \frac{(n-1)!}{[(n-1)-(n-1)]!\,(n-1)!}\right]$$

$$\therefore E(X) = \frac{n}{2^n}\left[{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + {}^{n-1}C_3 + \ldots + {}^{n-1}C_{n-1}\right] = \frac{n}{2^n}\cdot(1+1)^{n-1} = \frac{n}{2^n}\cdot 2^{n-1}$$

$$\therefore E(X) = \frac{n}{2^n}\cdot\frac{2^n}{2} = \frac{n}{2} \text{ — (2) for any } x$$

From (1) $E(S) = E(x_1) + E(x_2) + E(x_3) + E(x_4) + \ldots + E(x_m) = \frac{n}{2} + \frac{n}{2} + \frac{n}{2} + \ldots\ m \text{ times} = \frac{m\cdot n}{2}$

If we consider two tickets put $m = 2$

$\therefore E(S) = 2\cdot\frac{n}{2} = n$

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4.2.2 : Home Work Problems

Example-1 A player tosses three coins. He wins Rs.500 if 3 heads occur Rs. 300 if 2 head occur, Rs. 100 if one head occurs on the other hand he losses Rs. 1500 if 3 tail occur. Find the expectation of the player.

Solution: Let S be the sample space if we toss 3-coins simultaneously

$\therefore S = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}$

Let X be number of heads if we toss three coins simultaneously & Y be the income of person

$\therefore X$	0	1	2	3
$P(X=x)$ $P(Y=y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
Y	-1500	Rs.100	Rs. 300	Rs.500

$\therefore$ Net gain $= \Sigma\, y\, P(Y=y)$

$= (-1500 \times \frac{1}{8}) + (100 \times \frac{3}{8}) + (300 \times \frac{3}{8}) + (500 \times \frac{1}{8}) = \frac{200}{8} = 25$

$\therefore$ Expectation of player $= 25$

Example-2: What is the expectation of the number of failures proceeding the first success in an infinite series of independent trails with constant probabilities p and q of succes and failure respectively.

Solution: Let X denotes the number of failures before first success

Out come:	P	qP	qqP	qqqP	qqqqP	qqqqqP	
X	0	1	2	3	4	5	....

$\therefore E(X) = \Sigma\, x\, P(X=x) = 0 \cdot P + 1 \cdot qP + 2q^2P + 3q^3P + 4q^4P + 5q^5P + \cdots$

$\therefore E(X) = qP + 2q^2P + 3q^3P + 4q^4P + \cdots = qP[1 + 2q + 3q^2 + 4q^3 + \cdots]$

$\therefore E(X) = qP \frac{1}{(1-q)^2} = \frac{qP}{(P)^2} = \frac{q}{P}$

Example-3: what is the expectation and variance of the sum of points on the throw of n dice

Solution let x_i be the number on toss of ith dice

let S denotes the sum of the points of n dice

$$\therefore S = x_1 + x_2 + x_3 + x_4 + \dots + x_n \quad -①$$

$\because x_i$ denote the number on toss of ith dice

$x_i \to$	1	2	3	4	5	6
$P(X=x_i) \to$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\therefore E(x_i) = \sum_i x_i p(x_i) = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6}$$

$$\therefore E(x_i) = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2} \quad i=1,2,3\dots n \quad ②$$

$$E(x_i)^2 = \sum_i (x_i)^2 p(x_i) = 1^2\cdot\frac{1}{6} + 2^2\cdot\frac{1}{6} + 3^2\cdot\frac{1}{6} + 4^2\cdot\frac{1}{6} + 5^2\cdot\frac{1}{6} + 6^2\cdot\frac{1}{6}$$

$$= \frac{1}{6}[1+4+9+16+25+36] = \frac{91}{6}, \quad i=1,2,3\dots n \quad ③$$

$$\therefore var(x_i) = E(x_i)^2 - (E(x_i))^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

$$\therefore E(S) = E(x_1) + E(x_2) + E(x_3) + \dots + E(x_n)$$

$$\therefore E(S) = \frac{7}{2} + \frac{7}{2} + \frac{7}{2} + \dots \text{ n times} = n\frac{7}{2}$$

$$var(S) = var(x_1) + var(x_2) + var(x_3) + \dots + var(x_n)$$

$$= \frac{35}{12} + \frac{35}{12} + \frac{35}{12} + \dots \text{ n times} = n\cdot\frac{35}{12}$$

Example: A box contains 'n' tickets numbered 1, 2, 3, ..., n if m tickets are drawn at random from the box, what is the expectation of the sum of the number of tickets drawn

Solution:- let x_i be the number on the ith ticket drawn

$$\text{let } S = x_1 + x_2 + x_3 + \dots + x_m \quad -①$$

X	1	2	3	4	....	n
$P(X=x_i)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	....	$\frac{1}{n}$

$$\therefore E(x_i) = \sum_i x_i p(x_i) = \left(1\cdot\frac{1}{n}\right) + \left(2\cdot\frac{1}{n}\right) + \left(3\cdot\frac{1}{n}\right) + \left(4\cdot\frac{1}{n}\right) + \dots + \left(n\cdot\frac{1}{n}\right)$$

$$\therefore E(x_i) = \frac{1}{n}(1+2+3+4+\dots+n) = \frac{1}{n}\cdot\frac{n(n+1)}{2} = \frac{n+1}{2} \quad ② \quad i=1,2,3\dots n$$

$$\therefore E(S) = E(x_1) + E(x_2) + E(x_3) + \dots + E(x_m)$$

$$= \left(\frac{n+1}{2}\right) + \left(\frac{n+1}{2}\right) + \left(\frac{n+1}{2}\right) + \dots \text{ m times}$$

$$\therefore E(S) = m\left(\frac{n+1}{2}\right)$$

Moments, Moment Generating Functions

Example-1 $\boxed{\text{4.3: Class Work Problems}}$

Find M.G.F. and hence find first four central moments where a random variable X has a following p.d.f.

X :	-2	1	3
$P(X=x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Solution: $M_0(t) = E[e^{tx}] = \sum_x [e^{tx} p(x=x)] = e^{-2t} p(x=-2) + e^{1t} p(x=1) + e^{3t} p(x=3)$

$\therefore M_0(t) = e^{-2t}\frac{1}{3} + e^{1t}\frac{1}{2} + e^{3t}\frac{1}{6} = \frac{1}{6}[2e^{-2t} + 3e^{1t} + e^{3t}]$ —①

$\mu_1' = \frac{d}{dt}[M_0(t)]\Big|_{t=0} = \frac{1}{6}[-4e^{-2t} + 3e^{t} + 3e^{3t}]\Big|_{t=0} = \frac{1}{6}[-4+3+3] = \frac{2}{6} = \frac{1}{3}$ —②

$\mu_2' = \frac{d^2}{dt^2}[M_0(t)]\Big|_{t=0} = +\frac{1}{6}[8e^{-2t} + 3e^{t} + 9e^{3t}]\Big|_{t=0} = +\frac{1}{6}[8+3+9] = \frac{20}{6} = \frac{10}{3}$ —③

$\mu_3' = \frac{d^3}{dt^3}[M_0(t)]\Big|_{t=0} = \frac{1}{6}[-16e^{-2t} + 3e^{t} + 27e^{3t}]\Big|_{t=0} = \frac{1}{6}[-16+3+27] = \frac{14}{6} = \frac{7}{3}$ —④

$\mu_4' = \frac{d^4}{dt^4}[M_0(t)]\Big|_{t=0} = \frac{1}{6}[32e^{-2t} + 3e^{t} + 81e^{3t}]\Big|_{t=0} = \frac{1}{6}[32+3+81] = \frac{116}{6} = \frac{58}{3}$ —⑤

Now $\mu_2 = \mu_2' - (\mu_1')^2 = \frac{10}{3} - \left(\frac{1}{3}\right)^2 = \frac{10}{3} - \frac{1}{9} = \frac{30-1}{9} = \frac{29}{9}$ —⑥

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 = \frac{7}{3} - \left(3\times\frac{10}{3}\times\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)^3 = -\frac{25}{27}$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 = \frac{58}{3} - \left(4\times\frac{7}{3}\cdot\frac{1}{3}\right) + \left(6\times\frac{10}{3}\left(\frac{1}{3}\right)^2\right) - 3\left(\frac{1}{3}\right)^4 = \frac{497}{27}$

Example-2: ① Suppose a fair die is ~~known~~ thrown such that chance of occuring each face is equally likely. Find m.g.f. and using m.g.f. find first four moments

(ii) If X denotes the outcome when a fair die is tossed. Find M.G.F of X and hence find mean and variance of X

Solution:-: Probability distribution of random variable X is

X	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$\therefore M_0(t) = E(e^{tx}) = \sum e^{tx} p(x=x) = e^{1t}\frac{1}{6} + e^{2t}\frac{1}{6} + e^{3t}\frac{1}{6} + e^{4t}\frac{1}{6} + e^{5t}\frac{1}{6} + e^{6t}\frac{1}{6}$

$\therefore M_0(t) = \frac{1}{6}[e^{1t} + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}]$ —①

$\mu_1' = \frac{d}{dt}[M_0(t)]\Big|_{t=0} = \frac{1}{6}[e^{t} + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]\Big|_{t=0}$

$\mu_1' = \frac{1}{6}[1+2+3+4+5+6] = \frac{21}{6} = \frac{7}{2}$ —②

$\frac{d}{dt}[M_0(t)] = \frac{1}{6}[1e^{1t} + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]$

$\mu_2' = \frac{d^2}{dt^2}[M_0(t)]\Big/_{t=0} = \frac{1}{6}[1^2e^{1t} + 2^2e^{2t} + 3^2e^{3t} + 4^2e^{4t} + 5^2e^{5t} + 6^2e^{6t}]\Big/_{t=0}$

$\mu_2' = \frac{1}{6}[1^2+2^2+3^2+4^2+5^2+6^2] = \frac{91}{6}$

$\mu_3' = \frac{d^3}{dt^3}[M_0(t)]\Big/_{t=0} = \frac{1}{6}[1^3e^{1t} + 2^3e^{2t} + 3^3e^{3t} + 4^3e^{4t} + 5^3e^{5t} + 6^3e^{6t}]\Big/_{t=0}$

$\mu_3' = \frac{1}{6}[1^3+2^3+3^3+4^3+5^3+6^3] = \frac{441}{6}$

$\mu_4' = \frac{d^4}{dt^4}[M_0(t)] = \frac{1}{6}[1^4e^{1t} + 2^4e^{2t} + 3^4e^{3t} + 4^4e^{4t} + 5^4e^{5t} + 6^4e^{6t}]\Big/_{t=0}$

$\mu_4' = \frac{1}{6}[1^4+2^4+3^4+4^4+5^4+6^4] = \frac{2275}{6}$

$\therefore \mu_2 = \mu_2' - (\mu_1')^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \cancel{\frac{35}{12}} = \frac{35}{12}$

$\mu_3 = \mu_3' \, 3(\mu_2')(\mu_1') + (\mu_1')^3 = \frac{441}{6} - \left(3 \times \frac{91}{6} \times \frac{7}{2}\right) + \left(\frac{7}{2}\right)^3 = \cancel{-34} = -\frac{343}{8}$

Now mean $= \mu_1' = \frac{7}{2}$

Variance $= \mu_2' - (\mu_1')^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$

Example-3 probability of A hitting a target is 2/5, probability of B hitting a target is $\frac{1}{5}$ probability of C hitting a target is $\frac{4}{5}$. If they fire the target what is the probability that at least two ~~shots~~ should hit the target

Solution: By given $P(A) = \frac{2}{5}$, $P(B) = \frac{1}{5}$ $P(C) = \frac{4}{5}$

$\therefore P(\bar{A}) = \frac{3}{5}$ $P(\bar{B}) = \frac{4}{5}$, $P(\bar{C}) = \frac{1}{5}$

$\therefore$ probability of at least two should hit the target

$= P\cancel{(\bar{A} \cup \bar{B} \cup \bar{C})}$

$= P[(A \cap B \cap C) \cup (A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C)]$

$= P(A \cap B \cap C) + P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C)$

$= P(A) \cdot P(B) \cdot P(C) + P(A)P(B)P(\bar{C}) + P(A)P(\bar{B})P(C) + P(\bar{A})P(B)P(C)$

$= \left(\frac{2}{5} \times \frac{1}{5} \times \frac{4}{5}\right) + \left(\frac{2}{5} \times \frac{1}{5} \times \frac{1}{5}\right) + \left(\frac{2}{5} \times \frac{4}{5} \times \frac{4}{5}\right) + \left(\frac{3}{5} \times \frac{1}{5} \times \frac{4}{5}\right) = \frac{54}{125}$

: The first four moments of a frequency distribution about the value 4 are -1.5, 17, -30, and 108. Calculate the moments about the mean.

Solution we know $\mu_r' = \cancel{\frac{1}{n}\sum(x_i-A)^r} = E(x-A)^r$ & $\mu_r = E(x-\bar{x})^r$

$\therefore \mu_r' = E(x-4)^r$ ① & $\mu_r = E(x-\bar{x})^r$

and relation are $\mu_1 = 0$, $\mu_2 = \mu_2' - (\mu_1')^2$, $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

By given $\mu_1' = -1.5$, $\mu_2' = 17$, $\mu_3' = -30$ & $\mu_4' = 108$

$\therefore \mu_1 = 0$, $\mu_2 = 17 - (-1.5)^2 = \frac{59}{4} = 14.75$

$\mu_3 = (-30) - (3\times17\times(-1.5)) + 2(-1.5)^3 = \frac{159}{4} = 39.75$

$\mu_4 = 108 - (4\times(-30)\times(-1.5)) + (6\times17\times(-1.5)^2) - (3\times(-1.5)^4) = \frac{297}{2} = 148.5$

Now from ① $\mu_1' = E(x-4) = E(x) - E(4) = E(x) - 4$

$\therefore -1.5 = E(x) - 4 \Rightarrow E(x) = +4 - 1.5 = 2.5$

Example-5 If a random variable has moment generating function $M_0(t) = \frac{3}{3-t}$, obtain mean and standard deviation

Solution we know

$\mu_1' = \frac{d}{dt}[M_0(t)]\Big|_{t=0} = \frac{d}{dt}\left(\frac{3}{3-t}\right)\Big|_{t=0} = \frac{-3}{(3-t)^2}(-1)\Big|_{t=0} = \frac{+3}{(3)^2} = \frac{1}{3}$

$\mu_2' = \frac{d^2}{dt^2}[M_0(t)]\Big|_{t=0} = \frac{d}{dt}[3(3-t)^{-2}]\Big|_{t=0} = -6(3-t)^{-3}(-1)\Big|_{t=0} = 6(3)^{-3} = \frac{6}{27} = \frac{2}{9}$

$\therefore$ Mean $\mu_1' = \frac{1}{3}$ &

Variance $\mu_2 = \mu_2' - (\mu_1')^2 = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

~~Ex~~ **4.3: Home Work Problems**

Example-1: Find the m.g.f of a random variable x if r^{th} moment about mean is given by $\mu_r' = r!$ Also find mean & variance

Sol^n we know ~~$M_0(t) = E(e^{tx})$~~

$M_0(t) = E(e^{tx}) = E\left[1 + \frac{(tx)^1}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \frac{(tx)^4}{4!} \dots\right]$

$\therefore M_0(t) = E(1) + E(x)\frac{t^1}{1!} + E(x^2)\frac{t^2}{2!} + E(x^3)\cdot\frac{t^3}{3!} + E(x^4)\cdot\frac{t^4}{4!} \dots$

$\therefore M_0(t) = 1 + \mu_1'\frac{t^1}{1!} + \mu_2'\left(\frac{t^2}{2!}\right) + \mu_3'\left(\frac{t^3}{3!}\right) + \mu_4'\left(\frac{t^4}{4!}\right) \dots$

$\because \mu_r' = r!$: $\mu_1' = 1!$, $\mu_2' = 2!$ $\mu_3' = 3!$ $\mu_4' = 4! \dots$

$\therefore M_0(t) = 1 + 1!\,\frac{t^1}{1!} + 2!\,\frac{t^2}{2!} + 3!\,\frac{t^3}{3!} + \cdots$

$\therefore M_0(t) = 1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t}$

By given $\mu_r' = r!$

$\therefore \mu_1' = 1! = 1 \qquad \mu_2' = 2! = 2$

$\therefore$ mean $= \mu_1' = 1$ & variance $\mu_2 = \mu_2' - (\mu_1')^2 = 2 - (1)^2 = 2-1 = 1$

<u>Example-2</u> The first three moments of a distribution about value 2 of the variable are 1, 16, & −40, show that the mean = 3 the variance = 15, $\mu_3 = -86$

<u>Solution</u> we know $\mu_r' = E(x-A)^r = E(x-2)^r$ — ①

By given $\mu_1' = 1,\ \mu_2' = 16,\ \mu_3' = -40$

From $\mu_1' = E(x-2) = E(x) - E(2) = E(x) - 2$

$\therefore 1 = E(x) - 2 \Rightarrow E(x) = 1+2 = 3 =$ mean

Now variance $\mu_2 = \mu_2' - (\mu_1')^2 = 16 - (1)^2 = 15$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 = -40 - (3 \times 16 \times 1) + 2(1)^3 = -86$