caution : Nazar hutting marks cutting

University of Mumbai

Program: _First Year (All Branches) Engineering-SEM-II
Curriculum Scheme: Rev2019
Engineering Mathematics-II

Question Bank

MCQ

1.	The value of $\beta\left(\frac{5}{2},\frac{3}{2}\right)$ is equal to
Option A:	$\sqrt{\pi}$
Option B:	$\sqrt{2\pi}$
Option C:	π
Option D:	$\pi/16$

Solution:

$$\beta\left(\frac{5}{2},\frac{3}{2}\right)=\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)}=\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma 4}=\frac{\frac{3\pi}{8}}{3!}=\frac{\pi}{16}$$

2.	Length of the curve $y = logcosx$ from $x = 0\ to\ x = \frac{\pi}{3}$ is
Option A:	$\log(1+\sqrt{2})$
Option B:	$\log(2+\sqrt{3})$
Option C:	$\log 2$
Option D:	$\log 5$

Solution:

$y = \log\cos x$

$\frac{dy}{dx} = \frac{1}{\cos x} \times -\sin x$

$\frac{dy}{dx} = -\tan x$

$S = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$S = \int_0^{\pi/3} \sqrt{1+\tan^2 x}\, dx$

$S = \int_0^{\pi/3} \sec x\, dx$

$S = \left[\log(\sec x + \tan x)\right]_0^{\pi/3}$

$S = \log(2+\sqrt{3})$

3.	Integrating factor of $(12y + 4y^3 + 6x^2)dx + 3(x + xy^2)dy = 0$ is
Option A:	x^3
Option B:	x^2
Option C:	$logx$
Option D:	e^x

Solution:

We have,

$M = 12y + 4y^3 + 6x^2$ $\qquad$ $N = 3x + 3xy^2$

$\frac{\partial M}{\partial y} = 12 + 12y^2$ $\qquad$ $\frac{\partial N}{\partial x} = 3 + 3y^2$

Consider,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(12+12y^2)-(3+3y^2)}{3x+3xy^2} = \frac{9+9y^2}{3x(1+y^2)} = \frac{9(1+y^2)}{3x(1+y^2)} = \frac{3}{x} = f(x)$$

$$IF = e^{\int f(x)dx} = e^{\int \frac{3}{x}dx} = e^{3logx} = e^{\log(x^3)} = x^3$$

4.	The solution of differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ is
Option A:	$(c_1 + c_2x)c_3xe^{-x}$
Option B:	$(c_1 + c_2x)e^{-2x}$
Option C:	$(c_1 + c_2x)e^{2x}$
Option D:	$(c_1 + c_2x)c_3xe^{x}$

Solution:

We have, $D^2 + 4D + 4 = 0$

$$(D + 2)^2 = 0$$

$$D = -2, -2$$

Thus, $y = (c_1 + c_2x)e^{-2x}$

5.	Particular Integral (P.I.) of differential equation $(D^3 - 3D^2)y = e^{4x}$ is
Option A:	$P.I. = \frac{1}{16}e^{-4x}$
Option B:	$P.I. = \frac{1}{64}e^{-4x}$
Option C:	$P.I. = \frac{1}{16}e^{4x}$
Option D:	$P.I. = \frac{1}{64}e^{4x}$

Solution:

$$y = \frac{1}{D^3-3D^2}e^{4x} = \frac{e^{4x}}{4^3-3(4)^2} = \frac{e^{4x}}{16}$$

6.	Value of the integral $\int_0^\infty \int_0^\infty \int_0^\infty e^{-(x+y+z)} dx\, dy\, dz$ is
Option A:	∞
Option B:	0
Option C:	1

Solution:

$$I = \int_0^\infty \int_0^\infty \int_0^\infty e^{-x}.e^{-y}.e^{-z} dxdydz$$

$$I = \int_0^\infty e^{-x} dx \int_0^\infty e^{-y} dy \int_0^\infty e^{-z} dz$$

$$I = \Gamma(1).\Gamma(1).\Gamma(1) = 1$$

7.	Solution of the triple integral $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 r^4 \, sin\theta \, dr \, d\theta \, d\varphi$ is
Option A:	$\pi/10$
Option B:	$\pi/6$
Option C:	1
Option D:	π

Solution:

$$I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 r^4 sin\theta dr \, d\theta d\phi$$

$$I = \int_0^{\frac{\pi}{2}} sin\theta \, d\theta \int_0^{\frac{\pi}{2}} d\phi \int_0^1 r^4 dr$$

$$I = [-cos\theta]_0^{\frac{\pi}{2}} \times [\phi]_0^{\frac{\pi}{2}} \times \left[\frac{r^5}{5}\right]_0^1$$

$$I = 1 \times \frac{\pi}{2} \times \frac{1}{5} = \frac{\pi}{10}$$

8.	Integral $\int_0^\infty \int_0^\infty \frac{dxdy}{(1+x^2)(1+y^2)}$ is equal to
Option A:	$\frac{\pi}{8}$
Option B:	$\frac{\pi}{2}$
Option C:	$\frac{\pi^2}{4}$
Option D:	$\frac{\pi^2}{8}$

Solution:

$$I = \int_0^\infty \int_0^\infty \frac{dxdy}{(1+x^2)(1+y^2)}$$

$$I = \int_0^\infty \frac{dx}{1+x^2} \int_0^\infty \frac{dy}{1+y^2}$$

$$I = [\tan^{-1} x]_0^\infty [\tan^{-1} y]_0^\infty = \frac{\pi}{2}.\frac{\pi}{2} = \frac{\pi^2}{4}$$

9.	The value of $\int_0^1 \int_0^{\pi/2} r\, sin\theta\, dr\, d\theta$ is
Option A:	½
Option B:	$\pi/2$
Option C:	1/8
Option D:	$\pi/8$

Solution:

$$I = \int_0^1 \int_0^{\frac{\pi}{2}} r\, sin\theta\, dr\, d\theta$$

$$I = \int_0^1 r\, dr \times \int_0^{\frac{\pi}{2}} sin\theta\, d\theta$$

$$I = \left[\frac{r^2}{2}\right]_0^1 \times [-cos\theta]_0^{\frac{\pi}{2}}$$

$$I = \frac{1}{2} \times 1 = \frac{1}{2}$$

10.	Changing to polar co-ordinates the integral $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\, dydx$ will be
Option A:	$\int_0^{\pi/2}\int_0^a sin^2\theta\, r^4 drd\theta$
Option B:	$\int_0^{\pi/2}\int_0^a sin^4\theta\, r^3 drd\theta$
Option C:	$\int_0^{2\pi}\int_0^{a/2} sin^2\theta\, r^4 drd\theta$
Option D:	$\int_0^{a}\int_0^a sin^2\theta\, r^4 drd\theta$

Solution:

The region of integration is,

$x = 0$ ⇒ Y axis

$x = a$ ⇒a line parallel to Y axis passing through (a,0)

$y = 0$ ⇒ X axis

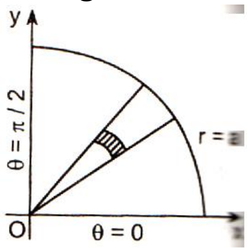
$y = \sqrt{a^2 - x^2}$

8 $x^2 + y^2 = a^2$ ⇒a circle with centre at (0,0) and radius a

Now changing it into polar co-ordinates i.e.

put $x = rcos\theta$ and $y = rsin\theta, dxdy = rdrd\theta$

we get the region of integration as, $r = 0$ & $r = a$, $\theta = 0$ & $\theta = \frac{\pi}{2}$ as shown in the figure.



$$I = \int_0^{\frac{\pi}{2}} \int_0^a r^2 \sin^2\theta \sqrt{r^2} rdrd\theta = \int_0^{\frac{\pi}{2}} \int_0^a r^4 \sin^2\theta\, drd\theta$$

11.	The Integrating Factor of DE $(x^2e^x - my)dx + mx\,dy = 0$ is given by
Option A:	$\frac{1}{y^2}$
Option B:	$\frac{1}{x^2}$
Option C:	$-\frac{1}{y^2}$
Option D:	$-\frac{1}{x^2}$

Solution:

We have,

$M = x^2e^x - my$ $\qquad$ $N = mx$

$\frac{\partial M}{\partial y} = -m$ $\qquad$ $\frac{\partial N}{\partial x} = m$

Consider,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-m-m}{mx} = -\frac{2m}{mx} = -\frac{2}{x} = f(x)$$

$$IF = e^{\int f(x)dx} = e^{\int \frac{-2}{x}dx} = e^{-2logx} = e^{logx^{-2}} = x^{-2} = \frac{1}{x^2}$$

12.	The DE $\frac{dr}{d\theta} = r\tan\theta - \frac{r^2}{\cos\theta}$ can be reduced to linear equation given by
Option A:	$\frac{dv}{d\theta} + \tan\theta\, v = \sec\theta$
Option B:	$\frac{dv}{d\theta} + \sec\theta\, v = -\tan\theta$
Option C:	$\frac{dv}{d\theta} + \tan\theta\, v = -\sec\theta$
Option D:	$\frac{dv}{d\theta} + \sec\theta\, v = \tan\theta$

Solution:

We have,

$$\frac{dr}{d\theta} - rtan\theta = -\frac{r^2}{cos\theta}$$

Dividing by r^2, we get

$$\frac{1}{r^2}\frac{dr}{d\theta} - \frac{1}{r}tan\theta = -sec\theta$$

Put $-\frac{1}{r} = V$, $\therefore \frac{1}{r^2}\frac{dr}{d\theta} = \frac{dV}{d\theta}$

$$\frac{dV}{d\theta} + tan\theta.V = -sec\theta$$

13.	The solution of $(D^3 - 2D + 4)\, y = 0$,where $D \equiv \frac{d}{dx}$ is given by
Option A:	$y = c_1 e^{-2x} + c_2 \cos x + c_3 \sin x$
-Option B:	$y = c_1 x\, e^{-2x} + c_2 \cos x + c_3 \sin x$
Option C:	$y = c_1 + c_2 \cos x + c_3 \sin x$
Option D:	$y = c_1 e^{-2x} + e^{x}(c_2 \cos x + c_3 \sin x)$

Solution:

We have $(D^3 - 2D + 4) = 0$

$D^3 + 0D^2 - 2D + 4 = 0$

$D = -2, 1 + i, 1 - i$

$y_c = c_1 e^{-2x} + e^{1x}[c_2 cosx + c_3 sinx]$

14.	The Particular Integral (P I) of the equation $(D^2 - 1)y = x\, e^x$ where $D \equiv \frac{d}{dx}$ is given by
Option A:	$\text{P I} = \frac{e^x}{4}(x^2 - x)$
Option B:	$\text{P I} = \frac{e^x}{4}(2x - 1)$
Option C:	$\text{P I} = \frac{e^x}{4}(x^2 + x)$
Option D:	$\text{P I} = \frac{e^x}{4}(2x + 1)$

Solution:

We have,

$$y = \frac{1}{D^2 - 1} x\, e^x$$

$$y = e^x \frac{1}{(D+1)^2 - 1} x$$

$$y = e^x \frac{1}{D^2 + 2D + 1 - 1} x$$

$$y = e^x \frac{1}{D^2 + 2D} x$$

$$y = \frac{e^x}{2D} \cdot \frac{1}{1 + \frac{D}{2}} \cdot x$$

$$y = \frac{e^x}{2D} \cdot \left[1 + \frac{D}{2}\right]^{-1} x$$

$$y = \frac{e^x}{2D} \cdot \left[1 - \frac{D}{2}\right] x$$

$$y = \frac{e^x}{2D}\left[x - \frac{1}{2}\right] = \frac{e^x}{2}\left[\frac{x^2}{2} - \frac{x}{2}\right] = \frac{e^x}{4}(x^2 - x)$$

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15.	The Value of $\int_0^\infty e^{-x^4}\, dx$is given by
Option A:	$\Gamma\left(\frac{1}{4}\right)$
Option B:	$\frac{1}{4}\Gamma\left(\frac{3}{4}\right)$
Option C:	$\frac{1}{4}\Gamma\left(\frac{1}{4}\right)$
Option D:	$\Gamma\left(\frac{3}{4}\right)$

Solution:

$$I = \int_0^\infty e^{-x^4} dx$$

Put $x^4 = t$

$$x = t^{\frac{1}{4}}$$

$$dx = \frac{1}{4} t^{-\frac{3}{4}} dt$$

$$I = \int_0^\infty e^{-t} . \frac{1}{4} t^{-\frac{3}{4}} dt$$

$$I = \frac{1}{4}\Gamma\left(-\frac{3}{4}+1\right)$$

$$I = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)$$

BT

16.	The length of the straight line $y = 2x + 5$ $from$ $x = 1$ to $x = 3$ is given by
Option A:	$\sqrt{5}$ units
Option B:	$3\sqrt{5}$units
Option C:	$4\sqrt{5}$units

Solution:

$$y = 2x + 5$$

$$\frac{dy}{dx} = 2$$

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^3 \sqrt{1+2^2}\, dx = \sqrt{5}[x]_1^3 = 2\sqrt{5}$$

17.	After changing the integral $I = \int_0^1 \int_0^{\sqrt{1-x^2}} x^2 y^2 \, dy \, dx$ into polar form, the integral can be given by
Option A:	$I = \int_0^{\pi/2} \int_{r=0}^{1} r^4 cos^2\theta \, sin^2\theta \, dr \, d\theta$
Option B:	$I = \int_0^{\pi/2} \int_{r=0}^{1} r^5 cos^2\theta \, sin^2\theta \, dr \, d\theta$
Option C:	$I = \int_0^{\pi} \int_{r=0}^{1} r^4 cos^2\theta \, sin^2\theta \, dr \, d\theta$
Option D:	$I = \int_0^{\pi} \int_{r=0}^{1} r^5 cos^2\theta \, sin^2\theta \, dr \, d\theta$

Solution:

The region of integration is

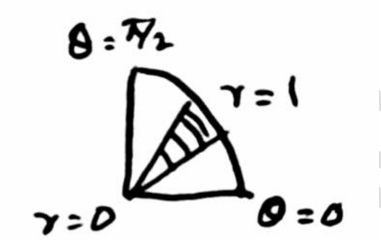
$x = 0$ $\Rightarrow$ Y axis $\Rightarrow r = 0, \theta = \frac{\pi}{2}$

$x = 1$ $\Rightarrow$ line parallel to Y axis $\Rightarrow r = \frac{1}{cos\theta}$

$y = 0$ $\Rightarrow$ X axis $\Rightarrow r = 0, \theta = 0$

$y = \sqrt{1 - x^2}$

$x^2 + y^2 = 1$ $\Rightarrow$ a standard circle with radius 1 $\Rightarrow r = 1$



$$I = \int_0^{\frac{\pi}{2}} \int_0^1 (rcos\theta)^2 (rsin\theta)^2 r dr d\theta = \int_0^{\frac{\pi}{2}} \int_0^1 r^5 \sin^2\theta \cos^2\theta \, dr d\theta$$

18.	The value of $I = \int_0^1 \int_0^1 \int_0^1 x\,y\,z \, dx \, dy \, dz$ is given by
Option A:	$-\frac{1}{8}$
Option B:	$\frac{1}{4}$
Option C:	$-\frac{1}{4}$
Option D:	$\frac{1}{8}$

Solution:

$$I = \int_0^1 x dx \int_0^1 y \, dy \int_0^1 z \, dz = \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_0^1 \left[\frac{z^2}{2}\right]_0^1 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

BT

19.	The value of $I = \int_0^1 (1 - \sqrt[5]{x})\, dx$ will be given by
Option A:	$-\frac{1}{6}$
Option B:	$\frac{1}{6}$
Option C:	$\frac{1}{30}$
Option D:	$-\frac{1}{30}$

Solution:

$I = \int_0^1 (1 - \sqrt[5]{x}) dx$

Put $\sqrt[5]{x} = t \Rightarrow x = t^5 \Rightarrow dx = 5t^4 dt$

$I = \int_0^1 (1-t)5t^4 dt = 5\,B(5,2) = 5 \times \frac{4!\times 1!}{6!} = \frac{1}{6}$

20.	The value of $I = \int_0^{\pi} \int_{r=0}^{a \sin\theta} dr\, d\theta$ is given by
Option A:	a
Option B:	$-a$
Option C:	$2a$
Option D:	$-2a$

Solution:

$I = \int_0^{\pi} [r]_0^{a sin\theta} d\theta = \int_0^{\pi} a\, sin\theta\, d\theta = a[-cos\theta]_0^{\pi} = 2a$

BT

21.	$\int_0^1 \frac{1}{\sqrt{-logx}} dx$ is
Option A:	π
Option B:	$\sqrt{\pi}/2$
Option C:	$\sqrt{\pi}$
Option D:	$2\sqrt{\pi}$

Solution:

Let $I = \int_0^1 \frac{dx}{\sqrt{-logx}}$

Put $logx = -t$, $\therefore x = e^{-t}$, $dx = -e^{-t} dt$

When $x = 0$, $t = \infty$

When $x = 1$, $t = 0$

$\therefore I = \int_{\infty}^{0} \frac{1}{\sqrt{t}} . -e^{-t} dt = \int_0^{\infty} e^{-t} t^{-\frac{1}{2}} dt = \Gamma\left(-\frac{1}{2} + 1\right) = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

22.	Find the complementary function of $\frac{d^4y}{dx^4}+6\frac{d^2y}{dx^2}+9y=e^{2x}$
Option A:	$(c_1+c_2x)cos\sqrt{3}x+(c_3+c_4x)sin\sqrt{3}x$
Option B:	$c_1cos\sqrt{3}x+c_2sin\sqrt{3}x$
Option C:	$c_1e^{\sqrt{3}x}+c_2e^{-\sqrt{3}x}$
Option D:	$(c_1+c_2x)e^{\sqrt{3}x}$

Solution:

$$D^4+6D^2+9=0$$
$$(D^2+3)^2=0$$
$$D^2=-3,-3$$
$$D=\pm\sqrt{3}i,\pm\sqrt{3}i$$
$$y=(c_1+c_2x)\cos\sqrt{3}x+(c_3+c_4x)\sin\sqrt{3}x$$

23.	*The value of* $\int_0^1\int_0^2\int_0^2 x^3yz\,dxdydz$ *is*
Option A:	1
Option B:	1/3
Option C:	2
Option D:	2/3

Solution:

$$I=\int_0^1 x^3dx\int_0^2 y\,dy\int_0^2 z\,dz$$
$$I=\left[\frac{x^4}{4}\right]_0^1\left[\frac{y^2}{2}\right]_0^2\left[\frac{z^2}{2}\right]_0^2=\frac{1}{4}\cdot\frac{4}{2}\cdot\frac{4}{2}=1$$

24.	The Order of the Differential Equation $\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^2+y^4=e^{-x}$ *is*
Option A:	1
Option B:	2
Option C:	3
Option D:	4

Solution:

Order is 2 and Degree is 3

25.	*The value of* $\int_0^{\pi/2}\int_0^{acos\theta} rsin\theta\,drd\theta$ is equal to
Option A:	$a^2/9$
Option B:	$a^2/12$
Option C:	$a^2/6$
Option D:	$a^2/3$

Solution:

$$I=\int_0^{\frac{\pi}{2}}\left[sin\theta\frac{r^2}{2}\right]_0^{acos\theta}d\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}a^2sin\theta\cos^2\theta\,d\theta=\frac{a^2}{2}\times\frac{1}{2}\mathrm{B}\left(\frac{1+1}{2},\frac{2+1}{2}\right)$$
$$I=\frac{a^2}{4}\times\frac{\Gamma1\times\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}=\frac{a^2}{6}$$

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26.	The value of $\Gamma\left(\frac{1}{4}\right)\Gamma(-\frac{1}{4})$
Option A:	$4\sqrt{2}\pi$
Option B:	$-4\sqrt{2}\pi$
Option C:	$\sqrt{2}\pi$
Option D:	$-\sqrt{2}\pi$

Solution:

We know that $\Gamma p\, \Gamma(1-p) = \frac{\pi}{\sin p\pi}$

If $p = \frac{5}{4}$ then $\Gamma\left(\frac{5}{4}\right)\Gamma\left(1-\frac{5}{4}\right) = \frac{\pi}{\sin\frac{5\pi}{4}}$

$$\Gamma\left(\frac{5}{4}\right)\Gamma\left(-\frac{1}{4}\right) = \frac{\pi}{-\frac{1}{\sqrt{2}}}$$

$$\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(-\frac{1}{4}\right) = -\sqrt{2}\pi$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(-\frac{1}{4}\right) = -4\sqrt{2}\pi$$

27.	Integrating factor of $(x^2+y^2+1)dx - 2xy\,dy = 0$ *is*
Option A:	$\frac{-1}{x^2}$
Option B:	$\frac{1}{x^2}$
Option C:	x^2
Option D:	$-x^2$

Solution:

$M = x^2 + y^2 + 1$ $\qquad$ $N = -2xy$

$\frac{\partial M}{\partial y} = 2y$ $\qquad$ $\frac{\partial N}{\partial x} = -2y$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{2y+2y}{-2xy} = \frac{4y}{-2xy} = -\frac{2}{x} = f(x)$$

$$IF = e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2logx} = x^{-2} = \frac{1}{x^2}$$

29.	Particular Integral of DE $(D^3+3D^2-4)y = e^x$ is
Option A:	$xe^x/9$

Solution:

$$y = \frac{1}{D^3+3D^2-4}e^x = x\frac{1}{3D^2+6D}e^x = \frac{xe^x}{9}$$

BT

30.	The value of $\int_0^{\pi} \frac{sin^4\theta}{(1+cos\theta)^2} d\theta$
Option A:	6π
Option B:	$3\pi/4$
Option C:	$\pi/2$
Option D:	$3\pi/2$

Solution:

$$I = \int_0^{\pi} \frac{\sin^4\theta}{(1+cos\theta)^2} d\theta \qquad sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}, 1 + cos\theta = 2\cos^2\frac{\theta}{2}$$

$$I = \int_0^{\pi} \frac{\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^4}{\left(2\cos^2\frac{\theta}{2}\right)^2} d\theta$$

$$I = \frac{16}{4}\int_0^{\pi} \sin^4\frac{\theta}{2} d\theta$$

Put $\frac{\theta}{2} = \phi$

$$\theta = 2\phi$$

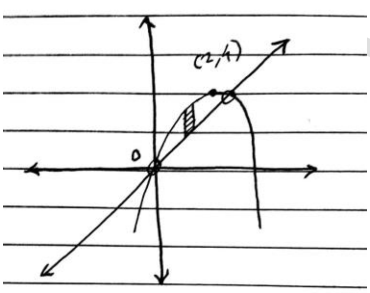
$$d\theta = 2d\phi$$

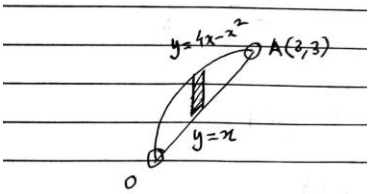
$$I = 4\int_0^{\frac{\pi}{2}} \sin^4\phi \, .2d\phi$$

$$I = 8 \times \frac{1}{2} B\left(\frac{4+1}{2}, \frac{0+1}{2}\right) = 4B\left(\frac{5}{2}, \frac{1}{2}\right) = 4.\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{1}{2}\right)} = \frac{4\times\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\times\sqrt{\pi}}{\Gamma(3)} = \frac{3\pi}{2!} = \frac{3\pi}{2}$$

31.	The area bounded by the line $y = x$ and $y = 4x - x^2$
Option A:	7/2
Option B:	9/2
Option C:	9/4
Option D:	11

Solution:





$$A = \int_0^3 \int_x^{4x-x^2} dxdy = \int_0^3 [4x - x^2 - x]dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{9}{2}$$

32.	The solution of DE $(D^3 - D)y = cosx$ is
Option A:	$(c_1 + c_2x + c_3x^2)e^x + (-\frac{1}{2})sinx$
Option B:	$(c_1 + c_2e^x + c_2e^{-x} + (-\frac{1}{2})cosx$
Option C:	$(c_1 + c_2e^x + c_2e^{-x} + (\frac{-1}{2})sinx$
Option D:	$(c_1 + c_2x + c_3x^2)e^x + (-\frac{1}{2})cosx$

Solution:

$$(D^3 - D) = 0$$

$$D(D^2 - 1) = 0$$

$$D = 0,1,-1$$

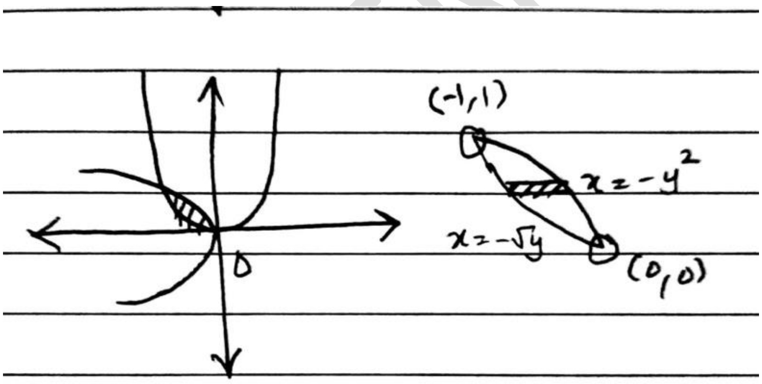
$$y_c = c_1 + c_2e^x + c_3e^{-x}$$

$$y_p = \frac{1}{D^3-D}cosx = \frac{1}{(-1)D-D}cosx = \frac{1}{-2D}cosx = \frac{sinx}{-2}$$

$$y = y_c + y_p = c_1 + c_2e^x + c_3e^{-x} - \frac{sinx}{2}$$

34.	The value of $\iint dxdy$ over the area bounded by $x^2 = y$ and $y^2 = -x$
Option A:	-1/6
Option B:	-1/12
Option C:	1/18
Option D:	1/9

Solution:



$$A = \int_0^1 \int_{-\sqrt{y}}^{-y^2} dxdy$$

$$A = \int_0^1 [-y^2 + \sqrt{y}]dy = \left[-\frac{y^3}{3} + \frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = -\frac{1}{3} + \frac{2}{3} = \frac{1}{3}$$

35.	The value of integral $\iiint \frac{dxdydz}{x^2+y^2+z^2}$ over the positive octant of the sphere $x^2+y^2+z^2=4$ is
Option A:	π
Option B:	2π
Option C:	$4\pi/3$
Option D:	$2\pi/3$

Solution:

Put $x = r\sin\theta\cos\emptyset,$

$y = r\sin\theta\sin\emptyset,$

$z = r\cos\theta$

$dxdydz = r^2\sin\theta drd\theta d\emptyset$

Also, $x^2+y^2+z^2 = r^2$

$$I = \int_{\theta=0}^{\frac{\pi}{2}}\int_{\emptyset=0}^{\frac{\pi}{2}}\int_{r=0}^{2}\frac{r^2\sin\theta drd\theta d\emptyset}{r^2}$$

$$I = \int_0^{\frac{\pi}{2}}\sin\theta d\theta \int_0^{\frac{\pi}{2}} d\emptyset \int_0^2 dr$$

$$I = \frac{1}{2}B\left(\frac{1+1}{2},\frac{0+1}{2}\right)\times[\emptyset]_0^{\frac{\pi}{2}}\times[r]_0^2$$

$$I = \frac{1}{2}.\frac{\Gamma(1).\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}.\frac{\pi}{2}.2$$

$$I = \frac{1}{2}.\frac{1}{\frac{1}{2}}.\frac{\pi}{2}.2 = \pi$$

BT

~~36.~~	The length of the cardioid $r = 2(1+\cos\theta)$ is
Option A:	16
Option B:	12
Option C:	8
Option D:	6

Solution:

$$s = 2\int_0^{\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\therefore s = 2\int_0^{\pi}\sqrt{2^2(1+\cos\theta)^2+2^2\sin^2\theta}\, d\theta$$

$$s = 4\int_0^{\pi}\sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}\, d\theta$$

$$s = 4\int_0^{\pi}\sqrt{2(1+\cos\theta)}\, d\theta$$

$$s = 4\int_0^{\pi}\sqrt{2.2\cos^2\frac{\theta}{2}}\, d\theta$$

$$s = 8\int_0^{\pi}\cos\frac{\theta}{2}\, d\theta$$

$$s = 8\left[\frac{\sin\frac{\theta}{2}}{\frac{1}{2}}\right]_0^{\pi} = 16$$

37.	The value of $\int_0^2 \int_0^{\sqrt{4-x^2}} xy\, dxdy$ is equal to
Option A:	4
Option B:	6
Option C:	2
Option D:	3

Solution:

$$I = \int_0^2 \int_0^{\sqrt{4-x^2}} xy\, dx\, dy = \int_0^2 \left[x\frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} dx = \int_0^2 \left[\frac{x(4-x^2)}{2}\right] dx = 2$$

38.	The Solution of $\frac{dy}{dx} = \frac{y+1}{(y+2)e^y - x}$ is
Option A:	$x.e^y - y - 1 = c$
Option B:	$(y+1)(x - e^y) = c$
Option C:	$ye^y + x - 1 = c$
Option D:	$xy - e^y + 1 = c$

Solution:

We have,

$$(y+1)dx - ((y+2)e^y - x)dy = 0$$

It is of the form, $Mdx + Ndy = 0$, where,

$M = y + 1$; $N = -(y+2)e^y + x$

$\frac{\partial M}{\partial y} = 1$ $\frac{\partial N}{\partial x} = 1$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus, the given differential equation is exact.

Its solution is given by,

$$\int M(treatingyconstant)dx + \int N(termsfreefromx)dy = C$$

$$\int [y+1]dx + \int -(y+2)e^y dy = C$$

$$xy + x - \left\{(y+2)\int e^y dy - \int \int e^y dy \frac{d}{dy}(y+2)dy\right\} = C$$

$$xy + x - \{(y+2)e^y - \int e^y\, dy\} = C$$

$$xy + x - \{ye^y + 2e^y - e^y\} = C$$

$$\boxed{xy + x - ye^y - e^y = C}$$

39.	Changing the order of integration in double integral $\int_0^2 \int_0^{2-\sqrt{4-y^2}} f(x,y)dxdy$ leads to $\int_a^b \int_c^d f(x,y)dxdy$ then value of 'c' is
Option A:	$\sqrt{4-x^2}$
Option B:	$\sqrt{-4x+x^2}$
Option C:	$\sqrt{4x-x^2}$
Option D:	$-\sqrt{4x-x^2}$

Solution:

The region is given by

$y = 0 \Rightarrow$ X axis

$y = 2 \Rightarrow$ line parallel to X axis

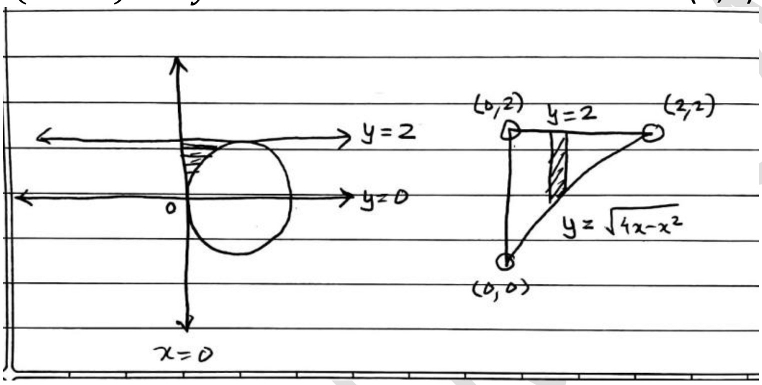
$x = 0 \Rightarrow$ Y axis

$x = 2 - \sqrt{4-y^2}$

$(x-2) = -\sqrt{4-y^2}$

$(x-2)^2 = 4 - y^2$

$(x-2)^2 + y^2 = 4 \Rightarrow$ a circle with centre at (2,0) and radius 2



$I = \int_0^2 \int_{\sqrt{4x-x^2}}^2 f(x,y)dxdy$

41.	Solution of the differential equations $3x^2y^4dx + 4x^3y^3dy = 0$ is
Option A:	$x^2y^4 = c$
Option B:	$x^3y^3 = c$
Option C:	$x^4y^3 = c$
Option D:	$x^3y^4 = c$

Solution:

$M = 3x^2y^4$ $\qquad$ $N = 4x^3y^3$

$\frac{\partial M}{\partial y} = 12x^2y^3$ $\qquad$ $\frac{\partial N}{\partial x} = 12x^2y^3$

Its solution, $\int Mdx + \int Ndy = C$

$\int 3x^2y^4dx + \int 0dy = C$

$x^3y^4 = C$

42.	The complementary function for $(D^2 + 2D + 5)y = 4e^{-x}\tan 2x + 5$, where $D=\frac{d}{dx}$ is given by
Option A:	$k_1e^{-x}\sin 2y + k_2e^{-x}\cos 2y$
Option B:	$k_1e^{x}\sin 2x + k_2e^{x}\cos 2x$
Option C:	$k_1e^{-x}\sin 2x - k_2e^{-x}\cos 2x$
Option D:	$k_1e^{x}\sin 2x + k_2e^{-x}\cos 2x$

Solution:

$$D^2 + 2D + 5 = 0$$

$$D = -1 + 2i, -1 - 2i$$

$$y_c = e^{-x}[k_1 cos2x + k_2 sin2x]$$

BT

~~43.~~	If $B(n,2) = \frac{1}{6}$, n is positive integer then value of n is
Option A:	3
Option B:	2
Option C:	1
Option D:	4

Solution:

$$B(n,2) = \frac{1}{6}$$

$$\frac{\Gamma n\ \Gamma 2}{\Gamma(n+2)} = \frac{1}{6}$$

$$\frac{\Gamma n \times 1!}{(n+1)n\Gamma n} = \frac{1}{6}$$

$$6 = (n+1)n$$

$$3 \times 2 = (n+1)n$$

$$\therefore n = 2$$

44.	The value of I = $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dy\, dx$ is
Option A:	$\frac{3}{35}$
Option B:	$\frac{3}{15}$
Option C:	$\frac{1}{35}$
Option D:	$\frac{3}{5}$

Solution:

$$I = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{\sqrt{x}} dx = \int_0^1 \left[x^2\sqrt{x} + \frac{x^{\frac{3}{2}}}{3} - x^3 - \frac{x^3}{3}\right] dx$$

$$I = \int_0^1 \left[x^{\frac{5}{2}} + \frac{x^{\frac{3}{2}}}{3} - \frac{4x^3}{3}\right] dx = \frac{3}{35}$$

45.	The region of integration in $\int_0^1 \int_0^{1-y} \int_0^{1-x-y} xyz\, dzdxdy$ represents
Option A:	Tetrahedron
Option B:	Cylinder
Option C:	Plane
Option D:	Sphere

Solution:

Tetrahedron

47.	If the differential equationsydx+x(1-3x²y²)dy=0 is non-exact then the integrating factor is
Option A:	$\frac{1}{3x^3y^2}$
Option B:	$\frac{1}{3x^3y^3}$
Option C:	$-\frac{1}{3x^3y^3}$
Option D:	$\frac{1}{3x^2y^3}$

Solution:

We have, $ydx + x(1-3x^2y^2)dy = 0$

It is of the form, $Mdx + Ndy = 0$, where,

$M = y$ $\qquad$ $N = x - 3x^3y^2$

$\frac{\partial M}{\partial y} = 1$ $\qquad$ $\frac{\partial N}{\partial x} = 1 - 9x^2y^2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Thus, the given differential equation is not exact. But it can be made exact by multiplying it with an integrating factor.

Now, I.F. $=\frac{1}{Mx-Ny} = \frac{1}{xy-(xy-3x^3y^3)} = \frac{1}{3x^3y^3}$

48.	The value of the particular integral $\frac{1}{(D^3-1)}(e^x+1)^2$ is
Option A:	$\frac{e^{2x}}{7} - \frac{2e^x}{3} + 1$
	$\frac{e^{2x}}{7} + \frac{2e^x}{3} - 1$
Option C:	$\frac{e^{2x}}{7} - \frac{2xe^x}{3} + 1$
Option D:	$\frac{e^{2x}}{7} + \frac{2xe^x}{3} - 1$

Solution:

$$y = \frac{1}{D^3-1}(e^{2x} + 2e^x + 1)$$

$$y = \frac{1}{D^3-1}e^{2x} + \frac{1}{D^3-1}2e^x + \frac{1}{D^3-1}e^{0x}$$

$$y = \frac{e^{2x}}{7} + x.\frac{1}{3D^2}2e^x + \frac{e^{0x}}{-1}$$

$$y = \frac{e^{2x}}{7} + \frac{2xe^x}{3} - 1$$

B T #Respect

49.	Value of $\Gamma\left(-\frac{8}{5}\right)$ is given by
Option A:	$\frac{25}{24}\Gamma\left(\frac{2}{5}\right)$
Option B:	$\frac{24}{25}\Gamma\left(\frac{1}{5}\right)$
Option C:	$\frac{25}{64}\Gamma\left(\frac{3}{5}\right)$
Option D:	$\frac{5}{8}\Gamma\left(\frac{13}{5}\right)$

Solution:

We have,

$\Gamma(n+1) = n\,\Gamma n$

$\Gamma n = \frac{\Gamma(n+1)}{n}$

Replace n by $n-1$

$\Gamma(n-1) = \frac{\Gamma n}{n-1}$ ...... (1)

Put $n = \frac{2}{5}$ in eqn (1)

$\Gamma\left(-\frac{3}{5}\right) = \frac{\Gamma\left(\frac{2}{5}\right)}{-\frac{3}{5}} = -\frac{5}{3}\Gamma\left(\frac{2}{5}\right)$

Put $n = -\frac{3}{5}$ in eqn (1)

$\Gamma\left(-\frac{8}{5}\right) = \frac{\Gamma\left(-\frac{3}{5}\right)}{-\frac{8}{5}} = -\frac{5}{8}\Gamma\left(-\frac{3}{5}\right) = -\frac{5}{8} \times -\frac{5}{3}\Gamma\left(\frac{2}{5}\right) = \frac{25}{24}\Gamma\left(\frac{2}{5}\right)$

50.	On changing the order of integration for $I = \int_0^1 \int_0^{1+\sqrt{1-x^2}} f(x,y)dy\,dx$, $I = \int_0^1 \int_a^1 f(x,y)dy\,dx$ where a=
Option A:	$1 + \sqrt{1 - y^2}$
Option B:	$\sqrt{2y - y^2}$
Option C:	$\sqrt{1 - y^2}$
Option D:	$\sqrt{y^2 - y}$

Solution:

The region of integration is

$x = 0 \Rightarrow$ Y axis

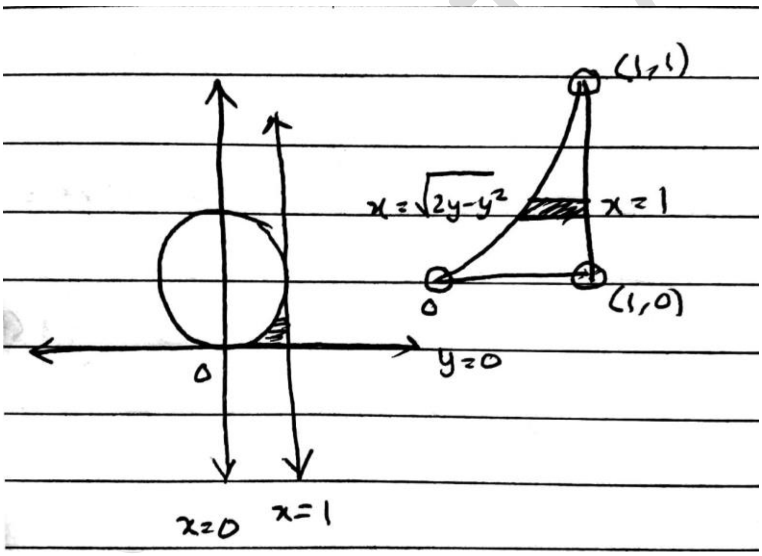
$x = 1 \Rightarrow$ line parallel to Y axis

$y = 0 \Rightarrow$ X axis

$y = 1 + \sqrt{1 - x^2}$

$(y - 1)^2 = 1 - x^2$

$x^2 + (y - 1)^2 = 1 \Rightarrow$ a circle with centre at (0,1) and radius 1



$I = \int_0^1 \int_{\sqrt{2y-y^2}}^1 f(x,y)dx\,dy$

51.	Evaluate: $\int_2^6 \int_{x^2-6x+3}^{2x-9} dydx$
Option A:	$\frac{34}{3}$
Option B:	$\frac{25}{3}$
Option C:	$\frac{32}{3}$
Option D:	$\frac{2}{3}$

Solution:

$$I = \int_2^6 [2x - 9 - x^2 + 6x - 3]dx = \left[-\frac{x^3}{3} + \frac{8x^2}{2} - 12x\right]_2^6 = \frac{32}{3}$$

53.	The degree and order of the differential equation $\frac{d^4y}{dx^4} - 3x(\frac{d^3y}{dx^3})^2 + \frac{dy}{dx} - 2y = 0$ are respectively
Option A:	4,1
Option B:	4,2
Option C:	2,4
Option D:	1,4

Solution:

The order is 4 and degree is 1

54.	The value of the particular integral $\frac{1}{(D^2+1)}(2^x + \sin x \sin 2x)$ is
Option A:	$\frac{4x\sin x - \cos 3x}{16} + \frac{2^x}{2\log 2 + 1}$
Option B:	$\frac{4x\sin x - \cos 3x}{16} + \frac{2^x}{\log 2 + 1}$
Option C:	$\frac{4x\sin x - \cos 3x}{16} + \frac{2^x}{(\log 2)^2 + 1}$
Option D:	$\frac{4x\sin x + \cos 3x}{16} + \frac{2^x}{2\log 2 + 1}$

Solution:

$$y = \frac{1}{D^2+1}2^x + \frac{1}{D^2+1}\sin x \sin 2x$$

$$y = \frac{1}{D^2+1}e^{x\log 2} + \frac{1}{D^2+1}.\frac{1}{2}(\cos(-x) - \cos(3x))$$

$$y = \frac{e^{x\log 2}}{(\log 2)^2+1} + \frac{1}{2}.\frac{1}{D^2+1}\cos x - \frac{1}{2}.\frac{1}{D^2+1}\cos 3x$$

$$y = \frac{2^x}{(\log 2)^2+1} + \frac{1}{2}x.\frac{1}{2D}\cos x + \frac{\cos 3x}{16}$$

$$y = \frac{2^x}{(\log 2)^2+1} + \frac{x\sin x}{4} + \frac{\cos 3x}{16}$$

#Paper

55.	Evaluate: $\int_0^1 \sqrt{1-\sqrt{1-\sqrt{x}}}\,dx$
Option A:	$\frac{308}{215}$
Option B:	$\frac{208}{315}$
Option C:	$\frac{215}{308}$
Option D:	$\frac{218}{305}$

Solution:

$$I = \int_0^1 \sqrt{1-\sqrt{1-\sqrt{x}}}\,dx$$

Put $\sqrt{x} = t$

$x = t^2$

$dx = 2t\,dt$

$$I = \int_0^1 \sqrt{1-\sqrt{1-t}}\;2t\,dt$$

Put $1 - t = y$

$t = 1 - y$

$dt = -dy$

$$I = \int_1^0 \sqrt{1-\sqrt{y}}.2(1-y)(-dy)$$

$$I = 2\int_0^1 \sqrt{1-\sqrt{y}}(1-y)dy$$

Put $\sqrt{y} = z$

$y = z^2$

$dy = 2zdz$

$$I = 2\int_0^1 \sqrt{1-z}\,(1-z^2)(2zdz)$$

$$I = 4\int_0^1 (1-z)^{\frac{1}{2}}(z-z^3)dz$$

$$I = 4\left\{\int_0^1 z(1-z)^{\frac{1}{2}}dz - \int_0^1 z^3(1-z)^{\frac{1}{2}}dz\right\}$$

$$I = 4\left\{B\left(1+1,\frac{1}{2}+1\right) - B\left(3+1,\frac{1}{2}+1\right)\right\}$$

$$I = 4\left\{B\left(2,\frac{3}{2}\right) - B\left(4,\frac{3}{2}\right)\right\}$$

$$I = 4\left[\frac{\Gamma 2\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} - \frac{\Gamma 4\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{11}{2}\right)}\right] = 4\left[\frac{1.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - \frac{3.2.1.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\right] = 4\left[\frac{4}{15} - \frac{32}{315}\right] = \frac{208}{315}$$

	Changing to polar co-ordinates $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ the value of integral is
Option A:	$\frac{\pi}{4}$
Option B:	$\frac{\pi}{2}$
Option C:	$\frac{1}{4}$
Option D:	π

Solution:

The region of integration is the first quadrant

Now changing it into polar co-ordinates

i.e. put $x = rcos\theta \; and \; y = rsin\theta, dxdy = rdrd\theta$

we get the region of integration as,

$r = 0 \,\&\, r = \infty\,, \theta = 0 \,\&\, \theta = \frac{\pi}{2}$

$I = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta$

Put $r^2 = t, \; \therefore rdr = \frac{dt}{2}$

$I = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-t} \frac{dt}{2} d\theta$

$I = \int_0^{\frac{\pi}{2}} \left[\frac{e^{-t}}{-2}\right]_0^\infty d\theta$

$I = \int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta$

$\boxed{I = \left[\frac{\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}}$

57.	Evaluate: $\iiint xyz dxdydz$ throughout the volume bounded by x=0,y=0,z=0,x+y+z=1
Option A:	$\frac{1}{5040}$
Option B:	$\frac{1}{720}$
Option C:	$\frac{1}{60}$
Option D:	$\frac{1}{144}$

Solution:

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz\, dxdydz$$

$$I = \int_0^1 x \int_0^{1-x} y \left[\frac{z^2}{2}\right]_0^{1-x-y} dxdy$$

$$I = \frac{1}{2}\int_0^1 x \int_0^{1-x} y[(1-x-y)^2]dydx$$

Put $y = (1-x)t\,, dy = (1-x)dt$;

$when\ y = 0, t = 0\ \&\ when\ y = 1-x, t = 1$

$$I = \frac{1}{2}\int_0^1 x \int_0^1 (1-x)t\left[\left((1-x)-(1-x)t\right)^2\right](1-x)dtdx$$

$$I = \frac{1}{2}\int_0^1 x(1-x)^4 \int_0^1 t(1-t)^2\, dtdx$$

$$I = \frac{1}{2}.B(2,5).B(2,3)$$

$$I = \frac{1}{2}.\frac{\Gamma(2)\Gamma(5)}{\Gamma(7)}.\frac{\Gamma(2)\Gamma(3)}{\Gamma(5)}$$

$$I = \frac{1}{2}.\frac{1!.4!}{6!}.\frac{1.2!}{4!} = \frac{1}{720}$$

59.	The particular integral $\frac{1}{D^6-64}\sin 2x =$
Option A:	$\frac{x\cos 2x}{16}$
Option B:	$\frac{\sin 2x}{128}$
Option C:	$\frac{-x\sin 2x}{128}$
Option D:	$\frac{-\sin 2x}{128}$

Solution:

$$y = \frac{1}{D^6-64}sin2x = \frac{1}{(D^2)^3-64}sin2x = \frac{sin2x}{(-2^2)^3-64} = -\frac{sin2x}{128}$$

60.	Evaluate: $\int_0^{log2}\int_0^{x}\int_0^{x-y} e^{x+y+z}\, dzdydx$
Option A:	$2\log 2-\frac{5}{4}$
Option B:	$2\log 2+\frac{5}{8}$
Option C:	$\log 2-\frac{5}{4}$

Solution:

$$I = \int_0^{log2}\int_0^{x}[e^{x+y+z}]_0^{x-y}dx\, dy$$

$$I = \int_0^{log2}\int_0^{x}[e^{2x} - e^{x+y}]dx\, dy$$

$$I = \int_0^{log2}[e^{2x}.y - e^{x+y}]_0^{x}dx$$

$$I = \int_0^{log2}[xe^{2x} - e^{2x} - 0 + e^{x}]dx$$

$$I = \left[x\left(\frac{e^{2x}}{2}\right) - (1)\left(\frac{e^{2x}}{4}\right) - \frac{e^{2x}}{2} + e^{x}\right]_0^{log2}$$

$$I = log2\left(\frac{e^{2log2}}{2}\right) - \frac{e^{2log2}}{4} - \frac{e^{2log2}}{2} + e^{log2} - 0 + \frac{e^0}{4} + \frac{e^0}{2} - e^0$$

$$I = 2log2 - 1 - 2 + 2 + \frac{1}{4} + \frac{1}{2} - 1$$

$$I = 2log2 - \frac{5}{4}$$

Descriptive Questions

1.	Using Beta function, Prove that $\int_0^\infty \frac{1}{1+x^2}dx = \frac{\pi}{2}$

Solution:

$I = \int_0^\infty \frac{1}{1+x^2}\, dx$

Put $x = \boxed{tan\theta}, dx = \sec^2\theta\, d\theta$

When $x = 0, \theta = 0$

When $x = 1, \theta = \frac{\pi}{2}$

$I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^2\theta} \sec^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} 1\, d\theta$

$I = \frac{1}{2} B\left(\frac{0+1}{2}, \frac{0+1}{2}\right)$

$I = \frac{1}{2} B\left(\frac{1}{2}, \frac{1}{2}\right)$

$I = \frac{1}{2}.\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$

$I = \frac{\pi}{2}$

OR

$I = \int_0^\infty \frac{1}{1+x^2}\, dx$

Put $x^2 = t, x = \sqrt{t}, dx = \frac{1}{2\sqrt{t}} dt$

$I = \int_0^\infty \frac{1}{1+t} \cdot \frac{dt}{2\sqrt{t}}$

$I = \frac{1}{2}\int_0^\infty \frac{t^{-\frac{1}{2}}}{1+t} dt$

$I = \frac{1}{2}\int_0^\infty \frac{t^{\frac{1}{2}-1}}{(1+t)^{\frac{1}{2}+\frac{1}{2}}} dt$

$I = \frac{1}{2} B\left(\frac{1}{2}, \frac{1}{2}\right)$ since $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$I = \frac{1}{2}.\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$

$I = \frac{\pi}{2}$

2.	Change the order of integration in the integral $$I = \int_{-a}^{a}\int_{0}^{\sqrt{(a^2-y^2)}} f(x,y)dxdy$$

Solution:

The region of integration is

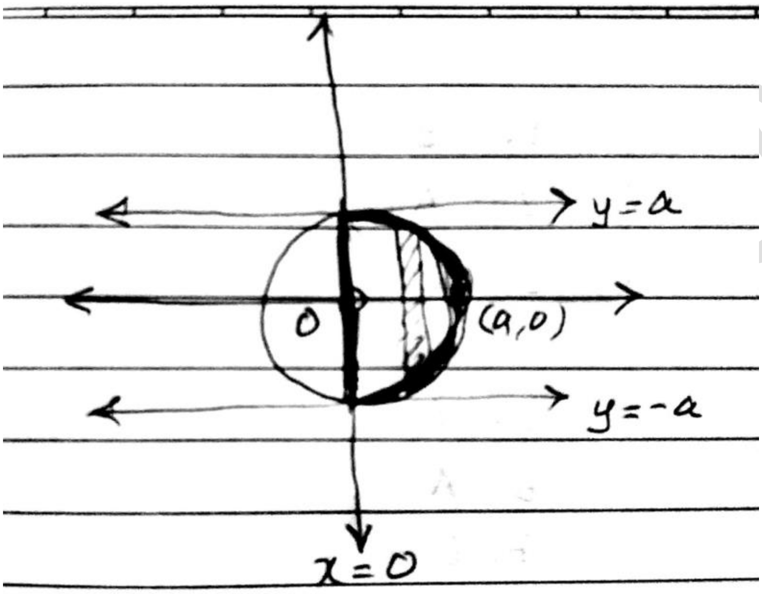
$y = -a \Rightarrow$ a line parallel to X axis

$y = a \Rightarrow$ a line parallel to X axis

$x = 0 \Rightarrow$ Y axis

$x = \sqrt{a^2 - y^2}$

$x^2 + y^2 = a^2 \Rightarrow$ a standard circle with radius a



$I = \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)dx\,dy$

3.	Using the method of variation of parameters, solve $$\frac{d^2y}{dx^2} + 4y = tan2x.$$

Solution:

C.F.:

Put $D^2 + 4 = 0$

$\therefore D = \pm 2i$

Thus, the C.F. can be written as

$y = c_1 cos2x + c_2 sin2x$

P.I.:

Let P.I. be $y = uy_1 + vy_2$

where $y_1 = cos2x, y_2 = sin2x, X = tan2x$

Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} cos2x & sin2x \\ -2sin2x & 2cos2x \end{vmatrix} = 2$$

$\therefore u = -\int \frac{y_2 X}{W} dx$

$\therefore u = -\int \frac{sin2x}{2} tan2xdx = -\frac{1}{2}\int \frac{\sin^2 2x}{cos2x} dx$

$\therefore u = -\frac{1}{2}\int \frac{1-\cos^2 2x}{cos2x} dx = -\frac{1}{2}\int sec2x - cos2x\, dx$

$\therefore u = -\frac{1}{2}\left[\frac{\log(sec2x+tan2x)}{2} - \frac{sin2x}{2}\right]$

$\therefore v = \int \frac{y_1 X}{W} dx$

$\therefore v = \int \frac{cos2x}{2} tan2xdx = \int \frac{sin2x}{2} dx = -\frac{cos2x}{4}$

Thus,

$$y = cos2x\left[-\frac{\log(sec2x+tan2x)}{4} + \frac{sin2x}{4}\right] + sin2x\left[-\frac{cos2x}{4}\right]$$

$\therefore y = -\frac{1}{4} cos2x\, .\log(sec2x + tan2x)$

G.S.:

$y = C.F. + P.I.$

$$\boxed{y = c_1 cos2x + c_2 sin2x - \frac{1}{4} cos2x. \log(sec2x + tan2x)}$$

HoD

4.	Solve $(D-2)^2 = 8(e^{2x} + \sin 2x + x^2)$.

Solution:

Direct hit

C.F.:

Put $(D-2)^2 = 0$

$\therefore D = 2,2$

Thus, the C.F. can be written as

$y = (c_1 + c_2 x)e^{2x}$

P.I.:

$$y = \frac{1}{f(D)}X$$

$$y = \frac{1}{(D-2)^2}8(e^{2x} + \sin 2x + x^2)$$

$$y = \frac{1}{(D-2)^2}8e^{2x} + \frac{1}{(D-2)^2}8\sin 2x + \frac{1}{(D-2)^2}8x^2$$

$$y = x.\frac{1}{2(D-2)}8e^{2x} + \frac{1}{D^2-4D+4}8\sin 2x + \frac{1}{D^2-4D+4}8x^2$$

$$y = x^2.\frac{1}{2}8e^{2x} + \frac{1}{-2^2-4D+4}8\sin 2x + \frac{1}{4}.\frac{1}{1-D+\frac{D^2}{4}}8x^2$$

$$y = 4x^2e^{2x} - 2\int \sin 2x dx + \frac{1}{4}\left[1-\left(D-\frac{D^2}{4}\right)\right]^{-1} 8x^2$$

$$y = 4x^2e^{2x} - 2\left[-\frac{\cos 2x}{2}\right] + \frac{1}{4}\left[1+\left(D-\frac{D^2}{4}\right)+\left(D-\frac{D^2}{4}\right)^2\right]8x^2$$

$$y = 4x^2e^{2x} + \cos 2x + \frac{1}{4}\left[1+D-\frac{D^2}{4}+D^2\right]8x^2$$

$$y = 4x^2e^{2x} + \cos 2x + \frac{1}{4}\left[8x^2+16x-\frac{16}{4}+16\right]$$

$$y = 4x^2e^{2x} + \cos 2x + \frac{1}{4}[8x^2+16x+12]$$

$$y = 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

G.S.:

$$y = C.F. + P.I.$$

$$\boxed{y = (c_1 + c_2 x)e^{2x} + 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3}$$

5.	Solve the Differential Equation $\frac{dy}{dx} + xsin2y = x^3cos^2y$.

Solution:

We have,

$$\frac{dy}{dx} + xsin2y = x^3 \cos^2 y$$

Dividing by$\cos^2 y$,

$$\sec^2 y \frac{dy}{dx} + x\frac{sin2y}{\cos^2 y} = x^3$$

$$\sec^2 y \frac{dy}{dx} + x.\frac{2sinycosy}{\cos^2 y} = x^3$$

$$\sec^2 y \frac{dy}{dx} + 2x.tany = x^3$$

Put $tany = V, \therefore \sec^2 y \frac{dy}{dx} = \frac{dV}{dx}$

$$\frac{dV}{dx} + 2x.V = x^3$$

It is of the form, $\frac{dV}{dx} + PV = Q$ where, $P = 2x$ & $Q = x^3$

$$\therefore I.F. = e^{\int 2xdx} = e^{x^2}$$

Its solution is given by,

$$V(I.F.) = \int Q(I.F.)dx + C$$

$$tany.e^{x^2} = \int x^3e^{x^2}dx + C$$

$$tany\ e^{x^2} = \int x^2e^{x^2}.xdx + C$$

Put $x^2 = t, \therefore xdx = \frac{dt}{2}$

$$tany\ e^{x^2} = \int te^t\frac{dt}{2} + C$$

$$tany\ e^{x^2} = \frac{1}{2}[te^t - e^t] + C$$

$$\boxed{tanye^{x^2} = \frac{1}{2}\left[x^2e^{x^2} - e^{x^2}\right] + C}$$

6.	Evaluate the integral $\int_0^2 \int_0^z \int_0^{yz} xyz\, dx\, dy\, dz$.

Solution:

$$I = \int_0^2 \int_0^z \int_0^{yz} xyzdxdydz$$

$$I = \int_0^2 \int_0^z yz \left[\frac{x^2}{2}\right]_0^{yz} dydz$$

$$I = \frac{1}{2}\int_0^2 \int_0^z y^3 z^3\, dydz$$

$$I = \frac{1}{2}\int_0^2 \left[\frac{y^4}{4} z^3\right]_0^z dz$$

$$I = \frac{1}{8}\int_0^2 z^7 dz$$

$$I = \frac{1}{8}\left[\frac{z^8}{8}\right]_0^2$$

$$\boxed{I = 4}$$

Imp

7.	Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}} dx\, dy\, dz.$

Solution:

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dydx$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \sin^{-1}(1)\, dydx$$

$$I = \frac{\pi}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} dydx = \frac{\pi}{2}\int_0^1 [y]_0^{\sqrt{1-x^2}} dx$$

$$I = \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\, dx$$

$$I = \frac{\pi}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x\right]_0^1$$

$$\boxed{I = \frac{\pi}{2}\left[\frac{1}{2}.\frac{\pi}{2}\right] = \frac{\pi^2}{8}}$$

8.	Solve the Differential Equation $(x^2e^x - my)dx + mx\,dy = 0$

Solution:

We have,

$M = x^2e^x - my$ $\qquad$ $N = mx$

$\frac{\partial M}{\partial y} = -m$ $\qquad$ $\frac{\partial N}{\partial x} = m$

Consider,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-m-m}{mx} = -\frac{2m}{mx} = -\frac{2}{x} = f(x)$$

$$IF = e^{\int f(x)dx} = e^{\int \frac{-2}{x}dx} = e^{-2logx} = e^{logx^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying the DE by IF to make it exact,

$$\left(\frac{x^2e^x}{x^2} - \frac{my}{x^2}\right)dx + \frac{mx}{x^2}dy = 0$$

$$(e^x - myx^{-2})dx + \frac{m}{x}dy = 0$$

Its solution,

$$\int Mdx + \int Ndy = C$$

$$\int (e^x - myx^{-2})dx + \int 0dy = C$$

$$e^x - my\frac{x^{-1}}{-1} = C$$

$$\boxed{e^x + \frac{my}{x} = c}$$

Imp

9.	Change to polar coordinates and Evaluate $\int_0^a \int_y^a \frac{x\,dx\,dy}{(x^2+y^2)}$.

Solution:

To change into polar coordinates

Put $x = rcos\theta, y = rsin\theta, dxdy = rdrd\theta$

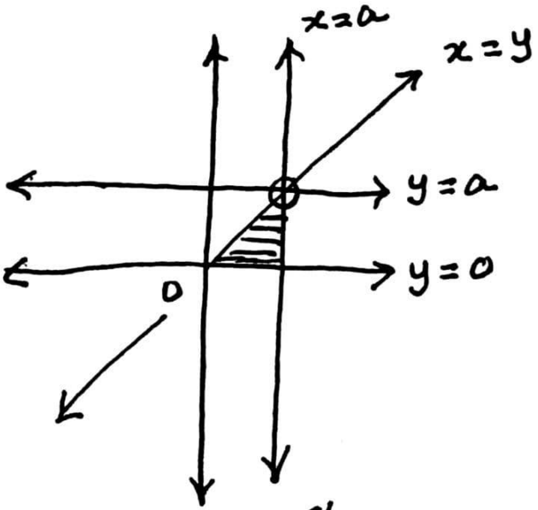
The region of integration is

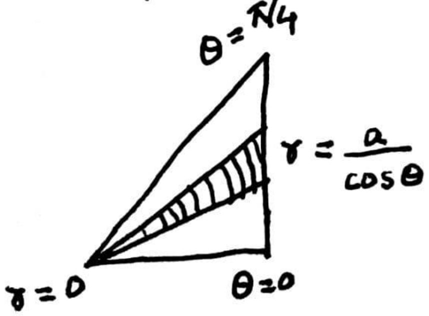
$y = 0 \Rightarrow$ X axis $\Rightarrow r = 0, \theta = 0$

$y = a \Rightarrow$ line parallel to X axis $\Rightarrow r = \frac{a}{sin\theta}$

$x = y \Rightarrow$ line passing through origin $\Rightarrow \theta = \frac{\pi}{4}$

$x = a \Rightarrow$ line parallel to Y axis $\Rightarrow r = \frac{a}{cos\theta}$





$$I = \int_0^{\frac{\pi}{4}} \int_0^{\frac{a}{cos\theta}} \frac{rcos\theta}{r^2}\ r\ dr\ d\theta = \int_0^{\frac{\pi}{4}} cos\theta [r]_0^{\frac{a}{cos\theta}} d\theta$$

$$I = \int_0^{\frac{\pi}{4}} cos\theta . \frac{a}{cos\theta} d\theta = [a\theta]_0^{\frac{\pi}{4}} = \frac{a\pi}{4}$$

10.	Assuming the validity of differentiation under the integral sign, Prove that $$\int_0^1 \frac{x^a - x^b}{logx} dx = log\frac{a+1}{b+1}$$

Solution:

We have, $I(a) = \int_0^1 \left(\frac{x^a - x^b}{logx}\right) dx \dots\dots\dots\dots\dots\dots\dots.(1)$

By the rule of DUIS, differentiating w.r.t a, we get

$$\frac{\partial I}{\partial a} = \int_0^1 \frac{1}{logx}(x^a logx)dx$$

$$\frac{\partial I}{\partial a} = \int_0^1 x^a dx$$

$$\frac{\partial I}{\partial a} = \left[\frac{x^a}{a+1}\right]_0^1$$

$$\frac{\partial I}{\partial a} = \frac{1}{a+1}$$

Integrating, we get

$$I = \log(a+1) + C$$

$$I(a) = \log(a+1) + C \dots\dots\dots\dots\dots\dots(2)$$

Put a = b in equation (1) & (2), we get

$I(b) = 0$ & $I(b) = \log(b+1) + C$

$\therefore C = -\log(b+1)$

$$\therefore I(a) = \log(a+1) - \log(b+1)$$

$$\boxed{I = \log\left(\frac{a+1}{b+1}\right)}$$

HOD

11.	Solve the Differential Equation $(D^3 + D)y = cosx$.

Solution:

Direct hit

CF:

Put $D^3 + D = 0$

$D(D^2 + 1) = 0$

$D = 0, +i, -i$

$y_c = c_1 e^{0x} + c_2 cosx + c_3 sinx$

PI:

$y = \frac{1}{f(D)} X$

$y_p = \frac{1}{D^3 + D} cosx$

$y_p = x.\frac{1}{3D^2 + 1} cosx$

$y_p = x.\frac{cosx}{3(-1^2) + 1}$

$y_p = -\frac{xcosx}{2}$

GS:

$y = y_c + y_p = c_1 + c_2 cosx + c_3 sinx - \frac{xcosx}{2}$

12.	Solve the Differential Equation $x\frac{dy}{dx} + 2y = y^2x^3$.

Solution:

$$x\frac{dy}{dx} + 2y = y^2x^3$$

$$\frac{dy}{dx} + \frac{2}{x}y = y^2x^2$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{2}{x}.\frac{1}{y} = x^2$$

Put $\frac{1}{y} = V$

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{dV}{dx}$$

$$\frac{1}{y^2}\frac{dy}{dx} = -\frac{dV}{dx}$$

$$-\frac{dV}{dx} + \frac{2}{x}V = x^2$$

$$\frac{dV}{dx} - \frac{2}{x}V = -x^2$$

It is of the form $\frac{dV}{dx} + PV = Q$

Where $P = -\frac{2}{x}, Q = -x^2$

$$IF = e^{\int Pdx} = e^{\int \frac{-2}{x}dx} = e^{-2logx} = e^{logx^{-2}} = x^{-2} = \frac{1}{x^2}$$

Solution,

$$V(IF) = \int Q(IF)dx + C$$

$$\frac{1}{y}.\frac{1}{x^2} = \int -x^2.\frac{1}{x^2}dx + C$$

$$\boxed{\frac{1}{x^2y} = -x + C}$$

BT 13.	Prove that $\int_0^1 \frac{x^2dx}{\sqrt{(1-x^4)}}.\int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\pi}{4\sqrt{2}}$.

Solution:

Let $I_1 = \int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx$

Put $x^4 = t$

$\boxed{x = t^{\frac{1}{4}}}$

$dx = \frac{1}{4}t^{\frac{-3}{4}}\,dt$

when $x = 0, t = 0$ and when $x = 1, t = 1$

$I_1 = \int_0^1 \frac{t^{\frac{2}{4}}}{\sqrt{1-t}}.\frac{1}{4}t^{\frac{-3}{4}}\,dt$

$I_1 = \frac{1}{4}\int_0^1 t^{\frac{-1}{4}}(1-t)^{\frac{-1}{2}}\,dt$ + Duplication

$I_1 = \frac{1}{4}\,B\left(-\frac{1}{4}+1,-\frac{1}{2}+1\right)$

$I_1 = \frac{1}{4}\,B\left(\frac{3}{4},\frac{1}{2}\right)$

Let $I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

Put $x^4 = t$

$x = t^{\frac{1}{4}}$

$dx = \frac{1}{4}t^{-\frac{3}{4}}dt$

$I_2 = \int_0^1 \frac{\frac{1}{4}t^{-\frac{3}{4}}dt}{(1+t)^{\frac{1}{2}}}$

$I_2 = \frac{1}{4}\int_0^1 \frac{t^{-\frac{3}{4}}}{(1+t)^{\frac{1}{2}}}dt$

$I_2 = \frac{1}{4}\int_0^1 \frac{t^{\frac{1}{4}-1}}{(1+t)^{\frac{1}{4}+\frac{1}{4}}}dt$

$I_2 = \frac{1}{4}\times\frac{1}{2}B\left(\frac{1}{4},\frac{1}{4}\right)$ since $\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx = \frac{1}{2}B(m,n)$

$I_2 = \frac{1}{8}\,B\left(\frac{1}{4},\frac{1}{4}\right)$

Thus, $I = I_1 \times I_2$

$I = \frac{1}{4}\,B\left(\frac{3}{4},\frac{1}{2}\right)\times\frac{1}{8}\,B\left(\frac{1}{4},\frac{1}{4}\right)$

$$I=\frac{1}{4}.\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}\times\frac{1}{8}.\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

$$I=\frac{1}{4}.\frac{\Gamma\left(\frac{3}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}\times\frac{1}{8}.\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$I=\frac{1}{8}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)=\frac{1}{8}\sqrt{2}\pi=\frac{\pi}{4\sqrt{2}}$$

14.	Solve the Differential Equation $(1+xy)ydx+(1-xy)xdy=0$

Solution:

We have,

$$M = y + xy^2 \qquad N = x - x^2y$$

$$\frac{\partial M}{\partial y} = 1 + 2xy \qquad \frac{\partial N}{\partial x} = 1 - 2xy$$

Not exact

Consider, $Mx - Ny = xy + x^2y^2 - xy + x^2y^2 = 2x^2y^2$

$$IF = \frac{1}{Mx-Ny} = \frac{1}{2x^2y^2}$$

Multiplying the DE by IF to make it exact

$$\left(\frac{y}{2x^2y^2} + \frac{xy^2}{2x^2y^2}\right)dx + \left(\frac{x}{2x^2y^2} - \frac{x^2y}{2x^2y^2}\right)dy = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0$$

Its solution,

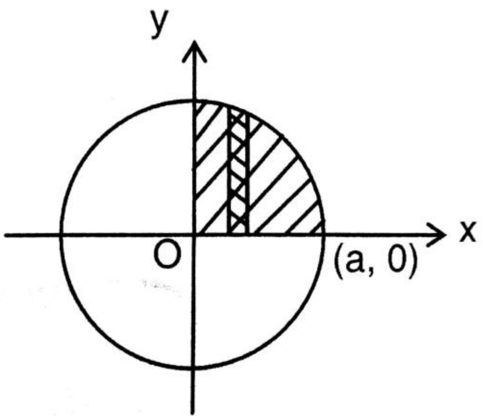
$$\int Mdx + \int Ndy = C$$

$$\int\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \int -\frac{1}{2y}dy = C$$

$$\boxed{-\frac{1}{2xy} + \frac{1}{2}logx - \frac{1}{2}logy = C}$$

15.	Evaluate $\int\int xy\, dxdy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$.

Solution:



$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dxdy$$

$$I = \int_0^a \left[x\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

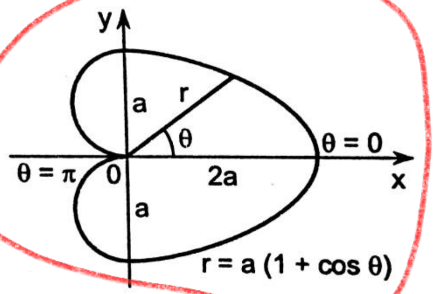
$$I = \int_0^a \left[\frac{x(a^2-x^2)}{2}\right] dx$$

$$I = \left[\frac{a^2x^2}{4} - \frac{x^4}{8}\right]_0^a$$

$$\boxed{I = \frac{a^4}{8}}$$

16.	Find the entire length of cardioid $r = a(1 + cos\theta)$

Solution:



$$s = 2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\therefore s = 2\int_0^{\pi} \sqrt{a^2(1 + cos\theta)^2 + a^2 sin^2\theta}\, d\theta$$

$$s = 2a\int_0^{\pi} \sqrt{1 + 2cos\theta + cos^2\theta + sin^2\theta}\, d\theta$$

$$s = 2a\int_0^{\pi} \sqrt{2(1 + cos\theta)}\, d\theta$$

$$s = 2a\int_0^{\pi} \sqrt{2.2cos^2\frac{\theta}{2}}\, d\theta$$

$$s = 4a\int_0^{\pi} cos\frac{\theta}{2}\, d\theta$$

$$s = 4a\left[\frac{sin\frac{\theta}{2}}{\frac{1}{2}}\right]_0^{\pi} = 8a$$

HOD

17. Solve $\frac{d^4x}{dt^4} + 4x = 0.$

Solution:

Put $D^4 + 4 = 0$

$D^4 + 4D^2 + 4 - 4D^2 = 0$

$(D^2 + 2)^2 - (2D)^2 = 0$

$(D^2 + 2 + 2D)(D^2 + 2 - 2D) = 0$

Consider, $D^2 + 2D + 2 = 0$

$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$D = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

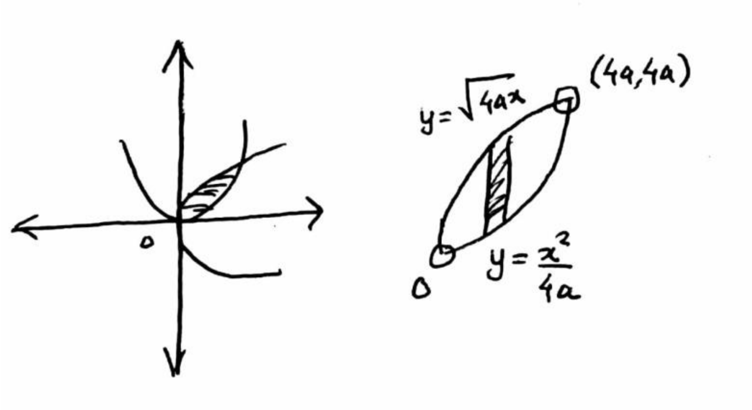
Similarly, $D^2 - 2D + 2 = 0$ will give

$D = 1 \pm i$

Thus, the solution is $\boxed{x = e^t[c_1 \cos t + c_2 sint] + e^{-t}[c_3 cost + c_4 sint]}$

18.	Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

Solution:



$$A = \int_0^{4a}\int_{\frac{x^2}{4a}}^{\sqrt{4ax}} dxdy$$

$$A = \int_0^{4a}\left[\sqrt{4ax} - \frac{x^2}{4a}\right] dx$$

$$A = \left[2\sqrt{a}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{12a}\right]_0^a = \frac{4\sqrt{a}}{3}(a)^{\frac{3}{2}} - \frac{a^3}{12a}$$

$$A = \frac{4a^2}{3} - \frac{a^2}{12} = \frac{5a^2}{4}$$

19.	Solve the DE $(2xy\cos x^2 - 2xy + 1)\,dx + (\sin x^2 - x^2)\,dy = 0$

DE

Solution:

$M = 2xy\cos x^2 - 2xy + 1$ $N = \sin x^2 - x^2$

$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x$ $\frac{\partial N}{\partial x} = \cos x^2 \times 2x - 2x$

Hence, exact

Its solution,

$\int M dx + \int N dy = C$

$\int (2xy\cos x^2 - 2xy + 1)dx + \int 0 dy = C$

$y\int 2x\cos x^2\,dx - \frac{2x^2}{2}y + x = C$

Put $x^2 = t, 2xdx = dt$

$y\int cost\ dt - x^2y + x = C$

$y\ sint - x^2y + x = C$

$\boxed{y\sin x^2 - x^2y + x = C}$

20.	Using Method of variation of parameters solve $$\frac{d^2y}{dx^2} + y = \tan x$$

HoD

Solution:

C.F.:

Put $D^2 + 1 = 0$

$\therefore D = \pm i$

Thus, the C.F. can be written as

$y = c_1 cosx + c_2 sinx$

P.I.:

Let P.I. be $y = uy_1 + vy_2$

where $y_1 = cosx, y_2 = sinx, X = tanx$

Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} cosx & sinx \\ -sinx & cosx \end{vmatrix} = 1$$

$\therefore u = -\int \frac{y_2 X}{W} dx$

$\therefore u = -\int \frac{sinx}{1} tanxdx = -\int \frac{\sin^2 x}{cosx} dx$

$\therefore u = -\int \frac{1-\cos^2 x}{cosx} dx = -\int secx - cosx\, dx$

$\therefore u = \boxed{-[\log(secx + tanx)} - sinx]$

$\therefore v = \int \frac{y_1 X}{W} dx$

$\therefore v = \int \frac{cosx}{1} tanxdx = \int sinxdx = -cosx$

Thus,

$y = cosx[-\log(secx + tanx) + sinx] + sinx[-cosx]$

$\therefore y = -cosx\,.\log(secx + tanx)$

G.S.:

$y = C.F. + P.I.$

$$\boxed{y = c_1 cosx + c_2 sinx - cosx\,.\log(secx + tanx)}$$

21	Change the order of integration $I = \int_0^a \int_{-a+\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\, dx\ dy$

Solution:

The region of integration is

$y = 0 \Rightarrow$ X axis

$y = a \Rightarrow$ line parallel to X axis

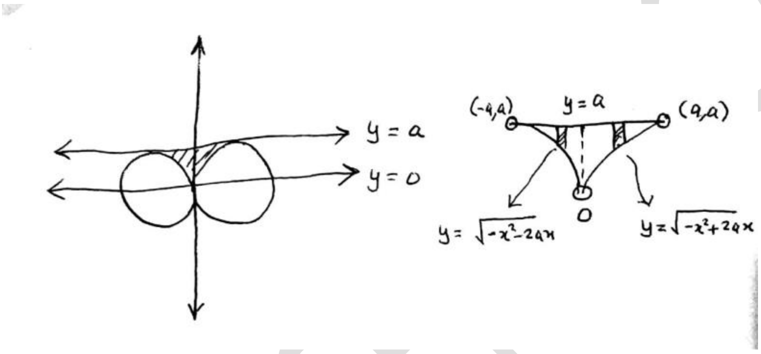
$x = -a + \sqrt{a^2 - y^2}$

$(x + a) = \sqrt{a^2 - y^2}$

$(x + a)^2 + y^2 = a^2 \Rightarrow$ a circle with centre at $(-a, 0)$ and radius a

$x = a + \sqrt{a^2 - y^2}$

$(x - a)^2 + y^2 = a^2 \Rightarrow$ a circle with centre at $(a, 0)$ and radius a



$$I = \int_{-a}^{0} \int_{\sqrt{-x^2-2ax}}^{a} f(x,y)dxdy + \int_0^a \int_{\sqrt{-x^2+2ax}}^{a} f(x,y)dx\, dy$$

BC

22. Prove that $\int_0^\infty \frac{e^{-\beta x} \sin \alpha x}{x}\, dx = \tan^{-1}\left(\frac{\alpha}{\beta}\right)$

Solution:

We have, $I(\beta) = \int_0^\infty \frac{e^{-\beta x} sin\alpha x}{x} dx \dots\dots\dots\dots (1)$

By the rule of DUIS, differentiating w.r.t β, we get

$$\frac{\partial I}{\partial \beta} = \int_0^\infty \frac{sin\alpha x}{x} e^{-\beta x}(-x)dx$$

$$\frac{\partial I}{\partial \beta} = -\int_0^\infty e^{-\beta x} sin\alpha x\, dx$$

$$\frac{\partial I}{\partial \beta} = -\left[\frac{e^{-\beta x}}{(-\beta)^2+(\alpha)^2}(-\beta sin\alpha x - \alpha cos\alpha x)\right]_0^\infty$$

$$\frac{\partial I}{\partial \beta} = -\frac{\alpha}{\beta^2+\alpha^2}$$

Integrating, we get

$$I(\beta) = -\alpha . \frac{1}{\alpha} \tan^{-1}\left(\frac{\beta}{\alpha}\right) + C$$

$$I(\beta) = -\tan^{-1}\left(\frac{\beta}{\alpha}\right) + C \dots\dots\dots\dots\dots\dots (2)$$

Put β = ∞ in equation (1) & (2), we get

$I(\infty) = 0$ & $I(\infty) = -\frac{\pi}{2} + C$

$\therefore C = \frac{\pi}{2}$

$$\therefore I(\beta) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

$$\boxed{\therefore \int_0^\infty \frac{e^{-\beta x} sin\alpha x}{x} dx = \frac{\pi}{2} - \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \tan^{-1}\left(\frac{\alpha}{\beta}\right)}$$

23	Prove that $\int\int\int(x+y+z)\,dx\,dy\,dz = \frac{1}{8}$, over the tetrahedron bounded by $x=0, y=0, z=0$ *and* $x+y+z=1$.

Solution:

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z)dzdydx$$

$$I = \int_0^1 \int_0^{1-x} \left[\frac{(x+y+z)^2}{2}\right]_0^{1-x-y} dydx$$

$$I = \int_0^1 \int_0^{1-x} \left[\frac{1}{2} - \frac{(x+y)^2}{2}\right] dy\,dx$$

$$I = \int_0^1 \left[\frac{y}{2} - \frac{(x+y)^3}{6}\right]_0^{1-x} dx$$

$$I = \int_0^1 \left[\frac{1-x}{2} - \frac{1}{6} - 0 + \frac{x^3}{6}\right] dx$$

$$I = \int_0^1 \left[\frac{x^3}{6} - \frac{x}{2} + \frac{1}{3}\right] dx$$

$$I = \left[\frac{x^4}{24} - \frac{x^2}{4} + \frac{x}{3}\right]_0^1 = \left[\frac{1}{24} - \frac{1}{4} + \frac{1}{3}\right]$$

$$\boxed{I = \frac{1}{8}}$$

24	Prove that $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta \,.\, \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\pi^2}{2}$

Solution:

$$I = \left[\int_0^{\frac{\pi}{2}} \sqrt{tan\theta} d\theta\right]\left[\int_0^{\frac{\pi}{2}} \sqrt{cot\theta} d\theta\right]$$

$$I = \left[\int_0^{\frac{\pi}{2}} \sqrt{\frac{sin\theta}{cos\theta}}\, d\theta\right]\left[\int_0^{\frac{\pi}{2}} \sqrt{\frac{cos\theta}{sin\theta}}\, d\theta\right]$$

$$I = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta \cos^{\frac{-1}{2}}\theta\, d\theta \,.\, \int_0^{\frac{\pi}{2}} \sin^{\frac{-1}{2}}\theta \cos^{\frac{1}{2}}\theta\, d\theta$$

$$I = \frac{1}{2} B\left(\frac{\frac{1}{2}+1}{2}, \frac{\frac{-1}{2}+1}{2}\right) \times \frac{1}{2} B\left(\frac{\frac{-1}{2}+1}{2}, \frac{\frac{1}{2}+1}{2}\right)$$

$$I = \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right) \times \frac{1}{2} B\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$I = \left[\frac{1}{2} \cdot \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)}\right]^2 \qquad \text{since } \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \sqrt{2}\,\pi$$

$$I = \left[\frac{1}{2} \cdot \frac{\sqrt{2}\pi}{\Gamma(1)}\right]^2 = \frac{2\pi^2}{4} = \frac{\pi^2}{2}$$

25.	Solve the DE $(D^2 - 4D + 4)y = 8\,(e^{2x} + \sin 2x)$, *where* $D \equiv \frac{d}{dx}$

HOD

Solution:

If (Basic formulas == clear)
{ printf (Direct hit);
}

CF:

Put $D^2 - 4D + 4 = 0$

$(D-2)^2 = 0$

$D = 2,2$

$y_c = (c_1 + c_2 x)e^{2x}$

PI:

$y = \frac{1}{f(D)}X$

$y_p = \frac{1}{D^2-4D+4}8(e^{2x} + sin2x)$

$y_p = \frac{1}{D^2-4D+4}8e^{2x} + \frac{1}{D^2-4D+4}8sin2x$

$y_p = x\frac{1}{2D-4}8e^{2x} + \frac{1}{-2^2-4D+4}8sin2x$

$y_p = x^2.\frac{1}{2}8e^{2x} + \frac{1}{-4D}8sin2x$

$y_p = 4x^2e^{2x} - \frac{1}{4}.\int 8sin2xdx$

$y_p = 4x^2e^{2x} - 2\left[-\frac{cos2x}{2}\right]$

$y_p = 4x^2e^{2x} + cos2x$

GS:

$y = y_c + y_p$

$\boxed{y = (c_1 + c_2 x)e^{2x} + 4x^2e^{2x} + cos2x}$

BT

~~26~~	Prove that $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$.

Solution:

$I = \int_0^\infty \frac{1}{1+x^4} dx$

Put $x^2 = tan\theta$

$x = \sqrt{tan\theta}$

$dx = \frac{1}{2\sqrt{tan\theta}} \sec^2\theta \, d\theta$

When $x = 0, \theta = 0$ and when $x = \infty, \theta = \frac{\pi}{2}$

$I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^2\theta} \frac{\sec^2\theta}{2\sqrt{tan\theta}} d\theta$

$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{tan\theta}}$

$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\theta \cos^{\frac{1}{2}}\theta \, d\theta$

$I = \frac{1}{2} \times \frac{1}{2} B\left(\frac{\frac{-1}{2}+1}{2}, \frac{\frac{1}{2}+1}{2}\right)$

$I = \frac{1}{2} \times \frac{1}{2} B\left(\frac{1}{4}, \frac{3}{4}\right)$

$I = \frac{1}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)}$

$I = \frac{1}{4} \cdot \sqrt{2}\pi = \frac{\pi}{2\sqrt{2}}$

~~27.~~	Solve the DE $\sin 2x\, \frac{dy}{dx} = y + \tan x$

Solution:

$$sin2x \frac{dy}{dx} = y + tanx$$

$$\frac{dy}{dx} - \frac{y}{sin2x} = \frac{tanx}{sin2x}$$

$$\frac{dy}{dx} - cosec2x\, y = \frac{\frac{sinx}{cosx}}{2sinx\, cosx}$$

$$\frac{dy}{dx} - cosec2x\, y = \frac{1}{2} . \frac{1}{\cos^2 x}$$

$$\frac{dy}{dx} - cosec2x\, y = \frac{1}{2} \sec^2 x$$

It is of the form, $\frac{dy}{dx} + Py = Q$, where,

$P = -cosec2x$ & $Q = \frac{1}{2} \sec^2 x$

Now,

$$I.F. = e^{\int Pdx} = e^{\int -cosec2x\, dx} = e^{-\frac{logtanx}{2}}$$

$$I.F. = e^{\log(tanx)^{-\frac{1}{2}}} = (tanx)^{-\frac{1}{2}} = \frac{1}{\sqrt{tanx}}$$

Its solution is given by,

$$y(I.F.) = \int Q\,(I.F.)dx + C$$

$$y\left(\frac{1}{\sqrt{tanx}}\right) = \int \frac{1}{2} \sec^2 x . \frac{1}{\sqrt{tanx}} dx + C$$

Put $tanx = t, \sec^2 x\ dx = dt$

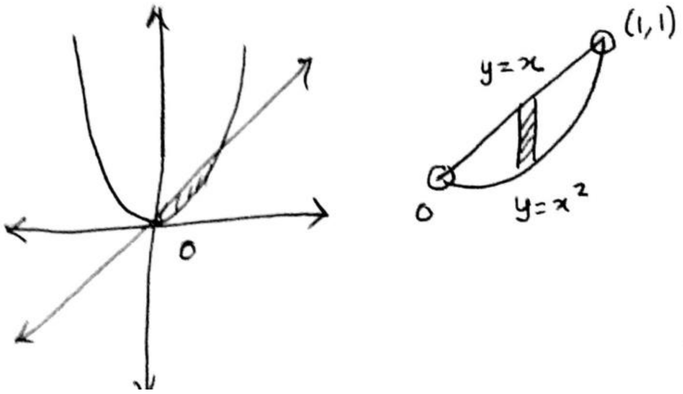
$$y\left(\frac{1}{\sqrt{tanx}}\right) = \frac{1}{2} \int \frac{1}{\sqrt{t}} dt + C$$

$$y\left(\frac{1}{\sqrt{tanx}}\right) = \frac{1}{2} . 2\sqrt{t} + C$$

$$\boxed{y\left(\frac{1}{\sqrt{tanx}}\right) = \sqrt{tanx} + C}$$

28.	Evaluate the integral $I = \int\int xy\,(x+y)\,dx\,dy$ over the region bounded by the curves $y = x^2$ & $y = x$.

Solution:



$$I = \int_0^1 \int_{x^2}^{x} xy(x+y)dx\,dy$$

$$I = \int_0^1 \left[x^2\frac{y^2}{2} + x\frac{y^3}{3}\right]_{x^2}^{x} dx$$

$$I = \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

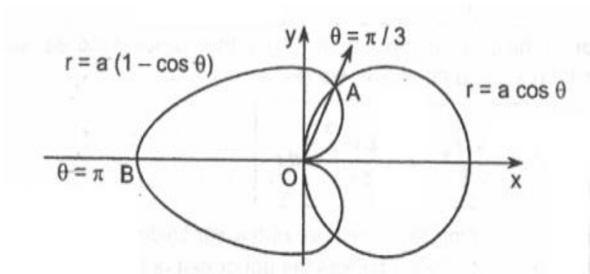
$$I = \left[\frac{x^5}{10} + \frac{x^5}{15} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$I = \frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24} = \frac{3}{56}$$

BT

29.	Show that the length of the cardioid $r = a\,(1 - \cos\theta)$ lies outside the circle $r = a\cos\theta$ is $4a\sqrt{3}$.

Solution:



We have,

$r = acos\theta$

$r^2 = a\,rcos\theta$

$x^2 + y^2 = ax$

$x^2 - ax + y^2 = 0$

$x^2 - ax + \frac{a^2}{4} + y^2 = \frac{a^2}{4}$

$\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$

Centre = $\left(\frac{a}{2}, 0\right)$ and radius = $\frac{a}{2}$

Solving for point of intersection

$a(1 - cos\theta) = acos\theta$

$2cos\theta = 1$

$cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

Now,

$r = a(1 - cos\theta)$

$\frac{dr}{d\theta} = a(0 - -sin\theta)$

$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 4a^2\sin^2\frac{\theta}{2}$

Cos2θ = 1 - 2sin²θ

$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$

$$l(AB) = \int_{\frac{\pi}{3}}^{\pi} 2a \sin\frac{\theta}{2}\, d\theta$$

$$l(AB) = 2a\left[-\frac{\cos\frac{\theta}{2}}{\frac{1}{2}}\right]_{\frac{\pi}{3}}^{\pi}$$

$$l(AB) = 2a\left[-2\cos\frac{\pi}{2} - -2\cos\frac{\pi}{6}\right]$$

$$l(AB) = 2a\left[-2(0) + 2\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$l(AB) = 2\sqrt{3}\, a$$

Thus, required length $= 2l(AB) = 4\sqrt{3}\, a$

30.	Evaluate the integral $\int\int\int \frac{dx\,dy\,dz}{x^2+y^2+z^2}$ over throughout the volume of the sphere $x^2+y^2+z^2=a^2$.

Solution:

Put $x = rsin\theta cos\emptyset,$

$y = rsin\theta sin\emptyset,$

$z = rcos\theta$

$dxdydz = r^2 sin\theta drd\theta d\emptyset$

Also, $x^2+y^2+z^2 = r^2$

$$I = 8\int_{\theta=0}^{\frac{\pi}{2}}\int_{\emptyset=0}^{\frac{\pi}{2}}\int_{r=0}^{a}\frac{r^2 sin\theta drd\theta d\emptyset}{r^2}$$

$$I = 8\int_0^{\frac{\pi}{2}} sin\theta d\theta \int_0^{\frac{\pi}{2}} d\emptyset \int_0^a dr$$

$$I = 8.\frac{1}{2}B\left(\frac{1+1}{2},\frac{0+1}{2}\right)\times[\emptyset]_0^{\frac{\pi}{2}}\times[r]_0^a$$

$$I = 4.\frac{\Gamma(1).\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}.\frac{\pi}{2}.a$$

$$I = 4.\frac{1}{\frac{1}{2}}.\frac{\pi}{2}.a$$

$$\boxed{I = 4\pi a}$$

DE

31.	Solve the DE $(xy - 2y^2)\,dx - (x^2 - 3xy)\,dy = 0$

Solution:

$M = xy - 2y^2$ $\quad$ $N = -x^2 + 3xy$

$\frac{\partial M}{\partial y} = x - 4y$ $\quad$ $\frac{\partial N}{\partial x} = -2x + 3y$

Not exact

Consider, $Mx + Ny = x^2y - 2xy^2 - x^2y + 3xy^2 = xy^2$

$IF = \frac{1}{Mx+Ny} = \frac{1}{xy^2}$

Multiplying the DE by IF to make it exact

$$\left(\frac{xy}{xy^2} - \frac{2y^2}{xy^2}\right)dx + \left(-\frac{x^2}{xy^2} + \frac{3xy}{xy^2}\right)dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(-\frac{x}{y^2} + \frac{3}{y}\right)dy = 0$$

Solution,

$$\int M dx + \int N dy = C$$

$$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}dy = C$$

$$\boxed{\frac{x}{y} - 2\log x + 3\log y = C}$$

Imp : Major Hating marks cutting

HOD

32.	Solve the DE $(D^2 - 2D + 1)y = x^2 e^{3x}$, where $D \equiv \frac{d}{dx}$

Solution:

Direct hit

C.F.:

Put $D^2 - 2D + 1 = 0$

$\therefore D = 1, 1$

Thus, the C.F. can be written as

$y = (c_1 x + c_2)e^x$

P.I.:

$$y = \frac{1}{f(D)} X$$

$$y = \frac{1}{D^2-2D+1} x^2 e^{3x}$$

$$y = e^{3x} \frac{1}{(D+3)^2-2(D+3)+1} x^2$$

$$y = e^{3x} \frac{1}{D^2+6D+9-2D-6+1} x^2$$

$$y = e^{3x} \frac{1}{D^2+4D+4} x^2$$

$$y = \frac{e^{3x}}{4} \cdot \frac{1}{1+D+\frac{D^2}{4}} x^2$$

$$y = \frac{e^{3x}}{4} \cdot \left[1 + D + \frac{D^2}{4}\right]^{-1} x^2$$

$$y = \frac{e^{3x}}{4} \left[1 - \left(D + \frac{D^2}{4}\right) + \left(D + \frac{D^2}{4}\right)^2\right] x^2$$

$$y = \frac{e^{3x}}{4} \left[1 - D - \frac{D^2}{4} + D^2\right] x^2$$

$$y = \frac{e^{3x}}{4} \left[x^2 - 2x - \frac{2}{4} + 2\right]$$

$$y = \frac{e^{3x}}{4} \left[x^2 - 2x + \frac{3}{2}\right]$$

G.S.:

$y = C.F. + P.I.$

$$\boxed{y = (c_1 x + c_2)e^x + \frac{e^{3x}}{4}\left[x^2 - 2x + \frac{3}{2}\right]}$$

33. Express into polar form and evaluate the integral

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dx\, dy$$

Solution:

The region of integration is,

$x = 0$ ⇒ Y axis

$x = a$ ⇒a line parallel to Y axis passing through (a,0)

$y = 0$ ⇒ X axis

$y = \sqrt{a^2 - x^2}$

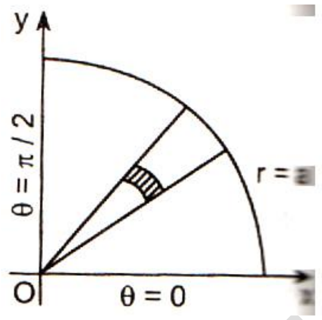
$x^2 + y^2 = a^2$ ⇒a circle with centre at (0,0) and radius a

Now changing it into polar co-ordinates

i.e. put $x = rcos\theta \; and \; y = rsin\theta, dxdy = rdrd\theta$

we get the region of integration as,

$r = 0 \;\&\; r = a\;, \theta = 0 \;\&\; \theta = \frac{\pi}{2}$ as shown in the figure.



$$I = \int_0^{\frac{\pi}{2}} \int_0^a e^{-r^2} r dr d\theta$$

Put $r^2 = t, 2rdr = dt$

When $r = 0, t = 0$ and when $r = a, t = a^2$

$$I = \int_0^{\frac{\pi}{2}} \int_0^{a^2} e^{-t} \frac{dt}{2}\, d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \left[\frac{e^{-t}}{-2}\right]_0^{a^2} d\theta$$

$$I = \left[\frac{e^{-a^2}}{-2} - \frac{e^0}{-2}\right] [\theta]_0^{\frac{\pi}{2}}$$

$$I = \left[\frac{1-e^{-a^2}}{2}\right]\left[\frac{\pi}{2}\right]$$

$$I = \frac{\pi\left(1-e^{-a^2}\right)}{4}$$

Imp

34.	Evaluate the integral $\int\int\int\sqrt{x^2+y^2}\,dx\,dy\,dz$ over the region bounded by $x^2+y^2=z^2\,,z>0\;\;and\;z=0\,,z=1.$

Solution:

$$I=\int\int\int\sqrt{x^2+y^2}\,dx\,dy\,dz$$

Since the region is bounded by $z=0$ and $z=1$

$$I=\int\int\int_{z=0}^{z=1}\sqrt{x^2+y^2}\,dxdydz$$

$$I=\int\int\sqrt{x^2+y^2}\,[z]_0^1 dx\,dy$$

$$I=\int\int\sqrt{x^2+y^2}\,dx\,dy$$

Now the region is bounded in the x-y plane by $x^2+y^2=1$ i.e. a standard circle with radius 1

Putting $x=r\cos\theta\,,y=r\sin\theta\,,dxdy=r\,dr\,d\theta$

$$I=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\sqrt{r^2}\,r\,dr\,d\theta$$

$$I=\int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^1 d\theta$$

$$I=\frac{1}{2}[\theta]_0^{2\pi}$$

$$I=\frac{1}{2}[2\pi]=\pi$$

BT

35.	Prove that $\int_0^\infty x\, e^{-x^8} dx \,.\int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}$

Solution:

Let $I_1 = \int_0^\infty x e^{-x^8} dx$

Put $x^8 = j$

$x = j^{\frac{1}{8}}$

$dx = \frac{1}{8} j^{-\frac{7}{8}} dj$

$I_1 = \int_0^\infty j^{\frac{1}{8}}.e^{-j}.\frac{1}{8} j^{-\frac{7}{8}} dj$

$I_1 = \frac{1}{8}\int_0^\infty e^{-j}.j^{-\frac{6}{8}} dj$

$I_1 = \frac{1}{8}\Gamma\left(-\frac{6}{8}+1\right)$

$I_1 = \frac{1}{8}\Gamma\left(\frac{1}{4}\right)$

let $I_2 = \int_0^\infty x^2 e^{-x^4} dx$

put $x^4 = k$

$x = k^{\frac{1}{4}}$

$dx = \frac{1}{4}.k^{-\frac{3}{4}} dk$

$I_2 = \int_0^\infty k^{\frac{2}{4}}.e^{-k}.\frac{1}{4}.k^{-\frac{3}{4}} dk$

$I_2 = \frac{1}{4}\int_0^\infty e^{-k}.k^{-\frac{1}{4}} dk$

$I_2 = \frac{1}{4}\Gamma\left(-\frac{1}{4}+1\right)$

$I_2 = \frac{1}{4}\Gamma\left(\frac{3}{4}\right)$

Thus,

$I = I_1 \times I_2$

$I = \frac{1}{8}\Gamma\left(\frac{1}{4}\right) \times \frac{1}{4}\Gamma\left(\frac{3}{4}\right)$

$I = \frac{1}{32}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

$I = \frac{1}{32}.\sqrt{2}\,\pi$

$I = \frac{\pi}{16\sqrt{2}}$

Duplic

① $\Gamma m\, \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma 2m$

② $\Gamma p\, \Gamma(1-p) = \frac{\pi}{\sin(p\pi)}$

36.	Show that the length of the parabola $x^2 = 4y$ which lies inside the circle $x^2 + y^2 = 6y$ is $2[\sqrt{6} + \log(\sqrt{2} + \sqrt{3})]$.

Solution:

Solving for the point of intersection of parabola $x^2 = 4y$ and

circle $x^2 + y^2 = 6y$, we get

$$4y + y^2 = 6y$$

$$y^2 + 4y - 6y = 0$$

$$y^2 - 2y = 0$$

$$y(y-2) = 0$$

$$y = 0, y = 2$$

$$x = 0, x = \pm\sqrt{8} = \pm 2\sqrt{2}$$

Thus, the points of intersection are $O(0,0), B(2\sqrt{2}, 2), A(-2\sqrt{2}, 2)$

Now,

$$x^2 + y^2 = 6y$$

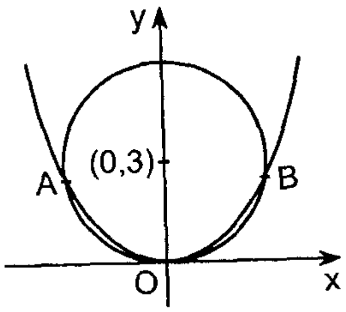
$$x^2 + y^2 - 6y = 0$$

$$x^2 + y^2 - 6y + 9 = 0 + 9$$

$$x^2 + (y-3)^2 = 9$$

$(\boldsymbol{x} - \boldsymbol{h})^2 + (\boldsymbol{y} - \boldsymbol{k})^2 = \boldsymbol{r}^2$ **where** $(\boldsymbol{h}, \boldsymbol{k})$ **is centre and** $\boldsymbol{r}$ **is the radius**

Centre is $(0,3)$ and radius is 3



Now,

$$x^2 = 4y$$

$$y = \frac{x^2}{4}$$

$$\frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2}$$

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,.dx$$

$$l(OB) = \int_0^{2\sqrt{2}} \sqrt{1 + \frac{x^2}{4}}\,.dx$$

$$l(OB) = \frac{1}{2}\int_0^{2\sqrt{2}} \sqrt{4 + x^2}.dx$$

$$l(OB) = \frac{1}{2}\left[\frac{x\sqrt{4+x^2}}{2} + \frac{4}{2}\log\left(x + \sqrt{4 + x^2}\right)\right]_0^{2\sqrt{2}}$$

$$l(OB) = \frac{1}{2}\left[\frac{2\sqrt{2}\sqrt{12}}{2} + 2\log\left(2\sqrt{2} + \sqrt{12}\right) - 0 - 2\log\left(0 + \sqrt{4}\right)\right]$$

$$l(OB) = \frac{1}{2}\left[\frac{4\sqrt{6}}{2} + 2\log\left(2\sqrt{2} + 2\sqrt{3}\right) - 2log2\right]$$

$$l(OB) = \frac{1}{2}\left[2\sqrt{6} + 2\log\left(\frac{2\sqrt{2}+2\sqrt{3}}{2}\right)\right]$$

$$l(OB) = \sqrt{6} + \log\left(\sqrt{2} + \sqrt{3}\right)$$

Thus,

Required length $= 2l(OB) = 2\left[\sqrt{6} + \log\left(\sqrt{2} + \sqrt{3}\right)\right]$

$$\int\sqrt{a^2 \pm x^2} = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\log\left|x + \sqrt{x^2 \pm a^2}\right| + c$$

37.	Solve the following Differential equation $$y\frac{dy}{dx}+\frac{4x}{3}-\frac{y^2}{3x}=0$$

Solution:

$$y\frac{dy}{dx}+\frac{4x}{3}-\frac{y^2}{3x}=0$$

$$y\frac{dy}{dx}-\frac{y^2}{3x}=-\frac{4x}{3}$$

Put $y^2=V$

$$2y\frac{dy}{dx}=\frac{dV}{dx}$$

$$y\frac{dy}{dx}=\frac{1}{2}\frac{dV}{dx}$$

Thus, the DE becomes

$$\frac{1}{2}\frac{dV}{dx}-\frac{V}{3x}=-\frac{4x}{3}$$

$$\frac{dV}{dx}-\frac{2}{3x}.V=-\frac{8x}{3}$$

It is of linear form,

$$P=-\frac{2}{3x},Q=-\frac{8x}{3}$$

$$IF=e^{\int Pdx}=e^{\int\frac{-2}{3x}dx}=e^{\frac{-2}{3}logx}=e^{\log x^{-\frac{2}{3}}}=x^{\frac{-2}{3}}$$

Its solution,

$$V(IF)=\int Q(IF)dx+C$$

$$y^2.x^{\frac{-2}{3}}=\int-\frac{8}{3}x.x^{\frac{-2}{3}}dx+C$$

$$\frac{y^2}{x^{\frac{2}{3}}}=-\frac{8}{3}\int x^{\frac{1}{3}}dx+C$$

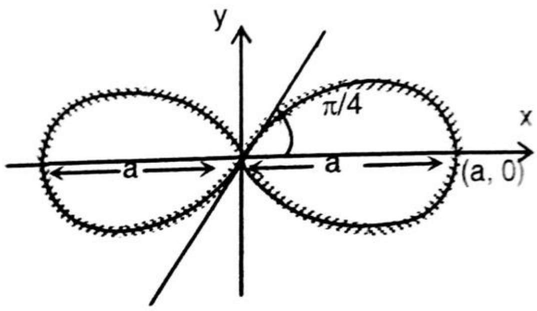
$$\frac{y^2}{x^{\frac{2}{3}}}=-\frac{8}{3}.\frac{x^{\frac{4}{3}}}{\frac{4}{3}}+C$$

$$\boxed{\frac{y^2}{x^{\frac{2}{3}}}=-2x^{\frac{4}{3}}+C}$$

38.	Find total length of $r^2 = 16cos2\theta$

Solution:

$r^2 = 16cos2\theta$ is of the form $r^2 = a^2cos2\theta$ (Bernoulli's lemniscate)



$$r^2 = a^2\ cos2\theta$$

$$r = a\sqrt{cos2\theta}$$

$$\frac{dr}{d\theta} = a.\frac{1}{2\sqrt{cos2\theta}} \times -sin2\theta \times 2 = -\frac{asin2\theta}{\sqrt{cos2\theta}}$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2 \cos 2\theta + \frac{a^2 \sin^2 2\theta}{cos2\theta}$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{a^2 \cos^2 2\theta + a^2 \sin^2 2\theta}{cos2\theta}$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{a^2}{cos2\theta}$$

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$s = \int_0^{\frac{\pi}{4}} \sqrt{\frac{a^2}{cos2\theta}}\ d\theta$$

$$s = a\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{cos2\theta}}\, d\theta$$

Put $\quad 2\theta = \phi$

$$\theta = \frac{\phi}{2}$$

$$d\theta = \frac{d\phi}{2}$$

$$s = a\int_0^{\frac{\pi}{2}} (cos\phi)^{-\frac{1}{2}}.\frac{d\phi}{2}$$

$$s = \frac{a}{2}\int_0^{\frac{\pi}{2}} \sin^0 \phi \ \cos^{-\frac{1}{2}} \phi \, d\phi$$

$$s = \frac{a}{2}.\frac{1}{2}B\left(\frac{0+1}{2}, \frac{-\frac{1}{2}+1}{2}\right)$$

$$s = \frac{a}{4} \ B\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$s = \frac{a}{4}.\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{4}\right)} = \frac{a\sqrt{\pi}}{4}.\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi$$

$$s = \frac{a\sqrt{\pi}}{4}.\frac{\Gamma\left(\frac{1}{4}\right)}{\frac{\sqrt{2}\pi}{\Gamma\left(\frac{1}{4}\right)}}$$

$$s = \frac{a\sqrt{\pi}}{4\sqrt{2}\pi}.\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

Total length $= 4s = \frac{a}{\sqrt{2\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$

Put $a = 4$ now,

Total length $= \frac{4}{\sqrt{2\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2 = 2\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$

HOD

39.	Solve $\frac{d^2y}{dx^2} + 4y = tan2x by Variation of parameters$

Solution:

(D^2y+4y)
$(D^2+4)y$

<u>C.F.:</u>

Put $D^2 + 4 = 0$

$\therefore D = \pm 2i$

Thus, the C.F. can be written as

$y = c_1 cos2x + c_2 sin2x$

<u>P.I.:</u>

Let P.I. be $y = uy_1 + vy_2$

where $y_1 = cos2x, y_2 = sin2x, X = tan2x$

Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} cos2x & sin2x \\ -2sin2x & 2cos2x \end{vmatrix} = 2$$

$$\therefore u = -\int \frac{y_2 X}{W} dx$$

$$\therefore u = -\int \frac{sin2x}{2} tan2x dx = -\frac{1}{2}\int \frac{\sin^2 2x}{cos2x} dx$$

$$\therefore u = -\frac{1}{2}\int \frac{1-\cos^2 2x}{cos2x} dx = -\frac{1}{2}\int sec2x - cos2x\, dx$$

$$\therefore u = -\frac{1}{2}\left[\frac{\log(sec2x+tan2x)}{2} - \frac{sin2x}{2}\right]$$

formula

$$\therefore v = \int \frac{y_1 X}{W} dx$$

$$\therefore v = \int \frac{cos2x}{2} tan2x dx = \int \frac{sin2x}{2} dx = -\frac{cos2x}{4}$$

Thus,

$$y = cos2x\left[-\frac{\log(sec2x+tan2x)}{4} + \frac{sin2x}{4}\right] + sin2x\left[-\frac{cos2x}{4}\right]$$

$$\therefore y = -\frac{1}{4} cos2x \,.\log(sec2x + tan2x)$$

<u>G.S.:</u>

$y = C.F. + P.I.$

$$\boxed{y = c_1 cos2x + c_2 sin2x - \frac{1}{4} cos2x.\log(sec2x + tan2x)}$$

40.	Change the order of integration for the integral $\int_0^8 \int_{(y-8)/4}^{y/4} f(x,y)$

Solution:

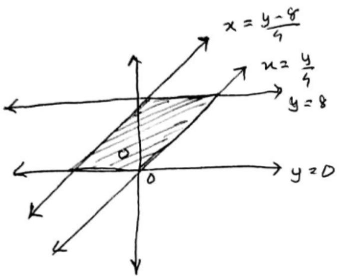
The region of integration is

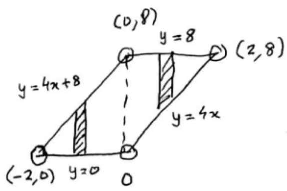
$y = 0 \Rightarrow$ X axis

$y = 8 \Rightarrow$ line parallel to X axis

$x = \frac{y-8}{4} \Rightarrow$ a straight line with intercepts -2 and 8

$x = \frac{y}{4} \Rightarrow$ line passing through origin





$$I = \int_{-2}^{0} \int_{0}^{4x+8} f(x,y)dxdy + \int_{0}^{2} \int_{4x}^{8} f(x,y)dxdy$$

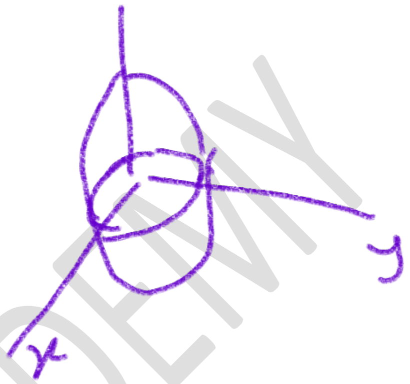
41.	Evaluate $\iiint dxdydz$ over the solid of the paraboloid $x^2 + y^2 = 4z$ cut off by the plane z = 4

Imp

Solution:

$I = \int \int \int dxdydz$

The region is given as $x^2 + y^2 = 4z$ and $z = 4$



$I = \int \int \int_{z=\frac{x^2+y^2}{4}}^{z=4} dxdydz$

$I = \int \int \left[4 - \frac{x^2+y^2}{4}\right] dxdy$

Now the region has become $x^2 + y^2 = 16$, a circle with centre at origin and radius 4

Putting $x = rcos\theta, y = rsin\theta, dxdy = rdrd\theta$

$I = \int_{\theta=0}^{2\pi} \int_{r=0}^{4} \left[4 - \frac{r^2}{4}\right] rdrd\theta$

$I = \int_0^{2\pi} \int_0^4 \left[4r - \frac{r^3}{4}\right] dr\, d\theta$

$I = [\theta]_0^{2\pi} \left[\frac{4r^2}{2} - \frac{r^4}{16}\right]_0^4$

$I = [2\pi][32 - 16]$

$I = 32\pi$

42.	Solve $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

Solution:

$$xy - \frac{dy}{dx} = y^3 e^{-x^2}$$

$$\frac{dy}{dx} - xy = -y^3 e^{-x^2}$$

$$\frac{1}{y^3}\frac{dy}{dx} - x.\frac{1}{y^2} = -e^{-x^2}$$

Put $\frac{1}{y^2} = V$

$$-\frac{2}{y^3}\frac{dy}{dx} = \frac{dV}{dx}$$

$$\frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dV}{dx}$$

$$-\frac{1}{2}\frac{dV}{dx} - x.V = -e^{-x^2}$$

$$\frac{dV}{dx} + 2x.V = 2e^{-x^2}$$

It is of the form linear

$$P = 2x, Q = 2e^{-x^2}$$

$$IF = e^{\int Pdx} = e^{\int 2xdx} = e^{x^2}$$

Solution,

$$V(IF) = \int Q(IF)dx + C$$

$$\frac{1}{y^2}e^{x^2} = \int 2e^{-x^2}e^{x^2}dx + C$$

$$\frac{e^{x^2}}{y^2} = \int 2dx + C$$

$$\frac{e^{x^2}}{y^2} = 2x + C$$

BT

43.	Evaluate $\int_0^\infty \frac{log(1+ax^2)}{x^2}dx$

Solution:

We have, $I(a) = \int_0^\infty \frac{log(1+ax^2)}{x^2}dx \ldots\ldots\ldots\ldots (1)$

By the rule of DUIS, differentiating w.r.t a, we get

$$\frac{\partial I}{\partial a} = \int_0^\infty \frac{1}{x^2}\frac{1}{1+ax^2}x^2dx = \int_0^\infty \frac{1}{1+ax^2}dx$$

$$\frac{\partial I}{\partial a} = \frac{1}{a}\int_0^\infty \frac{1}{\frac{1}{a}+x^2}dx$$

$$\frac{\partial I}{\partial a} = \frac{1}{a}\left[\frac{1}{\sqrt{\frac{1}{a}}}\tan^{-1}\left(\frac{x}{\sqrt{\frac{1}{a}}}\right)\right]_0^\infty$$

$$\frac{\partial I}{\partial a} = \frac{1}{\sqrt{a}}.\frac{\pi}{2}$$

Integrating, we get

$$I(a) = \frac{\pi}{2}.2\sqrt{a} + C \ldots\ldots\ldots\ldots\ldots (2)$$

Put a = 0 in equation (1) & (2), we get

$I(0) = 0$ & $I(0) = 0 + C$

$\therefore C = 0$

$\therefore \boxed{I(a) = \pi\sqrt{a}}$

44.	.Solve$(D^2-4D+1)y = e^{2x}sin5x$

HOP

Direct hit

Solution:

CF:

Put $D^2-4D+1=0$

$$D = 2+\sqrt{3}, 2-\sqrt{3}$$

$$y_c = c_1e^{(2+\sqrt{3})x} + c_2e^{(2-\sqrt{3})x}$$

PI:

$$y_p = \frac{1}{D^2-4D+1}\ e^{2x}sin5x$$

$$y_p = e^{2x}.\frac{1}{(D+2)^2-4(D+2)+1}sin5x$$

$$y_p = e^{2x}.\frac{1}{D^2+4D+4-4D-8+1}sin5x$$

$$y_p = e^{2x}.\frac{1}{D^2-3}sin5x$$

$$y_p = \frac{e^{2x}sin5x}{-5^2-3}$$

$$y_p = -\frac{e^{2x}sin5x}{28}$$

GS:

$$y = y_c + y_p = c_1e^{(2+\sqrt{3})x} + c_2e^{(2-\sqrt{3})x} - \frac{e^{2x}sin5x}{28}$$

45.	Change it to polar and Evaluate $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)dydx$

Solution:

The region of integration is,

$x = 0$ ⇒ Y axis

$x = 2a$ ⇒a line parallel to Y axis passing through (2a,0)

$y = 0$ ⇒ X axis

$y = \sqrt{2ax - x^2}$

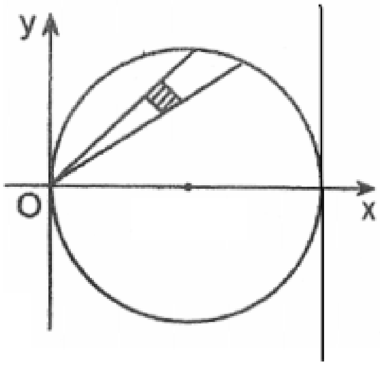
$(x-a)^2 + y^2 = a^2$ ⇒a circle with centre at (a,0) and radius a

Now changing it into polar co-ordinates

i.e. put $x = rcos\theta \ and \ y = rsin\theta, dxdy = rdrd\theta$

we get the region of integration as,

$r = 0 \ \& \ r = 2acos\theta \ , \theta = 0 \ \& \ \theta = \frac{\pi}{2}$as shown in the figure.



$$I = \int_0^{\frac{\pi}{2}}\int_0^{2acos\theta} r^2.rdrd\theta$$

$$I = \int_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^{2acos\theta} d\theta$$

$$I = \int_0^{\frac{\pi}{2}}\left[\frac{16a^4\cos^4\theta}{4}\right]d\theta$$

$$I = 4a^4 \times \frac{1}{2}B\left(\frac{0+1}{2},\frac{4+1}{2}\right)$$

$$I = 2a^4B\left(\frac{1}{2},\frac{5}{2}\right)$$

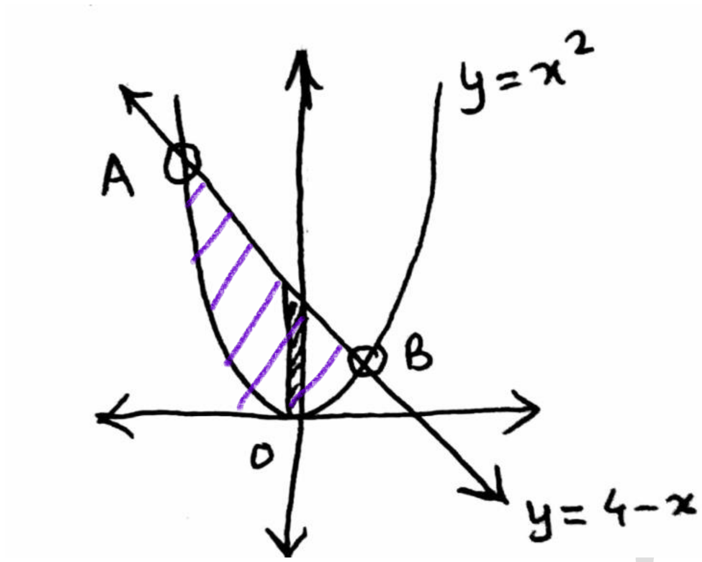
$$I = 2a^4.\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)}$$

$$I = 2a^4.\frac{\Gamma\left(\frac{1}{2}\right).\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2}$$

$$I = \frac{3\pi a^4}{4}$$

46.	Evaluate $\iint x dx dy$ throughout the area bounded by $y = x^2$ and $y = 4 - x$

Solution:



Solving for the point of intersections A and B, we get

$x^2 = 4 - x$

$x^2 + x - 4 = 0$

$x = \frac{-1+\sqrt{17}}{2}, \frac{-1-\sqrt{17}}{2}$

$y = \frac{9-\sqrt{17}}{2}, \frac{9+\sqrt{17}}{2}$

Thus, $A = \left(\frac{-1-\sqrt{17}}{2}, \frac{9+\sqrt{17}}{2}\right), B = \left(\frac{-1+\sqrt{17}}{2}, \frac{9-\sqrt{17}}{2}\right)$

Considering vertical strip, we get

$$I = \int_{x=\frac{-1-\sqrt{17}}{2}}^{x=\frac{-1+\sqrt{17}}{2}} \int_{y=x^2}^{y=4-x} x\, dx\, dy$$

$$I = \int_{\frac{-1-\sqrt{17}}{2}}^{\frac{-1+\sqrt{17}}{2}} [x.y]_{x^2}^{4-x} dx$$

$$I = \int_{\frac{-1-\sqrt{17}}{2}}^{\frac{-1+\sqrt{17}}{2}} [x(4-x) - x^3] dx$$

$$I = \left[4\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\right]_{\frac{-1-\sqrt{17}}{2}}^{\frac{-1+\sqrt{17}}{2}}$$

$$I=\left\{2\left(\frac{-1+\sqrt{17}}{2}\right)^2-\frac{\left(\frac{-1+\sqrt{17}}{2}\right)^3}{3}-\frac{\left(\frac{-1+\sqrt{17}}{2}\right)^4}{4}\right\}-\left\{2\left(\frac{-1-\sqrt{17}}{2}\right)^2-\frac{\left(\frac{-1-\sqrt{17}}{2}\right)^3}{3}-\frac{\left(\frac{-1-\sqrt{17}}{2}\right)^4}{4}\right\}$$

$$I=\left\{\frac{121-17\sqrt{17}}{24}\right\}-\left\{\frac{121+17\sqrt{17}}{24}\right\}$$

$$I=-\frac{17\sqrt{17}}{12}$$

47.	Solve: $\frac{dy}{dx} = \frac{y^3}{e^{2x} + y^2}$

Solution:

We have,

$$\frac{dy}{dx} = \frac{y^3}{e^{2x}+y^2}$$

$$e^{2x} + y^2 = \frac{dx}{dy} y^3$$

$$\frac{dx}{dy} - \frac{1}{y} = \frac{e^{2x}}{y^3}$$

Dividing by e^{2x},

$$e^{-2x}\frac{dx}{dy} - e^{-2x}.\frac{1}{y} = \frac{1}{y^3}$$

Put $e^{-2x} = V, \therefore -2e^{-2x}\frac{dx}{dy} = \frac{dV}{dy}, \therefore e^{-2x}\frac{dx}{dy} = -\frac{1}{2}\frac{dV}{dy}$

$$-\frac{1}{2}\frac{dV}{dy} - V.\frac{1}{y} = \frac{1}{y^3}$$

$$\frac{dV}{dy} + \frac{2}{y}V = -\frac{2}{y^3}$$

It is of the form, $\frac{dV}{dy} + PV = Q$ where, $P = \frac{2}{y}$ & $Q = -\frac{2}{y^3}$

$$\therefore I.F. = e^{\int \frac{2}{y} dy} = e^{2logy} = e^{logy^2} = y^2$$

Its solution is given by,

$$V(I.F.) = \int Q(I.F.)dy + C$$

$$e^{-2x}y^2 = \int -\frac{2}{y^3}.y^2 dy + C$$

$$e^{-2x}y^2 = -2\int \frac{1}{y}dy + C$$

$$\boxed{e^{-2x}y^2 = -2logy + C}$$

HOD

48.	Solve: $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$, using method of variation of parameters

Solution:

C.F.:

Put $D^2 - 1 = 0$

$\therefore D = 1, -1$

Thus, the C.F. can be written as

$y = c_1e^x + c_2e^{-x}$

P.I.:

Let P.I. be $y = uy_1 + vy_2$

where $y_1 = e^x, y_2 = e^{-x}, X = \frac{2}{1+e^x}$

Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1 - 1 = -2$$

$$\therefore u = -\int \frac{y_2X}{W} dx$$

$$\therefore u = -\int \frac{e^{-x}}{-2} \cdot \frac{2}{1+e^x} dx = \int \frac{e^{-x}}{1+e^x} dx = \int \frac{1}{e^x(1+e^x)} dx = \int \frac{1}{e^x} - \frac{1}{1+e^x} dx$$

$$\therefore u = \int e^{-x} dx - \int \frac{e^{-x}}{e^{-x}+1} dx = -e^{-x} + \log(e^{-x} + 1)$$

$$\therefore v = \int \frac{y_1X}{W} dx$$

$$\therefore v = \int \frac{e^x}{-2} \cdot \frac{2}{1+e^x} dx = -\int \frac{e^x}{1+e^x} dx = -\log(1 + e^x)$$

Thus,

$$y = [-e^{-x} + \log(e^x + 1)]e^x - \log(1 + e^x)\, e^{-x}$$

$$\therefore y = -1 + e^x \log(e^x + 1) - e^{-x} \log(e^x + 1)$$

G.S.:

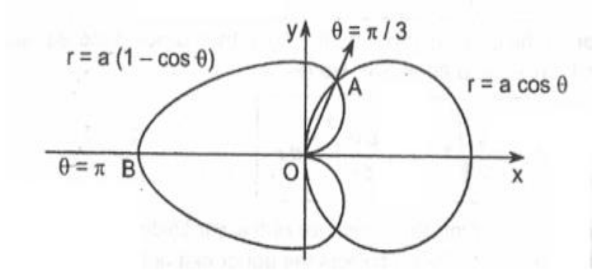
$$y = C.F. + P.I.$$

$$\boxed{y = c_1e^x + c_2e^{-x} - 1 + e^x \log(e^x + 1) - e^{-x} \log(e^x + 1)}$$

Direct hit

49.	Find the length of the cardioid r= a(1-cosθ) lying outside the circle r = acosθ.

Solution:



The circle and the cardiode are shown in the figure. They intersect where

$a(1-cos\theta) = acos\theta \quad i.e. 1 - cos\theta = cos\theta$

$cos\theta = \frac{1}{2} \quad \therefore \theta = \frac{\pi}{3}$

The length of the cardiode outside the circle is 2 arc AB where for B, $\theta = \pi$ and for A, $\theta = \frac{\pi}{3}$

$\therefore s = 2\int_{\frac{\pi}{3}}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$s = 2\int_{\frac{\pi}{3}}^{\pi} \sqrt{a^2(1-cos\theta)^2 + a^2sin^2\theta}\, d\theta$

$s = 2a\int_{\frac{\pi}{3}}^{\pi} \sqrt{1 - 2cos\theta + cos^2\theta + sin^2\theta}\, d\theta$

$s = 2a\int_{\frac{\pi}{3}}^{\pi} \sqrt{2(1-cos\theta)}\, d\theta$ — $Cos2\theta = 2cos^2\theta - 1$

$= 1 - 2sin^2\theta$

$= cos^2\theta - sin^2\theta$

$s = 2a\int_{\frac{\pi}{3}}^{\pi} \sqrt{2.2sin^2\frac{\theta}{2}}\, d\theta$

$s = 4a\int_{\frac{\pi}{3}}^{\pi} sin\frac{\theta}{2}\, d\theta$

$s = 4a\left[-\frac{cos\frac{\theta}{2}}{\frac{1}{2}}\right]_{\frac{\pi}{3}}^{\pi}$

$s = -8a\left[cos\frac{\pi}{2} - cos\frac{\pi}{6}\right]$

$s = 4\sqrt{3}a$

~~50~~	Change the order of integration and evaluate $\int_0^5 \int_{3-x}^{3+x} dx\, dy$

Solution:

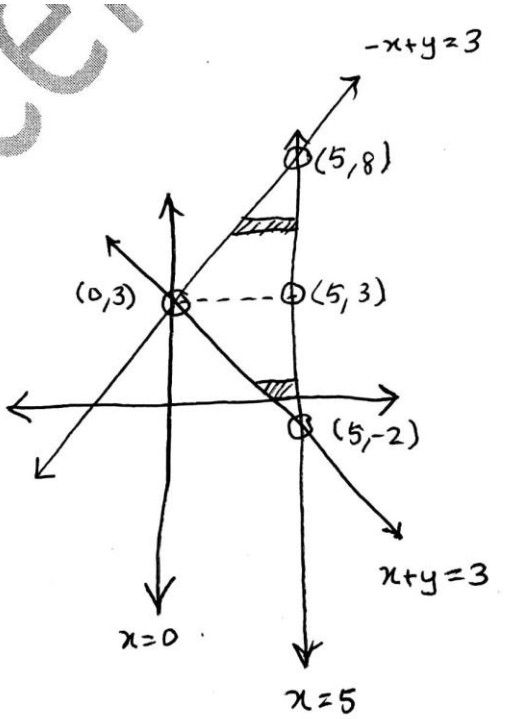
The region of integration is

$x = 0 \Rightarrow$ Y axis

$x = 5 \Rightarrow$ line parallel to Y axis

$y = 3 - x \Rightarrow$ line with intercepts 3 and 3

$y = 3 + x \Rightarrow$ line with intercepts -3 and 3



$I = \int_{-2}^{3} \int_{3-y}^{5} dxdy + \int_{3}^{8} \int_{y-3}^{5} dxdy$

$I = \int_{-2}^{3} [5 - (3 - y)]dy + \int_{3}^{8} [5 - (y - 3)]dy$

$I = \left[2\, y + \frac{y^2}{2}\right]_{-2}^{3} + \left[8\, y - \frac{y^2}{2}\right]_{3}^{8}$

$I = \left[6 + \frac{9}{2} + 4 - 2\right] + \left[64 - 32 - 24 + \frac{9}{2}\right]$

$I = 25$

	Evaluate: $\iiint \frac{dx\,dy\,dz}{x^2+y^2+z^2}$ throughout the volume of the sphere $x^2+y^2+z^2 = a^2$

Solution:

Put $x = r\sin\theta\cos\emptyset,$
$y = r\sin\theta\sin\emptyset,$
$z = r\cos\theta$
$dxdydz = r^2\sin\theta drd\theta d\emptyset$

Also, $x^2+y^2+z^2 = r^2$

$$I = 8\int_{\theta=0}^{\frac{\pi}{2}}\int_{\emptyset=0}^{\frac{\pi}{2}}\int_{r=0}^{a}\frac{r^2\sin\theta drd\theta d\emptyset}{r^2}$$

$$I = 8\int_0^{\frac{\pi}{2}}\sin\theta d\theta\int_0^{\frac{\pi}{2}}d\emptyset\int_0^{a}dr$$

$$I = 8.\frac{1}{2}B\left(\frac{1+1}{2},\frac{0+1}{2}\right)\times[\emptyset]_0^{\frac{\pi}{2}}\times[r]_0^{a}$$

$$I = 4.\frac{\Gamma(1).\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}.\frac{\pi}{2}.a$$

$$I = 4.\frac{1}{\frac{1}{2}}.\frac{\pi}{2}.a$$

$$\boxed{I = 4\pi a}$$

52.	Solve: $\frac{dx}{dy}+y^3\sin^2x+y\sin 2x=y^3$

Solution:

$$\frac{dx}{dy}+y^3\sin^2 x+y\,sin2x=y^3$$

$$\frac{dx}{dy}+y\,sin2x=y^3-y^3\sin^2 x$$

$$\frac{dx}{dy}+y\,sin2x=y^3(1-\sin^2 x)$$

$$\frac{dx}{dy}+y\,sin2x=y^3\cos^2 x$$

$$\frac{1}{\cos^2 x}\frac{dx}{dy}+y.\frac{2sinxcosx}{\cos^2 x}=y^3$$

$$\sec^2 x\frac{dx}{dy}+2y\,tanx=y^3$$

Put $tanx=V$

$$\sec^2 x\frac{dx}{dy}=\frac{dV}{dy}$$

Thus, $\frac{dV}{dy}+2y.V=y^3$

It is of the form linear

$$P=2y,Q=y^3$$

$$IF=e^{\int Pdy}=e^{\int 2ydy}=e^{y^2}$$

Solution,

$$V(IF)=\int Q(IF)dy+C$$

$$tanx.e^{y^2}=\int y^3e^{y^2}dy+C$$

$$tanx.e^{y^2}=\int y^2.e^{y^2}.ydy+C$$

Put $y^2=t,2ydy=dt,ydy=\frac{dt}{2}$

$$tanx.e^{y^2}=\int t.e^t\frac{dt}{2}+C$$

$$tanx.e^{y^2}=\frac{1}{2}[te^t-e^t]+C$$

$$tanx.e^{y^2}=\frac{1}{2}\left[y^2e^{y^2}-e^{y^2}\right]+C$$

HOD

53	Solve : $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = \frac{e^{-2x}}{x^5}$

Solution:

CF:

Put $D^2 + 4D + 4 = 0$

$(D+2)^2 = 0$

$D = -2, -2$

$y_c = (c_1 + c_2 x)e^{-2x}$

PI:

$y = \frac{1}{f(D)} X$

$y_p = \frac{1}{D^2+4D+4} . \frac{e^{-2x}}{x^5}$

$y_p = e^{-2x} . \frac{1}{(D-2)^2+4(D-2)+4} . \frac{1}{x^5}$

$y_p = e^{-2x} . \frac{1}{D^2-4D+4+4D-8+4} . x^{-5}$

$y_p = e^{-2x} . \frac{1}{D^2} x^{-5}$

$y_p = e^{-2x} . \int\int x^{-5} dx\, dx$

$y_p = e^{-2x} . \int \left[\frac{x^{-4}}{-4}\right] dx$

$y_p = e^{-2x} \left[\frac{x^{-3}}{12}\right]$

$y_p = \frac{e^{-2x}}{12x^3}$

GS:

$y = y_c + y_p$

$y = (c_1 + c_2 x)e^{-2x} + \frac{e^{-2x}}{12x^3}$

Direct hit

54.	Prove that $\int_0^\infty \frac{1-\text{coaax}}{\text{x}} e^{-x}\, dx = \frac{1}{2}\log(1+a^2)$, assuming the validity of differentiation under the integral sign.

Solution:

$$I(a) = \int_0^\infty \left(\frac{1-cosax}{x}\right) e^{-x}\, dx \qquad \text{....... (1)}$$

By DUIS, diff w.r.t a,

$$I'(a) = \int_0^\infty \frac{0+sinax \times x}{x} . e^{-x}\, dx$$

$$I'(a) = \int_0^\infty e^{-x} sinax\, dx$$

$$\int e^{ax} sinbx\, dx = \frac{e^{ax}}{a^2+b^2}[a\, sinbx - b\, cosbx]$$

$$I'(a) = \left[\frac{e^{-x}}{(-1)^2+a^2}(-1\, sinax - a\, cosax)\right]_0^\infty$$

$$I'(a) = \left[\{0\} - \left\{\frac{e^0}{1+a^2}(-1\, sin0 - a\, cos0)\right\}\right]$$

$$I'(a) = \frac{a}{1+a^2}$$

Integrating w.r.t a,

$$I(a) = \int \frac{a}{1+a^2} da = \frac{1}{2}\int \frac{2a}{1+a^2} da$$

$$I(a) = \frac{1}{2}\log(1+a^2) + C \qquad \text{..... (2)}$$

Put $a = 0$ in (1) & (2)

$I(0) = 0$ and $I(0) = \frac{1}{2}\log 1 + C$

$$0 = 0 + C$$

$$C = 0$$

Thus,

$$I(a) = \frac{1}{2}\log(1+a^2)$$

$\iint$

55. Evaluate : $\iint_R \frac{dx\,dy}{\sqrt{(1+x^2+y^2)^2}}$ over one loop of the lemniscate $(x^2+y^2)^2 = x^2 - y^2$

Note: I think there is a mistake in the question. The question must be as follows

Evaluate $\iint_R \frac{dxdy}{(1+x^2+y^2)^2}$ **over one loop of lemniscates** $(x^2+y^2)^2 = x^2 - y^2$

$(x^2+y^2)^2 = a^2(x^2-y^2)$

Solution:

The region of integration is given by,

$(x^2+y^2)^2 = x^2 - y^2$

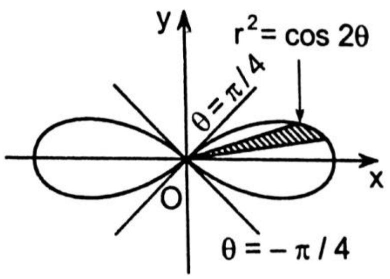
Put $x = r\cos\theta, y = r\sin\theta$ and $dxdy = rdrd\theta$

$(r^2\cos^2\theta + r^2\sin^2\theta)^2 = r^2\cos^2\theta - r^2\sin^2\theta$

$(r^2)^2 = r^2\cos 2\theta$

$r^2 = \cos 2\theta$

$r^2 = a^2\cos 2\theta$



$I = 2\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}} \frac{rdrd\theta}{(1+r^2)^2}$

Put $(1+r^2) = t$

$rdr = \frac{dt}{2}$

when $r = 0, t = 1$ and when $r = \sqrt{\cos 2\theta}, t = 1 + \cos 2\theta$

$I = 2\int_0^{\frac{\pi}{4}}\int_1^{1+\cos 2\theta} \frac{1}{t^2}\frac{dt}{2}.d\theta$

$I = \int_0^{\frac{\pi}{4}}\left[-\frac{1}{t}\right]_1^{1+\cos 2\theta} d\theta$

$I = -\int_0^{\frac{\pi}{4}}\frac{1}{1+\cos 2\theta} - 1.d\theta$

$I = -\int_0^{\frac{\pi}{4}}\left(\frac{1}{2\cos^2\theta} - 1\right)d\theta$

$I = -\int_0^{\frac{\pi}{4}}\frac{\sec^2\theta}{2} - 1.d\theta$

$I = -\left[\frac{\tan\theta}{2} - \theta\right]_0^{\frac{\pi}{4}} = -\left[\frac{1}{2} - \frac{\pi}{4}\right]$

$\boxed{I = \frac{\pi}{4} - \frac{1}{2}}$

SS

56.	Change to polar co-ordinates and evaluate $\int_0^1 \int_0^x x + y dy dx$

Solution:

Rnder $r = 0$ to $\sec(\theta)$

The region of integration is given by,

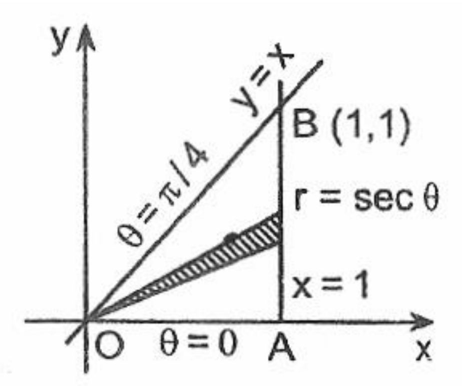
$x = 0$ ⇒ Y axis

$x = 1$ ⇒a line parallel to Y axis at (1,0)

$y = 0$ ⇒ X axis

$y = x$ ⇒a straight line passing through origin at 45^0

Now changing it into polar co-ordinates i.e. put $= r\cos\theta$ and $y = r\sin\theta$, we get the region of integration as, $r = 0$ & $r = \sec\theta$, $\theta = 0$ & $\theta = \frac{\pi}{4}$ as shown in the figure.



$$I = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} (r\cos\theta + r\sin\theta) r dr d\theta = \int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta) \int_0^{\sec\theta} r^2 dr\, d\theta$$

$$I = \int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta) \left[\frac{r^3}{3}\right]_0^{\sec\theta} d\theta = \frac{1}{3}\int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta)\sec^3\theta d\theta$$

$$I = \frac{1}{3}\left\{\int_0^{\frac{\pi}{4}} \sec^2\theta d\,\theta + \int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^3\theta} d\theta\right\}$$

Put $\cos\theta = t$, $\sin\theta d\theta = -dt$, thus when $\theta = 0$, $t = 1$ & when $\theta = \frac{\pi}{4}$, $t = \frac{1}{\sqrt{2}}$

$$I = \frac{1}{3}\left\{[\tan\theta]_0^{\frac{\pi}{4}} + \int_1^{\frac{1}{\sqrt{2}}} -\frac{1}{t^3} dt\right\}$$

$$I = \frac{1}{3}\left\{\left[\tan\frac{\pi}{4} - \tan 0\right] - \left[\frac{t^{-2}}{-2}\right]_1^{\frac{1}{\sqrt{2}}}\right\}$$

$$I = \frac{1}{3}\left\{1 + \frac{1}{2}[2 - 1]\right\}$$

$$\boxed{I = \frac{1}{2}}$$