Mathematical Foundations of Computer Science
Solutions to Homework Assignment 1
January 27, 2023

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1. Let p, q, and r be the following propositions.

p: You get an A on the final exam.

q: You do every exercise in the text book

r: You get an A in the course.

Express the following propositions using p, q, r and logical operators.

(a) You get an A in the course but you do not do every exercise in the text book.

(b) You get an A on the final exam, you do every exercise in the text book, and you get an A in the course.

(c) To get an A in the course it is necessary for you to get an A on the final exam.

(d) You get an A on the final, but you don't do every exercise in the text book; nevertheless, you get an A in the course.

(e) Getting an A on the final exam and doing every exercise in the text book is sufficient for getting an A in the course.

(f) You will get an A in the course if and only if you either do every exercise in the text book or you get an A on the final exam.

Solution.
(a) $r \wedge \neg q$
(b) $p \wedge q \wedge r$
(c) $\neg p \rightarrow \neg r \equiv r \rightarrow p$
(d) $p \wedge \neg q \wedge r$
(e) $(p \wedge q) \rightarrow r$
(f) $r \leftrightarrow (p \vee q)$

2. Rewrite the following formally using quantifiers and variables, and write a negation for the statement.

(a) Everybody loves somebody.

(b) Somebody loves everybody.

(c) Any even integer equals twice some other integer.

(d) There is a program that gives the correct answer to every question that is posed to it.

(e) There is a prime number between every integer and its double.

Solution.
(a) $\forall$ people p, $\exists$ a person q such that p loves q.
Negation: $\exists$ a person p such that $\forall$ people q, p does not love q.
(b) $\exists$ a person p such that $\forall$ people q, p loves q.
Negation: $\forall$ people p, $\exists$ a person q such that p does not love q.
(c) $\forall$ integers n, $\exists$ an integer l such that $n = 2l$.
Negation: $\exists$ an integer n such that $\forall$ integers l, $n \neq 2l$.
(d) $\exists$ a program p such that $\forall$ questions q posed to p, p gives a correct answer to q.
Negation: $\forall$ programs p, $\exists$ a question q that can be posed to p such that p does not give a correct answer to q.
(e) $\forall$ integers n, $\exists$ a prime number p such that $n \leq p \leq 2n$.
Negation: $\exists$ an integer n such that $\forall$ primes p, either $p < n$ or $p > 2n$.

3. Decide if the following proposition forms are a tautology using a truth table.

(a) $(p \vee q) \vee (\neg p \vee \neg q)$

(b) $(p \wedge q) \rightarrow (p \rightarrow q)$

Solution.
(a) The following truth table proves that the propositional form is a tautology.

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg p \vee \neg q$	$(p \vee q) \vee (\neg p \vee \neg q)$
T	T	F	F	T	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	T	T

(b) The following truth table shows that the propositional form is a tautology.

p	q	$p \wedge q$	$p \rightarrow q$	$(p \wedge q) \rightarrow (p \rightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

4. Prove or disprove the following.

(a) For every prime p, $p + 2$ is a prime.
(b) For all integers m and n, $m+n$ and $m-n$ are either both odd or both even.
(c) For any positive real numbers x and $y \leq x$, $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$.
(d) For all natural numbers x, $x^2 - x + 3$ is odd.
(e) For all natural numbers m, if m is even then m^7 is even

Solution. (a) The statement is false. $p = 7$ is a counterexample.
(b) The statement is true. We will prove this by contradiction. Without loss of generality assume that $m + n$ is even and $m - n$ is odd. Thus, for some integers k and l we have

$$\begin{aligned} m + n &= 2k \\ m - n &= 2l + 1 \end{aligned}$$

Adding the two equations we get

$$2m = 2(k + l) + 1$$

This is a contradiction since the left side is an even number and the right side is an odd number.
(c) The statement is false. Let $x = 3.1$ and $y = 2.9$. Then,

$$\lfloor x - y \rfloor = \lfloor 3.1 - 2.9 \rfloor = 0 \neq \lfloor x \rfloor - \lfloor y \rfloor = 1$$

(d) The statement is true. Here is the proof. Observe that

$$x^2 - x + 3 = x(x - 1) + 2 + 1$$

It is sufficient to show that $x(x - 1) + 2$ is even. Note that either x is even or $x - 1$ is even. Hence $x(x - 1)$ is even. Hence $x(x - 1) + 2$ is even.
(e) Let $m = 2k$ for some $k \in \mathbb{N}$. Thus $m^7 = (2k)^7 = 2(64m^7)$ which is even.

5. Suppose a, b, x, and y are integers. Prove that if $d|a$ and $d|b$, then $d|(ax + by)$.

Solution. Since $d|a$ and $d|b$ we can express a and b as $a = dk$ and $b = d\ell$, for some integers k and ℓ. Now we can write $ax + by$ as follows.

$$\begin{aligned} ax + by &= (dk)x + (d\ell)y \\ &= d(kx + \ell y) \\ &= dk' \end{aligned}$$

where $k' = kx + \ell y$ is an integer. This proves that $d|(ax + by)$.

6. Given any numbers x, y and z, if $x - y$ is odd and $y - z$ is even, is $x - z$ odd or even? Prove your claim.

Solution. Since $x - y$ is odd and $y - z$ is even, for some integers k and ℓ we can express these terms as

$$\begin{aligned} x - y &= 2k + 1 \\ x &= 2k + y + 1 && (1) \\ y - z &= 2\ell \\ z &= y - 2\ell && (2) \end{aligned}$$

Using (1) and (2) we can write $x - z$ as follows.

$$\begin{aligned} x - z &= (2k + y + 1) - (y - 2\ell) \\ &= 2(k + \ell) + 1 \\ &= 2k' + 1 \end{aligned}$$

where $k' = k + \ell$ is an integer. Thus $x - z$ has odd parity.

7. Let t be a positive integer. Prove the following statement by proving its contrapositive.

if r is irrational, then $r^{1/t}$ is irrational.

Be sure to state the contrapositive explicitly.

Solution. We will prove the claim by proving its contrapositive:

if $r^{1/t}$ is rational, then r is rational.

Since $r^{1/t}$ is assumed to be a rational we have

$$r^{1/t} = \frac{a}{b}, \text{where } a \text{ and } b \text{ are integers}$$

By raising both sides of the above equation to power of t we get

$$r = \frac{a^t}{b^t}$$

Since a^t and b^t are integers r is rational.

8. Prove that for all integers n, if $n - 3$ is divisible by 4 then $n^2 - 1$ is divisible by 8.

Solution. Since $n - 3$ is divisible by 4, we have

$$\begin{aligned} n - 3 &= 4k \qquad \text{for some integer } k \\ n &= 4k + 3 \\ n^2 - 1 &= (n + 1)(n - 1) \\ &= (4k + 3 + 1)(4k + 3 - 1) \\ &= (4k + 4)(4k + 2) \\ &= 4(k + 1) \cdot 2(2k + 1) \\ &= 8(k + 1)(2k + 1) \end{aligned}$$

Clearly, the R.H.S. is an integer that is divisible by 8.

9. Prove or disprove the following.

(a) For all integers n, $n^3 - n$ is divisible by 3.
(b) For all real numbers x, $2x^2 - 4x + 3 > 0$.

Solution.
(a) The proposition is true and we can prove it as follows. Let n be any particular but arbitrarily chosen integer. Then,

$$n^3 - n = n(n^2 - 1) = n(n-1)(n+1)$$

We have expressed $n^3 - n$ as a product of three consecutive integers. Observe that among any three consecutive integers one is divisible by 3. Hence, their product is also divisible by 3. Thus, $n^3 - n$ is also divisible by 3.

(b) The proposition is true and we prove it as follows. Let x be any particular but arbitrarily chosen real number. Then,

$$2x^2 - 4x + 3 = 2x^2 - 4x + 2 + 1 = 2(x^2 - 2x + 1) + 1 = 2(x-1)^2 + 1$$

Since $(x-1)^2$ is non-negative, $2(x-1)^2 + 1 \geq 1$. This proves the claim.