Example: Find singular value decomposition of $A=\begin{bmatrix}3&0\\4&5\end{bmatrix}$

Solution: By given $A=\begin{bmatrix}3&0\\4&5\end{bmatrix}$ & $A^T=\begin{bmatrix}3&4\\0&5\end{bmatrix}$

$\therefore AA^T=\begin{bmatrix}3&0\\4&5\end{bmatrix}\begin{bmatrix}3&4\\0&5\end{bmatrix}=\begin{bmatrix}9&12\\12&41\end{bmatrix}$

it's characteristic equation is

$\lambda^2-S_1\lambda+|A|=0$

$\therefore \lambda^2-50\lambda+225=0$

$\therefore (\lambda-45)(\lambda-5)=0$

$\therefore \lambda=\lambda_1=45,\ \lambda=\lambda_2=5$ be Eigen values of a matrix AA^T — (1) &

$\sigma_1=\sqrt{\lambda_1}=3\sqrt5,\ \sigma_2=\sqrt{\lambda_2}=\sqrt5$ — (2)

To find Eigen vector consider

$(AA^T-\lambda I)X=0$

$\therefore \begin{bmatrix}9-\lambda&12\\12&41-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ — (3)

Case-1

If $\lambda=\lambda_1=45$ $\begin{bmatrix}-36&12\\12&-4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$R_1\to R_1+3R_2,\ R_2\to \frac{1}{4}R_2$

$\begin{bmatrix}0&0\\3&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\therefore 3x_1-1x_2=0\Rightarrow 3x_1=1x_2\Rightarrow \frac{x_1}{1}=\frac{x_2}{3}=k=1$

$\therefore x_1=1,\ x_2=3\ \therefore$

For $\lambda=\lambda_1=45$ $X_1=\begin{bmatrix}1\\3\end{bmatrix}$ $\|X_1\|=\sqrt{10}$, $X_1'=\frac{X_1}{\|X_1\|}=\begin{bmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{bmatrix}$

Case-2

If $\lambda=\lambda_2=5$ $\begin{bmatrix}4&12\\12&36\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$R_2\to R_2-3R_1,\ R_1\to\frac{1}{4}R_1$

$\therefore \begin{bmatrix}1&3\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$1x_1+3x_2=0\Rightarrow 1x_1=-3x_2\Rightarrow\frac{x_1}{-3}=\frac{x_2}{1}=k=1$

$\therefore x_1=-3,\ x_2=1$

For $\lambda=\lambda_2=5$ $X_2=\begin{bmatrix}-3\\1\end{bmatrix}$ $\|X_2\|=\sqrt{10}$, $X_2'=\frac{X_2}{\|X_2\|}=\begin{bmatrix}-3/\sqrt{10}\\1/\sqrt{10}\end{bmatrix}$

$\therefore U=\begin{bmatrix}1/\sqrt{10}&-3/\sqrt{10}\\3/\sqrt{10}&1/\sqrt{10}\end{bmatrix}$

$A^TA=\begin{bmatrix}3&4\\0&5\end{bmatrix}\begin{bmatrix}3&0\\4&5\end{bmatrix}=\begin{bmatrix}25&20\\20&25\end{bmatrix}$

its characteristic equation is

$\lambda^2=S_1\lambda+|A|=0$

$\lambda^2-50\lambda+225=0$

$(\lambda-45)(\lambda-5)=0$

$\therefore \lambda=\lambda_1=45\ \ \lambda=\lambda_2=5$ be Eigen values of a matrix A^TA & — (1)

$\sigma_1=\sqrt{\lambda_1}=3\sqrt5,\ \sigma_2=\sqrt{\lambda_2}=5$ — (2)

To find Eigen vector consider

$(A^TA-\lambda I)=X$

$\begin{bmatrix}25-\lambda&20\\20&25-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ — (3)

Case-1

If $\lambda=\lambda_1=45$ $\begin{bmatrix}-20&20\\20&-20\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$R_2\to R_2+R_1,\ R_1\to\frac{1}{20}R_1$

$\begin{bmatrix}-1&1\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$-1x_1+1x_2=0\Rightarrow 1x_1=1x_2\Rightarrow\frac{x_1}{1}=\frac{x_2}{1}=k=1$

$x_1=1,\ x_2=1$

For $\lambda=\lambda_1=45$ $X_1=\begin{bmatrix}1\\1\end{bmatrix}$ $\|X_1\|=\sqrt2$, $X_1'=\frac{X_1}{\|X_1\|}=\begin{bmatrix}1/\sqrt2\\1/\sqrt2\end{bmatrix}$

Case-2

If $\lambda=\lambda_2=5$ $\begin{bmatrix}20&20\\20&20\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$R_2\to R_2-R_1,\ R_1\to\frac{1}{20}R_1$

$\therefore \begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\therefore 1x_1+1x_2=0\Rightarrow 1x_1=-1x_2\Rightarrow\frac{x_1}{-1}=\frac{x_2}{1}=k=1$

$\therefore x_1=-1,\ x_2=1$

For $\lambda=\lambda_2=5$ $X_2=\begin{bmatrix}-1\\1\end{bmatrix}$ $\|X_2\|=\sqrt2$, $\therefore X_2'=\frac{X_2}{\|X_2\|}=\begin{bmatrix}-1/\sqrt2\\1/\sqrt2\end{bmatrix}$

$V=\begin{bmatrix}1/\sqrt2&-1/\sqrt2\\1/\sqrt2&1/\sqrt2\end{bmatrix}$

$$\therefore A=UDV^T=\begin{bmatrix}\frac{1}{\sqrt{10}}&\frac{-3}{\sqrt{10}}\\\frac{3}{\sqrt{10}}&\frac{1}{\sqrt{10}}\end{bmatrix}\begin{bmatrix}3\sqrt5&0\\0&\sqrt5\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{bmatrix}$$