- Recitation on Saturday, Jan 21 - 9:30 - 10:30 pm IST.
  - For Zoom link see Piazza post @7.

- HW1 will be released today

- HW1 due: Before class starts next Friday.
  - no late submissions
  - make multiple submissions
    - latest will be saved.

Ex: ~~Let~~ x be an integer. If $x > 1$ then x^3+1 is composite.

Proof: We can write x^3+1 as

$$\boxed{x^3}\boxed{\left(1+\frac{1}{x^3}\right)}$$

not an integer.

Bogus!

Since $x > 1$, $x^3 \geq 1$.

Clearly, the second term > 1.

$\therefore x^3+1$ is composite.

$$x^3+1 = (x+1)(x^2-x+1)$$

Both the terms are integers.

$x+1$ is clearly > 1.

It remains to show that $x^2-x+1 > 1$

$$x > 1$$

$$x^2 > x$$

$$x^2 - x > 0$$

$$\boxed{x^2 - x + 1 > 1}$$

Ex: Prove that if x & y are integers where $x+y$ is even then x & y are both odd or both even.

Soln: $\quad p \Rightarrow q \equiv \bar{p} \vee q \equiv \bar{q} \Rightarrow \bar{p}$.

We will prove the claim by proving its contrapositive. That is, we will prove that if exactly one of x or y is even & the other is odd then $x+y$ is odd.

Without loss of generality, let x be odd & y be even.

By defn,

$x = 2k+1$, for some int k.

$y = 2l$, for some int l.

$$x + y = 2k+1 + 2l$$

$$= 2(k + l) + 1$$

$$= 2 \cdot z + 1,$$ where

$z = k + l$ is an integer.

$\therefore$ $x + y$ is odd.

Ex: Show that at least three of 25 days chosen must fall in the same month of the year.

<u>Soln</u>: We can rewrite the claim as follows:

if 25 days are chosen then atleast

three must fall in the same month of

the year.

[$p \Rightarrow q \equiv \bar{p} \vee q$.

Negation of $\bar{p} \vee q \equiv p \wedge \bar{q}$.

Contradiction: $p \equiv \bar{p} \Rightarrow c$]

Assume for contradiction that

25 days are chosen but at most 2 days fall in the same month of the year.

At most 24 days are chosen.

Thus we have

(25 days are chosen) $\wedge$ ($\leq$ 24 days are chosen),

which is a contradiction.

Ex: Prove that if $3n+2$ is odd then n is odd.

Proof: We will prove the claim by proving its contrapositive. That is, we will show that if n is even then $3n+2$ is even.

By defn, $n = 2k$, for some int k.

$$\therefore \quad 3n+2 = 3(2k)+2$$
$$= 2(3k+1)$$
$$= 2z, \text{ where } z = 3k+1 \text{ is an int.}$$

$\therefore$ $3n+2$ is even.

Ex: Prove that for all real numbers a & b, if the product ab is irrational then either a or b or both must be irrational.

Proof: We will prove the claim by proving its contrapositive. That is, we want to prove that if both a & b are rational then the product ab is rational.

By defn, let

$$a = \frac{p}{q} \quad \& \quad b = \frac{r}{s}, \text{ where}$$

p, q, r, s are integers & $q \neq 0, s \neq 0$.

$$\therefore \; ab = \frac{p}{q} \cdot \frac{r}{s}$$

$$= \frac{pr}{qs}$$

Since p & r are int, pr is an int. Similarly, if q & s are int then qs is an int.

Furthermore, since $q \neq 0$ & $s \neq 0$,

$qs \neq 0$.

$\therefore$ ab is a rational no.

Ex: Prove that the product of two odd nos is an odd number.

Proof: Let x & y be arbitrary, but particular odd nos.

By defn,

$x = 2k+1$, for some int k.

$y = 2l+1$, for some int l.

$$xy = (2k+1)(2l+1)$$

$$= 4kl + 2(k+l) + 1$$

$$= 2(2kl + k + l) + 1$$

$$= 2z + 1,$$ where

$z = 2kl + k + l$ is an integer.

Ex. Prove that $\sqrt{2}$ is irrational.

Proof: Assume for contradiction that $\sqrt{2}$ is rational. By defn,

$$\boxed{\sqrt{2} = \frac{a}{b}}$$

, where a & b are integers $b \neq 0$. & a & b do not have common factors. (relatively prime)

Squaring both sides, we get

$$2 = \frac{a^2}{b^2}.$$

$$\therefore a^2 = 2b^2 \quad \text{--- (1)}$$

a^2 is even.

______ (follows from 1

$\boxed{a \text{ is even}}$. (prev. claim)

Since a is even

$a = 2k$, for some int k.

Eqn (1) becomes

$$4k^2 = 2b^2$$

$$\therefore b^2 = 2k^2$$

b^2 is even

$\therefore \boxed{b \text{ is even}}$.

Thus a & b have a common factor of 2, a contradiction

$$\sqrt{2} = \frac{a'}{b'}, \quad \text{when} \quad \begin{aligned} a &= 2a' \\ b &= 2b' \end{aligned}$$

Repeat

$\downarrow$

$$\sqrt{2} = \frac{a''}{b''} \quad \ldots$$

$\downarrow$

$$\sqrt{2} = \frac{a'''}{b'''}$$

<u>Fundamental Theorem of Arithmetic.</u> or

<u>Unique Prime factorization theorem</u>

Every int $n > 1$ is a unique product of primes or it is a prime itself

$$84 = 2^2 \times 3^1 \times 7^1$$

Notation:

$S(m)$: # prime factors in the unique prime factorization of m.

$$S(84) = 4$$

Ex: Prove that $\sqrt{2}$ is irrational.

Proof: Assume for contradiction that $\sqrt{2}$ is rational.

By defn, $\sqrt{2} = \frac{a}{}$ where a & b are int

b' & $b \neq 0$.

Squaring both sides, we get

$$2 = \frac{a^2}{b^2}$$

$\therefore \; a^2 = 2b^2$

$S(a^2)$	$S(b^2)$
$= 2\,S(a) \rightarrow$ even	
$= 2\,S(b) + 1$	
$\hookrightarrow$ odd	
contradiction!	

Ex: Prove or disprove: The sum of 2 (positive) irrational nos is irrational.

Soln: The claim is false.

$$\underline{\underline{\sqrt{2}}} + (\underline{\underline{-\sqrt{2}}}) = \underline{\underline{0}}.$$

The second claim is also false.

$$(1+\sqrt{2}) + (\underline{2-\sqrt{2}})$$

Ex. Prove that there exists irrational nos x & y s.t. x^y is rational.

Soln: Let $x = \sqrt{2}$, $y = \sqrt{2}$.

$x^{y} = \sqrt{2}^{\sqrt{2}}$, which is ~~clearly rational~~.

Case I: $\sqrt{2}^{\sqrt{2}}$ is rational.

Done.

Case II: $\sqrt{2}^{\sqrt{2}}$ is irrational.

$x = \underline{\sqrt{2}^{\sqrt{2}}}$, $y = \underline{\sqrt{2}}$

$$x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$$

$$= \sqrt{2}^2 = \boxed{2}.$$

Ex: Prove that there are infinitely many primes.

Proof: Assume for contradiction that there are finitely many primes. Let P be the set of all primes.

$$P = \{2, 3, 5, 7, 11, 13, \ldots, q\}.$$

where q is the largest prime.

Consider the integer

$$n = (2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times q) + 1$$

Let $p \in P$ be a prime.

When n is divided by p, the remainder is 1.

By the unique factorization thm n is a prime.

But $n > q$. a contradiction!