4.4.2 Poisson Distribution

[4.4.2 : Class Work Problems]

① Prove that for a poisson Distribution $\sum_{x=0}^{\infty} P(x) = 1$

Solution we know p.d.f. of poisson distribution is

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x=0,1,2,\ldots\infty$$

$$\therefore \sum_{x=0}^{\infty} P(X=x) = \sum_{x=0}^{\infty} e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\left(\frac{\lambda^x}{x!}\right) = e^{-\lambda}\left[1+\frac{\lambda^1}{1!}+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}\ldots\right]$$

$$= e^{-\lambda} e^{\lambda} = e^{-\lambda+\lambda} = e^0 = 1$$

Thus $\sum_{x=0}^{\infty} P(X=x) = 1$

Example-2: Find mean and variance of poisson Distribution also find M.G.F. of Poisson Distribution.

Solution:- we know p.d.f. of poisson distribution is $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x=0,1,2,3-\infty$

Now $M_0(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} P(X=x) = \sum_{x=0}^{\infty} (e^t)^x \frac{e^{-\lambda}\lambda^x}{x!}$

$$\therefore M_0(t) = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{-\lambda+\lambda e^t} = e^{-\lambda(1-e^t)} = e^{\lambda(e^t-1)}$$

Thus moment generating function of poisson distribution is

$$M_0(t) = e^{\lambda(e^t-1)} \quad \text{—①}$$

Now $\mu_1' = \frac{d}{dt}[M_0(t)]\Big|_{t=0} = e^{\lambda(e^t-1)}\lambda e^t\Big|_{t=0} = e^{\lambda(e^0-1)}\lambda e^0 = e^{\lambda\cdot 0}\lambda\cdot 1 = \lambda$ —②

$$\mu_2' = \frac{d^2}{dt^2}[M_0(t)] = \frac{d}{dt}\left[\frac{d}{dt}M_0(t)\right]\Big|_{t=0} = \frac{d}{dt}\left[\lambda e^t e^{\lambda(e^t-1)}\right]\Big|_{t=0}$$

$$\mu_2' = \lambda\left[e^t. e^{\lambda(e^t-1)} + e^t. e^{\lambda(e^t-1)}\lambda e^t\right]\Big|_{t=0} = \lambda\left[e^0. e^{\lambda(e^0-1)} + e^0. e^{\lambda(e^0-1)}\lambda e^0\right]$$

$$\mu_2' = \lambda[1.e^0 + 1.e^0.\lambda] = \lambda[1+\lambda] = \lambda+\lambda^2 \quad \text{—③}$$

$$\therefore \text{mean} = \mu_1' = \frac{d}{dt}[M_0(t)] = \lambda$$

$$\text{Variance}(x) = \mu_2' - (\mu_1')^2 = \lambda^2+\lambda-(\lambda)^2 = \lambda$$

Example-3 Derive Poisson Distribution as a limiting case of binomial distribution

Solution:- We know p.d.f of Binomial distribution is

$p(X=x) = n_{C_x}\, p^x q^{n-x}$

$x = 0,1,2,3,\ldots n$

put $\lambda = np$ i.e $p = \frac{\lambda}{n}$, $\therefore q = 1-p = 1-\frac{\lambda}{n}$

$\therefore p(X=x) = n_{C_x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} = \frac{n!}{(n-x)!\,x!}\,\frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$

$\therefore P(X=x) = \frac{n(n-1)(n-2)(n-3)\cdots(n-(x-1))\cancel{(n-x)!}}{\cancel{(n-x)!}\;x!}\;\frac{\lambda^x}{n^x}\;\frac{\left[\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right]^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^x}$

$\therefore P(X=x) = \frac{\lambda^x}{x!}\;\frac{n(n-1)(n-2)\cdots[n-(x-1)]}{n\cdot n\cdot n\cdots x \text{ times}}\;\frac{\left[\left(1-\lambda/n\right)^{-\frac{n}{\lambda}}\right]^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^x}$

$\therefore P(X=x) = \frac{\lambda^x}{x!}\;1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\;\frac{\left[\left(1-\lambda/n\right)^{-\frac{n}{\lambda}}\right]^{-\lambda}}{\left(1-\lambda/n\right)^x}$

Now taking $\lim_{n\to\infty}$

$\lim_{n\to\infty} P(X=x) = \frac{\lambda^x}{x!}\lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right]\;\frac{\lim_{n\to\infty}\left[\left(1-\lambda/n\right)^{-\frac{n}{\lambda}}\right]^{-\lambda}}{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^x}$

$\therefore P(X=x) = \frac{\lambda^x}{x!}(1\cdot1\cdot1\cdots1)\,\frac{(e)^{-\lambda}}{1^x} = \frac{e^{-\lambda}\cdot\lambda^x}{x!}$ $x = 0,1,2,\ldots\infty$

which is p.d.f. of poission distribution

Example-5: Derive moment generating function $M_0(t)$ for Binomial and Poisson Distribution and find first two moments about origin in each case

Solution:-

(i) For Answer see Example-2 of 4.4.1: class work Problems page 166

(ii) For answer see Example-2 of 4.4.1: class work Problems page 167

Example-6-a: A poisson variate has standard deviation 2. Find $p(0)$ & $p(1)$

Solution:- we know p.d.f. of poisson distribution is $p(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x=0,1,2,3,\ldots\infty$

where variance$(x) = \lambda$ $\therefore$ S.D $= \sqrt{\lambda} = 2$ $\therefore \lambda = 4$

$\therefore P(X=x) = \frac{e^{-4}\cdot 4^x}{x!}$, $x = 0,1,2,\ldots\infty$

$\therefore p(x=0) = \frac{e^{-4}4^0}{0!} = e^{-4} = 0.018316$, $P(X=1) = \frac{e^{-4}4^1}{1!} = 0.073263$

Example-6-b: If the variance of poisson distribution is 3 Find (i) $p(x=2)$ (ii) $p(x\geq 4)$

Solution:- we know p.d.f. of poisson distribution is $p(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x=0,1,2,3,\ldots\infty$

where variance$(x) = \lambda = 3$ $\therefore P(X=x) = \frac{e^{-3}\cdot 3^x}{x!}$, $x = 0,1,2,\ldots\infty$

$\therefore P(X=2) = \frac{e^{-3}\cdot 3^2}{2!} = 0.224042$

$\therefore p(X\geq4) = 1 - p(X<4) = 1 - P(X=0,1,2,3) = 1 - \sum_{x=0}^{3} p(X=x) = 1 - \sum_{x=0}^{3}\left[\frac{e^{-3}\cdot3^x}{x!}\right]$

$\therefore P(X\geq4) = 0.352768$

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Example-7: If X is a Poisson variate such that $P(X=2) = 9P(X=4) + 90P(X=6)$. Find λ, the mean of poisson distribution.

Solution: we know p.d.f. of poisson distribution is $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x=0,1,2,\ldots$, $\lambda>0$ — ①

By given $P(X=2) = 9P(X=4) + 90P(X=6)$

$$\therefore \frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\therefore \frac{1}{2} = \frac{9}{24}\lambda^2 + \frac{90}{6\times5\times4}\lambda^4$$

$$\frac{1}{2} = \frac{3}{8}\lambda^2 + \frac{1}{8}\lambda^4 \quad \therefore \lambda^4 + 3\lambda^2 = 4$$

$$\therefore \lambda^4 + 3\lambda^2 - 4 = 0 \Rightarrow \lambda^4 + 4\lambda^2 - \lambda^2 - 4 = 0 \Rightarrow \lambda^2(\lambda^2+4) - 1(\lambda^2+4) = 0$$

$$\Rightarrow (\lambda^2+4)(\lambda^2-1) = 0 \Rightarrow (\lambda+2i)(\lambda-2i)(\lambda-1)(\lambda+1) = 0 \Rightarrow \lambda = 2i, -2i, 1, -1$$

$\because \lambda > 0 \Rightarrow \boxed{\lambda = 1}$ $\therefore$ p.d.f. of Poisson distribution is $P(X=x) = \frac{e^{-1}\cdot 1^x}{x!}$, $x=0,1,\ldots\infty$

Example-8: For a poisson distribution $P(X=2) = 9P(X=4) + 90P(X=6)$ find the mean and variance of distribution

Sol^n we know p.d.f of poisson distribution is $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x=0,1,\ldots\infty$, $\lambda>0$ — ①

By given $P(X=2) = 9P(X=4) + 90P(X=6)$

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\frac{1}{2} = \frac{9}{24}\lambda^2 + \frac{90}{6\times120}\lambda^4 = \frac{3}{8}\lambda^2 + \frac{\lambda^4}{8}$$

$$\therefore \lambda^4 + 3\lambda^2 = 4 \Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0 \Rightarrow \lambda^4 + 4\lambda^2 - \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2(\lambda^2+4) - 1(\lambda^2+4) = 0 \Rightarrow (\lambda^2+4)(\lambda^2-1) = 0 \Rightarrow (\lambda+2i)(\lambda-2i)(\lambda-1)(\lambda+1) = 0$$

$$\Rightarrow \lambda = -2i, \lambda = 2i, \lambda = 1, \lambda = -1 \Rightarrow \boxed{\lambda = 1}$$

$\therefore$ mean = variance = $\lambda = 1$

Example-9: show that recurrence relation for the moments of poisson distribution is $\mu_{r+1} = rm\mu_{r-1} + m\frac{d}{dm}(\mu_r)$

Solution:

$$\mu_r = E(x - E(x))^r = \sum_{x=0}^{\infty} P(X=x)(x-\lambda)^r = \sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{x!}(x-\lambda)^r$$

$$\frac{d}{d\lambda}(\mu_r) = \sum_{x=0}^{\infty}\left[-\frac{e^{-\lambda}\lambda^x}{x!}(x-\lambda)^r + \frac{e^{-\lambda}x\lambda^{x-1}}{x!}(x-\lambda)^r + \frac{e^{-\lambda}\lambda^x}{x!}r(x-\lambda)^{r-1}(1)\right]$$

$$\therefore \frac{d}{d\lambda}(\mu_r) = \sum_{x=0}^{\infty}\frac{(x-\lambda)^r}{x!}\left[-e^{-\lambda}\lambda^x + e^{-\lambda}x\lambda^{x-1}\right] - r\sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{x!}(x-\lambda)^{r-1}$$

$$\therefore \frac{d}{d\lambda}(\mu_r) = \sum_{x=0}^{\infty}\frac{e^{-\lambda}(x-\lambda)^r}{x!}[\lambda^{x-1}][-\lambda+x] - r\sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{x!}(x-\lambda)^{r-1}$$

$$\therefore \frac{d}{d\lambda}(\mu_r) = \frac{1}{\lambda}\sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{x!}(x-\lambda)^{r+1} - r\sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{x!}(x-\lambda)^{r-1}$$

$$\therefore \frac{d}{d\lambda}(\mu_r) = \frac{1}{\lambda}\mu_{r+1} - r\mu_{r-1}$$

$$\therefore \frac{\mu_{r+1}}{\lambda} = r\mu_{r-1} + \frac{d}{d\lambda}(\mu_r)$$

$$\therefore \mu_{r+1} = \lambda\left[r\mu_{r-1} + \frac{d}{d\lambda}(\mu_r)\right]$$

$$\therefore \mu_{r+1} = \lambda r\mu_{r-1} + \lambda\frac{d}{d\lambda}(\mu_r)$$

Example-10: show that in a poisson distribution with unit mean the mean deviation about the mean is $\frac{2}{e}$ times the standard deviation

Solution: we know p.d.f of poisson distribution is $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, x=0,1,2,\ldots\infty$

By given mean $=\lambda=1$

$\therefore P(X=x) = \frac{e^{-1}1^x}{x!} = \frac{1}{e\,x!}$

Now mean deviation about mean $= E|X-\bar{X}| = E|X-\lambda| = E|X-1|$

$$= \sum_{x=0}^{\infty} |x-1| \cdot P(X=x) = \sum_{x=0}^{\infty} |x-1| \cdot \frac{1}{e\,x!} = \frac{1}{e}\sum_{x=0}^{\infty}\frac{|x-1|}{x!}$$

$$= \frac{1}{e}\left[\frac{1}{0!} + 0 + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots\right]$$

$$= \frac{1}{e}\left[1 + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \cdots\right]$$

$$\because \frac{n}{(n+1)!} = \frac{(n+1)-1}{(n+1)!} = \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$$

$\therefore$ Mean deviation about mean

$$= \frac{1}{e}\left[1 + \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!} + \cdots\right)\right]$$

$$= \frac{1}{e}[1+1] = \frac{2}{e} = \frac{2}{e}\cdot 1 = \frac{2}{e}\cdot\text{standard deviation}$$

$\therefore$ Mean deviation about mean $= \frac{2}{e}$ times the standard deviation

Example-11 Fit a Poisson distribution to the following data

x	0	1	2	3	4	5	6
f	314	335	204	86	29	9	3

solution:

x	0	1	2	3	4	5	6	Total
f	314	335	204	86	29	9	3	$980 = N$
xf	0	335	408	258	116	45	18	$1180 = \Sigma x_i f_i$

$\therefore \lambda = \text{mean} = \frac{1}{N}\Sigma x_i f_i = \frac{1}{980}\times 1180 = \frac{59}{49}$

$\therefore$ p.d.f of poisson distribution $= \frac{e^{-\lambda}\lambda^x}{x!}$

$\therefore P(X=x) = e^{-(\frac{59}{49})}\frac{(59/49)^x}{x!}, x=0,1,2,3\ldots\infty$

$\therefore$ Expected frequency $= N \cdot P(X=x) = 980\, e^{-(59/49)} \times \frac{(59/49)^x}{x!}, x=0,1,2\ldots$

x	0	1	2	3	4	5	6
E_i	~~980~~ 293.96 ≈ 294	353.96 354	213.09 ≈ 213	85.529 ≈ 86	25.74 ≈ 26	6.2 ≈ 6	1.24 ≈ 1

Thus

x	0	1	2	3	4	5	6
E	294	354	213	86	26	6	1

Example-12 Fit a Poisson distribution to the following data:

x	0	1	2	3	4	5	6	7	8
f	56	156	132	92	37	22	4	0	1

Solution: we know mean $= \bar{x} = \lambda = \frac{1}{N}\sum x_i f_i = \frac{1}{500} \cdot 986 = \frac{493}{250} = 1.972$

we know p.d.f. of poisson distribution is $p(x=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-1.972}(1.972)^x}{x!}$, $x = 0,1,2,\ldots$

$\therefore$ Expected frequency $= N \cdot p(x=x) = 500 \times \frac{e^{-1.972}(1.972)^x}{x!}$, $x = 0,1,2,3,\ldots$

x	0	1	2	3	4	5	6	7	8	Total
Expected frequency	69.589 $\simeq$ 70	137.22 $\simeq$ 137	135.3 $\simeq$ 135	88.948 $\simeq$ 89	43.848 $\simeq$ 44	17.293 $\simeq$ 17	5.689 $\simeq$ 6	1.6012 $\simeq$ 2	0.2357 $\simeq$ 0	500

Example-13: If the probability that an individual suffers a bad reaction from a certain injection is 0.001, determine the probability that out of 2000 individuals (i) exactly 3 (ii) more than 2 will suffer a bad reaction

Solution: Let p be the probability that an individual suffer a bad reaction

$\therefore p = 0.001$, $n = 2000$

$\therefore$ mean $= \lambda = np = 2000 \times 0.001 = 2$

we know p.d.f. of a Poisson distribution is $p(x=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-2} \cdot 2^x}{x!}$, $x = 0,1,2,\ldots$

$\therefore$ p(exactly 3 suffer a bad reaction) $= p(x=3) = \frac{e^{-2} \cdot 2^3}{3!} = 0.180447044$

p(more than 2 suffer a bad reaction) $= p(x>2) = 1 - p(x \le 2) = 1 - \sum_{x=0}^{2} p(x=x)$

$= 1 - \sum_{x=0}^{2} \frac{e^{-2} 2^x}{x!} = 0.3233235838$

Example-14 If M.G.F. of a discrete random variate x is $e^{4(e^t - 1)}$ find $p(x = \mu + \sigma)$ where μ & σ are mean and standard deviation of X

Solution: By given M.G.F. is $M_0(t) = e^{4(e^t - 1)}$ — (1)

we know M.G.F. of ~~Binomial~~ Poisson distribution $M_0(t) = e^{\lambda(e^t - 1)}$ — (2)

From (1) & (2) $\lambda = 4$

But for poisson distribution mean = variance $= \lambda = 4$

$\therefore \mu = 4$, $\sigma^2 = 4 \therefore \sigma = 2$

we know p.d.f. of Poisson distribution is $p(x=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-4} \cdot 4^x}{x!}$, $x = 0,1,2,\ldots$

$\therefore p(x = \mu + \sigma) = p(x = 4+2) = p(x=6) = \frac{e^{-4} 4^6}{6!} = 0.1041956346$

Example-15: A skilled typist on routine work kept a record of a mistakes made per day during 300 working days. If shew mad 1 mistake on 143 day, 2 mistakes on 110 days. Find the number of days on which shew made 3 mistakes using Poisson distribution

Solution By given

x (No. of mistakes)	0	1	2	Total
f (No. of days)	47	143	110	300

we know mean $= \bar{x} = \lambda = \frac{1}{N}\sum x_i f_i = \frac{1}{300}(363) = 1.21$

we know p.d.f. of Poisson distribution $= p(x=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-1.21}(1.21)^x}{x!}$, $x = 0,1,2,3,\ldots$

p[no. of on which she made 3 mistake] $= p(x=3) = \frac{e^{-1.21}(1.21)^3}{3!} = 0.0880457842$