Wednesday, August 25, 2021 11:07 AM

* Laplace Transform *

* Some Useful formulae *

① $e^{iat} = \cos at + i \sin at$

② $e^{-iat} = \cos at - i \sin at$

③ $\cos at = \dfrac{e^{iat} + e^{-iat}}{2}$

④ $\sin at = \dfrac{e^{iat} - e^{-iat}}{2i}$

⑤ $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$

⑥ $\cosh at = \dfrac{e^{at} + e^{-at}}{2}$

⑦ $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

⑧ $\sin^2 t = \dfrac{1-\cos 2t}{2}$

$\cos^2 t = \dfrac{1+\cos 2t}{2}$

$\sin^3 t = \dfrac{3\sin t - \sin 3t}{4}$

$\cos^3 t = \dfrac{\cos 3t + 3\cos t}{4}$

⑨ $\int e^{At} \sin Bt \, dt = \dfrac{e^{At}}{A^2+B^2}(A \sin Bt - B\cos Bt)$

$\int e^{At} \cos Bt \, dt = \dfrac{e^{At}}{A^2+B^2}(A \cos Bt + B \sin Bt)$

⑩ $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

$\Gamma(n+1) = n\Gamma n$ $\boxed{\text{or}}$ $\Gamma n = (n-1)\Gamma(n-1)$

$\Gamma 1 = 1$

$\Gamma 2 = 1!$

$\Gamma 3 = 2!$

$\Gamma 4 = 3!$

$\vdots$

$\Gamma\frac{1}{2} = \sqrt{\pi}$

$\Gamma\frac{3}{2} = \frac{1}{2}\sqrt{\pi}$

$\Gamma\frac{5}{2} = \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}$

$\Gamma\frac{7}{2} = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}$

$\Gamma 4 = 3!$

$\vdots$

$\Gamma n = (n-1)!$ where n is positive integer

$\Gamma 7/2 = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}$

(11) $e^{\infty} = \infty$

$e^{-\infty} = 0$

$\cos\infty = [-1,1]$

$\sin\infty = [-1,1]$

(12) $\int_a^b uv\,dx = \left[u\int v\,dx\right]_a^b - \int_a^b \left(\frac{du}{dx}\int v\,dx\right)dx$

(13) $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

where $v_1 = \int v$, $v_2 = \int v_1$ & so on

$u' = \frac{du}{dx}$, $u'' = \frac{du'}{dx}$ & so on

Definition

Let $f(t)$ be a function of t, $t > 0$. Then the **Laplace Trasform** of $f(t)$, is denoted by $L\{f(t)\}$, is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

provided the integral exists, and it will be a function of the parameter s , which may be a real or complex number. $Lf(t)$ being clearly a function of s is briefly written as $f(s)$. thus,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = f(s)$$

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(1) find Laplace Transform of

$$f(t) = \begin{cases} 3, & 0 < t < 5 \\ 5, & t > 5 \end{cases}$$

sol:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^5 e^{-st}\, 3\,dt + \int_5^\infty e^{-st}\, 5\,dt$$

$$= 3\int_0^5 e^{-st}\,dt + 5\int_5^\infty e^{-st}\,dt$$

$$= 3\left[\frac{e^{-st}}{}\right]_0^5 + 5\left[\frac{e^{-st}}{}\right]_5^\infty$$

$$= 3\left[\frac{e^{-st}}{-s}\right]_0^5 + 5\left[\frac{e^{-st}}{-s}\right]_5^\infty$$

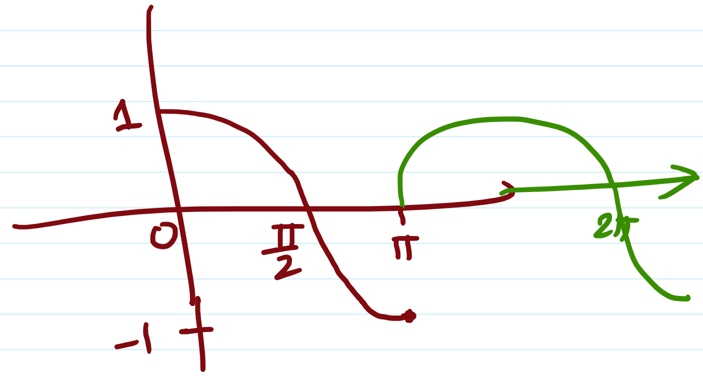
$$= 3\left[\frac{e^{-5s}}{-s} - \frac{1}{-s}\right] + 5\left[\{0\} - \left\{\frac{e^{-5s}}{-s}\right\}\right]$$

$$= 3\left(\frac{1-e^{-5s}}{s}\right) + 5\left(\frac{e^{-5s}}{s}\right)$$

$$= \frac{3-3e^{-5s}+5e^{-5s}}{s} = \frac{3+2e^{-5s}}{s}.$$

② Find Laplace Transform of

$$f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$$



Sol:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st} \cos t\, dt + \int_\pi^\infty e^{-st} \sin t\, dt$$

$$\int e^{At} \cos Bt\, dt = \frac{e^{At}}{A^2+B^2}(A\cos Bt + B\sin Bt)$$

$$\int e^{At} \sin Bt\, dt = \frac{e^{At}}{A^2+B^2}(A\sin Bt - B\cos Bt)$$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\right]_0^\pi + \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_\pi^\infty$$

$$= \left\{\frac{e^{-s\pi}}{s^2+1}(s+0)\right\} - \left\{\frac{1}{s^2+1}(-s+0)\right\} + \{0\} - \left\{\frac{e^{-s\pi}}{s^2+1}(0+1)\right\}$$

$$= \left\{ \frac{e^{-s\pi}}{s^2+1}(s+0) \right\} - \left\{ \frac{1}{s^2+1}(-s+0) \right\} + \{0\} - \left\{ \frac{e}{s^2+1}(0+1) \right\}$$

$$= \frac{s e^{-s\pi}}{s^2+1} + \frac{s}{s^2+1} - \frac{e^{-s\pi}}{s^2+1} = \frac{(s-1)e^{-s\pi} + s}{s^2+1}$$

H.W.

(3) find Laplace Transform of $f(t) = \begin{cases} \cos(t-a), & t>a \\ 0, & t<a \end{cases}$

H.W

(4) find Laplace Transform of $f(t) = \begin{cases} t, & 0<t<1/2 \\ t-1, & \frac{1}{2}<t<1 \\ 0, & t>1 \end{cases}$

Laplace Transform of standard function

① $L\{k\} = \frac{k}{s}$ where k is constant

② $L\{e^{at}\} = \frac{1}{s-a}$

③ $L\{e^{-at}\} = \frac{1}{s+a}$

④ $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

⑤ $L\{\sin at\} = \frac{a}{s^2+a^2}$

⑥ $L\{\cos at\} = \frac{s}{s^2+a^2}$

⑦ $L\{\sinh at\} = \frac{a}{s^2-a^2}$

⑧ $L\{\cosh at\} = \frac{s}{s^2-a^2}$

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① $L\{k\} = \frac{k}{s}$

Sol: Here $f(t) = k,\ t>0$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{k\} = \int_0^\infty e^{-st} k\,dt$$

$$= k\int_0^\infty e^{-st}\,dt$$

$$= k\left[\frac{e^{-st}}{-s}\right]_0^\infty = k\left[\{0\} - \left\{\frac{1}{-s}\right\}\right]$$

$$= k/s$$

② $L\{e^{at}\} = \frac{1}{s-a}$

Sol: $f(t) = e^{at},\ t>0$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\left\{\frac{1}{s-a}\right\} = \int_0^\infty e^{-st} e^{at}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \{0\} - \left\{\frac{1}{-(s-a)}\right\}$$

$$= \frac{1}{s-a}$$

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⑧ $L\{\cosh at\} = \frac{s}{s^2-a^2}$

Sol: $f(t) = \cosh at,\ t>0$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-a)t} + e^{-(s+a)t}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)}\right]^\infty$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$L\{\cosh at\} = \int_0^\infty e^{-st} \cosh at \, dt$$

$$= \int_0^\infty e^{-st} \left(\frac{e^{at} + e^{-at}}{2} \right) dt$$

$$= \frac{1}{2} \left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty$$

$$= \frac{1}{2} \left[\{0\} - \left\{ \frac{1}{-(s-a)} + \frac{1}{-(s+a)} \right\} \right]$$

$$= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[\frac{(s+a) + (s-a)}{s^2 - a^2} \right] = \frac{s}{s^2 - a^2}$$

Laplace Transform of elementary function

$$\boxed{\begin{aligned} a^x &= e^{\log a^x} \\ &= e^{x \log a} \end{aligned}}$$

1. $L\{5\} = \frac{5}{s}$

2. $L\{\pi\} = \frac{\pi}{s}$

3. $L\{\cos\alpha\} = \frac{\cos\alpha}{s}$

4. $L\{e^{3t}\} = \frac{1}{s-3}$

5. $L\{e^{3it}\} = \frac{1}{s-3i}$

6. $L\{2^t\} = L\{e^{\log 2^t}\}$

$$= L\{e^{t \log 2}\}$$

$$= \frac{1}{s - \log 2}$$

7. $L\{c^t\} = \frac{1}{s - \log c}$

8. $L\{\sin \pi t\} = \frac{\pi}{s^2 + \pi^2}$

$$L\{\sin \sqrt{3}\, t\} = \frac{\sqrt{3}}{s^2 + 3}$$

9. $L\{t^{-3}\} = \frac{\Gamma(-3+1)}{s^{-3+1}}$

$$= \frac{\Gamma(-2)}{s^{-2}} \quad \text{does not exist}$$

10. $L\{t^{-3/2}\} = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{s^{-3/2+1}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-1/2}} = \frac{-2\sqrt{\pi}}{s^{1/2}} = -2\sqrt{\pi s}$

$$\Gamma(n+1) = n\Gamma(n) \Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n}$$

put $n = -1/2$

$$\Rightarrow \Gamma\left(-\tfrac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = \frac{\sqrt{\pi}}{-\frac{1}{2}}$$

$$= -2\sqrt{\pi}$$

$$\boxed{\therefore \Gamma\left(-\tfrac{1}{2}\right) = -2\sqrt{\pi}}$$

* Properties of Laplace Transform *

Linearity:

$$L\{af_1(t) + bf_2(t)\} = aL\{f_1(t)\} + bL\{f_2(t)\}$$

① Find Laplace Transform of $4t^2 + \sin 3t + e^{2t}$

Sol: $$L\{4t^2 + \sin 3t + e^{2t}\} = 4L\{t^2\} + L\{\sin 3t\} + L\{e^{2t}\}$$

$$= 4\left(\frac{\Gamma 3}{s^3}\right) + \frac{3}{s^2+9} + \frac{1}{s-2}$$

$$= \frac{8}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2}$$

② Find L.T. of $\sin^3 t$

$$\sin 3A = 3\sin A - 4\sin^3 A$$

Sol: $$L\{\sin^3 t\} = L\left\{\frac{3\sin t - \sin 3t}{4}\right\}$$

$$\sin^3 A = \frac{3\sin A - \sin 3A}{4}$$

$$= \frac{1}{4} L\{3\sin t - \sin 3t\}$$

$$= \frac{1}{4}\left[3L\{\sin t\} - L\{\sin 3t\}\right]$$

$$= \frac{1}{4}\left[3\left(\frac{1}{s^2+1}\right) - \frac{3}{s^2+9}\right] = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

③ Find Laplace Transform of

ⓐ $\cos 3t \cos 2t \cos t$

ⓑ $\cosh^5 t$

ⓒ $\sin(\omega t + \alpha)$

ⓓ $\sin\sqrt{t}$

ⓔ $\frac{\cos\sqrt{t}}{\sqrt{t}}$

Sol: ⓐ $\cos 3t \cos 2t \cos t = \frac{1}{2}(2\cos 3t \cos 2t)\cos t$

$$= \frac{1}{2}(\cos 5t + \cos t)\cos t$$

$$= \frac{1}{4}(2\cos 5t \cos t + 2\cos t \cos t)$$

$$= \frac{1}{4}(\cos 6t + \cos 4t + \cos 2t + 1)$$

$$L\{\cos 3t \cos 2t \cos t\} = \frac{1}{4} L\{\cos 6t + \cos 4t + \cos 2t + 1\}$$

$$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4} + \frac{1}{s}\right]$$

ⓑ $\cosh^5 t = \left(\frac{e^t + e^{-t}}{2}\right)^5$

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

$$= \frac{1}{32}(e^t + e^{-t})^5$$

$$= \frac{1}{32}\left[e^{5t} + 5e^{4t}e^{-t} + 10e^{3t}e^{-2t} + 10e^{2t}e^{-3t} + 5e^{t}e^{-4t} + e^{-5t}\right]$$

$$= \frac{1}{32}\left[e^{5t} + 5e^{3t} + 10e^{t} + 10e^{-t} + 5e^{-3t} + e^{-5t}\right]$$

$$= \frac{1}{32}\left[\left(e^{5t} + e^{-5t}\right) + 5\left(e^{3t} + e^{-3t}\right) + 10\left(e^{t} + e^{-t}\right) \right]$$

$$= \frac{1}{32}\left[\left(2\cosh 5t\right) + 5\left(2\cosh 3t\right) + 10\left(2\cosh t\right) \right]$$

$$= \frac{1}{16}\left[\cosh 5t + 5\cosh 3t + 10\cosh t \right]$$

$$L\{\cosh^5 t\} = \frac{1}{16}\left[L\{\cosh 5t\} + 5L\{\cosh 3t\} + 10L\{\cosh t\} \right]$$

$$= \frac{1}{16}\left[\frac{s}{s^2-25} + 5\left(\frac{s}{s^2-9}\right) + 10\left(\frac{s}{s^2-1}\right) \right]$$

$$= \frac{1}{16}\left[\frac{s}{s^2-25} + \frac{5s}{s^2-9} + \frac{10s}{s^2-1} \right]$$

(c) $\sin(\omega t + \alpha)$

Sol: $L\{\sin(\omega t+\alpha)\} = L\{\sin\omega t \cos\alpha + \cos\omega t \sin\alpha\}$

constants

$$= \cos\alpha \, L\{\sin\omega t\} + \sin\alpha \, L\{\cos\omega t\}$$

$$= \cos\alpha \left(\frac{\omega}{s^2+\omega^2}\right) + \sin\alpha \left(\frac{s}{s^2+\omega^2}\right)$$

$$= \frac{\omega\cos\alpha + s\sin\alpha}{s^2+\omega^2}$$

(d)

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \cdots$$

$$= t^{1/2} - \frac{t^{3/2}}{\;} + \frac{t^{5/2}}{\;} - \frac{t^{7/2}}{\;} + \cdots$$

$$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \cdots$$

$$= t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots$$

$$L\{\sin\sqrt{t}\} = L\{t^{1/2}\} - \frac{1}{3!}L\{t^{3/2}\} + \frac{1}{5!}L\{t^{5/2}\} - \frac{1}{7!}L\{t^{7/2}\} + \cdots$$

$$= \frac{\Gamma 3/2}{s^{3/2}} - \frac{1}{3!}\left(\frac{\Gamma 5/2}{s^{5/2}}\right) + \frac{1}{5!}\left(\frac{\Gamma 7/2}{s^{7/2}}\right) - \frac{1}{7!}\left(\frac{\Gamma 9/2}{s^{9/2}}\right) + \cdots$$

$$= \frac{\Gamma 3/2}{s^{3/2}} - \frac{1}{3!}\left(\frac{\frac{3}{2}\Gamma 3/2}{s^{3/2}s^1}\right) + \frac{1}{5!}\left(\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma 3/2}{s^{3/2}s^2}\right) - \frac{1}{7!}\left(\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Gamma 3/2}{s^{3/2}s^3}\right) + \cdots$$

$$= \frac{\Gamma 3/2}{s^{3/2}}\left[1 - \frac{1}{3!}\frac{3}{2s} + \frac{1}{5!}\cdot\frac{5}{2}\cdot\frac{3}{2s^2} - \frac{1}{7!}\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2s^3} + \cdots\right]$$

$$= \frac{\Gamma 3/2}{s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{32s^2} - \frac{1}{384s^3} + \cdots\right]$$

$$= \frac{\frac{1}{2}\cdot\sqrt{\pi}}{s^{3/2}}\left[1 - \left(\frac{1}{4s}\right) + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \cdots\right]$$

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \cdots$$

$$e^{-\theta} = 1 - \theta + \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \cdots$$

$$= \frac{1}{2}\cdot\frac{\sqrt{\pi}}{s^{3/2}}\, e^{-1/4s} = \frac{\sqrt{\pi}\, e^{-1/4s}}{2s^{3/2}}$$

H.W.

(e) $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$

* Properties of Laplace Transform *

Change of scale:

If $L\{f(t)\} = \phi(s)$ then $L\{f(at)\} = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

eg. ① $L\{f(t)\} = \log\left(\frac{s+3}{s+1}\right)$ Here $a=2$

$$L\{f(2t)\} = \frac{1}{2}\log\left(\frac{\frac{s}{2}+3}{\frac{s}{2}+1}\right)$$

$$= \frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$$

② $L\{f(t)\} = \frac{2}{s^3}e^{-s}$

$$L\{f(3t)\} = \frac{1}{3}\left[\frac{2}{\left(\frac{s}{3}\right)^3}e^{-\frac{s}{3}}\right]$$ Here $a=3$

$$= \frac{18e^{-s/3}}{s^3}$$

③ $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}e^{-\frac{1}{4s}}$

$L\{\sin 2\sqrt{t}\} = ?$

Here $f(t) = \sin\sqrt{t}$

$$f(2t) = \sin\sqrt{2t} = \sin\sqrt{2}\sqrt{t}$$

$$f(4t) = \sin\sqrt{4t} = \sin 2\sqrt{t}$$

$\boxed{\therefore a=4}$ $\therefore L\{\sin 2\sqrt{t}\} = \frac{1}{4}\left[\frac{\sqrt{\pi}}{2\left(\frac{s}{4}\right)^{3/2}}e^{-\frac{1}{4\left(\frac{s}{4}\right)}}\right]$

$$= \frac{\sqrt{\pi}}{s^{3/2}}e^{-1/s}$$

(4) $L\{\operatorname{erf}\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}$

$\therefore L\{\operatorname{erf} 3\sqrt{t}\} = ?$

Here $a=9$

$\therefore L\{\operatorname{erf} 3\sqrt{t}\} = \frac{1}{9}\left[\frac{1}{\frac{s}{9}\sqrt{\frac{s}{9}+1}}\right] = \frac{3}{s\sqrt{s+9}}$

* Evaluation of Integral $\int_0^\infty e^{-st} f(t)\,dt$ *

(1) Evaluate $\int_0^\infty e^{-2t}\sin^3 t\,dt$

$\int_0^\infty e^{-st} (☺)\,dt = L\{☺\}$

Sol: $L\{\sin^3 t\} = L\left\{\frac{3\sin t - \sin 3t}{4}\right\}$

$= \frac{3}{4}L\{\sin t\} - \frac{1}{4}L\{\sin 3t\}$

$= \frac{3}{4}\left(\frac{1}{s^2+1}\right) - \frac{1}{4}\left(\frac{3}{s^2+9}\right)$

$\int_0^\infty e^{-st}\sin^3 t\,dt = \frac{3}{4}\left(\frac{1}{s^2+1}\right) - \frac{1}{4}\left(\frac{3}{s^2+9}\right)$

put $s=2$

$\int_0^\infty e^{-2t}\sin^3 t\,dt = \frac{3}{4}\left(\frac{1}{5}\right) - \frac{1}{4}\left(\frac{3}{13}\right) = \frac{3}{20} - \frac{3}{52}$

$= 6/65$

(2) Evaluate $\int_0^\infty e^{-3t} t^5\,dt$

Sol: $L\{t^5\} = \frac{\Gamma 6}{s^6} = \frac{5!}{s^6} = \frac{120}{s^6}$

∞

$$\int_0^\infty e^{-st} t^5 \, dt = \frac{120}{s^6}$$

put $s=3$

$$\int_0^\infty e^{-3t} t^5 \, dt = \frac{120}{3^6} = \frac{40}{243}$$

③ Evaluate $\int_0^\infty \cosh 3t \, dt$

Sol: $L\{\cosh 3t\} = \dfrac{s}{s^2-9}$

$$\int_0^\infty e^{-st} \cosh 3t \, dt = \frac{s}{s^2-9}$$

put $s=0$, we get

$$\int_0^\infty \cosh 3t \, dt = \frac{0}{0-9} = 0$$

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④ If $\int_0^\infty e^{-2t} \sin(t+\alpha)\cos(t-\alpha) \, dt = 3/8$ then find α

Sol: $L\{\sin(t+\alpha)\cos(t-\alpha)\} = L\{\frac{1}{2}(2\sin(t+\alpha)\cos(t-\alpha))\}$

$$= \frac{1}{2} L\{\sin 2t + \sin 2\alpha\}$$

$$= \frac{1}{2}\left[\frac{2}{s^2+4} + \frac{\sin 2\alpha}{s}\right]$$

$$\int_0^\infty e^{-st} \sin(t+\alpha)\cos(t-\alpha) \, dt = \frac{1}{2}\left[\frac{2}{s^2+4} + \frac{\sin 2\alpha}{s}\right]$$

put $s=2$

$$\int_0^\infty e^{-2t} \sin(t+\alpha)\cos(t-\alpha) \, dt = \frac{1}{2}\left[\frac{2}{8} + \frac{\sin 2\alpha}{2}\right]$$

$$\frac{3}{8} = \frac{1}{2}\left[\frac{1}{4} + \frac{\sin 2\alpha}{2}\right]$$

$$\frac{3}{4} = \frac{1}{4} + \frac{\sin 2\alpha}{2}$$

$$\frac{3}{4} - \frac{1}{4} = \frac{\sin 2\alpha}{2}$$

$$\frac{1}{2} = \frac{\sin 2\alpha}{2}$$

$$\sin 2\alpha = 1$$

$$2\alpha = \sin^{-1}(1)$$

$$= \frac{\pi}{2}$$

$$\boxed{\alpha = \frac{\pi}{4}}$$

* First Shifting Theorem *

If $L\{f(t)\} = \phi(s)$ then $L\{e^{at} f(t)\} = \phi(s-a)$

If $L\{f(t)\} = \phi(s)$ then $L\{e^{-at} f(t)\} = \phi(s+a)$

eg.

(1) $L\{\sin bt\} = \frac{b}{s^2+b^2}$ $\therefore L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$

(2) $L\{\cos bt\} = \frac{s}{s^2+b^2}$ $\therefore L\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2+b^2}$

(3) $L\{t^n\} = \frac{\overline{|n+1}}{s^{n+1}}$ $\therefore L\{e^{-pt} t^n\} = \frac{\overline{|n+1}}{(s+p)^{n+1}}$

(4) $L\{\cosh bt\} = \frac{s}{s^2-b^2}$ $\therefore L\{e^{-at}\cosh bt\} = \frac{s+a}{(s+a)^2-b^2}$

(1) Find L.T. of $e^{4t}\sin^2 t$

Sol:
$$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$$
$$= \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right]$$
$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$
$$L\{e^{4t}\sin^2 t\} = \frac{1}{2}\left[\frac{1}{(s-4)} - \frac{(s-4)}{(s-4)^2+4}\right]$$

(2) Find L.T. of $\cos 2t \sin t$

e^t

Sol: $L\{\cos 2t \sin t\} = \frac{1}{2} L\{2\cos 2t \sin t\}$

$$= \frac{1}{2} L\{\sin 3t - \sin t\}$$

$$= \frac{1}{2}\left[\frac{3}{s^2+9} - \frac{1}{s^2+1}\right]$$

$$L\{e^{-t}\cos 2t \sin t\} = \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]$$

$$\therefore L\left\{\frac{\cos 2t \sin t}{e^t}\right\} = \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]$$

③ Find L.T. of $\cosh 2t \cos 2t$

Sol: $\cosh 2t \cos 2t = \left(\frac{e^{2t}+e^{-2t}}{2}\right)\cos 2t$

$$= \frac{1}{2}\left(e^{2t}\cos 2t + e^{-2t}\cos 2t\right]$$

$$\therefore L\{\cosh 2t \cos 2t\} = \frac{1}{2}\left[L\{e^{2t}\cos 2t\} + L\{e^{-2t}\cos 2t\}\right] \quad \text{——— ①}$$

$$L\{\cos 2t\} = \frac{s}{s^2+4}$$

$$L\{e^{2t}\cos 2t\} = \frac{s-2}{(s-2)^2+4} \quad \text{——— ②}$$

$$L\{e^{-2t}\cos 2t\} = \frac{s+2}{(s+2)^2+4} \quad \text{——— ③}$$

put ② & ③ in eqn. ① $\Rightarrow$

$$= \frac{1}{2}\left[\frac{s-2}{\ldots} + \frac{s+2}{\ldots}\right]$$

put (2) & (3) in eqn. (1) $\Rightarrow$

$$L\{\cosh 2t \cos 2t\} = \frac{1}{2}\left[\frac{s-2}{(s-2)^2+4} + \frac{s+2}{(s+2)^2+4}\right]$$

(4) If $L\{f(t)\} = \frac{s}{s^2+s+4}$ find $L\{e^{-3t} f(2t)\}$

Sol: Here $a=2$

By change of scale Property

$$L\{f(2t)\} = \frac{1}{2}\left[\frac{s/2}{\left(\frac{s}{2}\right)^2+\frac{s}{2}+4}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{s}{2}}{\frac{s^2+2s+16}{4}}\right] = \frac{1}{2}\left[\frac{s}{2} \times \frac{4}{s^2+2s+16}\right]$$

$$= \frac{s}{s^2+2s+16}$$

$$L\{e^{-3t} f(2t)\} = \frac{s+3}{(s+3)^2+2(s+3)+16}$$

Property Preference

1. scaling
2. Multiplication / division by t
3. Derivative / Integration
4. first shifting

Linearity
Scaling
first shift
2nd shift X
multi by t^n
Divisi by t^n
L.T. of Derivat
L.T. of Integr

(5) Show that $L\{\sinh(\frac{t}{2})\sin(\frac{t}{2})\} = \frac{2s}{4s^4+1}$

⑤ Show that $L\{\sinh(\frac{t}{2})\sin(\frac{t}{2})\} = \frac{\quad}{4s^4+1}$

Sol: $L\{\sinh\frac{t}{2}\sin\frac{t}{2}\} = \frac{1}{2}\left[L\{e^{t/2}\sin\frac{t}{2}\} - L\{e^{-t/2}\sin\frac{t}{2}\}\right]$

$$= \frac{1}{2}\left[\frac{1/2}{(s-\frac{1}{2})^2+\frac{1}{4}} - \frac{1/2}{(s+\frac{1}{2})^2+\frac{1}{4}}\right]$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left[\frac{1}{s^2-s+\frac{1}{4}+\frac{1}{4}} - \frac{1}{s^2+s+\frac{1}{4}+\frac{1}{4}}\right]$$

$$= \frac{1}{4}\left[\frac{1}{(s^2+\frac{1}{2})-s} - \frac{1}{(s^2+\frac{1}{2})+s}\right]$$

$$= \frac{1}{4}\left[\frac{(s^2+\frac{1}{2})+s-((s^2+\frac{1}{2})-s)}{(s^2+\frac{1}{2})^2-s^2}\right]$$

$$= \frac{1}{4}\left[\frac{2s}{s^4+s^2+\frac{1}{4}-s^2}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^4+\frac{1}{4}}\right]$$

$$= \frac{1}{2}\left[\frac{4s}{4s^4+1}\right] = \frac{2s}{4s^4+1}$$

* Multiplication by t^n, $n>0$ *

If $L\{f(t)\} = \phi(s)$ then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\phi(s)$

eg ① $L\{te^{at}\}$

Sol: $L\{e^{at}\} = \frac{1}{s-a}$

$\therefore L\{t\,e^{at}\} = (-1)^1 \frac{d}{ds}\left(\frac{1}{s-a}\right)$

$= (-1)\left[\frac{-1}{(s-a)^2}\right] = \frac{1}{(s-a)^2}$

② $L\{t^2 \sin at\}$

$L\{\sin at\} = \frac{a}{s^2+a^2}$

$L\{t^2 \sin at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$

$= (1)\frac{d}{ds}\left[\frac{-a}{(s^2+a^2)^2}(2s)\right]$

$= -\frac{d}{ds}\left[\frac{2as}{(s^2+a^2)^2}\right]$

$= -\left[\frac{(s^2+a^2)^2(2a) - (2as)\left[2(s^2+a^2)(2s)\right]}{(s^2+a^2)^2}\right]$

③ $L\{t e^{-4t} \sin 3t\}$

Sol: $L\{\sin 3t\} = \frac{3}{s^2+9}$

$L\{t \sin 3t\} = (-1)^1 \frac{d}{ds}\left(\frac{3}{s^2+9}\right) = (-1)\left[\frac{-3}{(s^2+9)^2}(2s)\right]$

$= \frac{6s}{(s^2+9)^2}$

$L\{e^{-4t}\, t \sin 3t\} = \frac{6(s+4)}{((s+4)^2+9)^2} = \frac{6(s+4)}{(s^2+8s+25)^2}$

$\boxed{\text{OR}}$

$\boxed{\text{OR}}$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{e^{-4t}\sin 3t\} = \frac{3}{(s+4)^2+9}$$

$$L\{t\,e^{-4t}\sin 3t\} = (-1)^1 \frac{d}{ds}\left[\frac{3}{(s+4)^2+9}\right]$$

$$= (-3)\frac{d}{ds}\left[\frac{1}{s^2+8s+25}\right]$$

$$= (-3)\left(\frac{-1}{(s^2+8s+25)^2}(2s+8)\right)$$

$$= \frac{6(s+4)}{(s^2+8s+25)^2}$$

* Effect of division by t^n * where $n \in I^+$

If $L\{f(t)\} = \phi(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \phi(s)\,ds$

$$\therefore L\left\{\frac{f(t)}{t^2}\right\} = \int_s^\infty \left(\int_s^\infty \phi(s)\,ds\right) ds$$

* Some useful formulaes *

1. $\log(ab) = \log a + \log b$
2. $\log\left(\frac{a}{b}\right) = \log a - \log b$
3. $\log(a^k) = k\log a$
4. $\int \frac{1}{s}\,ds = \log s$
5. $\int \frac{1}{s\pm a}\,ds = \log(s\pm a)$
6. $\int \frac{a}{s^2+a^2}\,ds = \tan^{-1}\left(\frac{s}{a}\right)$
7. $\int \frac{s}{s^2\pm a^2}\,ds = \frac{1}{2}\log(s^2\pm a^2)$
8. $\int \frac{a}{s^2-a^2}\,ds = a\int \frac{1}{s^2-a^2}\,ds$
   $= a \times \frac{1}{2a}\log\left|\frac{s-a}{s+a}\right|$
   $= \frac{1}{2}\log\left|\frac{s-a}{s+a}\right|$
9. $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$
10. $\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left(\frac{x\pm y}{1\mp xy}\right)$

① Find L.T. of $\frac{1-\cos 2t}{t}$

Sol: $L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$

$$\therefore L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds = \left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[2\log s - \log(s^2+4)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\{0\} - \left\{\log\left(\frac{s^2}{s^2+4}\right)\right\}\right]$$

$$= -\frac{1}{2}\log\left(\frac{s^2}{s^2+4}\right) = \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$$

② Find L.T. of $\frac{1}{t}e^{-t}\sin t$

Sol: $L\{\sin t\} = \frac{1}{s^2+1}$

$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1}s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$

$L\left\{e^{-t}\frac{\sin t}{t}\right\} = \cot^{-1}(s+1)$

(3) Find L.T. of $\frac{1}{t}\left(e^{-at} - e^{-bt}\right)$

Sol: $L\{e^{-at} - e^{-bt}\} = \frac{1}{s+a} - \frac{1}{s+b}$

$L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$

$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$

$\log\left(\frac{s+a}{s+b}\right) = \log\left(\frac{1+a/s}{1+b/s}\right) = \log\left(1+\frac{a}{s}\right) - \log\left(1+\frac{b}{s}\right)$

$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty = \{0\} - \left\{\log\left(\frac{s+a}{s+b}\right)\right\}$

$= \log\left(\frac{s+b}{s+a}\right)$

(4) Find L.T. of $\frac{\sin at}{t}$. Does Laplace Transform of $\frac{\cos at}{t}$ exists.

Sol: $L\{\sin at\} = \frac{a}{s^2+a^2}$

$\therefore L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2}\,ds = \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}(s/a)$

$L\{\cos at\} = \frac{s}{s^2+a^2}$

$L\left\{\frac{\cos at}{t}\right\} = \int_s^\infty \frac{s}{s^2+a^2}\,ds = \left[\frac{1}{2}\log(s^2+a^2)\right]_s^\infty$

$\because \log(s^2+a^2) \longrightarrow \infty$ as $s \to \infty$

$\therefore L\left\{\frac{\cos at}{t}\right\}$ does not exists

(5) Find $L\left\{\frac{\sin^2 t}{t^2}\right\}$

Sol:

$$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$L\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \frac{1}{2}\left\{ \log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty$$

$$= \frac{1}{4}\left[\{0\} - \left\{\log\left(\frac{s^2}{s^2+4}\right)\right\}\right]$$

$$= \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

$$\therefore L\left\{\frac{\sin^2 t}{t^2}\right\} = \int_s^\infty \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right) ds$$

$$= \frac{1}{4}\int_s^\infty \underbrace{\log\left(\frac{s^2+4}{s^2}\right)}_{u} \cdot \underbrace{1}_{v}\, ds$$

$$\int_a^b uv\,dx = \left[u\int v\right]_a^b - \int_a^b \left(\frac{du}{dx}\int v\,dx\right) dx$$

$$\frac{d}{ds}\left[\log\left(\frac{s^2+4}{s^2}\right)\right] = \frac{d}{ds}\left[\log(s^2+4) - 2\log s\right]$$

$$= \frac{2s}{s^2+4} - \frac{2}{s} = \frac{2s^2 - 2s^2 - 8}{s(s^2+4)} = \frac{-8}{s(s^2+4)}$$

$$= \frac{1}{4}\left[\left[\log\left(\frac{s^2+4}{s^2}\right)\cdot s\right]_s^\infty - \int_s^\infty \frac{-8}{s(s^2+4)}\cdot s\, ds\right]$$

$$= \frac{1}{4}\left[\{0\} - \left\{s\log\left(\frac{s^2+4}{s^2}\right)\right\} + 8\int_s^\infty \frac{ds}{s^2+4}\right]$$

$$= \frac{1}{4}\left[-s\log\left(\frac{s^2+4}{s^2}\right) + 8\times\frac{1}{2}\left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty\right]$$

$$= \frac{1}{4}\left[-s\log\left(\frac{s^2+4}{s^2}\right) + 4\left(\frac{\pi}{2} - \tan^{-1}(s/2)\right)\right]$$

$$= -\frac{s}{4}\log\left(\frac{s^2+4}{s^2}\right) + \cot^{-1}(s/2) = \frac{s}{4}\log\left(\frac{s^2}{s^2+4}\right) + \cot^{-1}(s/2)$$

* Problems Based on Evaluation *

① Evaluate $\int_0^\infty \frac{\sin at}{t}\,dt$

Sol: $L\{\sin at\} = \frac{a}{s^2+a^2}$

$$\therefore L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2}\,ds = \left[\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$

$$\int_0^\infty e^{-st}\,\frac{\sin at}{t}\,dt = \cot^{-1}\left(\frac{s}{a}\right)$$

put $s=0$, we get

$$\int_0^\infty \frac{\sin at}{t}\,dt = \cot^{-1}(0) = \frac{\pi}{2}$$

② Evaluate $\int_0^\infty \left(\frac{\sin 2t + \sin 3t}{te^t}\right) dt$

Sol: $L\{\sin 2t + \sin 3t\} = \frac{2}{s^2+4} + \frac{3}{s^2+9}$

$$L\left\{\frac{\sin 2t + \sin 3t}{t}\right\} = \int_s^\infty \left(\frac{2}{s^2+4} + \frac{3}{s^2+9}\right) ds$$

$$= \left[\tan^{-1}\left(\frac{s}{2}\right) + \tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

$$= \left\{\frac{\pi}{2} + \frac{\pi}{2}\right\} - \left\{\tan^{-1}\left(\frac{s}{2}\right) + \tan^{-1}\left(\frac{s}{3}\right)\right\}$$

$$\int_0^\infty e^{-st}\left(\frac{\sin 2t + \sin 3t}{t}\right) dt = \pi - \left\{\tan^{-1}\left(\frac{s}{2}\right) + \tan^{-1}\left(\frac{s}{3}\right)\right\}$$

put $s=1$

$$\int_0^\infty \left(\frac{\sin 2t + \sin 3t}{te^t}\right) dt = \pi - \left\{\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)\right\}$$

$$= \pi - \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)$$

$$= \pi - \tan^{-1}(1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

* Laplace Transform of Derivatives *

If $L\{f(t)\} = \phi(s)$ then $L\{f'(t)\} = -f(0) + sL\{f(t)\}$

$$= -f(0) + s\phi(s)$$

$$L\{f''(t)\} = -f'(0) + sL\{f'(t)\}$$

$$= -f'(0) + s[-f(0) + s\phi(s)]$$

$$= -f'(0) - sf(0) + s^2\phi(s)$$

$$L\{f'''(t)\} = -f''(0) + sL\{f''(t)\}$$

① find L.T. of $\frac{d}{dt}\left(\frac{\sin t}{t}\right)$

Sol:

Let $f(t) = \frac{\sin t}{t}$

$$f(0) = \lim_{t \to 0} \frac{\sin t}{t} \quad \left[\frac{0}{0}\right]$$

$$= 1$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1}s\right]_s^\infty$$

$$= \cot^{-1}(s)$$

$$\therefore L\{f(t)\} = \cot^{-1}s$$

$$L\{f'(t)\} = -f(0) + sL\{f(t)\}$$

$$L\left\{\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right\} = -1 + s\cot^{-1}s$$

② find L.T. of $\frac{d}{dt}\left(\frac{1-\cos 2t}{t}\right)$

Sol:

$$f(t) = \frac{1-\cos 2t}{t}$$

$$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \frac{1}{2}\log\left(\frac{s^2+4}{\ldots}\right)$$

$$t$$

$$f(0) = \lim_{t \to 0} \frac{1-\cos 2t}{t} \left[\frac{0}{0}\right]$$

$$= \lim_{t \to 0} \frac{+2\sin 2t}{1}$$

$$= 0$$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$$

(prove!)

$$L\{f(t)\} = \frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$$

$$L\{f'(t)\} = -f(0) + sL\{f(t)\}$$

$$= -0 + s\left(\frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)\right)$$

$$= \frac{s}{2}\log\left(\frac{s^2+4}{s^2}\right)$$

H.W

(3) Given $f(t) = \begin{cases} t+1, & 0 \le t \le 2 \\ 3, & t > 2 \end{cases}$

find $L\{f(t)\}$, $L\{f'(t)\}$, $L\{f''(t)\}$

H.W

(4) Given $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$ then prove that

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

Sol

$$f(t) = \sin\sqrt{t}$$

$$f(0) = 0$$

$$L\{f(t)\} = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$$

$$L\{f'(t)\} = -f(0) + sL\{f(t)\}$$

$$L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = -0 + s\left(\frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}\right)$$

$$= \frac{\sqrt{\pi}}{2\sqrt{s}} e^{-1/4s}$$

$$\Downarrow$$

$$\frac{1}{2}L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{1}{2}\sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

* Laplace Transform of Integrals *

If $L\{f(t)\} = \phi(s)$ then $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\phi(s)$

Also $L\left\{\int_0^t \left(\int_0^t f(u)\,du\right) du\right\} = \frac{1}{s^2}\phi(s)$

Examples

① Find L.T. of $\int_0^t \sin 2u\,du$

Sol: $L\{\sin 2u\} = \frac{2}{s^2+4}$

$\therefore L\left\{\int_0^t \sin 2u\,du\right\} = \frac{1}{s}\left(\frac{2}{s^2+4}\right)$

② find L.T. of $\int_0^t u\cosh u\,du$

Sol: $L\{\cosh u\} = \frac{s}{s^2-1}$

$$L\{u\cosh u\} = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2-1}\right)$$

$$= -\left[\frac{(s^2-1)(1) - (s)(2s)}{(s^2-1)^2}\right]$$

$$= -\left[\frac{-s^2-1}{(s^2-1)^2}\right] = \frac{s^2+1}{(s^2-1)^2}$$

$\therefore L\left\{\int_0^t u\cosh u\,du\right\} = \frac{1}{s}\left(\frac{s^2+1}{(s^2-1)^2}\right)$

③ Find L.T. of $\int_0^t \sin u \cos 2u\,du$

Sol:
$$\sin u\cos 2u = \frac{1}{2}(2\cos 2u\sin u)$$
$$= \frac{1}{2}(\sin 3u - \sin u)$$
$$L\{\sin u\cos 2u\} = \frac{1}{2}\left[\frac{3}{s^2+9} - \frac{1}{s^2+1}\right]$$
$$\therefore L\left\{\int_0^t \sin u\cos 2u\,du\right\} = \frac{1}{2s}\left[\frac{3}{s^2+9} - \frac{1}{s^2+1}\right]$$

④ Find L.T. of $\int_0^t \frac{1-e^{au}}{u}\,du$

Sol:
$$L\{1-e^{au}\} = \frac{1}{s} - \frac{1}{s-a}$$
$$L\left\{\frac{1-e^{au}}{u}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-a}\right)ds \qquad \left(\text{Division by } t^n\right)$$
$$= \left[\ln s - \ln(s-a)\right]_s^\infty$$
$$= \left[\ln\left(\frac{s}{s-a}\right)\right]_s^\infty = \{0\} - \left\{\ln\left(\frac{s}{s-a}\right)\right\}$$
$$= \ln\left(\frac{s-a}{s}\right)$$
$$\therefore L\left\{\int_0^t \frac{1-e^{au}}{u}\,du\right\} = \frac{1}{s}\ln\left(\frac{s-a}{s}\right)$$

⑤ Find L.T. of $\int_0^t u^{-1} e^{-u}\sin u\,du$

Sol:
$$L\{\sin u\} = \frac{1}{s^2+1}$$
$$L\{u^{-1}\sin u\} = \int_s^\infty \frac{1}{s^2+1}\,ds \qquad \left(\text{Division by } t^n\right)$$

$$= \left[\tan^{-1} s\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}(s) = \cot^{-1} s$$

$$L\{e^{-u}\, u^{-1} \sin u\} = \cot^{-1}(s+1) \quad \text{(first shifting)}$$

$$L\left\{\int_0^t u^{-1} e^{-u} \sin u \, du\right\} = \frac{1}{s} \cot^{-1}(s+1) \quad \text{(L.T. of Integrals)}$$

⑥ Find L.T. of $t \int_0^t e^{-4u} \sin 3u \, du$

Sol:

$$L\{\sin 3u\} = \frac{3}{s^2+9}$$

$$L\{e^{-4u} \sin 3u\} = \frac{3}{(s+4)^2+9} = \frac{3}{s^2+8s+25}$$

$$L\left\{\int_0^t e^{-4u} \sin 3u \, du\right\} = \frac{1}{s}\left(\frac{3}{s^2+8s+25}\right) = \frac{3}{s^3+8s^2+25s}$$

$$L\left\{t \int_0^t e^{-4u} \sin 3u \, du\right\} = (-1)^1 \frac{d}{ds}\left(\frac{3}{s^3+8s^2+25s}\right)$$

$$= (-1)\left[\frac{(-3)(3s^2+16s+25)}{(s^3+8s^2+25s)^2}\right]$$

$$= \frac{3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}$$

⑦ Find L.T. of $e^{-4t} \int_0^t e^{u} \sin u \, du$

Sol:

$$L\{\sin u\} = \frac{1}{s^2+1}$$

$$L\{e^{u} \sin u\} = \frac{1}{(s-1)^2+1} = \frac{1}{s^2-2s+2}$$

$$L\left\{\int_0^t e^u \sin u\,du\right\} = \frac{1}{s}\left(\frac{1}{s^2-2s+2}\right) = \frac{1}{s^3-2s^2+2s}$$

$$L\left\{e^{-4t}\int_0^t e^u \sin u\,du\right\} = \frac{1}{(s+4)^3-2(s+4)^2+2(s+4)}$$

(8) Find L.T. of following

(a) $t^{-1}\int_0^t e^{-u}\sin u\,du$ (b) $\cosh t\int_0^t e^u \cosh u\,du$

Sol: (a) $L\{\sin u\} = \frac{1}{s^2+1}$

$$L\{e^{-u}\sin u\} = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}$$

$$L\left\{\int_0^t e^{-u}\sin u\,du\right\} = \frac{1}{s}\left(\frac{1}{s^2+2s+2}\right)$$

$$L\left\{t^{-1}\int_0^t e^{-u}\sin u\,du\right\} = \int_s^\infty \frac{1}{s}\cdot\left(\frac{1}{s^2+2s+2}\right)ds$$

$$= \int_s^\infty \frac{1}{2}\left[\frac{1}{s} - \frac{(s+2)}{s^2+2s+2}\right]ds$$

$$\int \frac{f'(x)}{f(x)}dx = \ln f(x)$$

$$= \frac{1}{2}\int_s^\infty\left(\frac{1}{s} - \frac{2(s+2)}{2(s^2+2s+2)}\right)ds$$

$$= \frac{1}{2}\int_s^\infty\left[\frac{1}{s} - \frac{2s+2}{2(s^2+2s+2)} - \frac{2}{2(s^2+2s+2)}\right]ds$$

$$= \frac{1}{2}\int_s^\infty\left(\frac{1}{s} - \frac{2s+2}{2(s^2+2s+2)} - \frac{1}{(s+1)^2+1}\right)ds$$

$$= \frac{1}{2}\left[\ln s - \frac{1}{2}\ln(s^2+2s+2) - \tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln\left(\frac{s^2}{\quad}\right) - 2\tan^{-1}(s+1)\right]^\infty$$

$$= \frac{1}{4}\left[\ln\left(\frac{s^2}{s^2+2s+2}\right) - 2\tan^{-1}(s+1) \right]_s^{\infty}$$

$$= \frac{1}{4}\left[\ln\left(\frac{s^2+2s+2}{s^2}\right) - 2\left(\frac{\pi}{2} - \tan^{-1}(s+1)\right) \right]$$

$$= \frac{1}{4}\left[\ln\left(\frac{s^2+2s+2}{s^2}\right) - 2\cot^{-1}(s+1) \right]$$

(b) $\cosh t \int_0^t e^u \cosh u \, du$

Sol: $$L\left\{\cosh t \int_0^t e^u \cosh u \, du\right\} = L\left\{ \frac{e^t + e^{-t}}{2} \int_0^t e^u \cosh u \, du \right\}$$

$$= \frac{1}{2} L\left\{ e^t \int_0^t e^u \cosh u \, du + e^{-t} \int_0^t e^u \cosh u \, du \right]$$

$$= \frac{1}{2}\left[L\left\{ e^t \int_0^t e^u \cosh u \, du \right\} + L\left\{ e^{-t} \int_0^t e^u \cosh u \, du \right\} \right]$$

$$L\{\cosh u\} = \frac{s}{s^2-1}$$

$$L\{e^u \cosh u\} = \frac{s-1}{(s-1)^2-1}$$

$$L\left\{\int_0^t e^u \cosh u\right\} = \frac{1}{s}\left(\frac{s-1}{(s-1)^2-1}\right)$$

$$L\left\{ e^t \int_0^t e^u \cosh u \, du \right\} = \frac{1}{s-1}\left(\frac{(s-2)}{(s-2)^2-1}\right)$$

$$L\left\{ e^{-t} \int_0^t e^u \cosh u \, du \right\} = \frac{1}{s+1}\left(\frac{s}{s^2-1}\right)$$

$$= \frac{1}{2}\left[\frac{1}{s-1}\left(\frac{(s-2)}{(s-2)^2-1}\right) + \frac{1}{s+1}\left(\frac{s}{s^2-1}\right) \right]$$

* Problems Based on Evaluation *

* Problems Based on Evaluation *

① Evaluate $\int_0^\infty e^{-2t}\left[\int_0^t \left(\frac{1-e^{-u}}{u}\right) du\right] dt$

$L\{☺\} = \int_0^\infty e^{-st} ☺ \, dt$

Sol:

$$L\{1-e^{-u}\} = \frac{1}{s} - \frac{1}{s+1}$$

$$L\left\{\frac{1-e^{-u}}{u}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right) ds = \left[\ln s - \ln(s+1)\right]_s^\infty = \ln\left(\frac{s+1}{s}\right)$$

$$L\left\{\int_0^t \frac{1-e^{-u}}{u} du\right\} = \frac{1}{s}\ln\left(\frac{s+1}{s}\right)$$

$$\int_0^\infty e^{-st}\left(\int_0^t \left(\frac{1-e^{-u}}{u}\right) du\right) dt = \frac{1}{s}\ln\left(\frac{s+1}{s}\right)$$

put $s=2$

$$\int_0^\infty e^{-2t}\left(\int_0^t \left(\frac{1-e^{-u}}{u}\right) du\right) dt = \frac{1}{2}\ln\left(\frac{3}{2}\right)$$ ✓

② Evaluate $\int_0^\infty e^{-t}\left(t\int_0^t e^{-4u}\cos u \, du\right) dt$

Sol:

$$L\{\cos u\} = \frac{s}{s^2+1}$$

$$L\{e^{-4u}\cos u\} = \frac{s+4}{(s+4)^2+1} = \frac{s+4}{s^2+8s+17}$$

$$L\left\{\int_0^t e^{-4u}\cos u \, du\right\} = \frac{1}{s}\left(\frac{s+4}{s^2+8s+17}\right) = \frac{s+4}{s^3+8s^2+17s}$$

$$L\left\{t\int_0^t e^{-4u}\cos u \, du\right\} = (-1)^1 \frac{d}{ds}\left(\frac{s+4}{s^3+8s^2+17s}\right)$$

$$= (-1)\left[\frac{(s^3+8s^2+17s) - (s+4)(3s^2+16s+17)}{(s^3+8s^2+17s)^2}\right]$$

$$\left[\quad (s^3+8s^2+1+s)^2 \quad \right]$$

$$\int_0^\infty e^{-st}\left[t\int_0^t e^{-4u}\cos 5u\,du\right]dt = \quad \text{—||——||—}$$

put $s=1$

$$\int_0^\infty e^{-t}\left[t\int_0^t e^{-4u}\cos 5u\,du\right]dt = (-1)\left[\frac{(26)-(5)(36)}{(26)^2}\right]$$

$$= \frac{77}{338}.$$

* Inverse Laplace Transform *

If $L\{f(t)\} = \phi(s)$ then $f(t)$ is called inverse Laplace Transform of $\phi(s)$ and it is denoted as

$$L^{-1}\{\phi(s)\} = f(t)$$

Table of Inverse Laplace Transform

1. $L\{1\} = \frac{1}{s} \Rightarrow L^{-1}\{\frac{1}{s}\} = 1$
2. $L\{e^{at}\} = \frac{1}{s-a} \Rightarrow L^{-1}\{\frac{1}{s-a}\} = e^{at}$
3. $L\{e^{-at}\} = \frac{1}{s+a} \Rightarrow L^{-1}\{\frac{1}{s+a}\} = e^{-at}$
4. $L\{\cos at\} = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\{\frac{s}{s^2+a^2}\} = \cos at$
5. $L\{\cosh at\} = \frac{s}{s^2-a^2} \Rightarrow L^{-1}\{\frac{s}{s^2-a^2}\} = \cosh at$
6. $L\{\sin at\} = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\{\frac{1}{s^2+a^2}\} = \frac{\sin at}{a}$
7. $L\{\sinh at\} = \frac{a}{s^2-a^2} \Rightarrow L^{-1}\{\frac{1}{s^2-a^2}\} = \frac{\sinh at}{a}$
8. $L\{t^{n-1}\} = \frac{\Gamma n}{s^n} \Rightarrow L^{-1}\{\frac{1}{s^n}\} = \frac{t^{n-1}}{\Gamma n}$

Some Basic Questions on I.L.T.

1. $L^{-1}\left\{\frac{3+2s+s^2}{s^3}\right\}$

Sol: $L^{-1}\left\{\frac{3+2s+s^2}{s^3}\right\} = L^{-1}\left\{\frac{3}{s^3} + \frac{2s}{s^3} + \frac{s^2}{s^3}\right\}$

$$= 3L^{-1}\left\{\frac{1}{s^3}\right\} + 2L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{1}{s}\right\}$$

$$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{\Gamma n}$$

$$= 3\left(\frac{t^2}{\lceil 3}\right) + 2\left(\frac{t^1}{\lceil 2}\right) + 1$$

$$= \frac{3t^2}{2} + 2t + 1$$

② Find $L^{-1}\left\{\frac{s+3}{s^2+4}\right\}$

Sol: $L^{-1}\left\{\frac{s+3}{s^2+4}\right\} = L^{-1}\left\{\frac{s}{s^2+4} + \frac{3}{s^2+4}\right\}$

$$= L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + 3L^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$, $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

$$= \cos 2t + 3\left(\frac{\sin 2t}{2}\right)$$

③ Find $L^{-1}\left\{\frac{1}{4s-5}\right\}$

Sol: $L^{-1}\left\{\frac{1}{4s-5}\right\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s-\frac{5}{4}}\right\} = \frac{1}{4}e^{\frac{5}{4}t}$

④ Find $L^{-1}\left\{\left(\frac{1-\sqrt{s}}{s^2}\right)^2\right\}$

Sol: $L^{-1}\left\{\left(\frac{1-\sqrt{s}}{s^2}\right)^2\right\} = L^{-1}\left\{\frac{1-2\sqrt{s}+s}{s^4}\right\}$

$$= L^{-1}\left\{\frac{1}{s^4} - \frac{2}{s^{7/2}} + \frac{1}{s^3}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^4}\right\} - 2L^{-1}\left\{\frac{1}{s^{7/2}}\right\} + L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= \frac{t^3}{\lceil 4} - 2\frac{t^{5/2}}{\lceil 7/2} + \frac{t^2}{\lceil 3} \checkmark$$

$$= \frac{t^3}{6} - \frac{16}{15\sqrt{\pi}}t^{5/2} + \frac{t^2}{2} \checkmark$$

⑤ If $L\{f(t)\} = \frac{s+2}{s^2+2}$. find $L\{f'(t)\}$

Sol: $L\{f'(t)\} = -f(0) + sL\{f(t)\}$

$\therefore f(t) = L^{-1}\left\{\frac{s+2}{s^2+2}\right\}$

$$\therefore\ f(t) = L^{-1}\left\{\frac{s+2}{s^2+2}\right\}$$

$$= L^{-1}\left\{\frac{s}{s^2+(\sqrt{2})^2}\right\} + 2\,L^{-1}\left\{\frac{1}{s^2+(\sqrt{2})^2}\right\}$$

$$= \cos\sqrt{2}t + 2\left(\frac{\sin\sqrt{2}t}{\sqrt{2}}\right)$$

$$= \cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t$$

$$\therefore\ f(0) = \cos 0 + \sqrt{2}\sin 0 = 1+0 = 1$$

$$\therefore\ L\{f'(t)\} = -f(0) + s\,L\{f(t)\}$$

$$= -1 + s\left(\frac{s+2}{s^2+2}\right) = -1 + \frac{s^2+2s}{s^2+2}$$

$$= \frac{2s-2}{s^2+2}$$

* Use of Shifting Theorem *

$$L^{-1}\{\phi(s+a)\} = e^{-at}\,L^{-1}\{\phi(s)\}$$

$$L^{-1}\{\phi(s-a)\} = e^{at}\,L^{-1}\{\phi(s)\}$$

How to apply ?

1. $L^{-1}\left\{\frac{1}{(s-b)^n}\right\} = e^{bt}\,L^{-1}\left\{\frac{1}{s^n}\right\} = e^{bt}\,\frac{t^{n-1}}{\lceil n}$ $\quad (s-b) \rightarrow b$

2. $L^{-1}\left\{\frac{(s-b)}{(s-b)^2+a^2}\right\} = e^{bt}\,L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = e^{bt}\cos at$

$(s-b) \longrightarrow b$

3. $L^{-1}\left\{\frac{(s+a)}{(s+a)^2-b^2}\right\} = e^{-at}\,L^{-1}\left\{\frac{s}{s^2-b^2}\right\} = e^{-at}\cosh bt$

4. $L^{-1}\left\{\frac{1}{(s+a)^2-b^2}\right\} = e^{-at}\,L^{-1}\left\{\frac{1}{s^2-b^2}\right\} = e^{-at}\,\frac{\sinh bt}{b}$

* Problems based on inverse shifting *

1. Find $L^{-1}\left\{\frac{s}{(s-2)^6}\right\}$

① find $L^{-1}\left\{\frac{s}{(s-2)^6}\right\}$

Sol: $L^{-1}\left\{\frac{s}{(s-2)^6}\right\} = L^{-1}\left\{\frac{(s-2)+2}{(s-2)^6}\right\}$

$= e^{2t} L^{-1}\left\{\frac{s+2}{s^6}\right\}$

$= e^{2t} L^{-1}\left\{\frac{1}{s^5}+\frac{2}{s^6}\right\}$

$= e^{2t}\left[\frac{t^4}{\lfloor 5}+2\left(\frac{t^5}{\lfloor 6}\right)\right]$

$= e^{2t}\left[\frac{t^4}{24}+\frac{t^5}{60}\right]$

② find $L^{-1}\left\{\frac{1}{\sqrt{2s+1}}\right\}$

Sol: $L^{-1}\left\{\frac{1}{\sqrt{2s+1}}\right\} = L^{-1}\left\{\frac{1}{\sqrt{2}\,(s+\frac{1}{2})^{1/2}}\right\}$

$= \frac{1}{\sqrt{2}} L^{-1}\left\{\frac{1}{(s+\frac{1}{2})^{1/2}}\right\}$

$= \frac{1}{\sqrt{2}} e^{-\frac{1}{2}t} L^{-1}\left\{\frac{1}{s^{1/2}}\right\}$

$= \frac{1}{\sqrt{2}} e^{-\frac{t}{2}} \frac{t^{\frac{1}{2}-1}}{\lfloor 1/2} = \frac{1}{\sqrt{2}} \frac{e^{-\frac{t}{2}}}{\sqrt{t}\sqrt{\pi}}$

How to write $s^2+as+b \longrightarrow (s\pm c)^2 \pm d^2$?

eg. ① $s^2-2s-3 = s^2-2s+\left(\frac{1}{2}\times \text{coeff.}s\right)^2-\left(\frac{1}{2}\times\text{coeff.}s\right)^2-3$

$= s^2-2s+1-1-3$

$= (s-1)^2-4 = (s-1)^2-2^2$

② $s^2+4s+8 = s^2+4s+\left(\frac{1}{2}\times\text{coeff.}s\right)^2-\left(\frac{1}{2}\times\text{coeff.}s\right)^2+8$

$= s^2+4s+4-4+8$

$(s+2)^2+4 = (s+2)^2+2^2$

① find $L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\}$

Sol: $L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s+2}{\underbrace{s^2-4s+4}\ \underbrace{-4+13}}\right\}$

$$= L^{-1}\left\{\frac{s+2}{(s-2)^2+3^2}\right\}$$

$$= L^{-1}\left\{\frac{(s-2)+4}{(s-2)^2+3^2}\right\}$$

$$= e^{2t}\, L^{-1}\left\{\frac{s+4}{s^2+3^2}\right\}$$

$$= e^{2t}\left[L^{-1}\left\{\frac{s}{s^2+3^2}\right\} + 4L^{-1}\left\{\frac{1}{s^2+3^2}\right\}\right]$$

$$= e^{2t}\left(\cos 3t + 4\,\frac{\sin 3t}{3}\right)$$

② find $L^{-1}\left\{\frac{4s+12}{s^2+8s+12}\right\}$

Sol: $L^{-1}\left\{\frac{4s+12}{s^2+8s+12}\right\} = L^{-1}\left\{\frac{4s+12}{(s+4)^2-2^2}\right\}$

$$= L^{-1}\left\{\frac{4(s+4)-4}{(s+4)^2-2^2}\right\}$$

$$= e^{-4t}\, L^{-1}\left\{\frac{4s-4}{s^2-2^2}\right\}$$

$$= e^{-4t}\left[L^{-1}\left\{\frac{4s}{s^2-2^2}\right\} - L^{-1}\left\{\frac{4}{s^2-2^2}\right\}\right]$$

$$= e^{-4t}\left[4\cosh 2t - 4\,\frac{\sinh 2t}{2}\right]$$

$$= e^{-4t}\left[4\cosh 2t - 2\sinh 2t\right]$$

③ find $L^{-1}\left\{\frac{2s-1}{\ldots}\right\}$

$s^2+4s+29$

Ans: $2e^{-2t}\cos 5t - \dfrac{5e^{-2t}\sin 5t}{5}$

$e^{-2t}(2\cos 5t - \sin 5t)$

④ find $L^{-1}\left\{\dfrac{s}{(2s+1)^2}\right\}$

sol: $L^{-1}\left\{\dfrac{s}{4\left(s+\frac{1}{2}\right)^2}\right\} = \dfrac{1}{4}L^{-1}\left\{\dfrac{\left(s+\frac{1}{2}\right)-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2}\right\}$

$= \dfrac{1}{4}e^{-\frac{t}{2}}L^{-1}\left\{\dfrac{s-\frac{1}{2}}{s^2}\right\}$

$= \dfrac{e^{-t/2}}{4}L^{-1}\left\{\dfrac{1}{s} - \dfrac{1}{2s^2}\right\}$

$= \dfrac{e^{-t/2}}{4}\left[1 - \dfrac{t}{2}\right]$

Tutorial

Tuesday, September 21, 2021 11:52 AM

Q3 (a) $t\sqrt{1+\sin t}$

Sol: $L\{t\sqrt{1+\sin t}\} = L\left\{t\sqrt{\cos^2\frac{t}{2}+\sin^2\frac{t}{2}+2\sin\frac{t}{2}\cos\frac{t}{2}}\right\}$

$$= L\left\{t\sqrt{\left(\cos\frac{t}{2}+\sin\frac{t}{2}\right)^2}\right\}$$

$$= L\left\{t\left(\cos\frac{t}{2}+\sin\frac{t}{2}\right)\right\}$$

$$L\left\{\cos\frac{t}{2}+\sin\frac{t}{2}\right\} = \frac{s}{s^2+\frac{1}{4}} + \frac{1/2}{s^2+1/4} = \frac{s+1/2}{s^2+\frac{1}{4}}$$

$$= \frac{2s+1}{2}\times\frac{4}{4s^2+1} = \frac{4s+2}{4s^2+1}$$

$$L\left\{t\left(\cos\frac{t}{2}+\sin\frac{t}{2}\right)\right\} = (-1)^1\frac{d}{ds}\left(\frac{4s+2}{4s^2+1}\right)$$

$$= (-1)\left[\frac{(4s^2+1)(4)-(4s+2)(8s)}{(4s^2+1)^2}\right]$$

$$= (-1)\left[\frac{16s^2+4-32s^2-16s}{(4s^2+1)^2}\right]$$

$$= \frac{16s^2+16s-4}{(4s^2+1)^2}$$

Q3 (b) $\int_0^\infty e^{-t}\sqrt{1+\sin t}\,dt$

Sol: $\sqrt{1+\sin t} = \sqrt{\cos^2\frac{t}{2}+\sin^2\frac{t}{2}+2\sin\frac{t}{2}\cos\frac{t}{2}}$

$= \cos\frac{t}{2}+\sin\frac{t}{2}$

$\therefore L\{\sqrt{1+\sin t}\} = L\{\cos\frac{t}{2}+\sin\frac{t}{2}\}$

$\Downarrow \qquad = \frac{s}{s^2+\frac{1}{4}} + \frac{1/2}{s^2+1/4} = \frac{s+1/2}{s^2+\frac{1}{4}}$

$$\int_0^\infty e^{-st}\sqrt{1+\sin t}\,dt = \frac{s+1/2}{s^2+1/4}$$

put $s=1$

$$\int_0^\infty e^{-t}\sqrt{1+\sin t}\,dt = \frac{1+1/2}{1+1/4} = \frac{3/2}{5/4} = \frac{3}{2}\times\frac{4}{5} = \frac{6}{5}$$

Method of Partial fraction :

Form of Rational Function	Form of the Partial Fraction
$\dfrac{p(x)+q}{(x-a)(x-b)} a \neq b$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}$
$\dfrac{p(x)+q}{(x-a)^2}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}$
$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$
$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{x-b}$
$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{x-a}+\dfrac{Bx+C}{x^2+bx+c}$
Where, x^2+bx+c cannot be factorised further	A, B, C are real numbers that are to be determined.

$$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)^2} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c} + \frac{Dx+E}{(x^2+bx+c)^2}$$

1. Find $L^{-1}\left\{\dfrac{s+29}{(s+4)(s^2+9)}\right\}$

Sol:

$$\frac{s+29}{(s+4)(s^2+9)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+9}$$

$$= \frac{A(s^2+9)+(s+4)(Bs+C)}{(s+4)(s^2+9)}$$

$$\therefore \quad s+29 = A(s^2+9)+(s+4)(Bs+C) \text{ ——— } (*)$$

put $s=-4$ in $(*)$

$$25 = A(25)+0 \quad \Rightarrow \boxed{A=1}$$

put $s=0$ in $(*)$

$$29 = 9A+4C$$
$$= 9(1)+4C \quad \Rightarrow 4C = 20$$
$$\boxed{C=5}$$

put $s=1$ in $(*)$

put $s=1$ in ⊛

$$30 = A(10) + (5)(B+C)$$
$$= 10 + (5)(B+5)$$
$$= 10 + 5B + 25 \Rightarrow 5B = -5$$
$$\boxed{B=-1}$$

$$\therefore \frac{s+29}{(s+4)(s^2+9)} = \frac{1}{s+4} + \frac{-s+5}{s^2+9}$$
$$= \frac{1}{s+4} - \frac{s}{s^2+9} + \frac{5}{s^2+9}$$

$$L^{-1}\left\{\frac{s+29}{(s+4)(s^2+9)}\right\} = L^{-1}\left\{\frac{1}{s+4}\right\} - L^{-1}\left\{\frac{s}{s^2+9}\right\} + 5L^{-1}\left\{\frac{1}{s^2+9}\right\}$$
$$= e^{-4t} - \cos 3t + \frac{5}{3}\sin 3t$$

② Find $L^{-1}\left\{\frac{s+2}{(s+3)(s+1)^3}\right\}$

Sol: Let $\frac{s+2}{(s+3)(s+1)^3} = \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$

$$\therefore \frac{(s+2)}{(s+3)(s+1)^3} = \frac{A(s+1)^3 + B(s+3)(s+1)^2 + C(s+3)(s+1) + D(s+3)}{(s+3)(s+1)^3}$$

$$\therefore s+2 = A(s+1)^3 + B(s+3)(s+1)^2 + C(s+3)(s+1) + D(s+3) \quad ⊛$$

put $s=-3$ in ⊛

$$\therefore -1 = A(-8) + 0 + 0 + 0 \qquad \therefore A = 1/8$$

put $s=-1$ in ⊛

$$\therefore 1 = 0 + 0 + 0 + D(2) \qquad \therefore D = \frac{1}{2}$$

put $s=0$ in ⊛

$$2 = A + 3B + 3C + 3D$$

$$= \frac{1}{8} + 3B + 3C + \frac{3}{2} \quad \Rightarrow 3B + 3C = 2 - \frac{1}{8} - \frac{3}{2} = \frac{3}{8}$$

$$\Rightarrow B + C = \frac{1}{8} \quad \text{—— (1)}$$

put $s = 1$ in (*)

$$3 = A(8) + B(4)(4) + C(4)(2) + D(4)$$

$$= 1 + 16B + 8C + 2$$

$$\Rightarrow 16B + 8C = 0$$

$$2B + C = 0 \quad \text{—— (2)}$$

(1) & (2) $\Rightarrow$

$B = -1/8$, $C = 1/4$

$\therefore A = \frac{1}{8}$, $B = -\frac{1}{8}$, $C = \frac{1}{4}$, $D = \frac{1}{2}$

$$\therefore \frac{s+2}{(s+3)(s+1)^3} = \frac{1/8}{s+3} + \frac{-1/8}{s+1} + \frac{1/4}{(s+1)^2} + \frac{1/2}{(s+1)^3}$$

$$\therefore L^{-1}\left\{\frac{s+2}{(s+3)(s+1)^3}\right\} = \frac{1}{8} L^{-1}\left\{\frac{1}{s+3}\right\} - \frac{1}{8} L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{4} L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= \frac{1}{8} e^{-3t} - \frac{1}{8} e^{-t} + \frac{1}{4} e^{-t} L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{2} e^{-t} L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= \frac{1}{8} e^{-3t} - \frac{1}{8} e^{-t} + \frac{1}{4} e^{-t}\, t + \frac{1}{2} e^{-t} \frac{t^2}{2}$$

$$= \frac{1}{8} e^{-3t} - e^{-t}\left(\frac{1}{8} - \frac{t}{4} - \frac{t^2}{4}\right)$$

(3) find $L^{-1}\left\{\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}\right\}$

Sol: $$L^{-1}\left\{\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}\right\} = L^{-1}\left\{\frac{(s+1)^2-5}{((s+1)^2+4)((s+1)^2+1)}\right\}$$

$$= e^{-t} L^{-1}\left\{ \frac{s^2-5}{\quad} \right\} \quad \text{—— (*)}$$

$$= e^{-t} L^{-1}\left\{\frac{s^2-5}{(s^2+4)(s^2+1)}\right\} \quad \text{—} (*)$$

Consider $\dfrac{s^2-5}{(s^2+4)(s^2+1)}$

Let $x = s^2$

$$\frac{x-5}{(x+4)(x+1)} = \frac{A}{x+4} + \frac{B}{x+1}$$

$$x-5 = A(x+1) + B(x+4)$$

put $x=-4 \Rightarrow -9 = A(-3)+0 \Rightarrow A = 3$

put $x=-1 \Rightarrow -6 = 0 + B(3) \Rightarrow B = -2$

$$\therefore \frac{x-5}{(x+4)(x+1)} = \frac{3}{x+4} + \frac{-2}{x+1}$$

put $x = s^2$

$$\therefore \frac{s^2-5}{(s^2+4)(s^2+1)} = \frac{3}{s^2+4} - \frac{2}{s^2+1}$$

$$\therefore L^{-1}\left\{\frac{s^2-5}{(s^2+4)(s^2+1)}\right\} = 3L^{-1}\left\{\frac{1}{s^2+4}\right\} - 2L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 3\frac{\sin 2t}{2} - 2\frac{\sin t}{1} \quad \text{—} (**)$$

from $(*)$ & $(**)$ $\Rightarrow$

$$L^{-1}\left\{\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}\right\} = e^{-t}\left(\frac{3\sin 2t}{2} - 2\sin t\right)$$

④ Find I.L.T. of

ⓐ $\dfrac{s}{(s^2+1)(s^2+4)(s^2+9)}$ ⓑ $\dfrac{2s}{s^4+4}$ ⓒ $\dfrac{s^2}{(s^2+1)(s^2+4)}$

(d) $\dfrac{2s-1}{s^4+s^2+1}$

Sol: (a) Consider $\dfrac{1}{(s^2+1)(s^2+4)(s^2+9)}$

Let $x=s^2$

$$\frac{1}{(x+1)(x+4)(x+9)} = \frac{A}{x+1}+\frac{B}{x+4}+\frac{C}{x+9}$$

$$1 = A(x+4)(x+9) + B(x+1)(x+9) + C(x+1)(x+4) \quad \text{———} \; (*)$$

put $x=-1$ in $(*)$

$$1 = A(3)(8)+0+0 \implies A = 1/24$$

put $x=-4$ in $(*)$

$$1 = 0+B(-3)(5)+0 \implies B = \frac{-1}{15}$$

put $x=-9$ in $(*)$

$$1 = 0+0+C(-8)(-5) \implies C = \frac{1}{40}$$

$$\frac{1}{(x+1)(x+4)(x+9)} = \frac{1/24}{x+1} + \frac{-1/15}{x+4} + \frac{1/40}{x+9}$$

put $x=s^2$

$$\frac{1}{(s^2+1)(s^2+4)(s^2+9)} = \frac{1}{24}\cdot\frac{1}{s^2+1} - \frac{1}{15}\,\frac{1}{s^2+4} + \frac{1}{40}\cdot\frac{1}{s^2+9}$$

multiply both sides by s

$$\frac{s}{(s^2+1)(s^2+4)(s^2+9)} = \frac{1}{24}\,\frac{s}{s^2+1} - \frac{1}{15}\,\frac{s}{s^2+4} + \frac{1}{40}\,\frac{s}{s^2+9}$$

$$L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right\} = \frac{1}{24}\cos t - \frac{1}{15}\cos 2t + \frac{1}{40}\cos 3t.$$

(b) $\dfrac{2s}{s^4+4}$

Sol: Middle Term $= 2\sqrt{\text{first term}} \times \sqrt{\text{const. term}}$

$$= 2\sqrt{s^4} \times \sqrt{4} = 2s^2(2) = 4s^2$$

$$\frac{2s}{s^4+4} = \frac{2s}{\underbrace{s^4+4s^2+4}\ -\underbrace{4s^2}}$$

$$= \frac{2s}{(s^2+2)^2-(2s)^2}$$

A^2-B^2
$=(A-B)(A+B)$

$$= \frac{2s}{(s^2+2-2s)(s^2+2+2s)}$$

$$\frac{2s}{s^4+4} = \frac{1}{2}\left[\frac{1}{s^2+2-2s} - \frac{1}{s^2+2+2s}\right]$$

$$L^{-1}\left\{\frac{2s}{s^4+4}\right\} = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}\right]$$

$$= \frac{1}{2}\left[e^{t}L^{-1}\left\{\frac{1}{s^2+1}\right\} - e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$$= \frac{1}{2}\left[e^{t}\sin t - e^{-t}\sin t\right]$$

$$= \frac{1}{2}\left(e^{t}-e^{-t}\right)\sin t = \left(\frac{e^{t}-e^{-t}}{2}\right)\sin t$$

$$= \sinh t \sin t$$

(c) $$\frac{s^2}{(s^2+1)(s^2+4)}$$

Let $x = s^2$

$$\frac{x}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4}$$

$$x = A(x+4) + B(x+1) \quad \text{——— } (*)$$

put $x=-1$ $\therefore$ $-1 = A(3)+0$ $\Rightarrow A = -1/3$

put $x=-4$ $\therefore -4 = 0 + B(-3) \Rightarrow B = 4/3$

$$\therefore \frac{x}{(x+1)(x+4)} = \frac{-1/3}{x+1} + \frac{4/3}{x+4}$$

put $x = s^2$

$$\frac{s^2}{(s^2+1)(s^2+4)} = \frac{-1}{3} \cdot \frac{1}{s^2+1} + \frac{4}{3}\frac{1}{s^2+4}$$

$$L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+4)}\right\} = -\frac{1}{3} L^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{4}{3} L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= -\frac{1}{3}\frac{\sin t}{1} + \frac{4}{3}\frac{\sin 2t}{2}$$

$$= \frac{1}{3}(2\sin 2t - \sin t)$$

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(d) $\dfrac{2s-1}{s^4+s^2+1}$

Sol: $\dfrac{2s-1}{s^4+s^2+1} = \dfrac{2s-1}{(s^4+2s^2+1) - s^2}$

$$= \frac{(2s-1)}{(s^2+1)^2 - s^2}$$

$$= \frac{2s-1}{(s^2+1-s)(s^2+1+s)}$$

$$\frac{2s-1}{s^4+s^2+\frac{1}{4}-\frac{1}{4}+1}$$

Let $\dfrac{2s-1}{(s^2-s+1)(s^2+s+1)} = \dfrac{As+B}{s^2-s+1} + \dfrac{Cs+D}{s^2+s+1}$ —— (*)

$$2s-1 = (As+B)(s^2+s+1) + (Cs+D)(s^2-s+1)$$

$$= (A+C)s^3 + (A+B-C+D)s^2 + (A+B+C-D)s$$

$+(B+D)$

Comparing both sides, we get

$$\left.\begin{aligned} A+C &= 0 \\ A+B-C+D &= 0 \\ A+B+C-D &= 2 \\ B+D &= -1 \end{aligned}\right\}$$

solving we get

$A=\frac{1}{2}, C=-\frac{1}{2}, B=\frac{1}{2}, D=-\frac{3}{2}$

$$\therefore \quad \frac{2s-1}{(s^2-s+1)(s^2+s+1)} = \frac{\frac{1}{2}s+\frac{1}{2}}{s^2-s+1} + \frac{-\frac{1}{2}s-\frac{3}{2}}{s^2+s+1}$$

$$= \frac{\frac{1}{2}(s+1)}{(s-\frac{1}{2})^2+\frac{3}{4}} - \frac{\frac{1}{2}(s+3)}{(s+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \frac{1}{2}\left[\frac{(s-\frac{1}{2})+\frac{3}{2}}{(s-\frac{1}{2})^2+3/4} - \frac{(s+\frac{1}{2})+\frac{5}{2}}{(s+\frac{1}{2})^2+3/4}\right]$$

$$L^{-1}\left\{-\text{''}-\right\} = \frac{1}{2}\left[e^{\frac{t}{2}} L^{-1}\left\{\frac{s+3/2}{s^2+3/4}\right\} - e^{-t/2} L^{-1}\left\{\frac{s+5/2}{s^2+3/4}\right\}\right]$$

$$= \frac{1}{2}\left[e^{t/2}\left(\cos\frac{\sqrt{3}}{2}t + \frac{3}{2}\times\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}\right) - e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t + \frac{5}{2}\,\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}\right)\right]$$

$$= \frac{1}{2}\left[e^{t/2}\left(\cos\frac{\sqrt{3}}{2}t + \sqrt{3}\sin\frac{\sqrt{3}}{2}t\right) - e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t + \frac{5}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)\right]$$

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$$\frac{2s-1}{(s^2-s+1)(s^2+s+1)} = \frac{2s}{(s^2-s+1)(s^2+s+1)} - \frac{1}{(s^2-s+1)(s^2+s+1)}$$

$$= \left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1}\right] -$$

Convolution Theorem

$$L^{-1}\{\phi_1(s) * \phi_2(s)\} = \int_0^t f_1(u)\, f_2(t-u)\, du$$

where $f_1(t) = L^{-1}\{\phi_1(s)\}$ & $f_2(t) = L^{-1}\{\phi_2(s)\}$

① Find $L^{-1}\left\{\frac{1}{s(s+a)}\right\}$ using Convolution theorem.

Sol: Let $\phi_1(s) = \frac{1}{s+a}$ | $\phi_2(s) = \frac{1}{s}$

$f_1(t) = L^{-1}\{\phi_1(s)\}$

$= L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

$\therefore f_1(u) = e^{-au}$

$f_2(t) = L^{-1}\{\phi_2(s)\}$

$= L^{-1}\left\{\frac{1}{s}\right\} = 1$

$f_2(u) = 1$

$\therefore f_2(t-u) = 1$

$$L^{-1}\{\phi_1(s) * \phi_2(s)\} = \int_0^t f_1(u) f_2(t-u) du$$

$$L^{-1}\left\{\frac{1}{s+a} * \frac{1}{s}\right\} = \int_0^t e^{-au} \cdot 1\, du$$

$$= \left[\frac{e^{-au}}{-a}\right]_0^t = \left\{\frac{e^{-at}}{-a}\right\} - \left\{\frac{1}{-a}\right\}$$

$$= \frac{e^{-at}}{-a} + \frac{1}{a} = \frac{1-e^{-at}}{a}$$

② Using convolution find $L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+4)}\right\}$

Sol: Let $\phi_1(s) = \frac{s}{s^2+1}$ | $\phi_2(s) = \frac{s}{s^2+4}$

Sol: Let $\phi_1(s) = \dfrac{s}{s^2+1}$

$f_1(t) = L^{-1}\{\phi_1(s)\} = L^{-1}\left\{\dfrac{s}{s^2+1}\right\}$

$= \cos t$

$f_1(u) = \cos u$

$\phi_2(s) = \dfrac{s}{s^2+4}$

$f_2(t) = L^{-1}\{\phi_2(s)\}$

$= L^{-1}\left\{\dfrac{s}{s^2+4}\right\} = \cos 2t$

$f_2(u) = \cos 2u$

$f_2(t-u) = \cos 2(t-u)$

$= \cos(2t-2u)$

$$L^{-1}\{\phi_1(s) * \phi_2(s)\} = \int_0^t f_1(u) f_2(t-u)\, du$$

$$L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+4)}\right\} = \int_0^t \cos u \cos(2t-2u)\, du$$

$$= \frac{1}{2}\int_0^t 2\cos u \cos(2t-2u)\, du$$

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$= \frac{1}{2}\int_0^t \cos(2t-u) + \cos(3u-2t)\, du$$

$$= \frac{1}{2}\left[\frac{\sin(2t-u)}{-1} + \frac{\sin(3u-2t)}{3}\right]_0^t$$

$$= \frac{1}{2}\left[\left\{\frac{\sin t}{-1} + \frac{\sin t}{3}\right\} - \left\{\frac{\sin 2t}{-1} + \frac{\sin(-2t)}{3}\right\}\right]$$

$$= \frac{1}{2}\left[-\frac{2}{3}\sin t + \frac{4}{3}\sin 2t\right]$$

$$= \frac{-\sin t + 2\sin 2t}{3}$$

③ Using convolution find $L^{-1}\left\{\frac{1}{s+a}\cdot\frac{1}{(s+b)^2}\right\}$

Sol: Let $\phi_1(s)=\frac{1}{(s+b)^2}$ | $\phi_2(s)=\frac{1}{s+a}$

$$f_1(t)=L^{-1}\{\phi_1(s)\}=L^{-1}\left\{\frac{1}{(s+b)^2}\right\}$$
$$=e^{-bt}L^{-1}\left\{\frac{1}{s^2}\right\}$$
$$=e^{-bt}t$$

$$f_1(u)=e^{-bu}u$$

$$f_2(t)=L^{-1}\{\phi_2(s)\}$$
$$=L^{-1}\left\{\frac{1}{s+a}\right\}$$
$$=e^{-at}$$

$$f_2(u)=e^{-au}$$
$$f_2(t-u)=e^{-a(t-u)}$$

$$L^{-1}\{\phi_1(s)\phi_2(s)\}=\int_0^t f_1(u)f_2(t-u)\,du$$

$$L^{-1}\left\{\frac{1}{s+a}\cdot\frac{1}{(s+b)^2}\right\}=\int_0^t e^{-bu}u\,e^{-a(t-u)}\,du$$
$$=\int_0^t e^{-bu}u\,e^{-at}e^{au}\,du$$
$$=e^{-at}\int_0^t e^{-bu}u\,e^{au}\,du$$
$$=e^{-at}\int_0^t u\,e^{(a-b)u}\,du$$
$$=e^{-at}\left[u\frac{e^{(a-b)u}}{(a-b)}-(1)\frac{e^{(a-b)u}}{(a-b)^2}\right]_0^t$$
$$=e^{-at}\left[\left\{t\frac{e^{(a-b)t}}{a-b}-\frac{e^{(a-b)t}}{(a-b)^2}\right\}-\left\{0-\frac{1}{(a-b)^2}\right\}\right]$$

$e^{-at}\left[\ t\,e^{(a-b)t}-e^{(a-b)t}\right.$

$$= e^{-at}\left[\frac{t\,e^{(a-b)t}}{a-b} - \frac{e^{(a-b)t}}{(a-b)^2} + \frac{1}{(a-b)^2}\right]$$

④ Using convolution Thm. find $L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$

Sol: $L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} = L^{-1}\left\{\frac{(s+1)^2+2}{((s+1)^2+1)\,((s+1)^2+4)}\right\}$

$$= e^{-t}\,L^{-1}\left\{\frac{s^2+2}{(s^2+1)(s^2+4)}\right\}$$

$$= e^{-t}\,L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+4)} + \frac{2}{(s^2+1)(s^2+4)}\right\}$$

$$= e^{-t}\left[L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+4)}\right\} + L^{-1}\left\{\frac{2}{(s^2+1)(s^2+4)}\right\}\right]$$

$$= e^{-t}\left[\int_0^t \cos u \cos(2t-2u)\,du + 2\int_0^t \sin u\,\frac{\sin(2t-2u)}{2}\,du\right]$$

① Find $L^{-1}\left\{\frac{1}{(s-a)(s+a)^2}\right\}$ using convolutions.

Sol: Let $\phi_1(s) = \frac{1}{(s+a)^2}$

$$f_1(t) = L^{-1}\{\phi_1(s)\}$$
$$= L^{-1}\left\{\frac{1}{(s+a)^2}\right\}$$
$$= e^{-at} L^{-1}\left\{\frac{1}{s^2}\right\}$$
$$= e^{-at}\, t$$

$$f_1(u) = e^{-au}\, u$$

$\phi_2(s) = \frac{1}{s-a}$

$$f_2(t) = L^{-1}\{\phi_2(s)\}$$
$$= L^{-1}\left\{\frac{1}{s-a}\right\}$$
$$= e^{at}$$

$$f_2(u) = e^{au}$$
$$f_2(t-u) = e^{a(t-u)}$$

∴ By convolution Theorem

$$L^{-1}\{\phi_1(s) * \phi_2(s)\} = \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$L^{-1}\left\{\frac{1}{(s-a)(s+a)^2}\right\} = \int_0^t e^{-au}\, u\, e^{a(t-u)}\, du$$

$$= \int_0^t e^{-au}\, u\, e^{at}\, e^{-au}\, du$$

$$= e^{at} \int_0^t u\, e^{-2au}\, du$$

$$= e^{at}\left[(u)\left(\frac{e^{-2au}}{-2a}\right) - (1)\left(\frac{e^{-2au}}{4a^2}\right)\right]_0^t$$

$$= e^{at}\left[\left\{\frac{t e^{-2at}}{-2a} - \frac{e^{-2at}}{4a^2}\right\} - \left\{0 - \frac{1}{4a^2}\right\}\right]$$

$$= \frac{t e^{-at}}{-2a} - \frac{e^{-at}}{4a^2} + \frac{e^{at}}{4a^2} = \frac{t e^{-at}}{-2a} + \frac{\sinh at}{2a^2}$$

② Find $L^{-1}\left\{\frac{1}{\ \ }\right\}$ using convolutions

② find $L^{-1}\left\{\frac{1}{(s+3)(s^2+2s+2)}\right\}$ using convolutions

Sol: $L^{-1}\left\{\frac{1}{(s+1+2)((s+1)^2+1)}\right\} = e^{-t}L^{-1}\left\{\frac{1}{(s+2)(s^2+1)}\right\}$ ——— (*)

Let $\phi_1(s) = \frac{1}{s^2+1}$

$f_1(t) = L^{-1}\{\phi_1(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= \sin t$

$f_1(u) = \sin u$

$\phi_2(s) = \frac{1}{s+2}$

$f_2(t) = L^{-1}\{\phi_2(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$

$f_2(u) = e^{-2u}$

$f_2(t-u) = e^{-2(t-u)}$

By convol. thm.

$$L^{-1}\{\phi_1(s) * \phi_2(s)\} = \int_0^t f_1(u)\, f_2(t-u)$$

$$L^{-1}\left\{\frac{1}{(s+2)(s^2+1)}\right\} = \int_0^t \sin u\; e^{-2(t-u)}\, du$$

$$= e^{-2t}\int_0^t e^{2u}\sin u\, du$$

$$\int e^{At}\sin Bt\, dt = \frac{e^{At}}{A^2+B^2}(A\sin Bt - B\cos Bt)$$

$$= e^{-2t}\left[\frac{e^{2u}}{5}(2\sin u - \cos u)\right]_0^t$$

$$= e^{-2t}\left[\left\{\frac{e^{2t}}{5}(2\sin t - \cos t)\right\} - \left\{\frac{1}{5}(0-1)\right\}\right]$$

$$= e^{-2t}\left[\frac{e^{2t}}{5}(2\sin t - \cos t) + \frac{1}{5}\right]$$

$$= \frac{1}{5}(2\sin t - \cos t) + \frac{e^{-2t}}{5}$$ ——— (**)

from (*) & (**), we get

from (*) & (**), we get

$$\bar{L}^1\left\{\frac{1}{(s+3)(s^2+2s+2)}\right\} = e^{-t}\left[\frac{1}{5}(2\sin t-\cos t)+\frac{e^{-2t}}{5}\right]$$

(3) find $\bar{L}^1\left\{\frac{s^2+s}{(s^2+1)(s^2+2s+2)}\right\}$ using convolutions

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Sol: $\phi_1(s) = \frac{(s+1)}{s^2+2s+2}$ | $\phi_2(s) = \frac{s}{s^2+1}$

$f_1(t) = \bar{L}^1\left\{\frac{s+1}{(s+1)^2+1}\right\}$

$= e^{-t}\cos t$

$f_1(u) = e^{-u}\cos u$

$f_2(t) = \bar{L}^1\left\{\frac{s}{s^2+1}\right\}$

$= \cos t$

$f_2(t-u) = \cos(t-u)$

By convolution Thm.

$$\bar{L}^1\{\phi_1(s) * \phi_2(s)\} = \int_0^t f_1(u) f_2(t-u)\,du$$

$$\bar{L}^1\left\{\frac{s^2+s}{(s^2+1)(s^2+2s+2)}\right\} = \int_0^t e^{-u}\cos u\cos(t-u)\,du$$

$$= \frac{1}{2}\int_0^t e^{-u}\, 2\cos u\cos(t-u)\,du$$

$\int e^{At}\cos(Bt+C)\,dt = \frac{e^{At}}{A^2+B^2}\left(A\cos(Bt+C)+B\sin(Bt+C)\right)$

$$= \frac{1}{2}\int_0^t e^{-u}\left[\cos t+\cos(2u-t)\right]du$$

$$= \frac{1}{2}\left[\cos t\int_0^t e^{-u}du + \int_0^t e^{-u}\cos(2u-t)\,du\right]$$

$$= \frac{1}{2}\left[\cos t\left[\frac{e^{-u}}{-1}\right]_0^t + \left[\frac{e^{-u}}{5}\left(-\cos(2u-t)+2\sin(2u-t)\right)\right]_0^t\right]$$

$$= \frac{1}{2}\left[\cos t\left(\frac{e^{-t}}{-1}-\frac{1}{-1}\right)+\left\{\frac{e^{-t}}{5}\left(-\cos t+2\sin t\right)\right.\right.$$

$$= \frac{1}{2}\left[\cos t\left(\frac{e^{-t}}{-1} - \frac{1}{-1}\right) + \left\{\frac{e^{-t}}{5}\left(-\cos t + 2\sin t\right) - \frac{1}{5}\left(-\cos(-t) + 2\sin(-t)\right)\right\}\right]$$

$$= \frac{1}{2}\left[\cos t\left(1 - e^{-t}\right) + \frac{e^{-t}}{5}\left(-\cos t + 2\sin t\right) + \frac{1}{5}\left(\cos t + 2\sin t\right)\right]$$

Method of differentiation of $\phi(s)$

$$L^{-1}\{\phi(s)\} = -\frac{1}{t} L^{-1}\{\phi'(s)\}$$

Applicable to problems involving foll. functions

$\log$, $\tanh^{-1}$, $\tan^{-1}$, $\cot^{-1}$

Note:

$$\boxed{\tanh^{-1}\theta = \frac{1}{2}\log\left(\frac{1+\theta}{1-\theta}\right)}$$

① Find $L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\}$

Sol:

$$L^{-1}\{\phi(s)\} = -\frac{1}{t} L^{-1}\{\phi'(s)\}$$

$$L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\log\left(\frac{s+a}{s+b}\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left(\log(s+a) - \log(s+b)\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\}$$

$$= -\frac{1}{t}\left(e^{-at} - e^{-bt}\right) = \frac{e^{-bt} - e^{-at}}{t}$$

② Find $L^{-1}\{\cot^{-1}(s+1)\}$

Sol:

$$L^{-1}\{\phi(s)\} = -\frac{1}{t} L^{-1}\{\phi'(s)\}$$

$$L^{-1}\{\cot^{-1}(s+1)\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\cot^{-1}(s+1)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{-1}{1+(s+1)^2}\right\}$$

$$= -\frac{1}{t} e^{-t} L^{-1}\left\{\frac{-1}{1+s^2}\right\}$$

$$= \frac{1}{t} e^{-t} L^{-1}\left\{\frac{1}{s^2+1}\right\} = \frac{1}{t} e^{-t}\sin t$$

③ Find $L^{-1}\{2\tanh^{-1}s\}$

sol:

$$\tanh^{-1}\theta = \frac{1}{2}\log\left(\frac{1+\theta}{1-\theta}\right)$$

$$\boxed{2\tanh^{-1}\theta = \log\left(\frac{1+\theta}{1-\theta}\right)}$$

$$L^{-1}\{\phi(s)\} = -\frac{1}{t} L^{-1}\{\phi'(s)\}$$

$$L^{-1}\{2\tanh^{-1}s\} = L^{-1}\left\{\log\left(\frac{1+s}{1-s}\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\log\left(\frac{1+s}{1-s}\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left(\log(1+s) - \log(1-s)\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{1}{1+s} - \frac{1}{1-s}(-1)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s-1}\right\}$$

$$= -\frac{1}{t}\left(e^{-t} - e^{t}\right) = \frac{1}{t}\left(e^{t} - e^{-t}\right) = \frac{2\sinh t}{t}$$

④ Find $L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$

Sol: $L^{-1}\{\phi(s)\} = -\frac{1}{t} L^{-1}\{\phi'(s)\}$

$$L^{-1}\left\{\tan^{-1}\left(\frac{2}{s^2}\right)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{1}{1+\left(\frac{2}{s^2}\right)^2}\left(\frac{-4}{s^3}\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{s^4}{s^4+4} \times \frac{-4}{s^3}\right\}$$

$$= \frac{1}{t} L^{-1}\left\{\frac{4s}{s^4+4}\right\}$$

$$= \frac{1}{t} L^{-1}\left\{\frac{4s}{s^4+4s^2+4-4s^2}\right\}$$

$$= \frac{1}{t} L^{-1}\left\{\frac{4s}{(s^2+2)^2-(2s)^2}\right\}$$

$$= \frac{1}{t} L^{-1}\left\{\frac{4s}{(s^2+2-2s)(s^2+2+2s)}\right\}$$

$$= \frac{1}{t} L^{-1}\left\{\frac{1}{s^2-2s+2} - \frac{1}{s^2+2s+2}\right\}$$

$$= \frac{1}{t} L^{-1}\left\{\frac{1}{(s-1)^2+1^2} - \frac{1}{(s+1)^2+1^2}\right\}$$

$$= \frac{1}{t}\left[e^{t}\sin t - e^{-t}\sin t\right]$$

$$= \frac{1}{t}\left(e^{t}-e^{-t}\right)\sin t = \frac{2\sinh t \sin t}{t}$$

⑤ find $L^{-1}\left\{ s\log\left(\frac{s+1}{s-1}\right) \right\}$

Sol: $L^{-1}\{\phi(s)\} = -\frac{1}{t} L^{-1}\{\phi'(s)\}$

$$\therefore L^{-1}\left\{ s\log\left(\frac{s+1}{s-1}\right) \right\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds} s\log\left(\frac{s+1}{s-1}\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left[s\log(s+1) - s\log(s-1)\right]\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{ s\times\frac{1}{s+1} + \log(s+1) - s\times\frac{1}{s-1} - \log(s-1) \right\}$$

$$= -\frac{1}{t} L^{-1}\left\{ \frac{s}{s+1} - \frac{s}{s-1} + \log(s+1) - \log(s-1) \right\}$$

$s^2 - s^3$

$$= -\frac{1}{t} L^{-1}\left\{ \frac{-2s}{s^2-1} + \log\left(\frac{s+1}{s-1}\right)\right\}$$

$$= -\frac{1}{t}\left[-2\cosh t + L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\}\right]$$

Again Apply method of diff of $\phi(s)$

$$= -\frac{1}{t}\left[-2\cosh t + \left(-\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\log\left(\frac{s+1}{s-1}\right)\right\}\right)\right]$$

$$= -\frac{1}{t}\left[-2\cosh t - \frac{1}{t} L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s-1}\right\}\right]$$

$$= -\frac{1}{t}\left[-2\cosh t - \frac{1}{t}\left(e^{-t} - e^{t}\right)\right]$$

$$= -\frac{1}{t}\left[-2\cosh t + \frac{2\sinh t}{t}\right] = \frac{2\cosh t}{t} - \frac{2\sinh t}{t^2}$$

⑥ Prove that Using convolution

$$L^{-1}\{\ldots\} = \int^{t} \frac{e^{-bu} - e^{-au}}{\ldots}\, du$$

$$L^{-1}\left\{\frac{1}{s}\log\left(\frac{s+a}{s+b}\right)\right\} = \int_0^t \frac{\quad}{u}$$

Sol: Let $\phi_1(s) = \log\left(\frac{s+a}{s+b}\right)$ | $\phi_2(s) = \frac{1}{s}$

$$f_1(t) = L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\log\left(\frac{s+a}{s+b}\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\}$$

$$= -\frac{1}{t}\left(e^{-at} - e^{-bt}\right)$$

$$= \frac{e^{-bt} - e^{-at}}{t}$$

$$f_2(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$f_2(t-u) = 1$$

$$\therefore f_1(u) = \frac{e^{-bu} - e^{-au}}{u}$$

$\therefore$ By convolution Theorem

$$L^{-1}\{\phi_1(s) * \phi_2(s)\} = \int_0^t f_1(u) f_2(t-u)\, du$$

$$L^{-1}\left\{\frac{1}{s}\log\left(\frac{s+a}{s+b}\right)\right\} = \int_0^t \frac{e^{-bu} - e^{-au}}{u} \cdot 1\, du$$

* Fourier series *

Prerequisites

① $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

② $\sin n\pi = 0$, $\sin 2n\pi = 0$, $n \in \mathbb{Z}$

$\sin(n+1)\pi = 0$

$\sin(\text{Integer} \times \pi) = 0$

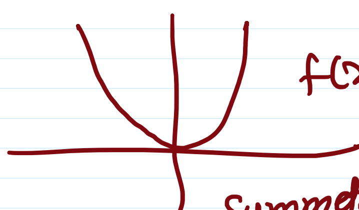
$\cos n\pi = (-1)^n$

$\cos 2n\pi = (-1)^{2n} = 1$

$\cos(n \pm 1)\pi = (-1)^{n+1}$

③ Even function

$f(x) = f(-x)$

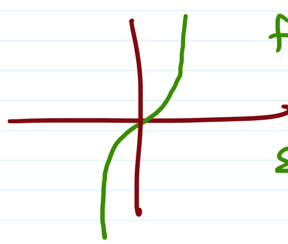


$f(x) = x^2$

Symmetric about y axis

④ ODD function

$f(x) = -f(-x)$



$f(x) = x^3$

Symm. about origin

⑤

$f(x)$	E	E	O	O
$g(x)$	E	O	E	O
$f(x) * g(x)$	E	O	O	E

if f(x) is even

$f(x)*g(x)$ | E | O | O | E (cut off at top)

(6) $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$$

(7) **Periodic function**

$$f(x+T) = f(x) \text{ where } T \text{ is period.}$$

(8) $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

(9) $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

(10) **Generalised Rule of Integration**

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\cdots$$

where $u' = \frac{du}{dx}$, $u'' = \frac{du'}{dx}$, $u''' = \frac{du''}{dx}$ & so on

$v_1 = \int v\,dx$, $v_2 = \int v_1\,dx$, $v_3 = \int v_2\,dx$ & so on

eg.

$$\int x^3 \sin 2x\,dx$$

$u = x^3$, $v = \sin 2x$

$u' = 3x^2$ $\quad v_1 = \int v\,dx = \int \sin 2x\,dx = -\frac{\cos 2x}{2}$

$u'' = 6x$ $\quad v_2 = \int v_1\,dx = \int -\frac{\cos 2x}{2}\,dx = -\frac{\sin 2x}{4}$

$u''' = 6$ $\quad V_3 = \int V_2 dx = \int -\frac{\sin 2x}{4} dx = \frac{\cos 2x}{8}$ $\quad$ & so on

$u^{iv} = 0$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4$$

$$\int x^3 \sin 2x\,dx = (x^3)\left(-\frac{\cos 2x}{2}\right) - (3x^2)\left(-\frac{\sin 2x}{4}\right) + (6x)\left(\frac{\cos 2x}{8}\right) - (6)\left(\frac{\sin 2x}{16}\right)$$

Drichlet's condition

A function $f(x)$ defined in $c_1 \le x \le c_2$ can be expressed as fourier series if

1. $f(x)$ and it's integrals are finite and single valued
2. $f(x)$ has finite no. of discontinuties.
3. $f(x)$ has finite no. of maxima & minima.

Fourier series in $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\,dx$$

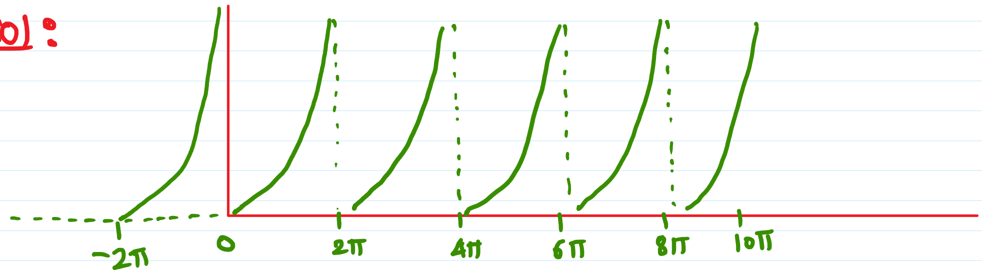
Parseval's Identity:

$$\frac{1}{\pi} \int_0^{2\pi} [f(x)]^2 dx = 2\left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

1. Obtain fourier series expansion of $f(x) = x^2$ in $(0, 2\pi)$ and $f(x+2\pi) = f(x)$. Hence deduce that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots$$

Sol :



$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{8\pi^3}{3}\right]$$

$$= \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(2x)\left(\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{4\pi\left(\frac{1}{n^2}\right)\right\} - \left\{0\right\}\right] = \frac{4}{n^2}$$

$$a_n = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x^2)\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(x^2)\left(-\frac{\cos nx}{n}\right) + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(x^2)\left(\frac{-\cos nx}{n}\right) + \frac{2\cos nx}{n^3} \right]_0$$

$$= \frac{1}{\pi}\left[\left\{(4\pi^2)\left(\frac{-1}{n}\right) + \frac{2}{n^3}\right\} - \left\{0 + \frac{2}{n^3}\right\}\right]$$

$$= -\frac{4\pi}{n}$$

$\therefore$ $a_0 = \frac{8\pi^2}{3}$, $a_n = \frac{4}{n^2}$, $b_n = -\frac{4\pi}{n}$

$\therefore$ The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$x^2 = \frac{\frac{8\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

$$x^2 = \frac{4\pi^2}{3} + \left(\frac{4}{1^2}\cos x - \frac{4\pi}{1}\sin x\right) + \left(\frac{4\cos 2x}{2^2} - \frac{4\pi}{2}\sin 2x\right) + \left(\frac{4\cos 3x}{3^2} - \frac{4\pi}{3}\sin 3x\right) + \text{--------}$$

__Deduction__

put $x = \pi$ in fourier series

$$\pi^2 = \frac{4\pi^2}{3} + \left(\frac{4}{1^2}(-1) - 0\right) + \left(\frac{4(1)}{2^2} - 0\right) + \left(\frac{4(-1)}{3^2} - 0\right) + \text{--------}$$

$$\pi^2 = \frac{4\pi^2}{3} - \frac{4}{1^2} + \frac{4}{2^2} - \frac{4}{3^2} + \text{----}$$

$$\pi^2 - \frac{4\pi^2}{3} = -\frac{4}{1^2} + \frac{4}{2^2} - \frac{4}{3^2} + \text{-----}$$

$$-\frac{\pi^2}{3} = -4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \text{----}\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \text{-------}$$

② Find fourier series expansion of $f(x) = e^{-x}, 0 < x < 2\pi$ and $f(x+2\pi) = f(x)$. Hence, deduce that

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{\pi}{2} \cdot \frac{1}{\sinh\pi}$$

Sol:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi}$$

$$= -\frac{1}{\pi}\left[e^{-x}\right]_0^{2\pi} = -\frac{1}{\pi}\left[e^{-2\pi} - 1\right] = \frac{1-e^{-2\pi}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\cos nx + n\sin nx\right)\right]_0^{2\pi}$$

$\cos 2\pi n = 1$
$\sin 2\pi n = 0$

$$= \frac{1}{\pi}\left[\left\{\frac{e^{-2\pi}}{1+n^2}(-1+0)\right\} - \left\{\frac{1}{1+n^2}(-1+0)\right\}\right]$$

$\cos 0 = 1$
$\sin 0 = 0$

$$= \frac{1}{\pi}\left[\frac{-e^{-2\pi}}{1+n^2} + \frac{1}{1+n^2}\right]$$

$$= \frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{e^{-2\pi}}{1+n^2}\left(0-n(1)\right)\right\} - \left\{\frac{1}{1+n^2}(0-n)\right\}\right]$$

$$\left[\frac{}{1+n^2} \right] \left[\right]$$

$$= \frac{1}{\pi}\left[\frac{-ne^{-2\pi}}{1+n^2} + \frac{n}{1+n^2} \right]$$

$$= \frac{n}{\pi(1+n^2)}\left(1-e^{-2\pi}\right)$$

$$a_0 = \frac{1-e^{-2\pi}}{\pi}, \quad a_n = \frac{1-e^{-2\pi}}{\pi(1+n^2)}, \quad b_n = \frac{n}{\pi(1+n^2)}\left(1-e^{-2\pi}\right)$$

The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$e^{-x} = \frac{\frac{1-e^{-2\pi}}{\pi}}{2} + \sum_{n=1}^{\infty}\left[\frac{1-e^{-2\pi}}{\pi(1+n^2)}\cos nx + \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}\sin nx \right]$$

$$e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\left[\sum_{n=1}^{\infty}\left(\frac{\cos nx}{1+n^2} + \frac{n \sin nx}{1+n^2} \right) \right]$$

$$\text{———} (*)$$

Deduction

put $x = \pi$ in $(*)$

$$e^{-\pi} = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\left[\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{1+n^2} + 0 \right) \right]$$

$\cos n\pi = (-1)^n, \quad \sin n\pi = 0$

$$e^{-\pi} = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2} \right]$$

$$e^{-\pi} = \frac{1-e^{-2\pi}}{2\pi} - \frac{(1-e^{-2\pi})}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$e^{-\pi} = \frac{1-e^{-2\pi}}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$e \;=\; \frac{1-e}{\pi} \sum_{n=2} \frac{\;}{1+n^2}$$

$$\frac{e^{\pi}}{e^{\pi}} \times \frac{\pi e^{-\pi}}{1-e^{-2\pi}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$$

$$\frac{\pi}{e^{\pi}-e^{-\pi}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$$

$$\frac{\pi}{2\sinh(\pi)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2} \qquad \left[\because \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}\right]$$

H.W obtain fourier series expansion for $f(x) = x\sin x$ in $[0, 2\pi]$

Sol:

Fourier series formula

Thursday, October 7, 2021 10:48 AM

Fourier Series Mind Map

Interval	Fourier Series	Parsevel's Identity
$(0, 2\pi)$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ **Where** $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$ $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx$ $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx$	$\frac{1}{\pi}\int_0^{2\pi}[f(x)]^2 dx = 2\left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty}({a_n}^2 + {b_n}^2)$
$(-\pi, \pi)$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ **Where** $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$ $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$ $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$	$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = 2\left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty}({a_n}^2 + {b_n}^2)$
$(0, 2L)$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$ **Where** $a_0 = \frac{1}{L}\int_0^{2L} f(x)\, dx$ $a_n = \frac{1}{L}\int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx$ $b_n = \frac{1}{L}\int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$	$\frac{1}{L}\int_0^{2L}[f(x)]^2 dx = 2\left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty}({a_n}^2 + {b_n}^2)$

For Even Function in $(-\pi, \pi)$ or Half Range Cosine series for f(x) in $(0, \pi)$	**For Odd Function in $(-\pi, \pi)$ or Half Range Sine series for f(x) in $(0, \pi)$**
The **Fourier series** is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ **Where** $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$ $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$	The **Fourier series** is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ **Where** $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$
The **Parsevel's Identity** is : $\frac{1}{\pi}\int_0^{\pi}[f(x)]^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} {a_n}^2$	The **Parsevel's Identity** is : $\frac{1}{\pi}\int_0^{\pi}[f(x)]^2 dx = \frac{1}{2}\sum_{n=1}^{\infty} {b_n}^2$

$(-L, L)$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$ **Where** $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$ $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$ $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$	$\frac{1}{L}\int_{-L}^{L}[f(x)]^2dx = 2\left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty}(a_n{}^2 + b_n{}^2)$	**For Even Function in $(-L, L)$ or Half Range Cosine series for f(x) in $(0, L)$**	**For Odd Function in $(-L, L)$ or Half Range Sine series for f(x) in $(0, L)$**
			The **Fourier series** is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ **Where** $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$ $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$	The **Fourier series** is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ **Where** $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$
			The **Parsevel's Identity** is : $\frac{1}{L}\int_0^L [f(x)]^2dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n{}^2$	The **Parsevel's Identity** is : $\frac{1}{L}\int_0^L [f(x)]^2dx = \frac{1}{2}\sum_{n=1}^{\infty} b_n{}^2$

Parsevel's Identity

- use for deduction purpose.
- when the series contain power of π more than or equal to 4.

eg ① $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$

② $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$

① Obtain fourier series expansion of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $[0, 2\pi]$, $f(x+2\pi) = f(x)$

Hence deduce that

ⓐ $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

ⓑ $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$

Sol:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi - x)^2\,dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{-\pi^3}{-3} - \frac{\pi^3}{-3}\right]$$

$$= \frac{1}{4\pi}\left[\frac{\pi^3}{3}+\frac{\pi^3}{3}\right] = \frac{1}{4\pi}\times\frac{2\pi^3}{3} = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^2\cos nx\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\underbrace{\left(\pi^2-2\pi x+x^2\right)}_{u}\underbrace{\cos nx}_{v}\,dx$$

$$= \frac{1}{4\pi}\left[\left(\pi^2-2\pi x+x^2\right)\left(\frac{\sin nx}{n}\right)-\left(-2\pi+2x\right)\left(\frac{-\cos nx}{n^2}\right)+(2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-2\pi+2x\right)\left(\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left\{(2\pi)\left(\frac{1}{n^2}\right)\right\}-\left\{(-2\pi)\left(\frac{1}{n^2}\right)\right\}\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2}+\frac{2\pi}{n^2}\right] = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^2\sin nx\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\left(\pi^2-2\pi x+x^2\right)\sin nx\,dx$$

$$\left[\left(\pi^2 \ldots x^2\right)\left(\frac{-\cos nx}{}\right)-\left(-2\pi+2x\right)\left(\frac{-\sin nx}{}\right)\right.$$

$$= \frac{1}{4\pi}\Big[(\pi^2 - 2\pi x + x^2)\Big(\frac{\quad}{n}\Big) - (\quad)\Big(\frac{\quad}{n^2}\Big) + (2)\Big(\frac{\cos nx}{n^3}\Big)\Big]_0^{2\pi}$$

$$= \frac{1}{4\pi}\Big[(\pi^2 - 2\pi x + x^2)\Big(-\frac{\cos nx}{n}\Big)\Big]_0^{2\pi}$$

$$= \frac{1}{4\pi}\Big[\Big\{\pi^2\Big(\frac{-1}{n}\Big)\Big\} - \Big\{\pi^2\Big(\frac{-1}{n}\Big)\Big\}\Big]$$

$$= 0$$

$$\therefore b_n = 0$$

The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\frac{\pi^2}{6}}{2} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos nx + 0 \sin nx\right)$$

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2} \longrightarrow (*)$$

Deduction

(a) put $x = 0$ in $(*)$

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{4}-\frac{\pi^2}{12} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

(b) $a_0 = \frac{\pi^2}{6}$

$$\frac{\pi^4}{90} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots$$

$a_n = \frac{1}{n^2}$

$b_n = 0$

By Parseval's Identity

$$\frac{1}{\pi}\int_0^{2\pi}[f(x)]^2\,dx = 2\left(\frac{a_0}{2}\right)^2+\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\frac{1}{\pi}\int_0^{2\pi}\left[\left(\frac{\pi-x}{2}\right)^2\right]^2 dx = 2\left(\frac{\frac{\pi^2}{6}}{2}\right)^2+\sum_{n=1}^{\infty}\left(\left(\frac{1}{n^2}\right)^2+0^2\right)$$

$$\frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^4}{16}\,dx = \frac{\pi^4}{72}+\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{16\pi}\left[\frac{(\pi-x)^5}{-5}\right]_0^{2\pi} = \frac{\pi^4}{72}+\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{16\pi}\left[\left\{\frac{-\pi^5}{-5}\right\}-\left\{\frac{\pi^5}{-5}\right\}\right] = \frac{\pi^4}{72}+\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{1}{16\pi}\left[\frac{2\pi^5}{5}\right] = \frac{\pi^4}{72}+\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{\quad}-\pi^4 = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots$$

$$\frac{\pi^4}{40} - \frac{\pi^4}{72} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\cdots$$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\cdots$$

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$$(-\pi,\pi) \Longrightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

~~NENO~~

All type

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

even

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = 0$$

Odd

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

① $f(x) = e^{x}$

a_0 ✓

a_n ✓

b_n ✓

② $f(x) = e^{|x|}$

Even function

a_0 ✓

a_n ✓

$b_n = 0$ ✗

③ $f(x) = e^{-|x|}$

odd function

$a_0 = 0$ ✗

$a_n = 0$ ✗

b_n ✓

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_0^{a} f(x) & \text{if } f(x) \text{ is even} \end{cases}$$

Fourier series in $(-\pi, \pi)$

① Find fourier series for $f(x) = e^x$ in $-\pi < x < \pi$

Sol:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right] = \frac{2\sinh(\pi)}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$\sin n\pi = 0$
$\cos n\pi = (-1)^n$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{e^{\pi}}{1+n^2}((-1)^n + 0)\right\} - \left\{\frac{e^{-\pi}}{1+n^2}((-1)^n + 0)\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{1+n^2}\left(e^{\pi} - e^{-\pi}\right)\right]$$

$$\left[\begin{array}{l} \sinh\theta = \dfrac{e^{\theta} - e^{-\theta}}{2} \\ 2\sinh\theta = e^{\theta} - e^{-\theta} \end{array}\right]$$

$$= \frac{1}{\pi}\,\frac{(-1)^n}{1+n^2}\,2\sinh(\pi)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{e^{\pi}}{1+n^2}(0 - n(-1)^n)\right\} - \left\{\frac{e^{-\pi}}{1+n^2}(0 - n(-1)^n)\right\}\right.$$

$$= \frac{1}{\pi}\left[\frac{n(-1)^n}{1+n^2}\left(-e^{\pi}+e^{-\pi}\right)\right]$$

$$= \frac{-1}{\pi}\frac{n(-1)^n}{1+n^2}\left(e^{\pi}-e^{-\pi}\right) = -\frac{1}{\pi}\frac{n(-1)^n}{1+n^2}\,2\sinh(\pi)$$

The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)$$

$$e^x = \frac{\frac{2\sinh(\pi)}{\pi}}{2} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n 2\sinh(\pi)}{\pi(1+n^2)}\cos nx - \frac{n(-1)^n\,2\sinh(\pi)}{\pi(1+n^2)}\sin nx\right]$$

$$e^x = \frac{\sinh\pi}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n\cos nx}{1+n^2} - \frac{n(-1)^n\sin nx}{1+n^2}\right]$$

② Find the fourier series for $f(x) = x^2$ in $-\pi \le x \le \pi$ & Hence, prove that

ⓐ $\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$ ⓑ $\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$

ⓒ $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Sol:

$$f(x) = x^2$$
$$f(-x) = (-x)^2 = x^2$$
$\therefore f(x) = f(-x) \Rightarrow f(x)$ is even function

$$b_n = 0$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int^{\pi} x^2\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[(2x)\left(\frac{\cos nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[(2\pi)\,\frac{(-1)^n}{n^2}\right] \qquad [\cos n\pi = (-1)^n]$$

$$= \frac{4(-1)^n}{n^2}$$

The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$x^2 = \frac{\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} \quad \text{------} \quad (**)$$

Deductions

① put $x = 0$ in $(**)$, we get

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \qquad (\cos 0 = 1)$$

$$-\frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} + \frac{-1}{3^2} + \frac{1}{4^2} + \frac{-1}{5^2} + \frac{1}{6^2} + \cdots\cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots \quad \text{------} \quad (A)$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

② put $x = \pi$ in $(**)$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1} \frac{\cos nx}{n^2} \qquad (-1)^n(-1)^n = 1$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2}$$

$$\frac{2}{3}\pi^2 = 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \text{———} \quad (B)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$$

(A) + (B) $\Rightarrow$

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right) + \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)$$

$$= 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots$$

(3) Obtain the fourier expansion of

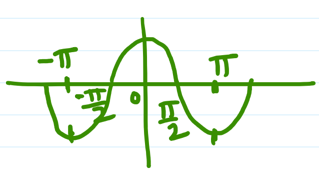
$$f(x) = \begin{cases} \cos x, & -\pi < x < 0 \\ -\cos x, & 0 < x < \pi \end{cases} \quad \& \quad f(x+2\pi) = f(x)$$

Sol:

$$f(-x) = \begin{cases} \cos(-x), & -\pi < -x < 0 \\ -\cos(-x), & 0 < -x < \pi \end{cases}$$

$\cos(-\theta) = \cos\theta$

$$= \begin{cases} \cos x, & \pi > x > 0 \\ -\cos x, & 0 > x > -\pi \end{cases}$$



$$= \begin{cases} \cos x, & 0 < x < \pi \\ -\cos x, & -\pi < x < 0 \end{cases} = \begin{cases} -\cos x, & -\pi < x < 0 \\ \cos x, & 0 < x < \pi \end{cases}$$

$$= -f(x)$$

$\therefore f(-x) = -f(x) \Longrightarrow f(x)$ is odd function

$a_n = 0$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} -\cos x \sin nx \, dx$$

$$= -\frac{1}{\pi}\int_0^{\pi} 2\sin nx \cos x \, dx$$

$$= -\frac{1}{\pi}\int_0^{\pi} \sin(n+1)x + \sin(n-1)x \, dx$$

$$= -\frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1} \right]_0^{\pi}$$

$$\cos(n\pm 1)\pi = \cos(n\pi \pm \pi)$$

$$= \cos n\pi \cos\pi \mp \sin n\pi \sin\pi$$

$$= (-1)^n(-1) \mp 0 \qquad = -(-1)^n$$

$$= -\frac{1}{\pi}\left[\left\{ \frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1} \right\} - \left\{ \frac{-1}{n+1} - \frac{1}{n-1} \right\} \right]$$

$$= -\frac{1}{\pi}\left[\frac{(-1)^n+1}{n+1} + \frac{(-1)^n+1}{n-1} \right]$$

$$= -\frac{1}{\pi}((-1)^n+1)\left[\frac{1}{n+1} + \frac{1}{n-1} \right]$$

$$= -\frac{1}{\pi}((-1)^n+1)\left(\frac{2n}{n^2-1}\right) =$$

$$b_1 = -\frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= -\frac{1}{\pi}\left[\frac{-\cos 2x}{2} \right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\cos 2\pi}{2} - \frac{\cos 0}{2} \right] = 0$$

$$b_1 = 0, \quad b_n = -\frac{1}{\pi}((-1)^n+1)\left(\frac{2n}{n^2-1}\right), \quad n \geq 2$$

The fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= 0 + \sum_{n=2}^{\infty} \left[-\frac{1}{\pi} \left((-1)^n + 1\right) \left(\frac{2n}{n^2-1}\right) \sin nx \right]$$

Q obtain Half Range sine series in $(0, \pi)$ for $f(x) = x(\pi - x)$ & Hence find value of $\sum \frac{(-1)^n}{(2n-1)^3}$

Sol:

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \sin nx \, dx$$

$$= \frac{2}{\pi} \left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[(-2) \frac{\cos nx}{n^3} \right]_0^{\pi} = -\frac{4}{\pi} \left[\frac{(-1)^n}{n^3} - \frac{1}{n^3} \right]$$

$$= \frac{4}{\pi} \left(\frac{1 - (-1)^n}{n^3} \right)$$

The Half Range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$x(\pi - x) = \sum_{n=1}^{\infty} \frac{4}{\pi} \left(\frac{1 - (-1)^n}{n^3} \right) \sin nx \quad //$$

Deduction

put $x = \frac{\pi}{2}$ in fourier series

$$\frac{\pi^2}{4} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin\left(\frac{n\pi}{2}\right)$$

$$\pi^3 = \sum^{\infty} 1 - (-1)^n \sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi^3}{16} = \sum_{n=1} \frac{\quad}{n^3}$$

$$\frac{\pi^3}{16} = \frac{2}{1^3}(1) + 0 + \frac{2}{3^3}(-1) + 0 + \frac{2}{5^3}(1) + 0 + \frac{2}{7^3}(-1) + \cdots$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

$$\frac{\pi^3}{32} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

Q find a cosine series of period 2π to represent $\sin x$ in $0 \le x \le \pi$

Sol:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx = \frac{2}{\pi}\left[-\cos x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(-\cos\pi) - (-\cos 0)\right] = \frac{2}{\pi}\left[-(-1) - (-1)\right] = \frac{2}{\pi}(2)$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\cos nx \sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - \left\{\frac{-1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$\cos(n\pm1)\pi = \cos n\pi \cos\pi \mp \sin n\pi \sin\pi$$

$$= -(-1)^n$$

$$= \frac{1}{\pi}\left[\left\{\frac{(-1)^n}{n+1} + \frac{-(-1)^n}{n-1}\right\} - \left\{\frac{-1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n+1}{n+1} - \frac{(-1)^n+1}{n-1}\right] = \frac{1}{\pi}\left((-1)^n+1\right)\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$n-1 - n+1$

$$= \frac{1}{\pi}\left[\frac{(-1)^n+1}{n+1} - \frac{(-1)^n+1}{n-1}\right] = \frac{1}{\pi}((-1)^n+1)\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}((-1)^n+1)\left(\frac{-2}{n^2-1}\right)$$

$$\therefore a_n = -\frac{2}{\pi}\frac{((-1)^n+1)}{n^2-1}, \quad n \geq 2$$

$$\therefore a_1 = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = \frac{-1}{2\pi}\left[\cos 2x\right]_0^{\pi}$$

$$= \frac{-1}{2\pi}[1-1] = 0$$

$\therefore$ The Half Range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$\sin x = \frac{\frac{4}{\pi}}{2} + 0 + \sum_{n=2}^{\infty} \frac{-2}{\pi}\frac{((-1)^n+1)}{n^2-1}\cos nx$$

$$= \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n+1}{n^2-1}\cos nx.$$

Fourier series in $(0, 2L)$

eg. $(0,8) \Rightarrow 2l = 8 \Rightarrow l = 4$

① Find Fourier series expansion of $f(x) = 4 - x^2$ in $(0,2)$
Graph the function.

Sol: We have $(0,2)$. Comparing it with $(0,2L)$ we get

$$2L = 2 \Rightarrow L = 1$$

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx = \frac{1}{1}\int_0^2 (4-x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 = \left[8 - \frac{8}{3}\right]$$

$$= \frac{16}{3}$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{1}\int_0^2 (4-x^2)\cos(n\pi x)\,dx \qquad [\because L = 1]$$

$$= \left[(4-x^2)\left(\frac{\sin(n\pi x)}{n\pi}\right) - (-2x)\left(\frac{-\cos(n\pi x)}{n^2\pi^2}\right) + (-2)\left(\frac{-\sin(n\pi x)}{n^3\pi^3}\right)\right]_0^2$$

$$= \left[-(2x)\left(\frac{\cos(n\pi x)}{n^2\pi^2}\right)\right]_0^2 = (-4)\left(\frac{1}{n^2\pi^2}\right) = \frac{-4}{n^2\pi^2}$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{1}\int_0^2 (4-x^2)\sin(n\pi x)\,dx$$

$$= \left[(4-x^2)\left(\frac{-\cos(n\pi x)}{n\pi}\right) - (-2x)\left(\frac{-\sin(n\pi x)}{n^2\pi^2}\right) + (-2)\left(\frac{\cos(n\pi x)}{n^3\pi^3}\right)\right]_0^2$$

$$= \left[\left\{0 - 0 + (-2)\left(\frac{1}{n^3\pi^3}\right)\right\} - \left\{(4)\left(\frac{-1}{n\pi}\right) - 0 + (-2)\left(\frac{1}{n^3\pi^3}\right)\right\}\right]$$

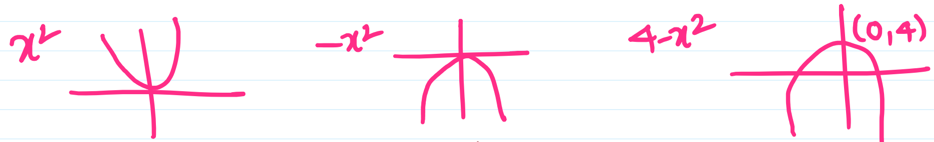
$$= \frac{-2}{n^3\pi^3} + \frac{4}{n\pi} + \frac{2}{n^3\pi^3} = \frac{4}{n\pi}$$

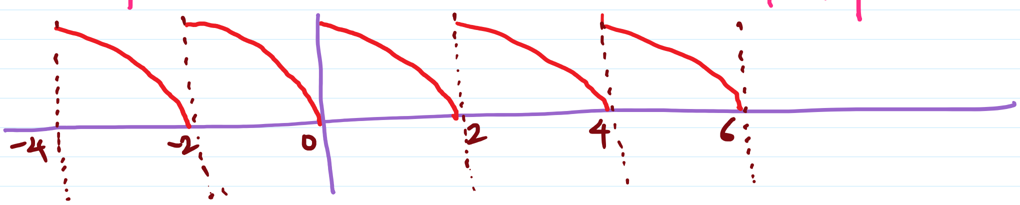
$\therefore$ The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

$$4-x^2 = \frac{\frac{16}{3}}{2} + \sum_{n=1}^{\infty} \frac{-4}{n^2\pi^2}\cos(n\pi x) + \frac{4}{n\pi}\sin(n\pi x)$$

$$4-x^2 = \frac{8}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2\pi^2}\cos(n\pi x) + \frac{4}{n\pi}\sin(n\pi x)$$





$f(x) = 4-x^2$, $f(x+2) = f(x)$

1

Fourier series in $(-l, l)$

① Find the fourier expansion of $f(x) = e^{ax}$ in $(-l, l)$

Sol:

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{ax}\,dx = \frac{1}{l}\left[\frac{e^{ax}}{a}\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{e^{al}}{a} - \frac{e^{-al}}{a}\right] = \frac{1}{la}\left(e^{al} - e^{-al}\right)$$

$$= \frac{2\sinh(al)}{al}$$

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

$$\boxed{2\sinh\theta = e^{\theta} - e^{-\theta}}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{ax}\cos\left(\frac{n\pi x}{l}\right)dx$$

$$\int e^{Ax}\cos Bx\,dx = \frac{e^{Ax}}{A^2+B^2}\left(A\cos Bx + B\sin Bx\right)$$

$$= \frac{1}{l}\left[\frac{e^{ax}}{a^2 + \frac{n^2\pi^2}{l^2}}\left(a\cos\left(\frac{n\pi x}{l}\right) + \frac{n\pi}{l}\sin\left(\frac{n\pi x}{l}\right)\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{\frac{e^{al}}{a^2 + \frac{n^2\pi^2}{l^2}}\left(a(-1)^n + 0\right)\right\} - \left\{\frac{e^{-al}}{a^2 + \frac{n^2\pi^2}{l^2}}\left(a(-1)^n + 0\right)\right\}\right]$$

$$= \frac{1}{l}\left[\frac{a(-1)^n}{a^2 + \frac{n^2\pi^2}{l^2}}\left(e^{al} - e^{-al}\right)\right]$$

$$= \frac{1}{l}\left[\frac{l^2 a(-1)^n}{a^2l^2 + n^2\pi^2}\left(2\sinh(al)\right)\right] = \frac{la(-1)^n\,2\sinh(al)}{a^2l^2 + n^2\pi^2}$$

$$b_n = \frac{1}{l}\int f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$\int e^{Ax}\sin Bx = \frac{e^{Ax}}{A^2+B^2}\left(A\sin Bx - B\cos Bx\right)$$

$$= \frac{1}{l}\int_{-l}^{l} e^{ax}\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\frac{e^{ax}}{a^2+\frac{n^2\pi^2}{l^2}}\left(a\sin\left(\frac{n\pi x}{l}\right) - \frac{n\pi}{l}\cos\left(\frac{n\pi x}{l}\right)\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{\frac{e^{al}}{a^2+\frac{n^2\pi^2}{l^2}}\left(0 - \frac{n\pi}{l}(-1)^n\right)\right\} - \left\{\frac{e^{-al}}{a^2+\frac{n^2\pi^2}{l^2}}\left(0 - \frac{n\pi}{l}(-1)^n\right)\right\}\right]$$

$$= \frac{1}{l}\left[\frac{1}{a^2+\frac{n^2\pi^2}{l^2}}\frac{n\pi}{l}(-1)^n\left(-e^{al} + e^{-al}\right)\right]$$

$$= \frac{-n\pi(-1)^n}{a^2l^2+n^2\pi^2}\left(e^{al} - e^{-al}\right)$$

$$= \frac{-n\pi(-1)^n}{a^2l^2+n^2\pi^2}\left(2\sinh(al)\right)$$

The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$e^{ax} = \frac{\frac{2\sinh(al)}{al}}{2} + \sum_{n=1}^{\infty}\left[\frac{la(-1)^n 2\sinh(al)}{a^2l^2+n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right) + \frac{-n\pi(-1)^n(2\sinh(al))}{a^2l^2+n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$= \frac{\sinh(al)}{al} + \sum_{n=1}^{\infty}\left[\text{——}\ ''\ \text{——}\ ''\ \text{——}\right]$$

(2) Obtain fourier series of $f(x) = x|x|$ in $(-1, 1)$

Sol: Here $l = 1$, $f(x) = x|x|$

$$f(-x) = -x\,|-x| \qquad [\,|-x| = |x|\,]$$
$$= -x|x|$$
$$= -f(x)$$

$\Rightarrow f(-x) = -f(x) \Rightarrow f(x)$ is odd function

$a_0 = 0$

$a_n = 0$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{1}\int_0^1 x|x|\sin(n\pi x)\,dx$$

$$= 2\int_0^1 x^2\sin(n\pi x)\,dx \qquad \left[\begin{array}{l}\because 0<x<1\\ \Rightarrow |x| = x\end{array}\right]$$

$$= 2\left[(x^2)\left(\frac{-\cos(n\pi x)}{n\pi}\right) - (2x)\left(\frac{-\sin(n\pi x)}{n^2\pi^2}\right) + (2)\left(\frac{\cos(n\pi x)}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[\left\{(1)\left(\frac{-(-1)^n}{n\pi}\right) - 0 + (2)\left(\frac{(-1)^n}{n^3\pi^3}\right)\right\} - \left\{0 - 0 + \frac{2}{n^3\pi^3}\right\}\right]$$

$$= 2\left[\frac{-(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]$$

The fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \sum_{n=1} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$x|x| = \sum_{n=1}^{\infty} 2\left[\frac{-(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right] \sin\left(n\pi x\right)$$

Q Obtain Half Range cosine series for $f(x) = e^x, 0 < x < 1$

Sol: Here $l = 1$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{1}\int_0^1 e^x\,dx = 2\left[e^x\right]_0^1$$

$$= 2\left[e^1 - 1\right]$$

$$= 2(e-1)$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{1}\int_0^1 e^x \cos(n\pi x)\,dx$$

$$= 2\left[\frac{e^x}{1+n^2\pi^2}\left(1\cos(n\pi x) + n\pi \sin(n\pi x)\right)\right]_0^1$$

$$= 2\left[\left\{\frac{e}{1+n^2\pi^2}\left((-1)^n + 0\right)\right\} - \left\{\frac{1}{1+n^2\pi^2}(1+0)\right\}\right]$$

$$= 2\left[\frac{1}{1+n^2\pi^2}\left((-1)^n e - 1\right)\right]$$

$\therefore$ The Half Range Cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$e^x = \frac{2(e-1)}{2} + \sum_{n=1}^{\infty} \frac{2\left((-1)^n e - 1\right)}{1+n^2\pi^2}\cos\left(n\pi x\right)$$

$$e^x = (e-1) + 2\sum_{n=1}^{\infty} \frac{(-1)^n e - 1}{1+n^2\pi^2}\cos(n\pi x)$$

Q Expand $f(x) = lx - x^2;\ 0 < x < l$ in a half Range sine series

Hence, from sine series deduce that

(a) $$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

(b) $$\frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots$$

Sol:

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{4l^2}{n^3\pi^3}\left(1-(-1)^n\right) \qquad \text{(prove!)}$$

The Half Range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$lx - x^2 = \sum_{n=1}^{\infty} \frac{4l^2}{n^3\pi^3}\left(1-(-1)^n\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$lx - x^2 = \frac{8l^2}{1^3\pi^3}\sin\left(\frac{\pi x}{l}\right) + 0 + \frac{8l^2}{3^3\pi^3}\sin\left(\frac{3\pi x}{l}\right) + 0$$
$$0 + \frac{8l^2}{5^3\pi^3}\sin\left(\frac{5\pi x}{l}\right) + \cdots$$

Deduction

put $x = l/2$

$$\frac{l^2}{2} - \frac{l^2}{4} = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right]$$

$$\frac{l^2}{4} \times \frac{\pi^3}{8l^2} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} + \cdots$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} + \cdots$$

Use Parseval's Identity — H.W

* Complex Variable *

$\mathbb{C}$ = set of complex No.

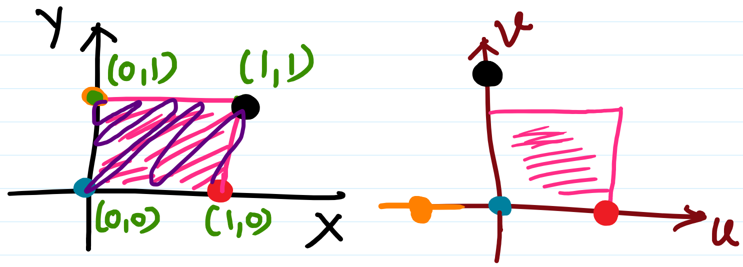
$f: R \to R$

$f(x) = x^2$

x	-1	-2	0	1	2
y	1	4	0	1	4

$f: \mathbb{C} \to \mathbb{C}$

$f(z) = z^2$ where $z \in \mathbb{C}$

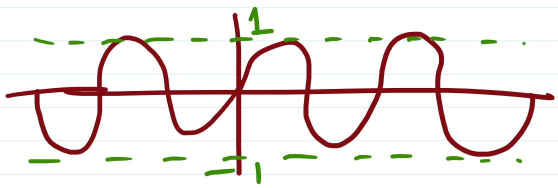


z-plane
Input

w-plane
output

$f: R \to R$

$f(x) = \sin x$



$[-1, 1]$

$f: \mathbb{C} \to \mathbb{C}$

$f(z) = \sin z$

This is not bounded

$-1 \le \sin z \le 1$

Absolute

Analytic functions

Analytic Function

If a single valued function $w = f(z)$ can be differentiated at each point of a domain D then it is called analytic or regular or holomorphic function in z in the domain D.

Cauchy-Riemann Equations in cartesian Coordinates

Theorem: The necessary ~~and sufficient~~ conditions for a continuous one valued function $w = f(z) = u(x,y) + iv(x,y)$ to be analytic in a region R are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in a region R and

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at each point of R. $[u_x = v_y \text{ \& } u_y = -v_x]$

Note:

(1) When $f(z)$ is analytic then its derivative is given by any one of the following expressions

$$f'(z) = u_x + iv_x \ , \cancel{f'(z) = v_y + iv_x}$$
$$\cancel{f'(z) = u_x - iu_y} \ , \cancel{f'(z) = v_y - iu_y}$$

(2) If $f(z)$ is analytic then it can be differentiated in usual manner.

(3) If $f(z) = u + iv$ and $f(z)$ is analytic then the function u and v of real variables x and y are called **conjugate functions**

(1) Show that the following functions are analytic and find their derivatives in terms of z.

(a) $f(z) = e^z$

Sol: $f(z) = e^z$

$= e^{x+iy}$

$= e^x e^{iy} = e^x(\cos y + i\sin y) = e^x\cos y + ie^x\sin y$

Here

$u = e^x\cos y \quad , \quad v = e^x\sin y$

$\frac{\partial u}{\partial x} = e^x\cos y \quad , \quad \frac{\partial v}{\partial x} = e^x\sin y$

$\frac{\partial u}{\partial y} = -e^x\sin y \quad , \quad \frac{\partial v}{\partial y} = e^x\cos y$

Also, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ f(z) satisfies C.R. eqns. $\therefore$ f(z) is analytic.

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= e^x\cos y + ie^x\sin y$

OR

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= e^x\cos y + ie^x\sin y$

$$= e^{x}\cos y + i e^{x}\sin y$$
$$= e^{x}(\cos y + i\sin y)$$
$$= e^{x} e^{iy} = e^{x+iy}$$
$$= e^{z}$$
$$\boxed{f'(z) = e^{z}}$$

$$= e^{x}\cos y + i e^{x}\sin y$$

put $x = z,\ y = 0$ (Milne Thompson Method)

$$= e^{z}\cos 0 + i e^{z}\sin 0$$
$$= e^{z}$$
$$\boxed{f'(z) = e^{z}}$$

(b) $\boxed{f(z) = \sinh z}$

$$\cos(i\theta) = \cosh\theta$$
$$\sin(i\theta) = i\sinh\theta$$
$$\cosh(i\theta) = \cos\theta$$
$$\sinh(i\theta) = i\sin\theta$$

Sol:

$$f(z) = \sinh z$$
$$u + iv = \sinh(x+iy)$$
$$= \sinh x \cosh iy + \cosh x \sinh iy$$
$$= \sinh x \cos y + i \cosh x \sin y$$

$$\frac{d}{dx}\sinh x = \cosh x$$
$$\frac{d}{dx}\cosh x = \sinh x$$

$\therefore\ u = \sinh x \cos y,\quad v = \cosh x \sin y$

$$\frac{\partial u}{\partial x} = \cosh x \cos y,\quad \frac{\partial v}{\partial x} = \sinh x \sin y$$

$$\frac{\partial u}{\partial y} = -\sinh x \sin y,\quad \frac{\partial v}{\partial y} = \cosh x \cos y$$

$$\therefore\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ \ \&\ \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ $f(z)$ satisfies CR. equation $\therefore$ $f(z)$ is analytic.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= \cosh x \cos y + i \sinh x \sin y$$

put $x = z,\ y = 0$ (Using Milne Thompson's Method)

$$= \cosh z \cos 0 + i \sinh z \sin 0$$

$$= \cosh z \qquad \left[\begin{array}{l} \sin 0 = 0 \\ \cos 0 = 1 \end{array}\right]$$

$$\boxed{\therefore\ f'(z) = \cosh z}$$

ⓒ $f(z) = x^2 - y^2 + i2xy$

Sol: $u = x^2 - y^2, \quad v = 2xy$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial y} = 2x$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ $f(z)$ satisfies cauchy Riemann eqn.

$\therefore$ $f(z)$ is analytic function.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= 2x + i2y$$

put $x = z, y = 0$ (using Milne Thompson's method)

$$\therefore f'(z) = 2z$$

$$\boxed{f'(z) = 2z}$$

(6) Show that if (a) $f(z) = \bar{z}$ (b) $f(z) = 2x + ixy^2$ then $f(z)$ is not analytic. **(M.U.2002)**

ⓐ $f(z) = \bar{z}$

$u + iv = x - iy \qquad \left[\begin{array}{l} z = x + iy \\ \bar{z} = x - iy \end{array}\right]$

$\therefore u = x, \quad v = -y$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = -1$$

$$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ $f(z)$ does not satisfy C-R-eqns.

ⓑ $f(z) = 2x + ixy^2$

$u + iv = 2x + ixy^2$

$\therefore u = 2x, \quad v = xy^2$

$$\frac{\partial u}{\partial x} = 2, \quad \frac{\partial v}{\partial x} = y^2$$

$$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial y} = 2xy$$

$$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial x} \ \& \ \frac{\partial u}{\partial y} \neq \frac{-\partial v}{\partial x}$$

$\therefore$ $f(z)$ does not satisfy C-R-eqns.

$\therefore$ $f(z)$ is not analytic

$\therefore$ $f(z)$ does not satisfy C-R-eqns.
$\therefore$ $f(z)$ is not analytic.

$\therefore$ $f(z)$ does not satisfy C-R-eqns.
$\therefore$ $f(z)$ is not analytic

(2) Find k such that $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{kx}{y}$ is analytic.

Sol:

$$u = \frac{1}{2}\log(x^2+y^2), \quad v = \tan^{-1}\frac{kx}{y}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{kx}{y}\right)^2}\left(\frac{-kx}{y^2}\right) = \frac{-kx}{y^2+k^2x^2}$$

$\because f(z)$ is analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{x}{x^2+y^2} = \frac{-kx}{y^2+k^2x^2}$$

Comparing both sides, we get $\boxed{k=-1}$

(4) Find the constants a, b, c, d if $f(z) = (x^2+2axy+by^2) + i(cx^2+2dxy+y^2)$ is analytic. **(M.U.D-12,D-13)** **Ans.** $[a=1, b=-1, c=-1, d=1]$

Sol:

$$u = x^2+2axy+by^2, \quad v = cx^2+2dxy+y^2$$

$$\frac{\partial u}{\partial x} = 2x+2ay, \quad \frac{\partial v}{\partial x} = 2cx+2dy$$

$$\frac{\partial u}{\partial y} = 2ax+2by \qquad \frac{\partial v}{\partial y} = 2dx+2y$$

$\because f(z)$ is analytic.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \Big| \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$2ax+2by = -(2cx+2dy)$$

$\frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial y}$

$2x+2ay = 2dx+2y$

$2d=2, \quad 2a=2$

$\boxed{d=1}, \quad \boxed{a=1}$

$2ax+2by = -(2cx+2dy)$

$2a=-2c, \quad 2b=-2d$

$a=-c, \quad b=-d$

$1=-c, \quad b=-1$

$\boxed{c=-1} \quad \boxed{b=-1}$

Homework

(3) Find the constants a,b,c,d,e if $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$ is analytic. **(M.U.2003,2007,2008,M-12,D-13)** **Ans.** $[a=1, b=-6, c=1, d=2, e=4]$

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* Cauchy Riemann Equations in (Polar form)

Cartesian form	Polar form
① $f(z) = u(x,y) + iv(x,y)$ (function)	$f(z) = u(r,\theta) + iv(r,\theta)$ (function)
② $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (C.R. eqn. in cartesian form)	$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ (C.R. eqn. in polar form)
③ $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ (derivative in cartesian form)	$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$ (derivative in polar)
Note: Milne Thompson's method is applicable.	Note: Milne Thompson's method is not applicable in polar form.

Cartesian form	Polar form
① $z = x + iy$	① $z = re^{i\theta}$
② $\bar{z} = x - iy$	② $\bar{z} = re^{-i\theta}$
③ $\lvert z\rvert = \sqrt{x^2 + y^2}$	③ $\lvert z\rvert = r$
④ $\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$	④ $\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{e^{-i\theta}}{r}$

Problems Based on Polar form

(1) Show that $\omega = \log z$ is analytic & hence find $\frac{d\omega}{dz}$

Sol:
$$\omega = \log z = \log(re^{i\theta}) = \log r + \log e^{i\theta} = \log r + i\theta$$

$\therefore u = \log r, \quad v = \theta$

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial v}{\partial r} = 0$$
$$\frac{\partial u}{\partial \theta} = 0, \quad \frac{\partial v}{\partial \theta} = 1$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$\therefore f(z)$ satisfies C-R-eqns

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = e^{-i\theta}\left(\frac{1}{r} + i(0)\right) = \frac{e^{-i\theta}}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

(2) Is $f(z) = \frac{z}{\bar{z}}$ analytic?

Sol:
$$f(z) = \frac{z}{\bar{z}} = \frac{re^{i\theta}}{re^{-i\theta}} = e^{i2\theta} = \cos 2\theta + i\sin 2\theta$$

$\therefore u = \cos 2\theta, \quad v = \sin 2\theta$

$$\frac{\partial u}{\partial r} = 0, \quad \frac{\partial v}{\partial \theta} = 2\cos 2\theta$$

$$\therefore \frac{\partial u}{\partial r} \neq \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$\therefore f(z)$ does not satisfy C-R-eqns. $\therefore f(z)$ is not analytic

(3) Find the value of k if $f(z) = r^3\cos k\theta + ir^k\sin k\theta$ is analytic

Sol: We have $u = r^3\cos k\theta, \quad v = r^k\sin k\theta$

$$\frac{\partial u}{\partial r} = 3r^2\cos k\theta \quad , \quad \frac{\partial v}{\partial \theta} = k r^k \cos k\theta$$

∂: $f(z)$ is analytic $\Rightarrow$ $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

$$\Rightarrow 3r^2\cos k\theta = \frac{1}{r}\left(k r^k \cos k\theta\right)$$

$$\Rightarrow 3r^2\cos k\theta = k\, r^{k-1} \cos k\theta$$

comparing both sides, we get

$$\boxed{k=3}$$

Harmonic Function

Any function of x, y which has continuous partial derivatives of the first and second order and satisfies laplace's equation $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ is called **Harmonic Function.**

eg. ① $\phi = x^2 - y^2$

$$\frac{\partial \phi}{\partial x} = 2x \quad , \quad \frac{\partial \phi}{\partial y} = -2y$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2 \qquad \frac{\partial^2 \phi}{\partial y^2} = -2$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2 - 2 = 0$$

$\Rightarrow$ ϕ is Harmonic function

① Show that real & Img. parts of an analytic function is Harmonic.

Sol: Let $f(z) = u + iv$ be analytic function.

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{——— } (1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{——— } (2)$$

To prove u is Harmonic function.

i.e. T.P.T. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

diff. eqn ① part. w.r.t. x we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{③}$$

diff. eqn. ② part w.r.t. y, we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad \text{④}$$

③ + ④ $\Rightarrow$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow$ u is Harmonic function.

Similarly we can prove v is Harmonic function.

Note:

If a function is not Harmonic then It can't be real or imaginary part of an analytic function.

Q State true or false with proper justification

"There doesn't exist an analytic function whose real part is $x^3 - 3x^2y - y^3$"

Sol: Let $u = x^3 - 3x^2y - y^3$

$$\frac{\partial u}{\partial x} = 3x^2 - 6xy \quad , \quad \frac{\partial u}{\partial y} = -3x^2 - 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = 6x - 6y \, , \quad \frac{\partial^2 u}{\partial y^2} = -6y$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (6x - 6y) + (-6y) \neq 0$$

$\Rightarrow$ u does not satisfy Laplace eqn.

$\Rightarrow$ u is not Harmonic

$\Rightarrow$ u can't be real part of an analytic function.

$\therefore$ Given statement is True.

Q2 Homework

Show that there does not exists an analytic function whose real part is $3x^2 + \sin x + y^2 + 5y + 4$

Laplace eqn.

Cartesian	Polar
$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$	$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$

(1) State Laplace's equation in polar and verify it for $u = r^2\cos 2\theta$ and also find v and $f(z)$.

OR

Show that $u = r^2\cos 2\theta$ is Harmonic. Also find its conjugate & analytic function

Sol: Laplace equation in polar form is given by

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

$$u = r^2\cos 2\theta$$

$$\frac{\partial u}{\partial r} = 2r\cos 2\theta, \quad \frac{\partial u}{\partial \theta} = -2r^2\sin 2\theta$$

$$\frac{\partial^2 u}{\partial r^2} = 2\cos 2\theta, \quad \frac{\partial^2 u}{\partial \theta^2} = -4r^2\cos 2\theta$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 2\cos 2\theta + \frac{1}{r}(2r\cos 2\theta) + \frac{1}{r^2}(-4r^2\cos 2\theta)$$

$$= 2\cos 2\theta + 2\cos 2\theta - 4\cos 2\theta$$

$$= 0$$

$\Rightarrow \nabla^2 u = 0$

$\Rightarrow$ u satisfies Laplace eqn.

$\Rightarrow$ u is Harmonic function.

To find v & $f(z)$

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$\left[\begin{array}{l} \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \end{array}\right]$$

$$= e^{-i\theta}\left(\frac{\partial u}{\partial r} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right)$$

$\because f(z)$ is analytic

$$= e^{-i\theta}\left(2r\cos 2\theta - \frac{i}{r}(-2r^2\sin 2\theta)\right)$$

$$= e^{-i\theta}(2r\cos 2\theta + i2r\sin 2\theta)$$

$$= e^{-i\theta}\, 2r\,(\cos 2\theta + i\sin 2\theta)$$

$$= 2r e^{-i\theta} e^{i2\theta} = 2re^{i\theta}$$

$$\therefore \; f'(z) = 2z \qquad (\because z = re^{i\theta})$$

$$f(z) = z^2 + c$$

To get v

$$f(z) = z^2 + c$$

$$= (re^{i\theta})^2 + c$$

$$= r^2 e^{i2\theta} + c$$

$$= r^2(\cos 2\theta + i\sin 2\theta) + c$$

$$= r^2\cos 2\theta + ir^2\sin 2\theta + c$$

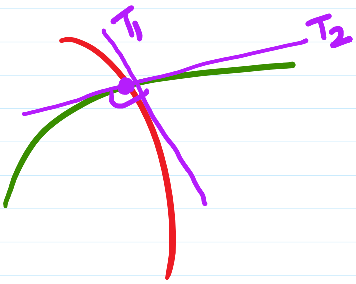
$$\boxed{\therefore \; v = r^2\sin 2\theta}$$

Homework

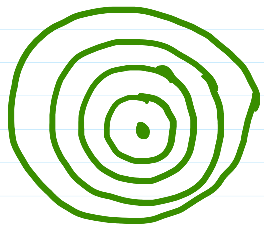
Q2 Verify Laplace's equation for $u = (r + \frac{a^2}{r})\cos\theta$ and also find v and $f(z)$.

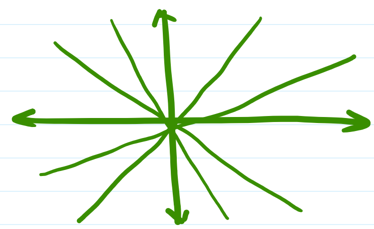
* Orthogonal family of functions *

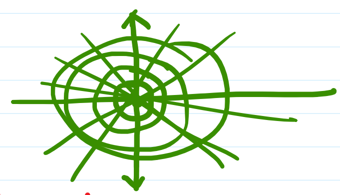
Any curve, **which cuts every member of a given family of curves at right angles**, is called an orthogonal trajectory of the family. It is not necessary that the curve should intersect every member of the family. But if they intersect, the angle between their tangents at every point of intersection is 90o.



$T_1 \angle T_2 = 90^\circ$







* If $f(z) = u + iv$ is analytic then $u = a$ & $v = b$ are orthogonal trajectories.

① $f(z) = z^2 = (x^2 - y^2) + i\,2xy$

$$x^2 - y^2 = a \quad , \quad 2xy = b$$

② $f(z) = \sin z$
$= \sin x \cosh y + i \cos x \sinh y$

$$\sin x \cosh y = a \quad , \quad \cos x \sinh y = b$$

③ $f(z) = e^z = e^{x+iy} = e^x \cos y + i\, e^x \sin y$

$$e^x \cos y = a \quad , \quad e^x \sin y = b$$

Given : u
To find : v & f(z)

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$
$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$
($\because$ f(z) is analytic)

$$dv = M\,dx + N\,dy$$

$$v = \underbrace{\int M\,dx}_{\text{Treat } y \text{ as const.}} + \int \begin{pmatrix}\text{terms of } N\\ \text{free from } z\end{pmatrix} dy$$

$$\therefore f(z) = u + iv$$
To get f(z) in terms of z put $x = z$, $y = 0$ (using M.T. method)

Given : v
To find : u & f(z)

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$
$$= \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$$
($\because$ f(z) is analytic)

$$du = M\,dx + N\,dy$$

$$u = \underbrace{\int M\,dx}_{\text{Treat } y \text{ as const.}} + \int \begin{pmatrix}\text{terms of } N\\ \text{free from } x\end{pmatrix} dy$$

$$\therefore f(z) = u + iv$$
To get f(z) in terms of z put $x = z$, $y = 0$ (using M.T. method)

1. Construct an analytic function whose real part is
$$x^4 - 6x^2y^2 + y^4$$
$\boxed{\text{OR}}$
Find the Harmonic conjugate of an analytic function whose real part is $x^4 - 6x^2y^2 + y^2$
$\boxed{\text{OR}}$
Find the orthogonal Trajectory of family of curves given by
$$x^4 - 6x^2y^2 + y^4 = a$$

Sol:
$$u = x^4 - 6x^2y^2 + y^4$$
$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2$$
$$\frac{\partial u}{\partial y} = -12x^2y + 4y^3$$
$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$
$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= -(-12x^2y+4y^3)\,dx + (4x^3-12xy^2)\,dy$$

$$v = \int (12x^2y - 4y^3)\,dx + \int 0\,dy$$
Treat y as const

$$= 4x^3y - 4xy^3 + C$$

$\therefore$ orthogonal Trajectory is given by $v = b$
$4x^3y - 4xy^3 = b$

$$f(z) = u + iv$$
$$= (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3) + C$$

put $x = z,\ y = 0$ using Milne Thompson's method

$$f(z) = z^4$$

(2) Find an analytic function $f(z)$ whose imaginary part is $e^{-x}(y\sin y + x\cos y)$

sol: $v = e^{-x}(y\sin y + x\cos y)$

$$\frac{\partial v}{\partial x} = e^{-x}(0+\cos y) + (y\sin y + x\cos y)\,e^{-x}(-1)$$
$$= e^{-x}\cos y - e^{-x}y\sin y - xe^{-x}\cos y$$

$$\frac{\partial v}{\partial y} = e^{-x}(y\cos y + \sin y - x\sin y)$$
$$= e^{-x}y\cos y + e^{-x}\sin y - e^{-x}x\sin y$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$
$$= \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$$
$$= (e^{-x}y\cos y + e^{-x}\sin y - e^{-x}x\sin y)\,dx - (e^{-x}\cos y - e^{-x}y\sin y - xe^{-x}\cos y)\,dy$$

$$u = \int (e^{-x}y\cos y + e^{-x}\sin y - e^{-x}x\sin y)\,dx + \int 0\,dy$$
Treat y as const

$$= \frac{e^{-x}}{-1}y\cos y + \frac{e^{-x}}{-1}\sin y + e^{-x}(x+1)\sin y$$
$$= -e^{-x}\left(y\cos y + \sin y - (x+1)\sin y\right)$$

$$\therefore f(z) = u + iv$$

$$= (\qquad) + i\,(\qquad)$$

put $x = z,\ y = 0$ using M.T. method

$$= 0 + i\left(e^{-z}(0 + z)\right)$$

$$= i z e^{-z}$$

(3) If $v = 3x^2y + 6xy - y^3$ then show that v is Harmonic and find it's corresponding analytic function.

Sol: $v = 3x^2y + 6xy - y^3$

$$\frac{\partial v}{\partial x} = 6xy + 6y, \quad \frac{\partial v}{\partial y} = 3x^2 + 6x - 3y^2$$

$$\frac{\partial^2 v}{\partial x^2} = 6y, \quad \frac{\partial^2 v}{\partial y^2} = -6y$$

$$\therefore \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6y + (-6y) = 0$$

$\therefore$ V satisfies Laplace eqn. $\therefore$ V is Harmonic function.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$= \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$$

$$= (3x^2 + 6x - 3y^2)dx - (6xy + 6y)\,dy$$

$$u = \int (3x^2 + 6x - 3y^2)dx + \int -6y\,dy$$

Treat y as const.

$$= x^3 + 3x^2 - 3xy^2 - 3y^2$$

$$f(z) = u + iv$$

$$= (x^3 + 3x^2 - 3xy^2 - 3y^2) + i\,(3x^2y + 6xy - y^3)$$

put $x = z,\ y = 0$ (Using Milne Thompson's method)

$$f(z) = z^3 + 3z^2$$

(4) Find the orthogonal trajectory of curves $e^{-x}\cos y + xy = \alpha$ where α is real constant in xy plane.

Sol: Let $u = e^{-x}\cos y + xy$

$$\frac{\partial u}{\partial x} = -e^{-x}\cos y + y \quad , \quad \frac{\partial u}{\partial y} = -e^{-x}\sin y + x$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \left[\because f(z) \text{ is analytic}\right]$$

$$= \left(e^{-x}\sin y - x\right)dx + \left(-e^{-x}\cos y + y\right)dy$$

$$v = \int \left(e^{-x}\sin y - x\right)dx + \int y\,dy$$

Treat y as const.

$$= \frac{e^{-x}\sin y}{-1} - \frac{x^2}{2} + \frac{y^2}{2}$$

$$= -e^{-x}\sin y - \frac{x^2}{2} + \frac{y^2}{2}$$

$\therefore$ Orthogonal Trajectory is given by

$$v = \beta'$$

$$-e^{-x}\sin y - \frac{x^2}{2} + \frac{y^2}{2} = \beta'$$

$$2e^{-x}\sin y + x^2 - y^2 = -2\beta'$$

$$2e^{-x}\sin y + x^2 - y^2 = \beta \quad \text{where } \beta = -2\beta'$$

Given: $au + bv$ where $a, b \in R$

To find: $f(z)$

Method:

Step I $\quad a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x}$ ——— (1)

Step II $\quad a\frac{\partial u}{\partial y} + b\frac{\partial v}{\partial y}$ ——— (2)

Step III $\quad a\left(-\frac{\partial v}{\partial x}\right) + b\left(\frac{\partial u}{\partial x}\right)$ ——— (3)

Solving (1) & (3), we get $\frac{\partial u}{\partial x}$ & $\frac{\partial v}{\partial x}$

step (4) $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

put $x=z, y=0$ using M.T. method

step (5) Integrate to find $f(z)$

(1) Find an analytic function $f(z) = u + iv$ where $u - v = e^x(\cos y - \sin y)$

sol:

$u - v = e^x(\cos y - \sin y)$ ——— $*$

diff. part. $(*)$ w.r.t. x

$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y)$ ——— (1)

diff. part. $*$ w.r.t. y

$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = e^x(-\sin y - \cos y)$ ——— (2)

$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = e^x(-\sin y - \cos y)$ ——— (3)

(1) + (3) $\Rightarrow$ $-2\frac{\partial v}{\partial x} = -2e^x \sin y$

$\frac{\partial v}{\partial x} = e^x \sin y$ ——— (4)

(3) & (4) $\Rightarrow$ $-\frac{\partial u}{\partial x} = -e^x\sin y - e^x\cos y + \frac{\partial v}{\partial x}$

$= -e^x\sin y - e^x\cos y + e^x\sin y$

$= -e^x\cos y$

$\frac{\partial u}{\partial x} = e^x\cos y$ ——— (5)

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= e^x\cos y + i e^x \sin y$

put $x=z, y=0$, using Thompson's method

$f'(z) = e^z$

$$\boxed{f(z) = e^z + c}$$

Q2 Find the analytic functin $f(z) = u + iv$ if $3u + 2v = y^2 - x^2 + 16xy$

Sol:

Q2 Find the analytic functin $f(z) = u + iv$ if $3u + 2v = y^2 - x^2 + 16xy$

Sol:

$$3u + 2v = y^2 - x^2 + 16xy \quad \text{——— } *$$

diff eqn. * part. w.r.t. x

$$3\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial x} = -2x + 16y \quad \text{——— } (1)$$

diff eqn. * part. w.r.t. y

$$3\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y} = 2y + 16x \quad \text{——— } (2)$$

$$\Updownarrow \qquad \Updownarrow$$

$$3\left(-\frac{\partial v}{\partial x}\right) + 2\left(\frac{\partial u}{\partial x}\right) = 2y + 16x \quad \text{——— } (3)$$

Solving (1) & (3) simultaneously, we get

$$\frac{\partial u}{\partial x} = 2x + 4y, \qquad \frac{\partial v}{\partial x} = -4x + 2y$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= (2x + 4y) + i(-4x + 2y)$$

put $x = z$, $y = 0$, using M.T. method

$$f'(z) = 2z + i(-4z)$$

$$= (2 - 4i)z$$

$$f(z) = (1 - 2i)z^2 + C$$

Homework

(2) Find an analytic function $f(z) = u + iv$ where $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$ when $f(\pi/2) = 0$ **(M.U.D-11,D-09,M-10,D-13)** **Ans.** $\left[\frac{1}{2}(1 - cot(z/2))e^{z} + c\right]$

(3) If $f(z) = u + iv$ is analytic and $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$ find $f(z)$ **(M.U.2004,06,07)** **Ans.** $\left[\frac{i}{1+i}\cot z + c\right]$

Doubt

Thursday, November 18, 2021 3:05 PM

* — Correlation & Regression — *

For example consider following data,

(i) Demand and Price of a certain commodity over a specified period of time.

(ii) Weight of a person and Blood Pressure of the person.

(iii) Quantity of water and crop yield.

(iv) Sales of cosmetics and advertisements

(v) Monthly income and expenditure of a family.

A set of observations made on two variables is called Bivariate Data. The two variables are denoted by X and Y respectively. Then observations on two variables X and Y can be represented by *n ordered pairs* (x_1,y_1), (x_2,y_2),...... (x_n,y_n),.. *The pair* (x_i,y_i), values of the variables for i^th^ observation. For example X denotes demand of the commodity and Y denotes price of the commodity then x_i *denotes demand of* i^{th} commodity and y_i *denotes price of i*th commodity.

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5.1 Concept of Correlation

In a bivariate data, we may be interested in finding if there is any relationship or association between the two variables."A correlation is a measure of association or relation". If we observe in the bivariate data the changes in one variable are accompanied by changes in the other variable then the two variables are said to be correlated. In this case we say that there is a correlation between two underlying variables.

For example,

i) Intelligence Quotient (IQ) and marks of a student.

ii) Demand and price of a commodity.

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5.2 Covariance:-

Covariance is a measure of joint variation between the two variables. If (x_1, y_1), (x_2, y_2), (x_n, y_n) are n ordered pairs of values of x and y, then covariance between X and Y is defined by

$$\operatorname{cov}(x, y) = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right),$$

where $\bar{x} = \frac{\sum x_i}{n}$ and $\bar{y} = \frac{\sum y_i}{n}$.

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$$\boxed{r = \frac{cov(x,y)}{\sigma x \, \sigma y}}$$

$$\boxed{r = \frac{\Sigma xy - N\bar{x}\bar{y}}{\sqrt{\Sigma x^2 - N\bar{x}^2}\sqrt{\Sigma y^2 - N\bar{y}^2}}}$$

$$r = \frac{cov(x,y)}{\sigma x \, \sigma y}$$

$$= \frac{\frac{1}{N}\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\frac{\Sigma(x-\bar{x})^2}{N}}\sqrt{\frac{\Sigma(y-\bar{y})^2}{N}}}$$

$$= \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2}\sqrt{\Sigma(y-\bar{y})^2}}$$

$$= \frac{\Sigma(xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y})}{\sqrt{\Sigma(x^2 - 2x\bar{x} + \bar{x}^2)}\sqrt{\Sigma(y^2 - 2y\bar{y} + \bar{y}^2)}}$$

$$= \frac{\Sigma xy - \bar{y}\Sigma x - \bar{x}\Sigma y + \bar{x}\bar{y}\Sigma 1}{\sqrt{\Sigma x^2 - 2\bar{x}\Sigma x + \bar{x}^2\Sigma 1}\sqrt{\Sigma y^2 - 2\bar{y}\Sigma y + \bar{y}^2\Sigma 1}}$$

$$\bar{x} = \frac{\Sigma x}{N} \quad , \quad \bar{y} = \frac{\Sigma y}{N}$$

$$\Sigma x = N\bar{x} \quad , \quad \Sigma y = N\bar{y} \quad , \quad \Sigma 1 = n$$

$$= \frac{\Sigma xy - \bar{y}(N\bar{x}) - \cancel{\bar{x}(N\bar{y})} + \cancel{\bar{x}\bar{y}N}}{\sqrt{\Sigma x^2 - 2\bar{x}(N\bar{x}) + \bar{x}^2 N}\sqrt{\Sigma y^2 - 2\bar{y}(N\bar{y}) + \bar{y}^2 N}}$$

$$= \frac{\Sigma xy - N\bar{x}\bar{y}}{}$$

$$\sqrt{\Sigma x^2 - N\bar{x}^2}\sqrt{\Sigma y^2 - N\bar{y}^2}$$

① Compute coefficient of correlation between x & y

x : 3, 6, 4, 5, 7
y : 2, 4, 5, 3, 6

Sol: $N = 5$

x	y	x^2	y^2	xy
3	2	9	4	6
6	4	36	16	24
4	5	16	25	20
5	3	25	9	15
7	6	49	36	42
$\Sigma x = 25$	$\Sigma y = 20$	$\Sigma x^2 = 135$	$\Sigma y^2 = 90$	$\Sigma xy = 107$

$$\bar{x} = \frac{\Sigma x}{N} = \frac{25}{5} = 5$$

$$\bar{y} = \frac{\Sigma y}{N} = \frac{20}{5} = 4$$

Karl Pearson's coefficient of correlation is given by

$$r = \frac{\Sigma xy - N\bar{x}\bar{y}}{\sqrt{\Sigma x^2 - N\bar{x}^2}\sqrt{\Sigma y^2 - N\bar{y}^2}} = \frac{107 - 5(5)(4)}{\sqrt{135 - 5(5)^2}\sqrt{90 - 5(4)^2}}$$

$$= \frac{107 - 100}{\sqrt{10}\sqrt{10}} = \frac{7}{10} = 0.7$$

$$\Rightarrow r = 0.7$$

$$\boxed{-1 \le r \le 1}$$

$\Rightarrow r = 0.7$

$\therefore$ positive correlation

$$\boxed{-1 \le r \le 1}$$

Note:

1. $-1 \le r \le 1$
2. $r = 0 \longrightarrow$ There is no correlation
3. $0 < r < 1 \Rightarrow$ strong positive correlation
4. $-1 < r < 0 \Rightarrow$ strong negative correlation
5. $r = 1 \Rightarrow$ Perfect positive correlation
6. $r = -1 \Rightarrow$ Perfect negative correlation

(2) compute the karl Pearson's coefficient of correlation

x: 23, 27, 28, 29, 30, 31, 33, 35, 36, 39

y: 18, 22, 23, 24, 25, 26, 28, 29, 30, 32

Sol: $N = 10$

x	y	x^2	y^2	xy
$\Sigma x = 311$	$\Sigma y = 257$	$\Sigma x^2 = 9875$	$\Sigma y^2 = 6763$	$\Sigma xy = 8171$

$\bar{x} = 31.1$

$\bar{y} = 25.7$

Karl Pearson's coefficient of correlation

$$r = \frac{\Sigma xy - N\bar{x}\bar{y}}{\sqrt{\Sigma x^2 - N\bar{x}^2}\sqrt{\Sigma y^2 - N\bar{y}^2}}$$

$$= \frac{8171 - 10(31.1)(25.7)}{}$$

$$= \frac{8171 - 10(31.1)(25.7)}{\sqrt{9875 - 10(31.1)^2}\sqrt{6763 - 10(25.7)^2}}$$

$$= 0.995$$

Q3. find coefficient of correlation for given data

House Price in $1000s (y)	Square Feet (x)
245	1400
312	1600
279	1700
308	1875
199	1100
219	1550
405	2350
324	2450
319	1425
255	1700

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$$\boxed{r = 0.76}$$

	No. of tanks	No. of soldier
America	2	4
Russia	1	3
china	3	1
India	6	2
Japan	5	5
Australia	4	6

$r \rightarrow$ Karl Pearson's coeff. of corr.

$R \rightarrow$ Spearman's Rank correlation

* Spearman's Rank correlation *

(a) When Ranks are not Repeated

$$R = 1 - \frac{6\sum d_i^2}{N^3 - N}$$

where N = pair of data
d_i = difference of Ranks

Note: $-1 \leq R \leq 1$

(1) Calculate Spearman's Rank correlation from foll. data.

X: 12 16 19 23 29
Y: 112 117 124 134 155

Sol:

X	Y	R_x	R_y	$d_i = R_x - R_y$	d_i^2
12	112	5	5	0	0
16	117	4	4	0	0
19	124	3	3	0	0
23	134	2	2	0	0
29	155	1	1	0	0
					$\sum d_i^2 = 0$

∴ Spearman's Rank correlation is given by

$$R = 1 - \frac{6\sum d_i^2}{N^3 - N}$$

$$= 1 - \frac{6(0)}{5^3 - 5} = 1$$

$\therefore R = 1$

$\therefore$ There is perfect positive correlation between X & Y

(2) Find the Spearman's Rank correlation from the following data.

X: 6, 24, 20, 15, 42

Y: 9, 36, 16, 25, 100

Sol:

X	Y	R_x	R_y	$d_i = R_x - R_y$	d_i^2
6	9	5	5	0	0
24	36	2	2	0	0
20	16	3	4	-1	1
15	25	4	3	1	1
42	100	1	1	0	0
					$\Sigma d_i^2 = 2$

Spearman's Rank correlation is given by

$$R = 1 - \frac{6 \Sigma d_i^2}{N^3 - N}$$

$$= 1 - \frac{6(2)}{5^3 - 5} = 1 - \frac{12}{120} = 0.9$$

$$\boxed{\therefore R = 0.9}$$

$\therefore$ There is positive correlation between X & Y.

(b) When Ranks are Repeated

$$R = 1 - \frac{6\left[\sum d_i^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \frac{1}{12}(m_3^3 - m_3) + \ldots\right]}{N^3 - N}$$

Where m_i is the No. of items having equal Ranks

① Obtain the Rank correlation from foll. data

x : 10 12 18 18 15 40

y : 12 18 25 25 30 30

Sol:

X	Y	R_x	R_y	$d_i = R_x - R_y$	d_i^2
10	12	6	6	0	0
12	18	5	5	0	0
18	25	2.5	3.5	−1	1
18	25	2.5	3.5	−1	1
15	30	4	1.5	2.5	6.25
40	30	1	1.5	0.5	0.25
					$\sum d_i^2 = 8.5$

$N = 6$ $\quad m_1 = 2$

$\sum d_i^2 = 8.5$ $\quad m_2 = 2$

$m_3 = 2$

∴ The Spearman's Rank correlation is given by

$$R = 1 - \frac{6\left[\sum d_i^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \frac{1}{12}(m_3^3 - m_3)\right]}{N^3 - N}$$

$$= 1 - 6\left[8.5 + \frac{1}{12}(2^3 - 2) + \frac{1}{12}(2^3 - 2) + \frac{1}{12}(2^3 - 2)\right]$$

$$= 1 - \frac{6\left[8.5 + \frac{1}{12}(2^3-2) + \frac{1}{12}(2^3-2) + \frac{1}{12}(2^3-2)\right]}{6^3-6}$$

$$= 1 - \frac{6\left[8.5 + 1.5\right]}{216-6}$$

$$= 1 - \frac{6(10)}{210} = 0.7142$$

$\therefore$ $R = 0.7142$

$\therefore$ There is strong positive corr. betn X & Y

Q2 find the rank correlation between x & y

X: 32, 55, 49, 60, 43, 37, 43, 49, 10, 20

Y: 40, 30, 70, 20, 50, 72, 60, 45, 25, 30

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Sol:

$m_1 = 2$

$m_2 = 2$

$m_3 = 2$

$R = -0.02424$

① Let $r_{x,y} = 0.4$, $cov(x,y) = 1.6$, $\sigma_y^2 = 25$
find σ_x

Sol:

$$r_{x,y} = \frac{cov(x,y)}{\sigma_x \sigma_y}$$

$$0.4 = \frac{1.6}{\sigma_x (5)}$$

$$\sigma_x = \frac{1.6}{(0.4)(5)} = \frac{4}{5} = 0.8$$

$$\boxed{\therefore \quad \sigma_x = 0.8}$$

② If $R_{x,y} = 0.143$ & the sum of the squares of difference between ranks is 48. find N.

Sol: Given $R_{x,y} = 0.143$

$$\Sigma d_i^2 = 48$$

$$N = ?$$

$$R_{x,y} = 1 - \frac{6 \Sigma d_i^2}{N^3 - N}$$

$$0.143 = 1 - \frac{6(48)}{N^3 - N}$$

$$\frac{(6)(48)}{N^3 - N} = 1 - 0.143$$

$$\frac{6 \times 48}{N^3 - N} = \frac{857}{1000}$$

$$N^3 - N = \frac{1000}{857} \times 6 \times 48$$

$$N^3 - N = \frac{288000}{857} = 336.056$$

$$\boxed{N \simeq 7}$$

③ Calculate the corr. coeff. between x & y from foll.
$N = 10$, $\Sigma x = 140$, $\Sigma y = 150$, $\Sigma (x-10)^2 = 180$
$\Sigma (y-15)^2 = 215$, $\Sigma (x-10)(y-15) = 60$

Sol:

$\Sigma(y-15)^2 = 215$, $\Sigma(x-10)(y-15) = 60$

Sol:

$$\boxed{\bar{x} = \frac{\Sigma x}{N} = \frac{140}{10} = 14}$$

$$\boxed{\bar{y} = \frac{\Sigma y}{N} = \frac{150}{10} = 15}$$

$$\Sigma(x-10)^2 = 180$$
$$\Sigma(x^2 - 20x + 100) = 180$$
$$\Sigma x^2 - 20\Sigma x + 100\Sigma 1 = 180$$
$$\Sigma x^2 - 20(140) + 100(10) = 180$$
$$\Sigma x^2 = 180 - 1000 + 2800$$
$$\boxed{\Sigma x^2 = 1980}$$

$$\Sigma(y-15)^2 = 215$$
$$\Sigma(y^2 - 30y + 225) = 215$$
$$\Sigma y^2 - 30\Sigma y + 225\Sigma 1 = 215$$
$$\Sigma y^2 - 30(150) + 225(10) = 215$$
$$\Sigma y^2 = 215 + 4500 - 2250$$
$$= 2465$$
$$\boxed{\therefore \Sigma y^2 = 2465}$$

$$\Sigma(x-10)(y-15) = 60$$
$$\Sigma(xy - 15x - 10y + 150) = 60$$
$$\Sigma xy - 15\Sigma x - 10\Sigma y + 150\Sigma 1 = 60$$
$$\Sigma xy - 15(140) - 10(150) + 150(10) = 60$$
$$\Sigma xy = 60 + (15)(140)$$
$$\boxed{\Sigma xy = 2160}$$

$$r = \frac{\Sigma xy - N\bar{x}\bar{y}}{\sqrt{\Sigma x^2 - N\bar{x}^2}\sqrt{\Sigma y^2 - N\bar{y}^2}} = \frac{2160 - 10(14)(15)}{\sqrt{1980 - 10(14)^2}\sqrt{2465 - 10(15)^2}}$$
$$= 0.9149$$

$$\boxed{r = 0.9149}$$

Aliter

$$\Sigma(x-10)^2 = 180$$
$$\Sigma(\underbrace{x-14}_{A} + \underbrace{4}_{B})^2 = 180$$
$$\Sigma[(x-14)^2 + 2(x-14)(4) + 16] = 180$$
$$\Sigma(x-14)^2 + 8\Sigma(x-14) + 16\Sigma 1 = 180$$
$$\Sigma(x-14)^2 + 8[\Sigma x - 14\Sigma 1] + 16\Sigma 1 = 180$$
$$\Sigma(x-14)^2 + 8[140 - 14\times 10] + 16\times 10 = 180$$
$$\Sigma(x-14)^2 = 180 - 160 = 20$$

$$\Sigma(x-14)^2 + 8[140 - 14\times10] + 16\times10 = 180$$

$$\Sigma(x-14)^2 = 180 - 160 = 20$$

$$\boxed{\Sigma(x-14)^2 = 20}$$

$$\Sigma(x-10)(y-15) = 60$$

$$\Sigma(x-14+4)(y-15) = 60$$

$$\Sigma[(x-14)(y-15) + 4(y-15)] = 60$$

$$\Sigma(x-14)(y-15) + 4\Sigma(y-15) = 60$$

$$\Sigma(x-14)(y-15) + 4[\Sigma y - 15\Sigma 1] = 60$$

$$\Sigma(x-14)(y-15) + 4[150 - 15(10)] = 60$$

$$\Sigma(x-14)(y-15) = 60$$

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2}\sqrt{\Sigma(y-\bar{y})^2}} = \frac{60}{\sqrt{20}\sqrt{215}} = 0.9149$$

* Regression *

① line of Regression of y on x is given by

$$y-\bar{y} = b_{yx}(x-\bar{x})$$

ie. $$y-\bar{y} = r\frac{\sigma_y}{\sigma_x}(x-\bar{x})$$

where $\bar{x}$ = mean of x
$\bar{y}$ = mean of y
b_{yx} = Regression coefficient

$$\boxed{b_{yx} = r\frac{\sigma_y}{\sigma_x}}$$

r = coeff. of correlation

② line of Regression of x on y is given by

$$x-\bar{x} = b_{xy}(y-\bar{y})$$

$$x-\bar{x} = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})$$

Where b_{xy} = Regression Coeff.

$$b_{xy} = r\frac{\sigma_x}{\sigma_y}$$

③ $$b_{xy} \times b_{yx} = r\cancel{\frac{\sigma_x}{\sigma_y}} \times r\cancel{\frac{\sigma_y}{\sigma_x}}$$

$$\therefore \boxed{b_{xy}\, b_{yx} = r^2}$$

$$\boxed{r = \sqrt{b_{xy} \cdot b_{yx}}}$$

④ Angle between the lines of Regression is given by

$$\tan\theta = \frac{1-r^2}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\right)$$

Note:

① If $r=0 \Rightarrow \tan\theta = \infty \Rightarrow \theta = \frac{\pi}{2}$
$\Rightarrow$ lines are perpendicular

② If $r = \pm 1 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$

$\Rightarrow$ lines are coincident.

⑤ The point of Intersection of Regression lines is $(\bar{x}, \bar{y})$

*** ⑥ If b_{xy} is Negative then b_{yx} is also negative

*** ⑦ If b_{yx} is Negative then b_{xy} is also negative

* Shortcut method to find lines of Regressions *

Line of Regression of y on x

Let $y = a + bx$ be required line

The Normal equations are

$$\Sigma y = na + b\Sigma x \quad \text{— ①}$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad \text{— ②}$$

Solve ① & ② simultaneously to get a & b

Line of Regression of x on y

Let $x = a + by$ be required line

The Normal equations are

$$\Sigma x = na + b\Sigma y \quad \text{— ①}$$

$$\Sigma xy = a\Sigma y + b\Sigma y^2 \quad \text{— ②}$$

Solve ① & ② simultaneously to get a & b

① Find the equation of the line of Regression for the foll. data

x:	5	6	7	8	9
y:	2	4	5	6	8

Also find y when $x = 7.5$

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Sol: Let line of Regression be $y = a + bx$

$\therefore$ The Normal equations are

$$\Sigma y = na + b\Sigma x \quad \text{——} ①$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad \text{——} ②$$

x	y	x^2	xy
5	2	25	10
6	4	36	24
7	5	49	35
8	6	64	48
9	8	81	72
$\Sigma x = 35$	$\Sigma y = 25$	$\Sigma x^2 = 255$	$\Sigma xy = 189$

$\therefore$ ① $\Rightarrow$ $25 = 5a + 35b$ —— ③

② $\Rightarrow$ $189 = 35a + 255b$ —— ④

Solving ③ & ④ we get

$$a = -4.8 \,,\; b = 1.4$$

$\therefore$ The line of Reg. is $y = a + bx$

$$y = -4.8 + 1.4x$$

When $x = 7.5$, $y = -4.8 + 1.4(7.5)$

$$= 5.7$$

Q2

② Height : 1.36 1.47 1.54 1.56 1.59 1.63 1.66 1.67 1.69 1.74 1.81

Weight : 52 50 67 62 69 74 59 87 77 73 67

Calculate the Regression equation of weight on height

Use Regression equation to estimate weight of some one whose height is 1.6m

Sol: Let the line of Regression of weight on height be

$y = a + bx$ where y = weight
x = Height

The Normal equations are

The Normal equations are

$$\Sigma y = na + b\Sigma x \quad \text{——} (1)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad \text{——} (2)$$

x	y	x^2	xy
$\Sigma x = 17.72$	$\Sigma y = 737$	$\Sigma x^2 = 28.705$	$\Sigma xy = 1196.1$

(1) $\Rightarrow$ $737 = 11a + 17.72b$ —— (3)

(2) $\Rightarrow$ $1196.1 = 17.72a + 28.705b$ —— (4)

Solving (3) & (4), we get

$a = -22.3764, \; b = 55.4821$

$\therefore$ line of Regression of x on y is

$$y = a + bx$$

$$y = -22.38 + 55.48x$$

When $x = 1.6$, $y = -22.38 + 55.48(1.6)$

$= 66.39$

Q3

The sales of a company (in million dollars) for each year are shown in the table below.

x (year)	2005	2006	2007	2008	2009
y (sales)	12	19	29	37	45

a) Find the least square regression line ~~y = a x + b~~ $y = a + bx$.

b) Use the least squares regression line as a model to estimate the sales of the company in 2012.

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Sol: Let $t = x - 2005$

t	y	t^2	ty

Let line be

Sol:

t	y	t^2	ty
0	12	0	0
1	19	1	19
2	29	4	58
3	37	9	111
4	45	16	180
$\Sigma t = 10$	$\Sigma y = 142$	$\Sigma t^2 = 30$	$\Sigma ty = 368$

Let line be

$$y = a + bt$$

The Normal eqn.s are

$$\Sigma y = na + b\Sigma t$$

$$\Sigma ty = a\Sigma t + b\Sigma t^2$$

$$142 = 5a + 10b \quad \text{--- (1)}$$

$$368 = 10a + 30b \quad \text{--- (2)}$$

Solving (1) & (2), we get $a = 11.6$, $b = 8.4$

∴ The line of Regression is

$$y = a + bt$$

$$\boxed{y = 11.6 + 8.4t}$$

(b) When $x = 2012$, $y = ?$

$$t = x - 2005$$

$$t = 2012 - 2005 = 7$$

$$y = 11.6 + (8.4)(7)$$

$$= 70.4$$

① Given lines of Regression

$$6y = 5x + 90, \quad 15x = 8y + 130, \quad \sigma_x^2 = 16$$

find ⓐ $\bar{x}, \bar{y}$ ⓑ r ⓒ σ_y^2

Sol: ⓐ To find $\bar{x}, \bar{y}$

$$5x - 6y = -90 \quad \text{—— ①}$$
$$15x - 8y = 130 \quad \text{—— ②}$$

Solving ① & ②, we get

$$\bar{x} = 30, \quad \bar{y} = 40$$

[∵ lines of Regression intersects at $(\bar{x}, \bar{y})$]

ⓑ Let line of Regression of y on x be

$$6y = 5x + 90$$
$$y = \frac{5}{6}x + 15$$
$$\therefore b_{yx} = \frac{5}{6} \quad \text{—— ①}$$

Let line of Regression of x on y be

$$15x = 8y + 130$$
$$x = \frac{8}{15}y + \frac{130}{15}$$
$$\therefore b_{xy} = \frac{8}{15} \quad \text{—— ②}$$

$$r = \sqrt{b_{xy} \cdot b_{yx}} = \sqrt{\frac{8}{15} \times \frac{5}{6}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$$

$$-1 \le r \le 1$$

ⓒ To find σ_y^2 given $\sigma_x^2 = 16$

$$b_{xy} = \frac{8}{15}$$

$$\left(\frac{2}{3}\right)^2 \times \frac{16}{\sigma_y^2} = \frac{64}{225}$$

$$\sigma_y^2 = \frac{4 \times 16 \times 225}{}$$

$$b_{xy} = \frac{8}{15}$$

$$r\frac{\sigma_x}{\sigma_y} = \frac{8}{15}$$

$$r^2\frac{\sigma_x^2}{\sigma_y^2} = \frac{64}{225}$$

$$\sigma_y^2 = \frac{4}{9} \times \frac{16 \times 225}{64}$$

$$= \frac{225}{9} = 25$$

② The lines of Regression of a sample are $x + 6y = 6$ & $3x + 2y = 10$

find ⓐ $\bar{x}, \bar{y}$ ⓑ r ⓒ estimate y when $x = 12$

ⓓ estimate x when $y = 5$

Sol:

ⓐ To find $\bar{x}$ & $\bar{y}$

$x + 6y = 6$ —— ①

$3x + 2y = 10$ —— ②

solving ① & ②, we get $\bar{x} = 3$, $\bar{y} = \frac{1}{2} = 0.5$

ⓑ Let the line of Reg. of y on x be

$$x + 6y = 6$$
$$6y = -x + 6$$
$$y = \frac{-1}{6}x + 1 \qquad \therefore b_{yx} = \frac{-1}{6}$$

$\therefore$ The line of Reg. of x on y is

$$3x + 2y = 10$$
$$3x = -2y + 10$$
$$x = \frac{-2}{3}y + \frac{10}{3}$$

$$\therefore b_{xy} = \frac{-2}{3}$$

$$r = \sqrt{b_{xy} \times b_{yx}} = \sqrt{\frac{-1}{6} \times \frac{-2}{3}} = \sqrt{\frac{1}{9}} = \frac{1}{3} = 0.3333$$

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ⓒ The line of Reg. of y on x is

$$y = \frac{-1}{6}x + 1$$
$$= \frac{-1}{6} \times 12 + 1 = -2 + 1 = -1$$

$$\boxed{y = -1}$$

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(d) The line of Reg. of x on y is

$$x = -\frac{2}{3}y + \frac{10}{3}$$
$$= -\frac{2}{3}\times 5 + \frac{10}{3}$$
$$= -\frac{10}{3} + \frac{10}{3} = 0$$
$$\boxed{\therefore x = 0}$$

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* Problems Based on Angle of Regression lines *

$$\tan\theta = \left(\frac{1-r^2}{r}\right)\left(\frac{\sigma_x \sigma_y}{\sigma_x^2+\sigma_y^2}\right)$$

(1) If the tgt. of angle made by lines of regression is 0.6 & $\sigma_y = 2\sigma_x$. Find r

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Sol: $\tan\theta = 0.6$

$\sigma_y = 2\sigma_x$, $r = ?$

$$\tan\theta = \left(\frac{1-r^2}{r}\right)\left(\frac{\sigma_x \sigma_y}{\sigma_x^2+\sigma_y^2}\right)$$

$$0.6 = \left(\frac{1-r^2}{r}\right)\left(\frac{\sigma_x (2\sigma_x)}{\sigma_x^2+(2\sigma_x)^2}\right)$$

$$= \left(\frac{1-r^2}{r}\right)\left(\frac{2\sigma_x^2}{\sigma_x^2+4\sigma_x^2}\right) = \left(\frac{1-r^2}{r}\right)\left(\frac{2}{5}\right)$$

$$0.6 = \left(\frac{1-r^2}{r}\right)\left(\frac{2}{5}\right)$$

$$\frac{6}{10}\times\frac{5}{2} = \frac{1-r^2}{r}$$

$$\frac{3}{2} = \frac{1-r^2}{r}$$

$$3r = 2-2r^2$$
$$2r^2+3r-2=0$$
$$2r^2+4r-r-2=0$$
$$2r(r+2)-1(r+2)=0$$
$$(2r-1)(r+2)=0$$

$$\frac{3}{2} = \frac{1-r^2}{r}$$

$$2r(r+2)-1(r+2)=0$$

$$(2r-1)(r+2)=0$$

$$r = \frac{1}{2}, -2$$

$$\boxed{\therefore r = \frac{1}{2}}$$

(2) find the angle between regression lines if
$n=10, \Sigma x = 270, \Sigma y = 630, \sigma_x = 4, \sigma_y = 5, r = 0.6$

Sol:

$$\tan\theta = \frac{1-r^2}{r}\left(\frac{\sigma_x \cdot \sigma_y}{\sigma_x^2+\sigma_y^2}\right)$$

$$\tan\theta = \frac{1-(0.6)^2}{0.6}\left(\frac{4\times5}{4^2+5^2}\right)$$

$$= \frac{1-0.36}{0.6}\left(\frac{20}{41}\right) = 0.52$$

$$\theta = \tan^{-1}(0.52)$$

(3) If the arithmetic mean of regression coefficient is p and their difference is $2q$, find the correlation coefficient (r).

Sol:

$$\frac{b_{xy}+b_{yx}}{2} = p$$

$$\therefore b_{xy}+b_{yx} = 2p \quad \text{——— (1)}$$

$$b_{xy}-b_{yx} = 2q \quad \text{——— (2)}$$

$(1)+(2) \Rightarrow$ $2b_{xy} = 2p+2q$

$$\boxed{b_{xy} = p+q}$$

$(1)-(2) \Rightarrow$ $2b_{yx} = 2p-2q$

$$\boxed{b_{yx} = p-q}$$

$$r = \sqrt{b_{xy} \cdot b_{yx}} = \sqrt{(p+q)\cdot(p-q)} = \sqrt{p^2-q^2}$$

Q Given the foll. results of weights X & heights Y of 1000 men

$\bar{X} = 150$ lbs $\quad \sigma_x = 20$ lbs

$\bar{Y} = 68$ inches $\quad \sigma_y = 2.5$ inches , $r = 0.6$

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John weighs 200 lbs. Smith is five feet tall. Estimate the height of John & weight of smith.

from the value of height of John estimate his weight. Why is it different from 200?

Sol:

Line of Reg. of y on x	Line of Reg. of x on y
$y-\bar{y} = b_{yx}(x-\bar{x})$	$x-\bar{x} = b_{xy}(y-\bar{y})$
$y-\bar{y} = r\frac{\sigma_y}{\sigma_x}(x-\bar{x})$	$x-\bar{x} = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})$
$y-68 = (0.6)\frac{2.5}{20}(x-150)$	$x-150 = 0.6\times\frac{20}{2.5}(y-68)$
$y-68 = \frac{15}{200}(x-150)$	$x-150 = \frac{24}{5}(y-68)$

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John, $X = 200$ lbs	Smith is 5 feet (60 inch) tall
$y-68 = \frac{15}{200}(200-150)$	$Y = 60$ inches $X = ?$
$y = 71.75$	$x-150 = \frac{24}{5}(60-68)$
	$x = 111.6$ lbs

from the value of height of John estimate his weight

$$x-150 = \frac{24}{5}(y-68)$$

$$x-150 = \frac{24}{5}(71.75-68)$$

$$x = 168$$

The difference is due to fact that we are using two different eqn.

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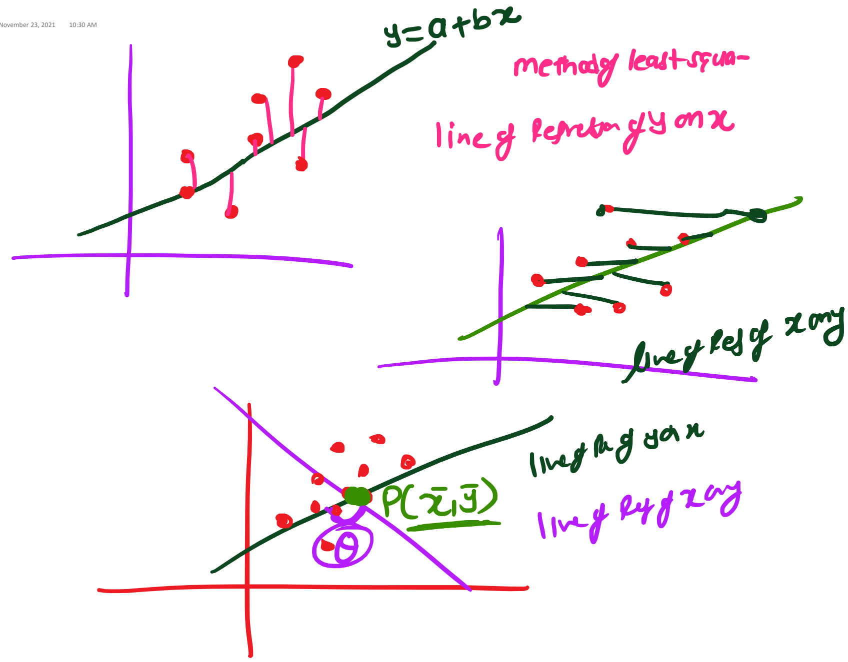

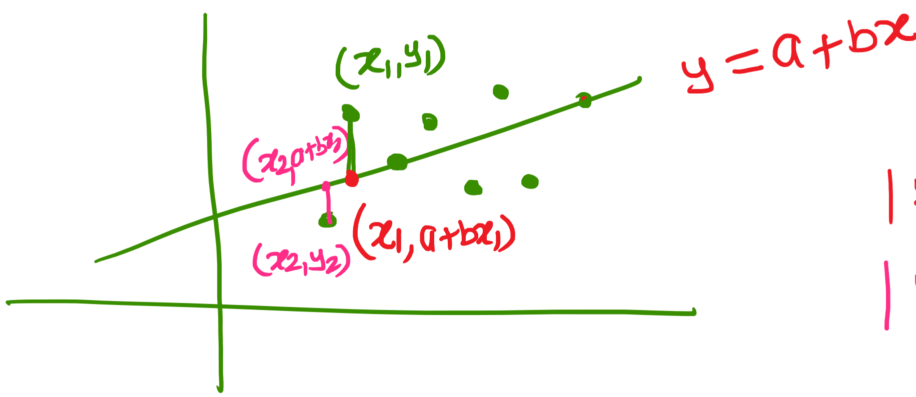

$|y_1-(a+bx_1)|$

$|y_2-(a+bx_2)|$

$\vdots$

$|y_n-(a+bx_n)|$

Let $S = (y_1-(a+bx_1))^2+(y_2-(a+bx_2))^2+\cdots\cdots+(y_n-(a+bx_n))^2$

$$S=\sum_{i=1}^{n}(y_i-(a+bx_i))^2$$

$$\frac{\partial S}{\partial a}=0 \Rightarrow \sum_{i=1}^{n}2(y_i-(a+bx_i))(0-1)=0$$

$$\Rightarrow \sum_{i=1}^{n}2[y_i-(a+bx_i)]=0$$

$$\Sigma y_i-\Sigma(a+bx_i)=0$$

$$\Sigma y_i-na-b\Sigma x_i=0 \quad (1)$$

$$\Sigma y_i - na - b\Sigma x_i = 0$$

$$\Sigma y_i = na + b\Sigma x_i \quad \text{——— (1)}$$

$$\frac{\partial S}{\partial b} = 0 \Rightarrow \sum_{i=1}^{n} 2\left[y_i - (a + bx_i)\right](-x_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} \left[y_i - (a + bx_i)\right](x_i) = 0$$

$$\Rightarrow \Sigma\left(x_i y_i - a x_i - b x_i^2\right) = 0$$

$$\Rightarrow \Sigma x_i y_i - a\Sigma x_i - b\Sigma x_i^2 = 0$$

$$\Rightarrow \Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 \quad \text{——— (2)}$$

$$\Sigma y_i = na + b\Sigma x_i$$

$$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2$$

$\Sigma x_i = \Sigma x$

$\Sigma y_i = \Sigma y$

y on x	x on y
y = ?, x given	x = ?, y is given

differ minute

* Probability *

* Random variable
* Types of Random variables (Discrete & continuous)

Discrete Random variable

Probability Distribution

X	x_1	x_2	x_3	x_4	-----	x_n
$P(X=x)$	P_1	P_2	P_3	P_4	-----	P_n

① $P_i \geq 0 \ \forall i$

② $\sum P_i = 1$

③ $\text{Mean} = \text{Average} = E[X] = \sum_{i=1}^{n} x_i P_i$

④ $E[X^2] = \sum_{i=1}^{n} x_i^2 P_i$

⑤ $V[X] = E[X^2] - (E[X])^2$

⑥ $S[X] = \sqrt{V[X]}$

* Cumulative Distribution function *

X	x_1	x_2	x_3	------	x_n
$P(X=x)$	P_1	P_2	P_3	------	P_n
$F(X=x)$	P_1	P_1+P_2	$P_1+P_2+P_3$	----------	$P_1+P_2+\dots+P_n = 1$

(1) A random Variable X has following probability distribution:

X	0	1	2	3	4	5	6
$P(X=x)$	k	3k	5k	7k	9k	11k	13k

Find i) k ii) $P(X<4)$ iii) $P(3<X\leq 6)$ **(M.U.**

Sol: (i) $\because$ It's Prob. dist.

$\sum P_i = 1$

$$k+3k+5k+7k+9k+11k+13k=1$$
$$49k=1 \quad \therefore k=1/49$$

(2) $P(X<4)=P(X=0,1,2,3)$
$$=P(X=0)+P(X=1)+P(X=2)+P(X=3)$$
$$=k+3k+5k+7k=16k=16\left(\frac{1}{49}\right)=\frac{16}{49}$$

(3) $P(3<X\le 6)=P(X=4,5,6)$
$$=P(X=4)+P(X=5)+P(X=6)$$
$$=9k+11k+13k$$
$$=33k$$
$$=33\times\frac{1}{49}=\frac{33}{49}$$

(2) A random Variable X has following probability distribution:

X	0	1	2	3	4	5	6	7
$P(X=x)$	0	c	2c	2c	3c	c^2	$2c^2$	$7c^2+c$

Find i) c ii) $P(X\geq 6)$ iii) $P(X<6)$ iv) Find k if, $P(X\leq k)>\frac{1}{2}$ where k is least positive integer
v) $P(1.5<X<4.5/x>2)$ **(M.U. 1996,2003,05)**

Sol: (1) $\Sigma p_i=1$
$$10c^2+9c=1$$
$$10c^2+9c-1=0$$
$$\boxed{c=\frac{1}{10}}$$

(2) $P(X\geq 6)=P(X=6,7)$
$$=P(X=6)+P(X=7)$$
$$=2c^2+7c^2+c$$
$$=9c^2+c$$
$$=\frac{9}{100}+\frac{1}{10}=\frac{19}{100}$$

(3) $P(X<6)=1-P(X\geq 6)=1-\frac{19}{100}=\frac{81}{100}$

(4)

X	0	1	2	3	4	5	6	7
$P(X=x)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$
$F(X=x)$	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{8}{10}$	$\frac{81}{100}$	$\frac{83}{100}$	1

$P(X\leq k)>1/2$

$P(X\leq 0)>1/2$ ✗

$P(X\leq 1)>1/2$ ✗

$P(X\leq 2)>1/2$ ✗

$P(X \le 2) > 1/2$ ✗

$P(X \le 3) > 1/2$ ✗

$P(X \le 4) > 1/2$ ✓ $\left(\because P(X \le 4) = \frac{8}{10}\right)$

$\therefore k = 4$

(v) $P(1.5 < X < 4.5 / X > 2)$

$$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}}$$

Let $A = 1.5 < X < 4.5$

$B = X > 2$

$A \cap B = (X = 3, 4)$

$$P(1.5 < X < 4.5 / X > 2) = \frac{P(1.5 < X < 4.5 \cap X > 2)}{P(X > 2)}$$

$$= \frac{P(X = 3, 4)}{P(X = 3, 4, 5, 6, 7)}$$

$$= \frac{P(X=3) + P(X=4)}{P(X = 3, 4, 5, 6, 7)}$$

$$= \frac{\frac{2}{10} + \frac{3}{10}}{\frac{2}{10} + \frac{3}{10} + \frac{1}{100} + \frac{2}{100} + \frac{17}{100}}$$

$$= \frac{5}{7}$$

(5) Find the probability distribution and the cumulative distribution function of a random variable X if X takes the values $1, 2, 3 \,\&\, 4$ such that $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4)$.

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Sol: $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = k$

$\therefore\ 2P(X=1)=k \Rightarrow P(X=1)=k/2$

$3P(X=2)=k \Rightarrow P(X=2)=k/3$

$P(X=3)=k \Rightarrow P(X=3)=k$

$5P(X=4)=k \Rightarrow P(X=4)=k/5$

X	1	2	3	4
$P(X=x)$	$k/2$	$k/3$	k	$k/5$

$\Sigma P_i = 1$

$\frac{k}{2}+\frac{k}{3}+k+\frac{k}{5}=1$

$\left(\frac{1}{2}+\frac{1}{3}+1+\frac{1}{5}\right)k=1$

$\left(\frac{61}{30}\right)k=1$ $\qquad \therefore k=\frac{30}{61}$

X	1	2	3	4
$P(X=x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$
$F(X=x)$	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	1

*Problems Based on mean, variance *

(4) A random Variable X has following probability distribution:

X	-2	-1	0	1	2	3
$P(X=x)$	0.2	k	0.1	2k	0.1	2k

Find i) k ii) Mean iii) Variance iii) Standard deviation

Sol (a) $\because$ It's Prob. dist.

$\Sigma P_i = 1$

$0.2+k+0.1+2k+0.1+2k=1$

$5k+0.4=1$

$5k=0.6$ $\qquad \therefore k=\frac{3}{25}$

(b) Mean $= E[X] = \Sigma x_i p_i$

$= (-2)(0.2)+(-1)(k)+(0)(0.1)+(1)(2k)+(2)(0.1)$

$$+3(2k)$$

$$= -0.4 - k + 0 + 2k + 0.2 + 6k$$

$$= 7k - 0.2 = 7\left(\frac{3}{25}\right) - 0.2 = \frac{16}{25}$$

(c) $E[X^2] = \sum x_i^2 p_i$

$$= (-2)^2(0.2) + (-1)^2(k) + (0)^2(0.1) + 1^2(2k) + (2^2)(0.1) + 3^2(2k)$$

$$= 0.8 + k + 0 + 2k + 0.4 + 18k$$

$$= 21k + 1.2 = 21\left(\frac{3}{25}\right) + 1.2 = \frac{93}{25}$$

$$V[X] = E[X^2] - (E[X])^2 = \frac{93}{25} - \left(\frac{16}{25}\right)^2 = \frac{2069}{625}$$

$$= 3.3104$$

(d) $S[X] = \sqrt{V[X]} = \sqrt{3.3104} = 1.8194$

(6) A random Variable X has following probability distribution:

X	0	1	2	3
$P(X = x)$	k	0.3	0.5	k

If If $Y = X^2 + 2X$, Find i) k ii) Mean Y iii) Variance Y iii) $P(1 < Y < 10)$

Sol. (a) $\sum p_i = 1$

$$k + 0.3 + 0.5 + k = 1$$

$$2k = 0.2$$

$$\boxed{k = 0.1}$$

(b)

$Y = X^2 + 2X$	0	3	8	15
$P(Y = y)$	0.1	0.3	0.5	0.1

(c) $E[Y] = \sum y_i p_i = (0)(0.1) + (3)(0.3) + (8)(0.5) + (15)(0.1)$

$$= 0 + 0.9 + 4 + 1.5 = 6.4$$

(d) $E[Y^2] = \sum y_i^2 p_i = (0)^2(0.1) + 3^2(0.3) + 8^2(0.5) + 15^2(0.1)$

$$= 57.2$$

(e) $V[Y] = E[Y^2] - (E[Y])^2$

$$= 57.2 - (6.4)^2 = 16.24$$

(f) $P(1 < Y < 10) = P(Y = 3, 8)$

$= P(Y=3) + P(Y=8)$

$= 0.3 + 0.5 = 0.8$

(9) If three coins are tossed find the expectation and variance of the number of heads.

Sol: Let X be No. of heads when 3 coins are tossed.

$S = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\}$

X	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$E[X] = \sum x_i p_i = (0)\left(\frac{1}{8}\right) + (1)\left(\frac{3}{8}\right) + (2)\left(\frac{3}{8}\right) + (3)\left(\frac{1}{8}\right)$

$= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}$

$E[X^2] = \sum x_i^2 p_i = (0)^2\left(\frac{1}{8}\right) + (1)^2(3/8) + (2)^2(3/8) + (3)^2(1/8)$

$= 0 + \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \frac{24}{8} = 3$

$V[X] = E[X^2] - (E[X])^2$

$= 3 - \left(\frac{3}{2}\right)^2 = 3/4$

Q A coin is tossed untill head appears. Find the expectation of tosses required?

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Sol: Let X be No. of tosses.

	H	TH	TTH	TTTH		
X	1	2	3	4	5	....
$P(X=x)$	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$	$\left(\frac{1}{2}\right)^5$	....

$E[X] = \sum x_i p_i$

$E[X] = (1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{2}\right)^2 + (3)\left(\frac{1}{2}\right)^3 + (4)\left(\frac{1}{2}\right)^4 + \ldots\ldots$ ——— (1)

$\frac{1}{2}E[X] = \left(\frac{1}{2}\right)^2 + (2)\left(\frac{1}{2}\right)^3 + (3)\left(\frac{1}{2}\right)^4 + (4)\left(\frac{1}{2}\right)^5 + \ldots\ldots$ ——— (2)

① − ② $\Rightarrow$

$$E[X]-\frac{1}{2}E[X] = \frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\cdots$$

$$\boxed{S_\infty = \frac{a}{1-r}}$$

$$\frac{E[X]}{2} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$$\boxed{E[X]=2}$$

Q A and B play a game of tossing a coin and who first throws a head wins the game. if A begins the game and each player wins an amount of money equal to the number of tosses required for the win;find the mathematical expectation of A. **Ans.**8/9

Sol.

	A	ABA	ABABA	----
X	1	2	3	---
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$	----

$$E[X] = (1)\left(\frac{1}{2}\right)+(2)\left(\frac{1}{2}\right)^3+(3)\left(\frac{1}{2}\right)^5+(4)\left(\frac{1}{2}\right)^7+\cdots \quad \text{——— ①}$$

$$\left(\frac{1}{2}\right)^2 E[X] = \left(\frac{1}{2}\right)^3+(2)\left(\frac{1}{2}\right)^5+(3)\left(\frac{1}{2}\right)^7+\cdots \quad \text{——— ②}$$

① − ② $\Rightarrow$

$$\frac{3}{4}E[X] = \frac{1}{2}+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^5+\left(\frac{1}{2}\right)^7+\cdots$$

$$= \frac{1/2}{1-1/4} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{1}{2}\times\frac{4}{3} = 2/3$$

$$E[X] = \frac{2}{3}\times\frac{4}{3} = \frac{8}{9}$$

X (weight)	f	Prob
80	100	0.5
100	20	0.1
120	40	0.2
130	40	0.2

$Y = X - 10$

Y	$P(Y=y)$
70	0.5
90	0.1
110	0.2
120	0.2

Continuous Random Variable

Let X be a continuous random variable that takes values in $[a,b]$.

A continuous function $y=f(x)$ such that

(a) $f(x)$ is integrable

(b) $f(x) \geq 0$

(c) $\int_a^b f(x)\,dx = 1$

(d) $P(\alpha \leq X \leq \beta) = \int_\alpha^\beta f(x)\,dx$ is called Prob. density function.

(1) $E[X] = \int_a^b x f(x)\,dx$

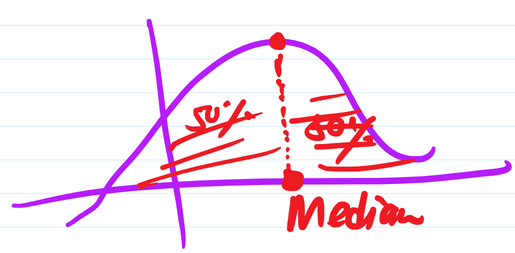
(2) $E[X^2] = \int_a^b x^2 f(x)\,dx$

(3) $V[X] = E[X^2] - (E[X])^2$

(4) $S[X] = \sqrt{V[X]}$

(5) Cumulative distribution function $F(x)$

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$



(6) Median

Let M be median

$$\int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x)\,dx = \frac{1}{2}$$

(1) The Probability density function of a continuous random variable X is given by $f(x) = kx(2-x)$, $0 \le x \le 2$. Find k, mean and variance. **Ans.** $[k = 3/4, Mean$
(M.U. 2005)

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Sol: (a) $\because f(x)$ is p.d.f.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^2 kx(2-x)\,dx = 1$$

$$k\int_0^2 (2x - x^2)\,dx = 1$$

$$k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$k\left[4 - \frac{8}{3}\right] = 1 \qquad \boxed{\therefore k = 3/4}$$

$$f(x) = \frac{3}{4}x(2-x), \quad 0 \le x \le 2$$

(b) Mean $= E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^2 x\left[\frac{3}{4}x(2-x)\right]dx$

$$= \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx$$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right] = \frac{3}{4} \times 16 \times \frac{1}{12} = 1$$

(c) $E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$$= \int_0^2 x^2\left[\frac{3}{4}x(2-x)\right]dx$$

$$= \frac{3}{4}\int_0^2 (2x^3 - x^4)\,dx = \frac{3}{4}\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{16}{2} - \frac{32}{5}\right] = \frac{3}{4}\left[\frac{8}{5}\right] = 6/5$$

$$V[X] = E[X^2] - (E[X])^2$$
$$= \frac{6}{5} - (1)^2 = \frac{1}{5}$$

(4) A continuous random variable X has p.d.f. $f(x)$ given by $f(x) = \begin{cases} 2ax + b, & 0 \le x < 2 \\ 0, & otherwise \end{cases}$

If the mean of the distribution is 3, find the constants a and b. **Ans.** $[a = 1.5,\ b = -2.5]$ **(M.U. 1996,2001)**

Sol: $\because$ It's p.d.f.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$
$$\int_0^2 (2ax+b)\,dx = 1$$
$$\left[ax^2 + bx\right]_0^2 = 1$$
$$4a + 2b = 1 \quad \text{——— } (1)$$

$$E[X] = 3$$
$$\int_{-\infty}^{\infty} x f(x)\,dx = 3$$
$$\int_0^2 x(2ax+b)\,dx = 3$$
$$\int_0^2 (2ax^2+bx)\,dx = 3$$
$$\left[2a\frac{x^3}{3} + b\frac{x^2}{2}\right]_0^2 = 3$$
$$\frac{16a}{3} + 2b = 3$$
$$16a + 6b = 9 \quad \text{——— } (2)$$

Solving (1) & (2), we get $a = 1.5,\ b = -2.5$

Homework

(8) A continuous random variable X has probability density function $f(x) = ke^{-x}x^3, 0 < x < \infty$. Find k, Mean and Variance of distribution. **Ans.** $\left[k = \frac{1}{6}, Mean = 4, Variance = 4\right]$ **(M.U. 1997)**

(9) If $f(x) = ke^{-|x|}, -\infty < x < \infty$ represents probability density function, find k, Mean and Variance of distribution. **Ans.** $\left[k = \frac{1}{2}, Mean = 0, Variance = 2\right]$ **(M.U. 1997)**