EM-3

LMR

Fourier series

(1) Radian

$[0, 2\pi]$

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n\cos(nx) + b_n\sin(nx)]$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

$[-\pi, \pi]$

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n\cos(nx) + b_n\sin(nx)]$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Even $f(-x) = f(x)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = 0$$

Odd $f(-x) = -f(x)$

$$f(x) = \sum_{n=1}^{\infty} b_n\sin(nx)$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$\pi \xrightarrow{\text{replace}} L$

$nx \xrightarrow{\text{replace}} n\pi x/L$

(2) General

[0, 2L]

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)]$$

$$a_0 = \frac{1}{2L}\int_0^{2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos(nx\pi/L)\,dx$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin(nx\pi/L)\,dx$$

[-L, L]

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx\pi/L) + b_n \sin(n\pi x/L)]$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)\,dx$$

Even $f(-x) = f(x)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_0 = \frac{1}{L}\int_0^{L} f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^{L} f(x)\cos(nx\pi/L)\,dx$$

$$b_n = 0$$

Odd $f(-x) = -f(x)$

$$\cancel{f(x) = a_0 + \sum_{n=1} a_n \cos(nx\pi/L)}$$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx\pi/L)$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{L}\int_0^{L} f(x)\sin(nx\pi/L)\,dx$$

(3)

$f(x)$	E	E	O	O
$g(x)$	E	O	E	O
$q(x)=f(x)\cdot g(x)$	E	O	O	E

(4) If n is integer

(i) $\sin(n\pi)=0$

(ii) $\sin(2n\pi)=0$

(iii) $\sin[(n\pm1)\pi]=0$

(iv) $\sin[(2n\pm1)\pi]=0$

(i) $\cos(n\pi) = (-1)^n$ — even $\rightarrow 1$, odd $\rightarrow -1$

(ii) $\cos(2n\pi)=1$

(iii) $\cos[(n\pm1)\pi] = (-1)^{n\pm1}$
$= (-1)^n(-1)^{\pm1}$
$= -(-1)^n$

(iv) $\cos[(2n\pm1)\pi] = -1$

(5) parseval's identity

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

(6) $$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx + b\cos bx\right]$$

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx - b\sin bx\right]$$

(7) Half range cosine series [$(0,\pi)$ / $(0,L)$]

Assume $f(x)$ is even function

$$b_n = 0 \qquad\qquad b_n = 0$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \qquad\qquad a_0 = \frac{1}{L}\int_0^{L} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx \qquad\qquad a_n = \frac{2}{L}\int_0^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nx)\right] \qquad\qquad f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\pi x/L)\right]$$

(8) Half range sine series [$(0,\pi)$ / $(0,L)$]

Assume $f(x)$ is odd function

$$a_n = 0 \qquad\qquad a_n = 0$$
$$a_0 = 0 \qquad\qquad a_0 = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \qquad\qquad b_n = \frac{2}{\pi}\int_0^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$f(x) = \sum_{n=1}^{\infty} b_n\sin(nx) \qquad\qquad f(x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)dx$$

Complex variable

1. Analytic function : $f(z) = u + iv$ is differentiable at each and every point.

2. Cauchy's Riemmon (CR) equation :

(A) If $f(z) = u + iv = f(x, y)$

(i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (ii) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(B) If $f(z) = f(r, \theta) = u(r, \theta) + iv(r, \theta)$

(i) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ (ii) $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

3. $|f'(z)|^2 = f'(z) \cdot f'(\bar{z})$

4. $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

~~5.~~ $\cancel{|f(z)|^2 = f(z) \cdot f'(z)}$

5. $|f(z)|^2 = f(z) \cdot f(\bar{z})$

6. Harmonic function : It should satisfy laplace equation

laplace equation : $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

7. Milne thomson theorem

If $f(z) = u + iv$ is an analytic function then

(i) Type I : u is given

step 1 ⇒ $f(z) = u + iv$ → analytic function

$u = u(x, y)$ ⇒ given

step 2 ⇒ $f'(z) = f'(x+iy) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

step 3 ⇒ by CR eq^n $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore f'(x+iy) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$

step 4 ⇒ by milne-Thomson theorem

put $x = z$ $y = 0$

$$f'(z+0) = \left(\partial u / \partial x\right)_{\substack{x=z \\ y=0}} - i \left(\frac{\partial u}{\partial y}\right)_{\substack{x=z \\ y=0}}$$

$$f'(z) = \phi_1(z) - i\phi_2(z)$$

step 5 ⇒ Integrate w.r.t. z

$$\int f'(z) = \int \phi_1(z) dz - i \int \phi_2(z) dz + c$$

$$\boxed{f(z) = \int \phi_1(z) dz - i \int \phi_2(z) dz + c}$$

(ii) Type II : v is given

Step 1 ⇒ $f(z) = u + iv$ ⇒ analytic function

Step 2 ⇒ $f'(z) = f'(x+iy) = \partial u/\partial x + i\,\partial v/\partial x$

Step 3 ⇒ by CR eq^n $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$f'(x+iy) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

Step 4 ⇒ By Milne - Thomson theorem
$x = z$ and $y = 0$

$$\therefore f'(z+0) = \left(\frac{\partial v}{\partial y}\right)_{\substack{x=z\\y=0}} + \left(\frac{\partial v}{\partial x}\right)_{\substack{x=z\\y=0}}$$

$$f'(z) = \varphi_1(z) + i\varphi_2(z)$$

Step 5 ⇒ $f(z) = \int \varphi_1(z)\,dz + i\int \varphi_2(z)\,dz$

(iii) Type III : $u-v$ is given (U is given)

Step 1 ⇒ $f(z) = u + iv$ ⇒ analytic function

Step 2 ⇒ multiply by i to both side
$i \cdot f(z) = iu - v$

$$(1+i)f(z) = (u-v) + i(u+v)$$

Assume $(1+i)f(z) = F(z)$

$u - v = U$

$u + v = V$

$F(z) = U + iV$

step 3 => $F'(z) = \frac{\partial U}{\partial x} + i \frac{\partial V}{\partial x}$

step 4 => by CR eq^n $\partial V/\partial x = -\partial U/\partial y$

$$F'(x+iy) = \left(\frac{\partial U}{\partial x}\right) - i\left(\frac{\partial U}{\partial y}\right)$$

by milne - thomson theorem $x = z$ and $y = 0$

$$F'(z) = \left(\frac{\partial U}{\partial x}\right)_{\substack{x=z\\y=0}} - i\left(\frac{\partial U}{\partial y}\right)_{\substack{x=z\\y=0}}$$

$$F'(z) = \phi_3(z) - i\phi_4(z)$$

$$F(z) = \int \phi_3(z)\,dz - i\int \phi_4(z)\,dz + C_1$$

$$(1+i)f(z) = F(z)$$

$$f(z) = \frac{F(z)}{(1+i)}$$

(iv) Step IV : $U+V$ is given (V is given)

Step 1 => $f(z) = u + iv$ => analytic function

Step 2 => multiply by i to both side

$$if(z) = iu - v$$

step 3 => $(1+i)f(z) = (u-v) + i(u+v)$

$$F(z) = U + iV$$

step 4 => $F'(z) = F'(x+iy) = \partial U/\partial x + i\,\partial V/\partial x$

by CR eq^n (I)

$\partial U/\partial x = \partial V/\partial y$ $\quad F'(z) = \partial V/\partial y + i\,\partial V/\partial x$

Step 5 => by milne thomson thm

$x = 2 \quad y = 0$

$$F'(z) = \left(\frac{\partial V}{\partial y}\right)_{\substack{x=z \\ y=0}} + i\left(\frac{\partial V}{\partial x}\right)_{\substack{x=z \\ y=0}}$$

$$\int F'(z)\,dz = \int \varphi_3(z)\,dz + \int \varphi_4(z)\,dz + C_2$$

$$(1+i)f(z) = F(z)$$

$$f(z) = \frac{F(z)}{1+i}$$

Statistical techniques

(1) Karl pearson's coefficient of correlation (r) :

$-1 \le r \le 1$

(i) Actual mean method

$$r = \frac{\Sigma[(x-\bar{x})(y-\bar{y})]}{\sqrt{\Sigma(x-\bar{x})^2} \cdot \sqrt{\Sigma(y-\bar{y})^2}}$$

(ii) Assumed mean method

$$r = \frac{\Sigma[(x-A)(y-B)] - \frac{\Sigma(x-A) \cdot \Sigma(y-B)}{N}}{\sqrt{\Sigma(x-A)^2 - \frac{[\Sigma(x-A)]^2}{N}} \sqrt{\Sigma(x-B)^2 - \frac{[\Sigma(x-B)]^2}{N}}}$$

(2) Spearmann's Rank correlation (R)

$-1 \le R \le 1$

(i) for non-repeated data

$$R = 1 - 6\left[\frac{\Sigma (R_1 - R_2)^2}{N^3 - N}\right]$$

R_1 = Rank of x
R_2 = Rank of y

(ii) for repeated data

$$R = 1 - 6\left[\frac{\Sigma (R_1 - R_2)^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \cdots + \frac{1}{12}(m_n^3 - m_n)}{N^3 - N}\right]$$

(3) Line of Regression of x on y

method 1 :

$x = a + by$ $\qquad$ $b = b_{xy}$

$\Sigma x = aN + b\Sigma y$

$\Sigma xy = a\Sigma y + b\Sigma y^2$

Method 2 ;

$(x - \bar{x}) = b_{xy}(y - \bar{y})$

$$b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y} = \frac{\Sigma (x - \bar{x})(y - \bar{y})}{\Sigma (y - \bar{y})^2}$$

(4) Line of regression y on x

Method 1 :

$$y = a + bx \qquad b = b_{yx}$$
$$\Sigma y = aN + b\Sigma x$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

Method 2 :

$$(y - \bar{y}) = b_{yx}(x - \bar{x})$$

$$b_{yx} = r \cdot \frac{\sigma y}{\sigma x} = \frac{\Sigma (x-\bar{x})(y-\bar{y})}{\Sigma (x-\bar{x})^2}$$

$$r = \sqrt{b_{xy} \cdot b_{yx}}$$

(5) Fitting of straight line

$$y = a + bx$$
$$\Sigma y = aN + b\Sigma x$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

(6) Fitting of parabola

$$y = a + bx + cx^2$$
$$\Sigma y = aN + b\Sigma x + c\Sigma x^2$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$
$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Probability

(1) Conditional probability :

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A/B) \cdot P(B)$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(B/A) \cdot P(A)$$

(2) Total probability :

$$P(E) = P(A_1 \cap E) + P(A_2 \cap E) + \cdots + P(A_n \cap E)$$
$$= P(A_1) \cdot P(E/A_1) + P(A_2) \cdot P(E/A_2) + \cdots + P(A_n) \cdot P(E/A_n)$$

(3) Bay's theorem :

$$P(A_1/E) = \frac{P(A_1 \cap E)}{P(E)}$$

$$= \frac{P(A_1) \cdot P(E/A_1)}{P(E)}$$

(4) Random variable

Discrete R.V	Continuous R.V
$P(x_i) \geq 0$	$f(x_i) \geq 0$
$\Sigma P(x_i) = 1$	$\int_{-\infty}^{\infty} f(x_i)\, dx = 1$

(5) Expectation

$$E(X) = \begin{cases} \xrightarrow{D\cdot R\cdot V} \sum x \cdot p(x) \\ \xrightarrow{C\cdot R\cdot V} \int_{-\infty}^{\infty} x \cdot f(x) \end{cases}$$

$$\cancel{E(X^n)} \; E(X^n) = \begin{cases} \xrightarrow{D\cdot R\cdot V} \sum x^n p(x) \\ \xrightarrow{C\cdot R\cdot V} \int_{-\infty}^{\infty} x^n f(x) \end{cases}$$

$E(c) = c$

$E(1) = 1$

(6) variance :

$$V(X) = E[(X-\bar{x})^2]$$
$$= E(X^2) - [E(X)]^2$$

$V(c) = 0$

$V(1) = 0$

$V(ax) = a^2 V(x)$

(7) Moments

Moments

→ Raw Moment

→ About anypoint (say a)

→ About origin

→ Central moment

about a mean

(i) r^{th} raw moment about any point a

$$u_r' = E[(x-a)^r]$$

$$\text{D.R.V} \swarrow \quad \searrow \text{C.R.V}$$

$$\sum (x-a)^r p(x) \qquad \int_{-\infty}^{\infty} (x-a)^r f(x)\,dx$$

(ii) r^{th} raw moment about origin

$$u_r' = E[(x-0)^r] = E[x^r]$$

$$\text{D.R.V} \swarrow \quad \searrow \text{C.R.V}$$

$$\sum x^r p(x) \qquad \int_{-\infty}^{\infty} x^r f(x)\,dx$$

(iii) ~~(3)~~ ~~central moment~~

r^{th} central moment about mean $(\bar{x})$

$$u_r = E[(x-\bar{x})^r]$$

$$\text{D.R.V} \swarrow \quad \searrow \text{C.R.V}$$

$$\sum (x-\bar{x})^r p(x) \qquad \int_{-\infty}^{\infty} (x-\bar{x})^r f(x)\,dx$$

(8) Moment generating function (MGF)

(i) M.G.F about a

$$M_a(t) = E[e^{t(x-a)}]$$

$$\begin{tikzcd} & M_a(t) = E\left[e^{t(x-a)}\right] \arrow[dl, "DRV"'] \arrow[dr, "CRV"] & \\ \sum e^{t(x-a)} p(x) & & \int_{-\infty}^{\infty} e^{t(x-a)} f(x) \end{tikzcd}$$

(ii) MGF about origin

$$\begin{tikzcd} & M_0(t) = E\left[e^{tx}\right] \arrow[dl, "DRV"'] \arrow[dr, "CRV"] & \\ \sum e^{tx} p(x) & & \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \end{tikzcd}$$

(iii) Mean $= \bar{x} = E(x) = u_1' = \frac{d}{dt}\left[M_0(t)\right]_{t=0}$

(iv) $E(x^2) = u_2' = \frac{d^2}{dt^2}\left[M_0(t)\right]_{t=0}$

(v) $var(x) = u_2' - (u_1')^2$