Mathematical Foundations of Computer Science
Solutions to Homework Assignment 3
February 10, 2023

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1. Suppose you take a piece of paper and draw a bunch of straight lines, no one exactly on top of another, that completely cross the paper. This divides the paper up into polygonal regions. Prove by induction that you can always color the various regions using only *two* colors, so that any two regions that share a boundary line have different colors. Regions that share only a boundary point are permitted to have the same color.

Solution. Let $P(n)$ be the following property.

> The regions formed by drawing n lines can be colored using two colors so that two regions sharing a boundary line are colored using different colors.

We want to prove that $P(n)$ is true for all $n \geq 1$. We will prove this using induction on n.
Base Case: $P(1)$ is clearly true as one line divides the plane into two regions that can be colored using two different colors.
Induction Hypothesis: Assume that $P(k)$ is true for some $k \geq 1$
Induction Step: We want to prove that $P(k+1)$ is true. Consider a set S of $k+1$ lines. Remove any line ℓ from the set S. Let S' be the resulting set of k lines. By induction hypothesis, the k lines in S' divide the plane into regions that can be colored red and blue such that regions sharing a boundary line are colored using different colors. Now we add the line ℓ to the set S' to obtain the set S. We invert the colors of regions to the right of line ℓ, i.e., regions that were colored red originally are now colored blue and vice-versa. The regions to the left of ℓ have valid coloring. The regions to the right of ℓ have valid coloring, just inverse of the original coloring. The regions sharing the boundary, by construction, will be colored using different colors. Thus the new coloring is a valid 2-coloring. Hence, $P(k+1)$ is true and this completes the induction proof.

2. There are n cities in a OneWayCountry (country in which every road is a One-Way road). Every pair of cities is connected by exactly one direct one-way road. Show that there exists a city which can be reached from every other city either directly or via at most one other city.

Solution. Let $P(n)$ be the following property.

> If there are n cities with exactly one direct road between every pair of cities then there exists a city which can be reached from every other city directly or via at most one other city.

We will prove that $P(n)$ is true for all $n > 0$.
Base Cases: $P(1)$ is clearly true as there is only one city that is reachable from itself. $P(2)$

is also true as if there are two cities a and b with a direct road from a to b then b is the city with the required property.
Induction Hypothesis: Assume that $P(k)$ is true for some $k > 1$.
Induction Step: We want to show that $P(k+1)$ is true. Let S be the set of $k+1$ cities and let v be a city in S. Let $S' = S \setminus \{v\}$. There are k cities in S'. By induction hypothesis, S' has a city , say u, that can be reached from every other city in S' directly or via at most one other city. Let $D_u \subseteq S'$ be the set of cities that are directly connected to u. Let $I_u = S' \setminus D_u$. Since $k \geq 2$, $|D_u| \geq 1$. We now consider two cases.

Case 1: There is a direct road from city v to city u or a direct road from v to some city in D_u. In this case u is also the city with the required property in S.

Case 2: There is no direct road from v to u and no direct road from v to any city in D_u. We claim that in this case v is the city with the required property. Observe that there is a direct road from u to v and a direct road to v from all cities in D_u. u can reach v using direct road and so can all cities in D_u. All cities in I_u can reach v via the same direct city that they used to reach v.

3. Prove, using induction, that every positive integer can be expressed as a sum of distinct powers of 2. For example, $13 = 2^3 + 2^2 + 2^0$.

Solution. Let $P(n)$ be the property that any integer n can be expressed as a sum of distinct powers of 2. We prove that $P(n)$ is true for all $n \geq 1$ using strong induction.

Base Case: $P(1)$ is true since $1 = 2^0$.
Induction Hypothesis: Assume that $P(z)$ is true for all $z \leq 2^k$, for some $k \geq 0$.
Induction Step: Let z' be any arbitrary but particular integer such that $2^k < z' \leq 2^k$. We want to prove that $P(z')$ is true. Let $y = z' - 2^k$. Note that $0 \leq y < 2^k$. By induction hypothesis, $y = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_y}$, where $k_1 < k_2 < \cdots < k_y < k$. Hence, $z' = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_y} + 2^k$, which is the sum of distinct powers of 2.

4. Prove that in any simple graph G with n vertices and m edges, $2m \leq n^2 - n$.

Solution. Let G_c be a complete graph on n vertices. Then,

$$2m = \sum_{v \in V(G)} deg(v) \leq \sum_{v \in V(G_c)} deg(v) = n(n-1) = n^2 - n$$

5. Prove or disprove the following. In any group of two or more people, there are always at least two people who have the same number of friends. Assume that if a person p is a friend of a person q then q is also a friend of p.

Solution. Consider a group of $n \geq 2$ people. We model the friendship relation among these n people as a graph $G = (V, E)$. Each vertex in the graph represents a distinct person from the group of n people, thus $|V| = n$. Also, edge $\{u, v\} \in E$ iff u and v are friends. To prove the claim we have to prove that there exists vertices p and q in G such that $deg(p) = deg(q)$. Assume otherwise, i.e., all vertices in G have different degrees. Since there are n vertices and $\Delta(G) \leq n-1$, the vertices must have degrees $0, 1, 2, \ldots, n-1$. This is a contradiction as G can not have one vertex with no neighbors and one vertex with $n-1$ neighbors at the same time. This completes the proof.

6. An r-regular graph is a graph in which the degree of each vertex is exactly r. Derive a simple algebraic relation between r, n, and m.

Solution. We know that

$$\begin{aligned} \sum_v deg(v) &= 2m \\ rn &= 2m \end{aligned}$$

7. Let G be a graph with $n \geq 2$ vertices. Prove that if $\delta(G) \geq \frac{n}{2}$, then G is connected.

Solution. Assume, for the sake of contradiction, that G is not connected. Then G must have at least two connected components. At least one of the connected components, say S, must have at most $n/2$ vertices (by pigeon-hole principle). Let u be a vertex in S. Note that all neighbors of u must be in S. Thus, $degree(u) \leq |V(S)| - 1 \leq n/2 - 1 < n/2$, a contradiction.

8. Show that in any simple graph G there is a path from any vertex of odd degree to another vertex of odd degree.

Solution. Let u be a vertex of odd degree in G. Let H be the connected component of G that contains u. We know that H has even number of vertices of odd degree. Hence there must be at least one vertex with odd degree other than u in H. Since H is connected there must be a path from u to v.

Another way to prove the claim is as follows. Let W be a longest walk starting from u in which no edge is traversed more than once. Let v be the last vertex in W. In W, an odd number of edges incident on u and an odd number of edges incident on v are visited. Note that $v \neq u$ since this would mean that u has an even degree which contradicts our assumption. We stopped at v because there were no edges incident on v that were not visited. This means that v has odd degree. A simple path from u to v can be constructed from W by "entering" each vertex in W through its incident edge that appears first in W and "leaving" the vertex through its incident edge that appears last in W.