- Exam 1 : Mon, Feb 20.

- No lecture on Feb 20.

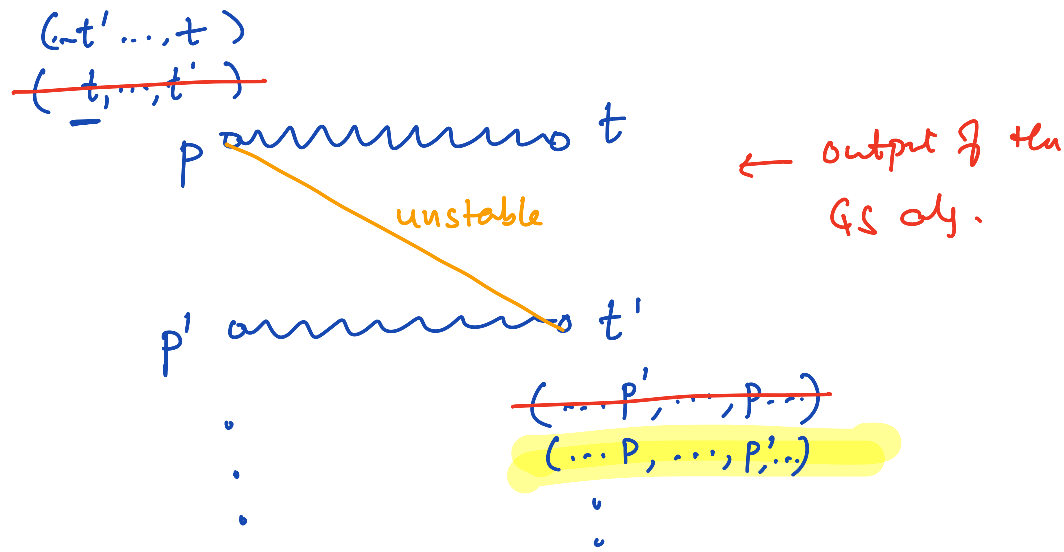
Stable Matching.

Last lecture: we proved that the GS algorithm output a perfect matching.

matching in which each person is matched to exactly one pet & each pet is matched with exactly one person.

Theorem: GS algorithm outputs a stable matching.

Proof: Assume for contradiction that the GS algorithm outputs a matching that has instability. Let (p, t') be an unstable pair.



- p will propose to t' before proposing to t in the GS alg.

- Since p is eventually paired with t, t' must have rejected p.

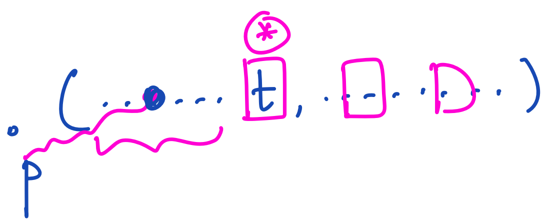
- By an earlier lemma, t''s partners only get better as the alg. progresses. Since p' is paired with t' at the end of the alg, p' must be ranked higher than p on t''s list, a contradiction!

Def^n.

$\text{valid}(p) = \{ t \in \text{Pets} \mid \exists$ a stable matching in which (p,t) is a pair $\}$

$\text{Best}(p) = t$ iff

- $t \in \text{valid}(p)$
- t has the highest rank among all pets in $\text{valid}(p)$.



$S^* = \{ (p, \text{Best}(p)) \mid p \in \text{People} \}$

Theorem: All executions of the GS alg will output S^*.

Proof: Assume for contradiction that in some execution of the GS alg, say $\mathcal{E}$, some people get rejected by their Best(·). Among these people let p be the first person to be rejected by $\text{Best}(p) =: t$. Why did t reject p? Because of p'.

By defn, there must be a stable matching S that contains (p, t) as a pair.

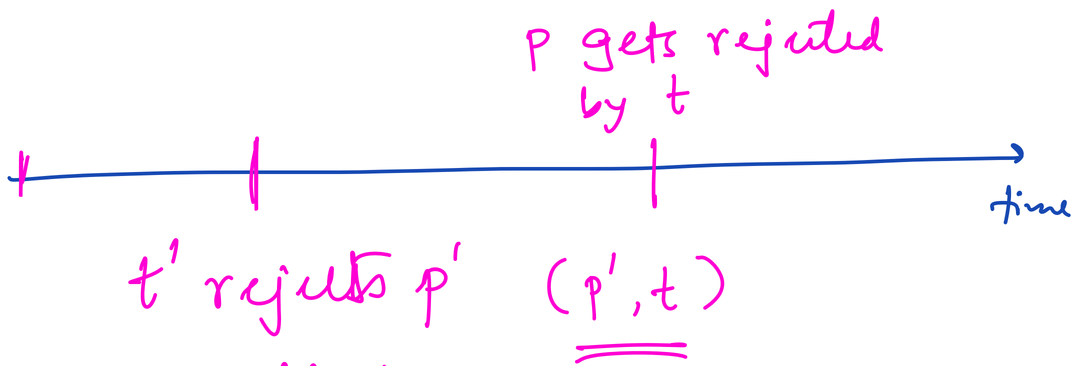
stable matching

S : $p \sim\sim\sim\sim\sim t$ $(\ldots p', \ldots, p, \ldots)$

$p' \sim\sim\sim\sim\sim t'$

$(\ldots t', \ldots, t, \ldots)$

GS alg. timeline.



Note that $t' \in \text{valid}(p')$

- Let h be the time when p gets rejected by t.

- Note that (p', t) is a pair at time h.

- In the GS alg, since people propose in $\searrow$ order of their preferences, it must be that p' must have proposed to t' before time h & must have got rejected; before time h.

- Since $t' \in \text{valid}(p')$, clearly, p is not the first person to be rejected by $\text{Best}(p)$, a contradiction!

<u>Defn</u> : $\text{valid}(t) = \{\, p \mid \exists$ a stable matching that contains (p, t) as a pair $\}$

$\text{worst}(t) = p$ iff

- $p \in \text{valid}(t)$
- p has the lowest rank among all valid partners of t.

$t\,(\,\cdots\,\square\;\square\;\square\,\cdots\,\boxed{p}\,\cdots\cdots)$

↳ cannot be a valid partner of t.

Thm: All executions of the GS alg will output

$$\{ (\text{worst}(t), t) \mid t \in \text{Pets} \}.$$

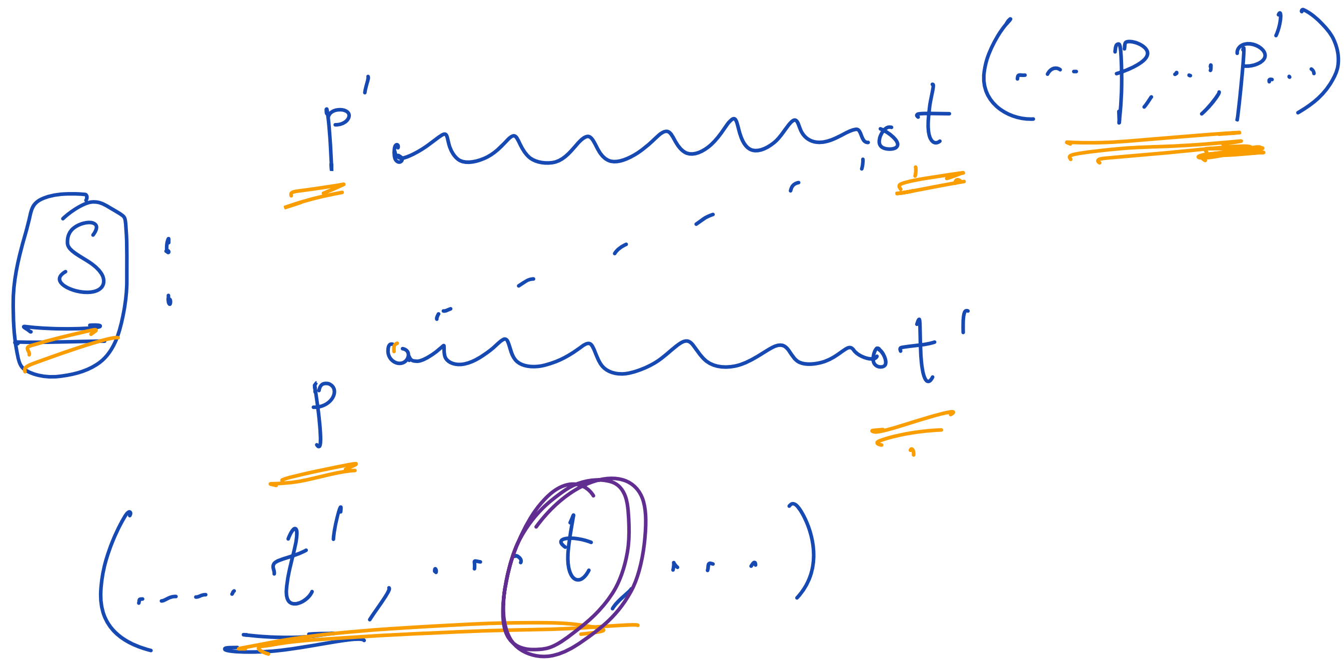
Proof: Assume for contradiction that some pet, say t, is paired with p in the GS alg & $p \neq \text{worst}(t)$. Let $p' = \text{worst}(t)$.

t : (. . . . . . . worst(t) . . . .)

p'

p?

There exists a stable matching S in which (p', t) is a pair.



We know the following:

- every person gets paired with Best(·).

- In the o/p of the GS alg
  (p, t) is a pair.

- Contradiction : $t \neq Best(\varphi)$.