Operation Research — L.P.P.

Linear Programming Problem

Note Linear programing problem is dealing with maximization of profit function with respect to few restrictions (called constraints)

EX A firm is engaged in manufacturing two products A, B. Each unit of product A requires 2 kg of row material and 4 hours of labour processing, whereas each unit of product B requires 3 kg of row material and 3 hours of Labour, of the same type. Every week the firm has availability of 60 kg of row material and 96 hours of labour. One unit of product A sold yields Rs 40 and one unit of product B yields Rs. 36 in profit.

Determine as to how many units of each of the products should be produced per week so that the firm can earn maximum profit. Formulate L.P.P. and solve it graphically.

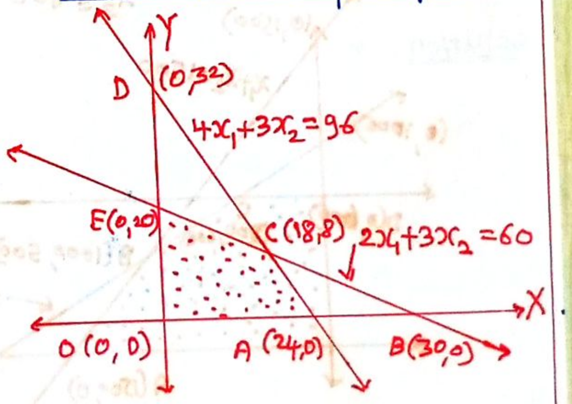
Solution: →

Suppose firm produces x_1 units of A and x_2 units of B

	A	B	
units	x_1	x_2	
Profit	40	36	limit
Row Me.	2	3	≤ 60
Labour	4	3	≤ 96

$$Z_{max} = 40x_1 + 36x_2$$

subject to
$$2x_1 + 3x_2 \le 60$$
$$4x_1 + 3x_2 \le 96$$
$$x_1, x_2 \ge 0$$



Feasible region is OACE.

Point	Value of $Z = 40x_1 + 36x_2$
O(0,0)	$Z = 0$
A(24,0)	$Z = 960$
C(18,8)	$Z = 720 + 288 = 1008$
E(0,20)	$Z = 720$

$\therefore Z_{max} = 1008$, $x_1 = 18$, $x_2 = 8$

EX A firm manufactures headche pills in two sizes A and B. Size A contains 2 grains of Asprin, 5 grains of Bicorbonate and one grain of codeine. Size B contains 1 grain of Asprin, 8 grains of bicarbonate and 6 grains of codeine. It is found by user that it requires at least 12 grains of Asprin, 74 grains of Bicarbonate and 24 grains of codeine for providing immediate effect. It is require to determine least number of pills a patient should take to get immediate relief. Formul[illegible]

Solution:- Let the firm produces x_1 pills of A and x_2 pills of B.

corresponding L.P.P.

$z_{min} = x_1 + x_2$

subject to, $2x_1 + x_2 \geq 12$

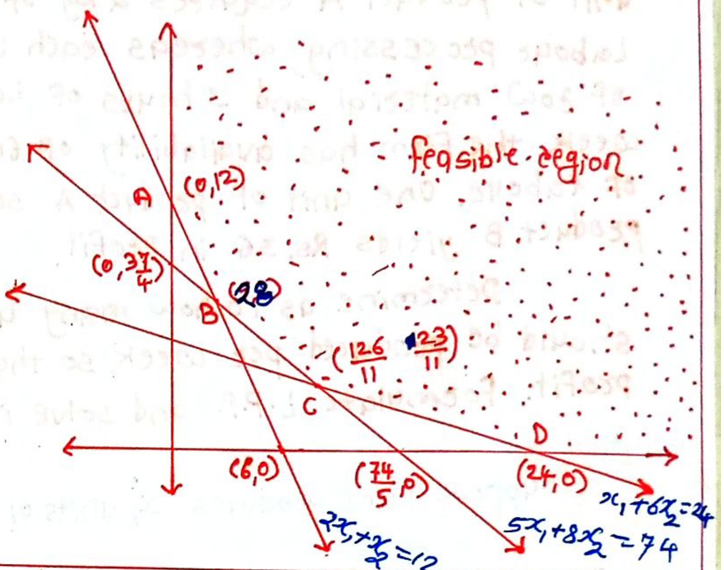
$5x_1 + 8x_2 \geq 74$

$x_1 + 6x_2 \geq 24$

$x_1, x_2 \geq 0$

	A	B	
units	x_1	x_2	
Profit	1	1	limit
Asprin	2	1	≥ 12
Bicarbonate	5	8	≥ 74
Codeine	1	6	≥ 24

Point	Value of z $z = x_1 + x_2$
A(0,12)	12
B(2,8)	10
C$(\frac{126}{11}, \frac{23}{11})$	$\frac{149}{11}$
D(24,0)	24



$\therefore z_{min} = 10, \quad x_1 = 2 \quad x_2 = 8$

Infinitely many solutions

Ex Solve graphically.

$z_{max} = x_1 + x_2$ subject to

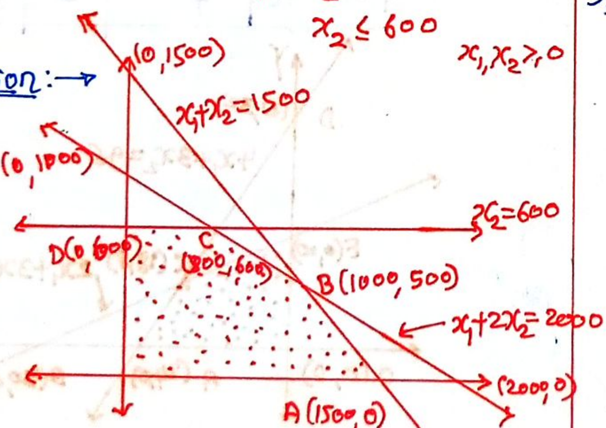
$x_1 + 2x_2 \leq 2000$

$x_1 + x_2 \leq 1500$

$x_2 \leq 600$

$x_1, x_2 \geq 0$

Solution: →



Point	$z = x_1 + x_2$
O(0 0)	z=0
A(1500,0)	1500
B(1000,500)	1500
C(800,600)	1400
D(0,600)	600

⇒ there are two points at which z is max.

⇒ any point on the line joining AB gives max.

⇒ L.P.P. Infinitely many solutions

$z_{max} = 1500$

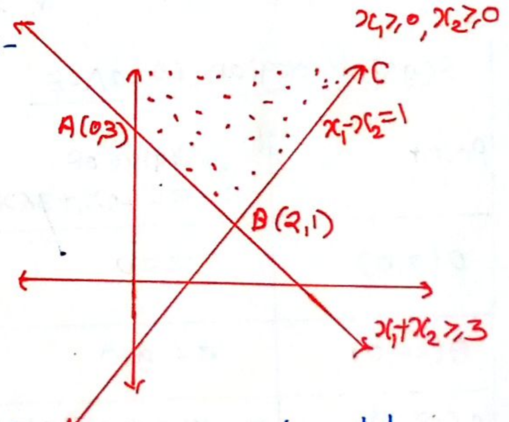
unbounded solution

Ex Solve $z_{max} = 3x_1 + 2x_2$

s.t. $x_1 - x_2 \leq 1$, $x_1 + x_2 \geq 3$

$x_1 \geq 0, x_2 \geq 0$

Solution:-



Feasible region is unbounded.

x_1 may be ∞, x_2 may be ∞

⇒ L.P.P. has unbounded solution.

Ex **no solution**

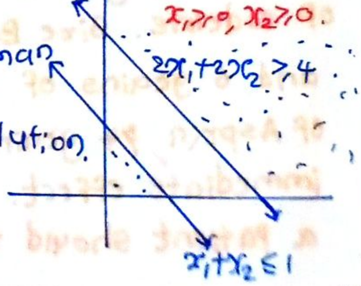
$z_{max} = 3x_1 - 2x_2$ s.t. $x_1 + x_2 \leq 1$

$2x_1 + 2x_2 \geq 4$

$x_1 \geq 0, x_2 \geq 0$.

Sol: no common region

⇒ no solution.



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Defination:- **Cannonical Form :→**

L.P.P. is said to be in cannonical form if

(i) it is of maximisation type

(ii) all constraints are of the type '≤'

(iii) all $x_i \geq 0$, $i = 1, 2 \ldots$

e.g. $Z_{max} = x_1 + x_2 - 3x_3$

s.t. $-x_1 + 2x_2 - x_3 \leq 3$

$2x_1 - x_2 - 3x_3 \leq -2$

$x_1, x_2, x_3 \geq 0$

Slack variables :→ If constraint is of '≤' type then in order to make it equality we have to add positive quantity to l.h.s. Such positive quantity is called slack variable.

e.g. $x_1 + x_2 \leq 3$ then $x_1 + x_2 + x_3 = 3$, where $x_3 \geq 0$ is slack variable.

Surplus variable :→ If constraint is of '≥' type then in order to make it equality we have to ~~add~~ subtract positive quantity from l.h.s. Such positive quantity is called surplus vari.

e.g. $x_1 + x_2 \geq 3$ then $x_1 + x_2 - x_3 = 3$

where $x_3 \geq 0$ is surplus variable.

Standard form of L.P.P.

Given L.P.P. is said to be in standard form if

(i) problem is of maximization type (if not multiply by -1)

(ii) r.h.s. of all constraints must be positive (if not multiply by (-1)) and add slack or surplus variables to create equality.

(iii) all variables must have non-negative values

(iv) If variable is unrestricted then replace it by $x_1' - x_2'$ where both x_1' and $x_2' \geq 0$

[1] Solution of L.P.P. :→ Any set of variables $\{x_1, x_2, \ldots x_{m+n}\}$ is solution of given L.P.P. if it satisfies constraints only.

[2] Feasible solution :→ any set of variables $\{x_1, x_2, \ldots x_{m+n}\}$ is called feasible solution of L.P.P. if it satisfies constraints and non-negativity restriction. (i.e. $x_i \geq 0$)

[3] Basic solution :→ Basic solution to the set of constraints is the solution obtained by setting 'n' variables (out of m+n) equal to zero and solving the system for remaining 'm' variables.

Such 'm' variables are called basic variables and remaining 'n' variables which are set equal to zero are called non-basic variables.

[4] **Basic Feasible solution** :→ Basic solution is said to be basic feasible if all basic variables are non-negative [i.e. $x_i \geq 0$]

[5] **Non-degenerate B.F.S.** :→ Basic feasible solution is said to be non-degenerate if all basic variable are greater than zero [i.e. $x_i > 0$].

[6] **degenerate basic feasible solution**:→ Basic feasible solution is said to be degenerate if at least one basic variable has zero valued.

[7] **Optimum basic feasible solution**:→ A basic feasible solution is said to be optimum if it optimizes (maximize or minimize) the objective function.

Limitations of LPP :→

[1] In LPP we deal with linear objective function and linear constraints but in real life constraints are not linear (business and industrial problem)

[2] There is no guarantee of getting integer valued solutions.

[3] LP model does not take into consideration the effect of time and uncertainity.

[4] Parameters in LP model are always constants but in real life they may not be.

[5] LP deals with single object but in real life situation problems come across with multiobjectives.

Advantages :→ [1] LP Technique helps us in making the optimum utilization of productive resouresess.

[2] The quality of decisions may be improved by user of LPP.

[3] LPP provides practically applicable solution.

[EX] Express the following LPP in standard form.

$$z_{max} = 3x_1 + 2x_2 + 5x_3$$

subject to,
$$2x_1 - 3x_2 \leq 3$$
$$x_1 + 2x_2 + 3x_3 \geq 5$$
$$3x_1 + 2x_3 \leq 2$$
$$, \; x_1, x_2 \geq 0 \qquad \text{— [2003}$$

Solution:- note that non-negativity restrictions are $x_1 \geq 0$, $x_2 \geq 0$

$x_3 \geq 0$ not given $\Rightarrow$ x_3 is unrestricted variable

$\therefore$ Let $x_3 = x_4 - x_5$ where $x_5 \geq 0$, $x_4 \geq 0$

$\therefore$ given LPP $z_{max} = 3x_1 + 2x_2 + 5(x_4 - x_5)$

subject to
$$2x_1 - 3x_2 \leq 3$$
$$x_1 + 2x_2 + 3[x_4 - x_5] \geq 5$$
$$3x_1 + 2(x_4 - x_5) \leq 2$$
$$x_1, x_2, x_4, x_5 \geq 0$$

∴ corresponding standard form

$$z_{max} = 3x_1+2x_2+5x_4-5x_5+0x_6+0x_7+0x_8-Ma_1$$

subject to, $2x_1-3x_2+x_6=3$

$x_1+2x_2+3x_4-3x_5-x_7+a_1=5$

$3x_1+2x_4-2x_5+x_8=2$

where x_6, x_8 are slack variables, x_7 surplus and a_1 artificial variable

after subtracting surplus variable add artificial variable to avoid infeasible solⁿ.

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coefficient of artificial variable in objective function is always $-M$

EX Express the following LPP in standard form.

minimize $z=2x_1+3x_2$

subject to $2x_1-3x_2-x_3=-4$

$3x_1+4x_2-x_4=-6$

$2x_1+5x_2+x_5=10$

$4x_1-3x_2+x_6=18$, $x_3, x_4, x_5, x_6 \geq 0$ —{2004

Solution:→ given $x_3, x_4, x_5, x_6 \geq 0 \Rightarrow x_1, x_2$ are unrestricted variable.

∴ Let $x_1 = x_1' - x_1''$ and $x_2 = x_2' - x_2''$

also eqⁿ ①, ② can be written as $-2x_1+3x_2+x_3=4$, $-3x_1-4x_2+x_4=6$

always add artificial variable for equality constraints.

∴ required std. form $z_{max} = -2(x_1'-x_1'')-3(x_2'-x_2'')-Ma_1-Ma_2-Ma_3-Ma_4$

s.t. $-2(x_1'-x_1'')+3(x_2'-x_2'')+x_3+a_1=4$

$-3(x_1'-x_1'')-4(x_2'-x_2'')+x_4+a_2=6$

$2(x_1'-x_1'')+5(x_2'-x_2'')+x_5+a_3=10$

$4(x_1'-x_1'')-3(x_2'-x_2'')+x_6+a_4=18$

* Basic Solutions *

EX Find all basic solutions to the following problem:-

$Z_{max} = x_1 + 3x_2 + 3x_3$

subject to $x_1 + 2x_2 + 3x_3 = 4$

$2x_1 + 3x_2 + 5x_3 = 7$ —{2000, 2015}

Also find which of the basic solutions are (i) basic feasible (ii) non-degenerate basic feasible (iii) degenerate basic feasible (iv) optimal basic feasible.

Solution:→ number of variables $m+n=3$

number of rows $= m = 2$ $\Rightarrow n=1$ to find basic solutions we have to set $n=1$ variables equal to zero.

non-basic variable	basic variable	values of basic var.	value of $z = x_1+3x_2+3x_3$	feasible	deg.	non-deg.	opt.
$x_1=0$	x_2, x_3	$2x_2+3x_3=4$, $3x_2+5x_3=7$ $\Rightarrow x_2=-1$, $x_3=2$	$0-3+6 = 3$	no	no	no	no
$x_2=0$	x_1, x_3	$x_1+3x_3=4$, $2x_1+5x_3=7$ $\Rightarrow x_1=1$, $x_3=1$	$1+0+3 = 4$	Yes	no	Yes	no
$x_3=0$	x_1, x_2	$x_1+2x_2=4$, $2x_1+3x_2=7$ $\Rightarrow x_2=1$, $x_1=2$	$2+3+0 = 5$	Yes	no	Yes	Yes

EX Find all basic solutions of the following LPP and classify them as feasible, infeasible, degenerate, optimal.

maximize $z = 2x_1 + 3x_2$

subject to $x_1 + 3x_2 \le 6$, $3x_1 + 2x_2 \le 6$ and $x_1 \ge 0$, $x_2 \ge 0$

—{2003, 2005}

Solution:→ given L.P.P can be written as

$Z_{max} = 2x_1 + 3x_2$

s.t. $x_1 + 3x_2 + x_3 = 6$

$3x_1 + 2x_2 + x_4 = 6$ x_3, x_4 are slack variables.

Here $m+n=4$ and no. of rows $= m = 2$

$\Rightarrow n=2 \Rightarrow$ we have to set two variables equal to zero.

Sr. N.	non-basic variable	Basic variable	values of basic variables	$z=2x_1+3x_2$	Feasible	inf.	deg.	optimal
1	$x_1=0$, $x_2=0$	x_3, x_4	$x_3=6$, $x_4=6$	0	Yes	no	no	no
2	$x_1=0$, $x_3=0$	x_2, x_4	$3x_2=6 \Rightarrow x_2=2$, $2x_2+x_4=6 \Rightarrow x_4=2$	6	Yes	no	no	no
3	$x_1=0$, $x_4=0$	x_2, x_3	$3x_2+x_3=6$, $2x_2=6$ $\Rightarrow x_2=3$, $x_3=-3$	9	no	Yes	no	no

Sr. no.	non-basic variable	basic variable	values of basic variables	$z=2x_1+3x_2$	feasible	infeasible	deg.	optimal
4	$x_2=0$, $x_3=0$	x_1, x_4	$\left.\begin{array}{l} x_1=6 \\ 3x_1+x_4=6 \end{array}\right\} \Rightarrow \begin{array}{l} x_1=6 \\ x_4=-12 \end{array}$	12	no	Yes	no	no
5	$x_2=0$, $x_4=0$	x_1, x_3	$\left.\begin{array}{l} x_1+x_3=6 \\ 3x_1=6 \end{array}\right\} \Rightarrow \begin{array}{l} x_1=2 \\ x_3=4 \end{array}$	4	Yes	no	no	no
6	$x_3=0$, $x_4=0$	x_1, x_2	$\left.\begin{array}{l} x_1+3x_2=6 \\ 3x_1+2x_2=6 \end{array}\right\} \Rightarrow \begin{array}{l} x_1=\frac{6}{7} \\ x_2=\frac{12}{7} \end{array}$	$\frac{12}{7}+\frac{36}{7}=\frac{48}{7}$	Yes	no	no	Yes

EX Find all the basic feasible solutions of the equations

$$2x_1+6x_2+2x_3+x_4=3$$
$$6x_1+4x_2+4x_3+6x_4=2$$

— {2008 Jan. 10}

Solutions:→ number of variables $m+n=4$

number of rows $=m=2 \Rightarrow n=2$

∴ to find basic solutions set $(n=2)$ variables equal to zero.

Sr. no.	non basic variables	basic variables	Values of basic variables	basic feasible solution.
1	$x_1=0$, $x_2=0$	x_3, x_4	$\left.\begin{array}{l} 2x_3+x_4=3 \\ 4x_3+6x_4=2 \end{array}\right\} \Rightarrow \begin{array}{l} x_4=-1 \\ x_3=2 \end{array}$	no
2	$x_1=0$, $x_3=0$	x_2, x_4	$\left.\begin{array}{l} 6x_2+x_4=3 \\ 4x_2+6x_4=2 \end{array}\right\} \Rightarrow \begin{array}{l} x_2=\frac{1}{2} \\ x_4=0 \end{array}$	Yes
3	$x_1=0$, $x_4=0$	x_2, x_3	$\left.\begin{array}{l} 6x_2+2x_3=3 \\ 4x_2+4x_3=2 \end{array}\right\} \Rightarrow \begin{array}{l} x_2=\frac{1}{2} \\ x_3=0 \end{array}$	Yes
4	$x_2=0$, $x_3=0$	x_1, x_4	$\left.\begin{array}{l} 2x_1+x_4=3 \\ 6x_1+6x_4=2 \end{array}\right\} \Rightarrow \begin{array}{l} x_4=-7/3 \\ x_1=8/3 \end{array}$	no
5	$x_2=0$, $x_4=0$	x_1, x_3	$\left.\begin{array}{l} 2x_1+2x_3=3 \\ 6x_1+4x_3=2 \end{array}\right\} \Rightarrow \begin{array}{l} x_3=7/2 \\ x_1=-2 \end{array}$	no
6	$x_3=0$, $x_4=0$	x_1, x_2	$\left.\begin{array}{l} 2x_1+6x_2=3 \\ 6x_1+4x_2=2 \end{array}\right\} \Rightarrow \begin{array}{l} x_2=\frac{1}{2} \\ x_1=0 \end{array}$	Yes

EX Find all basic solutions to the following problem. Which of them are basic feasible, non-degenerate, infeasible basic and optimal basic feasible solution.

$$\text{Max. } Z = x_1-2x_2+4x_3$$

subject to, $x_1+2x_2+3x_3=7$
$3x_1+4x_2+6x_3=15$ — {c-09 M}

Solution:- $m+n=3$, no. of rows $m=2$ $\Rightarrow n=1$

to find basic solutions we have to set $n=1$ variables equal to zero.

Sr. No.	non-basic variables	basic variables	values of basic variables	$z=x_1-2x_2+4x_3$	basic feasible	non-dege.	infeasible	optimal basic feasible.
1	$x_1=0$	x_2, x_3	$2x_2+3x_3=7$, $4x_2+6x_3=15$ ($\rho(A)=1 \neq 2=\rho(A\|B)$); x_2, x_3 does not exists	–	–	–	–	–
2	$x_2=0$	x_1, x_3	$x_1+3x_3=7$, $3x_1+6x_3=15$ $\Rightarrow x_3=2$, $x_1=1$	$z=9$	Yes	Yes	no	Yes
3	$x_3=0$	x_1, x_2	$x_1+2x_2=7$, $3x_1+4x_2=15$ $\Rightarrow x_1=1$, $x_2=3$	$z=-5$	Yes	Yes	no	no

EX find all basic feasible, infeasible, degenerate, non-degenerate solutions of

$x_1+2x_2+4x_3+x_4=7$, $2x_1-x_2+3x_3-2x_4=4$ {S-10 M}

solutions:- number of variables $=m+n=4$ $\Rightarrow$ set any two variables equal to zero.

no. of equations $m=2$ $\Rightarrow n=2$

Sr. No	non-basic variables	basic variables and its values	Feasible	infeasible	degenerate	non-degenerate
1	x_1, x_2	$4x_3+x_4=7$, $3x_3-2x_4=4$ $\Rightarrow x_3=\frac{18}{11}$, $x_4=\frac{5}{11}$	Yes	no	no	Yes
2	x_1, x_3	$2x_2+x_4=7$, $-x_2-2x_4=4$ $\Rightarrow x_2=6$, $x_4=-5$	no	yes	no	yes
3	x_1, x_4	$2x_2+4x_3=7$, $-x_2+3x_3=4$ $\Rightarrow x_3=\frac{3}{2}$, $x_2=\frac{1}{2}$	yes	no	no	yes
4	x_2, x_3	$x_1+x_4=7$, $2x_1-2x_4=4$ $\Rightarrow x_1=\frac{9}{2}$, $x_4=\frac{5}{2}$	yes	no	no	yes
5	x_2, x_4	$x_1+4x_3=7$, $2x_1+3x_3=4$ $\Rightarrow x_3=2$, $x_1=-1$	no	yes	no	yes
6	x_3, x_4	$x_1+2x_2=7$, $2x_1-x_2=4$ $\Rightarrow x_1=3$, $x_2=2$	yes	no	no	yes

Simplex Method

EX Solve the LPP : $\min z = x_1 - 3x_2 + 2x_3$

subject to: $3x_1 - x_2 + 3x_3 \le 7$

$-2x_1 + 4x_2 \le 12$

$-4x_1 + 3x_2 + 8x_3 \le 10$

$x_1, x_2, x_3 \ge 0$

Solution:- Given LPP can be written in std. form as

$Z_{max} = -x_1 + 3x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6$

subject to, $3x_1 - x_2 + 3x_3 + x_4 = 7$

$-2x_1 + 4x_2 + x_5 = 12$

$-4x_1 + 3x_2 + 8x_3 + x_6 = 10$

where x_4, x_5, x_6 slack variables.

Initial Basic feasible solution:- (I.B.F.S.)

Let $x_1 = 0, x_2 = 0, x_3 = 0 \Rightarrow x_4 = 7, x_5 = 12, x_6 = 10$

		C_j	-1	3	-2	0	0	0		
Basic var. B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	min.R.T. X_B/x_2	
x_4	0	7	3	-1	3	1	0	0	–	
x_5	0	12	-2	4 key	0	0	1	0	3	→ outgoing
x_6	0	10	-4	3	8	0	0	1	10/3	All $\Delta_j \not\le 0$
		$\Delta_j = C_j - C_B X_j$	-1	3 ↑ incom. var.	-2	0	0	0		∴ solⁿ is not opt.
									X_B/x_1	
x_4	0	10	5/2	0	3	1	1/4	0	4	→ outgoing
x_2	3	3	-1/2	1	0	0	1/4	0	–	
x_6	0	1	-5/2	0	8	0	-3/4	1	–	All $\Delta_j \not\le 0$
		$\Delta_j = C_j - C_B X_j$	$\frac{1}{2}$ ↑	0	-2	0	$-\frac{3}{4}$	0		∴ soⁿ is not opt.
x_1	-1	4	1	0	6/5	2/5	1/10	0		
x_2	3	5	0	1	3/5	1/5	3/10	0		
x_6	0	11	0	0	11	1	-1/2	1		
		$\Delta_j = C_j - C_B X_j$	0	0	-13/5	-1/5	-8/10	0		

All $\Delta_j \le 0 \Rightarrow$ solⁿ is optimal

$\therefore x_1 = 4, x_2 = 5, x_3 = 0$

$\therefore Z_{max} = -4 + 15 = 11$

$\Rightarrow Z_{min} = -11$

Note ① most positive Δ_j is incomming variable.
② For outgoing variable take ratio of X_B with incomming variable, avoid negative ratios.
③ create '1' in place of key element and using it create zeros in corresponding column.

EX Maximize $z = 4x_1 + 10x_2$ → 2ag no infinite solutions 8 mark

subject to $2x_1 + x_2 \le 50$ (10) show that LPP has infinitely many
$2x_1 + 5x_2 \le 100$ (20) Solutions. Hence find two alternate
$2x_1 + 3x_2 \le 90$ (18) Solutions.
$x_1, x_2 \ge 0$ {2003, 2007

Solution:- Given LPP can be written in std. form as

$$z_{max} = 4x_1 + 10x_2 + 0x_3 + 0x_4 + 0x_5$$

subject to, $2x_1 + x_2 + x_3 = 50$
$2x_1 + 5x_2 + x_4 = 100$
$2x_1 + 3x_2 + x_5 = 90$

I.B.F.S. :→ Let $x_1 = 0,\ x_2 = 0 \Rightarrow x_3 = 50,\ x_4 = 100,\ x_5 = 90$

		c_j	4	10	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	min. R.T. X_B/x_2
x_3	0	50	2	1	1	0	0	10
x_4	0	100	2	5 key	0	1	0	4 → out
x_5	0	90	2	3	0	0	1	6
		$\Delta_j = c_j - C_B X_j$	4	10 ↑in	0	0	0	X_B/x_1
x_3	0	30	8/5	0	1	$-\frac{1}{5}$	0	75/4 → out
x_2	10	20	2/5	1	0	1/5	0	50
x_5	0	30	4/5	0	0	-3/5	1	75/2
		$\Delta_j = c_j - C_B X_j$	0 ↑in	0	0	-2	0	

All $\Delta_j \le 0 \Rightarrow$ solution is optimal, $x_1 = 0,\ x_2 = 20$

and $z_{max} = 0 + 20(10) = 200$

Infinitely many solutions:-

x_1 is present in objective function but absent in the basic variables with corresponding Δ_j i.e. $\Delta_1 = 0$

$\Rightarrow$ above LPP has infinitely many solutions.

to find alternate solution take x_1 as incoming variable.

		c_j	4	10	0	0	0
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5
x_1	4	75/4	1	0	5/8	-1/8	0
x_2	10	25/2	0	1	-1/4	1/4	0
x_5	0	15	0	0	-1/2	-1/2	1
		$\Delta_j = c_j - C_B X_j$	0	0	0	-2	0

All $\Delta_j \le 0 \Rightarrow$ solution is optimal

$\therefore x_1 = \frac{75}{4},\ x_2 = \frac{25}{2} \Rightarrow z_{max} = 4\left(\frac{75}{4}\right) + 10\left(\frac{25}{2}\right) = 75 + 125 = 200$

V.M.Patil

Ex use simplex method to solve

maximize $z = 2x_1 + 4x_2 + x_3 + x_4$

subject to $x_1 + 3x_2 + x_4 \le 4$

$2x_1 + x_2 \le 3$

$x_2 + 4x_3 + x_4 \le 3 \qquad x_1, x_2, x_3 \ge 0$

Solution:- given LPP can written as,

$z_{max} = 2x_1 + 4x_2 + x_3 + x_4' - x_4'' + 0x_5 + 0x_6 + 0x_7$

subject to $x_1 + 3x_2 + x_4' - x_4'' + x_5 = 4$

$2x_1 + x_2 + x_6 = 3$

$x_2 + 4x_3 + x_4' - x_4'' + x_7 = 3$

$x_1, x_2, x_3 \ge 0$
$\Rightarrow x_4$ is unrestricted
$\Rightarrow x_4 = x_4' - x_4''$
where $x_4', x_4'' \ge 0$

I.B.F.S.:- Let $x_1 = 0, x_2 = 0, x_3 = 0, x_4' = 0, x_4'' = 0 \Rightarrow x_5 = 4, x_6 = 3, x_7 = 3$

		c_j	2	4	1	1	0	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4'	x_4''	x_5	x_6	x_7	min. R.T. X_B/x_2
x_5	0	4	1	3 key	0	1	-1	1	0	0	4/3 → out
x_6	0	3	2	1	0	0	0	0	1	0	3
x_7	0	3	0	1	4	1	-1	0	0	1	3
		$\Delta_j = c_j - C_B X_j$	2	4 ↑in	1	1	0	0	0	0	
x_2	4	4/3	1/3	1	0	1/3	-1/3	1/3	0	0	X_B/x_3
x_6	0	5/3	5/3	0	0	-1/3	1/3	-1/3	1	0	–
x_7	0	5/3	-1/3	0	4	2/3	-2/3	-1/3	0	1	–
		$\Delta_j = c_j - C_B X_j$	$\frac{2}{3}$	0	1 ↑in	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{4}{3}$	0	0	5/12 → out
x_2	4	4/3	1/3	1	0	1/3	-1/3	1/3	0	0	X_B/x_1
x_6	0	5/3	5/3	0	0	-1/3	-1/3	-1/3	1	0	4
x_3	1	5/12	$-\frac{1}{12}$	0	1	1/6	1/6	-1/12	0	$\frac{1}{4}$	1 → out
		$\Delta_j = c_j - C_B X_j$	$\frac{3}{4}$ ↑in	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{5}{4}$	0	$-\frac{1}{4}$	–
x_2	4	1	0	1	0	2/5	-2/5	2/5	-1/5	0	
x_1	2	1	1	0	0	-1/5	1/5	-1/5	3/5	0	
x_3	1	$\frac{1}{2}$	0	0	1	3/20	$-\frac{3}{20}$	$-\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{4}$	
		$\Delta_j = c_j - C_B X_j$	0	0	0	-7/20	-47/20	$\frac{-11}{10}$	-9/20	$-\frac{1}{4}$	

All $\Delta_j \le 0 \Rightarrow$ solution is optimal

$\therefore x_1 = 1, \; x_2 = 1, \; x_3 = \frac{1}{2}, \; x_4 = 0, \; z_{max} = 4 + 2 + \frac{1}{2} = \frac{13}{2}$

[EX] Solve the following LPP by simplex method.

$\max. Z = 10x_1 + x_2 + 2x_3$

subject to, $x_1 + x_2 - 3x_3 \le 10$

$4x_1 + x_2 + x_3 \le 20$

$x_1, x_2, x_3 \ge 0$

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Solution:- Given LPP can be written in std. form as

$Z_{max} = 10x_1 + x_2 + 2x_3 + 0x_4 + 0x_5$

Subject to, $x_1 + x_2 - 3x_3 + x_4 = 10$

$4x_1 + x_2 + x_3 + x_5 = 20$

For I.B.F.S.:- Let $x_1 = 0,\ x_2 = 0,\ x_3 = 0 \Rightarrow x_4 = 10,\ x_5 = 20$

		c_j	10	1	2	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	Min. R.T. X_B/x_1
x_4	0	10	1	1	-3	1	0	10
x_5	0	20	4	1	1	0	1	5 — outgoing variable.
		$\Delta_j = C_j - C_B X_j$	10 ↑ in	1	2	0	0	
x_4	0	5	0	3/4	-13/4	1	$-\frac{1}{4}$	
x_1	10	5	1	1/4	1/4	0	$\frac{1}{4}$	
		$\Delta_j = C_j - C_B X_j$	0	-3/2	-1/2	0	5/2	

All $\Delta_j \le 0 \Rightarrow$ Solution is optimal

$\therefore\ x_1 = 5,\ x_2 = 0,\ x_3 = 0$ and $Z_{max} = 50$

[EX] A Company produces two types of goods P and Q that requires gold and silver. Each unit of type P requires 2 gms of silver and 1 gms of gold while for Q 1 gm of silver and 2gms of gold. The company has only 90 gms of silver and 80 gms of gold. Each unit of type P gives a profit of Rs 400 and each unit of type Q gives a profit of Rs 500.

Formulate LPP and solve it by simplex method.

Solution:- Suppose company produces x_1 units of P and x_2 units of Q

	type P	type Q	
units	x_1	x_2	limit
Silver	2	1	≤ 90
gold	1	2	≤ 80
Profit	400	500	

$Z_{max} = 400x_1 + 500x_2$

subject to

$2x_1 + x_2 \le 90$

$x_1 + 2x_2 \le 80$

$x_1, x_2 \ge 0$

V.M.Patil

In std. form $z_{max} = 400x_1 + 500x_2 + 0x_3 + 0x_4$

subject to, $2x_1 + x_2 + x_3 = 90$

$x_1 + 2x_2 + x_4 = 80$, $x_1, x_2, x_3, x_4 \geq 0$

x_3, x_4 are slack variables

I.B.F.S. → Let $x_1 = 0$, $x_2 = 0 \Rightarrow x_3 = 90$, $x_4 = 80$

B	C_B	C_j / X_B	400 / x_1	500 / x_2	0 / x_3	0 / x_4	min.R.T
							X_B/x_2
x_3	0	90	2	1	1	0	90
x_4	0	80	1	2	0	1	40 → outgoing
		$\Delta_j = C_j - C_B X_j$	400	500 ↑ in	0	0	X_B/x_1
x_3	0	50	3/2	0	1	-1/2	100/3 → outgoing
x_2	500	40	1/2	1	0	1/2	80
		$\Delta_j = C_j - C_B X_j$	150	0	0	-250	
x_1	400	$\frac{100}{3}$	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	
x_2	500	$\frac{70}{3}$	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	
		$\Delta_j = C_j - C_B X_j$	0	0	-100	-200	

All $\Delta_j \leq 0 \Rightarrow$ solution is optimal

$\therefore x_1 = \frac{100}{3}$, $x_2 = \frac{70}{3}$ $\Rightarrow z_{max} = 400\left(\frac{100}{3}\right) + 500\left(\frac{70}{3}\right) = 25000$

Ex Use simplex method to Maximize $z = 3x_1 + 5x_2$

subject to, $3x_1 + 2x_2 \leq 18$

$x_1 \leq 4$

$x_2 \leq 6$, $x_1, x_2 \geq 0$

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Solution:- given LPP can be written in std. form as

$z_{max} = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5$

subject to,

$3x_1 + 2x_2 + x_3 = 18$

$x_1 + x_4 = 4$

$x_2 + x_5 = 6$ x_3, x_4, x_5 are slack variables

I.B.F.S:- Let $x_1 = 0$, $x_2 = 0$

$\Rightarrow x_3 = 18$, $x_4 = 4$

$x_5 = 6$

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		C_j	3	5	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	Min.R.T. X_B/x_2
x_3	0	18	3	2	1	0	0	9
x_4	0	4	1	0	0	1	0	–
x_5	0	6	0	1	0	0	1	6 $\rightarrow$ outgoing
		$\Delta_j = c_j - C_B X_j$	3	5 $\uparrow$	0	0	0	X_B/x_1
x_3	0	6	3	0	1	0	-2	2 $\rightarrow$ outgoing
x_4	0	4	1	0	0	1	0	4
x_2	5	6	0	1	0	0	1	–
		$\Delta_j = c_j - C_B X_j$	3	0	0	0	-5	
x_1	3	2	1	0	$\frac{1}{3}$	0	$-\frac{2}{3}$	
x_4	0	2	0	0	$-\frac{1}{3}$	1	$\frac{2}{3}$	
x_2	5	6	0	1	0	0	1	
		$\Delta_j = c_j - C_B X_j$	0	0	-1	0	-3	

All $\Delta_j \le 0 \Rightarrow$ solution is optimal

$\therefore x_1 = 2, \; x_2 = 6 \Rightarrow Z_{max} = 3(2) + 5(6) = 36$

EX Solve by simplex method Max $z = x_1 - x_2 + 3x_3$

subject to $x_1 + x_2 + x_3 \le 10, \; 2x_1 - x_3 \le 3, \; 2x_1 - 2x_2 + 3x_3 \le 0$

$x_1, x_2, x_3 \ge 0$

Solution:-

given LPP can be written as,

$Z_{max} = x_1 - x_2 + 3x_3 + 0x_4 + 0x_5 + 0x_6$

s.t. $x_1 + x_2 + x_3 + x_4 = 10$

$2x_1 + 0x_2 - x_3 + x_5 = 3$

$2x_1 - 2x_2 + 3x_3 + x_6 = 0$

I.B.F.S. :- Let $x_1 = 0, \; x_2 = 0, \; x_3 = 0$

$\therefore x_4 = 10, \; x_5 = 3, \; x_6 = 0$

		C_j	1	-1	3	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	Min.R.T. X_B/x_3
x_4	0	10	1	1	1	1	0	0	10
x_5	0	3	2	0	-1	0	1	0	–
x_6	0	0	2	-2	3	0	0	1	0 $\rightarrow$ out
		$\Delta_j = c_j - C_B X_j$	1	-1	3 $\uparrow$	0	0	0	X_B/x_2
x_4	0	10	1/3	5/3	0	1	0	-1/3	6 $\rightarrow$ out
x_5	0	3	8/3	-2/3	0	0	1	$\frac{1}{3}$	–
x_3	3	0	2/3	-2/3	1	0	0	1/3	–
		$\Delta_j = c_j - C_B X_j$	-1	1 $\uparrow$	0	0	0	-1	

B	C_B	C_j / X_B	1 x_1	-1 x_2	3 x_3	0 x_4	0 x_5	0 x_6	min. R.T.
x_2	-1	6	$\frac{1}{5}$	1	0	$\frac{3}{5}$	0	$-\frac{1}{5}$	
x_5	0	7	$\frac{14}{5}$	0	0	$\frac{2}{5}$	1	$\frac{1}{5}$	
x_3	3	4	$\frac{4}{5}$	0	1	$\frac{2}{5}$	0	$\frac{1}{5}$	
		$\Delta_j = c_j - C_B X_j$	$-\frac{6}{5}$	0	0	$-\frac{3}{5}$	0	$-\frac{4}{5}$	

All $\Delta_j \le 0 \Rightarrow$ solution is optimal

$\therefore x_1 = 0,\ x_2 = 6,\ x_3 = 4,\quad Z_{max} = -6+12 = 6$

EX Solve the following LPP by simplex method :-

Maximize $Z = 3x_1 + 2x_2 + 5x_3$

subject to; $x_1 + 2x_2 + x_3 \le 430$

$3x_1 + 2x_3 \le 460$

$x_1 + 4x_2 \le 420$

$x_1, x_2, x_3 \ge 0$ I.T.-2010

solution:-

Given L.P.P can be written in standard form as

$Z_{max} = 3x_1 + 2x_2 + 5x_3 + 0x_4 + 0x_5 + 0x_6$

s.t. $x_1 + 2x_2 + x_3 + x_4 = 430$

$3x_1 + 0x_2 + 2x_3 + x_5 = 460$

$x_1 + 4x_2 + 0x_3 + x_6 = 420$

I.B.F.S.:- Let $x_1 = x_2 = x_3 = 0 \Rightarrow x_4 = 430,\ x_5 = 460,\ x_6 = 420$

B	C_B	C_j / X_B	3 x_1	2 x_2	5 x_3	0 x_4	0 x_5	0 x_6	Min. R.T. X_B/x_3	
x_4	0	430	1	2	1	1	0	0	430	
x_5	0	460	3	0	2 key	0	1	0	230	→ outgoing
x_6	0	420	1	4	0	0	0	1	—	
		$\Delta_j = c_j - C_B X_j$	3	2	5 ↑ incomming	0	0	0	X_B/x_2	
x_4	0	200	-1/2	2	0	1	$-\frac{1}{2}$	0	100	→ outgoing
x_3	5	230	3/2	0	1	0	$\frac{1}{2}$	0	—	
x_6	0	420	1	4	0	0	0	1	105	
		$c_j - C_B X_j$	-9/2	2 ↑ incom	0	0	-5/2	0		
x_2	2	100	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0		
x_3	5	230	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0		
x_6	0	20	2	0	0	-2	1	1		
		$\Delta_j = c_j - C_B X_j$	-4	0	0	-1	-2	0		

All $\Delta_j \le 0$

$\Rightarrow$ soln optimal

$\therefore x_1 = 0, x_2 = 100$

$x_3 = 230$

$Z_{max} = 1350$

Big M-Method (Penalty-Method)

Note If the constraint is of '$\geq$' type then after using surplus variable add artificial variable and if constraint is of type '=' then add only artificial variable to the left side. It causes violation of the corresponding constraint. This difficulty is removed by introducing a condition which insures that artificial variables will be zero in final solution (Provided solution exists). In other words if final solution contains artificial variable then given problem don't have any solution. This is achieved by assigning a very large price to these variables in the objective function i.e. $-Ma_1$ where a_1 is artificial variables and M is the large price.

Note This method is used when constraint is of '$\geq$' or '=' type

Ex Use Big-M-method to solve,

minimize $z = 10x_1 + 3x_2$

subject to, $x_1 + 2x_2 \geq 3$

$x_1 + 4x_2 \geq 4$ $\quad x_1, x_2 \geq 0$ — S 2006, 18

Solution:-

Given LPP can be written in std. form as,

$$z_{max} = -10x_1 - 3x_2 + 0x_3 + 0x_4 - Ma_1 - Ma_2$$

subject to $x_1 + 2x_2 - x_3 + a_1 = 3$

$x_1 + 4x_2 - x_4 + a_2 = 4$

where, x_3, x_4 are surplus and a_1, a_2 are artificial variables.

I.B.F.S.:- let $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0 \Rightarrow a_1 = 3, a_2 = 4$

For I.B.F.S. set all vari. equal to zero except artificial and slack.

		c_j	-10	-3	0	0	$-M$	$-M$		
B	C_B	X_B	x_1	x_2	x_3	x_4	a_1	a_2	min.R.T. X_B/x_2	
a_1	$-M$	3	1	2	-1	0	1	0	$\frac{3}{2}$	
a_2	$-M$	4	1	4	0	-1	0	1	1	→ outgoing
		$\Delta_j = c_j - C_B X_j$	$-10+2M$	$-3+6M$ ↑ in	$-M$	$-M$	$\cancel{-M}$ 0	$\cancel{-M}$ 0	X_B/x_4	
a_1	$-M$	1	$\frac{1}{2}$	0	-1	$\frac{1}{2}$	1	$-\frac{1}{2}$	2	→ outgoing
x_2	-3	1	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	$-$	
		$\Delta_j = c_j - C_B X_j$	$\frac{-37}{4} + \frac{M}{2}$	0	$-M$	$\frac{M}{2} - \frac{3}{4}$ ↑ in	0	$\frac{-3M}{2} + \frac{3}{4}$		
x_4	0	2	1	0	-2	1	2	-1		
x_2	-3	$3/2$	$1/2$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0		
		$\Delta_j = c_j - C_B X_j$	$-\frac{17}{2}$	0	$-\frac{3}{2}$	0	$-M+\frac{3}{2}$	$-M$		

All $\Delta_j \leq 0 \Rightarrow$ solution is optimal

$\therefore x_1 = 0, x_2 = \frac{3}{2}$ and $z_{max} = 0 - 3(3/2) = \frac{-9}{2} \Rightarrow z_{min} = \frac{9}{2}$

EX Solve the following by Big-M-method.

$Z_{min} = 2y_1 + 3y_2$

Subject to $y_1 + y_2 \geq 5$

$y_1 + 2y_2 \geq 6$, $y_1, y_2 \geq 0$ — {2007

Solution:-

Given LPP can be written in std. form as

$Z_{max} = -2y_1 - 3y_2 + 0y_3 + 0y_4 - Ma_1 - Ma_2$

Subject to, $y_1 + y_2 - y_3 + a_1 = 5$
$y_1 + 2y_2 - y_4 + a_2 = 6$

where y_3, y_4 surplus and a_1, a_2 artificial variables.

I.B.F.S. :→ $y_1 = 0,\ y_2 = 0,\ y_3 = 0,\ y_4 = 0 \Rightarrow a_1 = 5,\ a_2 = 6$

		c_j	-2	-3	0	0	$-M$	$-M$	
B	C_B	X_B	y_1	y_2	y_3	y_4	a_1	a_2	Min.R.T. X_B/y_2
a_1	$-M$	5	1	1	-1	0	1	0	5
a_2	$-M$	6	1	2	0	-1	0	1	3 → outgoing
		$\Delta_j = c_j - C_B Y_j$	$-2+2M$	$-3+3M$ ↑	$-M$	$-M$	0	0	X_B/y_1
a_1	$-M$	2	$\frac{1}{2}$	0	-1	$\frac{1}{2}$	1	$-\frac{1}{2}$	4 → outgoing
y_2	-3	3	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	6
		$\Delta_j = c_j - C_B Y_j$	$-\frac{1}{2} + \frac{M}{2}$	0	$-M$	$\frac{M}{2} - \frac{3}{2}$	0	$-\frac{3M}{2} + \frac{3}{2}$	
y_1	-2	4	1	0	-2	1	2	-1	
y_2	-3	1	0	1	1	-1	-1	1	
		$\Delta_j = c_j - C_B Y_j$	0	0	-1	-1	$-M+1$	$-M+1$	

All $\Delta_j \leq 0 \Rightarrow$ solution is optimal

$y_1 = 4,\ y_2 = 1,\quad Z_{max} = -2(4) - 3(1) = -11 \Rightarrow Z_{min} = 11$

EX Solve using Big-M-method

$Z_{max} = -2x_1 - x_2$

subject to, $3x_1 + x_2 = 3$,
$4x_1 + 3x_2 \geq 6$
$x_1 + 2x_2 \leq 4$, $x_1, x_2 \geq 0$

solution:-

Introducing slack, surplus, artificial variables given LPP can be written as

$Z_{max} = -2x_1 - x_2 + 0x_3 + 0x_4 - Ma_1 - Ma_2$

Subject to
$3x_1 + x_2 + a_1 = 3$
$4x_1 + 3x_2 - x_3 + a_2 = 6$
$x_1 + 2x_2 + x_4 = 4$

} a_1, a_2 artificial vari. x_3 surplus and x_4 slack variable.

I.B.F.S. :- Let $x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ \Rightarrow a_1 = 3,\ a_2 = 6,\ x_4 = 4$

B	C_B	c_j / X_B	x_1 (−2)	x_2 (−1)	x_3 (0)	x_4 (0)	a_1 (−M)	a_2 (−M)	min.R.T. X_B/x_1
a_1	$-M$	3	3	1	0	0	1	0	1 → out
a_2	$-M$	6	4	3	-1	0	0	1	$\frac{3}{2}$
x_4	0	4	1	2	0	1	0	0	4
		$\Delta_j = c_j - C_B X_j$	$-2+7M$ ↑ in	$-1+4M$	$-M$	0	0	0	X_B/x_2
x_1	-2	1	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$	0	3
a_2	$-M$	2	0	$\frac{5}{3}$	-1	0	$-\frac{4}{3}$	1	$\frac{6}{5}$ → out
x_4	0	3	0	$\frac{5}{3}$	0	1	$-\frac{1}{3}$	0	$\frac{9}{5}$
		$\Delta_j = c_j - C_B X_j$	0	$-\frac{1}{3}+\frac{5M}{3}$	$-M$	0	$\frac{2}{3}-\frac{7M}{3}$	0	
x_1	-2	$3/5$	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$	
x_2	-1	$6/5$	0	1	$-\frac{3}{5}$	0	$-\frac{4}{5}$	$\frac{3}{5}$	
x_4	0	1	0	0	1	1	1	-1	
		$\Delta_j = c_j - C_B X_j$	0	0	$-\frac{1}{5}$	0	$-M+\frac{2}{5}$	$-M+\frac{1}{5}$	

All $\Delta_j \le 0 \Rightarrow$ solution is optimal

$\therefore x_1 = \frac{3}{5}, \quad x_2 = \frac{6}{5}, \quad z_{max} = -2\left(\frac{3}{5}\right) - 1\left(\frac{6}{5}\right) = \frac{-12}{5} \Rightarrow z_{min} = \frac{12}{5}$

EX Using penalty method solve the following LPP.

$$z_{min} = 4x_1 + x_2$$

subject to, $3x_1 + x_2 = 3$ — 8 marks M -{C-09

$4x_1 + 3x_2 \ge 6$

$x_1 + 2x_2 \le 4, \quad x_1, x_2 \ge 0$

Solution:- given LPP can be written in standard form as

$$z_{max} = -4x_1 - x_2 + 0x_3 + 0x_4 - Ma_1 - Ma_2$$

s.t. $3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - x_3 + a_2 = 6$

$x_1 + 2x_2 + x_4 = 4$

a_1, a_2 artificial, x_3 surplus and x_4 slack variable.

I.B.F.S.:- $x_1 = 0, x_2 = 0, x_3 = 0$

$\Rightarrow a_1 = 3, \quad a_2 = 6, \quad x_4 = 4$

B	C_B	C_j / X_B	x_1 (-4)	x_2 (-1)	x_3 (0)	x_4 (0)	a_1 (-M)	a_2 (-M)	Min. R.T. X_B/x_1
a_1	$-M$	3	3	1	0	0	1	0	1 $\rightarrow$ out
a_2	$-M$	6	4	3	-1	0	0	1	3/2
x_4	0	4	1	2	0	1	0	0	4
		$\Delta_j = c_j - C_B X_j$	$-4+7M$ $\uparrow$	$-1+4M$	$-M$	0	0	0	X_B/x_2
x_1	-4	1	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$	0	3
a_2	$-M$	2	0	$\frac{5}{3}$	-1	0	$-\frac{4}{3}$	1	6/5 $\rightarrow$ out
x_4	0	3	0	$\frac{5}{3}$	0	1	$-\frac{1}{3}$	0	9/5
		$\Delta_j = c_j - C_B X_j$	M	$\frac{1}{3}+\frac{5M}{3}$ $\uparrow$	$-M$	0	$-\frac{7M}{3}+\frac{4}{3}$	0	X_B/x_3
x_1	-4	$\frac{3}{5}$	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$	3
x_2	-1	$\frac{6}{5}$	0	1	$-\frac{3}{5}$	0	$-\frac{4}{5}$	$\frac{3}{5}$	—
x_4	0	1	0	0	1	1	1	-1	1 $\rightarrow$ out
		$\Delta_j = c_j - C_B X_j$	0	0	$\frac{1}{5}$ $\uparrow$	0	$-M+\frac{8}{5}$	$-M-\frac{1}{5}$	
x_1	-4	2/5	1	0	0	$-\frac{1}{5}$	$\frac{2}{5}$	0	
x_2	-1	9/5	0	1	0	3/5	$-\frac{1}{5}$	0	
x_3	0	1	0	0	1	1	1	-1	
		$\Delta_j = c_j - C_B X_j$	0	0	0	$-\frac{1}{5}$	$-M+\frac{7}{5}$	$-M$	

All $\Delta_j \le 0 \Rightarrow$ Solution is optimal

$\therefore\ x_1 = \frac{2}{5},\ x_2 = \frac{9}{5},\quad Z_{max} = -4\left(\frac{2}{5}\right) - 1\left(\frac{9}{5}\right) = \frac{-17}{5} \Rightarrow Z_{min.} = \frac{17}{5}$

EX Solve by using Big-M-method

$$Z_{max} = 5x_1 - 2x_2 + 3x_3$$

subject to,
$$2x_1 + 2x_2 - x_3 \ge 2$$
$$3x_1 - 4x_2 \le 3$$
$$x_2 + 3x_3 \le 5$$
$$x_1, x_2, x_3 \ge 0$$

{2004, 2005}

Solution:-

given LPP can be written as,

$$Z_{max} = 5x_1 - 2x_2 + 3x_3 + 0x_4 + 0x_5 + 0x_6 - Ma_1$$

subject to,
$$2x_1 + 2x_2 - x_3 - x_4 + a_1 = 2$$
$$3x_1 - 4x_2 + x_5 = 3$$
$$x_2 + 3x_3 + x_6 = 5$$

a_1 artificial variable
x_4 surplus variable
x_5, x_6 slack variables

I.B.F.S. :- set $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$

$\Rightarrow a_1 = 2,\ x_5 = 3,\ x_6 = 5$

		C_j	5	-2	3	0	0	0	-M	
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	a_1	Min.R.T. X_B/x_1
a_1	-M	2	2	2	-1	-1	0	0	1	1 → out
x_5	0	3	3	-4	0	0	1	0	0	1
x_6	0	5	0	1	3	0	0	1	0	–
		$\Delta_j = C_j - C_B X_j$	$5+2M$ ↑	$-2+2M$	$3-M$	$-M$	0	0	0	X_B/x_3
x_1	5	1	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	–
x_5	0	0	0	-7	$\frac{3}{2}$	$\frac{3}{2}$	1	0	$-\frac{3}{2}$	0 → outgoing
x_6	0	5	0	1	3	0	0	1	0	$\frac{5}{3}$
		$\Delta_j = C_j - C_B X_j$	0	-7	$\frac{11}{2}$ ↑	$\frac{5}{2}$	0	0	$-M-\frac{5}{2}$	X_B/x_2
x_1	5	1	1	$\frac{-4}{3}$	0	0	$\frac{1}{3}$	0	0	–
x_3	3	0	0	$\frac{-14}{3}$	1	1	$\frac{2}{3}$	0	-1	–
x_6	0	5	0	15	0	-3	-2	1	3	$\frac{1}{3}$ → outgoing
		$\Delta_j = C_j - C_B X_j$	0	$\frac{56}{3}$ ↑	0	-3	$\frac{-11}{3}$	0	$-M+3$	X_B/x_4
x_1	5	$\frac{13}{9}$	1	0	0	$\frac{-4}{15}$	$7/45$	$\frac{4}{45}$	$\frac{4}{15}$	–
x_3	3	$\frac{14}{9}$	0	0	1	$\frac{1}{15}$	$\frac{2}{45}$	$\frac{14}{45}$	$-\frac{1}{15}$	$\frac{70}{3}$ → outgoing
x_2	-2	$\frac{1}{3}$	0	1	0	$-\frac{1}{5}$	$-\frac{2}{15}$	$\frac{1}{15}$	$\frac{1}{5}$	–
		$\Delta_j = C_j - C_B X_j$	0	0	0	$\frac{11}{15}$ ↑	$\frac{-53}{45}$	$\frac{-56}{45}$	$-M-\frac{11}{15}$	
x_1	5	$\frac{23}{3}$	1	0	4	0	$\frac{1}{3}$	$\frac{4}{3}$	0	
x_4	0	$\frac{70}{3}$	0	0	15	1	$\frac{2}{3}$	$\frac{14}{3}$	-1	
x_2	-2	5	0	1	3	0	0	1	0	
		$\Delta_j = C_j - C_B X_j$	0	0	-11	0	$-\frac{5}{3}$	$\frac{-14}{3}$	-M	

M.R.T is Same but as a_1 is artificial take it as outgoing

All $\Delta_j \le 0 \Rightarrow$ Solution is optimal.

$x_1 = \frac{23}{3}$, $x_2 = 5$, $x_3 = 0$. $\Rightarrow Z_{max} = 5\left(\frac{23}{3}\right) - 2(5) + 0 = \frac{85}{3}$

EX Use Big M-Method to maximize $Z = 3x_1 - x_2$

subject to $2x_1 + x_2 \ge 2$

$x_1 + 3x_2 \le 3$

$x_2 \le 4$

$x_1, x_2 \ge 0$ — ε c-og D

Solution:- std. Form of LPP

$$Z_{max} = 3x_1 - x_2 + 0x_3 + 0x_4 + 0x_5 - Ma_1$$

s.t. $2x_1 + x_2 - x_3 + a_1 = 2$

$x_1 + 3x_2 + x_4 = 3$

$x_2 + x_5 = 4$

I.B.F.S.:- $x_1=0, x_2=0, x_3 \Rightarrow a_1=2, x_4=3, x_5=4$

B	C_B	C_j / X_B	3 / x_1	-1 / x_2	0 / x_3	0 / x_4	0 / x_5	-M / a_1	Min. R.T. X_B/x_1
a_1	-M	2	2	1	-1	0	0	1	1 → outgoing
x_4	0	3	1	3	0	1	0	0	3
x_5	0	4	0	1	0	0	1	0	–
		$\Delta_j = C_j - C_B X_j$	$3+2M$ ↑	$-1+M$	$-M$	0	0	0	X_B/x_3
x_1	3	1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	–
x_4	0	2	0	$\frac{5}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	4 → outgoing
x_5	0	4	0	1	0	0	1	0	–
		$\Delta_j = C_j - C_B X_j$	0	$-\frac{5}{2}$	$\frac{3}{2}$ ↑	0	0	$-M-\frac{3}{2}$	
x_1	3	3	1	3	0	1	0	0	
x_3	0	4	0	5	1	2	0	-1	
x_5	0	4	0	1	0	0	1	0	
		$\Delta_j = C_j - C_B X_j$	0	-10	0	-3	0	-M	

All $\Delta_j \le 0 \Rightarrow$ solution is optimal.

$x_1=3, x_2=0 \Rightarrow Z_{max}=9$

EX Use Big M method to solve $Z_{max} = x_1+4x_2$

subject to $3x_1+x_2 \le 3$

$2x_1+3x_2 \le 6$

$4x_1+5x_2 \ge 20$

$x_1, x_2 \ge 0$ $\quad \rightarrow \Sigma\, C-10$ M

Solution:- given LPP can be written as,

$$Z_{max} = x_1+4x_2+0x_3+0x_4+0x_5-Ma_1$$

s.t. $3x_1+x_2+x_3=3$

$2x_1+3x_2+x_4=6$

$4x_1+5x_2-x_5+a_1=20$

I.B.F.S.:-

Let $x_1=x_2=x_5=0$

$\Rightarrow x_3=3,$

$x_4=6$

$a_1=20$

B	C_B	c_j / X_B	1 / x_1	4 / x_2	0 / x_3	0 / x_4	0 / x_5	$-M$ / a_1	Min. R.T. X_B/x_2
x_3	0	3	3	1	1	0	0	0	3
x_4	0	6	2	3	0	1	0	0	2 $\longrightarrow$ outgoing
a_1	$-M$	20	4	5	0	0	-1	1	4
		$\Delta_j = c_j - C_B X_j$	$1+4M$	$4+5M$ ↑	0	0	$-M$	0	X_B/x_1
x_3	0	1	$7/3$	0	1	$-\frac{1}{3}$	0	0	$3/7$ $\longrightarrow$
x_2	4	2	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	3
a_1	$-M$	10	$\frac{2}{3}$	0	0	$\frac{-5}{3}$	-1	1	15
		$\Delta_j = c_j - C_B X_j$	$-5/3 + \frac{2M}{3}$ ↑	0	0	$-4/3 - 5M/3$	$-M$	0	
x_1	1	$\frac{3}{7}$	1	0	$\frac{3}{7}$	$-\frac{1}{7}$	0	0	
x_2	4	$\frac{12}{7}$	0	1	$\frac{-2}{7}$	$\frac{3}{7}$	0	0	
a_1	$-M$	$\frac{68}{7}$	0	0	$-\frac{2}{7}$	$\frac{-11}{7}$	-1	1	
		$\Delta_j = c_j - C_B X_j$	0	0	$\frac{5}{7} - \frac{2M}{7}$	$\frac{-3-11M}{7}$	$-M$	0	

All $\Delta_j \le 0 \Rightarrow$ solution is optimal

But presence of artificial variable in the basis indicates given LPP don't have any solution.

Special Case

Type-I Unbounded Solution

EX Show that the LPP

$$Z_{max} = 4x_1 + x_2 + 3x_3 + 5x_4$$

subject to,
$$4x_1 - 6x_2 - 5x_3 + 4x_4 \geq -20$$
$$-3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$$
$$-8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20 \qquad x_1, x_2, x_3, x_4 \geq 0$$

has unbounded solution.

Solution:-

First constraint can be written as $-4x_1 + 6x_2 + 5x_3 - 4x_4 \leq 20$

$\therefore$ Standard form of LPP,

$$Z_{max} = 4x_1 + x_2 + 3x_3 + 5x_4 + 0x_5 + 0x_6 + 0x_7$$

subject to,
$$-4x_1 + 6x_2 + 5x_3 - 4x_4 + x_5 = 20$$
$$-3x_1 - 2x_2 + 4x_3 + x_4 + x_6 = 10$$
$$-8x_1 - 3x_2 + 3x_3 + 2x_4 + x_7 = 20$$

x_5, x_6, x_7 are slack variables.

I.B.F.S.:-

Let $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$

$\Rightarrow x_5 = 20, \ x_6 = 10, \ x_7 = 20$

B	C_B	C_j →	4	1	3	5	0	0	0	
		X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	min. R.T. X_B/x_4
x_5	0	20	-4	6	5	-4	1	0	0	—
x_6	0	10	-3	-2	4	1	0	1	0	10 → out
x_7	0	20	-8	-3	3	2	0	0	1	10
		$\Delta_j = C_j - C_B X_j$	4	1	3	5 ↑	0	0	0	X_B/x_1
x_5	0	60	-16	-2	21	0	1	4	0	
x_4	5	10	-3	-2	4	1	0	1	0	
x_7	0	0	-2	1	-5	0	0	-2	1	
		$\Delta_j = C_j - C_B X_j$	19 ↑	11	-17	0	0	-5	0	

x_1 is incoming variable, but all elements in x_1 column are negative which indicates x_1 cannot get entry in the basis in other words variables in the basis can be increased indefinitely.

$\Rightarrow$ above LPP has unbounded solution.

EX Maximize $Z = 107x_1 + x_2 + 2x_3$

subject to, $14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$16x_1 + \frac{1}{2}x_2 - 6x_3 \le 5$

$3x_1 - x_2 - x_3 \le 0$, $x_1, x_2, x_3, x_4 \geq 0$ {2005}

Solution:-

Note In the objective function x_4 is absent and in the first constraint x_4 is present. Therefore divide this constraint by 3 and then treat x_4 as a slack variable.

$\therefore$ given LPP can be written as

$$Z_{max} = 107x_1 + x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$$

subject to $\frac{14}{3}x_1 + \frac{1}{3}x_2 - 2x_3 + x_4 = \frac{7}{3}$

$16x_1 + \frac{1}{2}x_2 - 6x_3 + x_5 = 5$

$3x_1 - x_2 - x_3 + x_6 = 0$

x_4, x_5, x_6 are slack variables.

I.B.F.S. :- $x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ \Rightarrow x_4 = \frac{7}{3},\ x_5 = 5,\ x_6 = 0$

		c_j	107	1	2	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	min. R.T. X_B/x_1
x_4	0	$\frac{7}{3}$	$\frac{14}{3}$	$\frac{1}{3}$	-2	1	0	0	$\frac{1}{2}$
x_5	0	5	16	$\frac{1}{2}$	-6	0	1	0	$\frac{5}{16}$
x_6	0	0	3	-1	-1	0	0	1	0 → outgoing
		$\Delta_j = c_j - C_B X_j$	107 ↑	1	2	0	0	0	
x_4	0	$\frac{7}{3}$	0	$\frac{17}{9}$	$-\frac{4}{9}$	1	0	$-\frac{14}{9}$	
x_5	0	5	0	$\frac{35}{6}$	$-\frac{2}{3}$	0	1	$-\frac{16}{3}$	
x_1	107	0	1	$-\frac{1}{3}$	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	
		$\Delta_j = c_j - C_B X_j$	0	$\frac{110}{3}$	$\frac{113}{3}$ ↑	0	0	$-\frac{107}{3}$	

all entries corresponding to incomming column are negative.

$\Rightarrow$ given LPP has unbounded solution.

OR. in first constrain add slack, use simplex method, we get same solution, 2nd simplex table gives final answer.

Type-II Infinitely many solutions

Ex Use Penalty method (Big-M-method) to solve the LPP

Maximize $Z = 6x_1 + 4x_2$

Subject to, $2x_1 + 3x_2 \le 30$

$3x_1 + 2x_2 \le 24$

$x_1 + x_2 \ge 3$, $x_1, x_2 \ge 0$

Is the solution unique? If not find another solution.

-Egr, 2008

Mech-2018 - 2nd table gives ans.

Solution:- Given LPP can be written as,

$$Z_{max} = 6x_1 + 4x_2 + 0x_3 + 0x_4 + 0x_5 - Ma_1$$

subject to
$2x_1 + 3x_2 + x_3 = 30$
$3x_1 + 2x_2 + x_4 = 24$
$x_1 + x_2 - x_5 + a_1 = 3$

where x_3, x_4 slack, x_5 surplus and a_1 artificial variable.

I.B.F.S. :- Let $x_1 = 0$, $x_2 = 0$, $x_5 = 0$ $\Rightarrow x_3 = 30$, $x_4 = 24$, $a_1 = 3$

		c_j	6	4	0	0	0	$-M$	
B	c_B	X_B	x_1	x_2	x_3	x_4	x_5	a_1	min.R.T. X_B/x_1
x_3	0	30	2	3	1	0	0	0	15
x_4	0	24	3	2	0	1	0	0	8
a_1	$-M$	3	1	1	0	0	-1	1	3 → outgoing
		$\Delta_j = c_j - C_B X_j$	$6+M$ ↑	$4+M$	0	0	$-M$	0	X_B/x_5
x_3	0	24	0	1	1	0	2	-2	12
x_4	0	15	0	-1	0	1	3	-3	5 → outgoing
x_1	6	3	1	1	0	0	-1	1	–
		$\Delta_j = c_j - C_B X_j$	0	-2	0	0	6 ↑	$-M-6$	X_B/x_2
x_3	0	14	0	$5/3$	1	$-\frac{2}{3}$	0	0	$\frac{42}{5}$ → outgoing
x_5	0	5	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	1	-1	–
x_1	6	8	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	0	12
		$\Delta_j = c_j - C_B X_j$	0	0 ↑	0	-2	0	$-M$	

All $\Delta_j \le 0$

$\therefore$ sol^n, $x_1 = 8$, $x_2 = 0$

$Z_{max} = 48$

x_2 is present in objective function and absent in the basis with corresponding $\Delta_2 = 0$

$\Rightarrow$ given LPP has infinitely many solutions

to find alternate solution take x_2 as incomming variable.

		c_j	6	4	0	0	0	$-M$	
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	a_1	min. R.T. $X_B/$
x_2	4	$\frac{42}{5}$	0	1	$\frac{3}{5}$	$-\frac{2}{5}$	0	0	
x_5	0	$\frac{39}{5}$	0	0	$\frac{1}{5}$	$\frac{1}{5}$	1	-1	
x_1	6	$\frac{12}{5}$	1	0	$-\frac{2}{5}$	$\frac{3}{5}$	0	0	
		$\Delta_j = c_j - C_B X_j$	0	0	0	-2	0	0	

All $\Delta_j \le 0 \Rightarrow$ solution is optimal.

$\therefore x_1 = \frac{12}{5},\ x_2 = \frac{42}{5} \Rightarrow Z_{max} = 6\left(\frac{12}{5}\right) + 4\left(\frac{42}{5}\right) = 48$

EX Show that the LPP $Z_{max} = x_1 + x_2$

subject to $x_1 + 2x_2 \le 2000$
$x_1 + x_2 \le 1500$
$x_2 \le 600$, $x_1, x_2 \ge 0$

has infinitely many solutions.
Hence find two different solutions.

Solution:- Standard form of above LPP is

$Z_{max} = x_1 + x_2 + 0x_3 + 0x_4 + 0x_5$

subject to, $x_1 + 2x_2 + x_3 = 2000$
$x_1 + x_2 + x_4 = 1500$
$x_2 + x_5 = 600$

x_3, x_4, x_5 are slack variables.

I.B.F.S.:- Let $x_1 = 0,\ x_2 = 0 \Rightarrow x_3 = 2000,\ x_4 = 1500,\ x_5 = 600$

		c_j	1	1	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	min. R.T. X_B/x_1
x_3	0	2000	1	2	1	0	0	2000
x_4	0	1500	1	1	0	1	0	1500 → outgoing
x_5	0	600	0	1	0	0	1	—
		$\Delta_j = c_j - C_B X_j$	1 ↑	1	0	0	0	X_B/x_2
x_3	0	500	0	1	1	-1	0	500 → outgoing
x_1	1	1500	1	1	0	1	0	1500
x_5	0	600	0	1	0	0	1	600
		$\Delta_j = c_j - C_B X_j$	0	0 ↑	0	-1	0	

All $\Delta_j \le 0 \Rightarrow$ solution is optimal

$\therefore x_1 = 1500,\ x_2 = 0 \Rightarrow Z_{max} = 1500$

x_2 is absent in basis but present in objective function with $\Delta_2=0$
$\Rightarrow$ infinitely many solution possible.
taking x_2 as incomming variable

		c_j	1	1	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	min. R.T. $X_B/$
x_2	1	500	0	1	1	-1	0	
x_1	1	1000	1	0	-1	2	0	
x_5	0	100	0	0	-1	1	1	
		$\Delta_j = c_j - C_B X_j$	0	0	0	-1	0	

All $\Delta_j \leq 0$
$\Rightarrow$ solution is optimal
$x_1 = 1000$, $x_2 = 500$
$Z_{max} = 1500$

All $\Delta_j \leq 0 \Rightarrow$ solution is optimal
$\therefore x_1 = 1000$, $x_2 = 500 \Rightarrow Z_{max} = 1000 + 500 = 1500$

Type-IV

Degenerate Solution

Ex Show that the LPP has degenerate solution.

$\max. Z = 3x_1 + 9x_2$

subject to, $x_1 + 4x_2 \le 8$

$x_1 + 2x_2 \le 4$

$x_1 \ge 0,\ x_2 \ge 0$ {2000

Solution:-

Given LPP can be written in std. form as,

$Z_{max} = 3x_1 + 9x_2 + 0x_3 + 0x_4$

subject to, $x_1 + 4x_2 + x_3 = 8$

$x_1 + 2x_2 + x_4 = 4$

x_3, x_4 are slack variables

I.B.F.S.:- Let $x_1 = 0, x_2 = 0 \Rightarrow x_3 = 8,\ x_4 = 4$

		c_j	3	9	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	min. R.T. X_B/x_2
x_3	0	8	1	4	1	0	2
x_4	0	4	1	2	0	1	2 → outgoing
		$\Delta_j = c_j - C_B X_j$	3	9	0	0	
x_3	0	0	-1	0	1	-2	
x_2	9	2	$\frac{1}{2}$	1	0	$\frac{1}{2}$	
		$\Delta_j = c_j - C_B X_j$	$\frac{-3}{2}$	0	0	$-9/2$	

All $\Delta_j \le 0 \Rightarrow$ solution is optimal

$\therefore x_1 = 0, x_2 = 2 \Rightarrow Z_{max} = 0 + 9(2) = 18$

$x_3 = 0$

as $x_3 = 0$ solution is degenerate

one of the basic variable must be zero valued.

Type-III No feasible solution

Ex Solve the LPP $Z_{max} = 3x_1 + 2x_2$

subject to, $2x_1 + x_2 \le 2$

$3x_1 + 4x_2 \ge 12$

$x_1, x_2 \ge 0$

Solution:-

Given LPP can be written in standard form as

$$Z_{max} = 3x_1 + 2x_2 + 0x_3 + 0x_4 - Ma_1$$

subject to, $2x_1 + x_2 + x_3 = 2$

$3x_1 + 4x_2 - x_4 + a_1 = 12$

a_1 artificial
x_3 slack
x_4 surplus variable

I.B.F.S.:- Let $x_1=0,\ x_2=0,\ x_4=0 \Rightarrow x_3=2,\ a_1=12$

		C_j	3	2	0	0	$-M$	
B	C_B	X_B	x_1	x_2	x_3	x_4	a_1	min. R.T. X_B/x_2
x_3	0	2	2	1	1	0	0	2 → outgoing
a_1	$-M$	12	3	4	0	-1	1	3
		$\Delta_j = C_j - C_B X_j$	$3+3M$	$2+4M$	0	$-M$	0	
x_2	2	2	2	1	1	0	0	
a_1	$-M$	4	-5	0	-4	-1	1	
		$\Delta_j = C_j - C_B X_j$	$-1-5M$	0	$-2-4M$	$-M$	0	

All $\Delta_j \le 0 \Rightarrow$ solution is optimal.

but artificial variable 'a_1' remains in the basis

$\Rightarrow$ it is not feasible.

$\therefore$ LPP don't have any feasible solution.

when '0' and ∞ look at the number $\frac{1}{x}$, they clearly see each other. so in some sense we say that 0, ∞ are duals of each other.

Duality

Duality in Linear Programming Problem states that every Linear P.P. has another LPP related to it and thus can be derived from it. The original LPP is called primal and derived is called dual.

Note-I If problem is of maximization type then all constraints must be of ≤ type.

How to write dual ?:- [1] maximization becomes minimization

[2] r.h.s. constants becomes coefficients in objective function and coefficients in objective function becomes r.h.s. constants.

[3] and constraints of the type '≤' becomes '≥' type

Note-II If problem is of minimization type then constraints must be of the type '≥'

How to write its dual ? :- [1] minimization becomes maximization.

[2] r.h.s. constants becomes coe. in objective function and vice-versa.

[3] and constraints of the type '≥' becomes '≤' type.

EX Write the dual of the following LPP

$$\text{Maximize } z = x_1 - x_2 + 3x_3$$

$$\text{Subject to } \quad x_1 + x_2 + x_3 \le 10$$
$$2x_1 - x_3 \le 2$$
$$2x_1 - 2x_2 + 3x_3 \le 6$$
$$x_1, x_2, x_3 \ge 0$$

(2 marks) — {2008}

Solution:→

$$\min Z = 10y_1 + 2y_2 + 6y_3$$

$$\text{s.t.} \quad y_1 + 2y_2 + 2y_3 \ge 1$$
$$y_1 + 0y_2 - 2y_3 \ge -1$$
$$y_1 - y_2 + 3y_3 \ge 3$$
$$y_1, y_2, y_3 \ge 0$$

EX Write the dual of the following problem:-

$$\text{Minimize } Z = 3x_1 - 2x_2 + 4x_3$$

$$\text{Subject to, } \quad 3x_1 + 5x_2 + 4x_3 \ge 7$$
$$6x_1 + x_2 + 3x_3 \ge 4$$
$$7x_1 - 2x_2 - x_3 \le 10$$
$$x_1 - 2x_2 + 5x_3 \ge 3$$
$$4x_1 + 7x_2 - 2x_3 \ge 2$$
$$x_1, x_2, x_3 \ge 0$$

(6 marks) — {2007}

solution:- given problem can be writen as

$$Z_{min} = 3x_1 - 2x_2 + 4x_3$$

subject to,

$$3x_1+5x_2+4x_3 \geq 7$$
$$6x_1+x_2+3x_3 \geq 4$$
$$-7x_1+2x_2+x_3 \geq -10$$
$$x_1-2x_2+5x_3 \geq 3$$
$$4x_1+7x_2-2x_3 \geq 2$$

$\therefore$ its dual $Z_{max} = 7y_1+4y_2-10y_3+3y_4+2y_5$

subject to, $3y_1+6y_2-7y_3+y_4+4y_5 \leq 3$
$$5y_1+y_2+2y_3-2y_4+7y_5 \leq -2$$
$$4y_1+3y_2+y_3+5y_4-2y_5 \leq 4$$
$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

Ex Find the dual of

minimize $z=2x_1+x_2+5x_3$

subject to, $x_1+x_2+x_3=10$
$$4x_1-x_2+2x_3 \geq 12$$
$$3x_1+2x_2-3x_3 \leq 6$$
$$x_1, x_2, x_3 \geq 0$$

— {2006 (6 marks)

Solution:- given LPP can be written as,

$Z_{min} = 2x_1+x_2+5x_3$

s.t.

subject to $x_1+x_2+x_3=10 \Rightarrow \left.\begin{array}{l} x_1+x_2+x_3 \leq 10 \\ x_1+x_2+x_3 \geq 10 \end{array}\right\} \Rightarrow \begin{array}{l} -x_1-x_2-x_3 \geq -10 \\ x_1+x_2+x_3 \geq 10 \end{array}$

$4x_1-x_2+2x_3 \geq 12 \Longrightarrow 4x_1-x_2+2x_3 \geq 12$

$3x_1+2x_2-3x_3 \leq 6 \Rightarrow -3x_1-2x_2+3x_3 \geq -6$

$\therefore$ its dual is $Z_{max} = -10y_1+10y_2+12y_3-6y_4$

subject to, $-y_1+y_2+4y_3-3y_4 \leq 2$
$$-y_1+y_2-y_3-2y_4 \leq 1$$
$$-y_1+y_2+2y_3+3y_4 \leq 5$$
$$y_1, y_2, y_3, y_4 \geq 0$$

Ex Construct the dual of the LPP

maximize $z = 3x_1+17x_2+9x_3$

subject to $x_1-x_2+x_3 \geq 3$
$$-3x_1+2x_3 \leq 1$$
$$2x_1+x_2-5x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

— {2003 M-2017

Solution:- given LPP can be written as,

$Z_{max} = 3x_1+17x_2+9x_3$

subject to,

$x_1 - x_2 + x_3 \geq 3 \qquad \Rightarrow -x_1 + x_2 - x_3 \leq -3$

$3x_1 + 2x_3 \leq 1 \qquad \Rightarrow -3x_1 + 0x_2 + 2x_3 \leq 1$

$$2x_1 + x_2 - 5x_3 = 1 \Rightarrow \left.\begin{array}{l} 2x_1 + x_2 - 5x_3 \leq 1 \\ 2x_1 + x_2 - 5x_3 \geq 1 \end{array}\right\} \Rightarrow \begin{array}{l} 2x_1 + x_2 - 5x_3 \leq 1 \\ -2x_1 + x_2 + 5x_3 \leq -1 \end{array}$$

$\therefore$ its dual is $Z_{min} = -3y_1 + y_2 + y_3 - y_4$

subject to, $-y_1 - 3y_2 + 2y_3 - 2y_4 \geq 3$

$y_1 + 0y_2 + y_3 - y_4 \geq 17$

$-y_1 + 2y_2 - 5y_3 + 5y_4 \geq 9$

$y_1, y_2, y_3, y_4 \geq 0$

EX Construct the dual of the following problem,

maximize $Z = 2x_1 + x_2 + x_3$

subject to $x_1 + x_2 + x_3 \geq 6$

$3x_1 - 2x_2 + 3x_3 = 3$

$-4x_1 + x_3 \leq 10$

x_1 and $x_3 \geq 0$, x_2 unrestricted. —S2000,

Solution: as x_2 is unrestricted let $x_2 = x_2' - x_2''$ where $x_2' \geq 0$, $x_2'' \geq 0$

$\therefore$ given problem $Z_{max} = 2x_1 + x_2' - x_2'' + x_3$

subject to, $x_1 + x_2' - x_2'' + x_3 \geq 6 \qquad \Rightarrow -x_1 - x_2' + x_2'' - x_3 \leq -6$

$$3x_1 - 2(x_2' - x_2'') + 3x_3 = 3 \Rightarrow \left.\begin{array}{l} 3x_1 - 2x_2' + 2x_2'' + 3x_3 \leq 3 \\ 3x_1 - 2x_2' + 2x_2'' + 3x_3 \geq 3 \end{array}\right\} \Rightarrow \begin{array}{l} 3x_1 - 2x_2' + 2x_2'' + 3x_3 \leq 3 \\ -3x_1 + 2x_2' - 2x_2'' - 3x_3 \leq -3 \end{array}$$

and $-4x_1 + x_3 \leq 10 \qquad \Rightarrow -4x_1 + 0x_2' + 0x_2'' + x_3 \leq 10$

$\therefore$ its dual, $Z_{min} = -6y_1 + 3y_2 - 3y_3 + 10y_4$

subject to, $-y_1 + 3y_2 - 3y_3 - 4y_4 \geq 2$

$-y_1 - 2y_2 + 2y_3 + 0y_4 \geq 1$

$y_1 + 2y_2 - 2y_3 + 0y_4 \geq -1$

$-y_1 + 3y_2 - 3y_3 + y_4 \geq 1$

with $y_1, y_2, y_3, y_4 \geq 0$

V.M.Patil

[EX] Construct the dual of the following problem :-

minimize $z = x_2 + 3x_3$

subject to $2x_1 + x_2 \leq 3$

$x_1 + 2x_2 + 6x_3 \geq 5$

$-x_1 + x_2 + 2x_3 = 2$

$x_1, x_2, x_3 \geq 0$ — {2002}

Solution:- given problem can be written as,

$z_{min} = 0x_1 + x_2 + 3x_3$

subject to $2x_1 + x_2 \leq 3 \implies -2x_1 - x_2 \geq -3 \implies -2x_1 - x_2 + 0x_3 \geq -3$

$x_1 + 2x_2 + 6x_3 \geq 5 \implies x_1 + 2x_2 + 6x_3 \geq 5$

$$-x_1 + x_2 + 2x_3 = 2 \Bigg\} \Rightarrow \begin{matrix} -x_1 + x_2 + 2x_3 \leq 2 \\ -x_1 + x_2 + 2x_3 \geq 2 \end{matrix} \Bigg\} \Rightarrow \begin{matrix} -x_1 - x_2 - 2x_3 \geq -2 \\ -x_1 + x_2 + 2x_3 \geq 2 \end{matrix}$$

$\therefore$ its dual, $z_{max} = -3y_1 + 5y_2 - 2y_3 + 2y_4$

subject to, $-2y_1 + y_2 + y_3 - y_4 \leq 0$

$-y_1 + 2y_2 - y_3 + y_4 \leq 1$

$0y_1 + 6y_2 - 2y_3 + 2y_4 \leq 3$

$y_1, y_2, y_3, y_4 \geq 0$

Fundamental Duality Theorem :→

[1] If the primal or dual has finite optimum solution then then other problem have finite and optimum solution.

[2] If either problem has unbounded solution, then other problem has no feasible solution at all.

[3] Both problem may be infeasible (i.e. may not have any solution)

Ex. Apply the principle of duality to solve the LPP

$$\text{Max. } Z = 3x_1 + 2x_2$$

Subject to,
$$x_1 + x_2 \geq 1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \leq 10$$
$$x_2 \leq 3, \quad x_1, x_2 \geq 0$$

7 marks — 2007

Hence find solution of primal.

Solution:-

given LPP

$$Z_{max} = 3x_1 + 2x_2$$

s.t.
$$-x_1 - x_2 \leq -1$$
$$x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \leq 10$$
$$0x_1 + x_2 \leq 3$$

It's dual LPP

$$Z_{min} = -y_1 + 7y_2 + 10y_3 + 3y_4$$

s.t.
$$-y_1 + y_2 + y_3 + 0y_4 \geq 3$$
$$-y_1 + y_2 + 2y_3 + y_4 \geq 2$$

to solve the dual :- standard form of the dual is

$$Z_{max} = y_1 - 7y_2 - 10y_3 - 3y_4 + 0y_5 + 0y_6 - Ma_1 - Ma_2$$

s.t.
$$-y_1 + y_2 + y_3 + 0y_4 - y_5 + a_1 = 3$$
$$-y_1 + y_2 + 2y_3 + y_4 - y_6 + a_2 = 2$$

y_5, y_6 are surplus, a_1, a_2 artificial variables.

I.B.F.S. :- Let $y_1 = 0, y_2 = 0, y_3 = 0, y_4 = 0, y_5 = 0, y_6 = 0 \Rightarrow a_1 = 3, a_2 = 2$

		c_j	1	-7	-10	-3	0	0	-M	-M	
B	C_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6	a_1	a_2	min. R.T. X_B/y_3
a_1	-M	3	-1	1	1	0	-1	0	1	0	3
a_2	-M	2	-1	1	2	1	0	-1	0	1	1 → out
		$\Delta_j = c_j - C_B X_j$	1-2M	-7+2M	-10+3M ↑	-3+M	-M	-M	0	0	X_B/y_2
a_1	-M	2	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$\frac{1}{2}$	1	$-\frac{1}{2}$	4
y_3	-10	1	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	2 → out
		$\Delta_j = c_j - C_B X_j$	$-\frac{M}{2}-4$	$\frac{M}{2}-2$ ↑	0	$-\frac{M}{2}+2$	-M	$\frac{M}{2}-5$	0	$-\frac{3M}{2}+5$	X_B/y_6
a_1	-M	1	0	0	-1	-1	-1	1	1	-1	1 → out
y_2	-7	2	-1	1	2	1	0	-1	0	1	–
		$\Delta_j = c_j - C_B X_j$	-6	0	-M+4	-M+4	-M	M-7 ↑	0	-2M+7	
y_6	0	1	0	0	-1	-1	-1	1	1	-1	
y_2	-7	3	-1	1	1	0	-1	0	1	0	
		$\Delta_j = c_j - C_B X_j$	-6	0	-3	-3	-7	0	-M+7	-M	

All $\Delta_j \leq 0 \Rightarrow$ Solution is optimal, $y_1 = 0, y_2 = 3 \Rightarrow Z_{max} = -21 \Rightarrow Z_{min} = 21$

to find solⁿ of original LPP i.e. primal :- surplus variables are y_5 and y_6 which gives idea about solution of primal. corresponding Δ's are $\Delta_5 = -7$, $\Delta_6 = 0$

$\therefore$ values of x_1 and x_2 are $x_1 = -\Delta_5 = 7$, $x_2 = -\Delta_6 = 0$ and $Z_{max} = 3(7) + 0 = 21$

EX Using principle of duality solve,

$$\text{Min } Z = 4x_1 + 14x_2 + 3x_3$$

subject to, $-x_1 + 3x_2 + x_3 \geq -3$

$2x_1 + 2x_2 - x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

Hence find solution of primal.

$\boxed{x_1 - 3x_2 - x_3 \leq -3 \quad C\text{-}0^M_g}$

(8 marks)

-{2006

Solution:- Dual of the problem,

$$Z_{max} = 3y_1 + 2y_2$$

subject to, $-y_1 + 2y_2 \leq 4$

$3y_1 + 2y_2 \leq 14$

$y_1 - y_2 \leq 3$, $\quad y_1 \geq 0,\ y_2 \geq 0$

standard form of above LPP is

$$Z_{max} = 3y_1 + 2y_2 + 0y_3 + 0y_4 + 0y_5$$

subject to $-y_1 + 2y_2 + y_3 = 4$

$3y_1 + 2y_2 + y_4 = 14$

$y_1 - y_2 + y_5 = 3$

y_3, y_4, y_5 are slack variable

I.B.F.S. :- $y_1 = 0,\ y_2 = 0 \Rightarrow y_3 = 4,\ y_4 = 14,\ y_5 = 3$

		C_j	3	2	0	0	0	
B	C_B	X_B	y_1	y_2	y_3	y_4	y_5	Min. R.T. X_B/y_1
y_3	0	4	-1	2	1	0	0	—
y_4	0	14	3	2	0	1	0	$\frac{14}{3}$
y_5	0	3	1	-1	0	0	1	3 → out
		$\Delta_j = C_j - C_B Y_i$	3 ↑	2	0	0	0	X_B/y_2
y_3	0	7	0	1	1	0	1	7
y_4	0	5	0	5	0	1	-3	1 → out
y_1	3	3	1	-1	0	0	1	—
		$\Delta_j = C_j - C_B Y_i$	0	5	0	0	-3	
y_3	0	6	0	0	1	$-\frac{1}{5}$	$\frac{8}{5}$	
y_2	2	1	0	1	0	$\frac{1}{5}$	$-\frac{3}{5}$	
y_1	3	4	1	0	0	$\frac{1}{5}$	$\frac{2}{5}$	
		$\Delta_j = C_j - C_B Y_i$	0	0	0	-1	0	

All $\Delta_j \leq 0 \Rightarrow$ solution is optimal. $y_1 = 4,\ y_2 = 1$, $\quad Z_{max} = 3(4) + 2(1) = 14$

to find solution of primal:-

slack variables y_3, y_4, y_5 gives idea about solution of primal.

corresponding Δ's are $\Delta_3 = 0,\ \Delta_4 = -1,\ \Delta_5 = 0$

$\therefore$ values of x_1, x_2, x_3 are 0, 1, 0 resp.

$\therefore\ x_1 = 0,\ x_2 = 1,\ x_3 = 0$

$$Z_{min} = 4(0) + 14(1) + 3(0) = 14$$

Ex Solve the following LPP by using its dual and hence find solution of primal.

$\max. Z = 2x_1 + x_2$

Subject to, $x_1 + 5x_2 \leq 10$

$x_1 + 3x_2 \leq 6$

$2x_1 + 2x_2 \leq 8$, $x_1, x_2 \geq 0$ — {2003,

Solution:- Dual of given LPP,

$Z_{min} = 10y_1 + 6y_2 + 8y_3$

subject to, $y_1 + y_2 + 2y_3 \geq 2$

$5y_1 + 3y_2 + 2y_3 \geq 1$

Standard form of dual

$Z_{max} = -10y_1 - 6y_2 - 8y_3 + 0y_4 + 0y_5 - Ma_1 - Ma_2$

s.t. $y_1 + y_2 + 2y_3 - y_4 + a_1 = 2$

$5y_1 + 3y_2 + 2y_3 - y_5 + a_2 = 1$

y_4, y_5 are surplus and a_1, a_2 artificial

I.B.F.S.:- $y_1 = 0, y_2 = 0, y_3 = 0, y_4 = 0, y_5 = 0 \Rightarrow a_1 = 2, a_2 = 1$

		c_j	-10	-6	-8	0	0	$-M$	$-M$	
B	C_B	X_B	y_1	y_2	y_3	y_4	y_5	a_1	a_2	Min.R.T. X_B/y_1
a_1	$-M$	2	1	1	2	-1	0	1	0	2
a_2	$-M$	1	5	3	2	0	-1	0	1	$\frac{1}{5}$ → out
		$\Delta_j = c_j - C_B Y_j$	$-10+6M$ ↑	$-6+4M$	$-8+4M$	$-M$	$-M$	0	0	X_B/y_3
a_1	$-M$	$\frac{9}{5}$	0	$\frac{2}{5}$	$\frac{8}{5}$	-1	$\frac{1}{5}$	1	$-\frac{1}{5}$	$9/8$
y_1	-10	$\frac{1}{5}$	1	$\frac{3}{5}$	$\frac{2}{5}$	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{1}{2}$ → out
		$\Delta_j = c_j - C_B Y_j$	0	$\frac{2M}{5}$	$\frac{8M}{5} - 4$ ↑	$-M$	$\frac{M}{5} - 2$	0	$-\frac{6M}{5} + 2$	X_B/y_5
a_1	$-M$	1	-4	-2	0	-1	1	1	-1	1 → out
y_3	-8	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{3}{2}$	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	—
		$\Delta_j = c_j - C_B Y_j$	$-4M + 10$	$-2M + 6$	0	$-M$	$M - 4$ ↑	0	$-2M + 4$	
y_5	0	1	-4	-2	0	-1	1	1	-1	
y_3	-8	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	
		$\Delta_j = c_j - C_B Y_j$	-6	-2	0	-4	0	$-M+4$	$-M$	

All $\Delta_j \leq 0 \Rightarrow$ Solution is optimal $y_1 = 0, y_2 = 0, y_3 = 1, Z_{max} = -8$

Solution of primal:- y_4, y_5 surplus variables $\Delta_4 = -4, \Delta_5 = 0$

$\Rightarrow x_1 = -\Delta_4 = 4, \quad x_2 = -\Delta_5 = 0$

$\therefore Z_{max} = 2(4) + 0 = 8$

Ex When the primal LPP is infeasible its dual is unbounded. Verify if this is true for the following problem using simplex method.

maximize $Z = 8x_1 + 6x_2$

subject to, $x_1 - x_2 \leq 0.6$

$x_1 - x_2 \geq 2$

[illegible] $x_2 \geq 0$ — {2004

Solution: to solve the primal

std. Form of LPP is $z_{max} = 8x_1 + 6x_2 + 0x_3 + 0x_4 - Ma_1$

Subject to, $x_1 - x_2 + x_3 = \frac{3}{5}$

$x_1 - x_2 - x_4 + a_1 = 2$

I.B.F.S.:- Let $x_1 = 0, x_2 = 0, x_4 = 0$

$\Rightarrow x_3 = \frac{3}{5}, \; a_1 = 2$

For I.B.F.S. set all variables equal to zero except slack and artificial

		c_j	8	6	0	0	$-M$	
B	C_B	X_B	x_1	x_2	x_3	x_4	a_1	Min.R.T. X_B/x_1
x_3	0	$3/5$	1	-1	1	0	0	$3/5$ → outgoing
a_1	$-M$	2	1	-1	0	-1	1	2
		$\Delta_j = c_j - C_B X_j$	$8+M$ ↑	$6-M$	0	$-M$	0	
x_1	8	$\frac{3}{5}$	1	-1	1	0	0	
a_1	$-M$	$\frac{7}{5}$	0	0	-1	-1	1	
		$\Delta_j = c_j - C_B X_j$	0	14 ↑	$-8-M$	$-M$	0	

All entries in the x_2 column are either zero or -ve

$\Longrightarrow x_2$ cannot enter in the basis

$\Longrightarrow$ artificial variable a_1 remains in the basis

$\Longrightarrow$ LPP has infeasible solution.

to solve the dual LPP :-

given problem can be written as

$z_{max} = 8x_1 + 6x_2$

subject to ; $x_1 - x_2 \le \frac{3}{5}$

$-x_1 + x_2 \le -2$

$\Rightarrow$ its dual $z_{min} = \frac{3}{5}y_1 - 2y_2$

subject to, $y_1 - y_2 \ge 8$

$-y_1 + y_2 \ge 6, \quad y_1, y_2 \ge 0$

std. Form of dual $z_{max} = -\frac{3}{5}y_1 + 2y_2 + 0y_3 + 0y_4 - Ma_1 - Ma_2$

subject to, $y_1 - y_2 - y_3 + a_1 = 8$

$-y_1 + y_2 - y_4 + a_2 = 6$

I.B.F.S.:- $y_1 = 0, y_2 = 0, y_3 = 0, y_4 = 0 \Rightarrow a_1 = 8, a_2 = 6$

		c_j	$-3/5$	2	0	0	$-M$	$-M$	
B	C_B	X_B	y_1	y_2	y_3	y_4	a_1	a_2	Min.R.T. X_B/y_2
a_1	$-M$	8	1	-1	-1	0	1	0	–
a_2	$-M$	6	-1	1	0	-1	0	1	6 → outgoing
		$\Delta_j = c_j - C_B Y_j$	$-3/5$	2 ↑	$-M$	$-M$	0	0	
a_1	$-M$	14	0	0	-1	-1	1	1	
y_2	2	6	-1	1	0	-1	0	1	
		$\Delta_j = c_j - C_B Y_j$	$7/5$ ↑	0	$-M$	$-M+2$	0	-2	

y_1 is incomming with corresponding entries zero or -ve

$\Rightarrow a_1$ remains in the basis

$\Rightarrow$ LPP is infeasible.

thus primal is infeasible then its dual also infeasible

$\therefore$ given statement

EX Apply the principle of duality to solve,

Maximize $z = 3x_1 + 4x_2$

subject to $x_1 - x_2 \le 1$

$x_1 + x_2 \ge 4$

$x_1 - 3x_2 \le 3$, $x_1, x_2 \ge 0$ { c-09, 10 D M }

Solution:- given LPP can be written as,

$$Z_{max} = 3x_1 + 4x_2$$
s.t. $x_1 - x_2 \le 1$
$-x_1 - x_2 \le -4$
$x_1 - 3x_2 \le 3$

$\Longrightarrow$ its dual,

$$Z_{min} = y_1 - 4y_2 + 3y_3$$
s.t. $y_1 - y_2 + y_3 \ge 3$
$-y_1 - y_2 - 3y_3 \ge 4$

std. form, $Z_{max} = -y_1 + 4y_2 - 3y_3 + 0y_4 + 0y_5 - Ma_1 - Ma_2$

s.t. $y_1 - y_2 + y_3 - y_4 + a_1 = 3$
$-y_1 - y_2 - 3y_3 - y_5 + a_2 = 4$

I.B.F.S.:- $y_1 = 0, y_2 = 0, y_3 = 0, y_4 = 0, y_5 = 0$

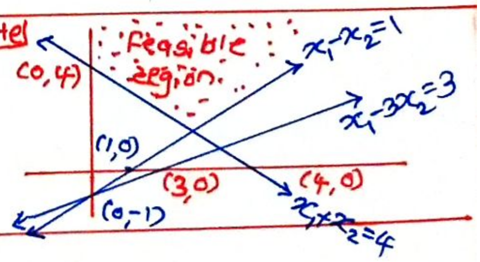
$\Rightarrow a_1 = 3, a_2 = 4$

B	C_B	C_j / X_B	y_1	y_2	y_3	y_4	y_5	a_1	a_2	Min. R.T $X_B/$
		C_j	-1	4	-3	0	0	$-M$	$-M$	
a_1	$-M$	3	1	-1	1	-1	0	1	0	
a_2	$-M$	4	-1	-1	-3	0	-1	0	1	
		$\Delta_j = C_j - C_B X_j$	-1	$4-2M$	$-3-2M$	$-M$	$-M$	0	0	

All $\Delta_j \le 0$ solution is optimal. But a_1, a_2 present in the basis $\Rightarrow$ no feasible solution.

$\Rightarrow$ original LPP has unbounded solution.

Note



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Dual Simplex Method

(Note) This method is opposite to that of simplex method.
It starts from optimality with infeasibility and works towards feasibility.

Steps involved in Dual simplex method.

1. Problem must be of maximization type and all constraints must be of $\le$ type.
2. Add slack variables (this method is free from surplus and artificial variables) to convert constraints into equalities. And obtain initial basic solution.
3. Compute $\Delta_j = c_j - c_B x_j$
   (i) If all $c_j - c_B x_j \le 0$ i.e. $\Delta_j \le 0$ and values of basic variables positive then solution is optimal basic feasible.
   (ii) if all $\Delta_j \le 0$ and at least one value of basic variable is negative then proceed to step 4
   (iii) if any $\Delta_j \ge 0$ then method fails.
4. Select the row containing most negative value of X_B, which is outgoing variable.
5. look at the elements of above row
   (i) if all entries are positive, then LPP does not have a feasible solut^n.
   (ii) If at least one element is negative, then find ratios of Δ_j with these elements. Ignore the ratios corresponding to positive or zero elements of above row. Variable corresponding to smallest ratio is incomming variable. Mark the key element.
6. Make this key element unity and Follow the same procedure.

EX Solve by dual simplex method

minimize $Z = 2x_1 + 2x_2 + 4x_3$

subject to; $2x_1 + 3x_2 + 5x_3 \ge 2$

$3x_1 + x_2 + 7x_3 \le 3$

$x_1 + 4x_2 + 6x_3 \le 5$

$x_1, x_2, x_3 \ge 0$ → {2008 Jan. 10, 15

2001, 03, 06, 09

Solution: → given LPP can be written in std. Form as

$Z_{max} = -2x_1 - 2x_2 - 4x_3 + 0x_4 + 0x_5 + 0x_6$

subject to $-2x_1 - 3x_2 - 5x_3 + x_4 = -2$

$3x_1 + x_2 + 7x_3 + x_5 = 3$

$x_1 + 4x_2 + 6x_3 + x_6 = 5$

I.B.F.S.:- Let $x_1 = 0$, $x_2 = 0$ [illegible] $x_4 = -2$

Let $x_1 = 0$, $x_2 = 0$, $x_3 = 0$
$\Rightarrow x_4 = -2$, $x_5 = 3$, $x_6 = 5$

		C_j	-2	-2	-4	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	-2	-2	-3	-5	1	0	0	→ outgoing
x_5	0	3	3	1	7	0	1	0	
x_6	0	5	1	4	6	0	0	1	
		$\Delta_j = C_j - C_B X_j$	-2	-2	-4	0	0	0	
	Min. R.T. →	$\frac{\Delta_j}{x_4}$	1	$\frac{2}{3}$ ↑	$\frac{4}{5}$	—	—	—	
x_2	-2	$\frac{2}{3}$	$\frac{2}{3}$	1	$\frac{5}{3}$	$-\frac{1}{3}$	0	0	
x_5	0	$\frac{7}{3}$	$\frac{7}{3}$	0	$\frac{16}{3}$	$\frac{1}{3}$	1	0	
x_6	0	$\frac{7}{3}$	$-\frac{5}{3}$	0	$-\frac{2}{3}$	$\frac{4}{3}$	0	1	
		$\Delta_j = C_j - C_B X_j$	$-\frac{2}{3}$	0	$-\frac{2}{3}$	$-\frac{2}{3}$	0	0	
	M.R.T. →	$\frac{\Delta_j}{(\ )}$							

All $\Delta_j \le 0$ and all values of basic variables are +ve

$\Rightarrow$ solution is optimal and feasible.

$\therefore x_1 = 0, \; x_2 = \frac{2}{3} \quad \Rightarrow Z_{max} = 0 - \frac{4}{3} = -\frac{4}{3} \quad \Rightarrow Z_{min} = \frac{4}{3}$

Ex When is dual simplex method applicable? Use dual simplex method to solve the following LPP.

Minimize $Z = 6x_1 + 7x_2 + 3x_3 + 5x_4$

subject to $5x_1 + 6x_2 - 3x_3 + 4x_4 \geq 12$

$x_2 + 5x_3 - 6x_4 \geq 10$

$2x_1 + 5x_2 + x_3 + x_4 \geq 8$ (8 marks)

$x_1, x_2, x_3, x_4 \geq 0$ — S 2007

Solution:- Dual Simplex method is applicable if all coefficients in objective function (max. type) are negative.

given LPP can be written as

$Z_{max} = -6x_1 - 7x_2 - 3x_3 - 5x_4$

subject to $-5x_1 - 6x_2 + 3x_3 - 4x_4 \leq -12$

$-x_2 - 5x_3 + 6x_4 \leq -10$

$-2x_1 - 5x_2 - x_3 - x_4 \leq -8$

std. form of above LPP is

$Z_{max} = -6x_1 - 7x_2 - 3x_3 - 5x_4 + 0x_5 + 0x_6 + 0x_7$

subject to

$-5x_1 - 6x_2 + 3x_3 - 4x_4 + x_5 = -12$

$-x_2 - 5x_3 + 6x_4 + x_6 = -10$

$-2x_1 - 5x_2 - x_3 - x_4 + x_7 = -8$

I.B.F.S. :- $x_1 = 0, x_2 = 0, x_3 = 0$ ... $= -12, \; x_6 = -10 \quad x_7 = -8$

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		c_j	-6	-7	-3	-5	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	0	-12	-5	-6	3	-4	1	0	0	$\rightarrow$ outgoing
x_6	0	-10	0	-1	-5	6	0	1	0	
x_7	0	-8	-2	-5	-1	-1	0	0	1	
		$\Delta_j = c_j - C_B X_j$	-6	-7	-3	-5	0	0	0	
		Min R.T. Δ_j / x_5	$\frac{6}{5}=1.2$	$\frac{7}{6}=1.16 \uparrow$	$-$	$\frac{5}{4}=1.25$	$-$	$-$	$-$	
x_2	-7	2	$\frac{6}{5}$	1	$-\frac{1}{2}$	$\frac{2}{3}$	$-\frac{1}{6}$	0	0	
x_6	0	-8	$\frac{5}{6}$	0	$\frac{-11}{2}$	$\frac{20}{3}$	$-\frac{1}{6}$	1	0	$\rightarrow$ outgoing
x_7	0	2	$\frac{13}{6}$	0	$\frac{-7}{2}$	$\frac{7}{3}$	$\frac{-5}{6}$	0	1	
		$\Delta_j = c_j - C_B X_j$	$-\frac{1}{6}$	0	$-\frac{13}{2}$	$-\frac{1}{3}$	$-\frac{7}{6}$	0	0	
		Min. R.T. Δ_j / x_6	$-$	$-$	$\frac{13}{11} \uparrow$	$-$	7	$-$	$-$	
x_2	-7	$\frac{30}{11}$	$\frac{25}{33}$	1	0	$\frac{2}{33}$	$\frac{-5}{33}$	$-\frac{1}{11}$	0	
x_3	-3	$\frac{16}{11}$	$\frac{-5}{33}$	0	1	$\frac{-40}{33}$	$\frac{1}{33}$	$\frac{-2}{11}$	0	
x_7	0	$\frac{78}{11}$	$\frac{18}{11}$	0	0	$\frac{-21}{11}$	$\frac{-8}{11}$	$\frac{-7}{11}$	1	
		$\Delta_j = c_j - C_B X_j$	$-\frac{38}{33}$	0	0	$-\frac{27}{33}$	$-\frac{32}{33}$	$-\frac{13}{11}$	0	
		Min. R.T. $\Delta_j / (\ )$								

All $\Delta_j \le 0$ and values of basic variables are positive

$\Rightarrow$ solution is optimal and feasible.

$x_1 = 0, \quad x_2 = \frac{30}{11}, \quad x_3 = \frac{16}{11}$ $\quad \Rightarrow Z_{max} = \frac{-210}{11} - \frac{48}{11} = \frac{-258}{11}$

$\Rightarrow Z_{min} = \frac{258}{11}$

$\boxed{\text{EX}}$ use Dual simplex method to solve,

$Z_{min} = 6x_1 + 3x_2 + 4x_3$

Subject to $\quad x_1 + 6x_2 + x_3 = 10$

$2x_1 + 3x_2 + x_3 = 15, \quad x_1, x_2, x_3 \ge 0$ [2008, 14, 18]

Solution:- given constraints

$$x_1 + 6x_2 + x_3 = 10 \Rightarrow \left.\begin{matrix} x_1 + 6x_2 + x_3 \le 10 \\ x_1 + 6x_2 + x_3 \ge 10 \end{matrix}\right] \Rightarrow \begin{matrix} x_1 + 6x_2 + x_3 \le 10 \\ -x_1 - 6x_2 - x_3 \le -10 \end{matrix}$$

$$2x_1 + 3x_2 + x_3 = 15 \Rightarrow \left.\begin{matrix} 2x_1 + 3x_2 + x_3 \le 15 \\ 2x_1 + 3x_2 + x_3 \ge 15 \end{matrix}\right] \Rightarrow \begin{matrix} 2x_1 + 3x_2 + x_3 \le 15 \\ -2x_1 - 3x_2 - x_3 \le -15 \end{matrix}$$

given LPP can be written in std. Form as,

$Z_{max} = -6x_1 - 3x_2 - 4x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7$

s.t. $\quad x_1 + 6x_2 + x_3 + x_4 = 10$

$-x_1 - 6x_2 - x_3 + x_5 = -10$

$2x_1 + 3x_2 + x_3 + x_6 = 15 \quad ; \quad -2x_1 - 3x_2 - x_3 + x_7 = -15$

I.B.F.S.:-

$x_1 = 0, x_2 = 0, x_3 = 0$

$\Rightarrow x_4 = 10$

$x_5 = -10$

$x_6 = 15$

$x_7 = -15$

B	C_B	c_j / X_B	-6 / x_1	-3 / x_2	-4 / x_3	0 / x_4	0 / x_5	0 / x_6	0 / x_7	
x_4	0	10	1	6	1	1	0	0	0	
x_5	0	-10	-1	-6	-1	0	1	0	0	
x_6	0	15	2	3	1	0	0	1	0	
x_7	0	-15	-2	-3	-1	0	0	0	1	→ outgoing
		$\Delta_j = c_j - C_B X_j$	-6	-3	-4	0	0	0	0	
		Min. R.T. Δ_j / x_7	3	1 ↑	4	-	-	-	-	
x_4	0	-20	-3	0	-1	1	0	0	2	→ outgoing
x_5	0	20	3	0	1	0	1	0	-2	
x_6	0	0	0	0	0	0	0	1	1	
x_2	-3	5	$\frac{2}{3}$	1	$\frac{1}{3}$	0	0	0	$-\frac{1}{3}$	
		$\Delta_j = c_j - C_B X_j$	-4	0	-3	0	0	0	-1	
		Min. R.T. Δ_j / x_4	$\frac{4}{3}$ ↑	-	3	-	-	-	-	
x_1	-6	$\frac{20}{3}$	1	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	$-\frac{2}{3}$	
x_5	0	0	0	0	0	1	1	0	0	
x_6	0	0	0	0	0	0	0	1	1	
x_2	-3	$\frac{5}{9}$	0	1	$\frac{1}{9}$	$\frac{2}{9}$	0	0	$\frac{1}{9}$	
		$\Delta_j = c_j - C_B X_j$	0	0	$-\frac{5}{3}$	$-\frac{4}{3}$	0	0	$-\frac{11}{3}$	
		Min. R.T. $\Delta_j / (\ )$								

All $\Delta_j \le 0$ and all basic variables are +ve

$\Rightarrow$ Solution is optimal and feasible. $\Rightarrow x_1 = \frac{20}{3},\ x_2 = \frac{5}{9},\ x_3 = 0$

$\therefore Z_{max} = \frac{-120}{3} - \frac{5}{3} = \frac{-125}{3} \Rightarrow Z_{min} = \frac{125}{3}$

EX Solve by using Dual simplex method.

$Z_{min} = x_1 + x_2$

subject to, $2x_1 + x_2 \ge 2$ { 2007, 2010, 2011, 14, D-18

$-x_1 - x_2 \ge 1$, $x_1, x_2 \ge 0$

Solution:-

given LPP

$Z_{max} = -x_1 - x_2$

s.t. $-2x_1 - x_2 \le -2$

$-x_1 + x_2 \le -1$

$\Rightarrow$ std. Form

$Z_{max} = -x_1 - x_2 + 0x_3 + 0x_4$

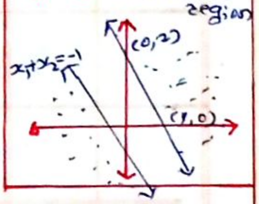
s.t. $-2x_1 - x_2 + x_3 = -2$

$x_1 + x_2 + x_4 = -1$

I.B.F.S.:- $x_1 = 0,\ x_2 = 0 \Rightarrow x_3 = -2,\ x_4 = -1$

		c_j	-1	-1	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	
x_3	0	-2	(-2)	-1	1	0	→ outgoing variable
x_4	0	-1	1	1	0	1	
		$\Delta_j = c_j - C_B X_j$	-1	-1	0	0	
		Min. R.T. Δ_j / x_3	$\frac{1}{2}$ ↑	1	—	—	
x_1	-1	1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	
x_4	0	-2	0	$\frac{1}{2}$	$\frac{1}{2}$	1	→ outgoing variable
		$\Delta_j = c_j - C_B X_j$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	
		min. R.T. Δ_j / x_4					

graphical method gives no feasible region



All elements of x_4 row are positive ⇒ no feasible solution.

EX Use the dual simplex method to solve the following LPP

Minimise $Z = 20x_1 + 16x_2$

subject to, $x_1 + x_2 \geq 12$

$2x_1 + x_2 \geq 17$

$x_1 \geq 2.5$

$x_2 \geq 6$, $x_1, x_2 \geq 0$ {C-09 M 2001}

Solution:-

given LPP can be written as

$Z_{max} = -20x_1 - 16x_2$

s.t. $-x_1 - x_2 \leq -12$

$-2x_1 - x_2 \leq -17$

$-x_1 \leq -5/2$

$-x_2 \leq -6$

⇒ std. Form

$Z_{max} = -20x_1 - 16x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$

subject to.

$-x_1 - x_2 + x_3 = -12$

$-2x_1 - x_2 + x_4 = -17$

$-x_1 + x_5 = -\frac{5}{2}$

$-x_2 + x_6 = -6$

I.B.F.S.:- Let $x_1 = 0$, $x_2 = 0$, ⇒ $x_3 = -12$, $x_4 = -17$, $x_5 = -\frac{5}{2}$, $x_6 = -6$

		c_j	-20	-16	0	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	0	-12	-1	-1	1	0	0	0	
x_4	0	-17	(-2)	-1	0	1	0	0	→ outgoing
x_5	0	$-\frac{5}{2}$	-1	0	0	0	1	0	
x_6	0	-6	0	-1	0	0	0	1	
		$\Delta_j = c_j - C_B X_j$	-20	-16	0	0	0	0	
		Min. R.T. Δ_j / x_4	10 ↑	16	—	—	—	—	

		c_j	-20	-16	0	0	0	0	
B	c_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	0	$-7/2$	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$	0	0	
x_1	-20	$\frac{17}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	
x_5	0	6	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	
x_6	0	-6	0	-1	0	0	0	1	$\rightarrow$ outgoing
		$\Delta_j = c_j - c_B x_j$	0	-6	0	-10	0	0	
		Min.R.T. Δ_j/x_6	–	6 $\uparrow$	–	–	–	–	
x_3	0	$-\frac{1}{2}$	0	0	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\rightarrow$ outgoing
x_1	-20	$\frac{11}{2}$	1	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	
x_5	0	3	0	0	0	$-\frac{1}{2}$	1	$\frac{1}{2}$	
x_2	-16	6	0	1	0	0	0	-1	
		$\Delta_j = c_j - c_B x_j$	0	0	0	-10	0	-6	
		Min.R.T. Δ_j/x_3	–	–	–	20	–	12 $\uparrow$	
x_6	0	1	0	0	-2	1	0	1	
x_1	-20	5	1	0	1	-1	0	0	
x_5	0	$\frac{5}{2}$	0	0	1	-1	1	0	
x_2	-16	7	0	1	-2	1	0	0	
		$\Delta_j = c_j - c_B x_j$	0	0	-12	-4	0	0	
		Min.R.T. $\Delta_j/(\ )$							

All $\Delta_j \le 0$ and values of all basic variables are positive

$\Rightarrow$ solution is optimal and feasible.

$x_1 = 5,\ x_2 = 7$ and $z_{max} = -100 - 112 \Rightarrow z_{max} = -212$

$\therefore z_{min} = 212$

EX Solve by using Dual simplex method $\text{Min } z = 2x_1 + x_2$

subject to $3x_1 + x_2 \ge 3$

$4x_1 + 3x_2 \ge 6$

$x_1 + 2x_2 \ge 3 \quad , x_1, x_2 \ge 0$

$\rightarrow$ § C-10 M. 18 $\rightarrow$ 3rd constraint $x_1 + 2x_2 \le 3$

19 same answer

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Solution:-

$$Z_{max} = -2x_1 - x_2$$

s.t. $-3x_1 - x_2 \le -3$

$-4x_1 - 3x_2 \le -6$

$-x_1 - 2x_2 \le -3$

$\Rightarrow$

$$Z_{max} = -2x_1 - x_2 + 0x_3 + 0x_4 + 0x_5$$

s.t. $-3x_1 - x_2 + x_3 = -3$

$-4x_1 - 3x_2 + x_4 = -6$

$-x_1 - 2x_2 + x_5 = -3$

I.B.F.S.:- $x_1 = 0,\ x_2 = 0$

$\Rightarrow x_3 = -3,\ x_4 = -6,\ x_5 = -3$

		c_j	-2	-1	0	0	0	
B	C_B	X_B	x_1	x_2	x_3	x_4	x_5	
x_3	0	-3	-3	-1	1	0	0	
x_4	0	-6	-4	-3	0	1	0	outgoing
x_5	0	-3	-1	-2	0	0	1	
		$\Delta_j = c_j - C_B X_j$	-2	-1	0	0	0	
		$\frac{\Delta_j}{\text{-ve elements of } x_4 \text{ row}}$	$\frac{1}{2}$	$\frac{1}{3}$ in	$-$	$-$	$-$	
x_3	0	-1	$-\frac{5}{3}$	0	1	$-\frac{1}{3}$	0	outgoing
x_2	-1	2	$\frac{4}{3}$	1	0	$-\frac{1}{3}$	0	
x_5	0	1	$-\frac{5}{3}$	0	0	$-\frac{2}{3}$	1	
		$\Delta_j = c_j - C_B X_j$	$-\frac{2}{3}$	0	0	$-\frac{1}{3}$	0	
		$\frac{\Delta_j}{\text{-ve elements of } x_3 \text{ row}}$	$\frac{2}{5}$ in	$-$	$-$	1	$-$	
x_1	-2	$\frac{3}{5}$	1	0	$-\frac{3}{5}$	$\frac{1}{5}$	0	
x_2	-1	$\frac{6}{5}$	0	1	$\frac{4}{5}$	$-\frac{3}{5}$	0	
x_5	0	0	0	0	-1	$-\frac{1}{3}$	1	
		$\Delta_j = c_j - C_B X_j$						

All $\Delta_j \le 0$ and values of basic variables are positive

$\Rightarrow$ solution is optimal and feasible

$\therefore x_1 = \frac{3}{5},\ x_2 = \frac{6}{5},\ Z_{max} = \frac{-6}{5} - \frac{6}{5} = \frac{-12}{5} \Rightarrow Z_{min} = \frac{12}{5}$

Ex Use dual simplex method to solve the LPP:

$Z_{min} = 6x_1 + x_2$, s.t. $2x_1 + x_2 \geq 3$, $x_1 - x_2 \geq 0$, $x_1, x_2 \geq 0$

[MU: 1997, 2008, 2016]

Solution:- given LPP can be written as

$Z_{max} = -6x_1 - x_2$

subject to $-2x_1 - x_2 \leq -3$, $-x_1 + x_2 \leq 0$

using slack variables it can be written as

$Z_{max} = -6x_1 - x_2 + 0x_3 + 0x_4$

s.t. $-2x_1 - x_2 + x_3 = -3$, $-x_1 + x_2 + x_4 = 0$

Initial solution:- Let $x_1 = 0$, $x_2 = 0 \Rightarrow x_3 = -3$, $x_4 = 0$

		c_j	-6	-1	0	0	
B	C_B	V_B	x_1	x_2	x_3	x_4	
x_3	0	-3	-2	-1	1	0	→ outgoing
x_4	0	0	-1	1	0	1	
		Δ_j	-6	-1	0	0	
		$\frac{\Delta_j}{x_3}$	3	$1 \uparrow$	$-$	$-$	
x_2	-1	3	2	1	-1	0	
x_4	0	-3	-3	0	1	1	→ outgoing
		Δ_j	-4	0	-1	0	
		Δ_j / x_4	$\frac{4}{3} \uparrow$	$-$	$-$	$-$	
x_2	-1	1	0	1	$-1/3$	$2/3$	
x_1	-6	1	1	0	$-\frac{1}{3}$	$-\frac{1}{3}$	
		Δ_j	0	0	$-7/3$	$-4/3$	

All $\Delta_j \leq 0$ and values are +ve $\Rightarrow$ soln is optimal and feasible

$\therefore$ $x_1 = 1$, $x_2 = 1$

$Z_{max} = -6 - 1 = -7$

$\Rightarrow Z_{min} = 7$

Non-Linear Programming

Lagrange's Method [equality constraint]

Type-I → one constraint

optimize $z = f(x_1, x_2)$ subject to $g(x_1, x_2) = b$, $x_1, x_2 \geq 0$

Let $h = g(x_1, x_2) - b$

and $L = f + \lambda h$

solve $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial \lambda} = 0$ to get values of x_1, x_2

To determine whether the solution yields maximum or minimum we find principle minors of the following ~~determinant~~ which is called as Hessian Matrix

$$H = \begin{bmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix}$$

* If there are n variables and 'm' constraints then we start with minor of order $(2m+1)$. Maximum or minimum is determined by the signs of last $(n-m)$ minors. If signs are alternate then the function is maximum and if all signs are same, the function is minimum

Type-II → two constraints

optimize $z = f(x_1, x_2, x_3)$ subject to $g_1(x_1, x_2, x_3) = b_1$, $g_2(x_1, x_2, x_3) = b_2$

Let $h_1 = g_1 - b_1$ and $h_2 = g_2 - b_2$

and $L = f + \lambda_1 h_1 + \lambda_2 h_2$

solve $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial x_3} = 0$, $\frac{\partial L}{\partial \lambda_1} = 0$, $\frac{\partial L}{\partial \lambda_2} = 0$ to get positive values of x_1, x_2, x_3

To determine whether the solution yields maximum or minimum we find principle minors of the following determinant.

$$H = \begin{bmatrix} 0 & 0 & \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ 0 & 0 & \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \frac{\partial h_1}{\partial x_1} & \frac{\partial h_2}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial h_1}{\partial x_2} & \frac{\partial h_2}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial h_1}{\partial x_n} & \frac{\partial h_2}{\partial x_n} & \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix}$$

H is called Hessian matrix.

If there are n variables and m constraints then we start with minor of order $(2m+1)$. Maximum or minimum is determined by the signs of last $(n-m)$ principle minors. If signs are alternate then the function is maximum and if all signs are same the function is minimum.

If $m=2$, $n=3$ $2m+1=5$ $n-m=1$

find Δ_5

if $\Delta_5 = +ve$ it is min

if $\Delta_5 = -ve$ max.

Ex Using method of Lagrange's multipliers solve N.L.P.P

optimize $z = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$

subject to ; $x_1 + x_2 + x_3 = 10$, $x_1, x_2, x_3 \geq 0$ — S 2008

Solution:- Let $f = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2$, $h = x_1 + x_2 + x_3 - 10$

consider $L = f + \lambda h = (12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2) + \lambda(x_1 + x_2 + x_3 - 10)$

$\therefore \frac{\partial L}{\partial x_1} = 0 \Rightarrow 12 - 2x_1 + \lambda = 0 \Rightarrow \lambda = 2x_1 - 12$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 8 - 2x_2 + \lambda = 0 \Rightarrow \lambda = 2x_2 - 8$

$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 6 - 2x_3 + \lambda = 0 \Rightarrow \lambda = 2x_3 - 6$

$\Rightarrow 2x_1 - 12 = 2x_2 - 8 \Rightarrow x_1 - x_2 = 2$ — ①

$\Rightarrow 2x_2 - 8 = 2x_3 - 6 \Rightarrow x_2 - x_3 = 1$ — ②

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x_1 + x_2 + x_3 - 10 = 0 \Rightarrow x_1 + x_2 + x_3 = 10$ — ③

using ①, ② in ③, ③ $\Rightarrow 2 + x_2 + x_2 + x_2 - 1 = 10 \Rightarrow 3x_2 = 9 \Rightarrow \boxed{x_2 = 3}$

$\therefore$ ① $\Rightarrow x_1 = 2 + x_2 \Rightarrow \boxed{x_1 = 5}$

② $\Rightarrow x_3 = x_2 - 1 \Rightarrow \boxed{x_3 = 2}$ and $\lambda = 10 - 12 \Rightarrow \boxed{\lambda = -2}$

consider the Hessian matrix

$$\begin{bmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}$$

Here $m = 1$
$\therefore 2m + 1 = 3$
and $n = 3$
$n - m = 2$
$\Rightarrow$ find Δ_3, Δ_4

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 0 - 1(-2) + 1(2) = 4, \quad \Delta_4 = |H| = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{vmatrix}$$

by $C_3 - C_2$, $C_4 - C_2$

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & 2 & 2 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 2 & 2 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = -1\{1(4) - 2(-2) + 2(2)\}$$

$\therefore \Delta_4 = -12$

as signs of Δ_3, Δ_4 are alternate, given function is maximum at

$x_1 = 5$, $x_2 = 3$, $x_3 = 2$

and $z_{max} = 12(5) + 8(3) + 6(2) - 25 - 9 - 4 - 23$

$\therefore z_{max} = 35$

EX Solve using Lagrange's multipliers

Optimize $z = 2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 196$

subject to $x_1 + x_2 + x_3 = 11$, $x_1, x_2, x_3 \geq 0$ —— § 2006

Solution:- Let $F = 2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 196$, $h = x_1 + x_2 + x_3 - 11$

consider, $L = f + \lambda h = (2x_1^2 + 2x_2^2 + 2x_3^2 - 24x_1 - 8x_2 - 12x_3 + 196) + \lambda(x_1 + x_2 + x_3 - 11)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 4x_1 - 24 + \lambda = 0 \Rightarrow \lambda = 24 - 4x_1$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 4x_2 - 8 + \lambda = 0 \Rightarrow \lambda = 8 - 4x_2$

$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 4x_3 - 12 + \lambda = 0 \Rightarrow \lambda = 12 - 4x_3$

$\Rightarrow 24 - 4x_1 = 8 - 4x_2 \Rightarrow x_1 = x_2 + 4$ — ①

$\Rightarrow 8 - 4x_2 = 12 - 4x_3 \Rightarrow x_3 = x_2 + 1$ — ②

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x_1 + x_2 + x_3 - 11 = 0 \Rightarrow x_1 + x_2 + x_3 = 11$ — ③

using ①, ② in eqⁿ ③, ③ $\Rightarrow x_2 + 4 + x_2 + x_2 + 1 = 11 \Rightarrow 3x_2 = 6 \Rightarrow \boxed{x_2 = 2}$

$\therefore$ ① $\Rightarrow \boxed{x_1 = 6}$, ② $\Rightarrow \boxed{x_3 = 3}$ and $\boxed{\lambda = 0}$

For maxima, minima :-

consider Hessian matrix,

$$\begin{bmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix}$$

$m = 1$
$\therefore 2m+1 = 3$
$n = 3$
$n - m = 2$
$\Rightarrow$ we have to find Δ_3, Δ_4

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix} = 0 - 1(4) + 1(-4) = -8 = -8$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix} \xrightarrow[C_4 - C_2]{C_3 - C_2} \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 4 & -4 & -4 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix} = (-1) \begin{vmatrix} 1 & -4 & -4 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix}$$

$$= -1\{1(16) + 4(4) - 4(-4)\}$$

$\therefore \Delta_4 = -48$

as signs of Δ_3 and Δ_4 are same in sign given function is minimum at $x_1 = 6$, $x_2 = 3$, $x_3 = 3$

and $z_{min} = 2(36) + 2(4) + 2(9) - 24(6) - 8(2) - 12(3) + 196$

$= 72 + 8 + 18 - 144 - 16 - 36 + 196$

$z_{min} = 98$

Ex Using method of Lagrange's multiplier solve the following NLPP.

optimize $z = x_1^2 + x_2^2 + x_3^2$

subject to $x_1 + x_2 + 3x_3 = 2$

$5x_1 + 2x_2 + x_3 = 5$ $\quad x_1, x_2, x_3 \geq 0$ — {2005, 2007

Solution:- Let $f = x_1^2 + x_2^2 + x_3^2$, $h_1 = x_1 + x_2 + 3x_3 - 2$, $h_2 = 5x_1 + 2x_2 + x_3 - 5$

consider Lagrangean $L = f - \lambda_1 h_1 - \lambda_2 h_2$

$$L = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$$

$\therefore \frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 - \lambda_1 - 5\lambda_2 = 0 \Rightarrow x_1 = \frac{\lambda_1 + 5\lambda_2}{2}$ — (1)

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 - \lambda_1 - 2\lambda_2 = 0 \Rightarrow x_2 = \frac{\lambda_1 + 2\lambda_2}{2}$ — (2)

$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2x_3 - 3\lambda_1 - \lambda_2 = 0 \Rightarrow x_3 = \frac{3\lambda_1 + \lambda_2}{2}$ — (3)

$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow -(x_1 + x_2 + 3x_3 - 2) = 0 \Rightarrow x_1 + x_2 + 3x_3 = 2$ — (4)

$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow -(5x_1 + 2x_2 + x_3 - 5) = 0 \Rightarrow 5x_1 + 2x_2 + x_3 = 5$ — (5)

using eq^n (1), (2), (3) in (4) and (5)

(4) $\Rightarrow \frac{\lambda_1 + 5\lambda_2}{2} + \frac{\lambda_1 + 2\lambda_2}{2} + \frac{3(3\lambda_1 + \lambda_2)}{3} = 2 \quad \Rightarrow 11\lambda_1 + 10\lambda_2 = 4$ — (6)

(5) $\Rightarrow 5\left(\frac{\lambda_1 + 5\lambda_2}{2}\right) + 2\left(\frac{\lambda_1 + 2\lambda_2}{2}\right) + \left(\frac{3\lambda_1 + \lambda_2}{2}\right) = 5 \quad \Rightarrow 10\lambda_1 + 30\lambda_2 = 10$ — (7)

3(6) − (7) $\Rightarrow 23\lambda_1 = 2 \Rightarrow \lambda_1 = \frac{2}{23}$

$\therefore \lambda_2 = \frac{7}{23}$

$\therefore$ (1) $\Rightarrow \boxed{x_1 = \frac{37}{46}}$, (2) $\Rightarrow \boxed{x_2 = \frac{16}{46}}$ (3) $\Rightarrow \boxed{x_3 = \frac{13}{46}}$

For maxima-minima:-

consider Hessian matrix

$$H = \begin{bmatrix} 0 & 0 & \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ 0 & 0 & \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \\ \frac{\partial h_1}{\partial x_1} & \frac{\partial h_2}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial h_1}{\partial x_2} & \frac{\partial h_2}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial h_1}{\partial x_3} & \frac{\partial h_2}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \\ 1 & 5 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

$m = 2$, $n = 3$ $\quad 2m+1 = 5$
we start with Δ_5 $\quad n - m = 1$ only one minor.

$$\Delta_5 = |H| = \begin{vmatrix} 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 5 & 2 & 1 \\ 1 & 5 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{vmatrix} \xrightarrow[C_5 - 3C_3]{C_4 - C_3} \begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & -3 & -14 \\ 1 & 5 & 2 & -2 & -6 \\ 1 & 2 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 2 \end{vmatrix} = (1) \begin{vmatrix} 0 & 0 & -3 & -14 \\ 1 & 5 & -2 & -6 \\ 1 & 2 & 2 & 0 \\ 3 & 1 & \text{[illegible]} & \text{[illegible]} \end{vmatrix}$$

R_3-R_2, R_4-3R_2

$$\begin{vmatrix} 0 & 0 & -3 & -14 \\ 1 & 5 & -2 & -6 \\ 0 & -3 & 4 & 6 \\ 0 & -14 & 6 & 20 \end{vmatrix} = (-1)\begin{vmatrix} 0 & -3 & -14 \\ -3 & 4 & 6 \\ -14 & 6 & 20 \end{vmatrix} = -\{0+3[-60+84]-14[-18+56]\}$$

$$= -\{72-532] = 460$$

$\Delta_5 = 460$ which is positive.

$\therefore$ F is minimum at $x_1 = \frac{37}{46}$, $x_2 = \frac{16}{46}$, $x_3 = \frac{13}{46}$

and $z_{min} = \left(\frac{37}{46}\right)^2 + \left(\frac{16}{46}\right)^2 + \left(\frac{13}{46}\right)^2 = 0.847$

[EX] Using the method of Lagrange's multipliers solve the following LPP.

optimize $z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

subject to, $x_1 + x_2 + x_3 = 15$

$2x_1 - x_2 + 2x_3 = 20$ — {2002, 2003, 2005}

Solution:- Let $F = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$, $h_1 = x_1 + x_2 + x_3 - 15$, $h_2 = 2x_1 - x_2 + 2x_3 - 20$

consider, $L = F + \lambda_1 h_1 + \lambda_2 h_2$

$$L = (4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2) + \lambda_1(x_1 + x_2 + x_3 - 15) + \lambda_2(2x_1 - x_2 + 2x_3 - 20)$$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 8x_1 - 4x_2 + \lambda_1 + 2\lambda_2 = 0 \Rightarrow \lambda_1 + 2\lambda_2 = -8x_1 + 4x_2$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 4x_2 - 4x_1 + \lambda_1 - \lambda_2 = 0$

$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2x_3 + \lambda_1 + 2\lambda_2 = 0 \Rightarrow \lambda_1 + 2\lambda_2 = -2x_3$

$-8x_1 + 4x_2 = -2x_3$

$-4x_1 + 2x_2 + x_3 = 0$ — (1)

$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x_1 + x_2 + x_3 - 15 = 0 \Rightarrow x_1 + x_2 + x_3 = 15$ — (2)

$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow 2x_1 - x_2 + 2x_3 - 20 = 0 \Rightarrow 2x_1 - x_2 + 2x_3 = 20$ — (3)

(1) $\Rightarrow x_3 = 4x_1 - 2x_2$ — (4)

$\therefore$ (2) $\Rightarrow x_1 + x_2 + 4x_1 - 2x_2 = 15$

$5x_1 - x_2 = 15$ — (5)

(3) $\Rightarrow 2x_1 - x_2 + 8x_1 - 4x_2 = 20$

$10x_1 - 5x_2 = 20$

$2x_1 - x_2 = 4$ — (6)

(5) − (6) $\Rightarrow 3x_1 = 11$

$\Rightarrow x_1 = \frac{11}{3}$

(6) $\Rightarrow x_2 = \frac{22}{3} - 4 = \frac{10}{3}$

$\therefore$ (4) $\Rightarrow x_3 = \frac{44}{3} - \frac{20}{3} = 8$

solving (1), (2), (3) $\boxed{x_1 = \frac{11}{3}}$, $\boxed{x_2 = \frac{10}{3}}$, $\boxed{x_3 = 8}$

for maxima/minima consider Hessian matrix.

$$\begin{bmatrix} 0 & 0 & \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ 0 & 0 & \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \\ \frac{\partial h_1}{\partial x_1} & \frac{\partial h_2}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial h_1}{\partial x_2} & \frac{\partial h_2}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial h_1}{\partial x_3} & \frac{\partial h_2}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{bmatrix}$$

$m = 2$, $n = 3$, $2m+1 = 5$ and $n - m = 3 - 2 = 1$

we start with minor of order 5 i.e. Δ_5

$$\xrightarrow[C_5-C_3]{C_4-C_3} \begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 1 & 2 & 8 & -12 & -8 \\ 1 & -1 & -4 & 8 & 4 \\ 1 & 2 & 0 & 0 & 2 \end{vmatrix} = (1) \begin{vmatrix} 0 & 0 & -3 & 0 \\ 1 & 2 & -12 & -8 \\ 1 & -1 & 8 & 4 \\ 1 & 2 & 0 & 2 \end{vmatrix} = -3 \begin{vmatrix} 1 & 2 & -8 \\ 1 & -1 & 4 \\ 1 & 2 & 2 \end{vmatrix}$$

$\therefore \Delta_5 = -3\{1(-10)-2(-2)-8(3)\} = 90$ which is positive

$\therefore$ F is minimum at $x_1 = \frac{11}{3}$, $x_2 = \frac{10}{3}$, $x_3 = 8$

and $F_{min} = 4\left(\frac{11}{3}\right)^2 + 2\left(\frac{10}{3}\right)^2 + (8)^2 - 4\left(\frac{11}{3}\right)\left(\frac{10}{3}\right) = \frac{484}{9} + \frac{200}{9} + 64 - \frac{440}{9}$

$F_{min} = \frac{820}{9}$

EX Using method of Lagrange's multipliers solve,

optimize $Z = 2x_1^2 + 3x_2^2 + x_3^2$

subject to $x_1 + x_2 + 2x_3 = 13$

$2x_1 + x_2 + x_3 = 10$, $x_1, x_2, x_3 \geq 0$ — {2008}

Solution:- Let $F = 2x_1^2 + 3x_2^2 + x_3^2$, $h_1 = x_1 + x_2 + 2x_3 - 13$, $h_2 = 2x_1 + x_2 + x_3 - 10$

consider, $L = F + \lambda_1 h_1 + \lambda_2 h_2$

$$L = 2x_1^2 + 3x_2^2 + x_3^2 + \lambda_1(x_1 + x_2 + 2x_3 - 13) + \lambda_2(2x_1 + x_2 + x_3 - 10)$$

$\therefore \frac{\partial L}{\partial x_1} = 0 \Rightarrow 4x_1 + \lambda_1 + 2\lambda_2 = 0 \Rightarrow x_1 = \frac{-\lambda_1 - 2\lambda_2}{4}$ — ①

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 6x_2 + \lambda_1 + \lambda_2 = 0 \Rightarrow x_2 = \frac{-\lambda_1 - \lambda_2}{6}$ — ②

$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2x_3 + 2\lambda_1 + \lambda_2 = 0 \Rightarrow x_3 = \frac{-2\lambda_1 - \lambda_2}{2}$ — ③

$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x_1 + x_2 + 2x_3 - 13 = 0 \Rightarrow x_1 + x_2 + 2x_3 = 13$ — ④

$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow 2x_1 + x_2 + x_3 - 10 = 0 \Rightarrow 2x_1 + x_2 + x_3 = 10$ — ⑤

using ①, ②, ③ in ④ and ⑤

④ $\Rightarrow \frac{(-\lambda_1 - 2\lambda_2)}{4} + \frac{(-\lambda_1 - \lambda_2)}{6} + 2\left(\frac{-2\lambda_1 - \lambda_2}{2}\right) = 13 \Rightarrow -29\lambda_1 - 20\lambda_2 = 156$ — ⑥

⑤ $\Rightarrow 2\frac{(-\lambda_1 - 2\lambda_2)}{4} + \frac{(-\lambda_1 - \lambda_2)}{6} + \frac{(-2\lambda_1 - \lambda_2)}{2} = 10 \Rightarrow -\lambda_1 - \lambda_2 = 6$ — ⑦

⑥ $- 20$⑦ $\Rightarrow -9\lambda_1 = 36 \Rightarrow \boxed{\lambda_1 = -4}$ ⑦ $\Rightarrow \lambda_2 = -\lambda_1 - 6 \Rightarrow \boxed{\lambda_2 = -2}$

$\therefore$ ① $\Rightarrow x_1 = \frac{4+4}{4} \Rightarrow \boxed{x_1 = 2}$

③ $\Rightarrow x_2 = \frac{4+2}{6} \Rightarrow \boxed{x_2 = 1}$

③ $\Rightarrow x_3 = \frac{8+2}{2} \Rightarrow \boxed{x_3 = 5}$

For maxima-minima :-

consider Hessian matrix.

$$H=\begin{bmatrix} 0 & 0 & \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ 0 & 0 & \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \\ \frac{\partial h_1}{\partial x_1} & \frac{\partial h_2}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial h_1}{\partial x_2} & \frac{\partial h_2}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial h_1}{\partial x_3} & \frac{\partial h_2}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 1 \\ 1 & 2 & 4 & 0 & 0 \\ 1 & 1 & 0 & 6 & 0 \\ 2 & 1 & 0 & 0 & 2 \end{bmatrix}$$

$m=2,\ n=3,\ 2m+1=5 \Rightarrow \Delta_5$

$n-m=3-2=1 \Rightarrow$ only one minor.

to Find Δ_5 :- $\xrightarrow[C_5-2C_3]{C_4-C_3} \begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & -3 \\ 1 & 2 & 4 & -4 & -8 \\ 1 & 1 & 0 & 6 & 0 \\ 2 & 1 & 0 & 0 & 2 \end{vmatrix} = (1)\begin{vmatrix} 0 & 0 & -1 & -3 \\ 1 & 2 & -4 & -8 \\ 1 & 1 & 6 & 0 \\ 2 & 1 & 0 & 2 \end{vmatrix}$

$\xrightarrow{C_4-3C_3} \begin{vmatrix} 0 & 0 & -1 & 0 \\ 1 & 2 & -4 & 4 \\ 1 & 1 & 6 & -18 \\ 2 & 1 & 0 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 2 & 4 \\ 1 & 1 & -18 \\ 2 & 1 & 2 \end{vmatrix} = -1\{1(20)-2(38)+4(-1)\}$

$\Delta_5 = 60$ which is positive

$\therefore$ F is minimum at $x_1=2,\ x_2=1,\ x_3=5$

and $z_{min} = F_{min} = 2(4)+3(1)+25$

$= 36$

EX Using method of Lagrangian multipliers solve the following problem

optimise $z=4x_1^2-x_2^2-x_3^2-4x_1x_2$ — mistake — $z=4x_1^2+2x_2^2+x_3^2-4x_1x_2$

subject to, $x_1+x_2+x_3=15$

$2x_1-x_2+2x_3=20$

$x_1, x_2, x_3 \geq 0$

(8 marks)

— ε C-09 M

Solution :- Let $F=4x_1^2-x_2^2-x_3^2-4x_1x_2$, $h_1=x_1+x_2+x_3-15$, $h_2=2x_1-x_2+2x_3-20$

Let $L=F+\lambda_1h_1+\lambda_2h_2$

$L=4x_1^2-x_2^2-x_3^2-4x_1x_2+\lambda_1(x_1+x_2+x_3-15)+\lambda_2(2x_1-x_2+2x_3-20)$

$\frac{\partial L}{\partial x_1}=0 \Rightarrow 8x_1-4x_2+\lambda_1+2\lambda_2=0$ — ①

$\frac{\partial L}{\partial x_2}=0 \Rightarrow -2x_2-4x_1+\lambda_1-\lambda_2=0$ — ②

$\frac{\partial L}{\partial x_3}=0 \Rightarrow -2x_3+\lambda_1+2\lambda_2=0 \Rightarrow \lambda_1+2\lambda_2=2x_3$ — ③

$\frac{\partial L}{\partial \lambda_1}=0 \Rightarrow x_1+x_2+x_3-15=0 \Rightarrow x_1+x_2+x_3=15$ — ④

$\frac{\partial L}{\partial \lambda_2}=0 \Rightarrow 2x_1-x_2+2x_3-20=0 \Rightarrow 2x_1-x_2+2x_3=20$ — ⑤

using ③ in ①

$8x_1-4x_2+2x_3=0$

$x_3=-4x_1+2x_2$ — ⑥

using ⑥ in ④ and ⑤

$$④ \Rightarrow x_1 + x_2 - 4x_1 + 2x_2 = 15 \Rightarrow -3x_1 + 3x_2 = 15 \Rightarrow -x_1 + x_2 = 5 \quad — ⑦$$

$$⑤ \Rightarrow 2x_1 - x_2 - 8x_1 + 4x_2 = 20 \Rightarrow -6x_1 + 3x_2 = 20 \Rightarrow -2x_1 + x_2 = \frac{20}{3} \quad — ⑧$$

$$⑦ - ⑧ \Rightarrow \boxed{x_1 = -\frac{5}{3}}$$

Paper setter is (mad)$^{\infty}$

$$⑦ \Rightarrow x_2 = 5 + x_1 = 5 - \frac{5}{3} \Rightarrow \boxed{x_2 = \frac{10}{3}}$$

$$⑥ \Rightarrow x_3 = -4\left(-\frac{5}{3}\right) + 2\left(\frac{10}{3}\right) \Rightarrow \boxed{x_3 = \frac{40}{3}}$$

$$\therefore ① \Rightarrow 8\left(-\frac{5}{3}\right) - 4\left(\frac{10}{3}\right) + \lambda_1 + 2\lambda_2 = 0 \Rightarrow \lambda_1 + 2\lambda_2 = \frac{80}{3}$$

$$② \Rightarrow -2\left(\frac{10}{3}\right) - 4\left(-\frac{5}{3}\right) + \lambda_1 - \lambda_2 = 0 \Rightarrow \lambda_1 - \lambda_2 = 0 \Rightarrow \lambda_1 = \lambda_2$$

$$\Rightarrow 3\lambda_2 = \frac{80}{3} \qquad \boxed{\therefore \lambda_1 = \lambda_2 = \frac{80}{9}}$$

For maxima-minima:-

$$H = \begin{bmatrix} 0 & 0 & \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ 0 & 0 & \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \\ \frac{\partial h_1}{\partial x_1} & \frac{\partial h_2}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial h_1}{\partial x_2} & \frac{\partial h_2}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial h_1}{\partial x_3} & \frac{\partial h_2}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & -2 & 0 \\ 1 & 2 & 0 & 0 & -2 \end{vmatrix}$$

$m = 2$, $n = 3$, $2m + 1 = 5$
we start with Δ_5
$n - m = 1 \Rightarrow$ only one minor

$$\Delta_5 = |H| = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & -2 & 0 \\ 1 & 2 & 0 & 0 & -2 \end{vmatrix} \xrightarrow{C_4 - C_3,\ C_5 - C_3} \begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 1 & 2 & 8 & -12 & -8 \\ 1 & -1 & -4 & 2 & 4 \\ 1 & 2 & 0 & 0 & -2 \end{vmatrix} = 1 \begin{vmatrix} 0 & 0 & -3 & 0 \\ 1 & 2 & -12 & -8 \\ 1 & -1 & 2 & 4 \\ 1 & 2 & 0 & -2 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 1 & 2 & -8 \\ 1 & -1 & 4 \\ 1 & 2 & -2 \end{vmatrix} = -3\left\{1(-6) - 2(-6) - 8(3)\right\} = -3(-18) = 54$$

Δ_5 is positive $\Rightarrow$ F is min. at $x_1 = -\frac{5}{3}$, $x_2 = \frac{10}{3}$, $x_3 = \frac{40}{3}$

EX Using the method of Lagrange's multipliers solve the NLPP

optimize $z = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$

subject to $x_1 + x_2 + x_3 = 7$, $x_i \geq 0$ → ε C-09 D

Solution:- Let $f = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3$, $h = x_1 + x_2 + x_3 - 7$

consider $L = f + \lambda h$

$$L = x_1^2 + x_2^2 + x_3^2 - 10x_1 - 6x_2 - 4x_3 + \lambda(x_1 + x_2 + x_3 - 7)$$

$\therefore \dfrac{\partial L}{\partial x_1} = 2x_1 - 10 + \lambda = 0 \Rightarrow \lambda = 10 - 2x_1$

$\dfrac{\partial L}{\partial x_2} = 2x_2 - 6 + \lambda = 0 \Rightarrow \lambda = 6 - 2x_2$

$\Rightarrow 10 - 2x_1 = 6 - 2x_2 \Rightarrow -x_1 + x_2 = -2$ — ①

$\dfrac{\partial L}{\partial x_3} = 2x_2 - 4 + \lambda = 0 \Rightarrow \lambda = 4 - 2x_2$

$\Rightarrow 6 - 2x_2 = 4 - 2x_2 \Rightarrow -2x_2 + 2x_3 = -2$

$-x_2 + x_3 = -1$ — ②

$\dfrac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 7 = 0 \Rightarrow x_1 + x_2 + x_3 = 7$ — ③

① + ② $\Rightarrow -x_1 + x_3 = -3 \Rightarrow x_3 = x_1 - 3$ — ④

① $\Rightarrow x_2 = x_1 - 2$ — ⑤

using ④, ⑤ in ③, $x_1 + x_1 - 2 - 3 + x_1 = 7 \Rightarrow x_1 = 4$

$\Rightarrow \boxed{x_1 = 4}$ ⑤ $\Rightarrow x_2 = 4 - 2 \Rightarrow \boxed{x_2 = 2}$

⑤ $\Rightarrow x_3 = 4 - 3$ $\boxed{x_3 = 1}$

For maxima-minima :-

$$H = \begin{bmatrix} 0 & \frac{\partial h_1}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

Here $m=1$
$\therefore 2m+1 = 3$
and $n = 3$
$\Rightarrow \Delta_3, \Delta_4$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 0 - (2-0) + 1(-2) = -4$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix} \xrightarrow[C_4 - C_2]{C_3 - C_2} \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & -2 & -2 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix} = -1 \begin{vmatrix} 1 & -2 & -2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -\{1(4) + 2(2) - 2(-2)\}$$

$\therefore \Delta_4 = -12$

as signs of Δ_3 and Δ_4 are same, given function is minimum at $x_1 = 4$, $x_2 = 2$, $x_3 = 1$.

and $Z_{min} = (16) + (4) + (1) - 10(4) - 6(2) - 4(1)$

$Z_{min} = -35$

Ex obtain the relative maximum or minimum (if any) of the function

$$z = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 10x_2 - 14x_3 + 103$$

D
—{C-09

Solution:- Let $L = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 10x_2 - 14x_3 + 103$

$m=0$, $n=3$
$2m+1=1 \Rightarrow$ start with Δ_1
$n-m=3$, calculate $\Delta_1, \Delta_2, \Delta_3$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 - 6 = 0 \Rightarrow \boxed{x_1 = 3}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 - 10 = 0 \Rightarrow \boxed{x_2 = 5}$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 3x_3 - 14 = 0 \Rightarrow \boxed{x_3 = \frac{14}{3}}$$

consider Hessian matrix:-

$$\begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$m=0$
$2m+1=1$
$n=3$
$n-m=3$
$\Rightarrow \Delta_1, \Delta_2, \Delta_3$

$$\Delta_1 = |2| = 2, \quad \Delta_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \quad \Delta_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$$

signs of $\Delta_1, \Delta_2, \Delta_3$ are same $\Rightarrow z$ is minimum at $x_1 = 3$, $x_2 = 5$, $x_3 = \frac{14}{3}$

Ex find the maximum or minimum of the Function

$$z = x_1 + 2x_3 + x_2 x_3 - x_1^2 - x_2^2 - x_3^2$$

M
—{C-10, 96, 98

Solution:- Let $L = x_1 + 2x_3 + x_2 x_3 - x_1^2 - x_2^2 - x_3^2$

$m=0$, $n=3$
$2m+1=1 \Rightarrow$ start with Δ_1
$n-m=3 \Rightarrow$ calculate $\Delta_1, \Delta_2, \Delta_3$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 1 - 2x_1 = 0 \Rightarrow \boxed{x_1 = \frac{1}{2}}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow x_3 - 2x_2 = 0 \Rightarrow x_3 = 2x_2 \quad -①$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2 + x_2 - 2x_3 = 0 \Rightarrow 2 + x_2 - 4x_2 = 0 \quad [\because ①]$$

$$\Rightarrow 3x_2 = 2 \Rightarrow \boxed{x_2 = \frac{2}{3}} \qquad ① \Rightarrow \boxed{x_3 = \frac{4}{3}}$$

consider Hessian matrix

$$H = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

Here $m=0$
$2m+1=1$, $n=3$
$n-m=3$
$\Rightarrow \Delta_1, \Delta_2, \Delta_3$

$$\Delta_1 = |-2| = -2, \quad \Delta_2 = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4, \quad \Delta_3 = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix} = -6$$

Signs are alternate $\Rightarrow z$ is maximum at $x_1 = \frac{1}{2}$, $x_2 = \frac{2}{3}$, $x_3 = \frac{4}{3}$

$$\therefore z_{max} = \frac{1}{2} + \frac{8}{3} + \frac{8}{9} - \frac{1}{4} - \frac{4}{9} - \frac{16}{9} \Rightarrow z_{max} = \frac{57}{36}$$

Ex Use the method of Lagrangian multipliers to solve the following problem.

$$\text{Minimize } z = 6x_1 + 8x_3 - x_1^2 - x_2^2$$

$8x_2$ Printing mistake

$$\text{subject to } 4x_1 + 3x_2 = 16$$
$$3x_1 + 5x_2 = 15, \quad x_1, x_2 \geq 0$$

— {C-10} M

Solution:-

$L = z + \lambda_1 h_1 + \lambda_2 h_2$, where $h_1 = 4x_1 + 3x_2 - 16$, $h_2 = 3x_1 + 5x_2 - 15$

$\therefore L = 6x_1 + 8x_2 - x_1^2 - x_2^2 + \lambda_1(4x_1 + 3x_2 - 16) + \lambda_2(3x_1 + 5x_2 - 15)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 6 - 2x_1 + 4\lambda_1 + 3\lambda_2 = 0$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 8 - 2x_2 + 3\lambda_1 + 5\lambda_2 = 0$

$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow 4x_1 + 3x_2 - 16 = 0 \Rightarrow 4x_1 + 3x_2 = 16$ — (1)

$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow 3x_1 + 5x_2 - 15 = 0 \Rightarrow 3x_1 + 5x_2 = 15$ — (2)

$\Rightarrow \boxed{x_1 = \frac{35}{11}} \quad \boxed{x_2 = \frac{12}{11}}$

consider Hessian Matrix:-

$$H = \begin{bmatrix} 0 & 0 & \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ 0 & 0 & \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \\ \frac{\partial h_1}{\partial x_1} & \frac{\partial h_2}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial h_1}{\partial x_2} & \frac{\partial h_2}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 & 3 \\ 0 & 0 & 3 & 5 \\ 4 & 3 & -2 & 0 \\ 3 & 5 & 0 & -2 \end{bmatrix}$$

$m = 2,\ n = 2$
$2m+1 = 5$
$\Rightarrow \Delta_5,\ n - m = 0$
How?

$$\Delta_5 = |H| = 4\begin{vmatrix} 0 & 0 & 5 \\ 4 & 3 & 0 \\ 3 & 5 & -2 \end{vmatrix} - 3\begin{vmatrix} 0 & 0 & 3 \\ 4 & 3 & -2 \\ 3 & 5 & 0 \end{vmatrix} = 4(5)\{11\} - 3(3)(11) = 121$$

z is minimum at $x_1 = \frac{35}{11}$, $x_2 = \frac{12}{11}$

Ex Find the relative maximum or minimum of the function

$$z = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 10x_2 - 14x_3 + 103$$

→ [2007, 2009]

Soln: Let $L = z = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 10x_2 - 14x_3 + 103$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 - 6 = 0 \Rightarrow x_1 = 3$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 - 10 = 0 \Rightarrow x_2 = 5$

$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2x_3 - 14 = 0 \Rightarrow x_3 = 7$

$m = 0,\ n = 3 \Rightarrow 2m+1 = 1,\quad n - m = 3 - 0 = 3 \Rightarrow$ find $\Delta_1, \Delta_2, \Delta_3$

Now
$$H = \begin{bmatrix} L_{x_1x_1} & L_{x_1x_2} & L_{x_1x_3} \\ L_{x_2x_1} & L_{x_2x_2} & L_{x_2x_3} \\ L_{x_3x_1} & L_{x_3x_2} & L_{x_3x_3} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$|\Delta_1| = |2| = 2$, $|\Delta_2| = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$, $|\Delta_3| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$

signs of all principal minors are same $\Rightarrow$ it gives minimum.

$\therefore$ soln is $x_1 = 3$, $x_2 = 5$, $x_3 = 7$

and $Z_{min} = 3^2 + 5^2 + 7^2 - 6(3) - 10(5) - 14(7) + 103$

$Z_{min} = 20$

Kuhn - Tucker conditions

inequality constraint

Type – I $\longrightarrow$ one constraint

$$\text{Max } z = f(x_1, x_2 \cdots x_n) \quad \text{subject to} \quad g(x_1, x_2, \cdots x_n) \le b \quad , \quad x_1, x_2, \cdots x_n \geq 0$$

Let $h = g(x_1, x_2, \cdots x_n) - b$

consider $L = f + \lambda h$

find positive values of $x_1, x_2, \cdots x_n$ such that

$$\left.\begin{array}{l} \frac{\partial L}{\partial x_1} = 0, \ \frac{\partial L}{\partial x_2} = 0, \ \cdots \frac{\partial L}{\partial x_n} = 0 \\ \lambda h = 0 \\ \text{and } h \le 0 \end{array}\right] \Rightarrow \text{K.T. conditions}$$

where $\lambda < 0$

For min $\lambda > 0$

Type – II $\longrightarrow$ two constraints

$$\text{Max } z = f(x_1, x_2, \cdots x_n) \quad \text{subject to,} \quad g_1(x_1, x_2, \cdots x_n) \le b_1$$
$$g_2(x_1, x_2, \cdots x_n) \le b_2$$

Let $h_1 = g_1(x_1, x_2, \cdots x_n) - b_1$, $h_2 = g_2(x_1, x_2, \cdots x_n) - b_2$

consider $L = f + \lambda_1 h_1 + \lambda_2 h_2$

find positive values of $x_1, x_2, \cdots x_n$ such that,

$$\left.\begin{array}{l} \frac{\partial L}{\partial x_1} = 0, \ \frac{\partial L}{\partial x_2} = 0, \ \cdots \frac{\partial L}{\partial x_n} = 0 \\ \lambda_1 h_1 = 0 \ , \ \lambda_2 h_2 = 0 \\ h_1 \le 0 \quad h_2 \le 0 \end{array}\right] \Rightarrow \text{K-T. conditions.}$$

$\lambda_1, \lambda_2 \le 0$

For min, $\lambda_1 \lambda_2 > 0$

EX Solve the following LPP using Kuhn-Tucker conditions

$$\text{Maximize } z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$$
$$\text{subject to } 2x_1 + x_2 \le 5 \ ; \ x_1 \geq 0, x_2 \geq 0 \ \text{—} \{2007$$

solution: $\longrightarrow$ $F = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$, $h = 2x_1 + x_2 - 5$

consider $L = f + \lambda h = (10x_1 + 4x_2 - 2x_1^2 - x_2^2) + \lambda(2x_1 + x_2 - 5)$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 10 - 4x_1 + 2\lambda = 0 \quad \text{— (1)}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 4 - 2x_2 + \lambda = 0 \quad \text{— (2)}$$

$$\lambda h = 0 \Rightarrow \lambda(2x_1 + x_2 - 5) = 0 \quad \text{— (3)}$$

$$h \le 0 \Rightarrow 2x_1 + x_2 - 5 \le 0 \quad \text{— (4)}$$

case (i) $\lambda = 0$ then (1) $\Rightarrow 10 - 4x_2 = 0 \Rightarrow x_2 = \frac{5}{2}$

(2) $\Rightarrow 4 - 2x_2 = 0 \Rightarrow x_2 = 2$

But this does not satisfy equation (4)

i.e. $5 + 2 - 5 = 2 \nleq 0$

case (ii) $\lambda \neq 0$, $③ \Rightarrow 2x_1+x_2-5=0 \Rightarrow 2x_1+x_2=5$ — ⑤

$$\left.\begin{aligned} ① &\Rightarrow \lambda = 2x_1-5 \\ ② &\Rightarrow \lambda = 2x_2-4 \end{aligned}\right\} \Rightarrow 2x_1-5 = 2x_2-4 \Rightarrow 2x_1-2x_2=1 \quad — ⑥$$

$$⑤-⑥ \Rightarrow 3x_2 = 4 \Rightarrow \boxed{x_2=\frac{4}{3}}$$

$$⑤ \Rightarrow 2x_1 = 5-\frac{4}{3} \Rightarrow \boxed{x_1=\frac{11}{6}}$$

$\lambda = -\frac{8}{3} < 0$

these values satisfies eqⁿ ④ i.e. $\frac{8}{3}+\frac{11}{6}-5 = \frac{16+11-30}{6} = -\frac{3}{6} < 0$

$② \Rightarrow \lambda = 2x_2-4 = \frac{8}{3}-4 = -\frac{4}{3}$ $\qquad z_{max} = 91/6$

EX Use Kuhn-Tucker conditions to solve the following problem.

$z_{max} = 8x_1+10x_2-x_1^2-x_2^2$ subject to $3x_1+2x_2 \le 6$, $x_1, x_2 \geq 0$ — {2006

Solution: → Let $f = 8x_1+10x_2-x_1^2-x_2^2$, $h = 3x_1+2x_2-6$

consider $L = f+\lambda h = (8x_1+10x_2-x_1^2-x_2^2)+\lambda(3x_1+2x_2-6)$

$$\frac{\partial L}{\partial x_1}=0 \Rightarrow 8-2x_1+3\lambda=0 \quad — ①$$

$$\frac{\partial L}{\partial x_2}=0 \Rightarrow 10-2x_2+2\lambda=0 \quad — ②$$

$$\lambda h=0 \Rightarrow \lambda(3x_1+2x_2-6)=0 \quad — ③$$

$$h \le 0 \Rightarrow 3x_1+2x_2-6 \le 0 \quad — ④$$

case (i) $\lambda=0$, $① \Rightarrow 8-2x_1=0 \Rightarrow x_1=4$

$② \Rightarrow 10-2x_2=0 \Rightarrow x_2=5$

But this does not satisfy equation ④ $\quad 12+10-6=16 \nleq 0$

case (ii) $\lambda \neq 0$ then $③ \Rightarrow 3x_1+2x_2-6=0 \Rightarrow 3x_1+2x_2=6$ — ⑤

$$\left.\begin{aligned} ① &\Rightarrow \lambda = \frac{2x_1-8}{3} \\ ② &\Rightarrow \lambda = x_2-5 \end{aligned}\right] \Rightarrow \frac{2x_1-8}{3} = x_2-5 \Rightarrow 2x_1-8 = 3x_2-15 \Rightarrow 2x_1-3x_2=-7 \quad — ⑥$$

$$3⑤+2⑥ \Rightarrow 9x_1+6x_2+4x_1-6x_2 = 18-14 \Rightarrow 13x_1=4$$

$$\Rightarrow \boxed{x_1=\frac{4}{13}}$$

$$\therefore ⑤ \Rightarrow 2x_2 = 6-3\left(\frac{4}{13}\right) = \frac{66}{13} \Rightarrow \boxed{x_2=\frac{33}{13}}$$

these values satisfy eqⁿ ④ $\frac{12}{13}+\frac{66}{13}-6 = 0 \le 0$

$$② \Rightarrow \lambda = x_2-5 = \frac{33}{13}-5, \quad \lambda = -\frac{32}{13} < 0$$

$$z_{max} = 8\left(\frac{4}{13}\right)+10\left(\frac{33}{13}\right)-\left(\frac{16}{169}\right)-\left(\frac{1089}{169}\right) \Rightarrow z_{max} = \frac{3601}{169}$$

Ex Using Kuhn-Tucker conditions solve the following LPP

Maximize $z = 2x_1 + 3x_2 - x_1^2 - x_2^2$

Subject to $x_1 + x_2 \le 1$

$2x_1 + 3x_2 \le 6$, $x_1, x_2 \ge 0$ — [2008, 09]

Sol^n:- Let $F = 2x_1 + 3x_2 - x_1^2 - x_2^2$, $h_1 = x_1 + x_2 - 1$, $h_2 = 2x_1 + 3x_2 - 6$

consider $L = F + \lambda_1 h_1 + \lambda_2 h_2 = (2x_1 + 3x_2 - x_1^2 - x_2^2) + \lambda_1(x_1 + x_2 - 1) + \lambda_2(2x_1 + 3x_2 - 6)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2 - 2x_1 + \lambda_1 + 2\lambda_2 = 0$ — (1)

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 3 - 2x_2 + \lambda_1 + 3\lambda_2 = 0$ — (2)

$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + x_2 - 1) = 0$ — (3)

$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(2x_1 + 3x_2 - 6) = 0$ — (4)

$h_1 \le 0 \Rightarrow x_1 + x_2 - 1 \le 0$ — (5)

$h_2 \le 0 \Rightarrow 2x_1 + 3x_2 - 6 \le 0$ — (6)

case (i):- $\lambda_1 = 0$, $\lambda_2 = 0$

(1) $\Rightarrow 2 - 2x_1 = 0 \Rightarrow x_1 = 1$

(2) $\Rightarrow 3 - 2x_2 = 0 \Rightarrow x_2 = \frac{3}{2}$

But these values violates condition (5) i.e. $1 + \frac{3}{2} - 1 = \frac{3}{2} \nleq 0$

case (ii):- $\lambda_1 \ne 0$, $\lambda_2 = 0$

$\lambda_1 \ne 0$ $\therefore$ (3) $\Rightarrow x_1 + x_2 - 1 = 0 \Rightarrow x_1 + x_2 = 1$ — (7)

(1) $\Rightarrow 2 - 2x_1 + \lambda_1 = 0 \Rightarrow \lambda_1 = 2x_1 - 2$

(2) $\Rightarrow 3 - 2x_2 + \lambda_1 = 0 \Rightarrow \lambda_1 = 2x_2 - 3$

$\Rightarrow 2x_1 - 2 = 2x_2 - 3 \Rightarrow 2x_1 - 2x_2 = -1$ — (8)

2(7) + (8) $\Rightarrow 4x_1 = 1 \Rightarrow \boxed{x_1 = \frac{1}{4}}$

(7) $\Rightarrow x_2 = 1 - \frac{1}{4} \Rightarrow \boxed{x_2 = \frac{3}{4}}$

these values satisfies equation (5) and (6)

i.e. $\frac{1}{4} + \frac{3}{4} - 1 = 0 \le 0$ and $\frac{2}{4} + \frac{9}{4} - 6 = \frac{-13}{4} \le 0$

case (iii) $\lambda_1 = 0$ and $\lambda_2 \ne 0$

eq^n (4) $\Rightarrow 2x_1 + 3x_2 - 6 = 0 \Rightarrow 2x_1 + 3x_2 = 6$ — (9)

(1) $\Rightarrow 2 - 2x_1 + 2\lambda_2 = 0 \Rightarrow \lambda_2 = x_1 - 1$

(2) $\Rightarrow 3 - 2x_2 + 3\lambda_2 = 0 \Rightarrow \lambda_2 = \frac{2x_2 - 3}{3}$

$\Rightarrow x_1 - 1 = \frac{2x_2 - 3}{3} \Rightarrow 3x_1 - 3 = 2x_2 - 3$

$\Rightarrow 3x_1 = 2x_2 \Rightarrow x_1 = \frac{2x_2}{3}$ — (10)

using (10) in (9) $\frac{4x_2}{3} + 3x_2 = 6 \Rightarrow x_2 = \frac{18}{13}$

$\therefore$ (10) $\Rightarrow x_1 = \frac{2}{3}\left(\frac{18}{13}\right) \Rightarrow x_1 = \frac{12}{13}$

But this does not satisfy eq^n (5), $\frac{12}{13} + \frac{18}{13} - 1 = \frac{17}{13} \nleq 0$

case (iv) $\lambda_1 \neq 0,\ \lambda_2 \neq 0$

$(3) \Rightarrow x_1 + x_2 - 1 = 0 \Rightarrow x_1 + x_2 = 1$ —— (11)

$(4) \Rightarrow 2x_1 + 3x_2 - 6 = 0 \Rightarrow 2x_1 + 3x_2 = 6$ —— (12)

$2(11) - (12) \Rightarrow -x_2 = -4 \Rightarrow x_2 = 4$

$(11) \Rightarrow x_1 = 1 - 4 \Rightarrow x_1 = -3$ which is not feasible.

$\therefore$ solution is $x_1 = \frac{1}{4},\ x_2 = \frac{3}{4}$

and $z_{max} = -\frac{1}{16} - \frac{9}{16} + 2\left(\frac{1}{4}\right) + 3\left(\frac{3}{4}\right) = \frac{34}{16} \Rightarrow z_{max} = \frac{17}{8}$

note that $\lambda_1 = \frac{3}{2},\ \lambda_2 = 0$

[Ex] Using Kuhn-Tucker condition solve the following LPP.

Maximize $z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

subject to, $x_1 + x_2 \leq 2$

$2x_1 + 3x_2 \leq 12,\ x_1, x_2 \geq 0$ — Summer 2002

Solution:- $F = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2,\quad h_1 = x_1 + x_2 - 2,\quad h_2 = 2x_1 + 3x_2 - 12$

consider $L = F + \lambda_1 h_1 + \lambda_2 h_2 = (-x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2) + \lambda_1(x_1 + x_2 - 2) + \lambda_2(2x_1 + 3x_2 - 12)$

$\therefore \frac{\partial L}{\partial x_1} = 0 \Rightarrow -2x_1 + 4 + \lambda_1 + 2\lambda_2 = 0$ —— (1)

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow -2x_2 + 6 + \lambda_1 + 3\lambda_2 = 0$ —— (2)

$\frac{\partial L}{\partial x_3} = 0 \Rightarrow -2x_3 = 0 \Rightarrow x_3 = 0$ —— (3)

$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + x_2 - 2) = 0$ —— (4)

$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(2x_1 + 3x_2 - 12) = 0$ —— (5)

$h_1 \leq 0 \Rightarrow x_1 + x_2 - 2 \leq 0$ —— (6)

$h_2 \leq 0 \Rightarrow 2x_1 + 3x_2 - 12 \leq 0$ —— (7)

case (i) Let $\lambda_1 = 0,\ \lambda_2 = 0$

$(1) \Rightarrow -2x_1 + 4 = 0 \Rightarrow x_1 = 2$

$(2) \Rightarrow -2x_2 + 6 = 0 \Rightarrow x_2 = 3$

but these values violates condition (6) $2 + 3 - 2 = 3 \nleq 0$

case (ii) $\lambda_1 \neq 0,\ \lambda_2 = 0$

$(4) \Rightarrow x_1 + x_2 - 2 = 0 \Rightarrow x_1 + x_2 = 2$ —— (8)

$(1) \Rightarrow -2x_1 + 4 + \lambda_1 = 0 \Rightarrow \lambda_1 = 2x_1 - 4$

$(2) \Rightarrow -2x_2 + 6 + \lambda_1 = 0 \Rightarrow \lambda_1 = 2x_2 - 6$

$\Rightarrow 2x_1 - 4 = 2x_2 - 6 \Rightarrow x_1 - x_2 = -1$ —— (9)

$(8) + (9) \Rightarrow 2x_1 = 1 \Rightarrow \boxed{x_1 = \frac{1}{2}}$

$(9) \Rightarrow x_2 = \frac{1}{2} + 1 \Rightarrow \boxed{x_2 = \frac{3}{2}}$

these values satisfy eq^n (6) and (7) $\frac{1}{2} + \frac{3}{2} - 2 = 0 \leq 0$

$2\frac{1}{2} + 3\frac{3}{2} - 12 = \frac{-13}{2} \leq 0$

case (iv) $\lambda_1 \neq 0,\ \lambda_2 \neq 0$

$$③ \Rightarrow x_1 + x_2 - 1 = 0 \Rightarrow x_1 + x_2 = 1 \quad \text{--- } (11)$$

$$④ \Rightarrow 2x_1 + 3x_2 - 6 = 0 \Rightarrow 2x_1 + 3x_2 = 6 \quad \text{--- } (12)$$

$$2(11) - (12) \Rightarrow -x_2 = -4 \Rightarrow x_2 = 4$$

$(11) \Rightarrow x_1 = 1 - 4 \Rightarrow x_1 = -3$ which is not feasible.

$\therefore$ solution is, $x_1 = \frac{1}{4},\ x_2 = \frac{3}{4}$

note that $\lambda_1 = -\frac{3}{2},\ \lambda_2 = 0$

and $z_{max} = -\frac{1}{16} - \frac{9}{16} + 2\left(\frac{1}{4}\right) + 3\left(\frac{3}{4}\right) = \frac{34}{16} \Rightarrow z_{max} = \frac{17}{8}$

Ex Using Kuhn-Tucker condition solve the following LPP.

$$\text{Maximize } z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$$

$$\text{Subject to } x_1 + x_2 \leq 2$$

$$2x_1 + 3x_2 \leq 12,\ x_1, x_2 \geq 0$$ — {2004, 2002

Solution:- $F = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$, $h_1 = x_1 + x_2 - 2$, $h_2 = 2x_1 + 3x_2 - 12$

consider $L = F + \lambda_1 h_1 + \lambda_2 h_2 = (-x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2) + \lambda_1(x_1 + x_2 - 2) + \lambda_2(2x_1 + 3x_2 - 12)$

$$\therefore \frac{\partial L}{\partial x_1} = 0 \Rightarrow -2x_1 + 4 + \lambda_1 + 2\lambda_2 = 0 \quad \text{--- } ①$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow -2x_2 + 6 + \lambda_1 + 3\lambda_2 = 0 \quad \text{--- } ②$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow -2x_3 = 0 \Rightarrow x_3 = 0 \quad \text{--- } ③$$

$$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + x_2 - 2) = 0 \quad \text{--- } ④$$

$$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(2x_1 + 3x_2 - 12) = 0 \quad \text{--- } ⑤$$

$$h_1 \leq 0 \Rightarrow x_1 + x_2 - 2 \leq 0 \quad \text{--- } ⑥$$

$$h_2 \leq 0 \Rightarrow 2x_1 + 3x_2 - 12 \leq 0 \quad \text{--- } ⑦$$

case (i) Let $\lambda_1 = 0,\ \lambda_2 = 0$

$$① \Rightarrow -2x_1 + 4 = 0 \Rightarrow x_1 = 2$$

$$② \Rightarrow -2x_2 + 6 = 0 \Rightarrow x_2 = 3$$

but these values violates condition ⑥ $2 + 3 - 2 = 3 \nleq 0$

case (ii) $\lambda_1 \neq 0,\ \lambda_2 = 0$

$$④ \Rightarrow x_1 + x_2 - 2 = 0 \Rightarrow x_1 + x_2 = 2 \quad \text{--- } ⑧$$

$① \Rightarrow -2x_1 + 4 + \lambda_1 = 0 \Rightarrow \lambda_1 = 2x_1 - 4$

$② \Rightarrow -2x_2 + 6 + \lambda_1 = 0 \Rightarrow \lambda_1 = 2x_2 - 6$

$\Rightarrow 2x_1 - 4 = 2x_2 - 6 \Rightarrow x_1 - x_2 = -1$ --- ⑨

$⑧ + ⑨ \Rightarrow 2x_1 = 1 \Rightarrow \boxed{x_1 = \frac{1}{2}}$

$⑨ \Rightarrow x_2 = \frac{1}{2} + 1 \Rightarrow \boxed{x_2 = \frac{3}{2}}$

these values satisfy eqn ⑤ and ⑦ $\frac{1}{2} + \frac{3}{2} - 2 = 0 \leq 0$

$$2\,\frac{1}{2} + 3\,\frac{3}{2} - 12 = \frac{-13}{2} \leq 0$$

case (iii) $\lambda_1=0$ and $\lambda_2\neq 0$

$(5) \Rightarrow 2x_1+3x_2-12=0 \Rightarrow 2x_1+3x_2=12$ —— (10)

$(1) \Rightarrow -2x_1+4+2\lambda_2=0 \Rightarrow \lambda_2 = x_1-2$

$(2) \Rightarrow -2x_2+6+3\lambda_2=0 \Rightarrow \lambda_2=\frac{2x_2-6}{3}$

$\Rightarrow x_1-2=\frac{2x_2-6}{3} \Rightarrow 3x_1-6=2x_2-6$

$\Rightarrow x_1=\frac{2x_2}{3}$ —— (11)

using (11) in (10)' $\frac{4x_2}{3}+3x_2=12 \rightarrow 13x_2=36 \Rightarrow x_2=\frac{36}{13}$

$\therefore (11) \Rightarrow x_1=\frac{2}{3}\left(\frac{36}{13}\right) \Rightarrow x_1=\frac{24}{13}$

But these values violates condition (6), $\frac{24}{13}+\frac{36}{13}-2=\frac{34}{13}\neq 0$

case (iv):- $\lambda_1\neq 0$, $\lambda_2\neq 0$

$\therefore (4) \Rightarrow x_1+x_2-2=0 \Rightarrow x_1+x_2=2$ —— (12)

$(5) \Rightarrow 2x_1+3x_2-12=0 \Rightarrow 2x_1+3x_2=12$ —— (13)

$2(12)-(13) \Rightarrow -x_2=-8 \Rightarrow x_2=8$

$\therefore (12) \Rightarrow x_1=2-x_2 \Rightarrow x_1=-6$ which is not feasible

$\therefore$ solution is $x_1=\frac{1}{2}$, $x_2=\frac{3}{2}$, $x_3=0$

and $z_{max}=-\frac{1}{4}-\frac{9}{4}-0+2+9=\frac{34}{4} \Rightarrow z_{max}=\frac{17}{2}$

note that $\lambda_1=-3$ $\lambda_2=0$

Ex What are the Kuhn-Tucker conditions for following LPP. Use these conditions to determine whether $(x_1,x_2)=(0,10)$ can be optimal

$z_{max}=x_1^4+2x_1^2+2x_1x_2+4x_2^2$

subject to $2x_1+x_2\geq 10$ (4 marks)

$x_1+2x_2\geq 10$, $x_1,x_2\geq 0$ —— 2005

Solution:- $h_1=2x_1+x_2-10$, $h_2=x_1+2x_2-10$

$L=(x_1^4+2x_1^2+2x_1x_2+4x_2^2)+\lambda_1(2x_1+x_2-10)+\lambda_2(x_1+2x_2-10)$

K.T. conditions are:-

$\frac{\partial L}{\partial x_1}=0 \Rightarrow 4x_1^3+4x_1+2x_2+2\lambda_1+\lambda_2=0$ —— (1)

$\frac{\partial L}{\partial x_2}=0 \Rightarrow 2x_1+\lambda_1+2\lambda_2+8x_2=0$ —— (2)

$\lambda_1h_1=0 \Rightarrow \lambda_1(2x_1+x_2-10)=0$ —— (3)

$\lambda_2h_2=0 \Rightarrow \lambda_2(x_1+2x_2-10)=0$ —— (4)

$h_1\leq 0 \Rightarrow 2x_1+x_2-10\leq 0$ —— (5)

$h_2\leq 0 \Rightarrow x_1+2x_2-10\leq 0$ —— (6)

given values $x_1=0$, $x_2=10$

these values satisfies eqn (5) and (6)

to find λ_1,λ_2:-

$(1) \Rightarrow 20+2\lambda_1+\lambda_2=0$ —— (7)

$(2) \Rightarrow 80+\lambda_1+2\lambda_2=0$ —— (8)

$2(7)-(8) \Rightarrow -40+3\lambda_1=0$

$\Rightarrow \lambda_1=\frac{40}{3}>0$

$(7) \Rightarrow \lambda_2=-20-\frac{80}{3}=\frac{-140}{3}<0$

$\therefore (0,10)$ does not give optimal solution.

$\lambda_1,\lambda_2\leq 0$ must

Ex Use Kuhn-Tucker conditions to determine if $(x_1, x_2) = \left(\frac{2}{\sqrt{3}}, \frac{3}{2}\right)$ is an optimal solution of NLPP :

Maximize $f(x_1, x_2) = 4x_1 + 6x_2 - x_1^3 - 2x_2^2$

subject to $x_1 + 3x_2 \le 8$ — 4 marks.

$5x_1 + 2x_2 \le 14$, $x_1, x_2 \ge 0$ — {2005,

Solution:- $h_1 = x_1 + 3x_2 - 8$, $h_2 = 5x_1 + 2x_2 - 14$

$$L = (4x_1 + 6x_2 - x_1^3 - 2x_2^2) + \lambda_1(x_1 + 3x_2 - 8) + \lambda_2(5x_1 + 2x_2 - 14)$$

K.T. conditions:-

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 4 - 3x_1^2 + \lambda_1 + 5\lambda_2 = 0$ ——— (1)

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 6 - 4x_2 + 3\lambda_1 + 2\lambda_2 = 0$ ——— (2)

$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + 3x_2 - 8) = 0$ ——— (3)

$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(5x_1 + 2x_2 - 14) = 0$ ——— (4)

$h_1 \le 0 \Rightarrow x_1 + 3x_2 - 8 \le 0$ ——— (5)

$h_2 \le 0 \Rightarrow 5x_1 + 2x_2 - 14 \le 0$ ——— (6)

given values $x_1 = \frac{2}{\sqrt{3}}$, $x_2 = \frac{3}{2}$

these values satisfies (5), (6)

$\frac{2}{\sqrt{3}} + \frac{9}{2} - 8 \le 0$ and $\frac{10}{\sqrt{3}} + 3 - 14 \le 0$

(1) $\Rightarrow 4 - 4 + \lambda_1 + 5\lambda_2 = 0 \Rightarrow \lambda_1 = -5\lambda_2$ — (7)

(2) $\Rightarrow 6 - 6 + 3\lambda_1 + 2\lambda_2 = 0 \Rightarrow \lambda_1 = -\frac{2\lambda_2}{3}$ — (8)

From (7), (8) $-5\lambda_2 = -\frac{2\lambda_2}{3}$

$\Rightarrow -5\lambda_2 + \frac{2\lambda_2}{3} = 0 \Rightarrow \frac{-13\lambda_2}{3} = 0$

$\therefore \lambda_2 = 0 \le 0$

(7) $\Rightarrow \lambda_1 = 0 \le 0$

$\therefore$ f is maximum at $\left(\frac{2}{\sqrt{3}}, \frac{3}{2}\right)$

Ex Using the Kuhn Tucker conditions solve NLPP

maximize $z = 7x_1^2 + 5x_2^2 + 6x_1$ subject to

$x_1 + 2x_2 \le 10$

$x_1 - 3x_2 \le 9$, $x_1, x_2 \ge 0$ — { D C-09

Solution:- Let $f = 7x_1^2 + 5x_2^2 + 6x_1$, $h_1 = x_1 + 2x_2 - 10$, $h_2 = x_1 - 3x_2 - 9$

$L = f + \lambda_1 h_1 + \lambda_2 h_2$

$$\therefore L = (7x_1^2 + 5x_2^2 + 6x_1) + \lambda_1(x_1 + 2x_2 - 10) + \lambda_2(x_1 - 3x_2 - 9)$$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 14x_1 + 6 + \lambda_1 + \lambda_2 = 0$ — (1)

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 10x_2 + 2\lambda_1 - 3\lambda_2 = 0$ — (2)

$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + 2x_2 - 10) = 0$ — (3)

$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(x_1 - 3x_2 - 9) = 0$ — (4)

$h_1 \le 0 \Rightarrow x_1 + 2x_2 - 10 \le 0$ — (5)

$h_2 \le 0 \Rightarrow x_1 - 3x_2 - 9 \le 0$ — (6)

Case (i) $\lambda_1 = 0, \lambda_2 = 0$

(1) $\Rightarrow 14x_1 + 6 = 0 \Rightarrow x_1 = -\frac{3}{7}$

(2) $\Rightarrow 10x_2 = 0 \Rightarrow x_2 = 0$

} which is not feasible $\Rightarrow$ case fails

case (ii) $\lambda_1 \neq 0,\ \lambda_2 = 0$

(3) $\Rightarrow x_1 + 2x_2 - 10 = 0$

i.e. $x_1 + 2x_2 = 10$ — (7)

(1) $\Rightarrow 14x_1 + 6 + \lambda_1 = 0 \Rightarrow \lambda_1 = -6 - 14x_1$

(2) $\Rightarrow 10x_2 + 2\lambda_1 = 0 \Rightarrow \lambda_1 = -5x_2$

$\therefore -6 - 14x_1 = -5x_2$

$\therefore -14x_1 + 5x_2 = 6$ — (8)

$14(7) + (8) \Rightarrow 14x_1 + 28x_2 - 14x_1 + 5x_2 = 140 + 6$

$\Rightarrow 33x_2 = 146 \Rightarrow \boxed{x_2 = \frac{146}{33}}$ $\quad x_2 = 4.42$

(7) $\Rightarrow x_1 = 10 - \frac{292}{33} \Rightarrow \boxed{x_1 = \frac{38}{33}}$ $\quad x_1 = 1.15$

these values satisfy equations (5) and (6) $\quad$ also $\lambda_1 = -10.12 < 0$

$f = 113.83$

case (iii) $\lambda_1 = 0,\ \lambda_2 \neq 0$

(1) $\Rightarrow 14x_1 + 6 + \lambda_2 = 0 \Rightarrow \lambda_2 = -14x_1 - 6$

(2) $\Rightarrow 10x_2 - 3\lambda_2 = 0 \Rightarrow \lambda_2 = \frac{10}{3}x_2$

$\Rightarrow -14x_1 - 6 = \frac{10}{3}x_2 \Rightarrow -42x_1 - 10x_2 = 18$ — (9)

(4) $\Rightarrow x_1 - 3x_2 - 9 = 0 \Rightarrow x_1 = 3x_2 + 9$ — (10)

using (10) in (9), $-42(3x_2 + 9) - 10x_2 = 18 \Rightarrow -136x_2 = 396 \Rightarrow \boxed{x_2 = -7.91}$

(10) $\Rightarrow \boxed{x_1 = 0.2647}$ $\quad$ not feasible

case (iv) $\lambda_1 \neq 0,\ \lambda_2 \neq 0$

(3) $\Rightarrow x_1 + 2x_2 = 10$ — (11)

(4) $\Rightarrow x_1 - 3x_2 = 9$ — (12)

(11) − (12) $\Rightarrow 5x_2 = 1 \Rightarrow \boxed{x_2 = \frac{1}{5}}$

(12) $\Rightarrow x_1 = 9 + 3\left(\frac{1}{5}\right) \Rightarrow \boxed{x_1 = \frac{48}{5}}$

above values satisfies all equations. $\quad$ also $\lambda_1 + \lambda_2 = -140.4$

$2\lambda_1 - 3\lambda_2 = -2$

$\Rightarrow \lambda_1 = -84.64 < 0$

$\lambda_2 = -55.76 < 0$

at $x_1 = \frac{38}{33}$ and $x_2 = \frac{146}{33}$

$z_{max} =$

at $x_2 = \frac{1}{5}$ and $x_1 = \frac{48}{5}$, $\quad z_{max} = 702.96$

Ex Using Kuhn-Tucker conditions Minimize $z = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$ subject to $x_1 + 3x_2 \le 6$, $5x_1 + 2x_2 \le 10$, $x_1, x_2 \ge 0$

→ EC-10 M

Solution:-

Let $f = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$, $\quad h_1 = x_1 + 3x_2 - 6$, $\quad h_2 = 5x_1 + 2x_2 - 10$

$L = f + \lambda_1 h_1 + \lambda_2 h_2 \Rightarrow L = 2x_1 + 3x_2 - x_1^2 - 2x_2^2 + \lambda_1(x_1 + 3x_2 - 6) + \lambda_2(5x_1 + 2x_2 - 10)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2 - 2x_1 + \lambda_1 + 5\lambda_2 = 0$ — (1)

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 3 - 4x_2 + 3\lambda_1 + 2\lambda_2 = 0$ — (2)

$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + 3x_2 - 6) = 0$ — (3)

$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(5x_1 + 2x_2 - 10) = 0$ — (4)

$h_1 \le 0 \Rightarrow x_1 + 3x_2 - 6 \le 0$ — (5)

$h_2 \le 0 \Rightarrow 5x_1 + 2x_2 - 10 \le 0$ — (6)

case (ii) $\lambda_1 \neq 0,\ \lambda_2 = 0$

(3) $\Rightarrow x_1 + 2x_2 - 10 = 0$

i.e. $x_1 + 2x_2 = 10$ — (7)

(1) $\Rightarrow 14x_1 + 6 + \lambda_1 = 0 \Rightarrow \lambda_1 = -6 - 14x_1$

(2) $\Rightarrow 10x_2 + 2\lambda_1 = 0 \Rightarrow \lambda_1 = -5x_2$

$\therefore -6 - 14x_1 = -5x_2$

$\therefore -14x_1 + 5x_2 = 6$ — (8)

14(7) + (8) $\Rightarrow 14x_1 + 28x_2 - 14x_1 + 5x_2 = 140 + 6$

$\Rightarrow 33x_2 = 146 \Rightarrow \boxed{x_2 = \frac{146}{33}}$ $\quad x_2 = 4.42$

(7) $\Rightarrow x_1 = 10 - \frac{292}{33} \Rightarrow \boxed{x_1 = \frac{38}{33}}$ $\quad x_1 = 1.15$

these values satisfy equations (5) and (6)

$f = 113.83$

also $\lambda_1 = -10.12 < 0$

case (iii) $\lambda_1 = 0,\ \lambda_2 \neq 0$

(1) $\Rightarrow 14x_1 + 6 + \lambda_2 = 0 \Rightarrow \lambda_2 = -14x_1 - 6$

(2) $\Rightarrow 10x_2 - 3\lambda_2 = 0 \Rightarrow \lambda_2 = \frac{10}{3}x_2$

$\Rightarrow -14x_1 - 6 = \frac{10}{3}x_2 \Rightarrow -42x_1 - 10x_2 = 18$ — (9)

(4) $\Rightarrow x_1 - 3x_2 - 9 = 0 \Rightarrow x_1 = 3x_2 + 9$ — (10)

using (10) in (9), $-42(3x_2 + 9) - 10x_2 = 18 \Rightarrow -136x_2 = 396 \Rightarrow \boxed{x_2 = -2.91}$

(10) $\Rightarrow \boxed{x_1 = 0.2647}$

not feasible

case (iv) $\lambda_1 \neq 0,\ \lambda_2 \neq 0$

(3) $\Rightarrow x_1 + 2x_2 = 10$ — (11)

(4) $\Rightarrow x_1 - 3x_2 = 9$ — (12)

(11) − (12) $\Rightarrow 5x_2 = 1 \Rightarrow \boxed{x_2 = \frac{1}{5}}$

(12) $\Rightarrow x_1 = 9 + 3\left(\frac{1}{5}\right) \Rightarrow \boxed{x_1 = \frac{48}{5}}$

above values satisfies all equations.

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Ex Using Kuhn-Tucker conditions Minimize $z = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$ subject to $x_1 + 3x_2 \le 6$, $5x_1 + 2x_2 \le 10$, $x_1, x_2 \ge 0$

— EC-10 M

Solution:-

Let $f = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$, $h_1 = x_1 + 3x_2 - 6$, $h_2 = 5x_1 + 2x_2 - 10$

$L = f + \lambda_1 h_1 + \lambda_2 h_2 \Rightarrow L = 2x_1 + 3x_2 - x_1^2 - 2x_2^2 + \lambda_1(x_1 + 3x_2 - 6) + \lambda_2(5x_1 + 2x_2 - 10)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2 - 2x_1 + \lambda_1 + 5\lambda_2 = 0$ — (1)

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 3 - 4x_2 + 3\lambda_1 + 2\lambda_2 = 0$ — (2)

$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + 3x_2 - 6) = 0$ — (3)

$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(5x_1 + 2x_2 - 10) = 0$ — (4)

$h_1 \le 0 \Rightarrow x_1 + 3x_2 - 6 \le 0$ — (5)

$h_2 \le 0 \Rightarrow 5x_1 + 2x_2 - 10 \le 0$ — (6)

case (i) $\lambda_1 = 0,\ \lambda_2 = 0$

(1) $\Rightarrow 2 - 2x_2 = 0 \Rightarrow \boxed{x_2 = 1}$

(2) $\Rightarrow 3 - 4x_2 = 0 \Rightarrow \boxed{x_2 = \frac{3}{4}}$

these values satisfies eqn (5) and (6)

$\because\ 1 + \frac{9}{4} - 6 \le 0$, and $5 + \frac{3}{2} - 10 \le 0$

and $z = 2 + \frac{9}{4} - 1 - \frac{9}{8}$ $H = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$

$\Delta_1 = -2,\ \Delta_2 = 8$

$\boxed{z = \frac{17}{8}}$ $\boxed{= 2.125}$ $z_{max} = 2.125$

case (ii)

$\lambda_1 = 0,\ \lambda_2 \ne 0$

(4) $\Rightarrow 5x_1 + 2x_2 = 10$ — (7)

(1) $\Rightarrow 2 - 2x_1 + 5\lambda_2 = 0 \Rightarrow 5\lambda_2 = 2x_1 - 2$

(2) $\Rightarrow 3 - 4x_2 + 2\lambda_2 = 0 \Rightarrow \lambda_2 = 2x_2 - \frac{3}{2}$

$\therefore 5(2x_2 - \frac{3}{2}) = 2x_1 - 2$

$\Rightarrow -2x_1 + 10x_2 = \frac{11}{2}$ — (8)

Solving (7), (8) we get

$\boxed{x_1 = \frac{89}{54}}$ $\boxed{x_2 = \frac{95}{108}}$, $\lambda_2 = 0.2870$

thes values satisfies (5) (6)

$\Rightarrow$ minimum value

$\boxed{\cancel{z = 1.65}}$ $z_{min} = 1.65$

case (iii) $\lambda_1 \ne 0,\ \lambda_2 = 0$

(3) $\Rightarrow x_1 + 3x_2 = 6$ — (9)

(1) $\Rightarrow 2 - 2x_1 + \lambda_1 = 0 \Rightarrow \lambda_1 = 2x_1 - 2$

(2) $\Rightarrow 3 - 4x_2 + 3\lambda_1 = 0 \Rightarrow 3\lambda_1 = 4x_2 - 3$

$\therefore 3(2x_1 - 2) = 4x_2 - 3$

$6x_1 - 4x_2 = 3$ — (10)

solving (9), (10) we get,

$x_1 = \frac{3}{2},\quad x_2 = \frac{3}{2}$

these values does not satisfies eqn (6)

$\because\ \frac{15}{2} + 3 - 10 \nleq 0$

case fails

case (iv)

$\lambda_1 \ne 0,\ \lambda_2 \ne 0$

(3) $\Rightarrow x_1 + 3x_2 = 6$ — (11)

(4) $\Rightarrow 5x_1 + 2x_2 = 10$ — (12)

Solving $\boxed{x_1 = \frac{18}{13}}$, $\boxed{x_2 = \frac{20}{13}}$

thesevalues satisfies eqn (5), (6)

$\because\ \frac{18}{13} + \frac{60}{13} - 6 \le 0$, $\frac{90}{13} + \frac{40}{13} - 10 \le 0$

and $\lambda_1 + 5\lambda_2 = -\frac{10}{13}$, $3\lambda_1 + 2\lambda_2 = -\frac{41}{13}$, $\lambda_1 > 0$, $\lambda_2 < 0$

Fails

$\therefore$ required solution, $x_1 = \frac{89}{54}$ $x_2 = \frac{95}{108}$, $z_{min} = 1.65$

Ex Using Kuhn-Tucker conditions, solve the Following NLPP

Maximize $z = x_1^2 + x_2^2$

s.t. $x_1 + x_2 - 4 \le 0,\quad 2x_1 + x_2 - 5 \le 0$ {Dec-2018}

Soln: Let $f = x_1^2 + x_2^2$, $h_1 = x_1 + x_2 - 4$, $h_2 = 2x_1 + x_2 - 5$

and $L = f + \lambda_1 h_1 + \lambda_2 h_2$

$\Rightarrow L = x_1^2 + x_2^2 + \lambda_1(x_1 + x_2 - 4) + \lambda_2(2x_1 + x_2 - 5)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 + \lambda_1 + 2\lambda_2 = 0 \longrightarrow (1)$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 + \lambda_1 + \lambda_2 = 0 \longrightarrow (2)$

$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + x_2 - 4) = 0 \longrightarrow (3)$ $h_1 \le 0 \Rightarrow x_1 + x_2 - 4 \le 0 \rightarrow (5)$

$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(2x_1 + x_2 - 5) = 0 \longrightarrow (4)$ $h_2 \le 0 \Rightarrow 2x_1 + x_2 - 5 \le 0 \rightarrow (6)$

case (i) $\lambda_1 = 0,\ \lambda_2 = 0$ (1) $\Rightarrow x_1 = 0,\ x_2 = 0$ trivial solution. case fails

case (ii) $\lambda_1 = 0,\ \lambda_2 \ne 0$, (4) $\Rightarrow 2x_1 + x_2 = 5 \rightarrow (7)$

(1) gives $\lambda_2 = -x_1$, (2) gives $\lambda_2 = -2x_2$ $\Big\} \Rightarrow -x_1 = -2x_2 \Rightarrow x_1 = 2x_2$, (5) $\Rightarrow 5x_2 = 5 \Rightarrow$

$\Rightarrow x_2 = 1,\ x_1 = 2$

(1) $\Rightarrow \lambda_2 = -2 < 0 \Rightarrow z$ is max value satisfies all conditions

$z_{max} = 5$

case (iii) $\lambda_1 \neq 0, \lambda_2 = 0$ (3) $\Rightarrow x_1 + x_2 = 4 \rightarrow$ (8)

(1) $\Rightarrow \lambda_1 = -2x_1$ } $\Rightarrow x_1 = x_2$ (8) $\Rightarrow$ $\boxed{x_2 = 2}$, $\boxed{x_1 = 2}$

(2) $\Rightarrow \lambda_1 = -2x_2$ } these values does not satisfy eqn (6)

$\therefore$ case fails.

case (iv) $\lambda_1 \neq 0, \lambda_2 \neq 0$

(3) $\Rightarrow x_1 + x_2 = 4 \rightarrow$ (9) , } solving, $x_1 = 1, \; x_2 = 3$

(4) $\Rightarrow 2x_1 + x_2 = 5 \rightarrow$ (10)

these values satisfies (5), (6)

Now ~~$\lambda_1 = 2$~~ using this in (1), (2)

$\lambda_1 + 2\lambda_2 = -2$ } solving $\lambda_2 = 4 > 0$, $\lambda_1 = -10 < 0$

$\lambda_1 + \lambda_2 = -6$ opposite sign case fails.

$\therefore$ Solution is $x_1 = 2, \; x_2 = 1, \; z_{max} = 5$

Ex Use the kuhn-Tucker conditions to solve

$z_{max} = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$

subject to, $x_1 + 3x_2 \leq 6$

$5x_1 + 2x_2 \leq 10, \quad x_1, x_2 \geq 0$ [97, 99, 03, 10]

Sol^n:

$L = z + \lambda_1 h_1 + \lambda_2 h_2$ gives, $h_1 = x_1 + 3x_2 - 6, \; h_2 = 5x_1 + 2x_2 - 10$

$L = 2x_1 + 3x_2 - x_1^2 - 2x_2^2 + \lambda_1(x_1 + 3x_2 - 6) + \lambda_2(5x_1 + 2x_2 - 10)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2 - 2x_1 + \lambda_1 + 5\lambda_2 = 0 \longrightarrow$ (1)

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 3 - 4x_2 + 3\lambda_1 + 2\lambda_2 = 0 \longrightarrow$ (2)

$\lambda_1 h_1 = 0 \Rightarrow \lambda_1(x_1 + 3x_2 - 6) = 0 \longrightarrow$ (3)

$\lambda_2 h_2 = 0 \Rightarrow \lambda_2(5x_1 + 2x_2 - 10) = 0 \longrightarrow$ (4)

$h_1 \leq 0 \Rightarrow x_1 + 3x_2 - 6 \leq 0 \longrightarrow$ (5)

$h_2 \leq 0 \Rightarrow 5x_1 + 2x_2 - 10 \leq 0 \longrightarrow$ (6)

case (i) $\lambda_1 = 0, \lambda_2 = 0$

(1) $\Rightarrow x_1 = 1$, ~~x_2~~ (2) $\Rightarrow x_2 = 3/4$

these values satisfies all eqns i.e. (5), (6)

[But these values can give minima as for minima $\lambda_1, \lambda_2 \geq 0$] equality included.

Hence consider Hessian matrix of objective function,

$$H = \begin{bmatrix} z_{x_1x_1} & z_{x_1x_2} \\ z_{x_2x_1} & z_{x_2x_2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$$

$\Delta_1 = |-2| = -2$

$\Delta_2 = \begin{vmatrix} -2 & 0 \\ 0 & -4 \end{vmatrix} = 8$

8

signs are alternate. $\Rightarrow$ Z is max at $x_1=1$, $x_2=3/4$

$Z_{max}=17/8$

case (ii) $\lambda_1=0$, $\lambda_2\neq0$ gives $4x_1-20x_2=-11$

$5x_1+2x_2=10$

$\Rightarrow x_1=1.648$, $x_2=0.880$

these values gives $\lambda_2=$ ~~-1.296~~. $\lambda_2=1.296>0$

$\Rightarrow$ case fails

case (iii) $\lambda_1\neq0$, and $\lambda_2=0$

$$\left.\begin{array}{l}6x_1-4x_2=3\\x_1+3x_2=6\end{array}\right\}\Rightarrow \quad x_1=3/2,\ x_2=3/2$$

do not satisfy (6) $\Rightarrow$ case Fails.

case (iv) $\lambda_1\neq0$, $\lambda_2\neq0$

$x_1+3x_2=6$, $5x_1+2x_2=10$

gives $x_1=1.538$, $x_2=1.385$

these values satisfy all constraints.

But $\lambda_1=0.81>0$, $\lambda_2=0.053>0$

case fails.