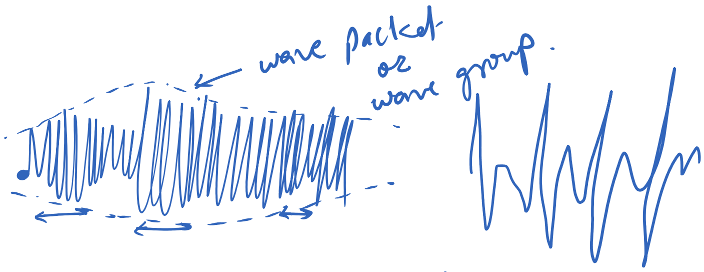
Matter waves



velocity of particle $\Rightarrow$ particle velocity
velocity of wave $\Rightarrow$ wave velocity or phase velocity.
velocity of wave packet $\Rightarrow$ Packet velocity or group velocity.

Properties of Matter waves

1. Lighter particle, greater is wavelength.
2. Lesser velocity of particle, greater is wavelength.
3. Matter waves are generated by motion of particles. (charged n uncharged!!!) matter waves are not *electromagnetic*.
4. Velocity of matter wave depends on velocity of material.
5. Wave nature of moving matter introduces uncertainty in position of particle.
6. Matter waves are characterized by quantity called *wave function* and denoted by letter

HW: Differentiate between matter waves and electromagnetic waves?

De-Broglie hypothesis:

$$E = h\nu = \frac{hc}{\lambda} \quad \text{— (1)}$$

$$E = mc^2 \quad \text{— (2)}$$

from eq^n (1) & (2)

$$mc^2 = \frac{hc}{\lambda}.$$

$$mc = \frac{h}{\lambda}$$

$$mc = \frac{h}{\lambda}$$

$$\lambda = \frac{h}{mc}$$

{mass × velocity = momentum}

$$\boxed{\lambda = \frac{h}{p}} \rightarrow \text{De-Broglie wavelength.}$$

Where, λ = De-Broglie wavelength

h = plank's const.

p = momentum.

*) De-Broglie wavelength in terms of K.E

$$K.E = \frac{1}{2}mv^2$$

$$E = \frac{1}{2}mv^2$$

$$= \frac{1}{2}mv^2 \times \frac{m}{m}$$

$$= \frac{m^2v^2}{2m}$$

$$= \frac{p^2}{2m}$$

$$p^2 = 2mE$$

$$p = \sqrt{2mE}$$

then from De-broglie wavelength

$$\lambda = \frac{h}{p}$$

$$\boxed{\lambda = \frac{h}{\sqrt{2mE}}}$$

*) De-Broglie wavelength in terms of accelerating potential energy.

$$K.E = P.E$$

$$\frac{1}{2}mv^2 = eV$$

$$\frac{1}{2}mv^2 \times \frac{m}{m} = eV$$

$$\frac{p^2}{2m} = eV$$

$$p^2 = 2m_e eV$$

{ $e = q$, $eV = qV$ }

$$p^2 = 2m_e qV$$

$$p = \sqrt{2mqV}$$

then De-Broglie wavelength

$$\lambda = \frac{h}{p}.$$

$$\boxed{\lambda = \frac{h}{\sqrt{2m_e qV}}}$$

<u>phase velocity</u> & <u>group velocity</u>

$V = n\lambda \rightarrow$ wavelength.

velocity, frequency

$$\boxed{V_{ph} = \nu\lambda} \quad -\ ①$$

from eq^n ① to 3.

$$V_{ph} = \frac{mc^2}{h} \times \frac{h}{mv_p} \quad \{p = mv_p$$

$$\boxed{V_{ph} = \nu\lambda} \quad -①$$

$E = mc^2$

$E = h\nu$

$h\nu = mc^2$

$$\boxed{\nu = \frac{mc^2}{h}} \quad -②$$

from De-Broglie hypothesis

$$\boxed{\lambda = \frac{h}{p}} \quad -③$$

$V_{ph} = \frac{mc^2}{h} \times \frac{h}{m v_p}$

$$\boxed{V_{ph} = \frac{c^2}{v_p}} \Rightarrow v_p \times V_{ph} = c^2$$

$V_{ph} > c$

phase velocity

Uncertainty in the velocity (ΔV)

Uncertainty in the position (Δx)

⇒ Uncertainty principle in 1D.

$\Delta x \cdot \Delta P_x \geq \hbar$ $\quad \left\{ \hbar = \frac{h}{2\pi} \right\}$

$\Delta y \cdot \Delta P_y \geq \hbar$

$\Delta z \cdot \Delta P_z \geq \hbar$ position - momentum relation

$\Delta E \cdot \Delta t \geq \hbar$ Energy - time relation

$\Delta L \cdot \Delta \theta \geq \hbar$ Linear momentum - angular momentum relation.

Kinetic energy

$K \cdot E = \frac{1}{2} m v^2$ $\quad \left\{ \frac{\Delta E}{\Delta V} = mV \right.$

$E = \frac{1}{2} m v^2$

Diff^ate w.r.to v

$\Delta E = m V \Delta V$

$= m V_x \frac{\Delta x}{\Delta t}$

$\Delta E \cdot \Delta t = \Delta P_x \, \Delta x$

$\Delta E \cdot \Delta t = \Delta x \cdot \Delta P_x$

But $\Delta x \cdot \Delta P_x \geq \hbar$

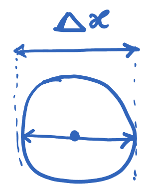
$$\boxed{\Delta E \cdot \Delta t = \Delta x \cdot \Delta P_x \geq \hbar}$$

Applications of uncertainty principle

1) Electron can not exist with in nucleus.

Δx

1) Electron can not exist with in nucleus.



Radius of nucleus is 5×10^{-15} m

$\Delta x = 2 \times 5 \times 10^{-15}$ m

$\Delta x \cdot \Delta P_x \geq \hbar$ $\left\{ \hbar = \frac{h}{2\pi} \right\}$

$\Delta P_x \geq \frac{\hbar}{\Delta x}$

$\geq \frac{h}{2\pi \Delta x}$

$\left\{ \begin{array}{l} h = 6.63 \times 10^{-34} \text{ Js} \\ \pi = 3.14 \\ \Delta x = 5 \times 2 \times 10^{-15} \text{ m} \end{array} \right.$

$\geq 1 \times 10^{-20}$ kg m/s ✓

$K \cdot E = mc^2 = (mc) \times c$

$= Pc$

$= Pc \geq 1 \times 10^{-20} \times 3 \times 10^{8}$

$E \geq 1 \times 10^{-20} \times 3 \times 10^{8}$ J

$\geq 3 \times 10^{-12}$ J

$\geq \frac{3 \times 10^{-12}}{1.6 \times 10^{-19}}$ eV

$\geq \frac{3}{1.6} \times 10^{7}$ eV

$\geq 1.875 \times 10^{7}$ eV

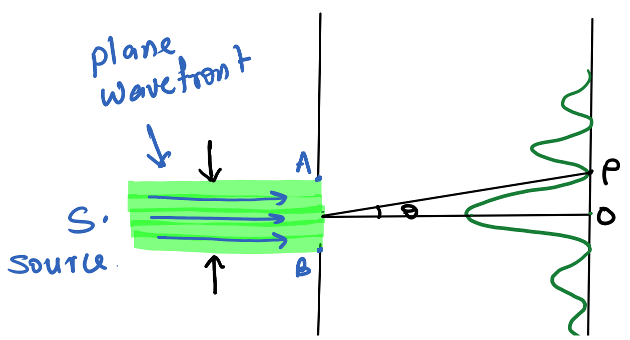
$\geq 19 \times 10^{6}$ eV

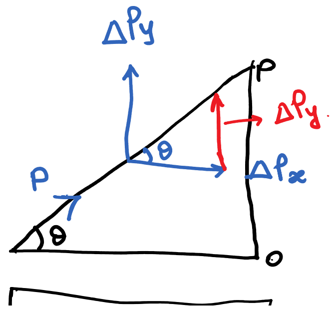
$E \geq 19$ MeV

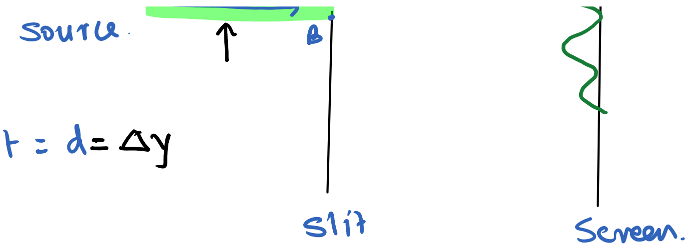
K.E of $e^{\ominus}$ must be greater than 19 MeV if it is remain present inside the nucleus. But it is not possible.

Hence, Electron can not exist with in nucleus.

2) Single slit diffraction using uncertainty principle.

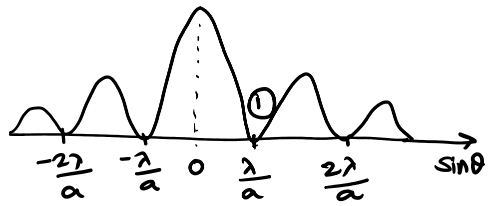






AB = width slit = d = Δy

$$\boxed{\sin\theta = \frac{\Delta P_y}{P}}$$



a = width slit

$$\sin\theta = \frac{\lambda}{a} = \frac{\lambda}{d} = \frac{\lambda}{\Delta y}$$

$$\boxed{\sin\theta = \frac{\lambda}{\Delta y}}$$

$$\boxed{a\sin\theta = n\lambda}$$

$$\sin\theta = \frac{n\lambda}{a} \quad n = \pm 0, 1, 2$$

$$\sin\theta = \frac{\lambda}{\Delta y} \quad -(1)$$

from diagram.

$\therefore$ $d = \Delta y$.

$$\therefore \boxed{\Delta y = \frac{\lambda}{\sin\theta}} \quad -(A)$$

$$\sin\theta = \frac{\Delta P_y}{P}$$

$$\boxed{\Delta P_y = P\sin\theta} \quad -(B)$$

from (A) & (B)

$$\Delta y \cdot \Delta P_y = P\sin\theta \times \frac{\lambda}{\sin\theta}$$

$$= P\lambda$$

Using De-broglie wavelength

$$\lambda = \frac{h}{P}$$

$$\Delta y \cdot \Delta P_y = P \cdot \frac{h}{P} = h$$

$$\Delta y \cdot \Delta P_y = h$$

$$\Delta y \cdot \Delta P_y > \frac{h}{2\pi}$$

$$\boxed{\Delta y \cdot \Delta P_y \geq h}$$

Imp

One dimensional schrodinger's time dependent equation

We know that

$$TE = K.E + P.E$$

$$E = \frac{1}{2}mv^2 + V(x)$$

$$E = \frac{1}{2}\frac{m^2v^2}{m} + V(x)$$

$$E = \frac{p^2}{2m} + V(x) \quad — (1)$$

wave function $\psi(x,t)$ is given as

$$\psi(x,t) = A\, e^{-i(Et - px)/\hbar} \quad — (2)$$

$$\boxed{E\,\psi(x,t) = \frac{p^2}{2m}\psi(x,t) + V(x)} \quad — (3)$$

Diff^{ate} eq^n (2) w.r.to 't'

$$\frac{\partial \psi(x,t)}{\partial t} = A\, e^{-i(Et-px)/\hbar}\left(-\frac{iE}{\hbar}\right)$$

$$= \psi(x,t)\,(-iE/\hbar)$$

$$\frac{\partial \psi(x,t)}{\partial t} = \frac{-i}{\hbar}\, E\,\psi(x,t)$$

$$\left\{ \frac{-i}{\hbar} = \frac{-i \times i}{i \times \hbar} = \frac{-i^2}{i\hbar} = \frac{1}{i\hbar} \right.$$

$$i^2 = -1$$

$$\boxed{E\,\psi(x,t) = i\hbar\,\frac{\partial \psi(x,t)}{\partial t}} \quad — (4)$$

Diff^{ate} eq^n (2) w.r.to x

$$\psi(x,t) = A\, e^{-i(Et-px)/\hbar}$$

$$\frac{\partial \psi(x,t)}{\partial x} = A\, e^{-i(Et-px)/\hbar}\left(\frac{ip}{\hbar}\right)$$

$$\frac{\partial \psi(x,t)}{\partial x} = \left(\frac{ip}{\hbar}\right) A\, e^{-i(Et-px)/\hbar}$$

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \left(\frac{ip}{\hbar}\right) A\, e^{-i(Et-px)/\hbar}\left(\frac{ip}{\hbar}\right)$$

$$= \left(\frac{i^2 p^2}{\hbar^2}\right)\psi(x,t)$$

$$\left\{ i^2 = -1 \right.$$

$$= \frac{-1}{\hbar^2}\, p^2\,\psi(x,t).$$

$$\boxed{p^2\,\psi(x,t) = -\hbar^2\,\frac{\partial^2 \psi(x,t)}{\partial x^2}} \quad — (5)$$

from eq^n (3), (4) & (5)

$$\boxed{i\hbar\,\frac{\partial \psi(x,t)}{\partial t} = \frac{-\hbar^2}{2m}\,\frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\,\psi(x,t)} \quad — (A)$$

... one dimensional schrodinger's time dependent

Above eq^n is known as one dimensional schrodinger's time dependent equation.

*) One dimensional schrodinger's time independent equation.

Reduce schrodiger's time dependent equation to independent form.

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\,\psi(x,t) \quad - (A)$$

$$\psi(x,t) = \psi(x)\,\phi(t) \quad - (1)$$

Then eq^n (A) becomes.

$$i\hbar \frac{\partial \psi(x)\phi(t)}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi(x)\phi(t)}{\partial x^2} + V(x)\,\psi(x)\,\phi(t)$$

$$i\hbar\, \psi(x) \frac{\partial \phi(t)}{\partial t} = \frac{-\hbar^2}{2m}\, \phi(t) \frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\,\psi(x)\,\phi(t) \quad - (2)$$

Divide eq^n (2) by $\psi(x)\,\phi(t)$

$$i\hbar \frac{\cancel{\psi(x)}}{\cancel{\psi(x)}\phi(t)} \frac{\partial \phi(t)}{\partial t} = \frac{-\hbar^2}{2m} \frac{\cancel{\phi(t)}}{\psi(x)\cancel{\phi(t)}} \frac{\partial^2 \psi(x)}{\partial x^2} + \frac{V(x)\,\psi(x)\,\phi(t)}{\psi(x)\,\phi(t)}$$

$$\underbrace{\frac{i\hbar}{\phi(t)} \frac{\partial \phi(t)}{\partial t}} = -\frac{\hbar^2}{2m} \times \frac{1}{\psi(x)} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x)$$

$\frac{i\hbar}{\phi(t)} \frac{\partial \phi(t)}{\partial t}$ will be const w.r.to x

$$E = \frac{-\hbar^2}{2m} \frac{1}{\psi(x)} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x). \quad - (3)$$

Multiply eq^n (3) by $\psi(x)$

$$\boxed{E\psi(x) = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\,\psi(x)} \quad - (B)$$

Above equation is known as one dimensional time independent equation.

from eq^n (B)

$$\frac{-\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\,\psi(x) = E\,\psi(x)$$

$$\therefore \left[\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x) = E\,\psi(x)$$ → Eigen

$$\therefore \underbrace{\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]}_{\text{Hamiltonia operator}} \underset{\text{Eigen function}}{\psi(x)} = \underset{\text{Eigen value}}{E}\, \psi(x) \rightarrow \text{Eigen function.}$$

*) Applications of schrodinger's equation.

Consider schrodinger's time independent equation

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$$

$$\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} - V(x)\psi(x) = -E\psi(x)$$

$$\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + (E - V(x))\psi(x) = 0$$

$$\frac{\partial^2 \psi(x)}{\partial x^2} + \frac{2m}{\hbar^2}(E - V(x))\psi(x) = 0 \qquad \left\{ \hbar = \frac{h}{2\pi} \right.$$

$$\boxed{\frac{\partial^2 \psi(x)}{\partial x^2} + \frac{8\pi^2 m}{h^2}(E - V(x))\psi(x) = 0}$$

A) Motion of free particle

Free particle means no force acted on particle. i.e PE is zero

$\Rightarrow V(x) = 0$

$$\frac{\partial^2 \psi(x)}{\partial x^2} + \frac{8\pi^2 m}{h^2} E\psi(x) = 0$$

$$\frac{\partial^2 \psi(x)}{\partial x^2} + K^2\psi(x) = 0$$

Where. $K^2 = \frac{8\pi^2 m}{h^2} E$

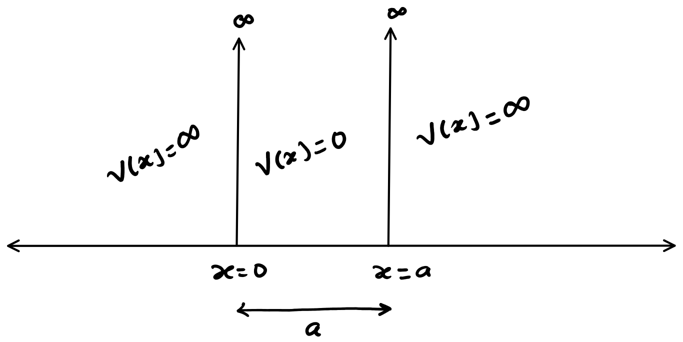
$$E = \frac{h^2}{8\pi^2 m} K^2$$

$$E = (\text{const})\, K^2$$

$$\boxed{E \propto K^2}$$

2) Particle in one dimensional potential well of infinite height

B) Particle in one dimensional potential well of infinite height
or
Particle in sq. well potential.



from fig.

$$V(x) = 0 \qquad 0<x<a$$
$$= \infty \qquad a<x<0$$

$$\frac{\partial^2 \psi(x)}{\partial x^2} + \frac{8\pi^2 m}{h^2}(E - v(x))\psi(x) = 0$$

With in potential well $v(x)=0$

$$\frac{\partial^2 \psi(x)}{\partial x^2} + \frac{8\pi^2 m}{h^2} E\psi(x) = 0$$

$$\boxed{\frac{\partial^2 \psi(x)}{\partial x^2} + K^2\psi(x) = 0} \quad \text{— (A)} \qquad \left\{ K^2 = \frac{8\pi^2 m}{h^2} E \right.$$

solution of eq^n (A) will be

$$\psi(x) = A\cos Kx + B\sin Kx \quad \text{— (B)}$$

i) $x=0 \qquad \psi(x)=0$

$$0 = A\cos 0^\circ + B\sin 0$$

$$\boxed{A=0}$$

ii) $x=a$, $\psi(x)=0$, $A=0$

$$0 = 0\cos ka + B\sin ka$$

$$\sin ka = 0$$

$$ka = n\pi \qquad \ldots n=0,1,2,3\ldots$$

$$k^2a^2 = n^2\pi^2$$

$$\frac{8\pi^2 mE}{h^2} a^2 = n^2\pi^2$$

$$E = \frac{h^2\pi^2 n^2}{8\pi^2 ma^2}$$

$$E = \left(\frac{h^2\pi^2}{8\pi^2 ma^2}\right) n^2$$

$$E = \frac{h^2}{8ma^2} n^2$$

$$\left\{ \frac{h^2}{8ma^2} = \text{constant} \right.$$

$$E = (\text{const})\, n^2$$

$$\boxed{E \propto n^2}$$

1) An $e^{\ominus}$ has a speed of 400 m/s with uncertainty of 0.01%. find accuracy in it's position.

→ Given data

$V = 400 m/s$

$\frac{\Delta V}{V} = 0.01\% = \frac{0.01}{100}$

$\Delta x = ?$

$m_e = 9.1 \times 10^{-31} kg$

formula

$\Delta x \cdot \Delta P_x \geq \hbar$

$\Delta x \cdot \Delta P_x \geq \frac{h}{2\pi}$

$\therefore \Delta x \geq \frac{h}{2\pi \times \Delta P_x}$

$\Delta x \geq \frac{6.63 \times 10^{-34}}{2 \times 3.14 \times 3.644 \times 10^{-32}}$

$$\boxed{\Delta x \geq 2.8971 \times 10^{-3} m}$$

$P = mV$

$= 9.1 \times 10^{-31} \times 400$

$= 3.644 \times 10^{-28} kg\, m/s$

$\Delta P_x = m \Delta V$

$= mV \frac{\Delta V}{V} = P \times \frac{\Delta V}{V}$

$= 3.644 \times 10^{-28} \times \frac{0.01}{100}$

$\Delta P_x = 3.644 \times 10^{-32} kg\, m/s$

2) Calculate De-broglie wavelength associated with α-particle, accelerated by potential difference of 200V ($m_\alpha = 6.68 \times 10^{-27} kg$)

→ Given data

$\lambda = ?$

α-particle

charge on particle $q = 2e$

$V = 200V$

$m_\alpha = 6.68 \times 10^{-27} kg$

$h = 6.63 \times 10^{-34} Js$

formula

$\lambda = \frac{h}{\sqrt{2 m_\alpha q V}}$

$= \frac{h}{\sqrt{4 \times m_\alpha e V}} = \frac{6.63 \times 10^{-34}}{\sqrt{4 \times 6.68 \times 10^{-27} \times 1.6 \times 10^{-19} \times 200}}$

$\lambda = 7.17 \times 10^{-13} m = 0.00717 Å$

3) find the energy of neutron in eV whose de-broglie wavelength is 1 Å (mass of neutron is $1.674 \times 10^{-27} kg$)

→ Given data

$E = ?$

$\lambda = 1 Å$

$m_n = 1.674 \times 10^{-27} kg.$

formula

$\lambda = \frac{h}{\sqrt{2 m_n E}}$

$E = \frac{h^2}{2 m_n \lambda^2}$

$E = 1.31 \times 10^{-20} J$

$= \frac{1.31 \times 10^{-20}}{1.6 \times 10^{-19}} eV$

$\left\{ \begin{array}{l} E = \frac{1}{2} mv^2 \\ = \frac{p^2}{2m} \\ P = \sqrt{2mE} \end{array} \right.$

$$= \frac{1.31 \times 10}{1.6 \times 10^{-19}} eV$$

$$\boxed{E = 0.081 \, eV}$$

4) An electron is accelerated through 1000V & is reflected from crystal. The 1st order reflection occurs when glancing angle is 70°. Calculate interplanar spacing in crystal.

→ Given data

$\theta = 70°$

$V = 1000V$

$n = 1$

$d = ?$

$e = 1.6 \times 10^{-19} C$

$m_e = 9.1 \times 10^{-31} kg$

formula

$$2d \sin\theta = n\lambda$$

$$d = \frac{n\lambda}{2\sin\theta}$$

$$\boxed{d = 2.0672 \times 10^{-11} m}$$

$$\lambda = \frac{h}{\sqrt{2 m_e eV}}$$

$$= 0.388 \, A°$$

5) Find minimum energy of neutron confined to nucleus of size of the order of 10^{-14} m. ($m_n = 1.675 \times 10^{-27}$ kg)

→ Given data

$E = ?$

$\Delta x = 10^{-14} m$

$m_n = 1.675 \times 10^{-27} kg$

formula

$$\Delta x \cdot \Delta p_x \geq \frac{h}{2\pi}$$

$$\Delta p_x \geq \frac{h}{2\pi \times \Delta x}$$

$$\geq 1.056 \times 10^{-20}$$

$$K.E = \frac{1}{2} m v^2 = \frac{p^2}{2m}$$

$$E = \frac{p^2}{2m_n} = \frac{(1.056 \times 10^{-20})^2}{2 \times 1.675 \times 10^{-27} kg}$$

$$\boxed{E = 3.328 \times 10^{-14} J}$$

6) An e⁻ is bound in one dimensional potential well of width 2A° & at infinite height. Find it's energy in ground state & in 1st two excited state

→ Given data

$a = 2 A°$

$n = 1, 2, 3$

$E_1 = ?$

formula

$$E_n = \frac{n^2 h^2}{8 m a^2}$$

$$E_1 = \frac{1^2 h^2}{\ldots} = \frac{h^2}{\ldots} = \frac{(6.63 \times 10^{-34})^2}{\ldots \times 10^{-31} \ldots \times 10^{-10})^2} = 1.5 \times 10^{-18} J = 9.43 eV$$

$n = 1, 2, 3$

$E_1 = ?$

$E_2 = ?$

$E_3 = ?$

$E_n = \frac{n^2 h^2}{8ma^2}$

$$E_1 = \frac{1^2 h^2}{8ma^2} = \frac{h^2}{8ma^2} = \frac{(6.63\times10^{-34})^2}{8\times9.1\times10^{-31}\times(2\times10^{-10})^2} = 1.5\times10^{-18}\,J = 9.43\,eV$$

$$E_2 = \frac{2^2 h^2}{8ma^2} = 4E_1 = 6\times10^{-18}\,J = 37.72\,eV$$

$$E_3 = \frac{3^2 h^2}{8ma^2} = 9E_1 = 13.5\times10^{-18}\,J = 84.375\,eV$$

7) What is the wavelength of beam of neutron having energy 0.025 eV & mass 1.676×10^{-27} kg.

→ Given data

$\lambda = ?$

$E = 0.025\,eV$

$m_n = 1.676\times10^{-27}\,kg$

$h = 6.63\times10^{-34}\,Js$

formula

$$\lambda = \frac{h}{\sqrt{2m_nE}}$$

$$= \frac{6.63\times10^{-34}}{\sqrt{2\times1.676\times10^{-27}\times0.025\times1.6\times10^{-19}}}$$

$$\boxed{\lambda = 1.810\,A^\circ}$$