Complex Integration

Def^n :- Let $F(z) = u + iv$ be a continuous function of complex variable $z = x + iy$.

$\boxed{\int_C F(z)\,dz}$ is called complex integration along a path C.

Ex ① Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along a parabola $2y^2 = x$

Solution :-

$$I = \int_0^{2+i} (\bar{z})^2 dz = \int_0^{2+i} (x - iy)^2 (dx + idy)$$

$$= \int_0^{2+i} [(x^2 - y^2) - 2ixy](dx + idy)$$

$$I = \int_{(0,0)}^{(2,1)} [(x^2 - y^2)dx + 2xy\,dy] + i[(x^2 - y^2)dy - 2xy\,dx]$$

given path is $\boxed{x = 2y^2 \implies dx = 4y\,dy \implies \text{limits } y=0, y=1}$

$$\therefore I = \int_0^1 [(4y^4 - y^2)4y\,dy + 2y \cdot 2y^2 dy] + i[(4y^4 - y^2)dy - 2y\,2y^2\,4y\,dy]$$

$$= \int_0^1 (16y^5 - 4y^3 + 4y^3)dy + i(4y^4 - y^2 - 16y^4)dy$$

$$= \int_0^1 16y^5 dy + i(-y^2 - 12y^4)dy$$

$$= \left[\left(\tfrac{16}{6}\right)y^6 + i\left(-\tfrac{1}{3}y^3 - \tfrac{12}{5}y^5\right)\right]_{y=0}^{y=1} = \frac{16}{6} + i\left(-\frac{1}{3} - \frac{12}{5}\right) - 0$$

$$\implies \boxed{I = 8/3 - 41i/15}$$

Ex ② Integrate xz along a straight line from A(1,1) to B(2,4) in the complex plane.

Solution :-

$$I = \int_A^B xz\,dz = \int_{(1,1)}^{(2,4)} x(x + iy)(dx + idy)$$

$$= \int_{(1,1)}^{(2,4)} (x^2 + ixy)(dx + idy)$$

$$I = \int_{(1,1)}^{(2,4)} [x^2 dx - xy\,dy] + i[x^2 dy + xy\,dx] \longrightarrow ①$$

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Equation of a straight line joining (1,1) to (2,4) is

$\frac{x-2}{2-1} = \frac{y-4}{4-1} \rightarrow \frac{x-2}{1} = \frac{y-4}{3} \rightarrow 3x-6 = y-4$

$\Longrightarrow$ $\boxed{y = 3x-2 \Longrightarrow dy = 3dx \quad \text{limits } x=1 \text{ to } x=2}$

① $\Longrightarrow I = \int_1^2 [x^2 dx - x(3x-2)3dx] + i[x^2 3dx + x(3x-2)dx]$

$= \int_1^2 (x^2 - 9x^2 + 6x)dx + i(3x^2 + 3x^2 - 2x)dx$

$= \left[(-\frac{8}{3}x^3 + 3x^2) + i(2x^3 - x^2)\right]_{x=1}^{x=2}$

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$= [(-64/3 + 12) + i(12)] - [(-8/3 + 3) + i]$

$\Longrightarrow$ $\boxed{I = -29/3 + 11i}$

Ex ③ Integrate a function $F(z) = x^2 + ixy$ from $A(1,1)$ to $B(2,4)$ along the curve $x=t,\ y=t^2$

Solution:-

$I = \int_A^B F(z)\,dz = \int_{(1,1)}^{(2,4)} (x^2 + ixy)(dx + idy)$

$I = \int_{(1,1)}^{(2,4)} [x^2 dx - xy\,dy] + i[x^2 dy + xy\,dx] \longrightarrow$ ①

$\boxed{\text{given path } x=t,\ y=t^2 \Longrightarrow dx = dt,\ dy = 2t\,dt}$

Limits: at $A(1,1)$ $x=t \Longrightarrow t=1$

at $B(2,4)$, $x=t \Longrightarrow t=2$

$I = \int_1^2 (t^2 dt - t t^2 2t dt) + i(t^2 2t dt + t \cdot t^2 dt)$

$= \int_1^2 [(t^2 - 2t^4)dt + 3it^3 dt] = \left[\frac{1}{3}t^3 - \frac{2}{5}t^5 + 3i\frac{t^4}{4}\right]_{t=1}^{t=2}$

$= \left[(\frac{8}{3} - \frac{64}{5}) + 12i\right] - \left[(\frac{1}{3} - \frac{2}{5}) + \frac{3i}{4}\right]$

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$\Longrightarrow$ $\boxed{I = -151/15 + 45i/4}$

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Note :- Method-II

$x = t, \ y = t^2 \Longrightarrow$ $\boxed{y = x^2, \quad dy = 2x\,dx \quad \text{limits}, \ x=1, \ x=2}$

use it in eqn ① and evaluate.

Ex ④ Evaluate $\int_0^{3+i} z^2\,dz$

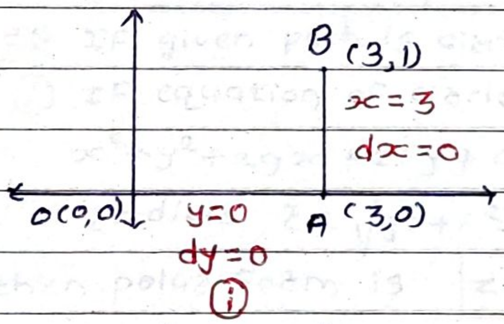
i) along the real line from 0 to 3 and then vertically to 3+i

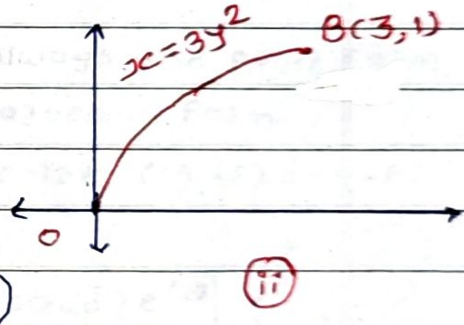
ii) along the parabola $x = 3y^2$

Solution :-

$$I = \int_0^{3+i} z^2\,dz = \int_{(0,0)}^{(3,1)} (x+iy)^2 (dx + i\,dy)$$

$$I = \int_{(0,0)}^{(3,1)} [(x^2-y^2) + 2ixy](dx + i\,dy) \longrightarrow ①$$





① $I = \int_{OA} + \int_{AB}$

$$= \int_{OA} [(x^2-y^2) + 2ixy](dx+i\,dy) + \int_{AB} [(x^2-y^2)+2ixy](dx+i\,dy)$$

$\boxed{y=0, \ dy=0}$ $\qquad$ $\boxed{x=3, \ dx=0}$

$$= \int_0^3 x^2\,dx + \int_0^1 [(9-y^2)+6iy]\,i\,dy$$

$$= \left[x^3/3\right]_0^3 + i\left[9y - \frac{y^3}{3} + 3iy^2\right]_0^1$$

$$= 9 - 0 + i\left[(9 - \tfrac{1}{3}) + 3i\right] - 0 \qquad = 9 + \frac{26i}{3} - 3$$

$$\Longrightarrow \boxed{I = 6 + \frac{26}{3}i}$$

(ii) given $\boxed{x = 3y^2 \Longrightarrow dx = 6y\,dy \text{ , limits } y=0, \text{ to } y=1}$

$\therefore$ (1) $\Longrightarrow I = \int_0^1 [(x^2-y^2)+2ixy](dx+idy)$

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$= \int_0^1 [(9y^4-y^2)+2i3y^2y](6y\,dy+idy)$

$= \int_0^1 (54\;y^5-6y^3-6y^3)\,dy+i(9y^4-y^2+36y^4)\,dy$

$= \Big[(9y^6-3y^4)+i(9y^5-\frac{y^3}{3})\Big]_{y=0}^{y=1} = (9-3)+i(9-\frac{1}{3})-0$

$\Longrightarrow \boxed{I = 6+\frac{26i}{3}}$

Examples based on circles

Note:- If given path is circle then always use polar form.

(i) If equation of circle is in cartesian form

$x^2+y^2+2gx+2fy+c=0$, centre $(-g,-f) = -g-fi$

radius $r = \sqrt{g^2+f^2-c}$,

then polar form is $\boxed{z-\text{centre} = (\text{rad.})\,e^{i\theta}}$

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(ii) If equation of circle is in modulus form $|z-a|=b$,

then its polar form is $\boxed{z-a=be^{i\theta}}$

Ex (5) Evaluate $\int_C (z-z^2)\,dz$, where C is upper half of circle, $|z-2|=3$

Solution :-

$I = \int_C (z-z^2)\,dz \longrightarrow (1)$

polar form of circle $|z-2|=3$ is $z-2=3e^{i\theta}$

$\longrightarrow$ $z = 2+3e^{i\theta} \Longrightarrow dz = 3ie^{i\theta}d\theta$

limits: for upper half $\theta=0$ to $\theta=\pi$

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$I = \int_{\theta=0}^{\theta=\pi} [(2+3e^{i\theta})-(2+3e^{i\theta})^2]\,3ie^{i\theta}\,d\theta$

$e^{3i\pi} = \cos 3\pi + i\sin 3\pi = -1 + i0 = -1$

$e^{2i\pi} = \cos 2\pi + i\sin 2\pi = 1 + 0i = 1$

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$$= \int_0^{\pi} (2+3e^{i\theta} - 4 - 12e^{i\theta} - 9e^{2i\theta})\, 3ie^{i\theta} d\theta$$

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$$= \int_0^{\pi} (-6ie^{i\theta} - 27ie^{2i\theta} - 27ie^{3i\theta})\, d\theta$$

$$= \left[\left(-\frac{6i}{i}\right)e^{i\theta} - \frac{27i}{2i}e^{2i\theta} - \frac{27i}{3i}e^{3i\theta} \right]_{\theta=0}^{\theta=\pi}$$

$$= \left[-6(-1) - \frac{27}{2}(1) - 9(-1)\right] - \left[-6 - \frac{27}{2} - 9\right]$$

$\Longrightarrow \boxed{I = 30}$

If n is integer, $e^{in\theta} = \cos n\theta + i\sin n\theta = (-1)^n$

Note :- ① what will be value of above integral along lower half of circle?

Ans: $I = -30$ limits $\theta = \pi$ to $\theta = 2\pi$

② Is there exists any other method to evaluate above integral.

Ex ⑥ Evaluate $\int_C (z^2 - 2\bar{z} + 1)dz$, where C is circle $x^2 + y^2 = 2$

Solution :- given circle is $x^2 + y^2 = \sqrt{2}^2$

$\Longrightarrow$ Centre $(0, 0) \equiv 0 + 0i = 0$ and radius $r = \sqrt{2}$

$\Longrightarrow$ polar form $z - 0 = \sqrt{2}e^{i\theta}$ (z - centre = (rad.) $e^{i\theta}$)

$\Longrightarrow \boxed{z = \sqrt{2}e^{i\theta} \Longrightarrow dz = \sqrt{2}e^{i\theta}\, i d\theta}$

For complete circle limits are $\theta = 0$, $\theta = 2\pi$

$$\therefore I = \int_C (z^2 - 2\bar{z} + 1)dz = \int_0^{2\pi} [2e^{2i\theta} - 2\sqrt{2}e^{-i\theta} + 1]\sqrt{2}ie^{i\theta}d\theta$$

$$= \int_0^{2\pi} [2\sqrt{2}ie^{3i\theta} - 4i + \sqrt{2}ie^{i\theta}]d\theta$$

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$$= \left[\frac{2\sqrt{2}i}{3i}e^{3i\theta} - 4i\theta + \frac{2i}{i}e^{i\theta} \right]_{\theta=0}^{\theta=2\pi}$$

$$= (2\sqrt{2}/3 - 8i\pi + 2) - (2\sqrt{2}/3 - 0 + 2)$$

$\Longrightarrow \boxed{I = -8i\pi}$

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Ex ⑦ Show that $\int_C \log z\, dz = 2\pi i$ where C is unit circle in z-plane.

Solution :- unit circle in z-plane is given by $|z|=1$

also given by $x^2+y^2=1$

$\Longrightarrow$ $\boxed{\text{polar Form } z = e^{i\theta}}$

$\Longrightarrow$ $\boxed{dz = e^{i\theta}\, i\, d\theta \quad \text{limits } \theta = 0 \text{ to } \theta = 2\pi}$

$$I = \int_C \log z\, dz = \int_0^{2\pi} (\log e^{i\theta})\, e^{i\theta}\, i\, d\theta$$

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$$= \int_0^{2\pi} i\theta \log e \; e^{i\theta}\, i\, d\theta \qquad = -\int_0^{2\pi} \theta e^{i\theta}\, d\theta$$

$$= -\left[\theta\left(e^{i\theta}/i\right) - (1)\left(e^{i\theta}/i^2\right)\right]_{\theta=0}^{\theta=2\pi}$$

$$= -\left[(2\pi/i + 1) - (0+1)\right]$$

$e^{2i\pi} = \cos 2\pi + i \sin 2\pi = 1 + 0i = 1$

$= -2\pi(-i)$ $\quad [\because \frac{1}{i} = -i]$

$\Longrightarrow$ $\boxed{I = 2\pi i}$

Home Work

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Ex ① Evaluate $\int_{1-i}^{1+i} (ix + iy + 1)\, dx$ along a straight line joining $(1-i)$ to $(1+i)$

Ans.: $I = 2(-1+i)$ $\qquad$ st. line joining $(1,-1)$, $(1,1)$ is $x = 1$

Ex ② Evaluate $\int_C F(z)\, dz$ along a square whose vertices are $(1,1)$, $(2,1)$, $(2,2)$, $(1,2)$ in anticlockwise direction where $F(z) = x - 2iy$

Ans : $I = 3i$

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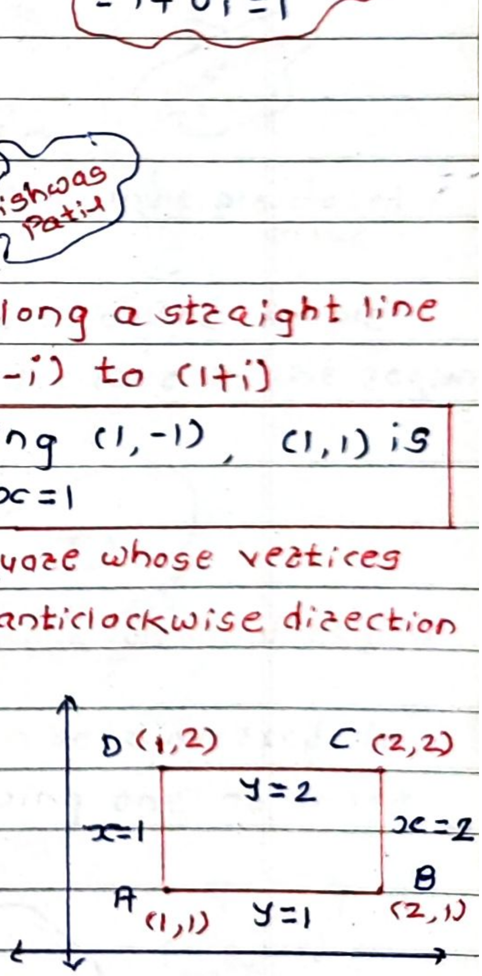


divide c into four parts.

Ex ③ Evaluate $\int_C \frac{dz}{z}$, where C is circle $|z| = 2$

Ans. $I = 2\pi i$ $\qquad$ put $z = 2e^{i\theta}$

Ex ④ Evaluate $\int_C (z-z^2)\,dz$, along upper half of circle $|z|=1$

$\boxed{\text{Ans.: } I = 2/3 \text{ , put } z=e^{i\theta}}$

Ex ⑤ Evaluate $\int_C z^2 dz$ from $P(1,1)$ to $Q(2,4)$ where C is i) $y=x^2$ ii) $y=3x-2$ iii) $x=t,\ y=t^2$

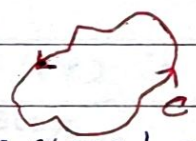
$\boxed{\text{Ans: along all paths } I = -86/3 - 6i \quad \text{why?}}$

Ex ⑥ $\int_C |z|^2 dz$, C is square of vertices $(0,0), (1,0), (0,1), (1,1)$

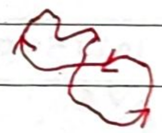
$\boxed{\text{Hint: } |z|^2 = x^2+y^2, \quad \text{Ans} = -1+i}$

Cauchy's Th^m

Simple closed curve :- If a closed curve do not intersect with itself, then it is called simple closed curve. Otherwise it is multiple closed curve.

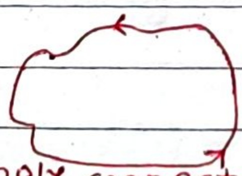


simple closed curve

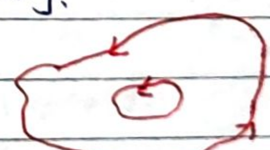


multiple closed curve

Simple connected region :- Region R is called simple connected region if every closed curve in the region encloses points of the region R only.

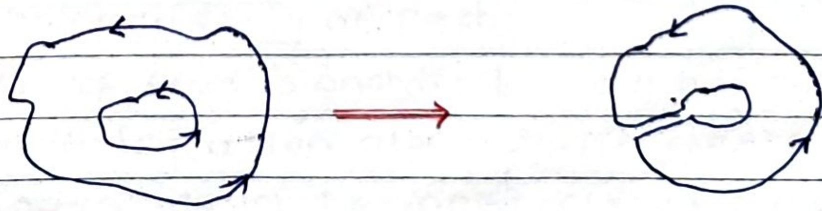


simply connected region



multiple connected region

Note :- multiple connected region can be converted to simple connected region by drawing one or more cuts.



Cauchy's Integral Theorem

Statement :- If $f(z)$ is analytic function and it's derivatives $f'(z)$ is continuous at each point within and on a simple closed curve C then

$$\boxed{\int_C f(z)\,dz = 0}$$

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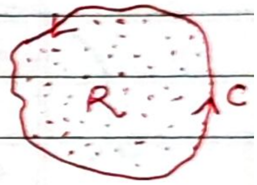
Proof :- Let $f(z) = u + iv$ be analytic function.

$$\Longrightarrow \boxed{u_x = v_y \ , \quad v_x = -u_y} \longrightarrow (1)$$

$$\therefore \int_C f(z)\,dz = \int_C (u+iv)(dx+idy) = \int_C (u\,dx - v\,dy) + i(v\,dx + u\,dy)$$

$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy \longrightarrow (2)$$



as $f'(z)$ is continuous, the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ are also continuous.

$\therefore$ By Green's Th^m $\displaystyle\int_C p\,dx + Q\,dy = \iint_R \left(\partial Q/\partial x - \partial P/\partial y\right) dx\,dy$

$$\therefore (2) \Longrightarrow \int_C f(z)\,dz = \iint_R (-v_x - u_y)\,dx\,dy + i\iint_R (u_x - v_y)\,dx\,dy$$

$$= \iint_R 0\,dx\,dy + \iint_R 0\,dx\,dy$$

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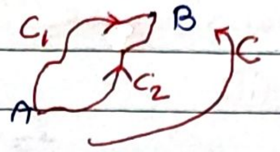
$$\Longrightarrow \boxed{\int_C f(z)\,dz = 0}$$

Note (Imp) :- (1) If $f(z)$ is analytic and C is a open path joining two points A and B then integration is $\boxed{\text{independent}}$ of path.

(2) If $f(z)$ is analytic between two curves C_1, C_2 then integration along outer closed curve is always equal to inner closed curve.

$$\boxed{\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz}$$

Proof:- (1) Let $F(z)$ be analytic everywhere



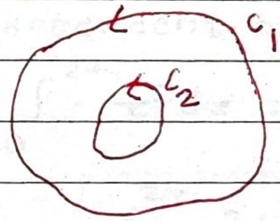
C is closed curve and $F(z)$ is analytic

$\Longrightarrow$ By C.I.T. $\int_C F(z)\,dz = 0$

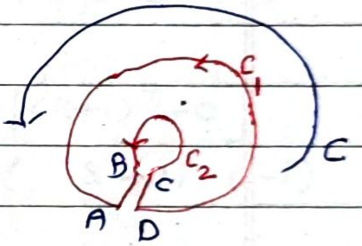
$$\Longrightarrow \int_{A\,(C_2)}^{B} F(z)\,dz + \int_{B\,(-C_1)}^{A} F(z)\,dz = 0$$

$$\Longrightarrow \int_{A\,(C_2)}^{B} F(z)\,dz = -\int_{B\,(-C_1)}^{A} F(z)\,dz \Longrightarrow \boxed{\int_{A\,(C_2)}^{B} F(z)\,dz = \int_{A\,(C_1)}^{B} F(z)\,dz}$$

(2)



Draw a cut AB, cut CD $\longrightarrow$



By C.I.T., as C is closed and $F(z)$ is analytic

$$\int_C F(z)\,dz = 0$$

$$\Longrightarrow \int_{C_1} F(z)\,dz + \int_A^B F(z)\,dz + \int_{-C_2} F(z)\,dz + \int_C^D F(z)\,dz = 0$$

$$\Longrightarrow \int_{C_1} F(z)\,dz = -\int_{-C_2} F(z)\,dz$$

$$\Longrightarrow \boxed{\int_{C_1} F(z)\,dz = \int_{C_2} F(z)\,dz}$$

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Ex ① Evaluate $\int_0^{2+i} z^2 dz$ along (i) $x = 2y$
(ii) along parabola $2y^2 = x$

Solution:- $f(z) = z^2$

$u + iv = (x+iy)^2 \Longrightarrow u + iv = (x^2 - y^2) + i(2xy)$

$\Longrightarrow \boxed{u = x^2 - y^2}$ $\boxed{v = 2xy}$

$\Longrightarrow u_x = 2x \longrightarrow ①$ $u_y = -2y \longrightarrow ②$

$v_x = 2y \longrightarrow ③$ $v_y = 2x \longrightarrow ④$

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From ①, ④ and ②, ③ we get

$u_x = v_y,\ v_x = -u_y \Longrightarrow f(z) = z^2$ is analytic
path is open curve

∴ by Cauchy's Integral Theorem, integration is independent of path.

$\therefore \int_0^{2+i} z^2 dz = \left[z^3/3 \right]_0^{z=2+i} = \frac{1}{3}\left[(2+i)^3 - 0\right]$

$= \frac{1}{3}(8 + 12i - 6 - i)$

$\longrightarrow \boxed{\int_0^{2+i} z^2 dz = \frac{1}{3}(2 + 11i)}$

Ex ② Evaluate $\int_C (z - z^2) dz$ along a circle $|z-2| = 3$

Solution:-

$f(z) = z - z^2 = (x+iy) - (x+iy)^2 = x + iy - (x^2 - y^2 + 2ixy)$

$\Longrightarrow u + iv = (x - x^2 + y^2) + i(y - 2xy)$

$\Longrightarrow \boxed{u = x - x^2 + y^2}$, $\boxed{v = y - 2xy}$

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$u_x = 1 - 2x \longrightarrow ①$ $u_y = 2y \longrightarrow ②$

$v_x = -2y \longrightarrow ③$ $v_y = 1 - 2x \longrightarrow ④$

From ①, ④ and ③, ②

$u_x = v_y,\ v_x = -u_y \Longrightarrow f(z) = z - z^2$ is analytic
and path is closed path.

∴ By C.I.T. $\boxed{\int_C (z - z^2) dz = 0}$

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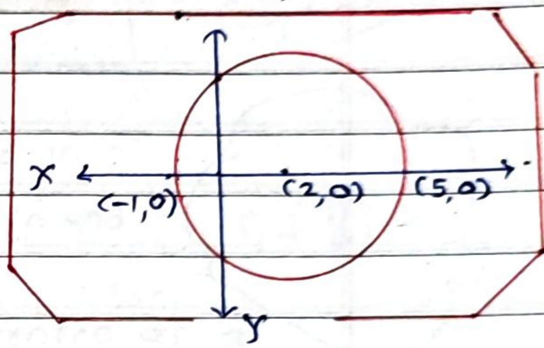
Ex ③ Evaluate $\int_C (z-z^2)\,dz$ along upper half of circle $|z-2|=3$

Solution :- $f(z) = z - z^2$ is analytic — ⟨refer Ex ②⟩

and path is open.

$|z-2| = 3$ is circle with centre $z=2=2+0i=(2,0)$, radius $= 3$

∴ Cauchy's Integral Theorem,



$$\int_C (z-z^2)\,dz = \int_{(5,0)}^{(-1,0)} (z-z^2)\,dz$$

$$= \left[\frac{z^2}{2} - \frac{z^3}{3}\right]_{z=5+0i}^{z=-1+0i}$$

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$$= \left(\frac{1}{2}+\frac{1}{3}\right) - \left(\frac{25}{2} - \frac{125}{3}\right)$$

$$\Longrightarrow \boxed{\int_C (z-z^2)\,dz = 30}$$

Note :- OR : put $z-2 = 3e^{i\theta} \Longrightarrow z = 2+3e^{i\theta}$, $dz = 3ie^{i\theta}d\theta$

limits $\theta = 0$, to $\theta = \pi$

Ex ④ Evaluate $\int_C (2z^3+8z+2)\,dz$ where C is arc of cycloid $x = a(\theta - \sin\theta)$

$y = a(1-\cos\theta)$ between the points $(0,0)$ and $(2\pi a, 0)$

Solution :- $f(z) = 2z^3+8z+2$ being a positive deg. poly. in z, is analytic function.

and curve is open $\Longrightarrow$ integration is independent of path.

$$\int_{(0,0)}^{(2\pi a,0)} (2z^3+8z+2)\,dz = \left[\frac{1}{2}z^4 + 4z^2 + 2z\right]_{z=(0,0)=0}^{z=(2\pi a,0)=2\pi a}$$

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$$= 16\pi^4 a^4/2 + 4(4\pi^2a^2) + 2(2\pi a) - 0$$

$$\Longrightarrow \boxed{\int_C (2z^3+8z+2)\,dz = 8\pi^4a^4 + 16\pi^2a^2 + 4\pi a}$$

Cauchy's Integral Formula

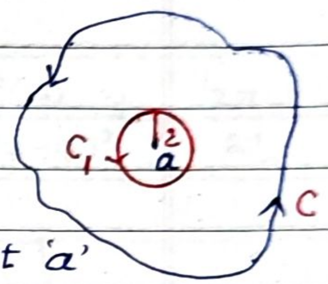
Statement :- If $f(z)$ is analytic within and on a closed curve 'C' containing point a inside C, then

$$\boxed{\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)}$$

Proof :- Out of syllabus :-

Given $f(z)$ is analytic within and on 'C'

$\Longrightarrow f(z)/(z-a)$ is also analytic within and on 'C' except at point $z=a$.



construct a small circle with centre at 'a' and radius 'r' so that circle lies completly inside 'C' as shown in figure.

$\Longrightarrow \dfrac{f(z)}{z-a}$ is analytic between two closed curve C_1, C.

$\Longrightarrow$ by Th^m $\displaystyle\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz \longrightarrow (1)$

as C_1 is circle with centre at 'a' radius 'r' its polar form $\Longrightarrow$ $\boxed{z-a = re^{i\theta} \Longrightarrow dz = re^{i\theta} i\,d\theta}$

$\boxed{\text{limits } \theta = 0 \text{ to } \theta = 2\pi}$

$$(1) \Longrightarrow \int_C \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, rie^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

as r is very very small taking $r \longrightarrow 0$ we get

$$\int_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta \qquad = if(a)\left[\theta\right]_0^{2\pi}$$

$$= if(a)(2\pi - 0)$$

$$\Longrightarrow \boxed{\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)}$$

Deduction of C.I. Formula :-

we know C.I.F. $\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$

d.w.r.t. 'a' using D.U.I.S. we get

$\int_C \frac{-f(z)(-1)}{(z-a)^2}\,dz = 2\pi i f'(a) \Longrightarrow \int_C \frac{f(z)}{(z-a)^2}\,dz = \frac{2\pi i f'(a)}{1!}$

again d.w.r.t. 'a' using D.U.I.S. we get,

$\int_C \frac{-2f(z)(-1)}{(z-a)^2}\,dz = \frac{2\pi i f''(a)}{1!} \Longrightarrow \int_C \frac{f(z)}{(z-a)^2}\,dz = \frac{2\pi i f''(a)}{2!}$

in general we get,

$$\boxed{\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i f^n(a)}{n!}}$$

Defⁿ: Pole :- Let $f(z)$ be a complex valued function. $z = a$ is said to be pole if $f(a) = \infty$

Procedure :- (I) Find the poles which are lying inside C.

(II) apart from this pole, remaining quantities will be value of $f(z)$

(III) Apply C.I.F. accordingly..

(IV) If more than two poles are inside, then use partial fraction method.

Ex (1) Evaluate, $\int_C \frac{e^{3z}}{z-i}\,dz$, where C is $|z-2|+|z+2| = 6$

Solution :- pole is $z = i$

given curve $C : |z-2|+|z+2| = 6$

at $z = i$, $|i-2|+|i+2| = \sqrt{4+1}+\sqrt{4+1} = 4.47 < 6$

$\Longrightarrow$ pole $z = i$ lies inside given curve

$\therefore$ by C.I.F. $\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$

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$$\therefore \int_C \frac{e^{3z}}{z-i}\,dz = 2\pi i f(i), \quad \text{where } f(z) = e^{3z}$$

$$= 2\pi i\, e^{3i} = 2\pi i[\cos 3 + i\sin 3]$$

$$\boxed{\int_C e^{3z}/_{z-i}\,dz = 2\pi i\cos 3 - 2\pi\sin 3}$$

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Ex ② Evaluate $\int_C z^2/(z^4-1)\,dz$, where C is

i) $|z| = 3/4$ ii) $|z+i| = 1$

Solution :-

$$I = \int_C \frac{z^2}{z^4-1}\,dz = \int_C \frac{z^2}{(z^2-1)(z^2+1)}\,dz = \int_C \frac{z^2}{(z-1)(z+1)(z-i)(z+i)}\,dz$$

$\longrightarrow$ ①

poles are $z=1$, $z=-1$, $z=i$, $z=-i$

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case ⓘ $|z| = 3/4$

if $z=1$, $|1| = 1 > 3/4 \Longrightarrow$ outside, if $z=-1$, $|-1| = 1 > 3/4 \Rightarrow$ outside

if $z=i$, $|i| = 1 > 3/4 \Longrightarrow$ outside, if $z=-i$, $|-i| = 1 > 3/4 \Rightarrow$ outside

$\Longrightarrow$ All poles are outside $|z| = 3/4$

$\Longrightarrow$ $f(z)$ is analytic within and on a circle $|z| = 3/4$

$\therefore$ by C.I.T. $\boxed{\int_C \frac{z^2}{z^4-1}\,dz = 0}$

case ⓘⓘ $|z+i| = 1$

Vishwas Patil

if $z=1$, $|1+i| = \sqrt{2} > 1 \Longrightarrow$ outside,

if $z=-1$, $|-1+i| = \sqrt{1+1} = \sqrt{2} \Longrightarrow$ outside

if $z=i$, $|i+i| = |0+2i| = \sqrt{0+4} = 2 > 1 \Longrightarrow$ outside

if $z=-i$, $|-i+i| = |0| = 0 < 1 \Longrightarrow$ pole $z=-i$ is inside C.

$$① \Longrightarrow \int_C \frac{z^2}{z^4-1}\,dz = \int_C \left[\frac{z^2}{(z-1)(z+1)(z-i)}\right] \Big/ (z+i)\,dz \qquad \int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$= 2\pi i f(-i), \quad f(z) = \frac{z^2}{(z^2-1)(z-i)}$$

$$= 2\pi i \frac{(-i)^2}{[(-i)^2-1](-2i)} = 2\pi i \frac{-1}{(-2)(-2i)}$$

$$\longrightarrow \boxed{\int_C \frac{z^2}{z^4-1}\,dz = \frac{-\pi}{2}}$$

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Ex (3) Evaluate $\int_C \frac{\sin^6 z}{(z-\pi/6)^3} dz$, where $C: |z|=1$ → [MU: 94, 95, 00, 04, 08, 11, 15, 19

Solution:-

pole is $z = \pi/6$ is of order 3

$C: |z|=1$, $\implies |\pi/6| = \sqrt{\pi^2/36 + 0} = \pi/6 = 0.52 < 1$

$\implies$ pole is lying inside C.

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$\therefore$ By C.I.F. $\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i f^n(a)}{n!}$

$$\implies \int_C \frac{\sin^6 z}{(z-\pi/6)^3} dz = \frac{2\pi i f^2(\pi/6)}{2!} \longrightarrow (2)$$

$f(z) = \sin^6 z$

$f'(z) = 6\sin^5 z \cos z$

$f''(z) = 6\sin^5 z(-\sin z) + 30\sin^4 z \cos z \cos z$

$$\implies f^2(z) = -6\sin^6 z + 30\sin^4 z \cos^2 z$$

$$\implies f^2(\pi/6) = -6\sin^6(\pi/6) + 30\sin^4(\pi/6)\cos^2(\pi/6)$$

$$= -(6/64) + 30(3/64) \implies \boxed{f^2(\pi/6) = 84/64}$$

using this in eqn (2) we get,

$$\boxed{\int_C \frac{\sin^6 z}{(z-\pi/6)^3} dz = \pi i\left(\frac{84}{64}\right) = \frac{21\pi i}{16}}$$

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Ex (4) Evaluate $\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{z^2+3z+2} dz$ C is (i) $|z|=1$ (ii) $|z|=3$

Solution:- $I = \int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z+2)(z+1)} dz \longrightarrow (1)$

poles are $z=-2$, $z=-1$

Vishwas Patil

Case (i) :- $C: |z|=1$

if $z=-2$, $|-2| = 2 > 1$ ⟶ pole is outside

if $z=-1$, $|-1| = 1 = 1$ ⟹ Pole is on the circle ⟹ $\boxed{\text{outside}}$

both poles are outside ⟹ $f(z)$ is analytic and curve circle is closed

$$\implies \text{by C.I.T.} \quad \boxed{\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z+2)(z+1)} dz = 0}$$

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case (ii) $|z|=3$

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if $z=-2$, $|-2|=2<3 \Longrightarrow$ inside C.

if $z=-1$, $|-1|=1<3 \Longrightarrow$ inside c.

both poles are inside $\Longrightarrow$ Use of partial fraction

$$\frac{1}{(z+2)(z+1)}=\frac{a}{z+1}+\frac{b}{z+2}$$

$$\boxed{a=\frac{1}{z+2}\Big|_{z=-1}=1,\quad b=\frac{1}{z+1}\Big|_{z=-2}=-1}$$

$$=\frac{1}{z+1}-\frac{1}{z+2}$$

multiplying by $\sin(\pi z^2)+\cos(\pi z^2)$ and integrating around C

$$I=\int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z+2)(z+1)}dz=\int_C \frac{\sin(\pi z^2)+\cos\pi z^2}{z+1}dz-\int_C\frac{\sin\pi z^2+\cos\pi z^2}{z+2}$$

$$=2\pi i f(-1)-2\pi i f(-2)$$

vishwas patil

$$\left[\because \text{by C.I.F.} \int_C \frac{f(z)}{z-a}dz=2\pi i f(a)\right]$$

$I=2\pi i f(-1)-2\pi i f(-2)$, where $f(z)=\sin\pi z^2+\cos\pi z^2$

$=2\pi i(-1)-2\pi i(1)$

$f(-1)=\sin\pi+\cos\pi=-1$

$f(-2)=\sin 4\pi+\cos 4\pi=1$

$$\boxed{I=-4\pi i}$$

Ex (5) IF $f(a)=\int_C \frac{4z^2+z+4}{z-a}dz$ and $c: 4x^2+9y^2=36$ then

Find $f(1)$, $f'(2+i)$, $f''(-i)$ $\longrightarrow$ {2006, 08, 14, 16, 20

Solution:- Given $f(a)=\int_C \frac{4z^2+z+4}{z-a}dz \longrightarrow (1)$

d.w.r.t. a, $f'(a)=\int_C \frac{(4z^2+z+4)}{(z-a)^2}dz \longrightarrow (2)$

d.w.r.t. 'a' again, $f''(a)=\int_C \frac{2(4z^2+z+4)}{(z-a)^3}dz \longrightarrow (3)$

To find f(1) :- put $a=1$ in equation (1)

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$$f(1)=\int_C \frac{4z^2+z+4}{z-1}dz$$

pole, $z=1=1+0i=x+iy \Rightarrow x=1, y=0$

$c: 4x^2+9y^2=36 \longrightarrow 4(1)+0=4<36 \Longrightarrow$ inside

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by C.I.F. $\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$

$\therefore f(1) = \int_C \frac{4z^2+z+4}{z-1} dz = 2\pi i f(1)$, $f(z) = 4z^2+z+4$

$= 2\pi i(4+1+4)$

$\boxed{\int_C \frac{4z^2+z+4}{z-1} dz = 18\pi i}$

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To find $f'(2+i)$:- put $a = 2+i$ in equation ② we get

$$f'(2+i) = \int_C \frac{4z^2+z+4}{[z-(2+i)]^2} dz$$

pole $z = 2+i = (2,1) = x+iy \Longrightarrow x=2,\ y=1$

$C: 4x^2+9y^2 = 36 \Longrightarrow 4(4)+9(1) = 25 < 36 \Longrightarrow$ Pole is inside

$\therefore$ by C.I.F. $\int_C \frac{f(z)}{(z-a)^{n+1}} dz = 2\pi i \frac{f'(a)}{1!}$

vishwas patil

$\therefore f'(2+i) = \int_C \frac{4z^2+z+4}{[z-(2+i)]^2} dz = 2\pi i \frac{f'(2+i)}{1!}$; $f(z) = 4z^2+z+4$, $f'(z) = 8z+1$

$= 2\pi i[8(2+i)+1]$

$f'(2+i) = \boxed{-16\pi + 34\pi i}$

To find $f''(-i)$:- put $a = -i$ in equation ③,

$$f''(-i) = \int_C \frac{2(4z^2+z+4)}{(z+i)^3} dz$$

pole $z = -i = 0-i = x+iy \Longrightarrow x=0,\ y=-1$

$C: 4x^2+9y^2 = 36 \Longrightarrow 0+9(1) = 9 < 36 \Longrightarrow$ inside C.

$\therefore$ by C.I.F. $f''(-i) = \int_C \frac{8z^2+2z+8}{(z+i)^3} dz$

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$= \frac{2\pi i f''(-i)}{2!}$; $f(z) = 8z^2+2z+8$, $f'(z) = 16z+2$, $f''(z) = 16$

$= \pi i(16)$

$\Longrightarrow \boxed{f''(-i) = 16\pi i}$

Ex ⑥ If C is circle $|z|=1$, using the integral $\int_C \frac{e^{kz}}{z} dz$, where K is real, show that

$$\int_0^{\pi} e^{k\cos\theta} \cos(k\sin\theta)\, d\theta = \pi$$

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solution:- $\int_C \frac{e^{kz}}{z}\,dz \qquad C: |z|=1$,

pole is $z=0$, $|0|=0<1 \Longrightarrow$ pole $z=0$ is inside C.

$\therefore$ C.I.F. $\int_C \frac{e^{kz}}{z}\,dz = 2\pi i f(0)$, $f(z)=e^{kz}$ $\quad \left(\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)\right)$

$= 2\pi i\, e^0$

$\Longrightarrow \int_C \frac{e^{kz}}{z}\,dz = 2\pi i \longrightarrow ①$

polar form of circle $|z|=1$ is $\boxed{z=e^{i\theta}}$

$\Longleftrightarrow \boxed{dz = e^{i\theta} i\,d\theta, \text{ limit } \theta=0 \text{ to } \theta=2\pi}$

$① \Longrightarrow \int_0^{2\pi} \frac{e^{ke^{i\theta}}}{e^{i\theta}} e^{i\theta} i\,d\theta = 2\pi i \Longrightarrow i\int_0^{2\pi} e^{k(\cos\theta + i\sin\theta)}\,d\theta = 2\pi i$

$\Longrightarrow 2\pi = \int_0^{2\pi} e^{k\cos\theta} e^{i(k\sin\theta)}\,d\theta$ vishwas patil

$= \int_0^{2\pi} e^{k\cos\theta}\left[\cos(k\sin\theta) + i\sin(k\sin\theta)\right]d\theta$

$2\pi + i0 = \int_0^{2\pi} e^{k\cos\theta}\cos(k\sin\theta)\,d\theta + i\int_0^{2\pi} e^{k\cos\theta}\sin(k\sin\theta)\,d\theta$

$\Longrightarrow 2\pi = \int_0^{2\pi} e^{k\cos\theta}\cos(k\sin\theta)\,d\theta$

$2\pi = 2\int_0^{\pi} e^{k\cos\theta}\cos(k\sin\theta)\,d\theta$ $\quad \left[\int_0^{2a} f(x)\,dx = 2\int_0^a f\,dx \text{ if } f(2a-x)=f(x)\right]$

$\Longrightarrow \boxed{\pi = \int_0^{\pi} e^{k\cos\theta}\cos(k\sin\theta)\,d\theta}$

Ex ⑦ Show that $\int_C \frac{dz}{z+1} = 2\pi i$ where C is circle $|z|=2$

Hence show that

$\int_C \frac{(x+1)dx + y\,dy}{(x+1)^2+y^2} = 0$; and $\int_C \frac{(x+1)dy - x\,dx}{(x+1)^2+y^2} = 2\pi$

Solution:- $\int_C \frac{dz}{z+1}$, pole is $z=-1$, $|-1|=1<2$

$\Longrightarrow z=-1$ is inside C.

$\therefore$ by C.I.F. $\int_C \frac{dz}{z+1} = 2\pi i f(-1)$, $\boxed{\text{where } f(z)=1, \; f(-1)=1}$

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$\int_C \frac{dz}{z+1} = 2\pi i$

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$$\Longrightarrow \int_C \frac{dx+idy}{x+iy+1} = 2\pi i$$

$$\Longrightarrow \int_C \frac{[(x+1)-iy]\,(dx+idy)}{[(x+1)+iy][(x+1)-iy]} = 2\pi i$$

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$$\Longrightarrow 2\pi i = \int_C \frac{[(x+1)dx+ydy]+i[(x+1)dy-ydx]}{(x+1)^2+y^2}$$

$$2\pi i+0 = \int_C \frac{(x+1)dx+ydy}{(x+1)^2+y^2} + i\int_C \frac{(x+1)dy-ydx}{(x+1)^2+y^2}$$

∴ comparing real and imaginary parts, we get,

$$\boxed{\int_C \frac{(x+1)dx+ydy}{(x+1)^2+y^2} = 0} \quad \text{and} \quad \boxed{\int_C \frac{(x+1)dy-ydx}{(x+1)^2+y^2} = 2\pi}$$

Ex ⑧ Evaluate $\int_C \frac{\sin z}{z^6}\,dz$, where $C: |z|=2$

Solution:- pole is $z=0$,

$C: |z|=2$, $|0|=0<2 \Longrightarrow z=0$ is inside C of order 6

B. C.I.F. $\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i\, f^n(a)}{n!}$

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$$\Longrightarrow \int_C \frac{\sin z}{z^6}dz = \frac{2\pi i\, f^5(0)}{5!}, \quad f(z)=\sin z$$

$$\Longrightarrow f^5(z) = \cos z$$

$$= 2\pi i/120 \qquad f^5(0) = \cos 0 = 1$$

$$\longrightarrow \boxed{\int_C \frac{\sin z}{z^6}dz = \frac{\pi i}{60}}$$

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Taylor's Series:- If $f(z)$ is analytic within and on a circle with centre at $z=a$, then $f(z)$ can be expanded as

$$\boxed{f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots}$$

Laurentz's series:- If C_1, C_2 be two concentric circles of radii r_1, r_2 with centre at $z=a$ and $f(z)$ is analytic between two annular regions between C_1 and C_2 then

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}}$$

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where
$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz, \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{-n+1}} dz$$

Note :-
1. Expansion about $z=a \Longrightarrow$ in powers of $(z-a)$
2. Taylor's series contains only positive powers of $z-a$
3. Laurentz's series contains negative powers or simultaneously both negative and positive powers of $(z-a)$

Remember the formula :-
1. $(1+p)^{-1} = 1 - p + p^2 - p^3 + \cdots$
2. $(1-p)^{-1} = 1 + p + p^2 + p^3 + \cdots$
3. $(1-p)^{-2} = 1 + 2p + 3p^2 + 4p^3 + \cdots$

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Type - I

Ex ① Expand $f(z) = \cos z$ about $z = \pi/3$

Solution:- We know Taylor's series about $z = a$ is

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

put $a = \pi/3$

$$\Rightarrow f(z) = f\left(\frac{\pi}{3}\right) + \frac{(z-\pi/3)}{1!} f'\left(\frac{\pi}{3}\right) + \frac{(z-\pi/3)^2}{2!} f''\left(\frac{\pi}{3}\right) + \cdots \quad \rightarrow ①$$

given $f(z) = \cos z$ $\quad$ $f(\pi/3) = \cos \pi/3 = 1/2$

$\therefore$ $f'(z) = -\sin z$ $\quad \Rightarrow \quad$ $f'(\pi/3) = -\sin \pi/3 = -\sqrt{3}/2$

$f''(z) = -\cos z$ $\quad$ $f''(\pi/3) = -1/2$

using these values in equation ① we get,

$$\cos z = \frac{1}{2} + \frac{(z-\pi/3)}{1!}\left(\frac{-\sqrt{3}}{2}\right) + \frac{(z-\pi/3)^2}{2!}\left(\frac{-1}{2}\right) + \cdots$$

$$\boxed{\cos z = 1/2 - \sqrt{3}(z-\pi/3)/2 - \frac{1}{4}(z-\pi/3)^2 + \cdots}$$

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Ex ② Expand $f(z) = e^z$ about $z = 1$

Solution:- Taylor's series about $z = 1$ is given by

$$f(z) = f(1) + \frac{(z-1)}{1!} f'(1) + \frac{(z-1)^2}{2!} f''(1) + \frac{(z-1)^3}{3!} f'''(1) + \cdots \quad \rightarrow ①$$

$f(z) = e^z$ $\quad$ $f(1) = e^1 = e$

$f'(z) = e^z$ $\quad$ $f'(1) = e^1 = e$

$f''(z) = e^z$ $\quad \Rightarrow \quad$ $f''(1) = e^1 = e$

$f'''(z) = e^z$ $\quad$ $f'''(1) = e^1 = e$

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using this in equation ① we get,

$$\boxed{e^z = e + \frac{(z-1)}{1!} e + \frac{(z-1)^2}{2!} e + \frac{(z-1)^3}{3!} e + \cdots}$$

Type - II

Note: ① In case of polynomial use partial fraction method

② Find region of convergence (ROC)

③ Expand using the formula $(1-p)^{-1}$, $(1+p)^{-1}$, $(1-p)^{-2}$ etc.

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Ex ① Expand $f(z)= \dfrac{(z-2)(z+2)}{(z+1)(z+4)}$, indicating region of convergence.

→ [M.U.:- 2004, 05, 08, 15, 17, 20,

Solution:- $f(z)= \dfrac{(z-2)(z+2)}{(z+1)(z+4)} = \dfrac{z^2-4}{z^2+5z+4}$

$$\Longrightarrow f(z)=1-\frac{5z+8}{z^2+5z+4}$$

$$= 1-\frac{5z+8}{(z+1)(z+4)}$$

$$=1-\left[\frac{a}{z+4}+\frac{b}{z+1}\right]$$

$$= 1-\left[4/(z+4)+1/(z+1)\right]$$

$$z^2+5z+4 \overline{\big)\, z^2-4 \,}\; 1$$
$$\underline{z^2+5z+4}$$
$$-5z-8$$

$$\Longrightarrow z^2-4 = 1(z^2+5z+4)-5z-8$$
$$\frac{z^2-4}{z^2+5z+4} = 1-\frac{(5z+8)}{z^2+5z+4}$$

$$\Longrightarrow \boxed{f(z)=1-\frac{4}{z+4}-\frac{1}{z+1}} \longrightarrow ①$$

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For ROC: $|z|<4$, $|z|>4$, $|z|<1$, $|z|>1$

i) $|z|<1$, $|z|$ ii) $1<|z|<4$ iii) $|z|>4$, $|z|>1$

Case ① $|z|<1$, $|z|<4$

$$① \Longrightarrow f(z)=1-\frac{4}{4(\frac{z}{4}+1)}-\frac{1}{z+1}$$

$$= 1-(1+z/4)^{-1}-(1+z)^{-1}$$

always take bigger quantity to create $1+p$

$(1+p)^{-1}=1-p+p^2-p^3+\dots$

$$\Longrightarrow f(z)=1-\left[1-\frac{z}{4}+\frac{z^2}{4^2}-\dots\right]-(1-z+z^2-\dots)$$

$$\boxed{f(z)=-1+\frac{5}{4}z-\frac{17}{16}z^2+\dots}$$ which is Taylor's series.

Case ⓘⓘ $1<|z|<4$

$$① \Longrightarrow f(z)=1-\frac{4}{4(\frac{z}{4}+1)}-\frac{1}{z(1+\frac{1}{z})}$$

vishwas patil

$$=1-(1+z/4)^{-1}-\frac{1}{z}(1+1/z)^{-1}$$

$$=1-\left(1-\frac{z}{4}+\frac{z^2}{4^2}-\dots\right)-\frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\dots\right)$$

$$\Longrightarrow \boxed{f(z)=\left(\frac{z}{4}-\frac{z^2}{16}+\dots\right)+\left(-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\dots\right)}$$

which is Laurentz's series.

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Case (iii) $|z|>1, \ |z|>4$

$$① \Longrightarrow f(z) = 1 - \frac{4}{z(1+4/z)} - \frac{1}{z(1+1/z)}$$

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$$= 1 - \frac{4}{z}\left(1+\frac{4}{z}\right)^{-1} - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= 1 - \frac{4}{z}\left(1-\frac{4}{z}+\frac{16}{z^2}-\cdots\right) - \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\cdots\right)$$

$$\Longrightarrow \boxed{f(z) = 1 - \frac{5}{z} + \frac{17}{z^2} - \cdots}$$ which is Laurentz's series.

Ex ② Find all possible Laurentz's series for,

$$f(z) = \frac{7z-2}{z(z-2)(z+1)} \quad \text{about } z=-1$$

→ [M.U. :- 1999, 2003, 06, 08, 09, 14, 18, 20

Solution :-

$$f(z) = \frac{7z-2}{z(z-2)(z+1)} = \frac{a}{z} + \frac{b}{z-2} + \frac{c}{z+1} \longrightarrow ①$$

$$\boxed{a = \left.\frac{7z-2}{(z-2)(z+1)}\right|_{z=0} = 1, \quad b = \left.\frac{7z-2}{z(z+1)}\right|_{z=2} = 2, \quad c = \left.\frac{7z-2}{z(z-2)}\right|_{z=-1} = -3}$$

$$\therefore ② \Longrightarrow f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

Expansion about $z=-1 \Longrightarrow$ in powers of $z+1$

$$\Longrightarrow \boxed{f(z) = \frac{1}{(z+1)-1} + \frac{2}{(z+1)-3} - \frac{3}{z+1}} \longrightarrow ②$$

For ROC :- $|z+1|<1, \ |z+1|>1, \ |z+1|<3, \ |z+1|>3$

$\Longrightarrow$ (i) $|z+1|<1, \ |z+1|<3$ (ii) $1<|z+1|<3$

(iii) $|z+1|>4, \ |z+1|>1$

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Case (i) $|z+1|<1, \ |z+1|<3$

$$① \Rightarrow f(z) = \frac{-1}{1-(z+1)} - \frac{2}{3\left[1-\left(\frac{z+1}{3}\right)\right]} - \frac{3}{(z+1)}$$

(420volt) we need z+1

$$= -\left[1-(z+1)\right]^{-1} - \frac{2}{3}\left[1-\left(\frac{z+1}{3}\right)\right]^{-1} - \frac{3}{(z+1)}$$

$$= -\left[1+(z+1)+(z+1)^2+\cdots\right] - \frac{2}{3}\left[1+\left(\frac{z+1}{3}\right)+\left(\frac{z+1}{3}\right)^2+\cdots\right]$$

$$- 3/(z+1)$$

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$$\Longrightarrow \boxed{f(z)=\left[-\frac{5}{3}-\frac{11}{9}(z+1)-\frac{29}{27}(z+1)^2-\cdots\right]-\frac{3}{(z+1)}}$$

-ve power of $z+1$

which is Laurentz's series.

vishwas patil

case (ii) $1<|z+1|<3$

$$(2)\Longrightarrow f(z)=\frac{1}{(z+1)\left[1-\left(\frac{1}{z+1}\right)\right]}-\frac{2}{3\left[1-\left(\frac{z+1}{3}\right)\right]}-\frac{3}{(z+1)}$$

$$=\frac{1}{(z+1)}\left[1-\left(\frac{1}{z+1}\right)\right]^{-1}-\frac{2}{3}\left[1-\left(\frac{z+1}{3}\right)\right]^{-1}-\frac{3}{(z+1)}$$

$$=\frac{1}{(z+1)}\left[1+\left(\frac{1}{z+1}\right)+\left(\frac{1}{z+1}\right)^2+\cdots\right]-\frac{2}{3}\left[1+\left(\frac{z+1}{3}\right)+\left(\frac{z+1}{3}\right)^2+\cdots\right]-\frac{3}{(z+1)}$$

$$\Longrightarrow \boxed{f(z)=\left[-\frac{2}{3}-\frac{2}{9}(z+1)-\frac{2}{27}(z+1)^2-\cdots\right]+\left[-\frac{2}{(z+1)}+\frac{1}{(z+1)^2}+\cdots\right]}$$

which is Laurentz's series.

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case (iii) $|z+1|>1,\ |z+1|>3$

$$(2)\Longrightarrow f(z)=\frac{1}{(z+1)\left[1-\left(\frac{1}{z+1}\right)\right]}+\frac{2}{(z+1)\left[1-\left(\frac{3}{z+1}\right)\right]}-\frac{3}{(z+1)}$$

$$=\frac{1}{(z+1)}\left[1-\left(\frac{1}{z+1}\right)\right]^{-1}+\frac{2}{(z+1)}\left[1-\left(\frac{3}{z+1}\right)\right]^{-1}-\frac{3}{z+1}$$

$$=\frac{1}{(z+1)}\left[1+\left(\frac{1}{z+1}\right)+\left(\frac{1}{z+1}\right)^2+\cdots\right]+\frac{2}{(z+1)}\left[1+\left(\frac{3}{z+1}\right)+\left(\frac{3}{z+1}\right)^2+\cdots\right]-\left(\frac{3}{z+1}\right)$$

$$\Longrightarrow \boxed{f(z)=\frac{7}{(z+1)^2}+\frac{19}{(z+1)^3}+\cdots}$$

which is Laurentz's series.

Ex (3) Expand $f(z)=\dfrac{1}{(z-1)(z-2)}$ in the region

i) $0<|z-1|<1$ (ii) $1<|z-3|<2$ (iii) $|z|<1$

Solution:- $f(z)=\dfrac{1}{(z-1)(z-2)}=\dfrac{a}{z-1}+\dfrac{b}{z-2}$

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$$\Longrightarrow \boxed{f(z)=\frac{-1}{z-1}+\frac{1}{z-2}} \longrightarrow (1)$$

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case (i): $0<|z-1|<1$ $\longrightarrow$ in powers of $z-1$

$$(1) \Rightarrow f(z) = \frac{-1}{z-1} + \frac{1}{(z-1)-1} = \frac{-1}{(z-1)} + \frac{-1}{1-(z-1)}$$

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$$= -(z-1)^{-1} - [1-(z-1)]^{-1}$$

$$\boxed{f(z) = \frac{-1}{(z-1)} - [1+(z-1)+(z-1)^2+(z-1)^3+\dots]}$$

which is Laurentz's series.

case (ii): $1<|z-3|<2$ $\Longrightarrow$ in powers of $z-3$

$$(1) \Rightarrow f(z) = \frac{-1}{(z-3)+2} + \frac{1}{(z-3)+1} = \frac{-1}{2[(\frac{z-3}{2})+1]} + \frac{1}{(z-3)[1+(\frac{1}{z-3})]}$$

$$= \frac{-1}{2}\left[1+\left(\frac{z-3}{2}\right)\right]^{-1} + \frac{1}{(z-3)}\left[1+\left(\frac{1}{z-3}\right)\right]^{-1}$$

$$\boxed{f(z) = \frac{-1}{2}\left[1-\left(\frac{z-3}{2}\right)+\left(\frac{z-3}{2}\right)^2+\dots\right] + \frac{1}{(z-3)}\left[1-\left(\frac{1}{z-3}\right)+\left(\frac{1}{z-3}\right)^2-\dots\right]}$$

which is Laurentz's series.

case (iii) $|z|<1$ $\Longrightarrow$ $|z|<2$ $\Longrightarrow$ in powers of z

$$(1) \Rightarrow f(z) = \frac{-1}{(z-1)} + \frac{1}{z-2} = \frac{1}{1-z} - \frac{1}{2(1-\frac{z}{2})} = (1-z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

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$$= (1+z+z^2+z^3+\dots) - \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\dots\right)$$

$$\boxed{f(z) = (1/2) + (3/4)z + (7/8)z^2 + (15/16)z^3 + \dots}$$

which is Taylor's series.

Ex (4) Expand $f(z) = \frac{1}{z^2(z-1)(z+2)}$ about $z=0$

indicating region of convergence.

Solution:- Note:- Expansion about $z=0$ $\longrightarrow$ in powers of z
$\Longrightarrow$ Keep $\frac{1}{z^2}$ as it is and use partial fraction for the rest.

consider, $$\frac{1}{(z-1)(z+2)} = \frac{a}{z-1} + \frac{b}{z+2} = \frac{1}{3(z-1)} - \frac{1}{3(z+2)}$$

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multiplying by $\frac{1}{z^2}$, $$\frac{1}{z^2(z-1)(z+2)} = \frac{1}{3z^2(z-1)} - \frac{1}{3z^2(z+2)}$$

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$$\Rightarrow \boxed{f(z)=\frac{1}{3z^2(z-1)}-\frac{1}{3z^2(z+2)}} \rightarrow (1)$$

ROC: i) $|z|<1,\ |z|<2$ ii) $1<|z|<2$ iii) $|z|>2,\ |z|>1$

case (i) $|z|<1,\ |z|<2$

$$f(z)=\frac{-1}{3z^2(1-z)}-\frac{1}{6z^2(\frac{z}{2}+1)}=\frac{-1}{3z^2}(1-z)^{-1}-\frac{1}{6z^2}(1+\frac{z}{2})^{-1}$$

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$$(1-p)^{-1}=1+p+p^2+\dots$$
$$(1+p)^{-1}=1-p+p^2-\dots$$

$$=-\frac{1}{3z^2}(1+z+z^2+z^3+\dots)-\frac{1}{6z^2}(1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\dots)$$

$$\boxed{f(z)=\frac{-3}{8}-\frac{1}{4z}-\frac{1}{2z^2}-\frac{5}{16}z-\dots} \rightarrow$$ which is Laurentz's series

case (ii) $1<|z|<2$

$$(1)\Rightarrow f(z)=\frac{1}{3z^3(1-\frac{1}{z})}-\frac{1}{6z^2(\frac{z}{2}+1)}=\frac{1}{3z^3}(1-\frac{1}{z})^{-1}-\frac{1}{6z^2}(1+\frac{z}{2})^{-1}$$

$$f(z)=\frac{1}{3z^3}(1+\frac{1}{z}+\frac{1}{z^2}+\dots)-\frac{1}{6z^2}(1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\dots)$$

$$\boxed{f(z)=(\frac{1}{3z^3}+\frac{1}{3z^4}+\frac{1}{3z^5}+\dots)+(-\frac{1}{6z^2}+\frac{1}{12z}-\frac{1}{24}+\frac{z}{48}+\dots)}$$

$\longrightarrow$ which is Laurentz's series.

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case (iii) $|z|>1,\ |z|>2$

$$(1)\Rightarrow f(z)=\frac{1}{3z^3(1-\frac{1}{z})}-\frac{1}{3z^3(1+\frac{2}{z})}=\frac{1}{3z^3}(1-\frac{1}{z})^{-1}-\frac{1}{3z^3}(1+\frac{2}{z})^{-1}$$

$$=\frac{1}{3z^3}(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\dots)-\frac{1}{3z^3}(1-\frac{2}{z}+\frac{4}{z^2}-\dots)$$

$$\boxed{f(z)=\frac{1}{z^4}-\frac{1}{z^5}+\dots} \rightarrow$$ which is Laurentz's series.

Note:- OR

$$f(z)=\frac{1}{z^2(z-1)(z+2)}=\frac{a}{z}+\frac{b}{z^2}+\frac{c}{z-1}+\frac{d}{z+2}$$

$$\Rightarrow \boxed{f(z)=\frac{-1}{4z}-\frac{1}{2z^2}+\frac{1}{3(z-1)}-\frac{1}{12(z+2)}}$$

Keep as it is | Expand

Ex ⑤ Find Laurentz's series for

$f(z) = \dfrac{1}{z(z-1)^2}$ about (i) $0<|z|<1$ (ii) $|z|>1$ (iii) $0<|z-1|<1$ (iv) $|z-1|>1$

Solution:- **Note:**- There is no need to use partial fraction why ?

case (i) $0<|z|<1$ $\Longrightarrow$ in powers of z only.

$$f(z) = \left(\frac{1}{z}\right)\frac{1}{(-1)^2(1-z)^2} = \frac{1}{z}(1-z)^{-2} = \frac{1}{z}(1+2z+3z^2+4z^3+\dots)$$

420 Volt.

$$\boxed{f(z) = \frac{1}{z}+2+3z^2+4z^3+\dots}$$

$(1-p)^{-2} = 1+2p+3p^2+\dots$

which is Laurentz's series.

case (ii) $|z|>1$ $\Longrightarrow$ in powers of z only

$$f(z) = \frac{1}{z\,\frac{1}{z^2}\left(1-\frac{1}{z}\right)^2} = z\left(1-\frac{1}{z}\right)^{-2} = z\left(1+\frac{2}{z}+\frac{3}{z^2}+\dots\right)$$

420 Volt.

$$\Longrightarrow \boxed{f(z) = z+2+\frac{3}{z}+\dots}$$

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which is Laurentz's series.

case (iii) $0<|z-1|<1$ $\Longrightarrow$ in powers of $z-1$ only

$$f(z) = \frac{1}{z(z-1)^2} = \frac{1}{(z-1)^2}\,\frac{1}{[(z-1)+1]} = \frac{1}{(z-1)^2}\,[1+(z-1)]^{-1}$$

420 Volt.

$$= \frac{1}{(z-1)^2}\,[1-(z-1)+(z-1)^2-\dots]$$

$$\Longrightarrow \boxed{f(z) = \frac{1}{(z-1)^2} - \frac{1}{(z-1)} + 1 - (z-1) + \dots}$$

which is Laurentz's series.

case (iv) $|z-1|>1$ $\Longrightarrow$ in powers of $z-1$ only.

$$f(z) = \frac{1}{z(z-1)^2} = \frac{1}{(z-1)^2\,[(z-1)+1]} = \frac{1}{(z-1)^3\left[1+\left(\frac{1}{z-1}\right)\right]}$$

420 Volt.

$$= \frac{1}{(z-1)^3}\left[1-\left(\frac{1}{z-1}\right)+\left(\frac{1}{z-1}\right)^2-\left(\frac{1}{z-1}\right)^3+\dots\right]$$

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$$\boxed{f(z) = \frac{1}{(z-1)^3} - \frac{1}{(z-1)^4} + \frac{1}{(z-1)^5} - \dots}$$

which is Laurentz's series.

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Home Work

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Ex (1) obtain two distinct Laurentz's series for $f(z)=\frac{2z-3}{z^2-4z+3}$ in powers of $(z-4)$ indicating region of convergence.

Ex (2) obtain two distict Laurentz's series for $f(z)=\frac{1}{z^2(2-z)}$ about $z=0$ indicating region of convergence.

Hint: ROC i) $|z|<2$ (ii) $|z|>2$ — no need of partial fraction.

Ex (3) Represent the function $f(z)=\frac{4z+3}{z(z-3)(z+2)}$

(i) within $|z|=1$ (ii) in the annular region between $|z|=2$, $|z|=3$

Ex (4) obtain Taylor's or Laurentz's series for $f(z)=\frac{1}{(1+z^2)(z+2)}$ when (i) $1<|z|<2$ (ii) $|z|>2$

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Hint: $f(z)=\frac{1}{(z+2)(z^2+1)}=\frac{a}{z+2}+\frac{bz+c}{z^2+1}$

$f(z)=\frac{1}{5(z+2)}-\frac{z}{5(z^2+1)}+\frac{2}{5(z^2+1)}$

Also note that $|z|>1 \Rightarrow |z^2|>1$ and $|z|<1 \Rightarrow |z^2|<1$

Ex (4) Find the expansion of $f(z)=\frac{1}{(1+z^2)(2+z^2)}$ in powers of in

(i) $|z|<1$ (ii) $1<|z|<\sqrt{2}$ (iii) $|z|>\sqrt{2}$

Hint: let $z^2=u$ $f(z)=\frac{1}{(1+u)(2+u)}=\frac{a}{u+1}+\frac{b}{u+2}=\frac{1}{u+1}-\frac{1}{u+2}$

$\Rightarrow f(z)=\frac{1}{(z^2+1)}-\frac{1}{(z^2+2)}$ Expand.

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Type-III : Expansion of a function involving, e^z, $\sin z$, $\cosh z$ etc.

Ex ① Expand $f(z) = \sin z / (z-\pi)$ about $z=\pi$

Solution:- Expansion about $z=\pi \Longrightarrow$ in powers of $z-\pi$

$$f(z) = \frac{\sin z}{z-\pi} = \frac{\sin[(z-\pi)+\pi]}{z-\pi} = \frac{-\sin(z-\pi)}{z-\pi}$$

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$$= \frac{-1}{(z-\pi)}\left[(z-\pi) - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} - \cdots\right]$$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$$\boxed{f(z) = -1 + \frac{(z-\pi)^2}{6} - \frac{(z-\pi)^4}{120} + \cdots}$$

Ex ② Find Laurentz's series expansion about $z=1$ for $f(z) = \frac{e^{3z}}{(z-1)^3}$

Solution:- expansion about $z=1 \Longrightarrow$ in powers of $z-1$

$$f(z) = \frac{e^{3z}}{(z-1)^3} = \frac{e^{3[(z-1)+1]}}{(z-1)^3} = \frac{e^3}{(z-1)^3} e^{3(z-1)}$$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$$= \frac{e^3}{(z-1)^3}\left[1 + \frac{3(z-1)}{1!} + \frac{9(z-1)^2}{2!} + \frac{27(z-1)^3}{3!} + \frac{81(z-1)^4}{4!} + \cdots\right]$$

$$\boxed{f(z) = e^3\left[\frac{1}{(z-1)^3} + \frac{3}{(z-1)^2} + \frac{9}{2(z-1)} + \frac{9}{2} + \frac{27}{8}(z-1) + \cdots\right]}$$

Ex ③ Find Laurentz's series for $f(z) = (z-2)\sin\left(\frac{1}{z+2}\right)$ about $z=-2$

Solution: expansion about $z=-2 \Longrightarrow$ in powers of $z+2$

$$f(z) = (z-2)\sin\left(\frac{1}{z+2}\right) = [(z+2)-4]\sin\left(\frac{1}{z+2}\right)$$

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$$= [(z+2)-4]\left[\frac{1}{(z+2)} - \frac{1}{(z+2)^3 3!} + \frac{1}{(z+2)^5 5!} - \cdots\right]$$

Ex ④ obtain Laurentz's series expansion of $f(z) = z^3 e^{1/z}$ about $z=0$

Solution:- $f(z) = z^3 e^{1/z}$ is not analytic at $z=0$, for $|z|>0$

$$f(z) = z^3 e^{\frac{1}{z}} = z^3\left[1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \frac{1}{4!\,z^4} + \cdots\right]$$

$$\boxed{f(z) = z^3 + z^2 + \frac{z}{2} + \frac{1}{6} + \frac{1}{24z} + \cdots}$$

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Residue

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Zero of an analytic function :-

Let the function $f(z)$ be analytic in a given domain D, then it can be expanded about $z=a$ as

$$f(z) = f(a) + (z-a)f'(a) + \frac{1}{2!} f''(a)(z-a)^2 + \cdots$$

$$\Longrightarrow \quad f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

If $a_0 = a_1 = \cdots = a_{m-1} = 0$ and $a_m \neq 0$ then first term of above expansion is $a_m (z-a)^m$, then $z=a$ is called zero of order m.

e.g. ① $f(z) = (z-1) + 2(z-1)^2 + 3(z-1)^3 + \cdots$

$\Longrightarrow f(1) = 0 \Longrightarrow z=1$ is pole of order 1

② $f(z) = (z-3)^5 + (z-3)^6 + \cdots$

$\Longrightarrow f(3) = 0 \Longrightarrow z=3$ is zero of order 5

Singular point :

If $f(z)$ is analytic at every point in the neighbourhood of a point $z=a$, except at $z=a$, then $z=a$ is called singular point

e.g.

$f(z) = \frac{z+3}{(z+1)(z+2)} \Longrightarrow z=-1, z=-2$ are singular points.

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$f(z) = e^{1/z} \Longrightarrow z=0$ is singular point.

$f(z) = \sin(1/z) \Rightarrow z=0$ singular point.

Isolated singularity :- Singular point $z=a$ is said to be isolated singularity, if there does not exists any other singularity in the neighbourhood of $z=a$.

$\Longrightarrow$ We can construct a circle with centre at $z=a$ and radius 'δ' which does not contain any other singular point other than $z=a$. $\boxed{\text{otherwise it is non-isolated}}$

e.g. $f(z) = \frac{z^2}{z-2} \Longrightarrow z=2$, isolated singularity.

$f(z) = \frac{1}{\sin z} \Longrightarrow z = 0, \pm\pi, \pm 2\pi, \cdots$ isolated singularity.

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Note:- if $f(z)$ has only $\boxed{\text{Finite}}$ number of singularities then they are necessarily $\boxed{\text{isolated.}}$

pole:- If $z=a$ is an isolated singularity then we can expand it about $z=a$ as

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$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} b_n (z-a)^{-n}}_{\text{principal part}}$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \frac{b_3}{(z-a)^3} + \cdots$$

in above series if $b_{n+1}=0, b_{n+2}=0, \ldots$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \cdots + \frac{b_n}{(z-a)^n}$$

then $z=a$ is called $\boxed{\text{pole}}$ of order n

If $b_2 = 0 = b_3 = \cdots$ then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a}$,

then, $z=a$ is a pole of order $n=1$ called $\boxed{\text{simple pole.}}$

Isolated essential singularity :-

If principal part does not terminate then $z=a$ is called isolated essential singularity.

$\Longrightarrow$ $\boxed{\text{pole of order } \infty}$

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Non-isolated essential singularity :- All poles must have same limit point.

e.g. $f(z) = \tan\left(\frac{1}{z}\right) = \frac{\sin(1/z)}{\cos(1/z)}$, For singularity $\cos(1/z)=0$

$\Longrightarrow \frac{1}{z} = \pm n\pi/2 \Longrightarrow \boxed{z = \pm 2/n\pi}$ $n=1,2,3,\ldots$

$\Longrightarrow$ above poles are non-isolated.

taking $\lim_{n\to\infty} \pm 2/n\pi = 0 \Longrightarrow z=0$ is limit point.

Removable singularity :- singularity $z=a$ is called removable if $\lim_{z\to a} f(z)$ exists.

OR If the expression of $f(z)$ does not contain negative powers of $z-a$

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e.g. $f(z) = \frac{\sin z}{z}$ $z=0$ is singularity,

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But $\lim_{z \to 0} \frac{\sin z}{z} = 1 \Longrightarrow$ removable singularity.

(or) $\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = \left(1 - \frac{z^2}{6} + \frac{z^4}{120} + \cdots\right)$

it is free from negative powers of $z \Rightarrow$ removable.

Note:-

Entire Function or Integral Function:-

A function which is analytic everywhere in the finite z-plane is called entire function or integral function.

e.g. $f(z) = e^z$, $f(z) = \sin z$, $f(z) = \cosh z$

Mermorphic Function:-

If a function is analytic everywhere in the z-plane except at finite number of points (which are poles) is called Mermorphic function.

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e.g. $f(z) = \frac{1}{(z-1)^2 z}$, $f(z) = \frac{3z}{(z-1)(z-3)^2}$

Ex: Determine the nature of singularities

① $\frac{e^z}{(z-1)^4}$ ② $(z+1)\sin\left(\frac{1}{z+2}\right)$ ③ $\frac{1}{z^2(e^z-1)}$ ④ $e^{1/z}$

⑤ $\frac{\cot \pi z}{(z-a)^3}$ ⑥ $\sin\left(\frac{1}{z-1}\right)$ ⑦ $\sec\left(\frac{1}{z}\right)$

Solution:- ① $f(z) = \frac{e^z}{(z-1)^4} \longrightarrow z=1$ is a pole of order 4

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② $f(z) = (z+1)\sin\left(\frac{1}{z+2}\right) = [(z+2)-1]\sin\left(\frac{1}{z+2}\right)$

$= [(z+2)-1]\left[\frac{1}{z+2} - \frac{1}{3!(z+2)^3} + \frac{1}{5!(z+2)^5} - \cdots\right]$

$= (z+2)\left[\left(\frac{1}{z+2}\right) - \frac{1}{3!(z+2)^3} + \frac{1}{5!(z+2)^5} - \cdots\right]$

$-\left[\left(\frac{1}{z+2}\right) - \frac{1}{3!(z+2)^3} + \frac{1}{5!(z+2)^5} - \cdots\right]$

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$f(z) = 1 - \frac{1}{z+2} - \frac{1}{3!(z+2)^2} + \frac{1}{3!(z+2)^3} \ldots$ infinite number of terms

$\Longrightarrow$ $\boxed{\text{essential isolated singularity}}$

③ $f(z) = \frac{1}{z^2(e^z-1)} \Longrightarrow$ for singularity $z^2(e^z-1)=0$

$z^2=0$, $e^z = 1 = e^{i2n\pi}$

$\Longrightarrow$ $z=0$ (order 2) and $z = 2n\pi i$, $n = 0, \pm1, \pm2, \ldots$

$\Longrightarrow$ $\boxed{z=0 \text{ is a pole of order 3 and other are simple poles.}}$

④ $f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots$ infinitely many terms

$\Longrightarrow$ $\boxed{z=0 \text{ is an isolated essential singularity.}}$

⑤ $f(z) = \frac{\cot\pi z}{(z-a)^3} = \frac{\cos\pi z}{\sin(\pi z)(z-a)^3}$

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for singularity, $\sin\pi z = 0$, $(z-a)^3 = 0$

$\Longrightarrow$ $\boxed{z = a, \text{ pole of order 3}, \quad \pi z = \pm n\pi \Rightarrow z = \pm n}$

$n = 0, 1, 2 \ldots$ simple poles

⑥ $f(z) = \sin\left(\frac{1}{z-1}\right) = \frac{1}{(z-1)} - \frac{1}{3!(z-1)^3} + \frac{1}{5!(z-1)^5} \ldots$

$\boxed{z=1 \text{ is issolated essential singularity.}}$

⑦ $f(z) = \sec\left(\frac{1}{z}\right) = \frac{1}{\cos(1/z)}$

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for singularity $\cos(1/z) = 0 \Longrightarrow \frac{1}{z} = \pm n\pi/2$, n is odd

$\Longrightarrow$ $z = \pm 2/n\pi$

$\boxed{\text{non-isolated essential singularity}}$

Note :-

If singularity contains 'n' in Denominator it is always non-isolated.

Ex : Determine nature of the singularity

i) $f(z) = \frac{1-\cos 2z}{z}$ ii) $f(z) = \frac{1-e^{2z}}{z^3}$

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Solution:-

① $f(z) = \frac{1-\cos 2z}{z} \implies z=0$ is singularity.

$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{1-\cos 2z}{z}$ [$\frac{1-1}{0} = \frac{0}{0}$ type]

$= \lim_{z\to 0} \frac{2\sin 2z}{1}$ [L Hospital's rule]

$= 0 \implies$ limit exists $\implies$ $\boxed{\text{removable singularity}}$

② $f(z) = \frac{1-e^{2z}}{z^3} \implies z=0$, singularity

$= \frac{1}{z^3}\left[1-\left(1+\frac{2z}{1!}+\frac{4z^2}{2!}+\frac{8z^3}{3!}+\frac{16z^4}{4!}+\cdots\right)\right]$

$= -\frac{2}{z^2} - \frac{2}{z} - \frac{4}{3} - \frac{2}{3}z - \cdots$

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$\implies z=0$ is a $\boxed{\text{pole of order } n=2}$

Residue

Note:- We know Laurentz's series expansion about $z=a$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + b_1/z-a + b_2/(z-a)^2 + \cdots$$

integrating around a closed curve C we get,

$$\int_C f(z)dz = \sum_{n=0}^{\infty} a_n \int_C (z-a)^n dz + \int_C b_1 \frac{dz}{z-a} + \int_C \frac{b_2 dz}{(z-a)^2} + \cdots$$

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By C.I.T → 0

$2\pi i f(a)$ where $f(z)=1$

0 [∵ C.I.F.] $\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i f^n(a)}{n!}$

$$\int_C f(z)\, dz = 0 + b_1 2\pi i + 0 + 0 + \cdots$$

b_1 is called residue.

$$\implies \boxed{\int_C f(z)dz = 2\pi i\, b_1}$$

$$\implies \boxed{\int_C f(z)dz = 2\pi i\,(\text{Residue at } z=a)}$$

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Method-I :-

Defⁿ :- Residue :- (Expansion method)

If $z=a$ is a pole then coefficient of $\boxed{\frac{1}{z-a}}$ in Laurentz's expansion is called residue.

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Method-II :- Limit Method :-

If $z=a$ is a pole of order m then

$$\boxed{\text{Residue} = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)}$$

proof :- (out of syllabus)

If $m=1$ i.e. $z=a$ is a pole of order one

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a}$$

$$\Longrightarrow (z-a) f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n+1} + b_1$$

taking Limit as $z \longrightarrow a$, we get,

$$\lim_{z\to a} (z-a) f(z) = b_1 \Longrightarrow \text{result is true for } m=1$$

If $z=a$ is a pole of order $m=2$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2}$$

multiplying by $(z-a)^2$ we get,

$$(z-a)^2 f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n+2} + b_1 (z-a) + b_2$$

d.w.r.t. z,

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$$\frac{d}{dz} (z-a)^2 f(z) = \sum_{n=0}^{\infty} a_n (n+2) (z-a)^{n+1} + b_1$$

taking Lim as $z \longrightarrow a$, we get,

$$\lim_{z\to a} \frac{d}{dz} (z-a)^2 f(z) = b_1 \Longrightarrow \text{result is true for } m=2$$

by Mathematical induction we can prove that result is true for all values of m.

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Method - II   Fraction method

If $z=a$ is a simple pole of $f(z)= P(z)/Q(z)$

then at $z=a$ $\boxed{\text{Residue} = \left.\dfrac{P(z)}{Q'(z)}\right|_{z=a}}$

Proof :- (out of syllabus)

given $z=a$ pole of $f(z)=\dfrac{P(z)}{Q(z)} \Longrightarrow Q(a)=0$

as pole is simple pole, by limit method,

$$\text{Res} = \lim_{z\to a} (z-a) f(z) = \lim_{z\to a} (z-a)\frac{P(z)}{Q(z)} \quad \left[\tfrac{0}{0} \text{ type}\right.$$

$$= \lim_{z\to a} \frac{(z-a)P'(z)+P(z)}{Q'(z)} \quad \left[\text{L-Hospital's rule}\right]$$

$$= \frac{P(a)}{Q'(a)}$$

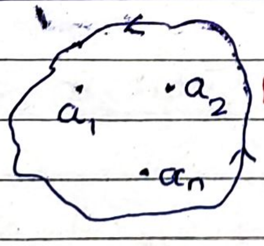
$\longrightarrow$ at $z=a$ $\text{Residue} = \left.\dfrac{P(z)}{Q'(z)}\right|_{z=a}$

Residue Th^m :-

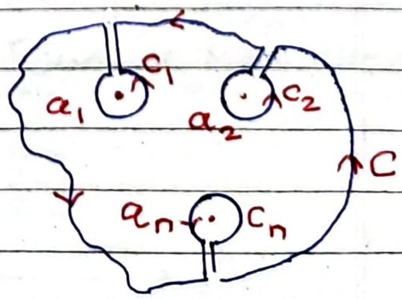
Statement :- If $f(z)$ is analytic within and on a closed curve, except at finite number of singularities (isolated) at $a_1, a_2, a_3 \ldots a_n$ lying inside c, then

$$\boxed{\int_C f(z)\,dz = 2\pi i\left[\text{sum of Residues at } a_1, a_2, \ldots a_n\right]}$$

Proof :-



draw small circles $c_1, c_2, \ldots c_n$ with centre at $a_1, a_2, \ldots a_n$ so that all circles lies inside c and do not intersect with each other (also consider their cross cut).



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By Cauchy's integral theorem,

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

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$$= 2\pi i\,(\text{Res. at } a_1) + 2\pi i\,(\text{Res. at } a_2) + \cdots + 2\pi i\,(\text{Res. at } a_n)$$

$$\Longrightarrow \boxed{\int_C f(z)\,dz = 2\pi i\,[\text{sum of Residues at } a_1, a_2, \ldots a_n]}$$

Steps :

I. For C.R.T. complete integrand is $f(z)$.
II. Find the singularities (poles) lying inside given C.
III. Find residues at above poles using proper method
IV. Apply C.R.T.

Ex ① Evaluate using Residue Theorem

$$\oint_C \frac{(1+z)}{z(2-z)}\,dz, \quad \text{where } C: |z| = 3$$

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Solution:- $f(z) = \dfrac{(1+z)}{z(2-z)} \Longrightarrow$ poles are $z=0$, $z=2$ which are simple poles

given $C: |z| = 3$

if, $z = 0$, $|0| = 0 < 3 \Longrightarrow z = 0$ lies inside

if, $z = 2$, $|2| = 2 < 3 \Longrightarrow z = 2$ lies inside

at $z = 0$, Residue $= \left.\dfrac{(1+z)}{(2-z) - z}\right|_{z=0} = \boxed{1/2}$

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at $z = 2$, Residue $= \left.\dfrac{1+z}{(2-z) - z}\right|_{z=2} = \boxed{\dfrac{-3}{2}}$

∴ By C.R.T. $\int_C f(z)\,dz = 2\pi i\,[\text{sum of Residues}]$

$$\Longrightarrow \int_C \frac{(1+z)}{z(2-z)}\,dz = 2\pi i\left[\frac{1}{2} - \frac{3}{2}\right]$$

$$= -2\pi i \longrightarrow \text{Ans.}$$

Note: If pole is of order greater than one, then fraction method fails. $\Longrightarrow$ use limit method

Ex ② Evaluate using Residue Theorem,

$$\oint_C \frac{z^3}{(z-1)^3(z-2)}\,dz, \quad \text{where } C: |z| = 2.5$$

Solution :- $f(z) = \dfrac{z^3}{(z-1)^3(z-2)} \Longrightarrow$ poles are $z=1$ order 3, $z=2$, simple pole

given curve $C: |z| = 2.5$

if $z=1$, $|1| = 1 < 2.5 \Longrightarrow z=1$ lies inside C

if $z=2$, $|2| = 2 < 2.5 \Longrightarrow z=2$ lies inside C.

at $z=1$, Residue $= \dfrac{1}{(m-1)!} \lim\limits_{z\to a} \dfrac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)$

$$= \frac{1}{2!} \lim_{z\to 1} \frac{d^2}{dz^2} (z-1)^3 \frac{z^3}{(z-1)^3(z-2)}$$

$$= \frac{1}{2} \lim_{z\to 1} \frac{d^2}{dz^2}\left[\frac{z^3}{(z-2)}\right] = \frac{1}{2} \lim_{z\to 1} \frac{d}{dz}\left[\frac{(z-2)3z^2 - z^3(1)}{(z-2)^2}\right]$$

$$= \frac{1}{2} \lim_{z\to 1} \frac{d}{dz}\left[\frac{2z^3 - 6z^2}{(z-2)^2}\right]$$

$$= \frac{1}{2} \lim_{z\to 1} \frac{(z-2)^2(6z^2-12z) - (2z^3-6z^2)\,2(z-2)}{(z-2)^4}$$

$$= \frac{1}{2}\left[\frac{(1)(-6) - (-4)(-2)}{1}\right]$$

$\Longrightarrow \boxed{\text{at } z=1, \text{ Res} = -7}$

at $z=2$ (Simple pole, $\Rightarrow$ Fraction method OR limit method)

$$\text{Residue} = \left.\frac{z^3}{(z-1)^3 + 3(z-1)^2(z-2)}\right|_{z=2} = \frac{8}{1+0}$$

$\Longrightarrow \boxed{\text{at } z=2 \text{ Res} = 8}$

$\therefore$ By C.R.T. $\displaystyle\int_C \frac{z^3}{(z-1)^3(z-2)}\,dz = 2\pi i(-7+8)$

$= 2\pi i \longrightarrow$ Ans.

Ex ③ Evaluate $\oint_C z^5 e^{1/z^2}\, dz$ where $C : |z-1| = 2$

Solution:- $f(z) = z^5 e^{1/z^2}$

Here $z=0$ is isolated essential singularity

$\Longrightarrow$ limit method and fraction method fails. $f(z) \neq P(z)/Q(z)$

$e^x = 1 + (x/1!) + (x^2/2!) + \cdots$

$\Longrightarrow f(z) = z^5 e^{1/z^2} = z^5\left[1 + \left(\frac{1}{z^2}\right)\frac{1}{1!} + \frac{1}{2!}\left(\frac{1}{z^4}\right) + \frac{1}{3!}\left(\frac{1}{z^6}\right) + \cdots\right]$

$f(z) = z^5 + z^3 + \frac{1}{2}z + \frac{1}{6z} + \frac{1}{24z^3} + \cdots$

By expansion method at $z=0$, Residue = coefficient of $\frac{1}{z-0}$

$\Longrightarrow$ at $z=0$, $\boxed{\text{Residue} = \frac{1}{6}}$

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By C.R.T. $\int_C f(z)\, dz = 2\pi i$ (sum of residues)

$\Longrightarrow \int_C z^5 e^{1/z^2} dz = 2\pi i\left(\frac{1}{3}\right) = \frac{\pi i}{3} \longrightarrow$ Ans.

Ex ④ Evaluate $\oint_C \frac{e^z}{\cos \pi z} dz$, where $C : |z| = 1$.

Solution:- $f(z) = \frac{e^z}{\cos \pi z}$, For poles $\cos \pi z = 0$

$\Longrightarrow \pi z = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2 \cdots$

$\Longrightarrow$ $\boxed{\text{poles are } z = \pm 1/2, \pm 3/2, \cdots \text{ infinitely many poles.}}$

all are simple poles.

given $C : |z| = 1$

if $z = \pm 1/2$, $|\pm 1/2| = 0.5 < 1 \Longrightarrow$ both poles are inside

if $z = \pm 3/2$, $|\pm 3/2| = 1.5 > 1 \Longrightarrow$ both poles are outside

similarly other poles will be outside.

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By fraction method,

at $z = \frac{1}{2}$, $\text{Res} = \left.\frac{e^z}{-\pi \sin(\pi z)}\right|_{z=1/2} = \frac{e^{1/2}}{-\pi \sin(\pi/2)} = \boxed{\frac{e^{1/2}}{-\pi}}$

at $z = \frac{-1}{2}$, $\text{Res} = \left.\frac{e^z}{-\pi \sin(\pi z)}\right|_{z=\frac{-1}{2}} = \frac{e^{-1/2}}{-\pi \sin(-\pi/2)} = \boxed{\frac{e^{-1/2}}{\pi}}$

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$\therefore$ By C.R.T. $\int_C f(z)\,dz = 2\pi i$ (sum of Residues)

$$\Longrightarrow \oint_C \frac{e^z}{\cos\pi z}\,dz = 2\pi i\left[\frac{-e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi}\right] = -\pi i = -2i\left(e^{1/2} - e^{-1/2}\right)$$

$$= -2i\, 2\sinh(1/2) \qquad \left[\because \sinh\theta = (e^{\theta} - e^{-\theta})/2\right.$$

$$= -4i\sinh(0.5) \longrightarrow \text{Ans.}$$

Ex (5) Evaluate $\oint_C \frac{dz}{z^2 \sin z}$, $C: |z| = 1$

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Solution:- $f(z) = \frac{1}{z^2 \sin z} \Longrightarrow$ For poles $z^2 \sin z = 0$

$z^2 = 0$, $\sin z = 0$

$\Longrightarrow z = 0$ (order 2) $\quad z = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$\Longrightarrow$ $\boxed{\text{poles are } z = 0 \text{ (order 3)}, z = \pm\pi, \pm 2\pi, \ldots \text{order one}}$

given $C: |z| = 1$

if $z = 0$, $|0| = 0 < 1 \Longrightarrow$ pole $z = 0$ of order 3 is inside.

if $z = \pm\pi$, $|\pm\pi| = 3.142 > 1 \Longrightarrow$ both poles are outside

Similarly all other poles are outside of given curve

To Find residue:-

By limit method

at $z = 0$ $\quad \text{Res} = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2} z^3 \frac{1}{z^2\sin z} = \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\frac{z}{\sin z}$

$$= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left[\frac{\sin z - z\cos z}{\sin^2 z}\right]$$

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$$= \frac{1}{2}\lim_{z\to 0}\left[\frac{\sin^2 z[\cos z - \cos z + z\sin z] - [\sin z - z\cos z]\,2\sin z\cos z}{\sin^4 z}\right.$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{z\sin^3 z - 2\sin^2 z\cos z + 2z\sin z\cos^2 z}{\sin^4 z}$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{z\sin^2 z - 2\sin z\cos z + 2z\cos^2 z}{\sin^3 z} \quad \left(\frac{0}{0}\text{ type}\right)$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{\sin^2 z + 2z\sin z\cos z - 2\cos^2 z + 2\sin^2 z + 2\cos^2 z - 4z\cos z\sin z}{3\sin^2 z\cos z}$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{3\sin^2 z - 2z\cos z\sin z}{3\sin^2 z\cos z} \qquad \text{cancelling } \sin z$$

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$$= \frac{1}{2} \lim_{z \to 0} \left[\frac{3\sin z - 2z\cos z}{3\sin z \cos z} \right] \quad \left[\frac{0}{0} \text{ type} \right]$$

$$= \frac{1}{2} \lim_{z \to 0} \left[\frac{3\cos z - 2\cos z + 2z\sin z}{3\cos^2 z - 3\sin^2 z} \right]$$

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$$= \frac{1}{2} \left[\frac{3-2+0}{3-0} \right]$$

$$\Longrightarrow \boxed{\text{Res} = 1/6}$$

OR using Expansion method :-

$$f(z) = \frac{1}{z^2 \sin z} = \frac{1}{z^2\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]}$$

$$= \frac{1}{z^3\left[1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots\right]} = \frac{1}{z^3}\left[1 - \left(\frac{z^2}{6} - \frac{z^4}{120} + \cdots\right)\right]^{-1}$$

$$= \frac{1}{z^3}\left[1 + \left(\frac{z^2}{6} - \frac{z^4}{120} + \cdots\right) + \left(\frac{z^2}{6} - \frac{z^4}{120} + \cdots\right)^2 + \cdots\right]$$

$$f(z) = \left[\frac{1}{z^3} + \frac{1}{6z} - \frac{z}{120} + \cdots\right] + \left[\frac{z}{36} + \cdots\right]$$

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By expansion method,

at $z = 0$, Residue = coefficient of of $\frac{1}{z}$

$$\Longrightarrow \boxed{\text{Residue} = 1/6}$$

$\therefore$ By C.R.T. $\int_C f(z)\,dz = 2\pi i$ (sum of Residues)

$$\therefore \oint_C \frac{dz}{z^2 \sin z} = 2\pi i\left(\frac{1}{6}\right) = \boxed{\frac{\pi i}{3}} \longrightarrow \text{Ans.}$$

Ex ⑥ Evaluate $\oint_C \frac{\sin z}{(4z^2 - 8iz)}\,dz$, where C is the boundaries of the squares with vertices $\pm 3, \pm 3i$

Solution :- $f(z) = \frac{\sin z}{4z^2 - 8iz} = \frac{\sin z}{4z(z-2i)} \Longrightarrow$ Poles are $z=0,\ z=2i$

C is boundary of square with vertex

$\pm 3 = \pm 3 + 0i \Longrightarrow (3,0), (-3,0)$

$\pm 3i = 0 \pm 3i \Longrightarrow (0,3), (0,-3)$

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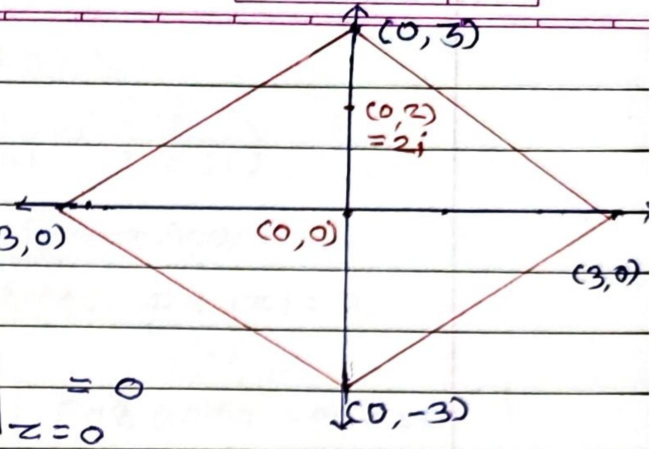
poles are $z = 0 = 0+0i = (0,0)$

$z = 2i = 0+2i = (0,2)$

$\Longrightarrow$ both poles are simple poles which are inside C.



To find Residue :-

at $z=0$ $\quad Res = \left.\dfrac{\sin z}{4(z-2i)+4z}\right|_{z=0} = 0$

at $z=2i$ $\quad Res = \left.\dfrac{\sin z}{4(z-2i)+4z}\right|_{z=2i} = \dfrac{\sin 2i}{0+8i} = \dfrac{i\sinh 2}{8i}$ $\quad [\because \sin i\theta = i\sinh\theta$

$Res = \dfrac{\sinh 2}{8}$

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by Residue Th^m, $\displaystyle\oint_C \frac{\sin z}{(4z^2-8iz)}\,dz = 2\pi i\left(0+\frac{\sinh 2}{8}\right)$

$= \dfrac{\pi i}{4}\sinh 2$

Ex (7) Evaluate $\displaystyle\oint_C \frac{z\sec z}{1-z^2}\,dz$, where C is ellipse $4x^2+\frac{9}{4}y^2 = 9$

Solution :- $f(z) = \dfrac{z\sec z}{1-z^2} \Longrightarrow f(z) = \dfrac{z}{(1-z^2)\cos z}$

for poles $z^2 = 1$, $\cos z = 0$

$\Longrightarrow$ $\boxed{z=1,\ z=-1,\ z=\pm\pi/2,\ z=\pm 3\pi/2,\ \ldots}$

all poles are simple poles

given C: $4x^2+\frac{9}{4}y^2 = 9$

if $z = 1 = 1+0i = (1,0) \Longrightarrow 4+0 = 4 < 9 \Longrightarrow$ pole is inside

if $z = -1 = -1+0i = (-1,0) \Longrightarrow 4+0 = 4 < 9 \Longrightarrow$ Pole is inside

if $z = \pm\pi/2 = \pm\pi/2 + 0i = (\pm\pi/2, 0) \Longrightarrow 4(1.52)^2+0 > 9 \Longrightarrow$ out

simillarly all other poles are outside.

at $z=1$, $\quad Res = \left.\dfrac{z}{-2z\cos z-(1-z^2)\sin z}\right|_{z=1} = \boxed{\dfrac{1}{-2\cos 1}}$

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at $z=-1$, $\quad Res = \left.\dfrac{z}{-2z\cos z-(1-z^2)\sin z}\right|_{z=-1} = \dfrac{-1}{2\cos(-1)} = \boxed{\dfrac{-1}{2\cos 1}}$

By Residue Theorem,

$$\oint_C \frac{z\sec z}{1-z^2}\,dz = 2\pi i\left(\frac{-1}{2\cos 1} - \frac{1}{2\cos 1}\right)$$

$$= \boxed{-2\pi i\sec 1} \longrightarrow \text{Ans.}$$

Ex (8) Evaluate $\oint_C \tan z\,dz$ where $c : |z| = 2$

Solution:- $f(z) = \dfrac{\sin z}{\cos z} \Longrightarrow$ For poles $\cos z = 0$, $z = \pm\pi/2, \pm 3\pi/2, \ldots$ All are simple poles

given $c : |z| = 2$

if $z = \pm\pi/2$, $|\pm\pi/2| = 1.52 < 2 \Longrightarrow$ both poles are inside

if $z = \pm 3\pi/2$, $|\pm 3\pi/2| = 4.56 > 2 \Longrightarrow$ both poles are outside

similarly all other poles will be outside.

at $z = \pi/2$, $\text{Res} = \left.\dfrac{\sin z}{-\sin z}\right|_{z=\pi/2} = \boxed{-1}$

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at $z = -\pi/2$, $\text{Res} = \left.\dfrac{\sin z}{-\sin z}\right|_{z=-\pi/2} = \boxed{-1}$

By Residue Theorem,

$$\oint_C \tan z\,dz = 2\pi i(-1-1) = \boxed{-4\pi i} \longrightarrow \text{Ans.}$$

Ex (9) Evaluate $\oint_C \dfrac{z+4}{z^2+2z+5}\,dz$, where $c : |z+1+i| = 2$

Solution:- $f(z) = \dfrac{z+4}{z^2+2z+5} \Longrightarrow$ For poles $z^2+2z+5 = 0$, $z = -1+2i$, $z = -1-2i$

given c is $|z+1+i| = 2$

if $z = -1+2i$, $|-1+2i+1+i| = |0+3i| = \sqrt{0+9} = 3 > 2 \Longrightarrow$ out

if $z = -1-2i$, $|-1-2i+1+i| = |0-i| = \sqrt{0+1} = 1 < 2 \Longrightarrow$ inside

at $z = -1-2i$ $\text{Res} = \left.\dfrac{z+4}{2z+2}\right|_{z=-1-2i} = \dfrac{-1-2i+4}{-2-4i+2} = \boxed{\dfrac{3-2i}{-4i}}$

By Residue Thm,

$$\oint_C \frac{z+4}{z^2+4z+5}\,dz = 2\pi i\left(\frac{3-2i}{-4i}\right) = \boxed{\frac{-\pi}{2}(3-2i)}$$

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Ex (10) using Cauchy's Residue Theorem, evaluate

$\oint_C e^{-1/z} \sin(\frac{1}{z}) dz$, where C is $|z|=1$

Solution :- $f(z) = e^{-1/z} \sin(1/z) \Longrightarrow z=0$ is the singularity

given $C: |z|=1$

If $z=0$, $|0| = 0 < 1 \Longrightarrow z=0$ lies inside C.

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To Find Residue :- $f(z) = e^{-1/z} \sin(\frac{1}{z})$

$$f(z) = \left(1 - \frac{1}{z} + \frac{1}{z^2 2!} - \frac{1}{z^3 3!} + \cdots\right)\left(\frac{1}{z} - \frac{1}{3! z^3} + \frac{1}{5! z^5} - \cdots\right)$$

$$= \left(\frac{1}{z} - \frac{1}{6z^3} + \frac{1}{120 z^5} - \cdots\right) + \left(-\frac{1}{z^2} - \frac{1}{6z^4} + \cdots\right) + \cdots$$

at $z=0$, Residue = coefficient of $\frac{1}{z-0}$ = coe. of $\frac{1}{z}$

$\Longrightarrow \boxed{Res = 1}$

By C.R.T. $\int_C f(z) dz = 2\pi i$ (sum of Residues)

$\therefore \int_C e^{-\frac{1}{z}} \sin(\frac{1}{z}) dz = 2\pi i (1) = \boxed{2\pi i} \longrightarrow$ ☺

Ex (11) Evaluate using Residue Theorem,

$\int_C \frac{e^z}{(z^2+\pi^2)^2} dz$ where $C: |z|=4$

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Solution :- $f(z) = \frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{[(z-i\pi)(z+i\pi)]^2}$

$f(z) = \frac{e^z}{(z-i\pi)^2 (z+i\pi)^2} \Longrightarrow$ poles are $z = i\pi$, $z=-i\pi$ with order 2

given $C: |z|=4$

if $z=i\pi$, $|i\pi| = \pi = 3.142 < 4 \Longrightarrow$ pole $z=i\pi$ is inside

if $z=-i\pi$, $|-i\pi| = \pi = 3.142 < 4 \Longrightarrow$ $z=-i\pi$ lies inside C.

By limit method,

at $z=i\pi$ $Res = \frac{1}{1!} \lim_{z\to i\pi} \frac{d}{dz} (z-i\pi)^2 \frac{e^z}{(z-i\pi)^2 (z+i\pi)^2}$

$$= \lim_{z\to i\pi} \frac{d}{dz} \left[\frac{e^z}{(z+i\pi)^2}\right]$$

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$$= \lim_{z \to i\pi} \left[\frac{(z+i\pi)^2 e^z - 2(z+i\pi) e^z}{(z+i\pi)^4} \right] = \frac{-4\pi^2 e^{i\pi} - 4i\pi e^{i\pi}}{16\pi^4}$$

$$= \frac{4\pi^2 + 4i\pi}{16\pi^4}$$ $[\because e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0i = -1]$

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$\Longrightarrow$ $\boxed{Res = (\pi + i)/4\pi^3}$

at $z = -i\pi$, $Res = \frac{1}{1!} \lim_{z \to -i\pi} \frac{d}{dz} (z+i\pi)^2 \frac{e^z}{(z-i\pi)^2 (z+i\pi)^2}$

$$Res = \lim_{z \to -i\pi} \frac{d}{dz} \frac{e^z}{(z-i\pi)^2}$$

$$= \lim_{z \to -i\pi} \left[\frac{(z-i\pi)^2 e^z - e^z 2(z-i\pi)}{(z-i\pi)^4} \right] = \frac{-4\pi^2 e^{-i\pi} + 4i\pi e^{-i\pi}}{16\pi^4}$$

$$= \frac{4\pi^2 - 4i\pi}{16\pi^4}$$ $[e^{-i\pi} = \cos\pi - i\sin\pi = -1 + i0 = -1]$

$\Longrightarrow$ $\boxed{Res = (\pi - i)/4\pi^3}$

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By Residue Thm,

$$\int_C F(z)\,dz = 2\pi i \text{ (sum of Residues)}$$

$$\therefore \int_C \frac{e^z}{(z+\pi^2)^2} dz = 2\pi i \left[\frac{\pi + i}{4\pi^3} + \frac{\pi - i}{4\pi^3} \right] = 2\pi i \left(\frac{2\pi}{4\pi^3} \right)$$

$= \boxed{i/\pi} \longrightarrow$ Ans.

Ex (12) Evaluate $\int_C \frac{dz}{z\sin z}$, where $c: x^2+y^2=1^2$

Solution:- $F(z) = \frac{1}{z\sin z} \Longrightarrow$ For poles $z\sin z = 0$

$z = 0, \sin z = 0,$

$\Longrightarrow$ $\boxed{z=0, \quad z=0, \quad z = \pm\pi, \quad z = \pm 2\pi \quad \cdots}$

$z = 0$ is a pole of order 2, other poles are simple poles.

given $x^2+y^2=1 \Longrightarrow c: |z|=1$

if $z=0$, $|0| = 0 < 1 \longrightarrow z=0$ lies inside c.

if $z = \pm\pi$, $|\pm\pi| = \pi = 3.142 > 1 \longrightarrow$ both poles are outside

similarly all other poles are outside.

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at $z=0$, $Res = \frac{1}{1!} \lim_{z \to 0} \frac{d}{dz} z^2 \frac{1}{z\sin z}$

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$$= \lim_{z\to 0} \frac{d}{dz}\left(\frac{z}{\sin z}\right) = \lim_{z\to 0}\left[\frac{\sin z - z\cos z}{\sin^2 z}\right] \quad \left[\frac{0}{0} \text{ type}\right]$$

$$= \lim_{z\to 0}\left[\frac{\cos z - \cos z + z\sin z}{2\sin z\cos z}\right] \quad \left[\text{By, L'Hospital's rule}\right]$$

$$= \lim_{z\to 0}\left[\frac{z\sin z}{2\sin z\cos z}\right] = \lim_{z\to 0} \frac{z}{2\cos z}$$

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$$\Longrightarrow \boxed{\text{Residue} = 0}$$

By Residue Theorem, $\int_C f(z)\,dz = 2\pi i\,(\text{sum of Residues})$

$$\therefore \int_C \frac{dz}{z\sin z} = 0$$

OR by expansion method :-

$$f(z) = \frac{1}{z\sin z} = \frac{1}{z\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]}$$

$$= \frac{1}{z^2\left[1 - \left(\frac{z^2}{6} - \frac{z^4}{120} + \cdots\right)\right]} = \frac{1}{z^2}\left[1 - \left(\frac{z^2}{6} - \frac{z^4}{120} + \cdots\right)\right]^{-1}$$

$$= \frac{1}{z^2}\left[1 + \left(\frac{z^2}{6} - \frac{z^4}{120} + \cdots\right) + \left(\frac{z^2}{6} - \frac{z^4}{120} + \cdots\right)^2 + \cdots\right]$$

$$\Longrightarrow f(z) = \left(\frac{1}{z^2} + \frac{1}{6} - \frac{z^2}{120} + \cdots\right) + \left(\frac{z^2}{6} - \cdots\right)$$

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$\Longrightarrow$ at $z = 0$, Residue $=$ coe $\left(\frac{1}{z}\right)$

$$\Longrightarrow \boxed{\text{Res} = 0}$$

Ex (13) Evaluate $\int_C \frac{dz}{\sinh z}$, where $C: x^2 + y^2 = 16$

Solution :- $f(z) = \frac{1}{\sinh z} = \frac{1}{-i\sin(iz)}$

$\sin i\theta = i\sinh\theta$
$\sinh\theta = \frac{1}{i}\sin i\theta$
$= -i\sin i\theta$

$\Longrightarrow$ For poles, $iz = 0, \pm\pi, \pm 2\pi, \ldots$

$$\Longrightarrow \boxed{z = 0, \mp i\pi, \mp 2\pi i, \ldots}$$

given $c: x^2 + y^2 = 4^2 \Longrightarrow |z| = 4$

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if $z=0$, $|0|=0<1 \Longrightarrow \boxed{z=0 \text{ lies inside } C}$

if $z=\mp\pi i$, $|\mp\pi i|=|0\mp\pi i|=\sqrt{0+\pi^2}=\pi=3.142<4 \Longrightarrow \boxed{\text{inside}}$

if $z=\mp 2\pi i$, $|\mp 2\pi i|=|0\mp 2\pi i|=\sqrt{0+4\pi^2}=2\pi=6.284>4$

$\Longrightarrow \boxed{\text{outside}}$

To Find Residue :-

at $z=0$, $\text{Res}=\left.\dfrac{1}{-i^2\cos(iz)}\right|_{z=0} \longrightarrow \boxed{\text{Res}=1}$

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at $z=\pi i$, $\text{Res}=\left.\dfrac{1}{-i^2\cos(iz)}\right|_{z=i\pi} = \dfrac{1}{\cos(-\pi)} \Longrightarrow \boxed{\text{Res}=-1}$

at $z=-\pi i$, $\text{Res}=\left.\dfrac{1}{-i^2\cos(iz)}\right|_{z=-\pi i} = \dfrac{1}{\cos(-\pi)} \Longrightarrow \boxed{\text{Res}=-1}$

By Residue Thm $\int_C f(z)\,dz = 2\pi i(\text{sum of Residues})$

$\therefore \int_C \dfrac{dz}{\sinh z} = 2\pi i(1-1-1) = \boxed{-2\pi i} \longrightarrow$ Ans.

Ex (14) Evaluate $\oint_C \dfrac{dz}{4z^2+1}$, where $C: |z|=1$

Solution:- $f(z)=\dfrac{1}{4z^2+1} \longrightarrow$ For poles $4z^2+1=0$

$\Longrightarrow z=-i/2, \quad z=i/2$ simple pole.

given $C: |z|=1$

if $z=\pm i/2$, $|\pm i/2|=|0\pm i/2|=\sqrt{0+1/4}=0.5<1 \Longrightarrow$ both poles inside

at $z=i/2$ $\text{Res}=\left.\dfrac{1}{8z}\right|_{z=i/2} \longrightarrow \boxed{\text{Res}=\dfrac{1}{4i}}$

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at $z=-i/2$ $\text{Res}=\left.\dfrac{1}{8i}\right|_{z=-i/2} \longrightarrow \boxed{\text{Res}=\dfrac{-1}{4i}}$

By Residue Theorem, $\int_C f(z)\,dz = 2\pi i(\text{sum of residues})$

$\therefore \oint_C \dfrac{1}{4z^2+1}\,dz = 2\pi i\left(\dfrac{1}{4i}-\dfrac{1}{4i}\right) = \boxed{0}$ — Ans

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Home Work

Evaluate using Residue Theorem,

① $\oint_C \left(\frac{\cos\pi z^2 + \sin\pi z^2}{z - z^2}\right) dz$, $C: |z-2| = 4$

Hint: poles $z = 0$, $z = 1$, Ans $= 4\pi i$

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② $\oint_C \frac{e^{2z}}{(z-\pi i)^3} dz$, where C is $|z-2i| = 2$

Hint: pole $z = \pi i$, Ans $= 4\pi i$

③ $\oint_C \frac{\sin z}{z^6} dz$, where $c: |z| = 1$

Hint: pole $z = 0$, Ans $= \pi i/60$

④ $\oint_C \operatorname{cosec} z\, dz$, where $C: |z| = 1$,

Hint: $f(z) = 1/\sin z$, Ans $= 2\pi i$

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⑤ $\oint_C \frac{15z+9}{z^3 - 9z} dz$, where C is $|z-1| = 3$

Ans $= 4\pi i$, poles $z = 0$, $z = 3$, $z = -3$

Ex ⑥ State true or false with proper justification

$$\oint_C z^3 dz = \oint_C \frac{dz}{z^3} \quad \text{where } c: |z-2i| = 1$$

$f(z) = z^3$ analytic

and c is closed

$\Longrightarrow$ by C.I.T $\oint_C z^3 dz = 0$

$f(z) = 1/z^3 \Longrightarrow$ pole is $z = 0$

$|0 - 2i| = \sqrt{0+4} = 2 > 1 \Longrightarrow$ outside

$\Longrightarrow f(z) = 1/z^3$ is analytic and

c is closed

$\Longrightarrow$ by C.I.F. $\oint_C \frac{dz}{z^3} = 0$

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$$\Longrightarrow \oint_C z^3 dz = \oint_C \frac{dz}{z^3} = 0 \Longrightarrow \text{True.}$$