HW 2 due today.

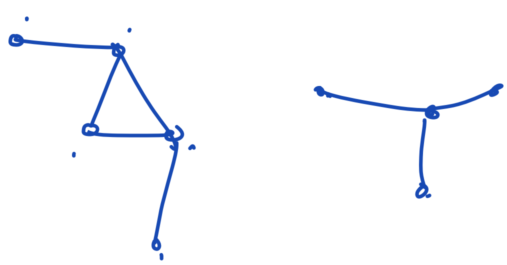
HW 3 will be released today.

Exam 1 in appx two weeks

↳ Feb 20.

Ex: Prove that every graph with n vertices & m edges has at least n-m connected components.

Soln:



$n = 9$

$m = 8$

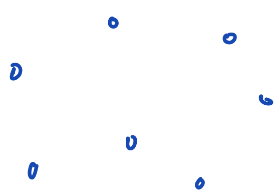
$n - m = 1$

We will prove the claim using induction on m.

IH: Let $k \geq 0$ be an arbitrary but

particular integer. Assume that the claim holds when $m = k$. In other words, assume that a graph with n vertices & k edges has at least $n-k$ connected comp.

BC : $m = 0$ $\quad \# cc = n \geq n - 0$ ✓.



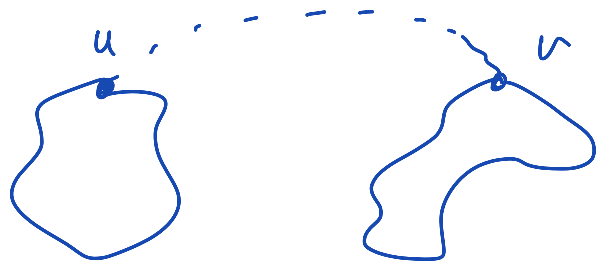
IS : We want to prove the claim when $m = k+1$. Let G be a graph with n vertices and $k+1$ edges. We want to prove that # conn. comp in $G \geq n - (k+1)$

$= n - k - 1$.

Let $G' = G - e$, where $e = (u,v)$ is an arbitrary edge in G.

Observe that G' has n vertices and exactly k edges. By IH, G has atleast $n-k$ connected components.

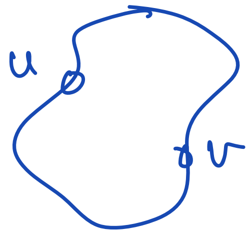
Case I: u & v belong to different connected components in G'.



Add e to G' to obtain G.

$$\# cc \text{ in } G = \# cc \text{ in } G' - 1$$
$$\geq n-k-1$$

Case II : u & v belong to the same cc in G'.



Add e to G' to obtain G.

$$\begin{aligned}\# \text{ cc in } G &= \# \text{ cc in } G' \\ &\geq n-k \\ &> n-k-1\end{aligned}$$ ✓

Ex: Prove that every connected graph with n vertices has at least $n-1$ edges.

Soln: # connected components in a connected graph = $\underline{\quad 1 \quad}$.

$\# cc(G) \geq n - m$

$1 \geq n - m$

$\therefore \boxed{m \geq n - 1}$

<u>Alternate proof</u> : We will prove the claim by proving its Contrapositive. That is, we will prove that if a graph G with n vertices has $\leq n-2$ edges then G is disconnected.

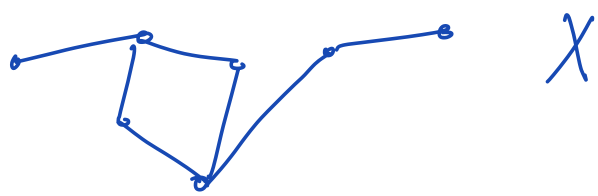
$\# cc$ in $G \geq n - m$, where $m = |E(G)|$

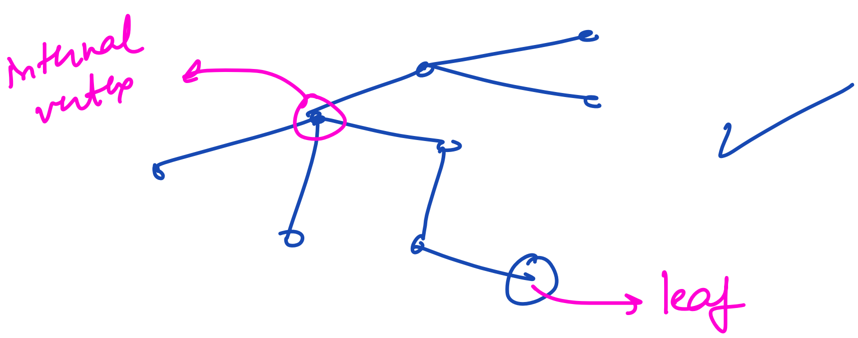
Since $m \le n-2$, we have

$\#\, cc$ in $G \geqslant n-(n-2) = 2.$, which

means that the graph G is not connected.

<u>Trees</u> : A tree is a <u>connected</u> <u>acyclic</u> graph.

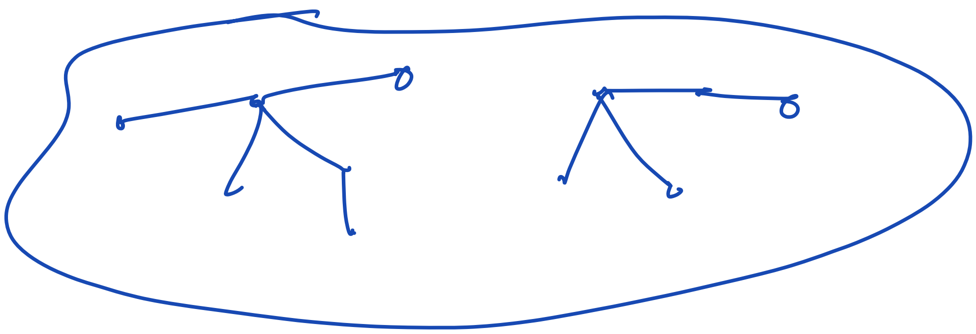




A vertex with degree 1 is called a leaf.

All other vertices are internal vertices.

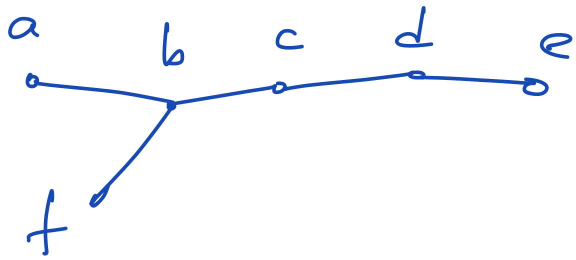
A *forest* is acyclic graph.



Ex: Prove that (every tree with at least two vertices has at least two leaves) and deleting a leaf from a n-vertex tree gives us another tree with $n-1$ vertices.

Proof: Let T be a tree. Let P be a maximal path in T. Let u & v be the end vertices of P.

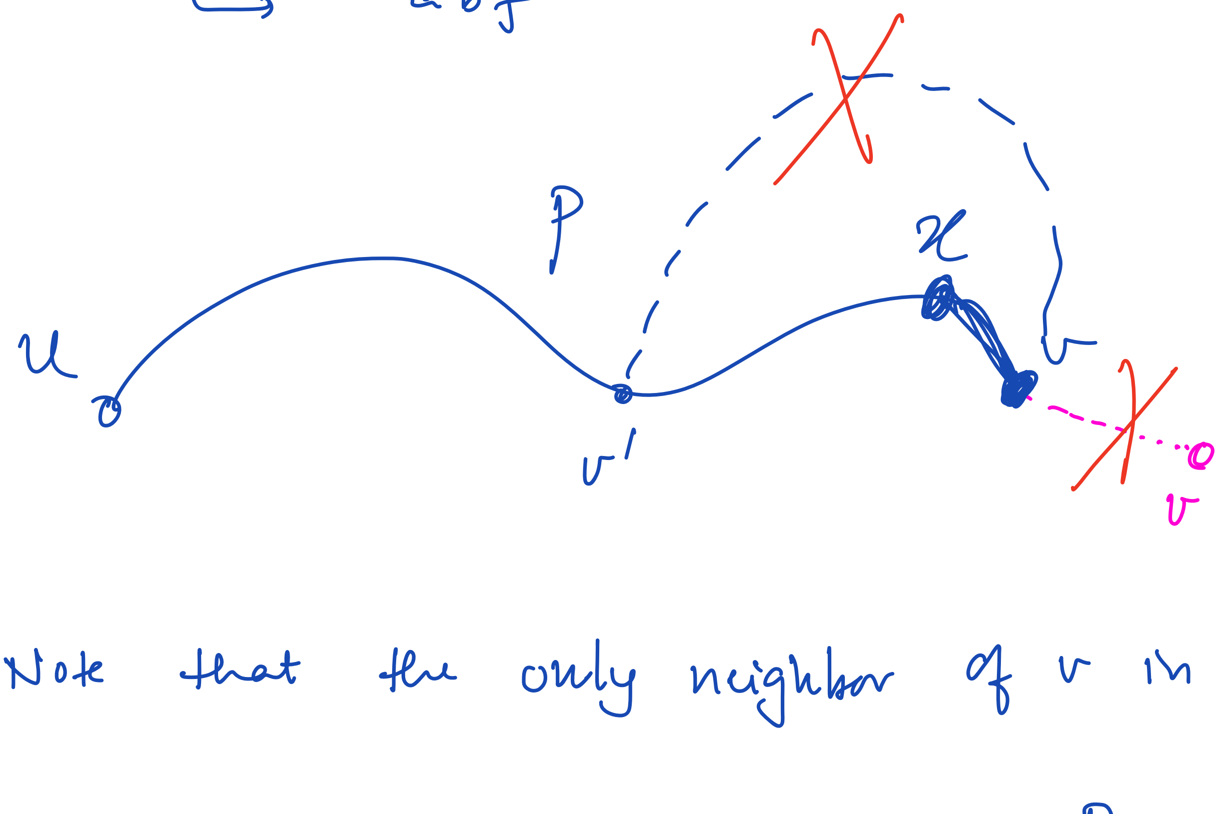
path that is not contained in a longer path.



maximum length path : a b c d e ✓

maximal path : a b c d e ✓

↳ a b f



Note that the only neighbor of v in the tree T is its neighbor in P.

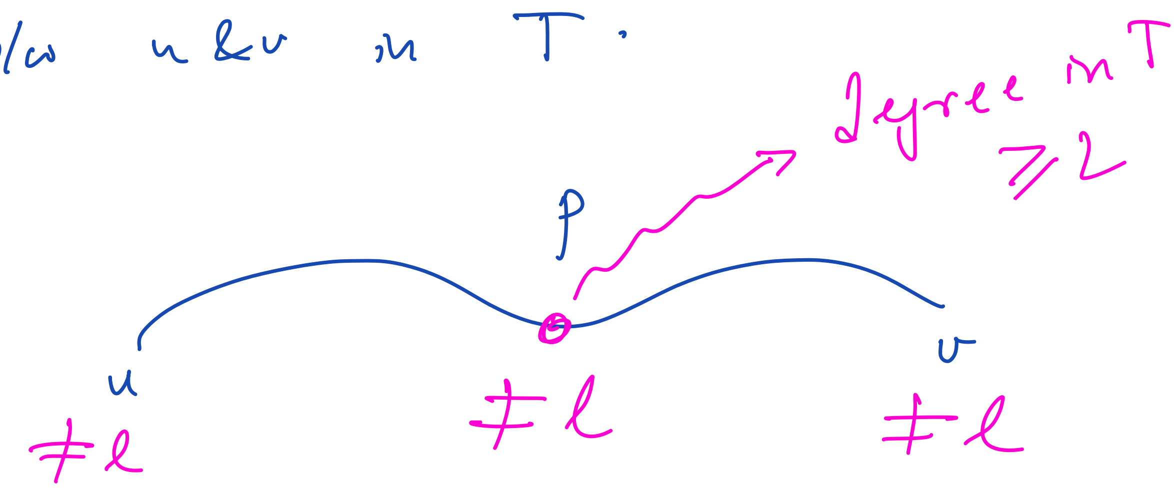
Thus v is a leaf in T.

The same argument can be applied to u. Thus T must have atleast two leaves.

Let l be a leaf in T. Let $T' = T - l$. Clearly, T' has exactly $n-1$ vertices. It remains to show that T' is a tree. Since T was acyclic, removing an edge from T cannot create a cycle. Thus

T' is acyclic.

To show that T' is connected, we need to show that for any two vertices u & v in T', there is a path b/w u & v. Consider a path P b/w u & v in T.



Note that P does not contain l.

This means that the

path P is intact in T'.

Thus T' is connected.

Ex: For a n-vertex graph G, the following are equivalent and characterize trees with n vertices.

(1) G is a tree. connected, acyclic

(2) G is connected & has exactly $n-1$ edges.

(3) G is minimally connected, i.e., G is connected but $G-e$ is disconnected for every edge $e \in G$.

(4) G contains no cycle but $G + \{x, y\}$ does for any two non-adjacent vertices $x, y \in G$.

(5) Any two vertices in G are linked by a unique path in G.

Proof : We will prove the claim by proving $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$

$$\Downarrow$$

$$(1).$$

$(1) \Rightarrow (2)$

Clearly T is connected. ✓.

It remains to show that T has exactly $n-1$ edges.

We will prove the claim using induction on n.

IH: Let $k \geq 1$ be an integer. Assume that the claim holds when $n=k$. That is, a tree with k vertices has exactly $k-1$ edges.

BC: $n=1$.

IS: We want to prove that the claim holds when $n = k+1$.

Let T be a tree with $k+1$ vertices. We want to show that T has exactly k edges.

Let $T' = T - l$, where l is a leaf in T.

T' has k vertices. By IH, T' has exactly $k-1$ edges.

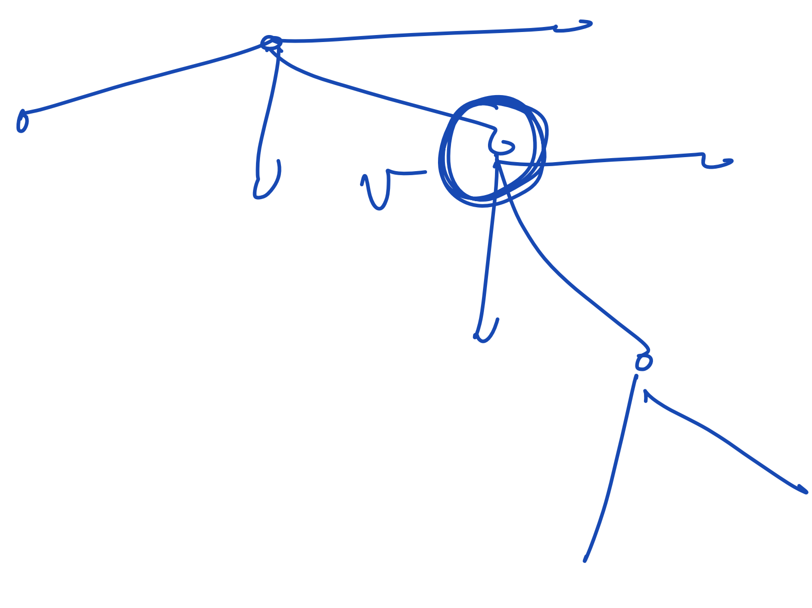
Add ℓ to T' to obtain T.

Since $\deg(\ell) = 1$, when we add ℓ to T', we are adding one edge.

Thus $\#$ edges in $T = \#$ edges in $T' + 1$

$$= k - 1 + 1$$

$$= k \checkmark .$$



$J' = J - v$ ✓