FEC101 - Engineering Mathematics-1 (EM-I) solution
Rev(2019) Date. 22-April-2021. Dr. Uday Kashid.

Q1. MCQ. 20 ques (2 marks each) 40 marks

Q.2 (A) $(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$

$\therefore \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta + i(6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta$
$+ 6\cos\theta\sin^5\theta) = \cos 6\theta + i\sin 6\theta$

By comparing real & Imaginary parts.

$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$

$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$

If we compare to given part.

$\cos 6\theta = a\cos^6\theta + b\cos^4\theta\sin^2\theta + c\cos^2\theta\sin^4\theta + d\sin^6\theta$

Then $a = 1,\ b = -15,\ c = 15,\ d = -1$.

(B) $\log[\sin(x+iy)] = a+ib$

$\sin(x+iy) = e^{a+ib} = e^a e^{ib}$

$\sin x\cosh y + i\cos x\sinh y = e^a(\cos b + i\sin b)$

$\Rightarrow \sin x\cosh y = e^a\cos b$ (I)

$\cos x\sinh y = e^a\sin b$ (II)

(I) $\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = e^{2a}$ (1)

$\sin^2 x\cosh^2 y + (1-\sin^2 x)(\cosh^2 y - 1) = e^{2a}$.

$\cosh^2 y - 1 + \sin^2 x = e^{2a}$

$\frac{1+\cosh 2y}{2} - 1 + \frac{1-\cos 2x}{2} = e^{2a}$

$\boxed{\cosh 2y - \cos 2x = 2e^{2a}}$

(II) $\frac{e^a\sin b}{e^a\cos b} = \frac{\cos x\sinh y}{\sin x\cosh y}$

$\boxed{\tan b = \cot x\tanh y}$

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(c)

$$A = \begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & 2 \\ 3 & 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$, $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$C_3 \to C_3 - C_1$, $C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + C_2$, $C_4 \to C_4 + C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ. \quad \therefore \rho(A) = 2$$

(b)

$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}$ is orthogonal Hence $AA' = A'A = I$

We have $AA' = I$

$$\frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{9}\begin{bmatrix} 5+a^2 & 4+ab & ac-2 \\ 4+ab & 5+b^2 & bc+2 \\ -2+ac & bc+2 & 8+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \frac{5+a^2}{9} = 1, \quad \frac{5+b^2}{9} = 1, \quad \frac{8+c^2}{9} = 1$

$\Rightarrow a^2 = 4 \quad b^2 = 4, \quad c^2 = 1$

$\Rightarrow a = \pm 2 \quad b = \pm 2 \quad c = \pm 1$

But soln are $(2,-2,1)$ & $(-2,2,-1)$

E let 3 parts of 24 be x, y & z. Hence

$$f(x,y,z) = xy^2z^3 \quad \text{—(I)}$$

& $$\phi(x,y,z) = x+y+z-24=0 \quad \text{(II)}$$

By Lagranges Fun

$$L(x,y,z,\lambda) = f(x,y,z) + \lambda\,\phi(x,y,z)$$

$$L = xy^2z^3 + \lambda(x+y+z-24) \quad \text{—(III)}$$

For stationary pts

(i) $$\frac{\partial L}{\partial x} = y^2z^3 + \lambda = 0 \quad \text{—(IV)}$$

(ii) $$\frac{\partial L}{\partial y} = 2xyz^3 + \lambda = 0 \quad \text{—(V)}$$

(iii) $$\frac{\partial L}{\partial z} = 3xy^2z^2 + \lambda = 0 \quad \text{—(VI)}$$

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From (IV) & (V)

$$y^2z^3 = -\lambda \quad \& \quad 2xyz^3 = -\lambda$$

$$\Rightarrow \quad y^2z^3 = 2xyz^3 \quad \Rightarrow \quad yz^3(y-2x)=0$$

$$\Rightarrow \quad y=2x \quad \text{as } y\neq 0,\ z\neq 0$$

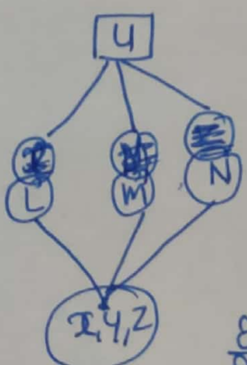
similarly $y^2z^2(z-3x)=0 \Rightarrow z=3x$

put $y=2x$ & $z=3x$ in (II)

$$x+y+z=24 \Rightarrow x+2x+3x = 6x = 24$$

$\boxed{x=4}$, and $y=2x=\boxed{8}$, $z=3x=\boxed{12}$

F



we have $u = f(x^2-y^2,\ y^2-z^2,\ z^2-x^2)$

let $l = x^2-y^2$, $m = y^2-z^2$, $n = z^2-x^2$

$$u = f(l,m,n) \rightarrow (x,y,z)$$

$$\frac{\partial u}{\partial x} = 2x\frac{\partial u}{\partial L} - 2x\frac{\partial u}{\partial n}$$

$$\frac{\partial u}{\partial y} = -2y\frac{\partial u}{\partial L} + 2y\frac{\partial u}{\partial m}$$

$$\frac{\partial u}{\partial z} = -2z\frac{\partial u}{\partial m} + 2z\frac{\partial u}{\partial n}$$

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 2\frac{\partial u}{\partial L} - 2\frac{\partial u}{\partial n} - 2\frac{\partial u}{\partial L} + 2\frac{\partial u}{\partial m} - 2\frac{\partial u}{\partial m} + 2\frac{\partial u}{\partial n}$$

$$= 0$$

Q3(A) $x^4 = 1+i = \sqrt{2}[\cos\pi/4 + i\sin\pi/4]$

$$x^4 = \sqrt{2}\, e^{i\pi/4 + 2n\pi i} = \sqrt{2}\, e^{i(2n\pi + \pi/4)}$$

$$x = (\sqrt{2})^{1/4} e^{i(2n\pi + \frac{\pi}{4})\frac{1}{4}}, \quad n = 0,1,2,3, = (\sqrt{2})^{1/4} e^{i\left(\frac{8n\pi+\pi}{16}\right)}$$

$$x = (\sqrt{2})^{1/4}\left[\cos\left(\frac{8n\pi+\pi}{16}\right) + i\sin\left(\frac{8n\pi+\pi}{16}\right)\right]$$

For $n=0$, $x_0 = (\sqrt{2})^{1/4}[\cos\pi/16 + i\sin\pi/16] = (\sqrt{2})^{1/4} e^{i\pi/16}$

$n=1$, $x_1 = (\sqrt{2})^{1/4} e^{i\frac{9\pi}{16}}$

$n=2$, $x_2 = (\sqrt{2})^{1/4} e^{i(17\pi/16)}$

$n=3$, $x_3 = (\sqrt{2})^{1/4} e^{i(25\pi/16)}$

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$$\therefore x_0 \cdot x_1 x_2 x_3 = [(\sqrt{2})^{1/4}]^4 e^{i(\pi + 9\pi + 17\pi + 25\pi)\frac{1}{16}}$$

$$= (\sqrt{2})\, e^{i(5\pi/4)} = \sqrt{2}[\cos(\pi + \pi/4) + i\sin(\pi + \pi/4)]$$

$$= \sqrt{2}[-\cos\pi/4 - i\sin\pi/4] = -\sqrt{2}\left(\frac{1}{\sqrt{2}} + i/\sqrt{2}\right) = -(1+i).$$

Q3(B) $e^z = \sin(u+iv)$

$$e^{x+iy} = e^x e^{iy} = \sin u\cosh v + i\cos u\sinh v$$

$$e^x\cos y = \sin u\cosh v, \quad e^x\sin y = \cos u\sinh v$$

$$e^{2x}\cos^2 y + e^{2x}\sin^2 y = \sin^2 u\cosh^2 v + \cos^2 u\sinh^2 v$$

$$e^{2x} = (1-\cos^2 u)\cosh^2 v + \cos^2 u(\cosh^2 v - 1)$$

$$= \cosh^2 v - \cos^2 u = \tfrac{1}{2}(1+\cosh 2v) - \tfrac{1}{2}(1+\cos 2u)$$

$$e^{2x} = \tfrac{1}{2}[\cosh 2v - \cos 2u]$$

$$\boxed{2e^{2x} = \cosh 2v - \cos 2u}$$

Q3(C) $A = \begin{bmatrix} 1+2i & 2 & 3-i \\ 2+3i & 2i & 1-2i \\ 1+i & 0 & 3+2i \end{bmatrix}$

$A = P + iQ = \frac{1}{2}(A + A^\theta) + i\frac{1}{2i}(A - A^\theta)$ Then Both P & Q are Hermitian matrices.

$$A^\theta = (\bar{A})' = \begin{bmatrix} 1-2i & 2-3i & 1-i \\ 2 & -2i & 0 \\ 3+i & 1+2i & 3-2i \end{bmatrix}$$

Hence $P=\frac{1}{2}(A+A^{\theta})=\frac{1}{2}\begin{bmatrix}2 & 4-3i & 4-2i\\ 4+3i & 0 & 1-2i\\ 4+2i & 1+2i & 6\end{bmatrix}$

and $Q=\frac{1}{2i}(A-A^{\theta})=\frac{1}{2i}\begin{bmatrix}4i & 3i & 2\\ 3i & 4i & 1-2i\\ -2 & -1-2i & 4i\end{bmatrix}$

Q3(D) $x=\cosh\left(\frac{\log y}{m}\right) \Rightarrow \cosh^{-1}(x)=\frac{1}{m}\log y$

$y=e^{m\cosh^{-1}x} \Rightarrow \log\left[x+\sqrt{x^2-1}\right]=\frac{1}{m}\log y$

~~$y_1 = e^{m\cosh^{-1}x}$~~ $\quad y=(x+\sqrt{x^2-1})^m$

$y_1=m(x+\sqrt{x^2-1})^{m-1}\left[1+\frac{2x}{2\sqrt{x^2-1}}\right]=m(x+\sqrt{x^2-1})^{m-1}\left[\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}\right]$

$(\sqrt{x^2-1})\,y_1=my$

$(x^2-1)\,y_1^2=m^2y^2$

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$(x^2-1)\,2y_1y_2+2xy_1^2-2m^2yy_1=0$

$(x^2-1)y_2+xy_1-m^2y=0$

By Leibnitz thm

$(x^2-1)y_{n+2}+n(2x)y_{n+1}+\frac{n(n-1)}{2}(2)y_n+xy_{n+1}+ny_n-m^2y_n=0$

$(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$

Q3(E) $u=\log r$ & $r^2=x^2+y^2$

$u=\log(\sqrt{x^2+y^2})$ is not homogeneous fun. But

$z=f(u)=e^u=\sqrt{x^2+y^2}$

putting $x=\lambda x$, & $y=\lambda y$ in z

$z(x,y)=f(u)=e^u=\sqrt{\lambda^2x^2+\lambda^2y^2}=\lambda\sqrt{x^2+y^2}=\lambda z$ is homogeneous fun of degree 1.

$\therefore\ x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\partial y}+y^2\frac{\partial^2u}{\partial y^2}=G(u)[G'(u)-1]$

where $G(u)=n\frac{f(u)}{f'(u)}=1\,\frac{e^u}{e^u}=1$ & $G'(u)=\frac{d}{du}G(u)=0$

$x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\partial y}+y^2\frac{\partial^2u}{\partial y^2}=1[0-1]=-1.$

Q3 (F) If $u = v + w$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \quad \& \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}$$

$v = \log\left(\frac{x+y}{\sqrt{x^2+y^2}}\right)$ is ~~not~~ homogeneous fun. ~~But~~ of degree '0'

$$z = e^v = \frac{x+y}{\sqrt{x^2+y^2}} = f(v) = \frac{\lambda x + \lambda y}{\lambda(\sqrt{x^2+y^2})} = \lambda^0 z \text{ is homogeneous}$$

~~Also~~ $$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1) = 0. \quad \text{--- (I)}$$

$w = \sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ is not homogeneous fun. But

$$z = \sin w = f(w) = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{\lambda(x+y)}{\sqrt{\lambda}(\sqrt{x}+\sqrt{y})} = \lambda^{1/2} z \text{ is}$$

homogeneous fun of degree $1/2$.

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} + y^2\frac{\partial^2 w}{\partial y^2} = G(w)[G'(w)-1]$$

$$G(w) = n\frac{f(w)}{f'(w)} = \frac{1}{2}\frac{\sin w}{\cos w} = \frac{1}{2}\tan w$$

$$G'(w) = \frac{1}{2}\sec^2 w$$

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x \partial y} + y^2\frac{\partial^2 w}{\partial y^2} = \frac{1}{2}\tan w\left[\frac{1}{2}\sec^2 w - 1\right] = \cancel{\frac{1}{2}\tan w}$$

$$= \frac{1}{2}\tan w\left[\frac{1}{2}(\tan^2 w + 1) - 1\right] = \frac{1}{2}\tan w\left[\frac{1}{2}\tan^2 w - \frac{1}{2}\right]$$

$$= \frac{1}{2}\frac{\sin w}{\cos w}\left[\frac{1}{2\cos^2 w} - 1\right] = \frac{1}{2}\frac{\sin w}{\cos w}\left[\frac{1-2\cos^2 w}{2\cos^2 w}\right]$$

$$= \frac{1}{4}\frac{\sin w}{\cos^3 w}\left[-\cos 2w\right] = -\frac{1}{4}\frac{\sin w \cos 2w}{\cos^3 w} \quad \text{--- (II)}$$

Adding (I) & (II)

$$x^2\frac{\partial^2}{\partial x^2}(v+w) + 2xy\frac{\partial^2}{\partial x \partial y}(v+w) + y^2\frac{\partial^2}{\partial y^2}(v+w) = 0 - \frac{1}{4}\frac{\sin w \cos 2w}{\cos^3 w}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{1}{4}\frac{\sin w \cos 2w}{\cos^3 w}.$$

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