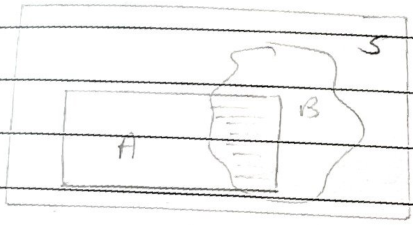
Module 6

Tuesday

Probability

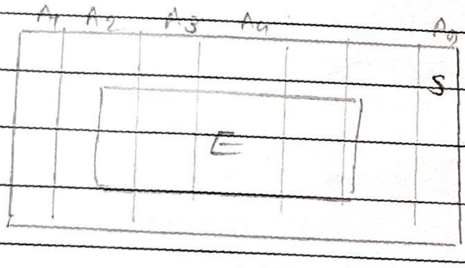
$S = \{$ Total no. of expected outcomes $\} = N$
$A = \{$ No. of favourable cases $\} = P$

$\therefore P(A) = P/N$



$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$ → conditional probability.

$\therefore P(A \cap B) = P(A) \cdot P(B/A)$



$S = A_1 \cup A_2 \cup A_3 \ldots \cup A_n$
$E = E \cap S$
$= E \cap [A_1 \cup A_2 \ldots \cup A_n]$
$= (E \cap A_1) \cup (E \cap A_2) \cup \ldots (E \cap A_n)$
$E = (A_1 \cap E) \cup (A_2 \cap E) \ldots \cup (A_n \cap E)$

$\therefore P(E) = P(A_1 \cap E) + P(A_2 \cap E) + \ldots P(A_n \cap E)$

Total prob of $E = P(A_1) \cdot P(E/A_1) + P(A_2) \cdot P(E/A_2) + \ldots + P(A_n) \cdot P(E/A_n)$

Bay's theorim

$$P(A_1/E) = \frac{P(A_1 \cap E)}{P(E)} \qquad \therefore P(A_1/E) = \frac{P(A_1) \cdot P(E/A_1)}{P(E)}$$

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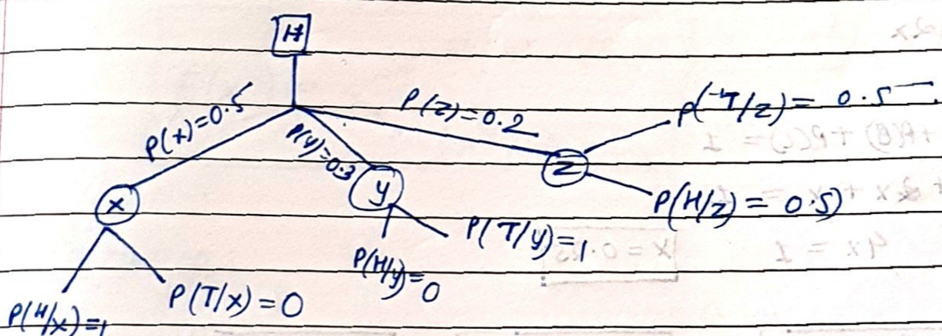
[x.1] A box contains 10 coins where 5 coins are 2 headed, 3 coins are 2-tailed and 2 coins are fair coins. A coin is selected at random and tossed. Find
i] Probability that head appears.
ii] If head appears, then find probability that coin is fair.

Sol^n ⇒ x = 2 headed coins $= 5 \Rightarrow P(x) = 1/2 = 0.5$
y = 2 tailed coins $= 3 \Rightarrow P(y) = 3/10 = 0.3$
z = Fair coins $= 2 \Rightarrow P(z) = 2/10 = 0.2$.



ⓐ $P(H) = P(x).\ P(H/x) + P(y).\ P(H/y) + P(z).\ P(H/z)$
$= (0.5)(1) + (0.3)(0) + (0.2)(0.5)$
$= 0.5 + 0.1$
$\boxed{P(H) = 0.6}$

ⓑ $P(Z/H) = \dfrac{P(Z \cap H)}{P(H)} = \dfrac{P(z).\ P(H/z)}{P(H)} = \dfrac{(0.2)(0.5)}{(0.6)}$

$\therefore \boxed{P(Z/H) = 1/6 = 0.1667}$

$\therefore$ 16.67 % chances.

Ex 2] A bolt is manufactured by 3 machines A, B and C. Machine A turns out twice as many times of B and machines B and C produce equal no of bolts. 3% of bolts produced by machine A and B are defective and 5% bolts produced by machine C are defective. If all bolts put into one box and one is selected randomly, then what is the probability that selected bolt is defective.

Sol^n ⇒ $B \Rightarrow x$ $\quad C = x$

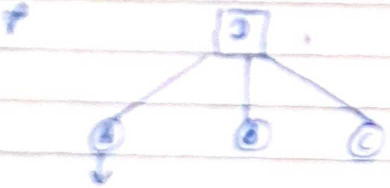
$A \Rightarrow 2x$

$P(A) + P(B) + P(C) = 1$

$2x + x + x = 1$

$4x = 1 \quad \boxed{x = 0.25}$

$\boxed{P(A) = 0.5} \quad \boxed{P(B) = 0.25} \quad \boxed{P(C) = 0.25}$



$P(D/A) =$

Random variable ⇒ A variable is used to denote outcomes of experiments in numbers; it is called random variable. There are 2 types of random variables:

1) Discrete random variable: It takes discrete (distinct) values

ex ⇒ let X is a no of heads obtained in tossing 2 coins simultaneously ⇒ {H = 0, 1, 2}

S = {HH, TH, HT, TT} ⇒

X	0	1	2
P(x)	1/4	2/4	1/4

} Probability Distribution Function [PDF]

Expectations

Expectation of $x = E(x) = \bar{x}$ = mean

$$\mu_1' = E(x) \begin{cases} \xrightarrow{\text{Discrete}} = \Sigma x \cdot P(x) \\ \xrightarrow{\text{continuous}} = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx \end{cases}$$

$$\mu_2' = E(x^2) \begin{cases} \xrightarrow{\text{D.R.V.}} = \Sigma x^2 P(x) \\ \xrightarrow{\text{C.R.V.}} = \int x^2 f(x) \cdot dx \end{cases}$$

$$\mu_r' = \therefore \boxed{E(x^r) = \begin{cases} \Sigma (x^r) \cdot P(x) \\ \int x^r \cdot f(x) \cdot dx \end{cases}}$$

[$\mu_r' \Rightarrow r^{th}$ moment about origin]

Properties:

1] $E[c] = E[cx^0] = \Sigma c x^0 = c \Sigma P(x) = c(1)$ $\quad \therefore \boxed{E[c] = c}$

2] $E[ax+b] = \Sigma (ax+b) \cdot P(x) = \Sigma[ax \cdot P(x) + b \cdot P(x)] = a \Sigma x \cdot P(x) + b\, \Sigma \cdot P(x)$

$= a\, E(x) + b(1)$

$\therefore E[ax \pm b] = a \cdot E(x) \pm E[b]$ $\quad \therefore \boxed{E[ax \pm b] = a \cdot E(x) \pm b}$

3] $E[ax] = a\, E(x)$

Variance

$$\boxed{V(x) = E\left[(x-\bar{x})^2\right]} \Rightarrow V(x) = E\left[x^2 - 2x\bar{x} + (\bar{x})^2\right]$$

$[\bar{x} = E(x)]$

$\Rightarrow V(x) = E(x^2) - 2\bar{x}E(x) + (\bar{x})^2 E(1)$ $\quad \therefore V(x) = E(x^2) - 2E(x) \cdot E(x) + [E(x)]^2$

$$\therefore \boxed{V(x) = E(x^2) - [E(x)]^2}$$

Properties:

1] $V(c) = V(c x^0) = E[(c x^0)^2] - [E(c x^0)]^2$
$= E[c^2] - [E(c)]^2$
$= c^2 - c^2$
$= 0$ $\quad \therefore \boxed{V(c) = 0}$

2] $V[ax] = E(ax)^2 - [E(ax)]^2$
$= a^2 E(x^2) - [a \cdot E(x)]^2$
$= a^2 E(x^2) - a^2 [E(x)]^2$
$= a^2 \{E(x^2) - [E(x)]^2\}$

$\therefore \boxed{V[ax] = a^2 \cdot V(x)}$

3] $V[ax \pm b] = V(ax) \pm V(b)$
$= a^2 V(x) \pm 0$

$\therefore \boxed{V[ax \pm b] = a^2 V(x) = V[ax]}$

ex. 1]

X	-2	-1	0	1	2	3
P(x)	0.1	K	0.1	2K	0.2	3K

Find K, mean & variance of X.

Sol^n ⇒ X is Discrete Random Variable

$\Sigma P(x) = 0.1 + K + 0.1 + 2K + 0.2 + 3K = 1$
$= 0.4 + 6K = 1$
$\therefore \boxed{K = 0.1}$

$E(x) = \bar{x} = \Sigma x P(x)$
$= (-2)(0.1) + (-1)(0.1) + 0(0.1) + (1)(0.2) + 2(0.2) + 3(0.3)$
$= -0.2 - 0.1 + 0 + 0.2 + 0.4 + 0.9$

$\boxed{\cancel{E(x) = 1.2}}$ $\quad \boxed{E(x) = 1.2}$

$V(x) = E(x^2) - [E(x)]^2 =$

$E(x^2) = \Sigma x^2 [P(x)]$

$= (4)(0.1) + (1)0.1) + 0 + (1)(0.2) + 4(0.2) + 9(0.3)$

$= 0.4 + 0.1 + 0 + 0.2 + 0.8 + 2.7$

$= 4.2$

$\therefore V(x) = 4.2 - (1.2)^2$

$= 4.2 - 1.44$

$\boxed{V(x) = 2.76}$

Ex. 2 A R.V. X has following PDF. Find K, mean & variance.

$$P(x) = \begin{cases} Kx^2 e^{-x} & , x>0, K>0 \\ 0 & \text{otherwise} \end{cases}$$

Sol $\Rightarrow$ X is Continuous Random Variable.

$\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

$\Rightarrow \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{\infty} K \cdot x^2 e^{-x} \cdot dx = 1$

$\Rightarrow 0 + K \int_0^\infty x^2 \cdot e^{-x} \cdot dx = 1 \Rightarrow [x^2][-e^{-x}]_0^\infty - [2x][e^{-x}]_0^\infty + 2[-e^{-x}]_0^\infty = 1$

$\Rightarrow K \int_0^\infty e^{-x} \cdot x^{3-1} \cdot dx = 1 \Rightarrow K \Gamma 3 = 1 \Rightarrow K(2!) = 1 \quad \therefore 2K = 1 \quad \boxed{K = 1/2}$

$E(x) = \int_{-\infty}^{\infty} x \cdot P(x) \cdot dx = \int_{-\infty}^{0} 0 \cdot x \cdot dx + \int_0^\infty x(K \cdot x^2 \cdot e^{-x}) dx$

$= 0 + K \int_0^\infty e^{-x} \cdot x^3 \cdot dx = K \int_0^\infty e^{-x} \cdot x^{4-1} \cdot dx = K \Gamma 4 = 6K$

$E(x) = 6(1/2) \quad \therefore \boxed{E(x) = 3}$

$E(x^2) = 0 + \int_0^\infty x^2 f(x) \cdot dx = K \int_0^\infty x^2 e^{-x} \cdot x^2 \cdot dx = K \int_0^\infty e^{-x} \cdot x^4 \cdot dx = K \int_0^\infty e^{-x} \cdot x^{5-1} \cdot dx$

$= K \Gamma 5 = K(24) \Rightarrow \boxed{E(x^2) = 12} \quad \therefore \boxed{V(x) = 3}$

$V(x) = E(x^2) - [E(x)]^2 = 12 - (3)^2 = 3$

ex. 3

x	8	12	16	20	24
$P(x)$	$1/8$	m	n	$1/4$	$1/12$

If ~~mean~~ expectation is 16, find m and n.

Sol^n $\Rightarrow$ $\Sigma P(x) = 1$

$\Rightarrow$ $1/8 + m + n + 1/4 + 1/12 = 1$

$\Rightarrow$ $m + n = 1 - 1/12 - 1/8 - 1/4$

$\Rightarrow$ $m + n = \dfrac{24-2-3-6}{24} = \dfrac{13}{24}$ .... (1)

$E(x) = 16$

$\Sigma x P(x) = 16$

$\Rightarrow$ $8(1/8) + 12(m) + 16(n) + 20(1/4) + 24(1/12) = 16$

$\Rightarrow$ $1 + 12m + 16n + 5 + 2 = 16$

$\Rightarrow$ $12m + 16n = 8$

$3m + 4n = 2$ .... (2)

$\therefore$ $3m + 4n = 2$

$3m + 3n = 39/24 = 13/8$

$n = 2 - 13/8$

$\boxed{n = 3/8}$ $\Rightarrow$ $m = 13/24 - n = 13/24 - 3/8$

$\therefore$ $m = \dfrac{13-9}{24}$ $\boxed{m = 1/6}$

$E(x^2) - \Sigma x^2 . P(x) = 64(1/8) + 144(1/6) + 256(3/8) + 400(1/4) + 576(1/12)$

$= 8 + 24 + 32(3) + 100 + 48$

$= 32 + 96 + 148$

$= 276$

$V(x) = E(x^2) - [E(x)]^2 = 276 - (256) = 20$ $\therefore$ $\boxed{V(x) = 20}$

Ex.4] A R.V X has PDF

X	0	1	2
P(X)	$3c^3$	$4c-10c^2$	$5c-1$

$C = 1, 2, 1$

Find const. c, mean & variance.
Find $P(X<2)$.

Sol.$\Rightarrow$

Ex. 3 Find expectation of no. of failures preceding the first success in an infinite series of independent trials with constant probabilities p and q of success and failure respectively.

Sol^n ⇒ x = No. of failures

p = Probability of success

q = Probability of failures

x	0	1	2	3 ....
$P(x)$	p	qp	q^2p	q^3p .....

$$E(x) = \Sigma x P(x)$$

$$= 0(p) + 1(qp) + 2(q^2p) + 3(q^3p)$$

$$= qp[1 + 2q + 3q^2 + \dots]$$

$$= qp[1-q]^{-2}$$

$$= qp[p]^{-2}$$

$$\boxed{E(x) = q/p}$$

★ Moments

Moments
- → Raw moments
  - → Raw moment about any pt 'a'
  - → Raw moment about origin
- → Central moments about the mean $(\bar{x})$

Ⓐ r^{th} raw moment about any pt 'a' $= \mu_r' = E\left[(x-a)^r\right]$

$$= \begin{cases} \text{DRV} & \sum (x-a)^r \cdot p(X=x) \\ \text{CRV} & \int_{-\infty}^{\infty} (x-a)^r \cdot f(X=x) \cdot dx \end{cases}$$

Ex. ⇒ $\mu_0' = E\left[(x-a)^0\right] = E[1] = 1$

$\mu_1' = E\left[(x-a)^1\right] =$

$\mu_2' = E\left[(x-a)^2\right]$

Ⓑ r^{th} raw moment about origin $(a=0)$ $= \mu_r' = E\left[(x-0)^r\right] = E\left[(x)^r\right]$

$$= \begin{cases} \text{DRV} & \sum x^r \cdot p(x) \\ \text{CRV} & \int_{-\infty}^{\infty} x^r \cdot f(x) \cdot dx \end{cases}$$

ex. ⇒ $\mu_0' = E\left[(x)^0\right] = 1.$

$\mu_1' = E\left[x^1\right] = E(x) = \bar{x} = \text{mean}$

$\mu_2' = E\left[x^2\right]$

© r^{th} central moment about mean $(\bar{x})$ $= \mu_r = E[(x-\bar{x})^r]$

$$\mu_r = E[(x-\bar{x})^r] = \begin{cases} \xrightarrow{DRV} \sum (x-\bar{x})^r \cdot P(x) \\ \xrightarrow{CRV} \int_{-\infty}^{\infty} (x-\bar{x})^r \cdot f(x) \cdot dx \end{cases}$$

Moment Generating Function (MGF)

1] Moment G.F. about any pt 'a'

$$M_a(t) = E[e^{t(x-a)}] = \begin{cases} \xrightarrow{DRV} \sum e^{t(x-a)} \cdot P(x) \\ \xrightarrow{CRV} \int_{-\infty}^{\infty} e^{t(x-a)} \cdot f(x) \cdot dx \end{cases}$$

2] Moment G.F about origin

$$M_0(t) = E[e^{tx}] \begin{cases} \xrightarrow{DRV} \sum e^{tx} \cdot P(x) \\ \xrightarrow{CRV} \int_{-\infty}^{\infty} e^{tx} \cdot f(x) \cdot dx \end{cases}$$

$$M_0(t) = E[e^{tx}] = \sum e^{tx} \cdot P(x) = \sum \left[1 + (tx) + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots \right] P(x)$$

$$= \sum P(x) + t \sum x \cdot P(x) + \frac{t^2}{2!} \sum x^2 \cdot P(x) + \dots \frac{t^r}{r!} \sum x^r \cdot P(x) + \dots$$

$$M_0(t) = 1 + \frac{t}{1!} \mu_1' + \frac{t^2}{2!} \mu_2' + \frac{t^3}{3!} \mu_3' + \dots \frac{t^r}{r!} \mu_r' + \dots$$

$$\left[\frac{d\,[M_0(t)]}{dt}\right]_{t=0} = 0 + \mu_1' + \frac{2t}{2!} \mu_2' + \frac{3t^2}{3!} \mu_3' = \mu_1' = \text{Mean}$$

$$\boxed{\left\{\frac{d^2[M_0(t)]}{dt^2}\right\}_{t=0} = \mu_2' = E[x^2]}$$

$$\boxed{V(X) = \mu_2' - (\mu_1')^2}$$

Summary : Mean and Variance by MGF method

Step 1 ⇒ $M_0(t) = E[e^{tx}]$

2 ⇒ $Mean = \bar{x} = E(x) = \mu_1' = \left\{\frac{d[M_0(t)]}{dt}\right\}_{t=0}$

3 ⇒ $E(x^2) = \mu_2' = \left\{\frac{d^2[M_0(t)]}{dt^2}\right\}_{t=0}$

4 ⇒ $V(X) = \mu_2' - (\mu_1')^2$

(7)

ex. 1] $P(x) = 1/2^x \quad , \quad x = 1, 2, 3, \ldots$

PDF of x is

x	1	2	3	4 ...
$P(x)$	$1/2$	$1/2^2$	$1/2^3$	$1/2^4$ ...

Sol.^{n} ⇒ $\Sigma P(X) = 1/2 + 1/2^2 + 1/2^3 + \ldots$

$= 1/2\,[1 + (1/2)^2 + (1/2)^3 + \ldots\ldots]$

$= 1/2\,[1 + x + x^2 + \ldots]$ (G.P).

$= 1/2\left[\frac{1}{1-x}\right] = \frac{1}{2}\frac{1}{(1-1/2)} = 1.$

Step 1: $M_0(t) = E[e^{tx}] = \sum_{x=1}^{\infty} e^{tx} \cdot P(x)$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot \frac{1}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots$$

$$= \frac{e^t}{2}\left[1 + e^t/2 + (e^t/2)^2 + \cdots\right]$$

$$= \frac{e^t}{2} \cdot \frac{1}{\left(1 - \frac{e^t}{2}\right)} = \frac{e^t}{2} \cdot \frac{1}{\frac{2-e^t}{2}} = \frac{e^t}{2-e^t} \quad \ldots (1)$$

Step 2: Mean $= E(x) = \mu_1' = \left\{\frac{d}{dt}[M_0(t)]\right\}_{t=0}$

Mean $= \left[\frac{d}{dt}\left(\frac{e^t}{2-e^t}\right)\right]_{t=0}$

$$= \frac{(2-e^t) \cdot e^t - e^t(-e^t)}{(2-e^t)^2} = \left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0} \quad \ldots (2)$$

$$= \frac{2(1)}{1} = 2.$$

Step 3: $E(x^2) = \mu_2' = \left\{\frac{d^2}{dt^2}[M_0(t)]\right\}_{t=0}$

$$\mu_2' = \left\{\frac{d}{dt}\left[\frac{2e^t}{(2-e^t)^2}\right]\right\}_{t=0} = 2\left[\frac{(2-e^t)^2(e^t) - e^t \cdot 2(2-e^t)(-e^t)}{(2-e^t)^4}\right]_{(t=0)}$$

$$\mu_2' = 2\left[\frac{(2-1)^2(1) + 1 \cdot 2(2-1)(1)}{(2-1)^4}\right] = 2[1+2] = 6$$

Step 4: $V(x) = \mu_2' - (\mu_1')^2 = 6 - (2)^2 = 6 - 4 = 2.$

⑥

ex. 2

x	0	1	2	3
$P(x)$	1/6	1/3	1/3	1/6

Sol^n ⇒ Method 1 : (General)

$$E(x) = \bar{x} = \Sigma x . P(x) = 0(1/6) + 1(1/3) + 2(1/3) + 3(1/6)$$

$$\mu_1' = 0 + 1 + 1/2 = 3/2$$

$$\mu_2' = E(x^2) = \Sigma(x^2).P(x)$$
$$= 0^2(1/6) + (1)^2(1/3) + (2)^2(1/3) + (3)^2(1/6)$$
$$= 0 + 1/3 + 4/3 + 9/6$$
$$= 5/3 + 9/6$$
$$= 19/6$$

$$V(x) = \mu_2' - (\mu_1')^2 = 19/6 - 9/4 = \frac{38-27}{12} = \frac{11}{12}.$$

Method 2 : (MGF)

$$M_0(t) = E[e^{tx}] = \sum_{x=0}^{3} e^{tx}.P(x) = e^{0t}.\frac{1}{6} + e^{1t}.\frac{1}{3} + e^{2t}.\frac{1}{3} + e^{3t}.\frac{1}{6}$$

$$M_0(t) = \frac{1}{6} + \frac{e^t}{3} + \frac{e^{2t}}{3} + \frac{e^{3t}}{6}$$

$$E(x) = \mu_1' = \left\{\frac{d[M_0(t)]}{dt}\right\}_{t=0} = \left[0 + \frac{e^t}{3} + \frac{2.e^{2t}}{3} + \frac{3.e^{3t}}{6}\right]_{t=0}$$

$$\mu_1' = [0 + 1/3 + 2/3 + 1/2] = 3/2$$

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$$E(x^2) = \mu_2' = \left\{\frac{d^2[M_0(t)]}{dt^2}\right\}_{t=0}$$

$$\mu_2' = \frac{d}{dt}\left[\frac{e^t}{3} + \frac{2e^{2t}}{3} + \frac{3 \cdot e^{3t}}{6}\right]_{t=0}$$

$$\mu_2' = \left[\frac{e^t}{3} + \frac{4 \cdot e^{2t}}{3} + \frac{9 \cdot e^{3t}}{6}\right]_{t=0}$$

$$\mu_2' = \left[1/3 + 4/3 + 3/2\right] = 5/3 + 3/2 = \frac{10+9}{6} = \frac{19}{6}$$

$$V(x) = \mu_2' - (\mu_1')^2 = 19/6 - (3/2)^2 = 19/6 - 9/4 = 11/12.$$

Ex. 3 Find mean and variance by MGF method of following PDF.

$$P(x) = 1/2 \quad , \quad -1 \le x \le 1$$
$$= 0 \quad , \quad \text{otherwise}$$

Sol.⇒ x is Continuous Random Variable (CRV).

$$M_0(t) = E[e^{tx}] = \int_{-1}^{x=1} e^{tx} \cdot P(x) \cdot dx = \int_{-1}^{1} e^{tx}(1/2) \cdot dx$$

$$= 1/2\left[\frac{e^{tx}}{t}\right]_{-1}^{1} = \frac{1}{2t}\left[e^t - e^{-t}\right] = \frac{1}{2t}\left[\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots\right) - \left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\dots\right)\right]$$

$$= \frac{1}{2t}\left[2 \cdot t + \frac{2 \cdot t^3}{3!} + \frac{2 \cdot t^5}{5!} + \dots\right]$$

~~$\mu_1' = \text{Mean} = d$~~

$$M_0(t) = \left[1 + t^2/3! + t^4/5! + t^6/7! + \dots\right]$$

$$\left\{\frac{d[M_0(t)]}{dt}\right\}_{t=0} = \left[0 + \frac{2t}{3!} + \frac{4t^3}{5!} + \frac{6t^5}{7!} + \dots\right]_{t=0} \quad \dots (1)$$

$$\therefore \boxed{\mu_1' = 0}$$

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$$E(x^2) = \mu_2' = \frac{d^2[M_0(t)]}{dt^2} = \left[\frac{2}{3!} + \frac{12t^2}{5!} + \frac{30t^4}{7!} + \dots \right]_{t=0}$$

$$\mu_2' = \frac{2}{3!} = \frac{1}{3}$$

$$V(x) = \mu_2' - (\mu_1')^2 = 1/3 - 0 = 1/3$$