① Find singular value decomposition of the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

Solution By given

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \quad A^T = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+9 & 0+6 \\ 0+6 & 0+4 \end{bmatrix}$$

$$\therefore AA^T = \begin{bmatrix} 13 & 6 \\ 6 & 4 \end{bmatrix} = M \text{ say}$$

it's characteristic equation is

$$\lambda^2 - S_1\lambda + |A| = 0$$

$$\lambda^2 - 17\lambda + 16 = 0$$

$$(\lambda-16)(\lambda-1) = 0$$

$$\therefore \lambda = \lambda_1 = 16 \quad \lambda = \lambda_2 = 1$$

$$A^TA = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix} = M'$$

it's characteristic equation is

$$\lambda^2 - S_1\lambda + |A| = 0$$

$$\therefore \lambda^2 - 17\lambda + 16 = 0$$

$$\therefore (\lambda-16)(\lambda-1) = 0$$

$\therefore \lambda_1 = \lambda_1 = 16, \ \lambda = \lambda_2 = 1$ — ① be the eigen values of $A^T A$

and $\sigma_1 = \sqrt{\lambda_1} = \sqrt{16} = 4$,

$\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$ — ②

To find Eigen vector $(AA^T - \lambda)X = 0$

$$\therefore \begin{bmatrix} 13-\lambda & 6 \\ 6 & 4-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 1 if $\lambda = \lambda_1 = 16$ $\quad \begin{bmatrix} -3 & 6 \\ 6 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$R_2 \to R_2 + 2R_1, \quad R_1 = \tfrac{1}{3}R_1$$

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-1x_1 + 2x_2 = 0$$

$$\therefore 1x_1 = 2x_2$$

$$\frac{x_1}{2} = \frac{x_2}{1} = k = 1$$

$$\therefore x_1 = 2, \ x_2 = 1$$

$$\therefore X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \therefore \|X_1\| = \sqrt{4+1} = \sqrt{5}, \quad \therefore X_1' = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

if $\lambda = \lambda_2 = 1$ $\quad \begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$R_1 \to R_1 - 2R_2, \quad R_2 \to \tfrac{1}{3}R_2$$

$$\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore 2x_1 + 1x_2 = 0$$

$$2x_1 = -1x_2$$

$$\frac{x_1}{-1} = \frac{x_2}{2} = k = 1$$

$$\therefore x_1 = -1, \ x_2 = 2$$

$\therefore X_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\|X_2\| = \sqrt{1+4} = \sqrt{5}$. $\therefore X_2' = \frac{X_2}{\|X_2\|} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

$$\therefore U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \quad \text{— (3)}$$

Now From eqn (1) $A^TA = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$ & $\lambda = \lambda_1 = 16$ $\lambda = \lambda_2 = 1$

To find Eigen vector consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

<u>Case-1</u> if $\lambda = \lambda_1 = 16$ $\begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_1 \to R_1 + 2R_2$, $R_2 \to \frac{1}{3} R_2$

$$\begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 - 1x_2 = 0$$

$$2x_1 = 1x_2$$

$$\frac{x_1}{1} = \frac{x_2}{2} = k = 1$$

$\therefore x_1 = 1, x_2 = 2$, $\therefore X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\|X_1\| = \sqrt{5}$, $X_1' = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

<u>Case-2</u> if $\lambda = \lambda_2 = 1$ $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_1 \to \frac{1}{3} R_1$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$1x_1 + 2x_2 = 0$$

$$1x_1 = -2x_2$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = k = 1$$

$x_1 = -2$, $x_2 = 1$, $\therefore X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\|X_2\| = \sqrt{5}$ $X_2' = \frac{X_2}{\|X_2\|} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$\therefore V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ $\therefore V^T = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

From (2) $D_{2\times 2} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore A = UDV^T \Rightarrow \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

Example-2 Find singular value decomposition of matrix

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \quad A^T = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}$$

AAT

$$AA^T = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$

it's characteristic equation;

$$\lambda^2 - S_1\lambda + |A| = 0$$

or $(\lambda - 32)(\lambda - 18) = 0$

$\therefore \lambda = \lambda_1 = 32,\ \lambda = \lambda_2 = 18$ be the eigen values of AA^T

$\therefore \sigma_1 = \sqrt{\lambda_1} = 4\sqrt{2},\ \sigma_2 = \sqrt{\lambda_2} = 3\sqrt{2}$ ②

To find Eigen vector consider

$$(AA^T - \lambda I)X = 0$$

$$\begin{bmatrix} 32-\lambda & 0 \\ 0 & 18-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{—②}$$

Case 1 $\lambda = \lambda_1 = 32$

$$\begin{bmatrix} 0 & 0 \\ 0 & -14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0x_1 + 14x_2 = 0$

$x_2 = 0$

take $x_1 = 1$

$\therefore$ For $\lambda = \lambda_1 = 32,\ X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\therefore \|X_1\| = 1,\ \therefore X_1' = \dfrac{X_1}{\|X_1\|} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Case-2 if $\lambda = \lambda_2 = 18$

$$\begin{bmatrix} 14 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$14x_1 + 0x_2 = 0$

$x_1 = 0$, take $x_2 = 1$

For $\lambda = \lambda_2 = 16,\ X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\|X_2\| = 1,\quad X_2' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\therefore U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

ATA

$$A^TA = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

it's characteristic equation;

$$\lambda^2 - S_1\lambda + |A| = 0$$

$$\lambda^2 - 50\lambda + 576 = 0$$

$\lambda = \lambda_1 = 32 \quad \lambda = \lambda_2 = 18$

To find Eigen vector consider

$$(A^TA - \lambda I)X = 0$$

$$\therefore \begin{bmatrix} 25-\lambda & 7 \\ 7 & 25-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Case 1 $\lambda = \lambda_1 = 32$

$$\begin{bmatrix} -7 & 7 \\ 7 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1,\ R_1 \to \frac{1}{7}R_1$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-1x_1 + 1x_2 = 0 \Rightarrow 1x_1 = 1x_2$

$\Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{1} = k = 1 \quad \therefore X_1 = 1,\ X_2 = 1$

$\therefore$ For $\lambda = \lambda_1 = 32,\ X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\|X_1\| = \sqrt{2},\ X_1' = \dfrac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Case-2 if $\lambda = \lambda_2 = 18$

$$\begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1,\ R_1 \to \frac{1}{7}R_1 \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$1x_1 + 1x_2 = 0 \Rightarrow 1x_1 = -1x_2$

$\Rightarrow \dfrac{x_1}{-1} = \dfrac{x_2}{1} = k = 1$

$\Rightarrow x_1 = -1,\ x_2 = 1$

$\therefore$ For $\lambda = \lambda_2 = 18,\ X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \|X_2\| = \sqrt{2}$

$\therefore X_2' = \dfrac{X_2}{\|X_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\therefore V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ & $D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix}$

$$\therefore A = UDV^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Example-3 Find the singular value decomposition of $A=\begin{bmatrix}1&1\\1&1\\1&-1\end{bmatrix}$

Solution By given $A=\begin{bmatrix}1&1\\1&1\\1&-1\end{bmatrix}$ & $A^T=\begin{bmatrix}1&1&1\\1&1&-1\end{bmatrix}$

$$\therefore AA^T=\begin{bmatrix}1&1\\1&1\\1&-1\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&-1\end{bmatrix}=\begin{bmatrix}2&2&0\\2&2&0\\0&0&2\end{bmatrix}$$

$\because$ AA^T be a square matrix of order 3×3

$\therefore$ it's characteristic equation is $\lambda^3-S_1\lambda^2+S_2\lambda-|A|=0$ - (1)

where $S_1=6$, $S_2=4+4+0=8$, $|A|=0$

$$\therefore \lambda^3-6\lambda^2+8\lambda-0=0 \Rightarrow \lambda(\lambda^2-6\lambda+8)=0$$

$$\Rightarrow \lambda(\lambda-2)(\lambda-4)=0 \Rightarrow \lambda=\lambda_1=4,\ \lambda=\lambda_2=2,\ \lambda=\lambda_3=0$$

$$\therefore \sigma_1=\sqrt{\lambda_1}=2,\ \sigma_2=\sqrt{\lambda_2}=\sqrt{2},\ \sigma_3=\sqrt{\lambda_3}=0 \quad (2)$$

To find Eigen vector consider $(A-\lambda I)X=0$

$$\begin{bmatrix}2-\lambda&2&0\\2&2-\lambda&0\\0&0&2-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad -(3)$$

Case-1 if $\lambda=\lambda_1=4$, $\begin{bmatrix}-2&2&0\\2&-2&0\\0&0&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\frac{x_1}{4}=\frac{-x_2}{-4}=\frac{x_3}{0}$$

$$\frac{x_1}{1}=\frac{x_2}{1}=\frac{x_3}{0}=k=1,\quad x_1=1, x_2=1, x_3=0 \therefore \text{For}$$

For $\lambda=\lambda_1=4$, $X_1=\begin{bmatrix}1\\1\\0\end{bmatrix}$ $\|X_1\|=\sqrt{2}$ $\therefore X_1'=\frac{X_1}{\|X_1\|}=\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}$

Case-2 if $\lambda=\lambda_2=2$ $\begin{bmatrix}0&2&0\\2&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\frac{x_1}{0}=\frac{-x_2}{0}=\frac{x_3}{4}$$

$$\frac{x_1}{0}=\frac{x_2}{0}=\frac{x_3}{1}=k=1 \therefore x_1=0, x_2=0, x_3=1$$

$\therefore$ For $\lambda=\lambda_2=2$ $X_2=\begin{bmatrix}0\\0\\1\end{bmatrix}$ $\|X_2\|=\sqrt{1}=1$ $\therefore X_2'=\frac{X_2}{\|X_2\|}=\begin{bmatrix}0\\0\\1\end{bmatrix}$

Case-3 if $\lambda=\lambda_3=0$, $\begin{bmatrix}2&2&0\\2&2&0\\0&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\frac{x_1}{4}=\frac{-x_2}{4}=\frac{x_3}{0}$$

$$\frac{x_1}{1}=\frac{x_2}{-1}=\frac{x_3}{0}=k=1 \therefore x_1=1, x_2=-1, x_3=0$$

For $\lambda=\lambda_3=0$ $X_3=\begin{bmatrix}1\\-1\\0\end{bmatrix}$ $\|X_3\|=\sqrt{2}$ $X_3'=\frac{X_3}{\|X_3\|}=\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}$

$$U = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

Now $A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

its characteristic eq^n is $\lambda^2 - s_1\lambda + |A| = 0 \Rightarrow \lambda^2 - 6\lambda + 8 = 0$

$(\lambda-2)(\lambda-4) = 0 \Rightarrow \lambda = \lambda_1 = 4, \quad \lambda = \lambda_2 = 2, \quad \sigma_1 = \sqrt{\lambda_1} = 2, \quad \sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$

To find Eigen vector consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (4)$$

If $\lambda = \lambda_1 = 4$ $\quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 + R_1 \quad \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-1x_1 + 1x_2 = 0 \Rightarrow 1x_1 = 1x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = k \Rightarrow$

$x_1 = 1, \; x_2 = 1$

For $\lambda = \lambda_1 = 4$, $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\|X_1\| = \sqrt{2}$, $X_1' = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

If $\lambda = \lambda_2 = 2$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - R_1 \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$1x_1 + 1x_2 = 0 \Rightarrow 1x_1 = -1x_2 \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = k \Rightarrow x_1 = -1, \; x_2 = 1$

For $\lambda = \lambda_2 = 2$, $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\|X_2\| = \sqrt{2}$, $X_2' = \frac{X_2}{\|X_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\therefore V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A_{3\times 2} = U_{3\times 3} D_{3\times 2} V^T_{2\times 2}$$

$$A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Example-4 Find singular value decomposion of $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$

Solution By given $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$ & $A^T = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

$\because$ AA^T be square matrix of order 2 $\therefore$ it's characteristic equation is $\lambda^2 - S_1\lambda + |A| = 0 \Rightarrow \lambda^2 - 22\lambda + 120 = 0$

$\Rightarrow (\lambda - 12)(\lambda - 10) = 0$

$\Rightarrow \lambda = \lambda_1 = 12, \ \lambda = \lambda_2 = 10$ (1) be the Eigen values of a matrix AA^T

$\therefore \ \sigma_1 = \sqrt{\lambda_1} = 2\sqrt{3} \quad \sigma_2 = \sqrt{\lambda_2} = \sqrt{10}$ - (2)

To find Eigen vector consider $(AA^T - \lambda I)X = 0$

$$\begin{bmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad - (3)$$

Case-1 If $\lambda = \lambda_1 = 12$ $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 + R_1 \quad \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-1x_1 + 1x_2 = 0$

$1x_1 = 1x_2$

$\frac{x_1}{1} = \frac{x_2}{1} = k = 1$

$\therefore x_1 = 1, \ x_2 = 1$

$\therefore$ For $\lambda = \lambda_1 = 12$, $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\|X_1\| = \sqrt{2}$, $\therefore X_1' = \frac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Case-2 If $\lambda = \lambda_2 = 10$, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - R_1 \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$1x_1 + 1x_2 = 0$

$1x_1 = -1x_2$

$\frac{x_1}{-1} = \frac{x_2}{1} = k = 1 \Rightarrow x_1 = -1, \ x_2 = 1$

For $\lambda = \lambda_2 = 10$ $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\|X_2\| = \sqrt{2}$, $X_2' = \frac{X_2}{\|X_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\therefore U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad - (4)$$

Now

$$A^TA = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

$\because$ A be a square matrix of order 3 $\therefore$ it's characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$ (5) when $S_1 = 22$, $S_2 = 4 + 16 + 100 = 120$

$|A| = 40 + 2(0 - 20) = 40 - 40 = 0$

Thus $S_1 = 22, \ S_2 = 120, \ |A| = 0$

$$\lambda^3 - 22\lambda^2 + 120\lambda - 0 = 0$$

$$\lambda(\lambda^2 - 22\lambda + 120) = 0 \Rightarrow \lambda(\lambda-10)(\lambda-12) = 0$$

$\therefore \lambda = \lambda_1 = 12,\ \lambda = \lambda_2 = 10,\ \lambda = \lambda_3 = 0$ — (6) be the Eigen value of a matrix A $\therefore \sigma_1 = \sqrt{\lambda_1} = 2\sqrt{3},\ \sigma_2 = \sqrt{\lambda_2} = \sqrt{10},\ \sigma_3 = \sqrt{\lambda_3} = 0$ — (7)

To find Eigen vector ~~ATA~~ consider $(A^TA - \lambda I)X = 0$

$$\therefore \begin{bmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -(8)$$

<u>Case 1</u> if $\lambda = \lambda_1 = 12 \Rightarrow$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{4} = \frac{-x_2}{-8} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1} = k = 1 \Rightarrow x_1 = 1,\ x_2 = 2,\ x_3 = 1$$

$\therefore$ For $\lambda = \lambda_1 = 12$, $X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $\|X_1\| = \sqrt{6}$, $X_1' = \dfrac{X_1}{\|X_1\|} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

<u>Case-2</u> if $\lambda = \lambda_2 = 10 \Rightarrow$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{-8} = \frac{-x_2}{-4} = \frac{x_3}{0} \Rightarrow \frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{0} = k = 1 \quad x_1 = -2,\ x_2 = 1,\ x_3 = 0$$

For $\lambda = \lambda_2 = 10$, $X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ $\|X_2\| = \sqrt{5}$ $X_2' = \dfrac{X_2}{\|X_2\|} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$

<u>Case-3</u> if $\lambda = \lambda_3 = 0 \Rightarrow$

$$\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{-20} = \frac{-x_2}{40} = \frac{x_3}{100} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{-2} = \frac{x_3}{5} = k = -1 \quad x_1 = 1,\ x_2 = 2,\ x_3 = -5$$

For $\lambda = \lambda_3 = 0$, $X_3 = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$ $\|X_3\| = \sqrt{30}$, $X_3' = \dfrac{X_3}{\|X_3\|} = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix}$

$$\therefore V = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{5} & 1/\sqrt{30} \\ 2/\sqrt{6} & 1/\sqrt{5} & 2/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix}$$

$$\therefore A_{2\times3} = UDV^T = U_{2\times2}\, D_{2\times3}\, V^T_{3\times3}$$

$$\therefore A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2\sqrt{3} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}_{2\times3}\begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 1/\sqrt{30} & 2/\sqrt{30} & -5/\sqrt{30} \end{bmatrix}$$