Mathematical Foundations of Computer Science
Homework Assignment 3

Given: February 3, 2023 **Due:** February 10, 2023

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1. Suppose you take a piece of paper and draw a bunch of straight lines, no one exactly on top of another, that completely cross the paper. This divides the paper up into polygonal regions. Prove by induction that you can always color the various regions using only *two* colors, so that any two regions that share a boundary line have different colors. Regions that share only a boundary point are permitted to have the same color.

2. There are n cities in a OneWayCountry (country in which every road is a One-Way road). Every pair of cities is connected by exactly one direct one-way road. Show that there exists a city which can be reached from every other city either directly or via at most one other city.

3. Prove, using induction, that every positive integer can be expressed as a sum of distinct powers of 2. For example, $13 = 2^3 + 2^2 + 2^0$.

4. Prove that in any simple graph G with n vertices and m edges, $2m \leq n^2 - n$.

5. Prove or disprove the following. In any group of two or more people, there are always at least two people who have the same number of friends. Assume that if a person p is a friend of a person q then q is also a friend of p.

6. An r-regular graph is a graph in which the degree of each vertex is exactly r. Derive a simple algebraic relation between r, n, and m.

7. Let G be a graph with $n \geq 2$ vertices. Prove that if $\delta(G) \geq \frac{n}{2}$, then G is connected.

8. Show that in any simple graph G there is a path from any vertex of odd degree to another vertex of odd degree.