Linear Recurrence Equation (kth order)

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \;\dots\dots + c_k a_{n-k} + \boldsymbol{f(n)}$$

If $\boldsymbol{f(n) = 0}$ Homogeneous Linear Recurrence equation

If $\boldsymbol{f(n) \neq 0}$ **Non-**Homogeneous Linear Recurrence equation

General solution $a_n = a_n^{(h)} + a_n^{(p)}$

<u>Process to find solution:</u>

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - c_3 a_{n-3} - \;\dots\dots - c_k a_{n-k} = \boldsymbol{f(n)} \quad \dots\dots\dots\dots\; (1)$$

<u>**For homogeneous solution**</u> $\boldsymbol{a_n^{(h)}}$

Consider homogenous equation

$a_n - c_1 a_{n-1} - c_2 a_{n-2} - c_3 a_{n-3} - \;\dots\dots - c_k a_{n-k} = 0$

Let $a_n^{(h)} = \alpha^n$

$\alpha^n - c_1 \alpha^{n-1} - c_2 \alpha^{n-2} - c_3 \alpha^{n-3} - \;\dots\dots - c_k \alpha^{n-k} = 0$

Divvied by α^{n-k} (in such way that power of α should be positive)

$\alpha^k - c_1 \alpha^{k-1} - c_2 \alpha^{k-2} - c_3 \alpha^{k-3} - \;\dots\dots - c_k = 0$

Solve equation and find solution let it be α_1 , α_2

Non-repeated : $a_n^{(h)} = c_1(\alpha_1)^n - c_2(\alpha_2)^n$

Repeated: $a_n^{(h)} = (c_1 + c_2 n)(\alpha_1)^n$

Where c_1, c_2 are arbitrary constant.

<u>**For particular solution**</u> $\boldsymbol{a_n^{(p)}}$

Check format of $\boldsymbol{f(n)}$

1. $\boldsymbol{f(n) = constant}$

   If $\boldsymbol{f(n) = c}$ then $a_n^{(p)} = \begin{cases} A & \alpha \neq 1 \\ An^m & \alpha = 1 \;\; with\;\; m\;\; time \end{cases}$

2. $\boldsymbol{f(n) = polynomial\, function\, (exculding\, constant)}$

   If $\boldsymbol{f(n) = an + b}$ then $\boldsymbol{a_n^{(p)} = An + B}$

   If $\boldsymbol{f(n) = an^2 + bn + c}$ then $\boldsymbol{a_n^{(p)} = An^2 + Bn + C}$

3. $\boldsymbol{f(n) = exponential}$

   If $\boldsymbol{f(n) = ab^n}$ then $\boldsymbol{a_n^{(p)} = Ab^n}$

4. $\boldsymbol{f(n) = linear\, polynomial\, .\, exponential}$

   If $\boldsymbol{f(n) = (an + b)\, c^n}$ then $\boldsymbol{a_n^{(p)} = (An + B)\, c^n}$

Put $a_n^{(p)}$ in equation-1 and find constant A, B, C

$\therefore$ **General solution** $a_n = a_n^{(h)} + a_n^{(p)}$

Now find arbitrary constant using initial condition $a_0 = ..$, $a_1 = ..$

Note: If $\boldsymbol{f(n) = 0}$ **then** $a_n^{(p)} = 0$

Self-Practice Problems

Type-1

1. **Solve the recurrence relation** $a_n = 4a_{n-1} + 5a_{n-2}$ **with the conditions** $a_1 = 2,\ a_2 = 6$

Solution:

$a_n - 4a_{n-1} - 5a_{n-2} = 0$

Let $a_n^{(h)} = \alpha^n$

$\alpha^n - 4\alpha^{n-1} - 5\alpha^{n-2} = 0$

Divided by α^{n-2}

$\alpha^2 - 4\alpha - 5 = 0 \quad \therefore\ \alpha = -1, 5$

$a_n^{(h)} = c_1(-1)^n + c_2(5)^n$

$\because\ f(n) = 0 \quad \therefore\ \boldsymbol{a_n^{(p)} = 0}$

$\therefore$ **General solution** $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = c_1(-1)^n + c_2(5)^n$ .... (1)

Given $a_1 = 2 \quad \therefore\ c_1(-1) + c_2(5) = 2$ .... (2)

$a_2 = 6 \quad \therefore\ c_1(1) + c_2(25) = 6$ .... (3)

Solve equation-2 & 3 simultaneously $c_1 = -\frac{2}{3}$ & $c_2 = \frac{4}{15}$

$a_n = -\frac{2}{3}(-1)^n + \frac{4}{15}(5)^n \quad {}_{Ans}$

....

2. **Solve** $\boldsymbol{a_{r+2} + 2a_{r+1} - 3a_r = 0}$ **that satisfies** $\boldsymbol{a_0 = 1,\ a_1 = 2}$

Solution:

$a_{r+2} + 2a_{r+1} - 3a_r = 0$

Let $a_n^{(h)} = \alpha^n$

$\alpha^{n+2} + 2\alpha^{n+1} - 3\alpha^n = 0$

Divided by α^n

$\alpha^2 + 2\alpha - 3 = 0 \quad \therefore\ \alpha = 1, -3$

$a_n^{(h)} = c_1(1)^n + c_2(-3)^n = c_1 + c_2(-3)^n$

$\because\ f(n) = 0 \quad \therefore\ \boldsymbol{a_n^{(p)} = 0}$

$\therefore$ **General solution** $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = c_1 + c_2(-3)^n$ .... (1)

Given $a_0 = 1 \quad \therefore\ c_1 + c_2 = 1$ .... (2)

$a_1 = 2 \quad \therefore\ c_1 + c_2(-3) = 2$ .... (3)

Solve equation-2 & 3 simultaneously $c_1 = \frac{5}{4}$ & $c_2 = \frac{-1}{4}$

$a_n = \frac{5}{4} - \frac{1}{4}(-3)^n \quad {}_{Ans}$

....

3. **Solve** $\boldsymbol{a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}}$ **with the conditions** $\boldsymbol{a_0 = 2,\ a_1 = 5,\ a_2 = 15}$

Solution:

$a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

Let $a_n^{(h)} = \alpha^n$

$\alpha^n - 6\alpha^{n-1} + 11\alpha^{n-2} - 6\alpha^{n-3} = 0$

Divided by α^{n-3}

$\alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0 \quad \therefore\ \alpha = 1, 2, 3$

$a_n^{(h)} = c_1(1)^n + c_2(2)^n + c_3(3)^n$

$\because\ f(n) = 0 \quad \therefore\ \boldsymbol{a_n^{(p)} = 0}$

$\therefore$ **General solution** $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = c_1 + c_2(2)^n + c_3(3)^n$ .... (1)

Given $a_0 = 2 \quad \therefore\ c_1 + c_2 + c_3 = 2$ .... (2)

$a_1 = 5 \quad \therefore\ c_1 + c_2(2) + c_3(3) = 5$ .... (3)

$a_2 = 15 \quad \therefore\ c_1 + c_2(4) + c_3(9) = 15$ .... (3)

Solve equation-2 & 3 simultaneously $c_1 = 1,\ c_2 = -1$ & $c_3 = 2$

$a_n = 1 - (2)^n + 2(3^n) \quad {}_{Ans}$

....

4. **Solve the recurrence relation** $\boldsymbol{a_n = 4a_{n-1} - 4a_{n-2}}$ **subject to the conditions** $\boldsymbol{a_0 = 1 = a_1}$

Solution:

$a_n - 4a_{n-1} + 4a_{n-2} = 0$

Let $a_n^{(h)} = \alpha^n$

$\alpha^n - 4\alpha^{n-1} + 4\alpha^{n-2} = 0$

Divided by α^{n-2}

$\alpha^2 - 4\alpha + 4 = 0 \quad \therefore\ \alpha = 2, 2$

$a_n^{(h)} = (c_1 n + c_2)(2)^n$

$\because\ f(n) = 0 \quad \therefore\ \boldsymbol{a_n^{(p)} = 0}$

$\therefore$ **General solution** $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = (c_1 n + c_2)(2)^n$ .... (1)

Given $a_0 = 1 \quad \therefore\ c_2 = 1$ .... (2)

$a_1 = 1 \quad \therefore\ (c_1 + c_2)(2) = 1 \quad \Rightarrow\ c_1 + 1 = \frac{1}{2} \quad \Rightarrow\ c_1 = \frac{-1}{2}$

$a_n = \left(1 - \frac{n}{2}\right)(2)^n \quad {}_{Ans}$

....

5. **Solve the recurrence relation** $\boldsymbol{a_n = -3(a_{n-1} + a_{n-2}) - a_{n-3}}$ **with** $\boldsymbol{a_0 = 5,\ a_1 = -9,\ a_2 = 15}$

$Ans\colon\ a_n = (n^2 + 3n + 5)(-1)^n$

Type-2

6. **Solve the recurrence relation** $\boldsymbol{a_{r+2} - a_{r+1} - 6a_r = 4}$

$Ans:\ a_n = c_1(-2)^n + c_2(3)^n - \frac{2}{3}$

Solution:

$a_{r+2} - a_{r+1} - 6a_r = 4$ .... (1)

For $\boldsymbol{a_n^{(h)}}$: $a_{r+2} - a_{r+1} - 6a_r = 0$

Let $a_n^{(h)} = \alpha^n$

$\alpha^{n+2} - \alpha^{n+1} - 6\alpha^n = 0$

Divided by α^n

$\alpha^2 - \alpha - 6 = 0 \quad \therefore\ \alpha = -2, 3$

$a_n^{(h)} = c_1(-2)^n + c_2(3)^n$ .... (2)

For $\boldsymbol{a_n^{(h)}}$: $\because\ f(n) = 4 \quad \therefore\ \boldsymbol{a_n^{(p)} = A}$

Put in equation-1

$A - A - 6A = 4 \quad \therefore\ A = \frac{-2}{3} \quad \therefore\ \boldsymbol{a_n^{(p)}} = \frac{-2}{3}$

$\therefore$ **General solution** $a_n = a_n^{(h)} + a_n^{(p)}$ $a_n = c_1(-2)^n + c_2(3)^n - \frac{2}{3}$ $_{Ans}$

....

7. **Solve the recurrence relation** $\boldsymbol{a_n - 2a_{n-1} + a_{n-2} = 6}$

$Ans:\ a_n = c_1 + c_2 n + 3n^2$

Solution:

$\boldsymbol{a_n - 2a_{n-1} + a_{n-2} = 6}$ .... (1)

For $\boldsymbol{a_n^{(h)}}$: $a_n - 2a_{n-1} + a_{n-2} = \boldsymbol{0}$

Let $a_n^{(h)} = \alpha^n$

$\alpha^n - 2\alpha^{n-1} + \alpha^{n-2} = 0$

Divided by α^{n-2}

$\alpha^2 - 2\alpha + 1 = 0 \quad \therefore\ \alpha = 1, 1$

$a_n^{(h)} = (c_1 + c_2 n)(1)^n = c_1 + c_2 n$ .... (2)

For $\boldsymbol{a_n^{(h)}}$: $\because\ f(n) = 6$ and $\alpha = 1\ \&\ m = 2 \quad \therefore\ \boldsymbol{a_n^{(p)} = An^2}$

Put in equation-1

$An^2 - 2A(n-1)^2 + A(n-2)^2 = 6$

$An^2 - 2A(n^2 - 2n + 1) + A(n^2 - 4n + 4) = 6$

$An^2 - 2An^2 + 4An - 2A + An^2 - 4An + 4A = 6$

$2A = 6 \quad \therefore\ A = 3 \quad \therefore\ \boldsymbol{a_n^{(p)}} = 3n^2$

$\therefore$ **General solution** $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = c_1 + c_2 n + 3n^2$ $_{Ans}$

....

8. **Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n$ with $a_0 = 13,\ a_1 = 29$**

Solution:

$a_n - 7a_{n-1} + 10a_{n-2} = 8n + 6$ .... (1)

For $a_n^{(h)}$: $a_n - 7a_{n-1} + 10a_{n-2} = 0$

Let $a_n^{(h)} = \alpha^n$

$\alpha^n - 7\alpha^{n-1} + 10\alpha^{n-2} = 0$

Divided by α^{n-2}

$\alpha^2 - 7\alpha + 10 = 0 \quad \therefore\ \alpha = 2, 5$

$a_n^{(h)} = c_1(2)^n + c_2(5)^n$ .... (2)

For $a_n^{(h)}$: $\because\ f(n) = 8n + 6 \quad \therefore\ a_n^{(p)} = An + B$

Put in equation-1

$An + B - 7[A(n-1) + B] + 10[A(n-2) + B] = 8n + 6$

$An + B - 7An + 7A - 7B + 10An - 20A + 10B = 8n + 6$

$4An - 13A + 4B = 8n + 6$

Compare $4A = 8$ & $-13A + 4B = 6$

$\therefore\ A = 2$ & $-26 + 4B = 6 \quad \therefore\ A = 8$

$\therefore\ a_n^{(p)} = 2n + 8$

$\therefore$ **General solution** $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = c_1(2)^n + c_2(5)^n + 2n + 8$

Given

$a_0 = 13, \quad \therefore\ c_1 + c_2 + 8 = 13 \quad \Longrightarrow \quad c_1 + c_2 = 5$

$a_1 = 29 \quad \therefore\ c_1(2) + c_2(5) + 2 + 8 = 29 \quad \Longrightarrow \quad c_1(2) + c_2(5) = 19$

Solve equations simultaneously $c_1 = 2$ & $c_2 = 3$

$a_n = 2(2)^n + 3(5)^n + 2n + 8 \quad _{Ans}$

....

9. **Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$**

Solution:

$a_n - 5a_{n-1} + 6a_{n-2} = 7^n$ .... (1)

For $a_n^{(h)}$: $a_n - 5a_{n-1} + 6a_{n-2} = 0$

Let $a_n^{(h)} = \alpha^n$

$\alpha^n - 5\alpha^{n-1} + 6\alpha^{n-2} = 0$

Divided by α^{n-2}

$\alpha^2 - 5\alpha + 6 = 0 \quad \therefore\ \alpha = 2, 3$

$a_n^{(h)} = c_1(2)^n + c_2(3)^n$ .... (2)

For $a_n^{(h)}$: $\because\ f(n) = 7^n \quad \therefore\ a_n^{(p)} = A7^n$

Put in equation-1

$A7^n - 5A7^{n-1} + 6A7^{n-2} = 7^n$

$\left(1 - \frac{5}{7} + \frac{6}{49}\right) A7^n = 7^n$

$\frac{20}{49}A = 1 \qquad \Longrightarrow \qquad A = \frac{49}{20} \qquad\qquad \therefore\ \boldsymbol{a_n^{(p)}} = \left(\frac{49}{20}\right) 7^n$

$\therefore$ **General solution** $\quad a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = c_1(2)^n + c_2(3)^n + \left(\frac{49}{20}\right) 7^n \quad {}_{Ans}$

....

10. Solve the equation $a_r + a_{r+1} = 3r.2^r$ with $a_0 = 11/3$.

$Ans:\ a_n = 3(-1)^n + \left(2n + \frac{2}{3}\right)(2)^n$