Recitation Guide - Week 3

Topics Covered: Graphs.

Problem 1: In this problem we illustrate a common trap that we can fall in when proving statements about graphs by induction on the number of vertices or the number of edges. Here is a *false statement*: "If every vertex in a simple graph G has strictly positive (> 0) degree, then G is connected".

(a) Prove that the statement is indeed false by providing a counterexample.

(b) Since the statement is false, there must be something wrong in the following "proof". Pinpoint the *first* logical mistake (unjustified step).

Buggy Proof:

We prove the statement by induction on the number of vertices. Let $P(n)$ be the following proposition: "for any graph with n vertices, if every vertex has strictly positive degree, then the graph is connected".

<u>Base Cases:</u> Notice that $P(1)$ is vacuously true. We also show that $P(2)$ is true. Notice that there is only one graph with two vertices of strictly positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

<u>Induction Hypothesis:</u> Assume that for some $k \geq 2$, $P(k)$ is true.

<u>Induction Step:</u>

Consider a graph G_{old} with k vertices in which every vertex has strictly positive degree. By the Induction Hypothesis this graph is connected. Now we add one more vertex, call it u, to obtain a graph G_{new} with $k + 1$ vertices.

All that remains is to check that in G_{new} there is a walk from u to every other vertex v. Since u has positive degree, there is an edge from u to some other vertex, say w. But w and v are in G_{old}, which is connected, and therefore there is a walk from w to v. This gives a walk $u - w - v$ in G_{new}. ✓

(c) Now consider the changed Induction Step and identify a mistake in this proof.

<u>Induction Step:</u>

Consider a graph G with $k + 1$ vertices in which every vertex has strictly positive degree. Remove an arbitrary vertex, call it u, and now we have a graph G' with k vertices. By the Induction Hypothesis this graph is connected. Now we add u back in to obtain a graph G with $k + 1$ vertices.

All that remains is to check that in G there is a walk from u to every other vertex v. Since u has positive degree, there is an edge from u to some other vertex, say w. But w and v are in G', which is connected, and therefore there is a walk from w to v. This gives a walk $u - w - v$ in G. ✓

Solution:

(a) Consider the graph $G = (V, E)$ where $V = \{a, b, c, d\}$ and $E = \{\{a, b\}, \{c, d\}\}$. Every vertex has degree one, however the graph is not connected (there is no path from a to c, for example).

(b) The logical mistake in the proof is where we "add one more vertex" in the induction step. It is certainly possible to add one more vertex to a graph such that all vertices have strictly positive degree, but this constructs a *particular* type of graph G_{new} with $k+1$ vertices, whereas we actually want to show $P(k+1)$, which is that the claim holds for *any* graph with $k+1$ vertices. In particular, there are graphs with $k+1$ vertices where all its vertices have strictly positive degree that cannot be constructed from graphs with k vertices that fulfill the same condition. For instance, there does not exist any graph with 3 vertices where all its vertices have strictly positive degree such that by adding a new vertex we obtain graph G in part (a). This highlights the importance of starting with an arbitrary graph with $k+1$ vertices, then deconstruct it to obtain a graph with k vertices to apply the IH to in graph induction proofs!

There are a couple of statements that may seem "bogus" but are actually not. They are as follows:

(a) "$P(1)$ is vacuously true": This is not "bogus", as a simple graph with 1 vertex must not have any edges, so it cannot have strictly positive degree.

(b) "Let k be an arbitrary integer such that $k \geq 2$": This is not "bogus", as we have an additional base case for $n = 2$, while $P(1)$ is proved separately.

(c) After removing a vertex, we have to make sure that in G', the properties specified in IH still exist. In this case, we have to make sure that after removing a vertex, every vertex still has a strictly positive degree to apply IH.

Consider the neighbors of u in G. If there was a neighbor x such that the degree of x in G was 1, since its only neighbor was removed, its degree in G' would be 0. Therefore, we cannot always apply IH to G'.

Problem 2: Let T be a tree where the maximum degree is Δ. Prove that T has at least Δ leaves by contradiction.

Solution:

Let us prove this by induction on the number of vertices in the graph n.

We formulate a proposition $P(n)$ which is: in a tree with n vertices and maximum degree Δ, the number of leaves in the tree is at least Δ.

Base Case (n= 1, 2 and 3): The case of $n = 1$ is trivial - a graph of just 1 node has maximum degree 0 and at least 0 leaves. There is only one possible tree when $n = 2$: $T = (V, E)$, $V = \{u, v\}$, $E = \{\{u, v\}\}$. Here $\Delta = 1$, and we have 2 leaves, so it checks out as required.

There is only one possible tree when $n = 3$: $T = (V, E)$, $V = \{u, v, w\}$, $E = \{\{u, v\}, \{v, w\}\}$. Here $\Delta = 2$, and we have 2 leaves, so it checks out as required.

We choose to show three base cases here to avoid a slightly unfortunate edge case in the Induction Step.

Induction Step: Assume that (IH) $P(k)$ is true, for some $k \in \mathbb{Z}^+, k \geq 3$. Consider an arbitrary tree $T = (V, E)$ such that $|V| = k+1$ and it has maximum degree Δ. Let $\ell \in V$ be an arbitrary leaf in T who has some neighbor a. Consider $T' = (V', E')$ where $V' = V \setminus \ell$ and $E' = E \setminus \{a, \ell\}$.

We know that $|V'| = k$ and is a tree (since removal of a leaf can never disconnect a tree), so we can apply the Induction Hypothesis on T'.

Note that there are two cases here:

1. a was the only vertex of degree Δ in T.

   It must be the case then that a has degree $\Delta - 1$ in T' and is of maximum degree. The Induction Hypothesis gives us that T' must have at least $\Delta - 1$ leaves.

   Further note if a is a leaf in T', then it must be the case that $n = 3$ (convince yourself of this), and that is already shown to be true by the base case. Hence, going forward we will operate under the assumption that a is not a leaf.

   Adding ℓ back to T' to reconstruct T increases the number of leaves by one (since a is not a leaf), so we have that T has at least Δ leaves.

2. There is some vertex in T' that has degree Δ.

   By the Induction Hypothesis, we have that T' must have Δ leaves.

   There are two more cases here:

   (a) a is a leaf in T'

   In this case, the addition of ℓ does not change the number of leaves, which means we have at least Δ leaves in T, as desired.

   (b) a is not a leaf in T'

   In this case, the addition of ℓ increases the number of leaves by 1, which means we have at least $\Delta + 1$ leaves in T, which proves our claim.

Problem 3: The *complement* of a graph G is a new graph formed by removing all the edges of G and replacing them by all possible edges that are not in G. Formally, consider a graph $G = (V, E)$. Then, the complement of the graph G is the graph $\overline{G} = (V, \overline{E})$, where

$$\overline{E} = \{\{x, y\} \mid x \neq y, \{x, y\} \notin E\}$$

Prove that for any graph G, G or $\overline{G}$ (or both) must be connected.

Solution:

We want to show that if G is disconnected then $\overline{G}$ is connected. Since G is disconnected there must be two vertices u and v such that u and v belong to different components. We want to show that any two vertices x and y are connected by a path in $\overline{G}$. We consider the following cases.

Case 1: In G, x and y belong to different components.
Then there is an edge between x and y in $\overline{G}$.

Case 2: In G, x and y belong to the same component. Then in $\overline{G}$, there are edges from both x and y to the same vertex, either u or v.