MODULE-1- COMPLEX NUMBERS (14 Marks)

10 February 2022 10:00

Dr. Uday Kashid , Ph.D.(Mathematics)

1. $z = x + iy$: *Caspar Wessel in 1799*
   Caspar Wessel (1745-1818), a Norwegian, was the first one to obtain and publish a suitable presentation of complex numbers. On march 10, 1797, Wessel presented his paper "On the Analytic Representation of Direction: An Attempt" to the Royal Danish Academy of Sciences.
2. Operations: $+, -, \times, \div$
3. Polar and Exponetial form:
4. Properties of Modulus and Arguments(Amplitude)
5. Examples 1) Find Modulus and Arguments : $z = -1 + i$
   2) **If** $|z-1| = |z+1|$ **, Then prove that Re(z)=0**
6. **De Moivre's theorem**

$\sqrt{4} = \pm 2$

$\sqrt{-4}$ $\quad i = \sqrt{-1}$ $\quad \boxed{i^2 = -1}$

$\sqrt{4i^2} = \pm 2i$

Note:- IF $z = x+iy$, $x, y \in \mathbb{R}$ $\quad \bar{z} = x - iy$ $\quad Re(z) = x$ and $Im(z) = y$

$|z| = \text{modulus of } z = \sqrt{x^2+y^2} = \sqrt{(Re)^2 + (Im)^2} = r$

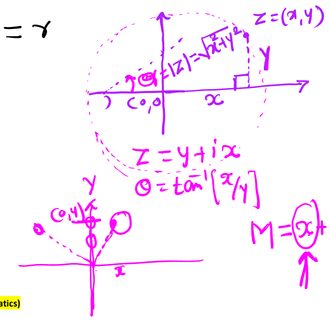
$z = x+iy = (x, y)$

argument or Amplitude of z is denoted by 'θ'

$\arg(z) = \text{amp}(z) = \theta = \tan^{-1}\left[\frac{Im}{Re}\right] = \tan^{-1}\left[\frac{y}{x}\right]$

$\sin\theta = \frac{y}{r} \Rightarrow \boxed{y = r\sin\theta}$ $\qquad \boxed{x^2 + y^2 = r^2}$

$\cos\theta = \frac{x}{r} \Rightarrow \boxed{x = r\cos\theta}$



$M = (x) + iy$

IF $z_1 = x_1 + iy_1$, $\quad z_2 = x_2 + iy_2$

$|z_1| = \sqrt{x_1^2 + y_1^2}$ $\qquad |z_2| = \sqrt{x_2^2 + y_2^2}$

$\theta_1 = \tan^{-1}\left[\frac{y_1}{x_1}\right]$ $\qquad \theta_2 = \tan^{-1}\left[\frac{y_2}{x_2}\right]$

① $z_1 \bar{z_1} = (x_1 + iy_1)(x_1 - iy_1) = x_1^2 - i^2 y_1^2 = x_1^2 + y_1^2$

$|z_1 \bar{z_1}| = x_1^2 + y_1^2 = |z_1|^2 = |\bar{z_1}|^2$

② $|z_1 z_2| = |z_1| \cdot |z_2|$

③ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

④ $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

⑤ $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Examples

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1. **Find Modulus and Arguments :** $z = -1 + i$
2. **If** $|z-1| = |z+1|$ **,** **Then prove that Re(z) = 0**
3. **Find the complex number** z **if** $\arg(z+1) = \frac{\pi}{6}$ **and** $\arg(z-1) = \frac{2\pi}{3}$ **[MU -Dec-11,May-08,12]**
4. **Find the complex number** z **if** $\arg(z+2i) = \frac{\pi}{4}$ **and** $\arg(z-2i) = \frac{3\pi}{4}$ **[Ans:** $z = x + iy = 2 + i0$**]**
5. **If** $|z^2 - 1| = |z|^2 + 1$ **, Then prove that complex no z is purely imaginary. [MU -Dec-07,16]**
6. **Show that** $\left|\frac{z}{|z|} - 1\right| \le |\arg(z)|$, where $z = x + iy$ $\qquad$ HW
7. **If** $a^2 + b^2 + c^2 = 1$ **and** $b + ic = (a+1)z$ **, then prove that** $\frac{1+iz}{1-iz} = \frac{a+ib}{1+c}$ **[MU -Dec-11]** $\qquad$ HW.

1) $z = -1 + i = x + iy \Rightarrow x = -1, \; y = 1$

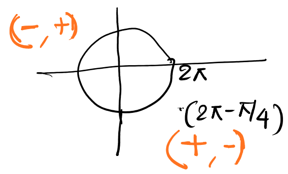
$|z| = \sqrt{x^2+y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

$\theta = \arg(z) = \tan^{-1}\left[\frac{y}{x}\right] = \tan^{-1}\left[\frac{1}{-1}\right] = \tan^{-1}(-1)$

$\Rightarrow \tan\theta = -1 = \tan(\pi - \pi/4) = -\tan(\pi/4) = -1$

$\tan\theta = -1 = \tan(3\pi/4)$

$\boxed{\theta = 3\pi/4}$



2. **If** $|z-1| = |z+1|$ **,** **Then prove that Re(z) = 0**

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$\rightarrow$ Let $z = x+iy$

$z+1 = (x+1) + iy$ $\qquad z-1 = (x-1) + iy$

$|z+1| = \sqrt{(x+1)^2 + y^2}$ — (I) $\qquad |z-1| = \sqrt{(x-1)^2 + y^2}$ — (II)

we have $|z+1| = |z-1|$

$|z+1|^2 = |z-1|^2$

$(x+1)^2 + y^2 = (x-1)^2 + y^2$

$\Rightarrow \cancel{x^2} + 2x + \cancel{1} = \cancel{x^2} - 2x + \cancel{1}$

$4x = 0$

$x = 0$

$\boxed{Re(z) = x = 0}$

3. **Find the complex number** z **if** $\arg(z+1) = \frac{\pi}{6}$ **and** $\arg(z-1) = \frac{2\pi}{3}$ **[MU -Dec-11,May-08,12]**

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$\rightarrow$ Let $z = x+iy$

$z+1 = (x+1) + iy$

$\arg(z+1) = \theta_1 = \tan^{-1}\left[\frac{y}{x+1}\right] = \frac{\pi}{6}$

$\Rightarrow \frac{y}{x+1} = \tan(\pi/6) = \tan(30^\circ) = \frac{1}{\sqrt{3}}$

$\Rightarrow \sqrt{3}y = x + 1$

$\Rightarrow \boxed{x - \sqrt{3}y = -1}$ — (I)

$z - 1 = (x-1) + iy$

$\arg(z-1) = \theta_2 = \tan^{-1}\left[\frac{y}{x-1}\right] = \frac{2\pi}{3}$

$\frac{y}{x-1} = \tan(2\pi/3) = \tan(\pi - \pi/3)$

$= -\tan(\pi/3) = -\tan(60^\circ)$

$\frac{y}{x-1} = -\sqrt{3}$

$y = -\sqrt{3}x + \sqrt{3}$

$\boxed{\sqrt{3}x + y = \sqrt{3}}$ — (II)

$\Rightarrow$ solve eq ① & (II)

$x - \sqrt{3}y = -1$

$+ \quad 3x + \sqrt{3}y = 3$ $\quad$ — (eq (II) multiple by $\sqrt{3}$)

$4x + 0 = 2$

$\boxed{x = 1/2}$

$y = -\sqrt{3} \cdot \frac{1}{2} + \sqrt{3} = \sqrt{3}(1/2)$

$z = x + iy = \frac{1}{2} + \frac{\sqrt{3}}{2}i = \boxed{\frac{(1+\sqrt{3}i)}{2}}$

1. **If** $|z^2 - 1| = |z|^2 + 1$ **, Then prove that complex no z is purely imaginary. [MU -Dec-07,16]**

$\rightarrow$ let $z = x+iy$ $\qquad z=a+ib$

$z^2 = (x+iy)^2 = x^2+2xyi+y^2i^2 = (x^2-y^2)+i(2xy)$

$z^2-1 = (x^2-y^2-1)+i(2xy)$

$|z^2-1| = \sqrt{(x^2-y^2-1)^2+(2xy)^2} \quad \text{--- (I)}$

$|z| = \sqrt{x^2+y^2} \Rightarrow |z|^2 = x^2+y^2$

$|z|^2+1 = x^2+y^2+1 \quad \text{--- (II)}$

Sem-I – Complex Nos

III $\rightarrow$ Complex variable

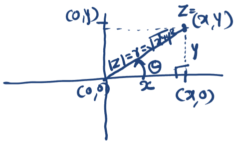
C.R. equ^n

IV Complex Integral

Residue Thm

Note:— Complex Nos.

1) Cartesian form :– $z = x+iy \qquad z=(x,y)$



2) Polar Form: -

$z = x+iy$ But $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} x^2+y^2 = r^2$

$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \Rightarrow \theta = \tan^{-1}\left[\frac{y}{x}\right]$

$\boxed{z = r[\cos\theta + i\sin\theta]} \rightarrow$ polar form of complex No. z

3) Exponential Form :-

We know $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Mod-6

Taylor series

$\rightarrow e^{x} = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots \quad \text{--- (1)}$

$e^{-x} = 1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^n}{n!}+\cdots$

If $y = f(x)$

$f(-x) = \begin{cases} f(x) \rightarrow \text{Even Fu}^n \\ -f(x) \rightarrow \text{odd Fu}^n \end{cases}$

$\cos(x) = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$

Ex. If $y = \cos x = f(x)$

$f(-x) = \cos(-x) = \cos x = f(x) \rightarrow$ Even

$\sin(x) = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$

Ex. If $y = \sin x = f(x)$

$f(-x) = \sin(-x) = -\sin x = -f(x) \rightarrow$ odd

put $x = i\theta$ in eq^n (1)

$e^{i\theta} = 1+\frac{i\theta}{1!}+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\cdots$

$e^{i\theta} = \left[1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\cdots\right]+i\left[\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots\right]$

$\boxed{e^{i\theta} = \cos(\theta)+i\sin(\theta)}$

$\rightarrow e^{-i\theta} = \frac{1}{e^{i\theta}} = \frac{1}{[\cos\theta+i\sin\theta]} \times \frac{[\cos\theta-i\sin\theta]}{[\cos\theta-i\sin\theta]} = \frac{\cos\theta-i\sin\theta}{\cos^2\theta-i^2\sin^2\theta}$

$\boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$

(or) We know $e^{i\theta} = \cos\theta + i\sin\theta$

put θ by $(-\theta)$

$e^{-i\theta} = \cos(-\theta)+i\sin(-\theta)$

$\boxed{e^{-i\theta} = \cos\theta - i\sin(\theta)}$

3) Exponential Form

$z = r[\cos\theta + i\sin\theta]$

$\boxed{z = r\,e^{i\theta}}$

Exa. P.T $\left|\frac{z}{|z|}-1\right| \leq |\arg z|$

$\rightarrow z = x+iy = r[\cos\theta+i\sin\theta]$

$|z| = \sqrt{x^2+y^2} = r$

$\frac{z}{|z|} = \frac{r[\cos\theta+i\sin\theta]}{r} = \cos\theta+i\sin\theta$

$\left(\frac{z}{|z|}-1\right) = (\cos\theta-1)+i\sin\theta \qquad 1-\cos\theta = 2\sin^2\theta/2$

$= -2\sin^2\theta/2 + i(2\sin\theta/2\cos\theta/2)$

$\ldots 2i\sin\theta/2\,[\cos\theta/2 + i\sin\theta/2]$

$$\frac{z}{|z|}-1 = (\cos\theta-1)+i\sin\theta \qquad 1-\cos\theta = 2\sin^2\theta/2$$

$$= -2\sin^2\theta/2 + i(2\sin\theta/2\cos\theta/2)$$

$$= 2i^2\sin^2\theta/2 + 2i\sin\theta/2\cos\theta/2 = 2i\sin\theta/2\left[\cos\theta/2+i\sin\theta/2\right]$$

$$\left|\frac{z}{|z|}-1\right| = \left|2i\sin\theta/2\left(\cos\theta/2+i\sin\theta/2\right)\right| \qquad |z_1 \cdot z_2 z_3| = |z_1|\cdot|z_2|\cdot|z_3|$$

$$= 2|i|\,|\sin\theta/2|\,|\cos\theta/2+i\sin\theta/2|$$

But $|i| = |0+i| = \sqrt{0^2+1^2} = 1 \Rightarrow \boxed{|i| = |-i| = 1}$

$$|\cos\theta/2+i\sin\theta/2| = \sqrt{\cos^2\theta/2+\sin^2\theta/2} = 1$$

$$\left|\frac{z}{|z|}-1\right| = 2|\sin\theta/2| \qquad \text{But } |\sin\theta| \leq |\theta|$$

$$\leq 2|\theta/2| \qquad |\sin\theta/2| \leq |\theta/2|$$

$$\leq |\theta|$$

$$\left|\frac{z}{|z|}-1\right| \leq |\arg z|$$

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If $z = r(\cos\theta + i\sin\theta)$ then by De – Moivre's theorem $z^n = r^n(\cos n\theta + i \sin n\theta)$

$\rightarrow z^n = r^n[\cos\theta + i\sin\theta]^n$

$= r^n[e^{i\theta}]^n = r^n e^{i(n\theta)}$

$\rightarrow z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$

$\boxed{z^n = r^n[\cos\theta + i\sin\theta]^n = r^n[\cos(n\theta) + i\sin(n\theta)]}$

$\bar{z} = r[\cos\theta + i\sin\theta]$

$\bar{z}^n = r^n[\cos\theta + i\sin\theta]^{-n}$

$\bar{z}^n = r^n[\cos(-n\theta) + i\sin(-n\theta)]$

$\boxed{\bar{z}^n = r^n[\cos(n\theta) - i\sin(n\theta)]}$

$r = 1$

Note: If $z = \cos\theta + i\sin\theta$ Then

1. $z^n + \frac{1}{z^n} = 2\cos(n\theta)$
2. $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$

Examples:

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1. If $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$, and $\bar{z}$ is the conjugate of z, then prove that $(z)^{10} + (\bar{z})^{10} = 0$. [MU -Dec-06]
2. If $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $\bar{z}$ is the conjugate of z, then prove that $(z)^{15} + (\bar{z})^{15} = -2$.
3. Prove that $\left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n = \cos\left(\frac{n\pi}{2} - n\alpha\right) + i\sin\left(\frac{n\pi}{2} - n\alpha\right)$, [MU -Dec-01,04,May-15]
4. If $\alpha = 1 + i$, $\beta = 1 - i$, and $\cot(\varphi) = x + 1$, then prove that $\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \sin(n\varphi)\,cosec^n(\varphi)$ [MU-Dec-02,04,11,14]
5. If $z = -1 + i\sqrt{3}$, Prove that $\left(\frac{z}{2}\right)^n + \left(\frac{2}{z}\right)^n = \begin{cases} -1, & \text{if } 'n' \text{ is not multiple of } 3 \\ 2 & \text{if } 'n' \text{ is multiple of } 3 \end{cases}$ [MU -Dec-15]
6. Prove that $\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^n + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^n = \begin{cases} -1, & \text{if } n = 3k \pm 1 \\ 2 & \text{if } n = 3k \end{cases}$, where k is an integer
7. Prove that $(1+i)^{100} + (1-i)^{100} = -2^{51}$
8. Prove that $\left(\frac{1+7i}{(2-i)^2}\right)^{4n} = (-4)^n$, where 'n' is positive integer [MU -Dec-04, 17]
9. If $x + iy = \sqrt[3]{a+ib}$, then $\frac{a}{x} + \frac{b}{y} = ?$

Thur – 2-3–phy
3-5–maths

$\boxed{\text{IF } r = 1}$

- Note: If $z = \cos\theta + i\sin\theta$ Then
  - $z^n + \frac{1}{z^n} = 2\cos(n\theta)$
  - $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$

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$z = r[\cos\theta + i\sin\theta] = 1[\cos\theta + i\sin\theta] = e^{i\theta}$

$\frac{1}{z} = \cos\theta - i\sin\theta = e^{-i\theta}$

(i) $z + \frac{1}{z} = [\cos\theta + i\sin\theta] + [\cos\theta - i\sin\theta]$

$\boxed{z + \frac{1}{z} = 2\cos\theta} \Rightarrow \cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right]$

$\Rightarrow \boxed{\cos\theta = \left[\frac{e^{i\theta} + e^{-i\theta}}{2}\right]}$

(ii) $\boxed{z - \frac{1}{z} = 2i\sin\theta}$

$\boxed{\sin\theta = \left[\frac{e^{i\theta} - e^{-i\theta}}{2i}\right]}$

$z^2 = [\cos\theta + i\sin\theta]^2 = \cos(2\theta) + i\sin(2\theta)$

$\frac{1}{z^2} = \cos(2\theta) - i\sin(2\theta)$

(i) $z^2 + \frac{1}{z^2} = 2\cos 2\theta \Rightarrow \cos(2\theta) = \left[\frac{e^{i2\theta} + e^{-i2\theta}}{2}\right]$

$z^2 - \frac{1}{z^2} = 2i\sin(2\theta) \Rightarrow \sin(2\theta) = \frac{e^{i(2\theta)} - e^{-i(2\theta)}}{2i}$

similary

$z^n = [\cos\theta + i\sin\theta]^n = \cos(n\theta) + i\sin(n\theta)$

$\frac{1}{z^n} = \cos(n\theta) - i\sin(n\theta)$

(i) $z^n + \frac{1}{z^n} = 2\cos(n\theta)$

(ii) $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$

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1. If $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$, and $\bar{z}$ is the conjugate of z, then prove that $(z)^{10} + (\bar{z})^{10} = 0$. [MU -Dec-06]

$\rightarrow z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = x + iy \Rightarrow x = 1/\sqrt{2},\ y = 1/\sqrt{2}$

$z = r[\cos\theta + i\sin\theta] \Rightarrow r = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(1/\sqrt{2}\right)^2} = 1$, $\theta = \tan^{-1}\left[\frac{y}{x}\right] = \tan^{-1}\left[\frac{1/\sqrt{2}}{1/\sqrt{2}}\right] = \pi/4$

$z = 1\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$ –(i)

$\bar{z} = \cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)$ –(ii)

$z^{10} = \left[\cos(\pi/4) + i\sin(\pi/4)\right]^{10} = \cos\left(\frac{10\pi}{4}\right) + i\sin\left(\frac{10\pi}{4}\right)$

$(\bar{z})^{10} = \left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]^{10} = \cos\left(\frac{10\pi}{4}\right) - i\sin\left(\frac{10\pi}{4}\right)$

Addition

$(z)^{10} + (\bar{z})^{10} = 2\cos\left(\frac{10\pi}{4}\right) = 2\cos\left(5\frac{\pi}{2}\right)$ $\cos[n\pi/2] = 0$, $n = odd$

$(z^{10}) + (\bar{z})^{10} = 0$.

$[\cos\theta + i\sin\theta]^n$

3. Prove that $\left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n = \cos\left(\frac{n\pi}{2} - n\alpha\right) + i\sin\left(\frac{n\pi}{2} - n\alpha\right)$, [MU -Dec-01,04,15] Dr. Uday Kashid, PhD (Mathematics)

$\rightarrow$ Let $z^n = \left[\frac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha}\right]^n$

$\cos\left[\frac{\pi}{2} - \alpha\right] = \sin\alpha$

$\sin\left[\frac{\pi}{2} - \alpha\right] = \cos\alpha$

$z^n = \left[\frac{1+\cos\left(\frac{\pi}{2}-\alpha\right) + i\sin\left(\frac{\pi}{2}-\alpha\right)}{1+\cos\left(\frac{\pi}{2}-\alpha\right) - i\sin\left(\frac{\pi}{2}-\alpha\right)}\right]^n$

$\therefore \begin{cases} 1 + \cos\theta = 2\cos^2(\theta/2) \\ 1 + \cos\left(\frac{\pi}{2}-\alpha\right) = 2\cos^2\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \\ \sin\theta = 2\sin(\theta/2)\cos(\theta/2) \end{cases}$

$z^n = \left[\frac{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) + i\,2\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) - i\,2\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}\right]^n$

$z = \cos\theta + i\sin\theta \Rightarrow z^n = [\cos(n\theta) + i\sin n\theta]$

$\therefore \bar{z} = \frac{1}{z} = \cos\theta - i\sin\theta$ $(\bar{z})^n = \frac{1}{z^n} = [\cos(n\theta) - i\sin n\theta]$

$\frac{1}{z} = e^{-i\theta} \Rightarrow \frac{1}{z^n} = e^{-in\theta} = \cos n\theta - i\sin n\theta$

$z^n = \left[\frac{\cancel{2\cos}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\left[\cos(\pi/4 - \alpha/2) + i\sin\left(\frac{\pi}{4} - \alpha/2\right)\right]}{\cancel{2\cos}\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\left[\cos(\pi/4 - \alpha/2) - i\sin(\pi/4 - \alpha/2)\right]}\right]^n$

By De-moivre's Thm

$z^n = \frac{\left[\cos\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right) + i\sin\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right)\right]}{\left[\cos\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right) - i\sin\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right)\right]} = \frac{e^{i\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right)}}{e^{-i\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right)}} = e^{i\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right)} \cdot e^{i\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right)}$

$z^n = e^{i2\left(\frac{n\pi}{4} - \frac{n\alpha}{2}\right)} = e^{i\left(\frac{n\pi}{2} - n\alpha\right)}$

$\boxed{z^n = \cos\left(\frac{n\pi}{2} - n\alpha\right) + i\sin\left(\frac{n\pi}{2} - n\alpha\right)}$

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4. If $\alpha = 1 + i$, $\beta = 1 - i$, and $\cot(\varphi) = x + 1$, then prove that $\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \sin(n\varphi)\,cosec^n(\varphi)$ [MU-Dec-02,04,11,14]

$\rightarrow \alpha = 1 + i$ –(i) $\beta = 1 - i$ –(ii) $x = \cot\phi - 1$ –(iii)

(i) $x + \alpha = [\cot\phi - 1] + [1 + i] = \cot\phi + i = \frac{\cos\phi}{\sin\phi} + i = \frac{\cos\phi + i\sin\phi}{\sin\phi}$

$(x+\alpha)^n = \frac{[\cos\phi + i\sin\phi]^n}{\sin^n\phi} = cosec^n\phi\,[\cos(n\phi) + i\sin(n\phi)]$ → By De-moivre's Thm –(iv)

(ii) $x + \beta = [\cot\phi - 1] + [1 - i] = \cot\phi - i = \frac{\cos\phi - i\sin\phi}{\sin\phi}$

$$= \operatorname{cosec}\phi\,[\cos(n\phi)+i\sin(n\phi)] \quad \text{—(IV)}$$
$\sin^n\phi$

(ii) $x+\beta = [\cot\phi + 1] + [1-i] = \cot\phi - i = \dfrac{\cos\phi - i\sin\phi}{\sin\phi}$

$$(x+\beta)^n = \frac{[\cos\phi - i\sin\phi]^n}{\sin^n\phi} = [\cos(n\phi) - i\sin(n\phi)]\operatorname{cosec}^n\phi \quad \text{—(V)}$$

(iii) $\alpha-\beta = (1+i)-(1-i) = 2i$ —(VI)

$$LHS = \frac{(x+\alpha)^n - (x+\beta)^n}{\alpha-\beta} = \frac{\operatorname{cosec}^n\phi\,[\cos n\phi + i\sin n\phi] - \operatorname{cosec}^n\phi\,[\cos n\phi - i\sin n\phi]}{2i}$$

$$LHS = \frac{\operatorname{cosec}^n\phi}{\cancel{2i}}\left[\cancel{2i}\sin(n\phi)\right] = \boxed{\operatorname{cosec}^n\phi\,\sin(n\phi)}$$

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5. If $z = -1 + i\sqrt{3}$, Prove that $\left(\frac{z}{2}\right)^n + \left(\frac{2}{z}\right)^n = \begin{cases} -1, & \text{if 'n' is not multiple of 3} \\ 2 & \text{if 'n' is multiple of 3} \end{cases}$ [MU -Dec-15]

$\rightarrow$ $z = x+iy = -1+i\sqrt{3} = r(\cos\theta + i\sin\theta)$ $\Rightarrow$ $x=-1$, $y=\sqrt{3}$, $z=(-1,\sqrt{3})$ lies in IInd quadrant.

$r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left[\frac{Im(z)}{Re(z)}\right] = \tan^{-1}\left[\frac{\sqrt{3}}{-1}\right] = \tan^{-1}[-\sqrt{3}]$ $\Rightarrow$ $\tan\theta = -\sqrt{3}$

$\tan(\pi - \pi/3) = -\tan(\pi/3) = -\sqrt{3}$

$\tan(2\pi/3) = -\sqrt{3}$

$\Rightarrow \boxed{\theta = 2\pi/3}$

$z = 2[\cos 2\pi/3 + i\sin 2\pi/3]$ —(1)

$\bar{z} = 2[\cos 2\pi/3 - i\sin(2\pi/3)]$

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$\rightarrow$ $\frac{z}{2} = \cos(2\pi/3) + i\sin(2\pi/3)$

$\frac{2}{z} = \frac{1}{\cos(2\pi/3)+i\sin(2\pi/3)} = \cos(2\pi/3) - i\sin(2\pi/3) = (\bar{z}/2)$

$LHS = \left(\frac{z}{2}\right)^n + \left(\frac{2}{z}\right)^n = [\cos(2\pi/3) + i\sin(2\pi/3)]^n + [\cos(2\pi/3) - i\sin(2\pi/3)]^n$

By De-moivre's thm

$= \cos(2n\pi/3) + i\sin(2n\pi/3) + \cos(2n\pi/3) - i\sin(2n\pi/3)$

$\boxed{\left(\frac{z}{2}\right)^n + \left(\frac{2}{z}\right)^n = 2\cos\left(\frac{2n\pi}{3}\right)}$ —(A) $n \in \pm$ Integer, Fraction

$n = \frac{4}{5}$

$k=0$, $k=1$, $k=2$, $k=3$, $k=4$

± 1, $2, 4$, $5, 7$, $8, 10$, $11, 13$

case-(I) If n is multiple of 3 i.e. $n = 3k$, $\pm k = 1,2,3,4,$

$\left(\frac{z}{2}\right)^n + \left(\frac{2}{z}\right)^n = 2\cos\left[\frac{2(3k)\pi}{3}\right] = 2\cos[2k\pi] = 2[1] = \boxed{2}$

For $k = 1,2,3,\ldots$
$\cos(2k\pi) = 1$

case-(II) If n is not multiple of 3 $\Rightarrow$ i.e. $n = 3k \pm 1$, $k = 0,1,2,3,\ldots$

$\left(\frac{z}{2}\right)^n + \left(\frac{2}{z}\right)^n = 2\cos\left[\frac{2(3k\pm1)\pi}{3}\right] = 2\cos\left[\frac{6\pi k \pm 2\pi}{3}\right] = 2\cos[2k\pi \pm 2\pi/3] = 2\cos(\pm 2\pi/3)$

$= 2\cos(2\pi/3) = 2\cos[\pi - \pi/3] = 2[-\cos(\pi/3)] = 2(-1/2) = \boxed{-1}$

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7. Prove that $(1+i)^{100} + (1-i)^{100} = -2^{51}$

$\rightarrow$ $z_1 = 1+i = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ $\Rightarrow x_1 = 1$, $y_1 = 1$ (Ist quadrant)

$r_1 = \sqrt{1^2+1^2} = \sqrt{2}$, $\theta_1 = \tan^{-1}\left[\frac{1}{1}\right] = \pi/4$

$z_1 = \sqrt{2}[\cos\pi/4 + i\sin\pi/4]$

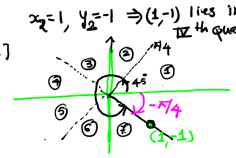
$z_1^{100} = (1+i)^{100} = (\sqrt{2})^{100}\left[\cos\left(\frac{100\pi}{4}\right) + i\sin\left(\frac{100\pi}{4}\right)\right]$ By De-moivre's Thm

$= 2^{50}[\cos(25\pi) + i\sin(25\pi)]$ —(I)

let $z_2 = 1-i = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ $x_2 = 1$, $y_2 = -1$ $\Rightarrow (1,-1)$ lies in IV th quadrant

$r_2 = \sqrt{1^2+(-1)^2} = \sqrt{2}$, $\theta_2 = \tan^{-1}\left[\frac{-1}{1}\right] = \tan^{-1}[-1]$

$\tan(\theta_2) = -1$



$\tan(2\pi - \pi/4) = -1$

$\tan(-\pi/4) = \tan[7\pi/4] = -1$

$\boxed{\theta_2 = -\pi/4}$

$z_2 = \sqrt{2}[\cos(-\pi/4) + i\sin(-\pi/4)] = \sqrt{2}[\cos(\pi/4) - i\sin(\pi/4)]$ —(II)

$z_2^{100} = (1-i)^{100} = (\sqrt{2})^{100}[\cos(\pi/4) - i\sin(\pi/4)]^{100} = 2^{50}[\cos(25\pi) - i\sin(25\pi)]$ $\rightarrow$ By De-moivre's Thm

Add (I) & (II)

$(1+i)^{100} + (1-i)^{100} = 2^{50}\{\cos(25\pi) + i\sin(25\pi) + \cos(25\pi) - i\sin(25\pi)\} = 2^{50}\cdot 2\cos(25\pi)$

$= 2^{51}(-1) = -\boxed{2^{51}}$

$(2i)^{50} + (-2i)^{50}$
$+ 2^{50}i^{50}$

$(1+3i)^{100} + (1-5i)^{100}$

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8. Prove that $\left(\frac{1+7i}{(2-i)^2}\right)^{4n} = (-4)^n$, where 'n' is positive integer [MU -Dec-04, 17]

$\rightarrow$ let $z = 1+7i = x+iy = \frac{(1+7i)}{} = \frac{(1+7i)}{} \times \frac{(3+4i)}{} = \frac{3+4i+21i-28}{} = \frac{-25+25i}{25}$

8. Prove that $\left(\frac{1+7i}{(2-i)^2}\right)^{4n} = (-4)^n$, where 'n' is positive integer [MU -Dec-04, 17]

$\rightarrow$ let $z = \frac{1+7i}{(2-i)^2} = x+iy = \frac{(1+7i)}{4-4i-1} = \frac{(1+7i)}{(3-4i)} \times \frac{(3+4i)}{(3+4i)} = \frac{3+4i+21i-28}{(9+16)} = \frac{-25+25i}{25}$

$z = -1+i \Rightarrow x=-1,\ y=1 \rightarrow$ IInd quadrant

$r = \sqrt{2},\quad \theta = \tan^{-1}\left[\frac{1}{-1}\right] = \tan^{-1}(-1) = 3\pi/4$

$z = \sqrt{2}\left[\cos 3\pi/4 + i\sin 3\pi/4\right] \Rightarrow (z)^{4n} = \left[\frac{1+7i}{(2-i)^2}\right]^{4n} = (\sqrt{2})^{4n}\left[\cos(3n\pi) + i\sin(3n\pi)\right]$

$z = 2^{2n}\left[\cos(3n\pi) + i\sin(3n\pi)\right] = (2^2)^n\left[(-1)^n + 0\right]$

$= 4^n(-1)^n = [4(-1)]^n = [-4]^n$

$\because$ $\sin(3n\pi) = 0$, $n = 0, 1, 2, 3 -$

$\cos(3n\pi) = (-1)^{3n}$, $n = 0, 1, 2, 3 -$ $= [(-1)^3]^n = (-1)^n$

$\cos[n\pi] = (-1)^n$

9. If $x + iy = \sqrt[3]{a+ib}$, then $\frac{a}{x} + \frac{b}{y} = ?$

$\rightarrow$ $(x+iy)^3 = a+ib$

$x^3 + 3x^2(iy) + 3(x)(iy)^2 + (iy)^3 = a+ib$

$[x^3 - 3xy^2] + i[3x^2y - y^3] = a+ib$

$\Rightarrow$ $x^3 - 3xy^2 = a$ & $3x^2y - y^3 = b$

divide by x divide by y

$x^2 - 3y^2 = \frac{a}{x}$ — (I) $3x^2 - y^2 = \frac{b}{y}$ — (II)

$\frac{a}{x} + \frac{b}{y} = (x^2 - 3y^2) + (3x^2 - y^2) = 4(x^2 - y^2)$

Examples based on expansion of $\cos^n(\theta), \sin^n(\theta)$ in terms of sine or cosine of multiple of θ

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Note: If $z = \cos\theta + i\sin\theta$, Then

1. $z + \frac{1}{z} = 2\cos(\theta) = (e^{i\theta} + e^{-i\theta})$
2. $z - \frac{1}{z} = 2i\sin(\theta) = (e^{i\theta} - e^{-i\theta})$

$r = 1$

Note: If $z = \cos\theta + i\sin\theta$, Then

1. $z^n + \frac{1}{z^n} = 2\cos(n\theta) = e^{in\theta} + e^{-in\theta}$
2. $z^n - \frac{1}{z^n} = 2i\sin(n\theta) = e^{in\theta} - e^{-in\theta}$

Pascal's Triangle:-

- $(a+b)^0 \rightarrow$ 1
- $(a+b)^1 \rightarrow$ 1 1
- $(a+b)^2 \rightarrow$ 1 2 1
- $(a+b)^3 \rightarrow$ 1 3 3 1
- $(a+b)^4 \rightarrow$ 1 4 6 4 1
- $(a+b)^5 \rightarrow$ 1 5 10 10 5 1
- $(a+b)^6 \rightarrow$ 1 6 15 20 15 6 1
- $(a+b)^7 \rightarrow$ 1 7 21 35 35 21 7 1

Examples:

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1. Show that $\sin^5\theta = \frac{1}{16}(\sin5\theta - 5\sin3\theta + 10\sin\theta)$ [MU -Dec-06, May-18] HW
2. Show that $\sin^7\theta = -\frac{1}{2^6}(\sin7\theta - 7\sin5\theta + 21\sin3\theta - 35\sin\theta)$ [MU -May-14]
3. Show that $\cos^7\theta = \frac{1}{2^6}(\cos7\theta + 7\cos5\theta + 21\cos3\theta + 35\cos\theta)$ [MU -Dec-12,14] HW
4. If $\sin^4\theta\cos^3\theta = a\cos\theta + b\cos3\theta + c\cos5\theta + d\cos7\theta$, Then find a,b,c, d. [MU -2000,02,09,17]
5. Prove that $\cos^6\theta - \sin^6\theta = \frac{1}{16}(\cos6\theta + 15\cos2\theta)$ [MU -Dec-07,16]
6. ✓ Prove that $\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos4\theta + 5)$ [MU -01,11,16]
7. ✓ If $\cos^6\theta + \sin^6\theta = (\alpha\cos4\theta + \beta)$, Then Prove that $\alpha + \beta = 1$ [MU -Dec-15, May-16]
8. Prove that $\cos^8\theta + \sin^8\theta = \frac{1}{64}(\cos8\theta + 28\cos4\theta + 35)$ [MU -01,11,16]
9. If $\sin^3\theta\cos^5\theta = -\frac{1}{2^7}(\sin8\theta + 2\sin6\theta - 2\sin4\theta - 6\sin2\theta)$

2. Show that $\sin^7\theta = -\frac{1}{2^6}(\sin7\theta - 7\sin5\theta + 21\sin3\theta - 35\sin\theta)$

$\rightarrow$ Let $z = \cos\theta + i\sin\theta$

$\frac{1}{z} = \bar{z} = \cos\theta - i\sin\theta$ $\Bigg\} \Rightarrow z - \frac{1}{z} = 2i\sin\theta$

$\boxed{z + \frac{1}{z} = 2\cos\theta}$; $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$

$(A+B)^n = {}^nC_0 A^n B^0 + {}^nC_1 A^{n-1} B^1 + \cdots$

$= {}^7C_0 \quad {}^7C_1 \quad {}^7C_2 \quad {}^7C_3$

$1 \quad 7 \quad \frac{7\times6}{2}, \quad \frac{7\times6\times5}{3\times2}$

$\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$

$$\sin^7\theta = LHS = \frac{1}{2^7 i^7}(z - \frac{1}{z})^7 = \frac{1}{2^7 (i^2)\cdot(i^2)(i^2)(i)}\left[z - \frac{1}{z}\right]^7$$

$$= \frac{1}{2^7(-i)}\left[z^7 - 7z^6\left(\frac{1}{z}\right) + 21z^5\left(\frac{1}{z^2}\right) - 35z^4\left(\frac{1}{z^3}\right) + 35z^3\left(\frac{1}{z^4}\right) - 21z^2\left(\frac{1}{z^5}\right) + 7z\left(\frac{1}{z^6}\right) - \frac{1}{z^7}\right]$$

$$= \frac{1}{2^7(-i)}\left[z^7 - 7z^5 + 21z^3 - 35z + 35\frac{1}{z} - 21\frac{1}{z^3} + 7\frac{1}{z^5} - \frac{1}{z^7}\right]$$

$$= \frac{-1}{2^7(i)}\left[\left(z^7 - \frac{1}{z^7}\right) - 7\left(z^5 - \frac{1}{z^5}\right) + 21\left(z^3 - \frac{1}{z^3}\right) - 35\left(z - \frac{1}{z}\right)\right]$$

But $z = \cos\theta + i\sin\theta \Rightarrow z^7 = \cos7\theta + i\sin7\theta$

$\frac{1}{z} = \cos\theta - i\sin\theta \Rightarrow \frac{1}{z^7} = \cos7\theta - i\sin7\theta$ $\Bigg\} z^7 - \frac{1}{z^7} = 2i\sin(7\theta)$

$z^n - \frac{1}{z^n} = 2i\sin(n\theta) \rightarrow$ Demoivre's Thm

$$\sin^7\theta = \frac{-1}{2^7 i}\left[(2i)\sin7\theta - 7(2i)\sin5\theta + 21(2i)\sin3\theta - 35(2i)\sin\theta\right]$$

$$= \frac{-1(2i)}{2^7(i)}\left[\sin7\theta - 7\sin5\theta + 21\sin3\theta - 35\sin\theta\right]$$

$$= -\frac{1}{2^6}\left[\sin7\theta - 7\sin5\theta + 21\sin3\theta - 35\sin\theta\right]$$

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4. If $\sin^4\theta\cos^3\theta = a\cos\theta + b\cos3\theta + c\cos5\theta + d\cos7\theta$, Then find a,b,c, d. [MU -2000,02,09,17]

$\rightarrow$ let $z = \cos\theta + i\sin\theta$ $\qquad \frac{1}{z} = \cos\theta - i\sin\theta$

$\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$ and $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$

$$LHS = \sin^4\theta\cos^3\theta = \frac{1}{2^4 i^4}(z - \frac{1}{z})^4 \cdot \frac{1}{2^3}(z + \frac{1}{z})^3 = \frac{1}{2^7}\left[z - \frac{1}{z}\right]\left[(z - \frac{1}{z})(z + \frac{1}{z})\right]^3$$

$$= \frac{1}{2^7}(z - \frac{1}{z})\left[z^2 - \frac{1}{z^2}\right]^3 = \frac{1}{2^7}(z - \frac{1}{z})\left[(z^2)^3 - 3(z^2)^2\left(\frac{1}{z^2}\right) + 3(z^2)^1\left(\frac{1}{z^2}\right)^2 - \left(\frac{1}{z^2}\right)^3\right]$$

$$LHS = \sin^4\theta\cos^3\theta = \frac{1}{2^4 i^4}\left(z-\frac{1}{z}\right)^4 \cdot \frac{1}{2^3}\left(z+\frac{1}{z}\right)^3 = \frac{1}{2^7}\left[\dots\right]$$

$$= \frac{1}{2^7}\left(z-\frac{1}{z}\right)\left[z^2-\frac{1}{z^2}\right]^3 = \frac{1}{2^7}\left(z-\frac{1}{z}\right)\left[(z^2)^3-3(z^2)^2\left(\frac{1}{z^2}\right)+3(z^2)^1\left(\frac{1}{z^2}\right)^2-\left(\frac{1}{z^2}\right)^3\right]$$

$$= \frac{1}{2^7}\left[\left(z^6-\frac{1}{z^6}\right)-3\left(z^2-\frac{1}{z^2}\right)\right]\left(z-\frac{1}{z}\right)$$

$$= \frac{1}{2^7}\left[z^7-\frac{1}{z^5}-z^5+\frac{1}{z^7}-3(z^3)+3\frac{1}{z}+3z-3\frac{1}{z^3}\right]$$

$$= \frac{1}{2^7}\left[\left(z^7+\frac{1}{z^7}\right)-\left(z^5+\frac{1}{z^5}\right)-3\left(z^3+\frac{1}{z^3}\right)+3\left(z+\frac{1}{z}\right)\right]$$

But $z^n+\frac{1}{z^n} = 2\cos(n\theta)$

$$LHS = \frac{1}{2^7}\left[2\cos7\theta - 2\cos5\theta - 3(2)\cos3\theta + 3(2)\cos(\theta)\right]$$

$$LHS = \frac{1}{2^6}\left[\cos7\theta - \cos5\theta - 3\cos3\theta + 3\cos\theta\right] = a\cos\theta + b\cos3\theta + c\cos5\theta + d\cos7\theta$$

$$a = \frac{3}{2^6}, \quad b = \frac{-3}{2^6}, \quad c = \frac{-1}{2^6}, \quad d = \frac{1}{2^6}$$

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7. If $\cos^6\theta + \sin^6\theta = (\alpha\cos4\theta + \beta)$, Then Prove that $\alpha + \beta = 1$

$\rightarrow$ Let $z = \cos\theta + i\sin\theta$ $\qquad \frac{1}{z} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) \qquad \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$

(I) $$\cos^6\theta = \frac{1}{2^6}\left(z+\frac{1}{z}\right)^6 = \frac{1}{2^6}\left[z^6+6z^5\frac{1}{z}+15z^4\frac{1}{z^2}+20z^3\frac{1}{z^3}+15z^2\frac{1}{z^4}+6z\frac{1}{z^5}+\frac{1}{z^6}\right]$$

$$= \frac{1}{2^6}\left[z^6+6z^4+15z^2+20+15\frac{1}{z^2}+6\frac{1}{z^4}+\frac{1}{z^6}\right]$$

$$= \frac{1}{2^6}\left[\left(z^6+\frac{1}{z^6}\right)+6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)+20\right]$$

$$\cos^6\theta = \frac{1}{2^6}\left[2\cos(6\theta)+6(2\cos4\theta)+15(2\cos2\theta)+20\right] \quad -\text{(I)}$$

(II) $$\sin^6\theta = \frac{1}{2^6 i^6}\left(z-\frac{1}{z}\right)^6 = \frac{1}{2^6(-1)}\left[z^6-6z^5\frac{1}{z}+15z^4\frac{1}{z^2}-20z^3\frac{1}{z^3}+15z^2\frac{1}{z^4}-6z\frac{1}{z^5}+\frac{1}{z^6}\right]$$

$$= \frac{-1}{2^6}\left[z^6-6z^4+15z^2-20+15\frac{1}{z^2}-6\frac{1}{z^4}+\frac{1}{z^6}\right]$$

$$= -\frac{1}{2^6}\left[\left(z^6+\frac{1}{z^6}\right)-6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)-20\right]$$

$$\sin^6\theta = -\frac{1}{2^6}\left[2\cos6\theta-6(2\cos4\theta)+15(2\cos2\theta)-20\right] \quad -\text{(II)}$$

Add (I) + (II)

$$\cos^6\theta+\sin^6\theta = \frac{1}{2^6}\left[\cancel{2\cos6\theta}+12\cos4\theta+\cancel{30\cos2\theta}+20-\cancel{2\cos6\theta}+12\cos4\theta-\cancel{30\cos2\theta}+20\right]$$

$$= \frac{1}{2^6}\left[24\cos4\theta+40\right] = \frac{3\times2\times2\times2}{2^6}\cos4\theta + \frac{5\times2\times2\times2}{2^6} = \frac{3}{8}\cos4\theta + \frac{5}{8} = \alpha\cos4\theta+\beta$$

$$\Rightarrow \alpha = 3/8, \quad \beta = 5/8 \quad \Rightarrow \alpha+\beta = 1$$

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Expansion of Cos(nθ), sin(nθ) in power of cos (θ), sin (θ)

Dr. Uday Kashid, PhD (Mathematics)

17 February 2022 13:28

z = cosθ + i sinθ

Note: By De Moivre's thm:. $(\cos\theta + i\sin\theta)^n = \boxed{\cos n\theta + i\sin n\theta}$

But by binomial expansion to LHS

$^nC_0\cos^n(\theta)(i\sin\theta)^0 + {}^nC_1\cos^{n-1}(\theta)(i\sin\theta)^1 + {}^nC_2\cos^{n-2}(\theta)(i\sin\theta)^2 + \cdots + {}^nC_n\cos^0(\theta)(i\sin\theta)^n = \cos n\theta + i\sin n\theta$

By equating Re & Im parts ...

- $\cos(n\theta) = {}^nC_0\cos^n(\theta) - {}^nC_2\cos^{n-2}(\theta)(\sin^2(\theta)) + {}^nC_4\cos^{n-4}(\theta)(\sin^4(\theta)) - \cdots$
- $\sin(n\theta) = {}^nC_1\cos^{n-1}(\theta)(\sin\theta) - {}^nC_3\cos^{n-3}(\theta)(\sin^3(\theta)) + {}^nC_5\cos^{n-5}(\theta)\left(\sin^5(\theta)\right) - \cdots$

Examples: Dr. Uday Kashid, PhD (Mathematics)

1. Prove that i) $\cos3\theta = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$, ii) $\sin3\theta = 3\sin(\theta)\cos^2(\theta) - \sin^3(\theta)$
2. HW Show that $\tan5\theta = \frac{5\tan\theta - 10\tan^3(\theta) + \tan^5(\theta)}{1 - 10\tan^2(\theta) + 5\tan^4(\theta)}$ [MU -Dec-03]
3. HW Show that $\tan7\theta = \frac{7\tan\theta - 35\tan^3(\theta) + 21\tan^5(\theta) - \tan^7(\theta)}{1 - 21\tan^2(\theta) + 35\tan^4(\theta) - 7\tan^6(\theta)}$ [MU -Dec-03,12]
4. HW Using De Moivre's thm, Express $\frac{\sin7\theta}{\sin\theta}$ in powers of $\sin\theta$ only. [MU -Dec-15]
5. If $\cos6\theta = a\cos^6(\theta) + b\cos^4(\theta)\sin^2(\theta) + c\cos^2(\theta)\sin^4(\theta) + d\sin^6(\theta)$, Then find a, b, c, d. [MU -April-21 online exam] ($^6C_0$, $-^6C_2$, $^6C_4$, $-^6C_6$)
6. HW If $\sin6\theta = a\cos^5(\theta)(\sin\theta) + b\cos^3(\theta)\sin^3(\theta) + c\cos(\theta)\sin^5(\theta)$, Then find a, b, c. [MU -95,05] ($^6C_1$, $-^6C_3$, $^6C_5$)
7. HW Using De Moivre's thm, Express $\frac{\sin6\theta}{\sin2\theta} = 16\cos^4(\theta) - 16\cos^2(\theta) + 3$ [MU -Dec-04,14]

$n=6$ $\sin6\theta = {}^6C_1\cos^5\theta\sin\theta - {}^6C_3\cos^3\theta\sin^3\theta + {}^6C_5\cos\theta\sin^5\theta$

$n=2$, $\sin2\theta = {}^2C_1\cos\theta\sin\theta = 2\cos\theta\sin\theta$

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1. $\cos3\theta = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$, ii) $\sin3\theta = 3\sin(\theta)\cos^2(\theta) - \sin^3(\theta)$

→ let $z = \cos\theta + i\sin\theta$ ⇒ $(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin3\theta$

$\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 = \boxed{\cos3\theta + i\sin3\theta}$ (Real, Im)

$[\cos^3\theta - 3\cos\theta\sin^2\theta] + i[3\cos^2\theta\sin\theta - \sin^3\theta] = \cos3\theta + i\sin3\theta$

By comparing Real & Im parts

$\boxed{\cos3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta}$, $\boxed{\sin3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta}$

$i^3 = -i$

or

If $z = \cos\theta + i\sin\theta$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$\cos n\theta = {}^nC_0\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \cdots$

$\cos3\theta = {}^3C_0\cos^3\theta - {}^3C_2\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \cdots$

$\sin3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

3. Show that $\tan7\theta = \frac{7\tan(\theta) - 35\tan^3(\theta) + 21\tan^5(\theta) - \tan^7(\theta)}{1 - 21\tan^2(\theta) + 35\tan^4(\theta) - 7\tan^6(\theta)}$ [MU -Dec-03,12] Dr. Uday Kashid, PhD (Mathematics)

→ $\tan7\theta = \frac{\sin7\theta}{\cos7\theta}$ — (I)

$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta + -$

$n=7$

$\sin7\theta = {}^7C_1\cos^6\theta\sin\theta - {}^7C_3\cos^4\theta\sin^3\theta + {}^7C_5\cos^2\theta\sin^5\theta - {}^7C_7\cos^0\theta\sin^7\theta$.

$^7C_1 = 7$, $^7C_3 = \frac{7\times6\times5}{3\times2} = 35$, $^7C_5 = \frac{\cancel{7\times6\times5}}{\cancel{5\times4}\times2\times1} = 21$

$\sin7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$ — (II)

$\cos n\theta = {}^nC_0\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + -$

$\cos7\theta = {}^7C_0\cos^7\theta - {}^7C_2\cos^5\theta\sin^2\theta + {}^7C_4\cos^3\theta\sin^4\theta - {}^7C_6\cos\theta\sin^6\theta$.

$^7C_0 = 1$, $^7C_2 = {}^7C_5 = 21$, $^7C_4 = {}^7C_3 = 35$, $^7C_6 = 7$

$\cos7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$ — (III)

put eq (II) & (III) in (I)

$$\tan(7\theta) = \frac{7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta}{\boxed{\cos^7\theta} - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta}$$

divide by $\cos^7\theta$ to N & D.

$$\tan7\theta = \frac{\frac{7\cos^6\theta\sin\theta}{\cos^7\theta} - \frac{35\cos^4\theta\sin^3\theta}{\cos^7\theta} + \frac{21\cos^2\theta\sin^5\theta}{\cos^7\theta} - \frac{\sin^7\theta}{\cos^7\theta}}{\frac{\cos^7\theta}{\cos^7\theta} - 21\frac{\cos^5\theta\sin^2\theta}{\cos^7\theta} + 35\frac{\cos^3\theta\sin^4\theta}{\cos^7\theta} - 7\frac{\cos\theta\sin^6\theta}{\cos^7\theta}}$$

$$\boxed{= \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}}$$

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Root of an equation

10 February 2022 11:56

Dr. Uday Kashid, PhD (Mathematics)

Note: De Moivre's thm can be used to find n^{th} root of a complex numbers.

- Let the equation be $z^n = \cos\theta + i\sin\theta$

$$z = (\cos\theta + i\sin\theta)^{\frac{1}{n}} = [\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)]^{\frac{1}{n}}$$

Where $\cos(2k\pi+\theta) = \cos\theta$ and $\sin(2k\pi+\theta) = \sin\theta$, for $k = 0,1,2,3,\ldots n\ldots\ldots$

By De Moivre's thm:-

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$$z = \cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right) \quad where\ k = 0,1,2,3,\ldots(n-1)$$

- *Note: Gate*

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$2k\pi$

1. $z^n = 1 = \cos(0) + i\sin(0) = \cos(2k\pi+0) + i\sin(2k\pi+0)$, for $k = 0,1,2,3,\ldots n\ldots\ldots$
2. $z^n = -1 = \cos(\pi) + i\sin(\pi) = \cos(2k\pi+\pi) + i\sin(2k\pi+\pi)$, for $k = 0,1,2,3,\ldots n\ldots\ldots$
3. $z^n = i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = \cos\left(2k\pi+\frac{\pi}{2}\right) + i\sin\left(2k\pi+\frac{\pi}{2}\right)$, for $k = 0,1,2,3,\ldots n\ldots\ldots$
4. $z^n = -i = \cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right) = \cos\left(2k\pi+\frac{\pi}{2}\right) - i\sin\left(2k\pi+\frac{\pi}{2}\right)$, for $k = 0,1,2,3,\ldots n\ldots\ldots$
5. $z^n = \beta = \beta(1) = \beta\{\cos(0) + i\sin(0)\} = \beta\{\cos(2k\pi+0) + i\sin(2k\pi+0)\}$
6. $z^n = -\beta = \beta(-1) = \beta\{\cos(\pi) + i\sin(\pi)\} = \beta\{\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)\}$

Examples:

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1. Solve the equation $x^7 + x^4 + i(x^3+1) = 0$ [MU -Dec-15]
2. Solve the equation $x^6 - i = 0$ [MU -Dec-13]
3. Solve the equation $x^6 + 1 = 0$ [MU -Dec-14]
4. Show that all the roots of $(x+1)^6 + (x-1)^6 = 0$ are given by $-i\cot\left[\frac{(2k+1)\pi}{12}\right]$, $k = 0,1,2,3,4,5$ [MU -Dec-14,17]
5. Show that all the roots of $(x+1)^7 = (x-1)^7$ are given by $\pm i\cot\left[\frac{(k)\pi}{7}\right]$, $k = 1,2,3$ [MU -Dec-08]
6. Solve the equation $(x+1)^8 + (x)^8 = 0$
7. Solve the equation $x^{10} + 11x^5 + 10 = 0$ [MU -05]
8. Solve the equation $x^7 + 64x^4 + 64(x^3+64) = 0$
9. Solve the equation $z^3 = (z+1)^3$, then show that $z = \frac{-1}{2} + \frac{i}{2}\cot\left[\frac{(k)\pi}{3}\right]$
10. Find the continued product of the roots of $x^4 = 1 + i$ [MU -April-21 online exam]

$\cos n\theta / \sin n\theta$

$\cos^n\theta / \sin^n\theta \to$ Conne

1. Solve the equation $x^7 + x^4 + i(x^3+1) = 0$ [MU -Dec-15]

$\to$ $x^4(x^3+1) + i(x^3+1) = 0$

$(x^3+1)(x^4+i) = 0$

$\Rightarrow \boxed{x^3+1=0}$ or $\boxed{x^4+i=0}$

(I) For $x^3+1=0 \Rightarrow x^3 = -1 = \cos(\pi) + i\sin(\pi) = \cos(2k\pi+\pi) + i\sin(2k\pi+\pi)$, $k=0,1,2,3,\ldots$

$\to$ $x = [\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)]^{1/3}$, $\boxed{k=0,1,2}$ $\boxed{3,4,5}$ $\boxed{6,7,8}$

$$\boxed{x = \cos\left(\frac{2k\pi+\pi}{3}\right) + i\sin\left(\frac{2k\pi+\pi}{3}\right)}$$

For $k=0$, $x_1 = \cos(\pi/3) + i\sin(\pi/3) = \boxed{\frac{1}{2} + i\frac{\sqrt{3}}{2}}$

$k=1$, $x_2 = \cos(\pi) + i\sin(\pi) = -1$

$k=2$, $x_3 = \cos(5\pi/3) + i\sin(5\pi/3) = \cos(2\pi - \pi/3) + i\sin(2\pi - \pi/3) = \cos(\pi/3) - i\sin(\pi/3)$

$x_3 = \boxed{\frac{1}{2} - i\frac{\sqrt{3}}{2}} = \boxed{\bar{x}_1}$

(II) For $x^4 + i = 0 \Rightarrow x^4 = -i = \cos(\pi/2) - i\sin(\pi/2) = \cos(2k\pi+\pi/2) - i\sin(2k\pi+\pi/2)$, $k=0,1,2,\ldots$

$x = \cos\left(2k\pi+\frac{\pi}{2}\right) - i\sin\left(2k\pi+\pi/2\right)^{\frac{1}{4}} \Rightarrow \boxed{k=0,1,2,3}$

$$x = \left\{\cos\left[\frac{4k\pi+\pi}{2}\right] - i\sin\left[\frac{4k\pi+\pi}{2}\right]\right\}^{\frac{1}{4}} = \cos\left[\frac{4k\pi+\pi}{8}\right] - i\sin\left[\frac{4k\pi+\pi}{8}\right], \boxed{k=0,1,2,3}$$

For $k=0$, $x_4 = \cos(\pi/8) - i\sin(\pi/8)$

$k=1$, $x_5 = \cos(5\pi/8) - i\sin(5\pi/8)$

$k=2$, $x_6 = \cos(9\pi/8) - i\sin(9\pi/8)$

$k=3$, $x_7 = \cos(13\pi/8) - i\sin(13\pi/8)$

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4. Show that all the roots of $(x+1)^6 + (x-1)^6 = 0$ are given by $-i\cot\left[\frac{(2k+1)\pi}{12}\right]$, $k = 0,1,2,3,4,5$

$\to$ $(x+1)^6 = -(x-1)^6$

$x^n = 1, -1, i, -i$

$\frac{(x+1)^6}{(x-1)^6} = -1 =$

$\left[\frac{x+1}{x-1}\right]^6 = -1 = \cos(2k\pi+\pi) + i\sin(2k\pi+\pi)$, $k=0,1,2,3,\ldots$

$\frac{x+1}{x-1} = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{\frac{1}{6}}$, $\boxed{k=0,1,2,3,4,5.}$

$\frac{x+1}{x-1} = \boxed{\cos(2k+1)\frac{\pi}{6} + i\sin(2k+1)\frac{\pi}{6}}$, $k=0,1,2,3,4,5.$

Let $(2k+1)\frac{\pi}{6} = \theta$ (say)

$\frac{x+1}{x-1} = \cos\theta + i\sin\theta = \frac{e^{i\theta}}{1}$

Note this step

By compodendo & dividendo Method $\left[\frac{N+D}{N-D}\right]$

$\frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{e^{i\theta}+1}{e^{i\theta}-1}$ $\Rightarrow \frac{2x}{2} = \frac{e^{i\theta}+1}{e^{i\theta}-1}$ $\Rightarrow x = \left[\frac{e^{i\theta}+1}{e^{i\theta}-1}\right] \times \frac{e^{-i\theta/2}}{e^{-i\theta/2}} = \frac{e^{i\theta/2} + e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}}$

$$= \frac{\cancel{2}\cos(\theta/2)}{\cancel{2}i\sin(\theta/2)} = (-i)\cot(\theta/2)$$

By componendo & dividendo $[N-D]$

$$\frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{e^{i\theta}+1}{e^{i\theta}-1} \Rightarrow \frac{2x}{2} = \frac{e^{i\theta}+1}{e^{i\theta}-1} \Rightarrow x = \left[\frac{e^{i\theta}+1}{e^{i\theta}-1}\right] \times \frac{e^{-i\theta/2}}{e^{-i\theta/2}} = \frac{e^{i\theta/2}+e^{-i\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}}$$

But $z = \cos\theta + i\sin\theta = e^{i\theta}$

$\frac{1}{z} = \bar{z} = \cos\theta - i\sin\theta = e^{-i\theta}$

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$$\boxed{\begin{aligned} e^{i\theta}+e^{-i\theta} &= 2\cos\theta \\ e^{i\theta}-e^{-i\theta} &= 2i\sin\theta \end{aligned}}$$

$$\Rightarrow \boxed{\frac{1}{i} = -i}$$

$$x = -i\cot(\theta/2) = -i\cot\left[\frac{(2k+1)\pi}{12}\right], \quad k = 0, 1, 2, 3, 4, 5$$

5. Show that all the roots of $(x+1)^7 = (x-1)^7$ are given by $\pm i\cot\left[\frac{(k)\pi}{7}\right]$, $k = 1, 2, 3$

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$\rightarrow \left[\frac{x+1}{x-1}\right]^7 = 1 = \cos(0) + i\sin 0 = \cos(2k\pi+0) + i\sin(2k\pi+0)$, $k = 0, 1, 2, 3, \ldots$

$\frac{x+1}{x-1} = [\cos(2k\pi) + i\sin(2k\pi)]^{\frac{1}{7}}$, $k = 0, 1, 2, 3, 4, 5, 6$.

$\frac{x+1}{x-1} = \cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right)$, $k = 0, 1, 2, 3, \ldots 6$.

For $k=0$: $\frac{x+1}{x-1} = 1 \Rightarrow x+1 = x-1 \Rightarrow (1 = -1) \rightarrow$ Never possible

$\boxed{k=0 \text{ is discard}}$.

$\frac{x+1}{x-1} = \cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right)$, $k = 1, 2, 3, 4, 5, 6$.

$\frac{x+1}{x-1} = \frac{e^{i\left(\frac{2k\pi}{7}\right)}}{1}$

By componendo & dividendo $\left[\frac{N+D}{N-D}\right]$

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$$\frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{\left[e^{i(2k\pi/7)}+1\right]}{\left[e^{i(2k\pi/7)}-1\right]}$$

$$\frac{2x}{2} = \frac{\left[e^{i(2k\pi/7)}+1\right]}{\left[e^{i(2k\pi/7)}-1\right]} \times \frac{e^{-i(k\pi/7)}}{e^{-i(k\pi/7)}} = \frac{e^{i(k\pi/7)} + e^{-i(k\pi/7)}}{e^{i(k\pi/7)} - e^{-i(k\pi/7)}}$$

$\frac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta$

$\frac{e^{i\theta}-e^{-i\theta}}{2i} = \sin\theta$

$$x = \frac{2\cos(k\pi/7)}{2i\sin(k\pi/7)} = -i\cot(k\pi/7), \quad k = 1, 2, 3, 4, 5, 6$$

For $k=1$, $x_1 = -i\cot(\pi/7)$

$k=2$, $x_2 = -i\cot(2\pi/7)$

$k=3$ $x_3 = -i\cot(3\pi/7)$

$k=4$, $x_4 = -i\cot(4\pi/7) = -i\cot\left(\pi - \frac{3\pi}{7}\right) =$

$x_4 = i\cot(3\pi/7) = \boxed{\overline{x_3}}$

$k=5$, $x_5 = -i\cot(5\pi/7) = -i\cot(\pi - 2\pi/7) = i\cot(2\pi/7) = \overline{x_2}$

$k=6$, $x_6 = -i\cot(6\pi/7) = -i\cot(\pi - \pi/7) = i\cot(\pi/7) = \overline{x_1}$

Hence Roots are $\boxed{x = \pm i\cot(k\pi/7), \ k = 1, 2, 3}$

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9. Solve the equation $z^3 = (z+1)^3$, then show that $z = \frac{-1}{2} + \frac{i}{2}\cot\left[\frac{(k)\pi}{3}\right]$

$\rightarrow \left(\frac{z}{z+1}\right)^3 = 1 = \cos(2k\pi+0) + i\sin(2k\pi+0)$ $k = 0, 1, 2, \ldots$

$\frac{z}{z+1} = \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)$, $k = 0, 1, 2$

For $k=0$, $\frac{z}{z+1} = 1 \Rightarrow z = z+1 \Rightarrow \boxed{0 = 1} \rightarrow$ Never possible

$\boxed{k=0 \text{ is not possible}}$.

$\frac{z}{z+1} = \cos\theta + i\sin\theta$, $k = 1, 2$ and $\theta = \left(\frac{2k\pi}{3}\right)$

$\frac{z}{z+1} = \frac{e^{i\theta}}{1}$

By dividendo $\left[\frac{N}{D-N}\right]$

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$$\frac{z}{(z+1)-z} = \left[\frac{e^{i\theta}}{1-e^{i\theta}}\right] \times \frac{e^{-i\theta/2}}{e^{-i\theta/2}}$$

$$z = \frac{e^{+i\theta/2}}{e^{-i\theta/2} - e^{i\theta/2}} = -\frac{e^{i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} = \frac{-e^{i\theta/2}}{2i\sin(\theta/2)} = \frac{-1[\cos\theta/2 + i\sin\theta/2]}{2i\sin\theta/2}$$

$$= \frac{(i)}{2}[\cot(\theta/2) + i] = \left[\frac{i\cot(k\pi/3)}{2} - \frac{1}{2}\right] \qquad \theta = (2k\pi/3)$$

$z^3 = (z+1)^3$

$\cancel{z^3} = \cancel{z^3} + 3z^2 + 3z + 1$

$\boxed{3z^2 + 3z + 1 = 0}$

8. Solve the equation $x^7 + 64x^4 + 64(x^3 + 64) = 0$

$\rightarrow x^4(x^3+64) + 64(x^3+64) = 0 \Rightarrow (x^4+64)(x^3+64) = 0$

$\Rightarrow \boxed{x^4 = -64}$ or $\boxed{x^3 = -64}$

(I) For $x^4 = -64$

(II) For $x^3 = -64$

$x^4 = 64(-1) = 64[\cos(2k+1)\pi + i\sin(2k+1)\pi]$ $k = 0, 1, 2, 3, 4, \ldots$

$x = (64)^{1/4}\left[\cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}\right]$, $k = 0, 1, 2, 3$.

For $k=0$, $x_1 = \sqrt{8}[\cos\pi/4 + i\sin\pi/4] = \sqrt{8}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$

$k=1$, $x_2 = \sqrt{8}[\cos 3\pi/4 + i\sin 3\pi/4] = \sqrt{8}[\cos(\pi-\pi/4) + i\sin(\pi-\pi/4)] = \sqrt{8}\left[-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$

$k=2$ $x_3 = \sqrt{8}[\cos 7\pi/4 + i\sin 7\pi/4] = \sqrt{8}[\cos(2\pi-\pi/4) + i\sin(2\pi-\pi/4)] = \sqrt{8}\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = \overline{x_1}$

$$= \frac{\cancel{2}\cos(\theta/2)}{\cancel{2}i\sin(\theta/2)} = (-1)\cot(\theta/2)$$

For

$k=0,\quad x_1 = \sqrt{8}\left[\cos \pi/4 + i\sin \pi/4\right] = \sqrt{8}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$

$k=1,\quad x_2 = \sqrt{8}\left[\cos 3\pi/4 + i\sin 3\pi/4\right] = \sqrt{8}\left[\cos(\pi - \pi/4) + i\sin(\pi - \pi/4)\right] = \sqrt{8}\left[-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$

$k=3,\quad x_3 = \sqrt{8}\left[\cos 7\pi/4 + i\sin \frac{7\pi}{4}\right] = \sqrt{8}\left[\cos(2\pi - \pi/4) + i\sin(2\pi - \pi/4)\right] = \sqrt{8}\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = \overline{x_1}$

$k=2,\quad x_4 = \sqrt{8}\left(\cos 5\pi/4 + i\sin 5\pi/4\right] = \sqrt{8}\left[\cos(\pi + \pi/4) + i\sin(\pi + \pi/4)\right) = \sqrt{8}\left[-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \overline{x_2}$

(11) For $x^3 = -64$ ↓ find $\boxed{x_5,\ x_6,\ \&\ x_7}$

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Hyperbolic Functions

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Dr. Uday Kashid, PhD (Mathematics)

Hyperbolic Functions

Definition

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\tanh x = \frac{\sinh x}{\cosh x}$

$\operatorname{csch} x = \frac{1}{\sinh x}$

$\operatorname{sech} x = \frac{1}{\cosh x}$

$\coth x = \frac{\cosh x}{\sinh x}$

Hyperbolic Identities

$\sinh(-x) = -\sinh x$

$\cosh(-x) = \cosh x$

$\cosh^2 x - \sinh^2 x = 1$

$1 - \tanh^2 x = \operatorname{sech}^2 x$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

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Derivatives of Hyperbolic Functions

$\frac{d}{dx}(\sinh x) = \cosh x$

$\frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Derivatives of Inverse Hyperbolic Functions

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, x > 1$

$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, |x| < 1$

$\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}, x \neq 0$

$\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}, 0 < x < 1$

$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}, |x| > 1$

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(i) $\sinh 2x = 2\sinh x \cosh x = \frac{2\tanh x}{1-\tanh^2 x}$

(ii) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1$

$= 1 + 2\sinh^2 x = \frac{1+\tanh^2 x}{1-\tanh^2 x}$

(iii) $2\cosh^2 x = \cosh 2x + 1$

(iv) $2\sinh^2 x = \cosh 2x - 1$

(v) $\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$

(vi) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

(vii) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

(viii) $\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$

(6) (i) $\cosh x + \sinh x = e^x$

(ii) $\cosh x - \sinh x = e^{-x}$

(iii) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

$\cosh(ix) = \frac{1}{2}[e^{ix} + e^{-ix}] = \cos x$

$\sinh(ix) = \frac{1}{2}[e^{ix} - e^{-ix}] = i\sin x$

$\cosh(x+iy) = \cosh(x)\cos(y) + i\sinh(x)\sin(y)$

$\sinh(x+iy) = \sinh(x)\cos(y) + i\cosh(x)\sin(y)$

$\tanh(ix) = i\tan x$

$\cosh x = \cos(ix)$

$\sinh x = -i\sin(ix)$

$\tanh x = -i\tan(ix)$

$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$

$\sinh(2x) = \frac{2\tanh(x)}{1-\tanh^2 x}$

$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

[illegible]

$\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$

$\sin(3x) = 3\sin(x) - 4\sin^3 x$

$\sinh(3x) = 3\sinh(x) + 4\sinh^3 x$

$\rightarrow \sin(ix) = -\frac{1}{i}\sinh(x) = i\sinh(x)$

$\tan(ix) = -\frac{1}{i}\tanh(x) = i\tanh(x)$

$\sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy)$

$\sinh(x+iy) = \sinh(x)\cosh(iy) + \cosh(x)\sinh(iy)$

$= \sinh(x)\cos(y) + i\cosh(x)\sin(y)$

Examples:

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1. If $x = \sqrt{3}$, then find the value of $\tanh(\log x)$
2. If $\tanh(x) = \frac{2}{3}$, then find the value x and $\cosh(2x)$ [MU -Dec-14]
3. If $7\cosh(x) + 8\sinh(x) = 1$, then find the real value x [MU -Dec-16]
4. Prove that $\left(\frac{1+\tanh(x)}{1-\tanh(x)}\right)^3 = \cosh(6x) + \sinh(6x)$ [MU -Dec-09,13]
5. If $\log(\tan x) = y$, then prove that 1) $\cosh(ny) = \frac{1}{2}(\tan^n x + \cot^n x)$ 2) $\sinh(n+1)y + \sinh(n-1)y = 2\sinh(ny)\operatorname{cosec}(2x)$
6. If $u = \log\left(\tan\left\{\frac{\pi}{4} + \frac{\theta}{2}\right\}\right)$, then prove that 1) $\cosh(u) = \sec(\theta)$ 2) $\sinh(u) = \tan(\theta)$ 3) $\tanh(u) = \sin(\theta)$ 4) $\tanh\left(\frac{u}{2}\right) = \tan\left(\frac{\theta}{2}\right)$
7. If $\tanh\left(\frac{u}{2}\right) = \tan\left(\frac{x}{2}\right)$, then prove that $u = \log\left(\tan\left\{\frac{\pi}{4} + \frac{x}{2}\right\}\right)$ [MU -May-18]

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B$

$\cosh(x+iy) = \cosh(x)\cosh(iy) + \sinh(x)\sinh(iy)$

$= \cosh(x)\cos(y) + i\sinh(x)\sin(y)$

Note:- Circular Fu^n :- $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

Hyperbolic Fu^n :- $\boxed{\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}}$, $\boxed{\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}}$

$i = \sqrt{-1}$ → complex No

$\boxed{\tanh\theta = \frac{\sinh\theta}{\cosh\theta} = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}}$, $\boxed{\operatorname{sech}\theta = \frac{1}{\cosh\theta} = \frac{2}{e^{\theta} + e^{-\theta}}}$

$\boxed{\operatorname{cosech}\theta = \frac{1}{\sinh\theta} = \frac{2}{e^{\theta} - e^{-\theta}}}$, $\boxed{\coth\theta = \frac{1}{\tanh\theta} = \frac{e^{\theta} + e^{-\theta}}{e^{\theta} - e^{-\theta}}}$

(i) $\cos(ix) = \cosh(x)$ (ii) $\sin(ix) = i\sinh(x)$ (iii) $\tan(ix) = i\tanh(x)$

$\rightarrow \cosh(ix) = \cos(x)$ $\sinh(ix) = i\sin(x)$ $\tanh(ix) = i\tan(x)$

$\rightarrow$ (i) $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \Rightarrow \cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh(x)$

$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} = \frac{1}{2}(-i)\left[e^{-x} - e^{x}\right]$

$\sin(ix) = i\left[\frac{e^{x} - e^{-x}}{2}\right] = i\sinh(x)$

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(iv) $\cot(ix) = -i\coth(x)$ (v) $\operatorname{cosec}(ix) = \frac{1}{\sin(ix)} = \frac{1}{i\sinh(x)} = -i\operatorname{cosech}(x)$

(vi) $\sec(ix) = \frac{1}{\cos(ix)} = \frac{1}{\cosh x} = \operatorname{sech} x$

(ii) $\frac{d}{dx}[\cosh x] = \frac{d}{dx}\left[\frac{e^{x} + e^{-x}}{2}\right] = \frac{1}{2}\left[e^{x} - e^{-x}\right] = \sinh(x)$

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Hind :— $\sin(x)\sin(x) \rightarrow$ change the sign in hyperbolic fun

$\boxed{\cos^2 x + \sin^2 x = 1}$ $\cos^2(ix) + \sin^2(ix) = 1$

$\boxed{\cosh^2 x - \sinh^2 x = 1}$ $\cosh^2(x) - \sinh^2 x = 1$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\boxed{\cosh^2 x - \sinh^2 x = 1}$

$\cos^2 x + \sin^2 x = 1$

(ii) $1 + \tan^2 x = \sec^2 x \Rightarrow \boxed{1 - \tanh^2 x = \operatorname{sech}^2 x}$

(iii) $1 + \cot^2 x = \operatorname{cosec}^2 x$

$\boxed{1 - \coth^2 x = -\operatorname{cosech}^2 x}$

1) $\sin(3x) = 3\sin x - 4\sin^2 x \sin x$

$\boxed{\sinh(3x) = 3\sinh(x) + 4\sinh^3 x}$

2) $\cos 2x = \cos^2 x - \sin^2 x$

$\boxed{\cosh(2x) = \cosh^2 x + \sinh^2 x}$

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1. If $x = \sqrt{3}$, then find the value of $\tanh(\log x)$

$\rightarrow \tanh(y) = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$, put $y = \log x \Rightarrow \tanh[\log x] = \frac{e^{\log x} - e^{-\log x}}{e^{\log x} + e^{-\log x}}$

$\therefore \boxed{e^{\log x} = x}$

$\boxed{e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}}$

$\tanh(\log x) = \frac{x - \frac{1}{x}}{x + \frac{1}{x}} = \boxed{\frac{x^2 - 1}{x^2 + 1}}$ put $x = \sqrt{3}$

$= \frac{3-1}{3+1} = \frac{2}{4} = 0.5$

2. If $\tanh(x) = \frac{2}{3}$, then find the value x and $\cosh(2x)$ [MU -Dec-14]

$\rightarrow$ HW -

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3. If $7\cosh(x) + 8\sinh(x) = 1$, then find the real value x [MU -Dec-16]

$\rightarrow 7\left[\frac{e^{x} + e^{-x}}{2}\right] + 8\left[\frac{e^{x} - e^{-x}}{2}\right] = 1$

$7e^x+7e^{-x}+8e^x-8e^{-x}=2 \Rightarrow 15e^x-e^{-x}=2 \Rightarrow 15e^x-\frac{1}{e^x}=2$

$\Rightarrow 15(e^x)^2-1=2e^x \Rightarrow \boxed{15(e^x)^2-2e^x-1=0} \rightarrow$ quadratic eq^n

$a=15,\ b=-2,\ c=-1 \leftarrow ax^2+bx+c=0$

$e^x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{4-4(15)(-1)}}{2(15)}=\frac{2\pm 8}{30}=\frac{10}{30},\frac{-6}{30} \Rightarrow e^x=\frac{1}{3}$ and $e^x=-\frac{1}{5}$

$\Rightarrow \log e^x=\log(1/3)$ and $\log(e^x)=\log(-1/5) \rightarrow$ does not exist

$\Rightarrow x=\log(1/3)$; $\boxed{e^x=-1/5 \text{ is discard}}$

$\Rightarrow \boxed{x=-\log 3}$

4. Prove that $\left(\frac{1+\tanh(x)}{1-\tanh(x)}\right)^3=\cosh(6x)+\sinh(6x)$ [MU-Dec-09,13]

$\rightarrow$ LHS $=\left[\frac{1+\tanh x}{1-\tanh x}\right]^3=\left[\frac{1+\frac{e^x-e^{-x}}{e^x+e^{-x}}}{1-\frac{(e^x-e^{-x})}{(e^x+e^{-x})}}\right]^3=\left[\frac{e^x+e^{-x}+e^x-e^{-x}}{e^x+e^{-x}-e^x+e^{-x}}\right]^3=\left[\frac{2e^x}{2e^{-x}}\right]^3=\boxed{e^{6x}}$

RHS $=\cosh(6x)+\sinh(6x)=\left(\frac{e^{6x}+e^{-6x}}{2}\right)+\left(\frac{e^{6x}-e^{-6x}}{2}\right)=\boxed{e^{6x}}$

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LHS = RHS.

5. If $\log(\tan x)=y$ **, then prove that** 1) $\cosh(ny)=\frac{1}{2}(\tan^n x+\cot^n x)$

2) $\sinh(n+1)y+\sinh(n-1)y=2\sinh(ny)\operatorname{cosec}(2x)$

$\rightarrow \log(\tan x)=y$

$e^{\log(\tan x)}=e^y$

$\boxed{\tan x=e^y}$

$\boxed{\cot x=e^{-y}}$

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$\cosh(ny)=\frac{e^{ny}+e^{-ny}}{2}=\frac{(e^y)^n+(e^{-y})^n}{2}$

1) $\boxed{\cosh(ny)=\frac{1}{2}\left[\tan^n x+\cot^n x\right]}$

2) We know that $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

$\sinh(ny+y)=\sinh(ny)\cosh(y)+\cosh(ny)\sinh(y)$ — (I)

$\sinh(ny-y)=\sinh(ny)\cosh(y)-\cosh(ny)\sinh(y)$ — (II)

Add (I) & (II)

$\sinh(ny+y)+\sinh(ny-y)=2\sinh(ny)\cosh(y)=\cancel{2}\sinh(ny)\left[\frac{e^y+e^{-y}}{\cancel{2}}\right]=\sinh(ny)\left[\tan x+\cot x\right]=$

$=\sinh(ny)\left[\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right]=\sinh(ny)\left[\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}\right]=2\sinh(ny)\frac{1}{2\sin x\cos x}=2\sinh(ny)\frac{1}{\sin(2x)}=2\sinh(ny)\operatorname{cosec}(2x)$

6. If $u=\log\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)$**, then prove that** 1) $\cosh(u)=\sec(\theta)$ 2) $\sinh(u)=\tan(\theta)$ 3) $\tanh(u)=\sin(\theta)$ 4) $\tanh\left(\frac{u}{2}\right)=\tan\left(\frac{\theta}{2}\right)$

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$\rightarrow u=\log\left[\tan(\pi/4+\theta/2)\right]$

$e^u=\tan(\pi/4+\theta/2)=\frac{\tan\pi/4+\tan\theta/2}{1-\tan\pi/4\tan\theta/2}=\boxed{\frac{1+\tan\theta/2}{1-\tan\theta/2}}$ — (I)

$e^u=\frac{1+\frac{\sin\theta/2}{\cos\theta/2}}{1-\frac{\sin\theta/2}{\cos\theta/2}}=\frac{[\cos\theta/2+\sin\theta/2]}{[\cos\theta/2-\sin\theta/2]}\times\frac{[\cos\theta/2+\sin\theta/2]}{[\cos\theta/2+\sin\theta/2]}=\frac{1+2\sin\theta/2\cos\theta/2}{\cos^2\theta/2-\sin^2\theta/2}$

$e^u=\frac{1+\sin\theta}{\cos\theta}=\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}=\boxed{\sec\theta+\tan\theta}$ — (II)

$e^{-u}=\frac{1}{[\sec\theta+\tan\theta]}=\frac{[\sec\theta-\tan\theta]}{[\sec\theta+\tan\theta][\sec\theta-\tan\theta]}=\frac{\sec\theta-\tan\theta}{(\sec^2\theta-\tan^2\theta)}=\boxed{\sec\theta-\tan\theta}$ — (III)

(I) $\cosh u=\frac{e^u+e^{-u}}{2}=\frac{(\sec\theta+\tan\theta)+(\sec\theta-\tan\theta)}{2}=\boxed{\sec\theta}$

(II) $\sinh u=\frac{e^u-e^{-u}}{2}=\frac{(\sec\theta+\tan\theta)-(\sec\theta-\tan\theta)}{2}=\boxed{\tan\theta}$

(III) $\tanh(u)=\frac{\sinh u}{\cosh u}=\frac{\tan\theta}{\sec\theta}=\frac{\sin\theta}{\cos\theta}\times\cos\theta=\boxed{\sin\theta}$

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(IV) $\tanh(u/2)=\frac{e^{u/2}-e^{-u/2}}{e^{u/2}+e^{-u/2}}=\frac{e^{u/2}-\frac{1}{e^{u/2}}}{e^{u/2}+\frac{1}{e^{u/2}}}=\frac{e^u-1}{e^u+1}=\frac{\frac{1+\tan\theta/2}{1-\tan\theta/2}-1}{\frac{1+\tan\theta/2}{1-\tan\theta/2}+1}=\frac{(1+\tan\theta/2)-1+\tan\theta/2}{(1+\tan\theta/2)+1-\tan\theta/2}=\tan\theta/2$

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7. If $\tanh\left(\frac{u}{2}\right)=\tan\left(\frac{x}{2}\right)$**, then prove that** $u=\log\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)$ [MU-May-18]

Inverse Hyperbolic Functions

07 February 2022 18:39

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- **Note : (useful for GATE exam)**
  1) $\sinh^{-1}(x) = \log\left(x + \sqrt{x^2+1}\right)$
  2) $\cosh^{-1}(x) = \log\left(x + \sqrt{x^2-1}\right)$
  3) $\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Examples:

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1. Prove that $\tanh^{-1}(x) = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, [MU -Dec-01,02,06,09, May-11,13]
2. Prove that $\operatorname{sech}^{-1}(\sin\theta) = \log\left[\cot\left(\frac{\theta}{2}\right)\right]$, [MU- May-14]
3. Prove that $\cosh^{-1}\left(\sqrt{x^2+1}\right) = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$
4. Prove that $\cosh^{-1}\left(\sqrt{x^2+1}\right) = \sinh^{-1}(x)$ [MU-Dec-02,04, May-09]
5. Prove that $\sinh^{-1}(\tan\theta) = \log\left(\tan\left\{\frac{\pi}{4}+\frac{\theta}{2}\right\}\right)$
6. Separate into real and Im parts for $\cos^{-1}\left(\frac{3i}{4}\right)$ [MU -Dec-03, 11, May-17] $= x+iy$ $\quad \operatorname{cosec}\theta + i\,0 = \sin(x+iy)$
7. Separate into real and Im parts for $\sin^{-1}(\operatorname{cosec}\theta)$

- 1) $\sinh^{-1}(x) = \log\left(x + \sqrt{x^2+1}\right)$ Dr. Uday Kashid, PhD (Mathematics)

$\to$ Let $\sinh^{-1}(x) = y$

$\Rightarrow x = \sinh(y) = \frac{e^y - e^{-y}}{2}$

$\Rightarrow 2x = e^y - \frac{1}{e^y} = \frac{(e^y)^2 - 1}{e^y}$

$\Rightarrow \boxed{(e^y)^2 - 2x\,e^y - 1 = 0}$

$ax^2+bx+c=0$ $\quad a=1,\ b=-2x,\ c=-1$

$e^y = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2x \pm \sqrt{4x^2-4(1)(-1)}}{2(1)} = \frac{2x \pm 2\sqrt{x^2+1}}{2}$

$e^y = \left(x + \sqrt{x^2+1}\right)$ and $e^y = \left(x - \sqrt{x^2+1}\right) < 0 \Rightarrow \left[\sqrt{x^2} - \sqrt{x^2+1}\right] < 0$

$\Rightarrow y = \log\left[x + \sqrt{x^2+1}\right]$ and $e^y = \left(x - \sqrt{x^2+1}\right)$ is discarded.

$\boxed{\sinh^{-1}(x) = \log\left[x + \sqrt{x^2+1}\right]}$

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2) $\cosh^{-1}(x) = \log\left(x + \sqrt{x^2-1}\right)$ HW

let $\cosh^{-1}(x) = y$

$x = \cosh y = \frac{e^y + e^{-y}}{2}$

3. $\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

$\to$ let $\tanh^{-1}(x) = y \quad \Rightarrow x = \tanh(y) = \frac{e^y - e^{-y}}{e^y + e^{-y}}$

$\frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{x}{1}$

By dividendo & compodendo $\quad \frac{D+N}{D-N}$

$\frac{(e^y+e^{-y}) + (e^y - e^{-y})}{(e^y+e^{-y}) - (e^y - e^{-y})} = \frac{1+x}{1-x}$

$\frac{2e^y}{2e^{-y}} = \frac{1+x}{1-x} \quad \Rightarrow e^{2y} = \frac{1+x}{1-x} \quad \Rightarrow 2y = \log\left[\frac{1+x}{1-x}\right]$

$\Rightarrow y = \boxed{\tanh^{-1}(x) = \frac{1}{2}\log\left[\frac{1+x}{1-x}\right]}$

1. Prove that $\tanh^{-1}(x) = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, [MU -Dec-01,02,06,09, May-11,13] Dr. Uday Kashid, PhD (Mathematics)

$\to$ let $\tanh^{-1}(x) = y \quad \Rightarrow x = \tanh(y)$

$1 - x^2 = 1 - \tanh^2 y$

$1 - x^2 = \operatorname{sech}^2 y$

$\therefore$ $\boxed{\begin{array}{l} 1+\tan^2\theta = \sec^2\theta \\ 1-\tanh^2\theta = \operatorname{sech}^2\theta \end{array}}$

$\therefore \frac{x}{\sqrt{1-x^2}} = \frac{\tanh y}{\sqrt{\operatorname{sech}^2 y}} = \frac{\tanh y}{\operatorname{sech} y} = \frac{\sinh y}{\cosh y} \times \cosh y = \sinh y$

$\Rightarrow y = \boxed{\sinh^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right] = \tanh^{-1}(x)}$

(or)

$RHS = \sinh^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right] = \log\left[\left(\frac{x}{\sqrt{1-x^2}}\right) + \sqrt{\frac{x^2}{(1-x^2)} + 1}\right]$ $\qquad \sinh^{-1}(x) = \log\left[x + \sqrt{x^2+1}\right]$

$= \log\left[\frac{x}{\sqrt{1-x^2}} + \sqrt{\frac{x^2+1-x^2}{1-x^2}}\right] = \log\left[\frac{(1+x)}{\sqrt{1-x^2}}\right] = \log\left[\frac{(1+x)}{\sqrt{1-x}\,\sqrt{1+x}}\right]$

$= \log\left[\frac{\sqrt{1+x}}{\sqrt{1-x}}\right] = \boxed{\frac{1}{2}\log\left[\frac{1+x}{1-x}\right] = \tanh^{-1}(x)}$

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2. Prove that $\operatorname{sech}^{-1}(\sin\theta) = \log\left[\cot\left(\frac{\theta}{2}\right)\right]$, [MU- May-14]

$\to$ Let $\operatorname{sech}^{-1}(\sin\theta) = y \quad \Rightarrow \sin\theta = \operatorname{sech}(y)$

$\Rightarrow \frac{1}{\sin\theta} = \frac{1}{\operatorname{sech}(y)} \quad \Rightarrow \operatorname{cosec}\theta = \cosh(y)$

$\Rightarrow \boxed{y = \cosh^{-1}\left[\operatorname{cosec}\theta\right]}$

$\operatorname{sech}^{-1}[\sin\theta] = \cosh^{-1}[\operatorname{cosec}\theta]$ $\qquad \boxed{\cosh^{-1}[x] = \log\left[x + \sqrt{x^2-1}\right]}$

$= \log\left[\operatorname{cosec}\theta + \sqrt{\operatorname{cosec}^2\theta - 1}\right]$

$= \log\left[\operatorname{cosec}\theta + \cot\theta\right] = \log\left[\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right] = \log\left[\frac{1+\cos\theta}{\sin\theta}\right]$

$= \log\left[\frac{2\cos^2\theta/2}{2\sin\theta/2\,\cos\theta/2}\right] = \log\left[\cot(\theta/2)\right]$

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4. Prove that $\cosh^{-1}\left(\sqrt{x^2+1}\right) = \sinh^{-1}(x)$ [MU-Dec-02,04, May-09]

$\to \boxed{\cosh^{-1}[\theta] = \log\left[\theta + \sqrt{\theta^2-1}\right]}$, put $\theta = \sqrt{x^2+1}$

4. Prove that $\cosh^{-1}\left(\sqrt{x^2+1}\right) = \sinh^{-1}(x)$ [MU-Dec-02,04, May-09]

$\rightarrow$ $\boxed{\cosh^{-1}[\theta] = \log[\theta + \sqrt{\theta^2 - 1}]}$, put $\theta = \sqrt{x^2+1}$

$\cosh^{-1}[\sqrt{x^2+1}] = \log[\sqrt{x^2+1} + \sqrt{(x^2+1)-1}] = \log[\sqrt{x^2+1} + x] = \sinh^{-1}[x]$

3. Prove that $\cosh^{-1}\left(\sqrt{x^2+1}\right) = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$\rightarrow$ let $\cosh^{-1}[\sqrt{x^2+1}] = y$ $\quad \Rightarrow \sqrt{x^2+1} = \cosh(y)$

$\Rightarrow x^2+1 = \cosh^2(y)$

$\Rightarrow x^2 = \cosh^2(y) - 1$ $\qquad$ But $\cosh^2 y - \sinh^2 y = 1$

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$\Rightarrow \boxed{x = \sinh y}$

$\Rightarrow y = \sinh^{-1}[x]$

Now $\dfrac{x}{\sqrt{x^2+1}} = \dfrac{\sinh y}{\cosh y} = \tanh(y) \quad \Rightarrow \boxed{y = \tanh^{-1}\left[\dfrac{x}{\sqrt{x^2+1}}\right]}$

$$\boxed{\cosh^{-1}[\sqrt{x^2+1}] = y = \tanh^{-1}\left[\frac{x}{\sqrt{x^2+1}}\right]}$$

5. Prove that $\sinh^{-1}(\tan\theta) = \log\left(\tan\left\{\frac{\pi}{4} + \frac{\theta}{2}\right\}\right)$

$\rightarrow$ $\sinh^{-1}[\tan\theta] = \log[\tan\theta + \sqrt{\tan^2\theta + 1}] = \log[\tan\theta + \sec\theta] = \log\left[\dfrac{\sin\theta + 1}{\cos\theta}\right]$

Note this step

$= \log\left[\dfrac{1 + \cos(\pi/2 - \theta)}{\sin(\pi/2 - \theta)}\right] = \log\left[\dfrac{2\cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\cos\left(\pi/4 - \theta/2\right)}\right] = \log\left[\cot\left(\pi/4 - \theta/2\right)\right]$

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$= \log\left\{\tan\left[\frac{\pi}{2} - \left(\frac{\pi}{4} - \theta/2\right)\right]\right\} = \log\left[\tan\left(\frac{\pi}{4} + \theta/2\right)\right]$

6. Separate into real and Im parts for $\cos^{-1}\left(\frac{3i}{4}\right)$ [MU -Dec-03, 11, May-17]

$\rightarrow$ Let $\cos^{-1}\left[\frac{3i}{4}\right] = z = x + iy$

$\frac{3i}{4} = \cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy)$

$0 + \frac{3i}{4} = \cos x \cosh(y) - i \sin x \sinh(y)$

By comparing Real & Im parts

$0 = \cos x \cosh(y)$ $\qquad$ and $\qquad \frac{3}{4} = -\sin(x)\sinh y$

But $\cosh(y) = \dfrac{e^y + e^{-y}}{2} \neq 0$, $\forall y$

$\Rightarrow \cos(x) = 0$ $\qquad$ and $-\frac{3}{4} = \sin(\pi/2)\sinh(y)$

$\boxed{x = \pi/2}$

$\sinh(y) = -3/4$

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$y = \sinh^{-1}[-3/4] = \log\left[-\frac{3}{4} + \sqrt{(-3/4)^2 + 1}\right]$

$y = \log\left[-\frac{3}{4} + \sqrt{\frac{9}{16} + 1}\right] = \log\left[-\frac{3}{4} + \sqrt{25/16}\right]$

$y = \log\left[-\frac{3}{4} + \frac{5}{4}\right] = \log[1/2] = -\log(2)$

$\cos^{-1}\left[\frac{3i}{4}\right] = \boxed{\frac{\pi}{2} - i\log 2}$

2. $\cosh^{-1}(x) = \log\left(x + \sqrt{x^2-1}\right)$

$\rightarrow$ let $\cosh^{-1}(x) = y$ $\quad \Rightarrow x = \cosh(y) = \dfrac{e^y + e^{-y}}{2} = \dfrac{(e^y)^2 + 1}{2e^y}$

$\Rightarrow (e^y)^2 + 1 = 2xe^y$

$\boxed{(e^y)^2 - 2xe^y + 1 = 0}$ $\qquad ax^2 + bx + c = 0$, $a = 1$, $b = -2x$, $c = 1$

$e^y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2(a)} = \dfrac{2x \pm \sqrt{4x^2 - 4(1)(1)}}{2(1)} = \dfrac{2x \pm 2\sqrt{x^2-1}}{2}$

$e^y = (x + \sqrt{x^2-1}) > 0$ $\qquad$ and $\qquad e^y = (x - \sqrt{x^2-1}) > 0$

Take log to B.S. $\qquad\qquad = (\sqrt{x^2} - \sqrt{x^2-1}) > 0$

$y = \log[x + \sqrt{x^2-1}]$ — (1) $\qquad$ and $\qquad y = \log[x - \sqrt{x^2-1}]$

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$y = \log\left[\dfrac{(x - \sqrt{x^2-1})(x + \sqrt{x^2-1})}{(x + \sqrt{x^2-1})}\right]$

$y = \log\left[\dfrac{x^2 - (x^2-1)}{(x + \sqrt{x^2-1})}\right] = \log\left[\dfrac{1}{(x + \sqrt{x^2-1})}\right]$

$y = \log[x + \sqrt{x^2-1}]$ $\qquad$ and $\qquad y = -\log[(x + \sqrt{x^2-1})]$

$\Rightarrow y = \pm\log[(x + \sqrt{x^2-1})]$

$\Rightarrow \cosh^{-1}[x] = \pm\log[x + \sqrt{x^2-1}]$

$x = \cosh[\pm\log(x + \sqrt{x^2-1})]$ $\qquad$ $\sinh(-x) = -\sinh(x)$

$= \cosh[\log(x + \sqrt{x^2-1})]$ $\qquad$ $\cosh(-x) = \cosh(x)$

$$\boxed{\cosh^{-1}(x) = \log[x + \sqrt{x^2-1}]}$$

$$= \cosh\{\log(x+\sqrt{x^2-1})\}$$

$$\boxed{\cosh^{-1}(x) = \log\{x+\sqrt{x^2-1}\}}$$

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Logarithm form of complex numbers

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$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

Take log to both sides

- $\log z = \log(re^{i\theta}) = \log(r) + \log(e^{i\theta})$
- $\log z = \log(r) + i\,\theta$
- $\log(x+iy) = \log\left(\sqrt{x^2+y^2}\right) + i\tan^{-1}\left(\frac{y}{x}\right)$ $\rightarrow$ *Principal value of* $\log(x+iy)$
- *General value of* $\log(x+iy)$: –
- $\text{Log}(x+iy) = \log\left(\sqrt{x^2+y^2}\right) + i\,(2k\pi + \theta)$, *for* $k = 0,1,2,3,\ldots.n\ldots\ldots$

small 'i' (I) $z = x+iy = re^{i\theta}$

$\log[z] = \log(x+iy) = \log(re^{i\theta})$

$= \log r + \log e^{i\theta}$

$\log(z) = \log r + i\theta$

$\boxed{\log(x+iy) = \log(\sqrt{x^2+y^2}) + i\tan^{-1}(y/x)}$ $\rightarrow$ principal value of $\log(x+iy)$

(II) General value

$z = x+iy = r[\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)]$

capital 'L' $z = re^{i(2k\pi+\theta)}$, $k=0,1,2,3\ldots$

$\text{Log}(x+iy) = \log[re^{i(2k\pi+\theta)}]$

$= \log r + i(2k\pi+\theta) = [\log r + i\theta] + i2k\pi$

$= \log(x+iy) + i2k\pi$

Examples:

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1. **Prove that** $\log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right] = 2i\tan^{-1}(\cot x\tanh y)$ **[MU -Dec-05, 17]**
2. **Show that** $\log(e^{i\alpha} + e^{i\beta}) = \log\left[2\cos\left(\frac{\alpha-\beta}{2}\right)\right] + i\left(\frac{\alpha+\beta}{2}\right)$ **[MU -May-16]**
3. **If** $\tan[\log(x+iy)] = a + ib$, *where* $a^2 + b^2 \neq 1$, **then prove that** $\tan[\log(x^2+y^2)] = \frac{2a}{1-(a^2+b^2)}$
4. **If** $\sin^{-1}(x+iy) = \log(a+ib) = \alpha + i\beta$, **then prove that** 1) $\frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\alpha} = 1$ 2) $a^2 + b^2 = e^{2\alpha}$
5. **Prove that** $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$ **[MU -Dec-2000,02, 14]**
6. **If** $\log[\,Sin(x+iy)] = a + ib$, **Then 1)** $Cosh(2y) - Cos(2x) = 2e^{2a}$, **2)** $\tan(b) = \cot(x)\tanh(y)$ **[MU -April-21 online exam]**
7. **Considering only principal values, separate into real and imaginary parts** $i^{\log(1+i)}$
8. **If** $i^{i^{i^{\cdots\infty}}} = \alpha + i\beta$, **then show that** $\alpha^2 + \beta^2 = e^{-(4k+1)\pi\beta}$, $k = 0,1,2,3\ldots.$ **[MU -May-15]**
9. **Find the value of** $\log_{(-3)}(-2)$

1. **Prove that** $\log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right] = 2i\tan^{-1}(\cot x\tanh y)$ **[MU -Dec-05, 17]**

$\rightarrow \log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right] = \log[\sin(x+iy)] - \log[\sin(x-iy)]$ —(I)

(1) $\log[\sin(x+iy)] = \log[\sin(x)\cos(iy) + \cos x\sin(iy)] = \log[\sin x\cosh(y) + i\cos x\sinh y] = \log[a_1+ib_1]$

$= \log(\sqrt{a_1^2+b_1^2}) + i\tan^{-1}[b_1/a_1] = \log[\sqrt{\sin^2x\cosh^2y + \cos^2x\sinh^2y}] + i\tan^{-1}\left[\frac{\cos x\sinh y}{\sin x\cosh y}\right]$

$= \log[\sqrt{\sin^2x\cosh^2y + \cos^2x\sinh^2y}] + i\tan^{-1}[\cot x\tanh y]$ —(II)

(2) $\log[\sin(x-iy)] = \log[\sin x\cos iy - \cos x\sin(iy)] = \log[\sin x\cosh y - i\cos x\sinh y] = \log[a_2+ib_2]$

$= \log[\sqrt{\sin^2x\cosh^2y + (-\cos x\sinh y)^2}] + i\tan^{-1}\left[-\frac{\cos x\sinh y}{\sin x\cosh y}\right]$

$\tan^{-1}(-x) = -\tan^{-1}(x)$

$= \log[\sqrt{\sin^2x\cosh^2y + \cos^2x\sinh^2y}] - i\tan^{-1}[\cot x\tanh y]$ —(III)

put substract eqn (II) – (III)

$\text{LHS} = \log[\sin(x+iy)] - \log[\sin(x-iy)] = 0 + 2i\tan^{-1}[\cot x\tanh y]$

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2. **Show that** $\log(e^{i\alpha} + e^{i\beta}) = \log\left[2\cos\left(\frac{\alpha-\beta}{2}\right)\right] + i\left(\frac{\alpha+\beta}{2}\right)$ **[MU -May-16]**

$\rightarrow \log[e^{i\alpha} + e^{i\beta}] = \log[(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta)] = \log[(\cos\alpha+\cos\beta) + i(\sin\alpha+\sin\beta)]$

But $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\log[e^{i\alpha} + e^{i\beta}] = \log\left[2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + i\,2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]$

$= \log\left\{2\cos\left(\frac{\alpha-\beta}{2}\right)\left[\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right]\right\} = \log\left[2\cos\left(\frac{\alpha-\beta}{2}\right).\,e^{i\left(\frac{\alpha+\beta}{2}\right)}\right]$

$= \log\left\{2\cos\left(\frac{\alpha-\beta}{2}\right)\left[\cos\left(\frac{\alpha+\beta}{2}\right)+i\sin\left(\frac{\alpha+\beta}{2}\right)\right]\right\} = \log\left[2\cos\left(\frac{\alpha-\beta}{2}\right)\cdot e^{i\left(\frac{\alpha+\beta}{2}\right)}\right]$

$= \log\left[2\cos\left(\frac{\alpha-\beta}{2}\right)\right] + \log\left[e^{i\left(\frac{\alpha+\beta}{2}\right)}\right] = \boxed{\log\left[2\cos\left(\frac{\alpha-\beta}{2}\right)\right] + i\left(\frac{\alpha+\beta}{2}\right)}$

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3. If $\tan[\log(x+iy)] = a+ib$ **, where** $a^2+b^2 \neq 1$ **, then prove that** $\tan[\log(x^2+y^2)] = \frac{2a}{1-(a^2+b^2)}$

$\log[(x+iy)(x-iy)] = \log(x+iy) + \log(x-iy)$

$\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left[\frac{A+B}{1-AB}\right]$

→ $\tan[\log(x+iy)] = a+ib$

$\log(x+iy) = \tan^{-1}[a+ib]$ —(I)

$\log(x-iy) = \tan^{-1}[a-ib]$ —(II)

Add (I) & (II)

$\log(x+iy) + \log(x-iy) = \tan^{-1}(a+ib) + \tan^{-1}(a-ib)$

$\log[(x+iy)(x-iy)] = \tan^{-1}\left[\frac{(a+ib)+(a-ib)}{1-(a+ib)(a-ib)}\right]$

$\log(x^2+y^2) = \tan^{-1}\left[\frac{2a}{1-(a^2+b^2)}\right]$

$\boxed{\tan[\log(x^2+y^2)] = \frac{2a}{1-(a^2+b^2)}}$, $a^2+b^2 \neq 1$

4. If $\sin^{-1}(x+iy) = \log(a+ib) = \alpha+i\beta$ **, then prove that** 1) $\frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\alpha} = 1$ 2) $a^2+b^2 = e^{2\alpha}$

→ We have $\sin^{-1}(x+iy) = \alpha+i\beta$ and $\log(a+ib) = \alpha+i\beta$

$x+iy = \sin(\alpha+i\beta)$

$x+iy = \sin(\alpha)\cos(i\beta) + \cos(\alpha)\sin(i\beta) = \sin(\alpha)\cosh(\beta) + i\cos(\alpha)\sinh\beta$

⇒ By comparing Real & Im. parts

⇒ $x = \sin\alpha\cosh\beta$ ⇒ $\frac{x}{\sin\alpha} = \cosh\beta$ —(I)

⇒ $y = \cos\alpha\sinh\beta$ ⇒ $\frac{y}{\cos\alpha} = \sinh\beta$ —(II)

From (I) & (II)

$\boxed{\frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\alpha} = \cosh^2\beta - \sinh^2\beta = 1}$

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(II) $\log(a+ib) = \alpha+i\beta$

⇒ $(a+ib) = e^{\alpha+i\beta} = e^{\alpha}\cdot e^{i\beta}$

⇒ $(a-ib) = e^{\alpha-i\beta} = e^{\alpha}\cdot e^{-i\beta}$

$(a+ib)(a-ib) = e^{\alpha}\, e^{i\beta}\cdot e^{\alpha}\cdot e^{-i\beta}$

$\boxed{a^2+b^2 = e^{2\alpha}}$

or

$\log(\sqrt{a^2+b^2}) + i\tan^{-1}(b/a) = \alpha+i\beta$ —(I)

→ $\log(a-ib) = \alpha-i\beta$

$\log(\sqrt{a^2+b^2}) + i\tan^{-1}\left[\frac{-b}{a}\right] = \alpha+i\beta$

$\log(\sqrt{a^2+b^2}) - i\tan^{-1}(b/a) = \alpha-i\beta$ —(II)

Add (I) & (II)

$2\log(\sqrt{a^2+b^2}) = 2\alpha$

$\log\left[(a^2+b^2)^{1/2}\right]^{2} = 2\alpha$

$\boxed{a^2+b^2 = e^{2\alpha}}$

5. Prove that $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$ **[MU -Dec-2000,02, 14]**

→ Let $a+ib = re^{i\theta}$ —(I), $r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}(b/a)$

⇒ $\boxed{\tan\theta = b/a}$ —(II)

$a-ib = re^{-i\theta}$ —(III)

We have

$\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \tan\left[i\log\left(\frac{re^{-i\theta}}{re^{i\theta}}\right)\right] = \tan\left[i\log\left(e^{-2i\theta}\right)\right]$

$= \tan[i(-2i\theta)] = \tan[2\theta] = \tan(\theta+\theta) = \frac{\tan\theta+\tan\theta}{1-\tan^2\theta}$

$\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \tan\left[i\log\left(\frac{re^{-i\theta}}{re^{i\theta}}\right)\right] = \tan\left[i\log\left(e^{-2i\theta}\right)\right]$

$= \tan[i(-2i\theta)] = \tan[2\theta] = \tan(\theta+\theta) = \frac{\tan\theta+\tan\theta}{1-\tan^2\theta}$

$= \frac{2\left(\frac{b}{a}\right)}{1-\frac{b^2}{a^2}} = \frac{\frac{2b}{a}}{\frac{(a^2-b^2)}{a^2}} = \frac{2ab}{a^2-b^2}$

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6. If $\log[\,Sin(x+iy)] = a+ib$, Then 1) $Cosh(2y) - Cos(2x) = 2e^{2a}$, 2) $\tan(b) = \cot(x)\tanh(y)$ [MU -April-21 online exam]

$\rightarrow$ $\log[\sin(x+iy)] = a+ib$

$\sin(x+iy) = e^{a+ib} = e^a e^{ib}$

$\boxed{\sin(x)\cosh(y) + i\cos x\sinh y} = e^a[\cos b + i\sin b]$

Comparing Real & Im parts

$\sin x\cosh y = e^a\cos b \quad \text{—(I)}$

$\cos x\sinh y = e^a\sin b \quad \text{—(II)}$

(I) square and add eqn's (I) & (II)

$\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = e^{2a}[\cos^2 b+\sin^2 b] = e^{2a}$

$\boxed{(1-\cos^2 x) = \sin^2 x}$

$\cosh^2 y - \sinh^2 y = 1 \Rightarrow \boxed{\cosh^2 y - 1 = \sinh^2 y}$

$(1-\cos^2 x)\cosh^2 y + \cos^2 x[\cosh^2 y - 1] = e^{2a}$

$\cosh^2 y - \cancel{\cos^2 x\cosh^2 y} + \cancel{\cos^2 x\cosh^2 y} - \cos^2 x = e^{2a}$

$\cosh^2 y - \cos^2 x = e^{2a}$

But $1+\cos(2x) = 2\cos^2(x)$

$\boxed{1+\cosh(2x) = 2\cosh^2(x)}$

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multiple by 2

$2\cosh^2 y - 2\cos^2(x) = 2e^{2a}$

$[1+\cosh(2y)] - [1+\cos(2x)] = 2e^{2a}$

$\boxed{\cosh(2y) - \cos(2x) = 2e^{2a}}$

(II) divide eqn (II) by (I)

$\frac{e^a\sin b}{e^a\cos b} = \frac{\cos x\sinh y}{\sin x\cosh y} \Rightarrow \boxed{\tan(b) = \cot x\tanh y}$

7. Considering only principal values, separate into real and imaginary parts $i^{\log(1+i)}$

$\rightarrow$ Let $z = i^{\log(1+i)}$

$\log(z) = \log\left[i^{\log(1+i)}\right] = \log(1+i)\log(i)$

(I) $\log(1+i) = \log\left(\sqrt{1^2+1^2}\right) + i\tan^{-1}(1/1) = \log(\sqrt{2}) + i\frac{\pi}{4}$

(II) $\log(i) = \log\left[e^{i\pi/2}\right] = i\frac{\pi}{2}$ $\quad \boxed{i = e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2}$

$\log(z) = \left[\log(\sqrt{2}) + i\frac{\pi}{4}\right]\frac{\pi i}{2} = -\frac{\pi^2}{8} + i\frac{\pi}{2}\log(\sqrt{2}) = \left(-\frac{\pi^2}{8} + \frac{i\pi}{2}\frac{1}{2}\log(2)\right)$

$z = \boxed{e^{-\frac{\pi^2}{8} + i\frac{\pi}{4}\log 2}}$

$x+iy = e^{-\pi^2/8}\left[e^{i\frac{\pi}{4}\log 2}\right] = e^{-\frac{\pi^2}{8}}\left[\cos\left(\frac{\pi}{4}\log 2\right) + i\sin\left(\frac{\pi}{4}\log 2\right)\right]$

$\Rightarrow$ $x = e^{-\pi^2/8}\cos\left(\frac{\pi}{4}\log 2\right)$

$y = e^{-\pi^2/8}\sin\left(\frac{\pi}{4}\log 2\right)$

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$$\Rightarrow \boxed{\begin{aligned} x &= e^{-\pi^2/8}\cos\left(\frac{\pi}{4}\log 2\right) \\ y &= e^{-\pi^2/8}\sin\left(\frac{\pi}{4}\log 2\right)\end{aligned}}$$

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8. If $i^{i^{i^{\cdots\infty}}} = \alpha + i\beta$, then show that $\alpha^2+\beta^2 = e^{-(4k+1)\pi\beta}$, $k = 0,1,2,3\ldots$. [MU -May-15]

→ By suppresing 'i', $i^{\alpha+i\beta} = \alpha + i\beta$

$\text{Log}\left[i^{\alpha+i\beta}\right] = \text{Log}(\alpha+i\beta)$ ∵ [General value of $\text{Log}(i^{\alpha+i\beta})$ because in prove that we have general value.]

$(\alpha+i\beta)\,\text{Log}(i) = L[\alpha+i\beta]$ —(I)

But $\text{Log}[i] = \text{Log}[0+i] = \log(\sqrt{0^2+1^2}) + i\left(2k\pi + \frac{\pi}{2}\right)$, $r = \sqrt{1} = 1$ and $\theta = \tan^{-1}\left[\frac{1}{0}\right] = \pi/2$

$\text{Log}(i) = \log(1) + i(4k+1)\frac{\pi}{2} = i(4k+1)\frac{\pi}{2}$, $k = 0,1,2,3,\ldots$ (II)

Also $\text{Log}(\alpha+i\beta) = \log(\sqrt{\alpha^2+\beta^2}) + i\left[2n\pi + \tan^{-1}(\beta/\alpha)\right]$, $n = 0,1,2,3,\ldots$ $\theta = \tan^{-1}[\beta/\alpha]$ —(III)

put eqn (II) & (III) in eqn (I)

$(\alpha+i\beta)\, i(4k+1)\frac{\pi}{2} = \frac{1}{2}\log(\alpha^2+\beta^2) + i\left[2n\pi + \tan^{-1}(\beta/\alpha)\right]$

$i\frac{\alpha\pi}{2}(4k+1) - \beta(4k+1)\frac{\pi}{2} = \frac{1}{2}\log(\alpha^2+\beta^2) + i\left[2n\pi + \tan^{-1}(\beta/\alpha)\right]$

By comparing Real part, $-(4k+1)\frac{\pi\beta}{\cancel{2}} = \frac{1}{\cancel{2}}\log(\alpha^2+\beta^2)$

$-(4k+1)\pi\beta = \log(\alpha^2+\beta^2)$

$\Rightarrow \boxed{e^{-(4k+1)\pi\beta} = \alpha^2+\beta^2}$, $k = 0,1,2,3,\ldots$

9. Find the value of $\log_{(-3)}(-2)$

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→ We know that $\log_a b = \dfrac{\log_e b}{\log_e a}$

$\therefore \log_{(-3)}(-2) = \dfrac{\log_e(-2)}{\log_e(-3)} = \dfrac{\log_e(-2+i0)}{\log_e(-3+i0)}$ —(I)

But $\log(-2+i0) = \log(\sqrt{(-2)^2+0^2}) + i\tan^{-1}\left(\frac{0}{-2}\right) = \log(2) + i\tan^{-1}(-0)$

Similary $\log(-3+i0) = \log(\sqrt{(-3)^2+0^2}) + i\tan^{-1}\left(\frac{0}{-3}\right) = \log(3) + i\tan^{-1}(-0)$

But $\tan^{-1}(-0) = -\tan^{-1}(0) = \tan^{-1}(\pi - 0)$ $\Rightarrow$ $(-2+i0)$ or $(-3+i0)$ lies in IInd quadrant Hence

$\tan(\pi - 0) = -\tan(0) = -0$

$\Rightarrow \tan^{-1}(-0) = \pi - 0 = \pi$

Hence $\log(-2+i0) = \log 2 + i\pi$ and $\log(-3+i0) = \log 3 + i\pi$ put in (I)

$\log_{(-3)}(-2) = \dfrac{(\log 2 + i\pi)}{(\log 3 + i\pi)} \times \dfrac{(\log 3 - i\pi)}{(\log 3 - i\pi)}$

$$\boxed{\log_{(-3)}(-2) = \frac{\left[(\log 2 + i\pi)(\log 3 - i\pi)\right]}{\left[(\log 3)^2 + \pi^2\right]}}$$

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