# CSC 402
## Practice Problems for Exam 1 Solutions
### February 16, 2023

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**1.** Prove that for any prime $p$, $\sqrt{p}$ is irrational.

**Solution.** Assume for the purpose of contradiction that $\sqrt{p}$ is rational. Then there are integers $a$ and $b$ ($b \neq 0$) such that

$$\sqrt{p} = \frac{a}{b}$$

Squaring both sides of the above equation gives

$$\begin{aligned} p &= \frac{a^2}{b^2} \\ a^2 &= pb^2 \end{aligned}$$

Let $S(m)$ be the sum of the number of times each prime factor occurs in the unique factorization of $m$. Note that $S(a^2)$ and $S(b^2)$ is even. This is because the number of times that each prime factor appears in the prime factorization of $a^2$ and $b^2$ is exactly twice the number of times that it appears in the prime factorization of $a$ and $b$. Then, $S(pb^2) = 1 + S(b^2)$ must be odd. This is a contradiction as $S(a^2)$ is even and the prime factorization of a positive integer is unique.

**2.** For sets $A, B, C$, and $D$, suppose that $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$ then $x \in B$.

**Solution.** We will prove the claim by proving the contrapositive. Suppose that $A \setminus B \subseteq C \cap D$ and $x \in A$ but $x \notin B$. Since $x \notin B$ and $x \in A$, it must be that $x \in A \setminus B$ and hence $x \in C \cap D$. Thus $x \in D$.

**3.** Prove that if for some integer $a$, $a \geq 3$, then $a^2 > 2a + 1$.

**Solution.** First we assume that $a \geq 3$. We can note also that $3a > 2a + 1$ since $a > 1$. So if we can show that $a^2 \geq 3a$, then we're done. Note that $a^2 = a * a$, and $3a = 3 * a$. Since we know that $a \geq 3$, we can conclude $a * a \geq 3 * a$. Hence our proof is complete.

**4.** Consider an undirected graph $G$ with minimum degree $\delta(G) \geq 2$. Prove that $G$ has a path of length $\delta(G)$ and a cycle with at least $\delta(G) + 1$ vertices.

**Solution.** Consider a maximal path $P$ in $G$. Let $u$ and $v$ be the end vertices of $P$. Consider vertex $v$. Since $P$ is maximal, $v$ does not have a neighbor that is not on the path $P$. Thus all of $v$'s neighbors are vertices in $P$. Since $G$ is a simple graph, $P$ must have at least $\deg(v) \geq \delta(G)$ vertices other than $v$. Thus $P$ has at least $\delta(G) + 1$ vertices and hence has a length of at least $\delta(G)$.

Let $\deg(v) = k$. Traversing $P$ from $u$ to $v$, let $x_1, x_2, \ldots, x_k$ be the neighbors of $v$. Note that the edge $(v, x_1)$ along with the path from $x_1$ to $v$ forms a cycle of length $k + 1$. Since $k \geq \delta(G)$, the claim that $G$ has a cycle of length $\delta(G) + 1$ follows.

**5.** Let $G$ be a connected graph where all vertices are of even degree. Prove that $G$ has no *cut edges*. A *cut edge* is an edge, that if removed, would increase the number of connected components of the graph.

**Solution.** Suppose, for the sake of contradiction, that $G$ does have a cut edge $e = (u, v)$. Since $G$ is connected, $G - e$ has exactly two connected components. Note that each vertex in $G - e$ has the same degree as in $G$ except $u$ and $v$, whose degrees are each one less than in $G$. Thus, $u$ is the only vertex of odd degree in its connected component in $G - e$. Therefore, there are an odd number of odd-degree vertices in that connected component. However, we know that there must be an even number of odd degree vertices because each connected component of a graph is itself a graph, and hence contains even number of odd degree vertices. Thus, $G$ has no cut edges.

**6.** An angel tells you in a dream that every connected graph has a connected subgraph that is a tree, which retains all the vertices of the original graph (called a *spanning tree*). The angel also tells you a procedure that allows you to find that exact subgraph given any connected graph, $G$. The following is a procedure: We will keep adding edges to a subgraph $H$ of $G$ so that at the end $H$ is a spanning tree of $G$. Initially $H$ has no edges and $V(H) := V(G)$. While $H$ has more than 1 component, find an edge in $G$ that has endpoints in two different components of $H$ and add it to $H$. Prove the following properties:

a. If $H$ has more than 1 component, there is some edge in $G$ whose endpoints lie in different components of $H$.

b. At all times $H$ is an acyclic graph.

c. When this procedure terminates, $H$ will be a spanning tree of $G$.

**Solution.**

a. Suppose $H$ has more than one component and each edge in $G$ has endpoints in the same component of $H$. Then there are no paths in $G$ between vertices in different components of $H$, contradicting the fact that $G$ is connected. So we must have edges in $G$ with endpoints in different components.

b. The proof is by induction on the number of edges $n$ added to $H$.

   <u>Induction hypothesis</u>: The subgraph $H$ is acyclic after the addition of $n = k$ edges, for some $k \geq 0$.
   <u>Base Case</u>: When $n = 0$, the subgraph $H$ has no edges in it, and therefore is acyclic.
   <u>Induction Step</u>: Consider the subgraph $H$ after the addition of $k + 1$ edges. Note that by the induction hypothesis, the subgraph was acyclic prior to the addition of the $k+1^{th}$ edge. Now let us consider the addition of the $k+1^{th}$ edge $(u, v)$. Suppose for contradiction that $(u, v)$ causes a cycle to be formed in $H$. This cycle must include the edge $(u, v)$ because there was no cycle before it was added. Then before the addition of $(u, v)$, there must have been a path in $H$ between $u$ and $v$. However, $u$ and $v$ were in different components of $H$, meaning that there was no such path, a contradiction. Thus the addition of the $k + 1^{th}$ edge does not cause a cycle and we have proved the inductive step.
   **Remark:** Note that we use a proof by induction here even though we are not proving a statement for an infinite set of cases. We will only add finitely many edges to $H$, but the proof by induction is still valid.

c. As long as $H$ contains more than one component, we will find an edge in $G$ to add to $H$ by part (a). Thus we will only stop when $H$ becomes a single component, at which point it will be connected and acyclic by part (b), i.e., a tree.