Lecture 02: Markov Chain

SOLVED PROBLEMS OF THE MARKOV CHAIN USING TRANSITION PROBABILITY MATRIX (Part I):

(1) P of states (2) P after n steps (3) P of chain.

eq $P(X_2=3)$ eq $P(X_3=4 \mid X_1=2)$ eq. $P(X_3=2, X_2=3, X_1=1, X_0=4)$

Notations:

q_0 : Initial prob. of the states.

q_n : Prob. of the states after n-time period.

P : TPM after 1 time period.

P^n : TPM " n " "

Type 1: How to calculate prob. of the states?

Notation → $P(X_n=a) = q_n(a)$

Formula: $\boxed{q_n = q_0 P^n}$ OR $\boxed{q_{n+1} = q_n P^1}$

* q_0 and P are known to you.

$n-0=n$ so power of P is n

$n+1-n=1$ so power of P is 1

eg. $q_3 = q_0 P^3 = q_1 P^2 = q_2 P^1$

eg1 A man either uses his car or takes a bus or a train to work each day. The TPM of the Markov chain (MC) with these 3 states — 1 (car), 2 (Bus), 3 (Train) is

$$P = \text{past}\left\{\begin{array}{c|c} & \begin{array}{ccc} C & B & T \end{array} \\ \begin{array}{c} C \\ B \\ T \end{array} & \underbrace{\begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}}_{\text{future}} \end{array}\right.$$

And the initial probability is $(0.7, 0.2, 0.1)$.
Calculate $P(X_2 = 3)$

→ Ga

$$\begin{array}{cccc} & C & B & T \\ q_0 = & (0.7, & 0.2, & 0.1) \end{array}$$

There is 70% chance that a person uses car on 1st day.
" " 20% " " " " " Bus " "
" " 10% " " " " " Train " "

$P(X_2 = 3) = ?$ — X_2: 2nd day (time); 3: Train

What is the p that a man will use a train to go to work on a 2nd day?

$P(X_2 = 3)$ (1) $= q_2(3)$? (2)

$q_1 p^1$

$q_0 p^2$ (3)

q_0 Given ✓

$= q_0 P^2 (3)$

$= q_0 \underbrace{P \times P}_{(4)} (3)$

$P^2 = P \times P$ (5) $= \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$

$$= \begin{array}{cc} & \begin{array}{ccc} C & B & T \end{array} \\ \begin{array}{c} C \\ B \\ T \end{array} & \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix} \end{array}$$

$\therefore q_0 P^2 =$ (6) $\begin{bmatrix} 0.7 & 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$

$$= \begin{array}{c} \begin{bmatrix} 0.385 & 0.336 & 0.279 \end{bmatrix} \\ \begin{array}{ccc} C & B & T \end{array} \end{array}$$ ✓

(4) $\boxed{P(X_2 = 3) = 0.279.}$

eq2. The TPM of MC with states 1,2,3 is

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.1 & 0.5 & 0.4 \\ 2 & 0.6 & 0.2 & 0.2 \\ 3 & 0.3 & 0.4 & 0.3 \end{array}$$

And the initial probability is $(0.7, 0.2, 0.1)$

Calculate

(i) $P(X_2 = 1)$

(ii) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$ — Type 3

→ (i) $P(X_2 = 1) = q_2(1)$

$q_2 = q_0 P^2$

$$P^2 = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

$q_0 = [0.7 \quad 0.2 \quad 0.1]$

$q_0 P^2 = [0.385 \quad 0.336 \quad 0.279]$

$\therefore \boxed{P(X_2 = 1) = 0.385}$

eq3. lec 1 eq 3 cont.

If the initial probability distribution of three states A, B, C is 0.3, 0.4, 0.3 resp. Find

(i) TM → done

(ii) $P(X_2 = B)$

(iii) $P(X_3 = B, X_2 = C, X_1 = B, X_0 = A)$ — Type 3

(iv) the distribution of the balls after two rounds

→ (ii) $P(X_2 = B) = q_2(B)$

$q_2 = q_0 P^2$

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}, \quad q_0 = [0.3 \ 0.4 \ 0.3]$$

$q_2 = q_0 P^2 = [0.20 \ \ 0.35 \ \ 0.45]$

$q_2(B) = 0.35$

$\therefore \boxed{P(X_2 = B) = 0.35}$

eg4. A man either drives his car or takes a train to work each day. Suppose he never takes the train two days in a row, but if he drives to work, then the next day he is just as likely to drive again as he is to take the train. At the first day he tosses a coin and if head comes then he takes a train to work, ow he drives to work. Find the TPM. Also, find the probability that he will use the car to go to work after two days?

→

$$P = \begin{array}{c} \\ \text{train} \\ \text{car} \end{array} \begin{array}{c} \begin{array}{cc} \text{train} & \text{car} \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$$

$q_0 = [0.5 \ 0.5]$

T C

$P(X_2 = C) = q_2(C)$ $\qquad$ $q_2 = q_0 P^2$

$$P^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

$$q_2 = q_0 P^2 = \begin{bmatrix} 3/8 & 5/8 \end{bmatrix}$$

$$q_2(C) = 5/8$$

$$\therefore \boxed{P(X_2 = C) = \frac{5}{8}}$$