Recitation Guide - Week 2

Topics Covered: Induction.

Problem 1: Prove using induction that for any positive integer n and for any integers $d_0, d_1, \ldots, d_{n-1} \in [0..9]$ we have:

$$\sum_{j=0}^{n-1} d_j \cdot 10^j < 10^n$$

Solution:

<u>Base Case:</u> $n = 1$.

$$\sum_{j=0}^{0} d_j \cdot 10^j = d_0 \cdot 10^0 = d_0 < 10 = 10^1$$

Note that d_j can only take values from 0 through 9, thus $d_0 < 10$. This concludes the Base Case.

<u>Induction Hypothesis:</u> Assume that the claim is true when $n = k$, for some integer $k \geq 1$ such that:

$$\sum_{j=0}^{k-1} d_j \cdot 10^j < 10^k$$

<u>Induction Step:</u> We now want to show that our claim holds when $n = k+1$. In other words, we seek to show that:

$$\sum_{j=0}^{k} d_j \cdot 10^j < 10^{k+1}$$

We see that we can show this as follows:

$$\begin{aligned}
\sum_{j=0}^{k} d_j \cdot 10^j &= d_k \cdot 10^k + \sum_{j=0}^{k-1} d_j \cdot 10^j && \text{(splitting the sum)} \\
&< d_k \cdot 10^k + 10^k && \text{(by IH)} \\
&= (d_k + 1) \cdot 10^k \\
&\leq (9+1) \cdot 10^k \\
&= 10^{k+1}
\end{aligned}$$

Thus, we have shown our claim is true when $n = k+1$, concluding our Induction Step and completing our proof.

Problem 2: All the sheep in Bethany's flock have the same color! Bethany claims that she can use induction to prove all sheep in the world have the same color. Find the fault in her reasoning.

<u>Base Case:</u> size = 1. One sheep, one color. ✓

<u>Induction Hypothesis:</u> Assume that in a flock of size k, where $k \in \mathbb{Z}^+$, all sheep have the same color.

Induction Step: We want to prove the claim is true for a flock of size $k+1$. Take one sheep, let's call her Sana, out. What remains is a flock of size k, so by IH, they all share the same color. Now put Sana back in and take out another sheep, let's call him Ethan, out. By IH, what remains is a flock of size k, so by IH, they all share the same color. Sana and Ethan must share the same color as they both are the same color as the other $k-1$ sheep. Thus we arrive at the conclusion that all $k+1$ sheep share the same color.

Solution:

The problem lies in the step where we try to show that having one sheep of the same color implies that two sheep must have the same color (That is, when $k=1$ and $k+1=2$). It is possible that the sheep don't share the same color. Note that in the induction step when we remove one of the sheep (say Sana), Ethan would be the only sheep left behind since there are $k-1=0$ sheep left over. Thus, we can only conclude that Ethan has the same color as himself. Similarly, after the other removal, we would only be able to conclude the Sana is the same color as herself. Thus, we can't say anything about Ethan or Sana sharing the same color as another sheep, so our logic for the induction step does not work.

Thus, the induction used in the question is not valid.

(Note: For this proof, we may have wanted to attempt two bases to avoid this problem (i.e. proving when $n=2$ that we are good), but as shown above, the claim is not necessarily true in this example, so the claim is false.)

Problem 3: Prove by induction that the number of diagonals in a convex polygon is $\frac{n(n-3)}{2}$, where n is the number of sides of the polygon.

Solution:

We prove by induction on n that the number of diagonals in a convex polygon is $\frac{n(n-3)}{2}$ where n is the number of sides of the polygon.

Notice that the polygon with the least number of sides is a triangle.

(BASE CASE) $n=3$. The number of diagonals in a triangle is $\frac{n(n-3)}{2} = \frac{3(3-3)}{2} = 0$. ✓

(INDUCTION STEP) Let $k \in \mathbb{N}, k \geq 3$ be arbitrary. Assume that (IH) any convex polygon with k sides has $\frac{k(k-3)}{2}$ diagonals. We want to show that any convex polygon with $k+1$ sides has $\frac{(k+1)(k-2)}{2}$ diagonals.

Consider an arbitrary convex polygon with $k+1$ sides and label its vertices in clockwise order. If we draw a diagonal from vertex 1 to vertex 3, the resulting partitioned polygon has k sides. Then, by IH, the partitioned part of the polygon has $\frac{k(k-3)}{2}$ diagonals. The number of diagonals from the missing vertex 2 is just $k-2$, since it has a diagonal to every vertex other than 1 and 3. The edge we drew from vertex 1 to vertex 3 is also a diagonal in the original polygon. Then, the total number of diagonals in this polygon would be

$$
\begin{aligned}
&= \frac{k(k-3)}{2} + k - 2 + 1 \\
&= \frac{k^2 - 3k + 2k - 4 + 2}{2} \\
&= \frac{k^2 - k - 2}{2} \\
&= \frac{(k+1)(k-2)}{2}
\end{aligned}
$$

thus concluding our Induction Step and the proof.