Mathematical Foundations of Computer Science
Lecture Outline
January 30, 2022

Example. Prove that, for any positive integer n, if $x_1, x_2, \ldots, x_n$ are n distinct real numbers, then no matter how the parenthesis are inserted into their product, the number of multiplications used to compute the product is $n-1$.

Solution. Let $P(n)$ be the property that "If $x_1, x_2, \ldots, x_n$ are n distinct real numbers, then no matter how the parentheses are inserted into their product, the number of multiplications used to compute the product is $n-1$".
Induction Hypothesis: Assume that $P(j)$ is true for all j such that $1 \leq j \leq k$.
Base Case: $P(1)$ is true, since x_1 is computed using 0 multiplications.
Induction Step: We want to prove $P(k+1)$. Consider the product of $k+1$ distinct factors, $x_1, x_2, \ldots, x_{k+1}$. When parentheses are inserted in order to compute the product of factors, some multiplication must be the final one. Consider the two terms, of this final multiplication. Each one is a product of at most k factors. Suppose the first and the second term in the final multiplication contain f_k and s_k factors. Clearly, $1 \leq f_k, s_k \leq k$. Thus, by induction hypothesis, the number of multiplications to obtain the first term of the final multiplication is $f_k - 1$ and the number of multiplications to obtain the second term of the final multiplication is $s_k - 1$. It follows that the number of multiplications to compute the product of $x_1, x_2, \ldots, x_k, x_{k+1}$ is

$$(f_k - 1) + (s_k - 1) + 1 = f_k + s_k - 1 = k + 1 - 1 = k$$

Example. The game of NIM is played as follows: Some positive number of sticks are placed on the ground. Two players take turns, removing one, two or three sticks. The player to remove the last stick loses.

A winning strategy is a rule for how many sticks to remove when there are n left. Prove that the first player has a winning strategy iff the number of sticks, n, is not $4k+1$ for any $k \in \mathbb{N}$.

Solution. We will show that if $n = 4k+1$ then player 2 has a strategy that will force a win for him, otherwise, player 1 has a strategy that will force a win for him.

Let $P(n)$ be the property that if $n = 4k+1$ for some $k \in N$ then the first player loses, and if $n = 4k, 4k+2$, or $4k+3$, the first player wins. This exhausts all possible cases for n.
Induction Hypothesis: Assume that for some $z \geq 1$, $P(j)$ is true for all j such that $1 \leq j \leq z$.
Base Case: $P(1)$ is true. The first player has no choice but to remove one stick and lose.
Induction Step: We want to prove $P(z+1)$. We consider the following four cases.
Case I: $z + 1 = 4k + 1$, for some k. We have already handled the base case, so we can assume that $z + 1 \geq 5$. Consider what the first player might do to win: he can remove 1, 2, or 3 sticks. If he removes one stick then the remaining number of sticks $n = 4k$. By

strong induction, the player who plays at this point has a winning strategy. So the player who played first loses. Similarly, if the first player removes two sticks or three sticks, the remaining number of sticks is $4(k-1)+3$ and $4(k-1)+2$ respectively. Again, the first player loses (using induction hypothesis). Thus, in this case, the first player loses regardless of what move he/she makes.
Case II: $z+1 = 4k$, or $z+1 = 4k+2$, or $z+1 = 4k+3$. If the first player removes three sticks in the first case, one stick in the second case, and two sticks in the third case then the second player sees $4(k-1)+1$ sticks in the first case and $4k+1$ sticks in the other two cases. By induction hypothesis, in each case the second player loses.

Graphs

A *graph* consists of two sets, a non-empty set, V, of vertices or nodes, and a possibly empty set, E, of 2-element subsets of V. Such is graph is denoted by $G = (V, E)$. Each element of E is called an *edge*. We say that an edge $\{u, v\} \in E$ *connects* vertices u and v. Two nodes u and v are *adjacent* if $\{u, v\} \in E$. Nodes adjacent to a vertex u are called *neighbors* of u. The number of neighbors of a vertex v is called the *degree* of v and is denoted by $deg(v)$. The value $\delta(G) = \min_{v \in V}\{deg(v)\}$ is the *minimum degree* of G, the value $\Delta(G) = \max_{v \in V}\{deg(v)\}$ is the *maximum degree* of G. An edge that connects a node to itself is called a *loop* and multiple edges between the same pair of nodes are called *parallel* edges. Graphs without loops and parallel edges are called *simple* graphs, otherwise they are called *multigraphs.* Unless specified otherwise, we will only deal with simple graphs.

Example. Prove that the sum of degrees of all nodes in a graph is twice the number of edges.

Solution. Since each edge is incident to exactly two vertices, each edge contributes two to the sum of degrees of the vertices. The claim follows.

Example. In any graph there are an even number of vertices of odd degree.

Solution. Let V_e and V_o be the set of vertices with even degree and the set of vertices with odd degree respectively in a graph $G = (V, E)$. Then,

$$\sum_{v \in V} deg(v) = \sum_{v \in V_e} deg(v) + \sum_{v \in V_o} deg(v)$$

The first term on R.H.S. is even since each vertex in V_e has an even degree. From the previous example, we know that L.H.S. of the above equation is even. Thus the second term on the R.H.S. must be even. Let $|V_o| = \ell$. We want to show that ℓ is even. Since each vertex in V_o has odd degree, we have

$$\begin{aligned}(2k_1+1)+(2k_2+1)+\cdots+(2k_\ell+1) &\text{ is an even number}\\ 2(k_1+k_2+\cdots+k_\ell)+\ell &\text{ is an even number}\\ \therefore \ell &\text{ is an even number}\end{aligned}$$

This proves the claim.

A *walk* in G is a non-empty sequence $v_0e_0v_1e_1 \dots e_{k-1}v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If the vertices in a walk are all distinct, we call it a *path* in G. Thus, a *path* in G is a sequence of distinct vertices $v_0, v_1, v_2, \dots v_k$ such that for all i, $0 \leq i < k$, $\{v_i, v_{i+1}\} \in E$. The *length* of the walk (path) is k, the number of edges in the walk (resp. path). Note that the length of the walk (path) is one less than the number of vertices in the walk (path) sequence. If $v_o = v_k$, the walk (path) is *closed.* A closed path is called a *cycle*.
The graph $H = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A graph G is *connected* if there is a path in G between its every pair of vertices. A graph H is a *connected component*("island") of G if (a) H is a subgraph of G, (b) H is connected, and (c) H is maximal, i.e., H is not contained in any other connected subgraph of G. In short, H is a connected component of G if H is a maximal subgraph of G that is connected.

We say that H is an *induced subgraph* of a graph G if the vertex set of H is a subset of the vertex set of G, and if u and v are vertices in H, then (u, v) is an edge in H iff (u, v) is an edge in G.