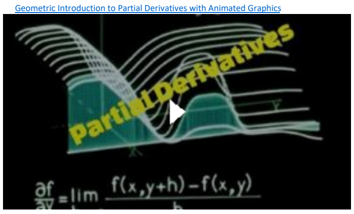
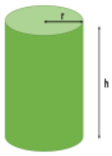
PARTIAL DIFFERENTIATION (20 Marks)

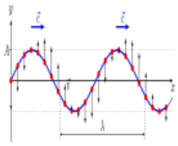
15 January 2022 09:54

Dr. Uday Kashid , Ph.D.(Mathematics)

Geometric Introduction to Partial Derivatives with Animated Graphics







$$\frac{\partial^2 u}{\partial t^2} = k^2 \frac{\partial^2 u}{\partial x^2}$$

Examples based on Cartesian Co-Ordinates

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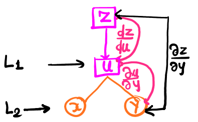
1. If $z = \log(x^2 + y^2)$, Then prove that $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$
2. If $z = \left(x^2 y + e^{xy^2}\right)$, Then prove that $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$ [MU -Dec-2013]
3. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, Then find $\frac{\partial^2 u}{\partial x\, \partial y}$
4. If $u = \log(\tan x + \tan y + \tan z)$, Then prove that $\sin(2x)\frac{\partial u}{\partial x} + \sin(2y)\frac{\partial u}{\partial y} + \sin(2z)\frac{\partial u}{\partial z} = 2$ [MU -Dec-09. May-10]
5. If $u = z\tan^{-1}\left(\frac{x}{y}\right)$, Then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ [MU -Dec-2010]
6. If $z = (1 - 2xy + y^2)^{-\frac{1}{2}}$, Then prove that $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = y^2 z^3$ [MU -May-2008]
7. If $z = \tan(y + ax) + (y - ax)^{3/2}$, Then prove that $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$ [MU -May-2009, Dec-11, 17]
8. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, Then show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$ [MU - Dec-07, 14]
9. If $u = \log(x^3 + y^3 - x^2 y - xy^2)$, Then show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = \frac{-4}{(x+y)^2}$ [MU - Dec-02, May-07]
10. If $z = (x^y + y^x)$, Then prove that $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$
11. If $z = e^{x^y}$, Then find $\frac{\partial^2 z}{\partial y\, \partial x}$
12. If $u\,(x + y) = x^2 + y^2$ show that $\left[\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right]^2 = 4\left[1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right]$
13. If $u = x^y$ then show that: $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$
14. If $u = \theta^n \cdot e^{-\frac{r^2}{4\theta}}$ then find the value of n, so that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial \theta}$ [MU - Dec-93, 02]

1. If $z = \log(x^2 + y^2)$, Then prove that $\boxed{\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}}$

→ let $u = x^2 + y^2 \Rightarrow u = f(x, y)$

$z = \log(u) \rightarrow z = f(u)$



$$LHS = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] \quad \text{— (I)}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du}\cdot\frac{\partial u}{\partial y} = \frac{1}{u}\cdot\frac{\partial}{\partial y}[x^2+y^2]$$

$$= \frac{1}{u}[0+2y] = \frac{2y}{(x^2+y^2)}$$

put in (I)

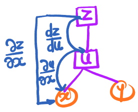
$u = x^2+y^2$

$$LHS = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{2y}{(x^2+y^2)}\right] = 2y\,\frac{\partial}{\partial x}\left[(x^2+y^2)^{-1}\right]$$

$$= 2y\left[(-1)\,u^{-2}\right]\cdot\frac{\partial u}{\partial x} = 2y\left[(-1)(x^2+y^2)^{-2}\right]\frac{\partial}{\partial x}(x^2+y^2)$$

$$LHS = 2y\frac{(-1)}{(x^2+y^2)^2}[2x] = \frac{-4xy}{(x^2+y^2)^2} \quad \text{—— (A)}$$

$$RHS = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] \quad \text{— (II)}$$



$$\frac{\partial z}{\partial x} = \frac{dz}{du}\cdot\frac{\partial u}{\partial x} = \frac{1}{u}\cdot\frac{\partial}{\partial x}[x^2+y^2]$$

$$= \frac{1}{u}[2x+0] = \left(\frac{2x}{x^2+y^2}\right)$$

put in (II)

$$RHS = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial}{\partial y}\left[\frac{2x}{(x^2+y^2)}\right] = 2x\cdot\frac{\partial}{\partial y}\left[\frac{1}{(x^2+y^2)}\right] = 2x\frac{(-1)}{(x^2+y^2)^2}\cdot\frac{\partial}{\partial y}[x^2+y^2]$$

$$= 2x\frac{(-1)}{(x^2+y^2)^2}[0+2y] = \frac{-4xy}{(x^2+y^2)^2} \quad \text{—— (B)}$$

By eqⁿ (A) & (B)

LHS = RHS $\Rightarrow \boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$ → commutative prop

Matrix Tut-

class-Test-1

10.10 - 10.80 am

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MCQ

10

10 10/10 → Tut-1

10 10/10 ← Tut- 2

0/10 07/10 Tut 3 {class test}1

05/10 10 tut 4

10 5

16/10 9/10 10 6 —

tut 7 ← class-test-2

4. If $u = \log(\tan x + \tan y + \tan z)$, Then prove that $\sin(2x)\frac{\partial u}{\partial x} + \sin(2y)\frac{\partial u}{\partial y} + \sin(2z)\frac{\partial u}{\partial z} = 2$ [MU -Dec-09. May-10]

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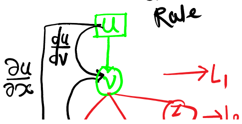
chain Rule

→ let $v = \tan x + \tan y + \tan z \Rightarrow v = f(x, y, z)$

$u = \log(v) \Rightarrow u = f(v)$



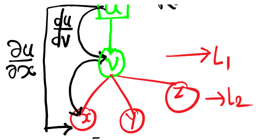
(1) $\frac{\partial u}{\partial x} = \frac{du}{dv}\cdot\frac{\partial v}{\partial x} = \frac{d}{dv}[\log v]\cdot\frac{\partial}{\partial x}[\tan x + \tan y + \tan z]$

$u = \log(v) \quad \Rightarrow \quad u = f(v)$

(i) $\frac{\partial u}{\partial x} = \frac{du}{dv}\cdot\frac{\partial v}{\partial x} = \frac{d}{dv}[\log v]\cdot\frac{\partial}{\partial x}[\tan x + \tan y + \tan z]$

$\frac{\partial u}{\partial x} = \frac{1}{v}\cdot[\sec^2 x + 0 + 0] = \frac{\sec^2 x}{v}$

$\sin 2x \frac{\partial u}{\partial x} = \sin 2x\cdot\frac{\sec^2 x}{v} = \frac{2\sin x\cos x\cdot\frac{1}{\cos^2 x}}{v} = \frac{2\tan x}{v}$ — (I)

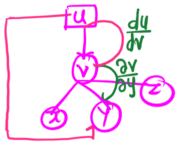


(ii) $\frac{\partial u}{\partial y} = \frac{du}{dv}\cdot\frac{\partial v}{\partial y} = \frac{d}{dv}[\log v]\cdot\frac{\partial}{\partial y}[\tan x + \tan y + \tan z]$

$= \frac{1}{v}[0 + \sec^2 y + 0] = \frac{\sec^2 y}{v}$

$\sin(2y)\frac{\partial u}{\partial y} = \sin 2y\,\frac{\sec^2 y}{v} = \frac{2\sin y\cos y\,\frac{1}{\cos^2 y}}{v}$

$= \frac{2\tan y}{v}$ — (II)



(iii) $\frac{\partial u}{\partial z} = \frac{du}{dv}\cdot\frac{\partial v}{\partial z} = \frac{d}{dv}[\log v]\,\frac{\partial}{\partial z}[\tan x + \tan y + \tan z]$

$= \frac{1}{v}[0 + 0 + \sec^2 z] = \frac{\sec^2 z}{v}$

$\sin(2z)\frac{\partial u}{\partial z} = \sin 2z\,\frac{\sec^2 z}{v} = \frac{2\sin z\cos z\,\frac{1}{\cos^2 z}}{v} = \frac{2\tan z}{v}$ — (III)

Add eq (I) + (II) + (III)

$\sin(2x)\frac{\partial u}{\partial x} + \sin(2y)\frac{\partial u}{\partial y} + \sin(2z)\frac{\partial u}{\partial z} = \frac{2\tan x}{v} + \frac{2\tan y}{v} + \frac{2\tan z}{v}$

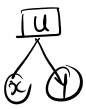
$= \frac{2[\tan x + \tan y + \tan z]}{[\tan x + \tan y + \tan z]} = \boxed{2}$

5 If $u = z\tan^{-1}\left(\frac{x}{y}\right)$, Then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ [MU -Dec-2010]

If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, Then find $\frac{\partial^2 u}{\partial x\,\partial y}$

$\rightarrow \frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right]$ — (1)

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\partial x} = \frac{\partial^2 u}{\partial x\partial y}$



$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[x^2\tan^{-1}(y/x)\right] - \frac{\partial}{\partial y}\left[y^2\tan^{-1}(x/y)\right] \rightarrow *$

$= x^2\frac{\partial}{\partial y}\left[\tan^{-1}\left(\frac{y}{x}\right)\right] - \frac{\partial}{\partial y}[y^2].\tan^{-1}(x/y) - y^2\frac{\partial}{\partial y}[\tan^{-1}(x/y)]$

$= x^2\left(\frac{1}{1+\theta^2}\cdot\frac{\partial}{\partial y}[\theta]\right)$

$= x^2\,\frac{1}{1+(y/x)^2}\cdot\frac{\partial}{\partial y}\left[\frac{y}{x}\right] - 2y.\tan^{-1}(x/y) - y^2\,\frac{1}{1+(x/y)^2}\cdot\frac{\partial}{\partial y}[x/y]$

$= x^2\,\frac{1}{\frac{(x^2+y^2)}{x^2}}\left[\frac{1}{x}\right] - 2y\tan^{-1}(x/y) - y^2\,\frac{1}{\frac{(x^2+y^2)}{y^2}}\left[x\frac{(-1)}{y^2}\right]$

$= x^2\,\frac{x^2}{(x^2+y^2)}\cdot\frac{1}{x} - 2y\tan^{-1}(x/y) - y^2\cdot\frac{y^2}{(x^2+y^2)}\left(\frac{-x}{y^2}\right)$

$\frac{\partial u}{\partial y} = \frac{x^3}{x^2+y^2} - 2y\tan^{-1}(x/y) + \frac{xy^2}{x^2+y^2} = \frac{x(x^2+y^2)}{(x^2+y^2)} - 2y\tan^{-1}(x/y) = x - 2y\tan^{-1}(x/y)$

$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial x}[x] - 2y.\frac{\partial}{\partial x}\left[\tan^{-1}((x/y))\right] = 1 - 2y\,\frac{1}{1+x^2/y^2}\cdot\frac{\partial}{\partial x}[x/y]$

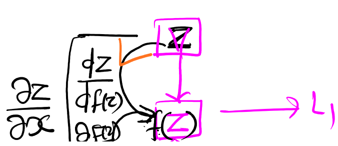
$= 1 - 2y\,\frac{y^2}{(x^2+y^2)}\left[\frac{1}{y}\right] = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$

2. If $z = (1-2xy+y^2)^{-\frac{1}{2}}$, Then prove that $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = y^2z^3$ [MU -May-2008]

$\rightarrow \quad z^{-2} = \left[(1-2xy+y^2)^{-1/2}\right]^{-2} = 1-2xy+y^2$ — (1)

$z = f(z) = z^{-2} = 1-2xy+y^2$ diff w.r.t x

$(-2)\,z^{-2-1}\,\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[1-2xy+y^2]$



$Z = f(z) = Z^{-2} = 1 - 2xy + y^2$ diff w.r.t x

$(-2) Z^{-2-1} \cdot \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[1 - 2xy + y^2]$

$-\cancel{2} \frac{1}{z^3} \frac{\partial z}{\partial x} = 0 - \cancel{2}y + 0$

$$\boxed{\frac{\partial z}{\partial x} = yz^3}$$

$x \frac{\partial z}{\partial x} = xyz^3$ — (A)

diff eqn (1) w.r.t y

$(-2) Z^{-3} \cdot \frac{\partial z}{\partial y} = 0 - 2x + 2y$

$-\cancel{2} \frac{\partial z}{\partial y} \cdot \frac{1}{z^3} = -\cancel{2}(x - y)$

$\frac{\partial z}{\partial x}$ $\frac{dz}{df(z)}$ $\frac{\partial f(z)}{\partial x}$ $f(z)$ $\to L_1$, x, y $\to L_2$

$\frac{\partial z}{\partial y} = z^3(x - y) = xz^3 - yz^3$

$y \frac{\partial z}{\partial y} = xyz^3 - y^2 z^3$ — (II)

substract eqn (I) & (II)

$x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = \cancel{xy}z^3 - \cancel{xy}z^3 + y^2 z^3$

$$\boxed{= y^2 z^3}$$

$Z = (1-2xy \ldots$

$u = 1 - 2 \ldots$

$Z = u^{-1/2}$

$\frac{\partial z}{\partial x}$ $\frac{dz}{du}$ $\frac{\partial u}{\partial x}$

$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \ldots$

$= \frac{d}{du}(u^{-1/2})$

$= (-\frac{1}{2}) u^{-3/2} (-2 \ldots$

$x \frac{\partial z}{\partial x} = xy\, u^{-3/2} = x \ldots$

$\frac{\partial z}{\partial y} = \frac{dz}{du} \frac{\partial u}{\partial y}$

1. If $z = \tan(y + ax) + (y - ax)^{3/2}$, Then prove that $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$ **[MU -May-2009, Dec-11, 17]**

Dr. Uday Kashid, Ph.D.(Mathematics)

8. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, Then show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$ **[MU -Dec-07, 14]**

Note:— (I) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right] u$

$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right]$

Sem-III Laplace eqn $\nabla^2 u = 0$ Harmonic Fun

Assume $v = \left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right]$

$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) v = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}$ — (A)

OR

(II) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right]$

$= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right]$

$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 u}{\partial z \partial y} + \frac{\partial^2 u}{\partial z^2}$

$$\boxed{\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial x \partial z} + 2\frac{\partial^2 u}{\partial y \partial z}}$$

8. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, Then show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$ **[MU -Dec-07, 14]**

Ans-(8) Let $v = x^3 + y^3 + z^3 - 3xyz$ $\to v = f(x, y, z)$

$u = \log(v)$ $\to u = f(v)$

$\frac{\partial u}{\partial x}$ $\frac{du}{dv}$ $\frac{\partial v}{\partial x}$ u, v $\to L_1$, x, y, z $\to L_2$

We know that

$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right]$ — (I)

$\frac{\partial u}{\partial x} = \frac{du}{dv} \cdot \frac{\partial v}{\partial x} = \frac{d}{dv}[\log v] \cdot \frac{\partial}{\partial x}[x^3 + y^3 + z^3 - 3xyz]$

$= \frac{1}{v}[3x^2 + 0 + 0 - 3yz] = \frac{3x^2 - 3yz}{v}$ — (II)

$\frac{\partial u}{\partial y} = \frac{du}{dv} \cdot \frac{\partial v}{\partial y} = \frac{1}{v} \cdot [0 + 3y^2 + 0 - 3xz] = \frac{3y^2 - 3xz}{v}$ — (III)

$\frac{\partial u}{\partial z} = \frac{du}{dv} \frac{\partial v}{\partial z} = \frac{1}{v}[0 + 0 + 3z^2 - 3xy] = \frac{3z^2 - 3xy}{v}$ — (IV)

$3x^2 + 3y^2 + 3z^2 - 3yz - 3xz - 3xy$

$$\text{-}y)^{-1/2}$$

$$xy+y^2$$

$$\partial y / \partial x$$

$$\cdot \frac{\partial}{\partial x}(1-2xy+y^2)$$

$$y\left[(1-2xy+y^2)^{-1/2}\right]^3 = xyz^3$$

$$\frac{\partial u}{\partial z} = \frac{du}{dv}\frac{\partial v}{\partial z} = \frac{1}{v}\left[0+0+3z^2-3xy\right] = \frac{3z^2-3xy}{v}$$

Let $w = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2+3y^2+3z^2-3yz-3xz-3xy}{v}$

$$= \frac{3[x^2+y^2+z^2-xy-xz-yz]}{x^3+y^3+z^3-3xyz} = \frac{3[x^2+y^2+z^2-xy-yz-xz]}{(x+y+z)[x^2+y^2+z^2-xy-yz-xz]}$$

$$w = \frac{3}{x+y+z}$$

put (1)

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{3(-1)}{(x+y+z)^2}\frac{\partial}{\partial x}(x+y+z) + \frac{3(-1)}{(x+y+z)^2}\frac{\partial}{\partial y}(x+y+z) + \frac{3(-1)}{(x+y+z)^2}\frac{\partial}{\partial z}(x+y+z)$$

$$\boxed{= \frac{-9}{(x+y+z)^2}}$$

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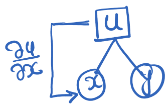
12. If $u(x+y) = x^2+y^2$ show that $\left[\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right]^2 = 4\left[1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right]$

→ We have $u = \frac{(x^2+y^2)}{(x+y)}$ —(I) $\quad u \to f(x,y)$



$$\frac{\partial u}{\partial x} = \frac{(x+y)\frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial x}(x+y)}{(x+y)^2} \quad \because d\left(\frac{u}{v}\right)$$

$$\frac{\partial u}{\partial x} = \frac{(x+y)(2x)-(x^2+y^2)(1)}{(x+y)^2} = \frac{2x^2+2xy-x^2-y^2}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2} \quad \text{—(II)}$$

$$\frac{\partial u}{\partial y} = \frac{(x+y)\frac{\partial}{\partial y}(x^2+y^2)-(x^2+y^2)\frac{\partial}{\partial y}(x+y)}{(x+y)^2} = \frac{(x+y)(2y)-(x^2+y^2)(1)}{(x+y)^2} = \frac{2xy+2y^2-x^2-y^2}{(x+y)^2} = \frac{y^2+2xy-x^2}{(x+y)^2} \quad \text{(III)}$$

substract & square eq (II) & (III)

$$LHS = \left[\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right]^2 = \left[\frac{x^2+2xy-y^2-y^2-2xy+x^2}{(x+y)^2}\right]^2 = \left[\frac{2x^2-2y^2}{(x+y)^2}\right]^2 = \frac{4(x^2-y^2)^2}{(x+y)^4}$$

$$LHS = 4\frac{(x-y)^2(x+y)^2}{(x+y)^2(x+y)^2} = \boxed{\frac{4(x-y)^2}{(x+y)^2}} \quad \text{—(A)}$$

$$RHS = 4\left[1-\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\right] = 4\left[1 - \frac{(x^2+2xy-y^2)}{(x+y)^2} - \frac{(y^2+2xy-x^2)}{(x+y)^2}\right] = 4\left[\frac{(x+y)^2 - x^2-2xy+y^2-y^2-2xy+x^2}{(x+y)^2}\right]$$

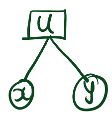
$$RHS = 4\left[\frac{x^2+2xy+y^2-4xy}{(x+y)^2}\right] = 4\left[\frac{x^2-2xy+y^2}{(x+y)^2}\right] = \boxed{\frac{4(x-y)^2}{(x+y)^2}} \quad \text{—(B)}$$

→ From (A) & (B) $\boxed{LHS = RHS}$

13. If $u = x^y$ then show that: $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

→ $u = x^y \quad u = f(x,y)$

Dr. Uday Kashid , Ph.D.(Mathematics)



$$LHS = \frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\right] \quad \text{—(A)}$$

$\frac{d}{dx}[a^x] = a^x \log a$

(I) $\frac{\partial u}{\partial y} = x^y \log x$

$\frac{\partial}{\partial y}[x^y] = x^y \log x$ (const)

(II) $\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial x}\left[x^y \cdot \log x\right] = \frac{\partial}{\partial x}(x^y)\log x + x^y\frac{\partial}{\partial x}(\log x)$

$$= y x^{y-1}\log x + x^y \frac{1}{x} = y x^{y-1}\log x + x^{y-1}$$

$\frac{\partial}{\partial x}(x^n) = n x^{n-1}$

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] = x^{y-1}\left[1 + y\log x\right]$$

(III) $LHS = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\right] = \frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left[x^{y-1}\left[1+y\log x\right]\right]$

$\frac{\partial}{\partial x}\left[\frac{\partial y}{\partial y}\right] = x^{y-1}[1+y\log x]$

(iii) LHS $= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial y}\right)\right] = \frac{\partial^3 y}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left[x^{y-1}[1+y\log x]\right]$

$= \frac{\partial}{\partial x}(x^{y-1})\cdot[1+y\log x] + x^{y-1}\cdot\frac{\partial}{\partial x}[1+y\log x]$

LHS $= (y-1)\,x^{y-2}[1+y\log x] + x^{y-1}\left[0+y\frac{1}{x}\right] = (y-1)x^{y-2}[1+y\log x] + y\,x^{y-2}$ —— (A)

(iv) RHS $= \frac{\partial^3 y}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial x}\right)\right]$ —— (ii)

$u = x^y$

(v) $\frac{\partial y}{\partial x} = y\,x^{y-1}$

(vi) $\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial y}\left[y\,x^{y-1}\right] = \frac{\partial}{\partial y}(y)\,x^{y-1} + y\,\frac{\partial}{\partial y}(x^{y-1})$

$= 1\,x^{y-1} + y\,x^{y-1}\log x = x^{y-1}[1+y\log x]$

(vii) RHS $= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial x}\right)\right] = \frac{\partial}{\partial x}\left[x^{y-1}(1+y\log x)\right]$

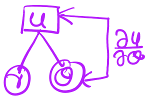
$= \frac{\partial}{\partial x}(x^{y-1})\,[1+y\log x] + x^{y-1}\,\frac{\partial}{\partial x}[1+y\log x]$

$= (y-1)\,x^{y-2}(1+y\log x) + x^{y-1}\left[0+y\frac{1}{x}\right] = (y-1)x^{y-2}(1+y\log x) + y\,x^{y-2}$ —— (B)

From eq (A) & (B) $\boxed{\text{LHS} = \text{RHS}}$

14. If $u = \theta^n \cdot e^{-\frac{r^2}{4\theta}}$ **then find the value of** n, **so that** $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial \theta}$ **[MU - Dec-93, 02]**

→ RHS $= \frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}(\theta^n)\,e^{-r^2/4\theta} + \theta^n\frac{\partial}{\partial \theta}\left[e^{-r^2/4\theta}\right]$



$= n\,\theta^{n-1}e^{-r^2/4\theta} + \theta^n e^{-r^2/4\theta}\cdot\frac{\partial}{\partial \theta}\left[-r^2/4\theta\right]$ ***

$= n\,\theta^{n-1}e^{-r^2/4\theta} + \theta^n e^{-r^2/4\theta}\left[-\frac{r^2}{4}\frac{(-1)}{\theta^2}\right]$

$= n\,\theta^{n-1}e^{-r^2/4\theta} + \theta^{n-2}e^{-r^2/4\theta}\,\frac{r^2}{4}$

Dr. Uday Kashid , Ph.D.(Mathematics)

RHS $= \theta^{n-2}e^{-r^2/4\theta}\left[n\theta + \frac{r^2}{4}\right]$ —— (A)

(i) $\frac{\partial u}{\partial r} = \theta^n\frac{\partial}{\partial r}\left[e^{-r^2/4\theta}\right] = \theta^n e^{-r^2/4\theta}\cdot\frac{\partial}{\partial r}\left[\frac{-r^2}{4\theta}\right] = \theta^n e^{-r^2/4\theta}\left[\frac{-2r}{4\theta}\right] = -\theta^{n-1}e^{-r^2/4\theta}\,\frac{r}{2}$

(ii) $r^2\frac{\partial u}{\partial r} = -\theta^{n-1}\frac{r^3}{2}e^{-r^2/4\theta}$

(iv) $\frac{\partial}{\partial r}\left[r^2\frac{\partial u}{\partial r}\right] = -\frac{\theta^{n-1}}{2}\frac{\partial}{\partial r}\left[r^3 e^{-r^2/4\theta}\right] = -\frac{\theta^{n-1}}{2}\left[3r^2 e^{-r^2/4\theta} + r^3 e^{-r^2/4\theta}\cdot\left(\frac{-2r}{4\theta}\right)\right]$

(v) LHS $= \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left[r^2\frac{\partial u}{\partial r}\right]\right) = -\frac{\theta^{n-1}}{2}\left[\frac{3}{\theta}\cdot\theta\, e^{-r^2/4\theta} - \frac{r^2}{2\theta}e^{-r^2/4\theta}\right]$

$= -\frac{\theta^{n-2}}{2}e^{-r^2/4\theta}\left[3\theta - \frac{r^2}{2}\right]$

(a) But we have LHS = RHS

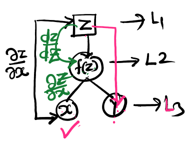
$\cancel{\theta^{n-2}e^{-r^2/4\theta}}\left[\frac{r^2}{4} - \frac{3\theta}{2}\right] = \cancel{\theta^{n-2}e^{-r^2/4\theta}}\left[n\theta + \frac{r^2}{4}\right]$

$\cancel{\frac{r^2}{4}} - \frac{3\theta}{2} = \cancel{\frac{r^2}{4}} + n\theta$

$\boxed{-\frac{3}{2} = n}$

11. If $z = e^{x^y}$, **Then find** $\frac{\partial^2 z}{\partial y\,\partial x}$

→ $\log z = \log e^{x^y} = x^y$



x^{n} $= n x^{n-1}$

$\frac{1}{z}\cdot\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[x^y\right] = y\,x^{y-1}$

$\frac{\partial z}{\partial x} = yz\,x^{y-1}$

$z \to f(x, y)$

$\frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial}{\partial y}\left[yz\,x^{y-1}\right]$

$= \frac{\partial}{\partial y}[y]\,[z\,x^{y-1}] + \frac{\partial}{\partial y}[z]\,[y\,x^{y-1}] + \frac{\partial}{\partial y}(x^{y-1})\,[yz]$

$\frac{\partial^2 z}{\partial y\partial x} = 1\,[z\,x^{y-1}] + \left(\frac{\partial z}{\partial y}\right)[y\,x^{y-1}] + x^{y-1}\log x\,[yz]$ —— (1)

$$= \frac{\partial}{\partial y}[y]\,[z x^{y-1}] + \frac{\partial}{\partial y}[z]\,[y x^{y-1}] + \frac{\partial}{\partial y}[x^{y-1}]\,[yz]$$

$$\frac{\partial^2 z}{\partial y \partial x} = 1[z x^{y-1}] + \left(\frac{\partial z}{\partial y}\right)[y x^{y-1}] + x^{y-1}\log x\,[yz] \quad \text{—(I)}$$

We have $\log z = x^y$

diff w.r.t y

$$\frac{1}{z}\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[x^y] = x^y \log x$$

$$\boxed{\frac{\partial z}{\partial y} = z x^y \log x} \quad \text{—(II)}$$

Dr. Uday Kashid , Ph.D.(Mathematics)

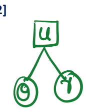
eqn (II) put in (I)

$$\frac{\partial^2 z}{\partial y \partial x} = z x^{y-1} + y x^{y-1}(z x^y \log x) + x^{y-1} y z \log x$$

$$= x^{y-1} z\,[1 + y x^y \log x + y \log x] \;//$$

14. If $u = \theta^n \cdot e^{-\frac{r^2}{4\theta}}$ **then find the value of** n, **so that** $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial \theta}$ **[MU - Dec-93, 02]**

$\rightarrow$ (I) $\text{RHS} = \frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}\left[\theta^n \cdot e^{-r^2/4\theta}\right]$ $\rightarrow d(uv)$



$t = -r^2/4\theta$

$\frac{\partial}{\partial \theta}[e^t] = e^t \cdot \frac{\partial t}{\partial \theta}$

$$= \frac{\partial}{\partial \theta}[\theta^n]\, e^{-r^2/4\theta} + \theta^n \frac{\partial}{\partial \theta}\left[e^{-r^2/4\theta}\right]$$

$$= n\theta^{n-1} e^{-r^2/4\theta} + \theta^n e^{-r^2/4\theta} \cdot \frac{\partial}{\partial \theta}\left[-r^2/4\theta\right] \quad ***$$

$$= n\theta^{n-1} e^{-r^2/4\theta} + \theta^n e^{-r^2/4\theta}\left[-\frac{r^2}{4}\frac{(-1)}{\theta^2}\right]$$

$$\text{RHS} = n\theta^{n-1} e^{-r^2/4\theta} + \frac{r^2}{4}\theta^{n-2} e^{-r^2/4\theta} = \theta^{n-2} e^{-r^2/4\theta}\left[n\theta + \frac{r^2}{4}\right] \quad \text{—(A)}$$

(II) $\frac{\partial u}{\partial r} = \theta^n \frac{\partial}{\partial r}\left[e^{-r^2/4\theta}\right] = \theta^n e^{-r^2/4\theta} \cdot \frac{\partial}{\partial r}\left[-\frac{r^2}{4\theta}\right]$ ** $u = \theta^n e^{-r^2/4\theta}$

$$= \theta^n e^{-r^2/4\theta}\left[-\frac{2r}{4\theta}\right] = -\frac{r}{2}\theta^{n-1} e^{-r^2/4\theta}$$

(III) $r^2 \frac{\partial u}{\partial r} = -\frac{r^3}{2}\theta^{n-1} e^{-r^2/4\theta}$

(IV) $\frac{\partial}{\partial r}\left[r^2 \frac{\partial u}{\partial r}\right] = -\frac{\theta^{n-1}}{2}\frac{\partial}{\partial r}\left[r^3 e^{-r^2/4\theta}\right] = -\frac{\theta^{n-1}}{2}\left[3r^2 e^{-r^2/4\theta} + r^3 e^{-r^2/4\theta} \cdot \frac{\partial}{\partial r}\left[-\frac{r^2}{4\theta}\right]\right]$

$$= -\frac{\theta^{n-1}}{2}\left[3r^2 e^{-r^2/4\theta} - \frac{r^3}{4\theta} e^{-r^2/4\theta}[2r]\right]$$

$$= -\frac{\theta^{n-1}}{2} e^{-r^2/4\theta}\left[\frac{\theta}{\theta}3r^2 - \frac{r^4}{2\theta}\right]$$

$$= -\frac{\theta^{n-1} e^{-r^2/4\theta}}{2\theta}\left[3\theta r^2 - \frac{r^4}{2}\right] = -\frac{\theta^{n-2}}{2} e^{-r^2/4\theta}\left[3\theta r^2 - \frac{r^4}{2}\right]$$

(V) $\text{LHS} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial u}{\partial r}\right] = -\frac{\theta^{n-2}}{2} e^{-r^2/4\theta}\left[3\theta - \frac{r^2}{2}\right]$ —(B)

But LHS = RHS (given)

$$\frac{\cancel{\theta^{n-2}}}{2}\cancel{e^{-r^2/4\theta}}\left[\frac{r^2}{2} - 3\theta\right] = \cancel{\theta^{n-2}}\,\cancel{e^{-r^2/4\theta}}\left[n\theta + \frac{r^2}{4}\right]$$

$$\cancel{\frac{r^2}{4}} - \frac{3\theta}{2} = n\theta + \cancel{\frac{r^2}{4}}$$

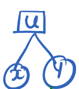
$$-\frac{3\cancel{\theta}}{2} = n\cancel{\theta}$$

$$\boxed{n = -3/2}$$

Dr. Uday Kashid , Ph.D.(Mathematics)

12. If $u(x + y) = x^2 + y^2$ **show that** $\left[\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right]^2 = 4\left[1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right]$

$\rightarrow$ $u = \frac{x^2+y^2}{(x+y)}$ —(I) $u = f(x, y)$



(I) $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\frac{(x^2+y^2)}{(x+y)}\right] = \frac{(x+y)\frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial x}(x+y)}{(x+y)^2}$ $\quad d\left[\frac{u}{v}\right] = \frac{v\,du - u\,dv}{v^2}$

$$\frac{\partial u}{\partial x} = \frac{(x+y)(2x+0) - (x^2+y^2)(1+0)}{(x+y)^2} = \frac{2x^2+2xy-x^2-y^2}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2} \quad \text{—(II)}$$

(II) $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x^2+y^2}{(x+y)}\right] = \frac{(x+y)\frac{\partial}{\partial y}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial y}(x+y)}{(x+y)^2}$

$$= \frac{(x+y)(0+2y) - (x^2+y^2)(0+1)}{(x+y)^2} = \frac{2xy+2y^2-x^2-y^2}{(x+y)^2} = \frac{y^2+2xy-x^2}{(x+y)^2} \quad \text{—(III)}$$

(III) $\text{LHS} = \left[\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right]^2 = \left[\frac{x^2+2xy-y^2}{(x+y)^2} - \frac{(y^2+2xy-x^2)}{(x+y)^2}\right]^2$

$$= \left[\frac{x^2+\cancel{2xy}-y^2-y^2-\cancel{2xy}+x^2}{(x+y)^2}\right]^2 = \left[\frac{2x^2-2y^2}{(x+y)^2}\right]^2 = \left[\frac{2(x^2-y^2)}{(x+y)^2}\right]^2$$

(iii) $LHS = \left[\frac{\partial u}{\partial x} - \frac{\partial y}{\partial y}\right]^2 = \left[\frac{\quad}{(x+y)^2} - \frac{\quad}{(x+y)^2} \right]^2$

$$= \left[\frac{x^2 + \cancel{2xy} - y^2 - y^2 - \cancel{2xy} + x^2}{(x+y)^2}\right]^2 = \left[\frac{2x^2 - 2y^2}{(x+y)^2}\right]^2 = \left[\frac{2(x^2-y^2)}{(x+y)^2}\right]^2$$

$$= \left[\frac{2(x-y)\cancel{(x+y)}}{(x+y)\cancel{(x+y)}}\right]^2 = \boxed{\frac{4(x-y)^2}{(x+y)^2}} = LHS \quad \text{— (A)}$$

(iv) $RHS = 4\left[1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right] = 4\left[1 - \frac{(x^2+2xy-y^2)}{(x+y)^2} - \frac{(y^2+2xy-x^2)}{(x+y)^2}\right]$

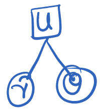
$$= 4\left[\frac{(x+y)^2 - \cancel{x^2} - 2xy + \cancel{y^2} - \cancel{y^2} - 2xy + \cancel{x^2}}{(x+y)^2}\right] = 4\left[\frac{x^2+2xy+y^2-4xy}{(x+y)^2}\right]$$

$$RHS = 4\left[\frac{x^2-2xy+y^2}{(x+y)^2}\right] = \frac{4(x-y)^2}{(x+y)^2} \quad \text{— (B)}$$

From (A) & (B) $\boxed{LHS = RHS}$ $\boxed{\left[\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right]^2 = 4\left[1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right]}$

14. If $u = \theta^n \cdot e^{-\frac{r^2}{4\theta}}$ then find the value of n, so that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial \theta}$ [MU - Dec-93, 02]

method I



$$\log u = \log \theta^n + \log e^{-r^2/4\theta}$$

$$\log u = n \log \theta - \frac{r^2}{4\theta} \quad \text{— (I)}$$

diff w.r.t θ

$$\frac{1}{u}\frac{\partial u}{\partial \theta} = n\frac{1}{\theta} - \frac{r^2}{4}\frac{\partial}{\partial \theta}\left[\frac{1}{\theta}\right]$$

$$= n\frac{1}{\theta} - \frac{r^2}{4}\left(\frac{-1}{\theta^2}\right)$$

$$RHS = \frac{\partial u}{\partial \theta} = \left[\frac{n}{\theta} + \frac{r^2}{4\theta^2}\right]u \quad \text{— (II)}$$

diff eq (I) w.r.t r

$$\frac{1}{u}\frac{\partial u}{\partial r} = 0 - \frac{1}{4\theta}(2r) \quad \Rightarrow \quad \frac{\partial u}{\partial r} = -\frac{r}{2\theta}u \quad \text{— (III)}$$

$$\left[r^2 \frac{\partial u}{\partial r}\right] = -\frac{r^3}{2\theta}u$$

diff. again w.r.t r ***

$u \to r, \theta$

$d(uv) = du \cdot v + u \cdot dv$

$$\frac{\partial}{\partial r}\left[r^2 \frac{\partial u}{\partial r}\right] = -\frac{1}{2\theta}\frac{\partial}{\partial r}\left[r^3 u\right]$$

$$= -\frac{1}{2\theta}\left[3r^2 u + r^3 \frac{\partial u}{\partial r}\right]$$

divided by r^2

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial u}{\partial r}\right] = -\frac{1}{2\theta}\left[3u + r\frac{\partial u}{\partial r}\right]$$

$$LHS = -\frac{1}{2\theta}\left[3u + r\left(\frac{-ru}{2\theta}\right)\right] \quad \because \text{By eq (III)}$$

$$= -\frac{1}{2\theta}\left[3u - \frac{r^2 u}{2\theta}\right] = -\frac{1}{2\theta}u\left[3 - \frac{r^2}{2\theta}\right] \quad \text{— (IV)}$$

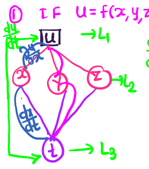
But eq (II) & (IV) are equal $\boxed{LHS = RHS}$

$$-\frac{1}{2\theta}\cancel{u}\left[3 - \frac{r^2}{2\theta}\right] = \left[\frac{n}{\theta} + \frac{r^2}{4\theta^2}\right]\cancel{u}$$

$$\frac{-3}{2\theta} + \cancel{\frac{r^2}{4\theta^2}} = \frac{n}{\theta} + \cancel{\frac{r^2}{4\theta^2}}$$

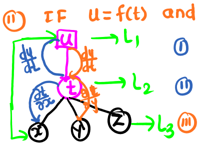
$$\frac{-3}{2\cancel{\theta}} = \frac{n}{\cancel{\theta}} \quad \Rightarrow \quad \boxed{n = -3/2}$$

(i) IF $u=f(x,y,z)$ and $x=f_1(t)$, $y=f_2(t)$ & $z=f_3(t)$



$$\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}$$ → chain Rule of composite fun.

(ii) IF $u=f(t)$ and $t=f(x,y,z)$

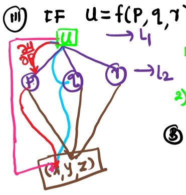


(i) $\frac{\partial u}{\partial x}=\frac{du}{dt}\cdot\frac{\partial t}{\partial x}$

(ii) $\frac{\partial u}{\partial y}=\frac{du}{dt}\cdot\frac{\partial t}{\partial y}$

(iii) $\frac{\partial u}{\partial z}=\frac{du}{dt}\cdot\frac{\partial t}{\partial z}$

(iii) IF $u=f(p,q,r)$ and $p=f_1(x,y,z)$, $q=f_2(x,y,z)$, $r=f_3(x,y,z)$



1) $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial u}{\partial q}\frac{\partial q}{\partial x}+\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$

2) $\frac{\partial u}{\partial y}=\frac{\partial u}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial u}{\partial q}\frac{\partial q}{\partial y}+\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$

3) $\frac{\partial u}{\partial z}=\frac{\partial u}{\partial p}\frac{\partial p}{\partial z}+\frac{\partial u}{\partial q}\frac{\partial q}{\partial z}+\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$

} chain of Rule.

Dr. Uday Kashid , Ph.D.(Mathematics)

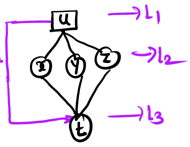
Examples based on Composite function by chain rule

1. If $u=x^3y\,e^z$, where $x=t$, $y=t^2$ and $z=\log t$, then find $\frac{du}{dt}$ at $t=2$
2. If $u=\sin^{-1}(x-y)$, and $x=3t$, $y=4t^3$, then find $\frac{du}{dt}$ [MU -Dec-11]
3. If $z=f(x,y)$, where $x=e^u cos(v)$ and $y=e^u sin(v)$, then show that $y\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}=e^{2u}\frac{\partial z}{\partial y}$ [MU -May-2008, 09]
4. If $z=f(x,y)$, where $x=e^u+e^{-v}$ and $y=e^{-u}-e^{-v}$, then show that $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$ [MU -Dec-09, May-13]
5. If $x=r\cos(\theta)$ and $y=r\sin(\theta)$, Where r and θ are the functions of t then prove that $x\frac{dy}{dt}-y\frac{dx}{dt}=r^2\frac{d\theta}{dt}$
6. If $u=f[(x-y),(y-z),(z-x)]$, then show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$, [MU -Dec-14]
7. If $u=f\left(\frac{y-x}{xy},\frac{z-x}{xz}\right)$, then show that $x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=0$, [MU -May-14]
8. If $g=f(x,y,z)$, and $x=\sqrt{vw}$, $y=\sqrt{uw}$, $z=\sqrt{uv}$, then show that $u\frac{\partial g}{\partial u}+v\frac{\partial g}{\partial v}+w\frac{\partial g}{\partial w}=x\frac{\partial g}{\partial x}+y\frac{\partial g}{\partial y}+z\frac{\partial g}{\partial z}$, [MU -Dec-14] HW →
9. If $u=f[(x^2-y^2),(y^2-z^2),(z^2-x^2)]$, then show that $\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial y}+\frac{1}{z}\frac{\partial u}{\partial z}=0$, [MU -April-21 online exam.]

Dr. Uday Kashid , Ph.D.(Mathematics)

1. If $u=x^3y\,e^z$, where $x=t$, $y=t^2$ and $z=\log t$, then find $\frac{du}{dt}$ at $t=2$

→ $$\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$



$$=3x^2ye^z\cdot(1)+x^3(1)e^z(2t)+x^3ye^z\cdot\frac{1}{t}$$

$$=3t^2t^2(t)+t^3(t)2(t)+t^3t^2t\cdot\frac{1}{t}$$ $z=\log t$, $\boxed{e^z=t}$

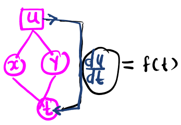
$$\frac{du}{dt}=3t^5+2t^5+t^5=t^5(6)$$

At $t=2$

$$\left(\frac{du}{dt}\right)_{t=2}=(2)^5 6=32\times6=\boxed{192}$$

2. If $u=\sin^{-1}(x-y)$, and $x=3t$, $y=4t^3$, then find $\frac{du}{dt}$ [MU -Dec-11]

→ $$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$$ → chain Rule (1)



$u=f(t)$

But $u=\sin^{-1}(x-y)$

** $$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-(x-y)^2}}\cdot\frac{\partial}{\partial x}[x-y]=\frac{1}{\sqrt{1-(x-y)^2}}\ (1)$$

$$\frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-(x-y)^2}}\cdot\frac{\partial}{\partial y}(x-y)=\frac{(-1)}{\sqrt{1-(x-y)^2}}$$

$$\frac{du}{dt}=\frac{1}{\sqrt{1-(x-y)^2}}(3)-\frac{1}{\sqrt{1-(x-y)^2}}(12t^2)=\frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}}=\frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

Dr. Uday Kashid , Ph.D.(Mathematics)

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$$=\frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}}=\frac{3(1-4t^2)}{\sqrt{1-8t^2+16t^4-t^2+8t^4-16t^6}}$$

$$=\frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2-t^2(1-8t^2+16t^4)}}=\frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2-t^2(1-4t^2)^2}}$$

$$=\frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2[1-t^2]}}=\frac{3(1-4t^2)}{(1-4t^2)\sqrt{1-t^2}}=\frac{3}{\sqrt{1-t^2}}=\frac{du}{dt}$$

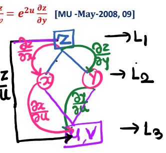
3. If $z=f(x,y)$, where $x=e^u cos(v)$ and $y=e^u sin(v)$, then show that $y\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}=e^{2u}\frac{\partial z}{\partial y}$ [MU -May-2008, 09]

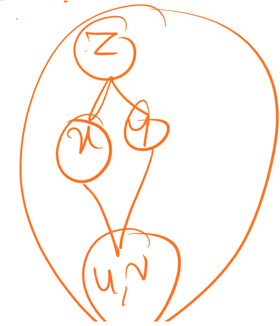
$x^2+y^2=e^{2u}\to(v)$

$\frac{y}{x}=\tan v\to(u)$

→ (i) $\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$ — (I)

(ii) $\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$ — (II)





But $x=e^u\cos v$ and $y=e^u\sin v$

$\frac{\partial x}{\partial u}=e^u\cos v=x$, $\frac{\partial y}{\partial u}=e^u\sin v=y$

$\frac{\partial x}{\partial v}=-e^u\sin v=-y$, $\frac{\partial y}{\partial v}=e^u\cos v=x$

$$\frac{\partial z}{\partial u}=\left(\frac{\partial z}{\partial x}\right)e^u\cos v+\left(\frac{\partial z}{\partial y}\right)e^u\sin v$$

Dr. Uday Kashid , Ph.D.(Mathematics)

multiple by y

$$y\frac{\partial z}{\partial u}=y\left(\frac{\partial z}{\partial x}\right)e^u\cos v+y\left(\frac{\partial z}{\partial y}\right)e^u\sin v$$ — (A)

$$\frac{\partial z}{\partial v}=\left(\frac{\partial z}{\partial x}\right)(-e^u\sin v)+\left(\frac{\partial z}{\partial y}\right)e^u\cos v$$

multiple by x

$y\frac{\partial z}{\partial u} = y\left(\frac{\partial z}{\partial x}\right)e^u\cos v + y\left(\frac{\partial z}{\partial y}\right)\ldots$

$\frac{\partial z}{\partial v} = \left(\frac{\partial z}{\partial x}\right)(-e^u\sin v) + \left(\frac{\partial z}{\partial y}\right)e^u\cos v$

multiple by x

$x\frac{\partial z}{\partial v} = x\left(\frac{\partial z}{\partial x}\right)(-e^u\sin v) + x\left(\frac{\partial z}{\partial y}\right)e^u\cos v$ — (B)

Add (A) & (B)

$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left[e^u\sin v\cdot e^u\cos v - e^u\cos v\cdot e^u\sin v\right] + \frac{\partial z}{\partial y}\left[e^u\sin v\cdot e^u\sin v + e^u\cos v\cdot e^u\cos v\right]$

$= 0 + \frac{\partial z}{\partial y}\left[e^{2u}\sin^2 v + e^{2u}\cos^2 v\right] = e^{2u}(\sin^2 v+\cos^2 v)\frac{\partial z}{\partial y}$

$$\boxed{y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}}$$

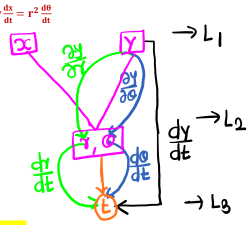
5. If $x = r\cos(\theta)$ and $y = r\sin(\theta)$, Where r and θ are the functions of t then prove that $x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt}$

→ (I) $\frac{dy}{dt} = \left(\frac{\partial y}{\partial r}\right)\cdot\left(\frac{dr}{dt}\right) + \left(\frac{\partial y}{\partial \theta}\right)\cdot\left(\frac{d\theta}{dt}\right)$ — (I)

(II) $\frac{dx}{dt} = \left(\frac{\partial x}{\partial r}\right)\left(\frac{dr}{dt}\right) + \left(\frac{\partial x}{\partial \theta}\right)\left(\frac{d\theta}{dt}\right)$ — (II)

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$



Dr. Uday Kashid , Ph.D.(Mathematics)

$\frac{dy}{dt} = (\sin\theta)\left(\frac{dr}{dt}\right) + r(\cos\theta)\frac{d\theta}{dt}$

Multiple by x i.e. $\boxed{x = r\cos\theta}$

$x\frac{dy}{dt} = (r\cos\theta)\sin\theta\left(\frac{dr}{dt}\right) + (r\cos\theta)\, r\cos\theta\frac{d\theta}{dt} = r\sin\theta\cos\theta\frac{dr}{dt} + r^2\cos^2\theta\frac{d\theta}{dt}$ — (A)

$\frac{dx}{dt} = (\cos\theta)\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}$

multiple by y i.e. $\boxed{y = r\sin\theta}$

$y\frac{dx}{dt} = (r\sin\theta)\cos\theta\frac{dr}{dt} - (r\sin\theta)\, r\sin\theta\frac{d\theta}{dt} = r\sin\theta\cos\theta\frac{dr}{dt} - r^2\sin^2\theta\frac{d\theta}{dt}$ — (B)

Substract eq (A) − (B)

$x\frac{dy}{dt} - y\frac{dx}{dt} = r\sin\theta\cos\theta\frac{dr}{dt} + r^2\cos^2\theta\frac{d\theta}{dt} - r\sin\theta\cos\theta\frac{dr}{dt} + r^2\sin^2\theta\frac{d\theta}{dt}$

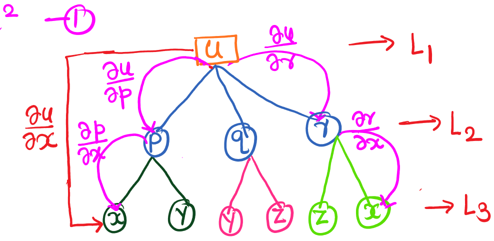
$= r^2\frac{d\theta}{dt}\left[\cos^2\theta + \sin^2\theta\right] = \boxed{r^2\frac{d\theta}{dt}}$

9. If $u = f[(x^2-y^2),(y^2-z^2),(z^2-x^2)]$, then show that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$, **[MU -April-21 online exam.]**

→ Let $p = x^2 - y^2$, $q = y^2 - z^2$, $r = z^2 - x^2$ — (I)

$u = f(p, q, r)$ — (II)

Dr. Uday Kashid , Ph.D.(Mathematics)



(I) $\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial p}\right)\left(\frac{\partial p}{\partial x}\right) + \left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial r}{\partial x}\right)$

$= \left(\frac{\partial u}{\partial p}\right)(2x) + \left(\frac{\partial u}{\partial r}\right)(-2x)$

$\frac{1}{x}\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial p} - 2\frac{\partial u}{\partial r}$ — (III)

(II) $\frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial p}\right)\left(\frac{\partial p}{\partial y}\right) + \left(\frac{\partial u}{\partial q}\right)\left(\frac{\partial q}{\partial y}\right)$ → Chain Rule :-

$= \left(\frac{\partial u}{\partial p}\right)(-2y) + \left(\frac{\partial u}{\partial q}\right)(2y)$

$\frac{1}{y}\frac{\partial u}{\partial y} = -2\frac{\partial u}{\partial p} + 2\frac{\partial u}{\partial q}$ — (IV)

(III) $\frac{\partial u}{\partial z} = \left(\frac{\partial u}{\partial q}\right)\left(\frac{\partial q}{\partial z}\right) + \left(\frac{\partial u}{\partial r}\right)\left(\frac{\partial r}{\partial z}\right)$

$\frac{\partial u}{\partial z} = \left(\frac{\partial u}{\partial q}\right)(-2z) + \left(\frac{\partial u}{\partial r}\right)(2z)$ ⟹ $\frac{1}{z}\frac{\partial u}{\partial z} = -2\frac{\partial u}{\partial q} + 2\frac{\partial u}{\partial r}$ — (V)

Add eqn (III) + (IV) + (V)

$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 2\frac{\partial u}{\partial p} - 2\frac{\partial u}{\partial r} - 2\frac{\partial u}{\partial p} + 2\frac{\partial u}{\partial q} - 2\frac{\partial u}{\partial q} + 2\frac{\partial u}{\partial r} = 0$

7. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$, **[MU -May-14]**

→ Let $p = \frac{y-x}{xy} = \frac{y}{xy} - \frac{x}{xy} = \frac{1}{x} - \frac{1}{y}$ — (I) {Imp step}

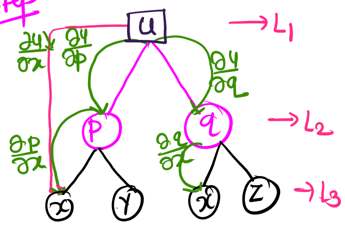
u → L1

7. If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then show that $x^2\frac{\sigma u}{\partial x} + y^2\frac{\sigma u}{\partial y} + z^2\frac{\sigma u}{\partial z} = 0$, **[MU -May-14]**

→ Let $p = \frac{y-x}{xy} = \frac{y}{xy} - \frac{x}{xy} = \frac{1}{x} - \frac{1}{y}$ —(I)

$q = \frac{z-x}{xz} = \frac{z}{xz} - \frac{x}{xz} = \frac{1}{x} - \frac{1}{z}$ —(II)

Imp step



$u = f(p, q)$

Dr. Uday Kashid , Ph.D.(Mathematics)

(I) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x}$

$= \frac{\partial u}{\partial p}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial q}\left(-\frac{1}{x^2}\right)$

$x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial p} - \frac{\partial u}{\partial q}$ —(III)

(II) $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} = \frac{\partial u}{\partial p}\left(\frac{1}{y^2}\right)$

$y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}$ —(IV)

(III) $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} = \frac{\partial u}{\partial q}\left(\frac{1}{z^2}\right)$ ⇒ $z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial q}$ —(V)

Add eq^n (III) + (IV) + (V)

$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial p} - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} = 0$

Dr. Uday Kashid , Ph.D.(Mathematics)

Homogeneous function of degree 'n' :

A function $f(x_1, x_2, x_3, \ldots\ldots x_k)$ is said to be Homogeneous function of degree 'n' (n is a rational number) if it satisfies the following indentity.

For $x_1 = \lambda X_1,\ x_2 = \lambda X_2,\ x_3 = \lambda X_3, \ldots\ldots, x_k = \lambda X_k$

$f(x_1, x_2, x_3, \ldots\ldots x_k) = \lambda^n f(X_1, X_2, X_3, \ldots\ldots X_k)$ where $\lambda \neq 0$ and it is called as parameter.

Examine where following functions are not homogeneous or homogeneous and if yes , find its degree.

Homogeneous of deg = $-7/12$

1. $u = \dfrac{x^{2/3} - y^{2/3}}{x^{5/4} + y^{5/4}} = f(x,y) \Rightarrow x = \lambda x,\ y = \lambda y\quad u = \dfrac{(\lambda x)^{2/3} - (\lambda y)^{2/3}}{(\lambda x)^{5/4} + (\lambda y)^{5/4}} = \dfrac{\lambda^{2/3}[x^{2/3} - y^{2/3}]}{\lambda^{5/4}[x^{5/4} + y^{5/4}]} = \lambda^{\frac{2}{3} - \frac{5}{4}}\left[\dfrac{x^{2/3} - y^{2/3}}{x^{5/4} + y^{5/4}}\right] = \lambda^{-\frac{7}{12}} u = \lambda^n u \Rightarrow u$ is Homo. function of degree $= -\frac{7}{12}$

Non-Homogeneous

2. $u = \dfrac{x^3 - y^3}{x^{2/3} + y^{3/2}} = \begin{cases} x = \lambda x \\ y = \lambda y \end{cases} \Rightarrow u = \dfrac{\lambda^3 x^3 - \lambda^3 y^3}{\lambda^{2/3} x^{2/3} + \lambda^{3/2} y^{3/2}} = \dfrac{\lambda^3 (x^3 - y^3)}{\lambda^{2/3}[x^{2/3} + \lambda^{5/6} y^{3/2}]} = \lambda^{\frac{7}{3}} \dfrac{(x^3 - y^3)}{[x^{2/3} + \lambda^{5/6} y^{3/2}]} \neq \lambda^n u$

Non-Homo

3. $u = \tan^{-1}\left(\dfrac{x^3 - y^3}{x + y}\right) = \tan^{-1}\left[\dfrac{\lambda^3 x^3 - \lambda^3 y^3}{\lambda x + \lambda y}\right] = \tan^{-1}\left[\dfrac{\lambda^3 (x^3 - y^3)}{\lambda (x + y)}\right] = \tan^{-1}\left[\lambda^2 \dfrac{(x^3 - y^3)}{(x + y)}\right] \neq \lambda^0 u$

Homogen deg = n = 0

4. $u = \sin^{-1}\left(\dfrac{x - y}{\sqrt{x^2 + y^2}}\right) = \sin^{-1}\left[\dfrac{\lambda (x - y)}{\lambda \sqrt{x^2 + y^2}}\right] = \lambda^0 \sin^{-1}\left[\dfrac{x - y}{\sqrt{x^2 + y^2}}\right] = \lambda^n u$

Non Homo

5. $u = \dfrac{x^2 y^2 z^2}{x^2 + y^2 + z^2} + \cos\left(\dfrac{xy + yz}{x^2 + y^2 + z^2}\right) = v + w = \lambda^4 v + \lambda^0 w \neq \lambda^n u$

Dr. Uday Kashid , Ph.D.(Mathematics)

$u = f(x, y, z) \quad u = x^n f\left(\frac{y}{x}, \frac{z}{x}\right) = y^n f\left(\frac{x}{y}, \frac{z}{y}\right)$

Note : Gate exam.

1. If $u = f(x, y)$ be a homogeneous function of degree 'n', then 'u' can be written as $u = x^n f\left(\frac{y}{x}\right)$ or $u = y^n f\left(\frac{x}{y}\right)$

Euler's Theorem on Homogeneous functions : (IMP)

Dr. Uday Kashid , Ph.D.(Mathematics)

Thm-1 - If $u = f(x, y)$ be a homogeneous function of degree 'n', then $\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = n\,u$

Thm-2- If $u = f(x, y)$ be a homogeneous function of degree 'n', then $\left(x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2 \frac{\partial^2 u}{\partial y^2}\right) = n(n-1)u$

Thm-3- If z is a a homogeneous function of x and y of degree 'n', and $z = f(u)$, then

$u =$ non Homo

- $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n\dfrac{f(u)}{f'(u)} = G(u)$
- $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = G(u)\,[G'(u) - 1]$

Note → $u = \sin^{-1}, \cos^{-1}, \tan^{-1}, \cot^{-1}(z)$, $\log$, $\operatorname{cosec}^{-1}$, $\sec^{-1}$,

$z = f(u) \Rightarrow u = f^{-1}(z)$ Inverse

$G'(u) = \dfrac{d}{du} G(u)$

1. If $u = \dfrac{\sqrt{x} + \sqrt{y}}{x + y}$, Then find $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ [MU -May-13]

① $\dfrac{\partial u}{\partial x} = \dfrac{(x+y)\frac{1}{2\sqrt{x}} - (\sqrt{x} + \sqrt{y})(1)}{(x+y)^2}$

$x\dfrac{\partial u}{\partial x} = \dfrac{(x+y)\frac{\sqrt{x}}{2} - (\sqrt{x} + \sqrt{y})x}{(x+y)^2}$

$\dfrac{\partial u}{\partial y} = \dfrac{(x+y)\frac{1}{2\sqrt{y}} - (\sqrt{x} + \sqrt{y})(1)}{(x+y)^2} =$

$y\dfrac{\partial u}{\partial y} = \dfrac{(x+y)\frac{\sqrt{y}}{2} - y(\sqrt{x} + \sqrt{y})}{(x+y)^2}$

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{\frac{(x+y)}{2}(\sqrt{x} + \sqrt{y}) - (\sqrt{x} + \sqrt{y})(x+y)}{(x+y)^2} = \dfrac{(\sqrt{x} + \sqrt{y})\left(\frac{-(x+y)}{2}\right)}{(x+y)^2} = -\dfrac{1}{2}\dfrac{(\sqrt{x} + \sqrt{y})}{(x+y)} = \boxed{-\frac{1}{2}u}$

By Euler's Thm

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$

$u = \dfrac{\sqrt{\lambda x} + \sqrt{\lambda y}}{(\lambda x + \lambda y)} = \dfrac{\sqrt{\lambda}(\sqrt{x} + \sqrt{y})}{\lambda (x+y)} = \lambda^{-1/2} u$

u is Homo. fun of deg $= n = -1/2$

$\boxed{x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\frac{1}{2}u}$

Dr. Uday Kashid , Ph.D.(Mathematics)

Thm-1 - If $u = f(x, y)$ be a homogeneous function of degree 'n', then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,u$

→ We have given $u = f(x, y)$ is homogeneous fun of degree $= n$

$\Rightarrow \boxed{u = x^n f(y/x)}$ —①

① $\dfrac{\partial u}{\partial x} = \dfrac{\partial}{\partial x}\left[x^n f(y/x)\right] = \dfrac{\partial}{\partial x}(x^n) f(y/x) + x^n \dfrac{\partial}{\partial x}[f(y/x)]$

$= n x^{n-1} f(y/x) + x^n f'(y/x) \cdot \dfrac{\partial}{\partial x}\left(\dfrac{y}{x}\right)$

$= n x^{n-1} f(y/x) + x^n f'(y/x)\left[\dfrac{-y}{x^2}\right]$

$\dfrac{\partial}{\partial x}[f(y/x)]$ put $t = y/x \Rightarrow f(x, y)$

$f(y/x) = f(t)$

$\dfrac{\partial}{\partial x}[f(y/x)]$

multiple by x to B.S.

$x\dfrac{\partial u}{\partial x} = n x^n f(y/x) - x^{n-1} y f'(y/x)$ —Ⓐ

(ii) $\dfrac{\partial u}{\partial y} = \dfrac{\partial}{\partial y}\left[x^n f(y/x)\right] = x^n \cdot \dfrac{\partial}{\partial y}[f(y/x)] = x^n f'(y/x) \cdot \dfrac{\partial}{\partial y}[y/x]$

$= x^n f'(y/x)\left(\dfrac{1}{x}\right) = x^{n-1} f'(y/x)$

$$= x^n f'(y/x)\left(\frac{1}{x}\right) = x^{n-1} f'(y/x)$$

multiple by y to B.S.

$$y\frac{\partial u}{\partial y} = x^{n-1}\cdot y\, f'(y/x) \quad \text{—} (B)$$

Add (A) + (B)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\, x^n f(y/x) - \cancel{x^{n-1} y\, f'(y/x)} + \cancel{x^{n-1} y\, f'(y/x)}$$

Dr. Uday Kashid , Ph.D.(Mathematics)

$$= n\, x^n f(y/x)$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu} \longrightarrow \text{Euler's Thm for Homogeneous Fun of degree } 'n'. \text{ of First order}$$

— (C)

Thm-2- If $u = f(x,y)$ be a homogeneous function of degree 'n', then $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

→ From equ^n (C) $\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$ — (C)

$u = f(x,y) = y^2 \sin x$

$\frac{\partial u}{\partial x} = \phi(x,y) = y^2\cos x$

(1) diff. eq^n (C) w.r.t x to B.S

$$\frac{\partial}{\partial x}\left[x\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}\left[y\frac{\partial u}{\partial y}\right] = n\frac{\partial}{\partial x}[u]$$

$$1\frac{\partial u}{\partial x} + x\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] + y\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] = n\frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = n\frac{\partial u}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = n\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} = (n-1)\frac{\partial u}{\partial x}$$

Multiple by x to B.S.

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} = (n-1)\,x\frac{\partial u}{\partial x} \quad \text{—} (D)$$

(2) Diff equ^n (C) w.r.t. y to B.S

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{—} (C)$$

$u = f(x,y)$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \rightarrow x, y$

$$\frac{\partial}{\partial y}\left[x\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[y\frac{\partial u}{\partial y}\right] = n\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial y\partial x} + 1\frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} = (n-1)\frac{\partial u}{\partial y}$$

multiple by y to B.S.

$$xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\,y\frac{\partial u}{\partial y} \quad \text{—} (E)$$

Add equ^n (D) + (E)

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} + xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\,x\frac{\partial u}{\partial x} + (n-1)\,y\frac{\partial u}{\partial y}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

Dr. Uday Kashid , Ph.D.(Mathematics)

$$= (n-1)\,nu = n(n-1)u \quad \rightarrow \text{By eq}^n \text{ (C)}$$

Thm-3- If z *is a* a homogeneous function of x *and* y of degree 'n', and $z = f(u)$, then

- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = G(u)$

Dr. Uday Kashid , Ph.D.(Mathematics)

- $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = G(u)\,[G'(u) - 1]$

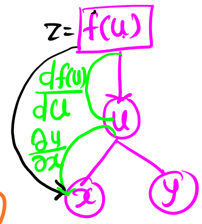
→ We have $z = f(u)$ and z is homo. fun in x & y with degree $= n$.

→ We have $z = f(u)$ and $\boxed{z \text{ is homo. fun in } x \,\&\, y \text{ with degree } n}$

By Euler's Thm (I)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \quad \text{— (I)}$$

$$x\frac{\partial f(u)}{\partial x} + y\frac{\partial f(u)}{\partial y} = n f(u) \qquad \because z = f(u)$$



$$x\frac{df(u)}{du}\cdot\frac{\partial u}{\partial x} + y\frac{df(u)}{du}\cdot\frac{\partial u}{\partial y} = n f(u)$$

$$x f'(u)\frac{\partial u}{\partial x} + y f'(u)\frac{\partial u}{\partial y} = n f(u)$$

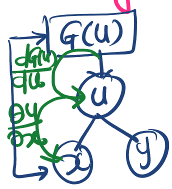
$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = G(u)} \quad \text{— (II)}$$

✓

Diff eqn (II) w.r.t. x

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \to (x, y)$$

$$\frac{\partial}{\partial x}\left[x\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}\left[y\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial x}G(u)$$



$$1\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{dG(u)}{du}\cdot\frac{\partial u}{\partial x}$$

multiple by x to B.S.

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = G'(u)\,x\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial x}$$

$$= [G'(u) - 1]\,x\frac{\partial u}{\partial x} \quad \text{— (III)}$$

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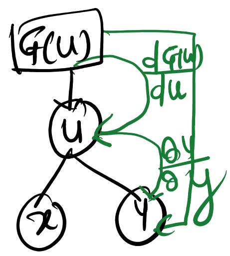
diff eqn (II) w.r.t. y



$$\frac{\partial}{\partial y}\left[x\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[y\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial y}G(u)$$

$$x\frac{\partial^2 u}{\partial y \partial x} + 1\frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{dG(u)}{du}\cdot\frac{\partial u}{\partial y}$$

multiple by y to B.S.

$$xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} = G'(u)\,y\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial y} = [G'(u) - 1]\,y\frac{\partial u}{\partial y} \quad \text{— (IV)}$$

Add eqn (III) + (IV)

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = [G'(u) - 1]\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$= [G'(u) - 1]\,G(u) \qquad \because \text{By eqn (II)}$$

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = G(u)[G'(u) - 1]} \qquad \therefore G(u) = n\frac{f(u)}{f'(u)}$$

Dr. Uday Kashid , Ph.D.(Mathematics)

Examples based on Euler's Thm

20 January 2022 12:07

Examples based on Cartesian Co-Ordinates

Dr. Uday Kashid , Ph.D.(Mathematics)

1. If $u = \frac{\sqrt{x}+\sqrt{y}}{x+y}$, Then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ [MU -May-13]
2. If $u = (8x^2 + y^2)(logx - logy)$ Then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ $= 2u$ [MU -Dec-15]
3. If $u = f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{logx-logy}{x^2}$, Then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + 2u = 0$
4. If $u = f\left(\frac{y}{x}\right) + \sqrt{x^2 + y^2}$, Then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ [MU -May-08]
5. If $u = log(x^2 + y^2) + \frac{x^2+y^2}{x+y} - 2log(x + y)$, Then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ [MU -May-11]
6. If $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + x^4 \tan^{-1}\left(\frac{y}{x}\right)$, Then find $\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right) + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$ at $x = 1$ & $y = 1$ [MU -May10]
7. If $u = \frac{(x^2+y^2)^n}{2n(2n-1)} + xf\left(\frac{y}{x}\right) + \emptyset\left(\frac{x}{y}\right)$, Then show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (x^2 + y^2)^n$
8. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, $x \neq y$, Then show that 1) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = sin2u$
9. If $u = \tan^{-1}\left(\frac{x^3+y^3}{2x+3y}\right)$, Then show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1 - 4sin^2u)sin2u$ [MU -May-09]
10. If $u = \operatorname{cosec}^{-1}\sqrt{\left(\frac{\frac{1}{x^{2}}+\frac{1}{y^{2}}}{\frac{1}{x^{3}}+\frac{1}{y^{3}}}\right)}$, Then show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}[tan^2u + 13]$ [MU -May-07,08 Dec-09,18]
11. If $u = \sin^{-1}\sqrt{\left(\frac{x^2+y^2}{x+y}\right)}$, Then show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{4}[tan^2u - 1]$ [MU -Dec-09,18]
12. If $u = log\left(\frac{x+y}{\sqrt{x^2+y^2}}\right) + \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ Then show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{sinw\cos 2w}{4cos^3w}$ where $w = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ [MU -April-21 online exam.]

$u = (8x^2+y^2)\log(x/y) = \lambda^2[8x^2+y^2]\log\left[\frac{\lambda x}{\lambda y}\right] = \lambda^2 u$

$\Rightarrow u = v + w \Rightarrow v = f(y/x) = \lambda^0 f(y/x) \rightarrow$ Homo. Fun of deg $= n_1 = 0$

$w = \sqrt{x^2+y^2} \Rightarrow \lambda\sqrt{x^2+y^2} \Rightarrow$ Homo. fun of deg $n_2 = 1$

$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 0$ — (I)

$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = nw = w$ — (II)

$x\frac{\partial}{\partial x}(v+w) + y\frac{\partial}{\partial y}(v+w) = 0 + w$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sqrt{x^2+y^2}$

$u = v + w$ $= \lambda^3 v + \lambda^4 w$ $\neq \lambda^n . u \rightarrow$

If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, $x \neq y$, Then show that 1) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = sin2u$

$\rightarrow$ $u = \tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$ $\Rightarrow x = \lambda x,\ y = \lambda y$ $u = \tan^{-1}\left[\frac{\lambda^3(x^3+y^3)}{\lambda(x-y)}\right] = \lambda^{\circ}\tan^{-1}\left[\lambda^2\frac{(x^3+y^3)}{(x-y)}\right] \neq \lambda^n u$

But Assume $z = f(x,y) = \frac{x^3+y^3}{x-y}$ By $x = \lambda x$ & $y = \lambda y$

$z = \frac{\lambda^3(x^3+y^3)}{\lambda(x-y)} = \lambda^2 . z$ $\Rightarrow$ z is homo. fun of deg $= n = 2$

$u = \tan^{-1}[z]$ $\Rightarrow$ $\boxed{z = \tan u = f(u)} \rightarrow$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 2\frac{\tan u}{\sec^2 u} = 2\frac{\sin u}{\cos u}\cos^2 u = 2\sin u\cos u$
$= \sin(2u)$

Thm-3
(1) $z = f(u)$
$u = f^{-1}(z) \Rightarrow$ Non Homo
(2) $z = f(x,y) \rightarrow$ Homo. Fun of deg $= n$

1. If $u = \frac{\sqrt{x}+\sqrt{y}}{x+y}$, Then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ [MU -May-13]

Dr. Uday Kashid , Ph.D.(Mathematics)

$\rightarrow$ put $x = \lambda x$ & $y = \lambda y$

$u = \frac{\sqrt{\lambda}\sqrt{x} + \sqrt{\lambda}\sqrt{y}}{\lambda x + \lambda y} = \frac{\sqrt{\lambda}(\sqrt{x}+\sqrt{y})}{\lambda(x+y)} = \lambda^{-1/2}\frac{(\sqrt{x}+\sqrt{y})}{(x+y)} = \lambda^{-1/2}u \Rightarrow$ Homo. Fun of deg $= n = -1/2$

By Euler's Thm-1 $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = -\frac{1}{2}u$

By Euler's Thm-2 $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = -\frac{1}{2}\left(-\frac{1}{2}-1\right)u = \frac{3}{4}u$

2. $u = (8x^2 + y^2)(logx - logy)$ Then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ [MU -Dec-15]

$\rightarrow$ $u = (8x^2+y^2)\log\left(\frac{x}{y}\right) \Rightarrow u = \lambda^2(8x^2+y^2)\log\left[\frac{\lambda x}{\lambda y}\right] = \lambda^2 u \Rightarrow$ Homo. Fu of deg $= n = 2$

$\Rightarrow$ $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 2u$

Dr. Uday Kashid , Ph.D.(Mathematics)

3. If $u = f(x, y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{logx-logy}{x^2}$, Then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + 2u = 0$

$\rightarrow$ $u = \frac{1}{\lambda^2 x^2} + \frac{1}{\lambda x\lambda y} + \frac{\log\left(\frac{\lambda x}{\lambda y}\right)}{\lambda^2 x^2} = \frac{1}{\lambda^2}\left[\frac{1}{x^2} + \frac{1}{xy} + \frac{\log(x/y)}{x^2}\right] = \lambda^{-2}u = \lambda^n u$

u is homo. Fun of deg $= n = -2$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = -2u$ $\Rightarrow$ $\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + 2u = 0}$

4. If $u = f\left(\frac{y}{x}\right) + \sqrt{x^2 + y^2}$, Then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ [MU -May-08]

$\rightarrow$ let $u = v + w = f\left(\frac{\lambda y}{\lambda x}\right) + \sqrt{\lambda^2 x^2 + \lambda^2 y^2} = \lambda^0 f(y/x) + \lambda\sqrt{x^2+y^2} = \lambda^0 v + \lambda w$

$\neq \lambda^n u$ $\rightarrow$ u is non-homo. Fun

But v and w are homo. funs of degree 0 and 1 resp.

By Euler's Thm 1 $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 0v = 0$ —(I)

Dr. Uday Kashid , Ph.D.(Mathematics)

$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=nw=1w=w$ — (ii)

Add (i) & (ii)

$x\frac{\partial}{\partial x}(v+w)+y\frac{\partial}{\partial y}(v+w)=0+w$

$$\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=w=\sqrt{x^2+y^2}}$$

5. If $u=log(x^2+y^2)+\frac{x^2+y^2}{x+y}-2log(x+y)$, Then find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$ **[MU -May-11]**

$\rightarrow u=\log\left[\frac{(x^2+y^2)}{(x+y)^2}\right]+\frac{x^2+y^2}{x+y}=v+w$

$=\log\left[\frac{\lambda^2(x^2+y^2)}{\lambda^2(x+y)^2}\right]+\frac{\lambda^2(x^2+y^2)}{\lambda(x+y)}=\lambda^0\log\left[\frac{x^2+y^2}{x+y}\right]+\lambda\frac{(x^2+y^2)}{x+y}=\lambda^0v+\lambda w$

Dr. Uday Kashid , Ph.D.(Mathematics)

$\neq\lambda^n u$

$\Rightarrow$ u is non-homo Fun but v and w are homo. Funs of degree 0 & 1 resp.

By Euler's Thm (1) $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=nv=0$ — (i)

$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=nw=1w=w$ — (ii)

Add (i) & (ii) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0+w=\left(\frac{x^2+y^2}{x+y}\right)$

6. If $u=x^3\sin^{-1}\left(\frac{y}{x}\right)+x^4\tan^{-1}\left(\frac{y}{x}\right)$, Then find $\left(x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\,\partial y}+y^2\frac{\partial^2u}{\partial y^2}\right)+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$, at $x=1$ & $y=1$ **[MU -May10]**

$\rightarrow u=v+w \Rightarrow u=\lambda^3x^3\sin^{-1}\left(\frac{\lambda y}{\lambda x}\right)+\lambda^4x^4\tan^{-1}\left[\frac{\lambda y}{\lambda x}\right]$

$u=\lambda^3v+\lambda^4w \neq\lambda^n u$

$\Rightarrow$ u is non-homo. Fun. But v and w are homo Funs of degree 3 and 4 resp.

By Euler's (1) Thm $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=nv=3v$ — (i)

Dr. Uday Kashid , Ph.D.(Mathematics)

$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=nw=4w$ — (ii)

Add (i) & (ii) $\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3v+4w}$ — (iii)

By Euler's Thm (2) $x^2\frac{\partial^2v}{\partial x^2}+2xy\frac{\partial^2v}{\partial x\partial y}+y^2\frac{\partial^2v}{\partial y^2}=n(n-1)v=3(3-1)=6v$ — (iv)

$x^2\frac{\partial^2w}{\partial x^2}+2xy\frac{\partial^2w}{\partial x\partial y}+y^2\frac{\partial^2w}{\partial y^2}=n(n-1)w=4(4-1)w=12w$ — (v)

Add (iv) & (v)

$x^2\frac{\partial^2}{\partial x^2}(u)+2xy\frac{\partial^2u}{\partial x\partial y}+y^2\frac{\partial^2u}{\partial y^2}=6v+12w$ — (vi)

Add (iii) & (vi)

$x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\partial y}+y^2\frac{\partial^2u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=6v+12w+3v+4w$

$$\boxed{=9v+16w=9x^3\sin^{-1}(y/x)+16x^4\tan^{-1}(y/x)}$$

At $x=1$ & $y=1$

$=9\sin^{-1}(1)+16\tan^{-1}(1)=9\left(\frac{\pi}{2}\right)+16\frac{\pi}{4}$

$$\boxed{=\frac{17\pi}{2}}$$

Dr. Uday Kashid , Ph.D.(Mathematics)

9. If $u=\tan^{-1}\left(\frac{x^3+y^3}{2x+3y}\right)$, Then show that $x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\,\partial y}+y^2\frac{\partial^2u}{\partial y^2}=(1-4sin^2u)sin2u$

9. If $u=\tan^{-1}\left(\frac{x^3+y^3}{2x+3y}\right)$, Then show that $x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\,\partial y}+y^2\frac{\partial^2u}{\partial y^2}=(1-4\sin^2u)\sin2u$

→ put $x=\lambda x$ & $y=\lambda y$

$$u=\tan^{-1}\left[\frac{\lambda^3(x^3+y^3)}{\lambda(2x+3y)}\right]=\tan^{-1}\left(\lambda^2\left(\frac{x^3+y^3}{2x+3y}\right)\right)\neq\lambda^n u$$

Assume $z=f(x,y)=\frac{x^3+y^3}{2x+3y}$ is homo. Fun of deg $=n=2$

Then $u=\tan^{-1}(z)\Rightarrow \boxed{z=\tan u}=f(u)$ — (I)

Hence, By Euler's Thm ③

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}=2\frac{\tan u}{\sec^2u}=2\frac{\sin u}{\cos u}\times\cos^2u$$

$$=2\sin u\cos u=\sin(2u)=G(u)=n\frac{f(u)}{f'(u)}\quad\text{— (II)}$$

Also $x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\partial y}+y^2\frac{\partial^2u}{\partial y^2}=G(u)[G'(u)-1]$

$\boxed{G'(u)=\frac{d}{du}G(u)}$

$$=\sin(2u)[2\cos2u-1]$$

$\boxed{1-\cos2u=2\sin^2u}$

$\boxed{\because\ \cos2u=1-2\sin^2u}$

$$=\sin(2u)[2(1-2\sin^2u)-1]$$

$$=\sin(2u)[2-4\sin^2u-1]$$

$$\boxed{=\sin(2u)[1-4\sin^2u]}$$

Ans

Dr. Uday Kashid , Ph.D.(Mathematics)

7. If $u=\frac{(x^2+y^2)^n}{2n(2n-1)}+xf\left(\frac{y}{x}\right)+\emptyset\left(\frac{x}{y}\right)$, Then show that $x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\,\partial y}+y^2\frac{\partial^2u}{\partial y^2}=(x^2+y^2)^n$

→ $u=p+q+r \Rightarrow u=\frac{(\lambda^2x^2+\lambda^2y^2)^n}{2n(2n-1)}+\lambda x f\left(\frac{\lambda y}{\lambda x}\right)+\phi\left(\frac{\lambda x}{\lambda y}\right)$

$$u=\lambda^{2n}\frac{(x^2+y^2)^n}{2n(2n-1)}+\lambda\, x f(y/x)+\lambda^0\phi(x/y)\neq\lambda^n u$$

$$u=\lambda^{2n}p+\lambda q+\lambda^0 r$$

⇒ u is non homo-Fun but $p, q, \&\ r$ are homo fun of degree $2n$, 1 and 0 resp.

By Euler's Thm ②

$$x^2\frac{\partial^2p}{\partial x^2}+2xy\frac{\partial^2p}{\partial x\partial y}+y^2\frac{\partial^2p}{\partial y^2}=n(n-1)p=2n(2n-1)p\quad\text{— (I)}$$

$$x^2\frac{\partial^2q}{\partial x^2}+2xy\frac{\partial^2q}{\partial x\partial y}+y^2\frac{\partial^2q}{\partial y^2}=n(n-1)q=1(1-1)q=0\quad\text{— (II)}$$

$$x^2\frac{\partial^2r}{\partial x^2}+2xy\frac{\partial^2r}{\partial x\partial y}+y^2\frac{\partial^2r}{\partial y^2}=n(n-1)r=0(0-1)r=0\quad\text{— (III)}$$

Add (I) + (II) + (III)

$$x^2\frac{\partial^2}{\partial x^2}(u)+2xy\frac{\partial^2u}{\partial x\partial y}+y^2\frac{\partial^2u}{\partial y^2}=2n(2n-1)p+0+0$$

$$=2n(2n-1)\frac{(x^2+y^2)^n}{2n(2n-1)}=(x^2+y^2)^n$$

10. If $u=\operatorname{cosec}^{-1}\sqrt{\left(\frac{x^{\frac12}+y^{\frac12}}{x^{\frac13}+y^{\frac13}}\right)}$, Then show that $x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\,\partial y}+y^2\frac{\partial^2u}{\partial y^2}=\frac{\tan u}{144}[\tan^2u+13]$ [MU -Dec-09,18]

If $u=\log\left(\frac{x+y}{\sqrt{x^2+y^2}}\right)+\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ Then show that $x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\,\partial y}+y^2\frac{\partial^2u}{\partial y^2}=-\frac{\sin w\cos2w}{4\cos^3w}$ where $w=\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ [MU -April-21 online

→ $u=v+w\Rightarrow u=\log\left[\frac{\lambda(x+y)}{\lambda\sqrt{x^2+y^2}}\right]+\sin^{-1}\left[\frac{\lambda(x+y)}{\sqrt{\lambda}(\sqrt{x}+\sqrt{y})}\right]$

$\to u = v + w \Rightarrow u = \log\left[\frac{\lambda(x+y)}{\lambda\sqrt{x^2+y^2}}\right] + \sin^{-1}\left[\frac{\lambda(x+y)}{\sqrt{\lambda}(\sqrt{x}+\sqrt{y})}\right]$

$u = \lambda^0 \log\left[\frac{x+y}{\sqrt{x^2+y^2}}\right] + \sin^{-1}\left[\lambda^{1/2}\frac{(x+y)}{\sqrt{x}+\sqrt{y}}\right]$

u is non-Homo Fun But v is Homo. Fun of deg $= n_1 = 0$

By Euler-Thm-② $x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)v = 0(0-1)v = 0$ — ① — (I)

Let $w = \sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

Dr. Uday Kashid , Ph.D.(Mathematics)

Let $z = \frac{x+y}{\sqrt{x}+\sqrt{y}} = f(x,y)$ and Homo. Fun of deg $= n_2 = \frac{1}{2}$

$w = \sin^{-1}[z] \Rightarrow z = f(w) = \sin w \to$

By Euler's Thm-③ $x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = G(w)[G'(w)-1]$ — (II)

where $G(w) = n\frac{f(w)}{f'(w)} = \frac{1}{2}\frac{\sin w}{\cos w} = \frac{1}{2}\tan w$

$G'(w) = \frac{d}{dw}G(w) = \frac{1}{2}\sec^2 w = \frac{1}{2}(\tan^2 w + 1)$

put in (II)

$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = \frac{1}{2}\frac{\sin w}{\cos w}\left[\frac{1}{2}\tan^2 w + \frac{1}{2} - 1\right] = \frac{1}{2}\frac{\sin w}{\cos w}\left[\frac{1}{2}\frac{\sin^2 w}{\cos^2 w} - \frac{1}{2}\right]$

$= \frac{1}{4}\frac{\sin w}{\cos w}\left[\frac{\sin^2 w - \cos^2 w}{\cos^2 w}\right] = \frac{1}{4}\frac{\sin w}{\cos^3 w}\left[-\cos 2w\right] = -\frac{1}{4}\frac{\sin w}{\cos^3 w}\cos(2w)$ — (III)

Now add eq (I) & (II)

$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} + x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = 0 - \frac{1}{4}\frac{\sin w\cos(2w)}{\cos^3 w}$

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin w\cos(2w)}{4\cos^3 w}}$$

Dr. Uday Kashid , Ph.D.(Mathematics)

10. If $u = \operatorname{cosec}^{-1}\sqrt{\left(\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}\right)}$, Then show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}[\tan^2 u + 13]$ [MU -Dec-09,18] HW.

SUMMARY

1) Partial derivative
2) composite Rule :-
3) Euler's Thm ①, ② & ③ with proofs
4) Examples on Euler's Thm.