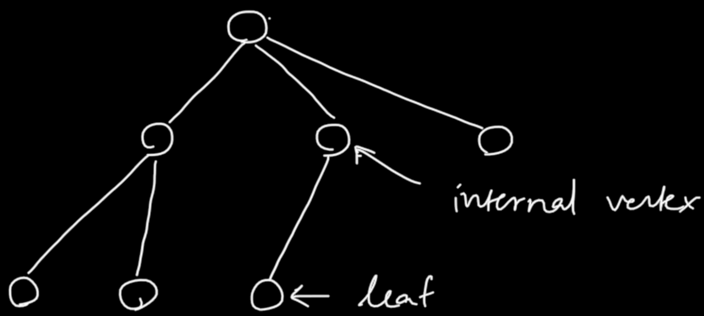
Trees

A tree is a <u>connected</u> <u>acyclic</u> graph.



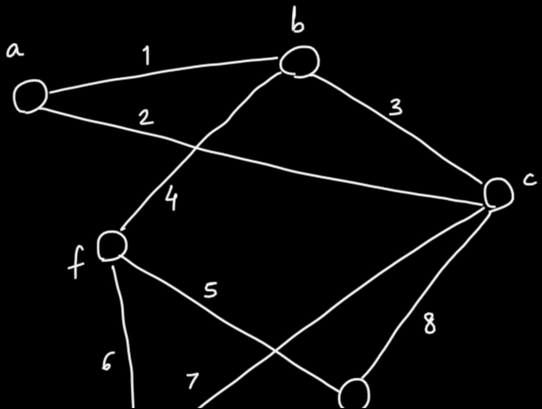
A forest is an <u>acyclic</u> graph

For a n-vertex graph G, the following are equivalent and characterize trees with n vertices

✓(1) G is a tree.
✓(2) G is connected and has exactly $n-1$ edges.
✓(3) G is minimally connected i.e. G is connected but $G-\{e\}$ is disconnected for every edge $e \in G$.
✓(4) G contains no cycle but $G+\{x,y\}$ does, for any two non-adjacent vertices $x, y \in G$.
✓(5) Any two vertices of G are linked by a unique path in G.

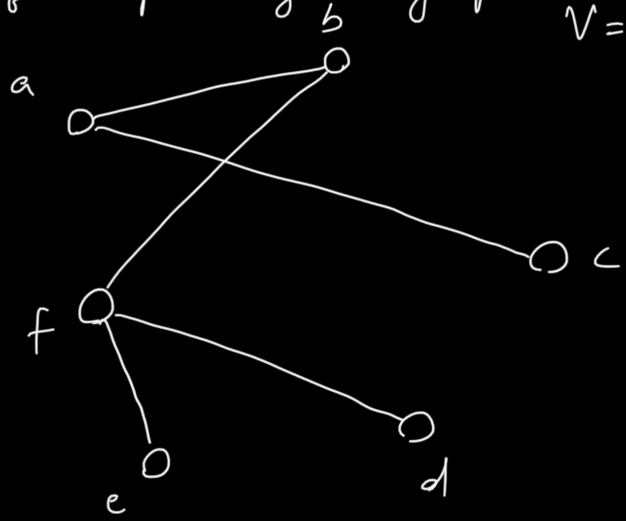
Spanning Trees

A spanning subgraph of a graph G is a subgraph with vertex set $V(G)$. A spanning tree is a spanning subgraph that is tree.



G
$V = \{a, b, c, d, e, f\}$

example of a spanning subgraph: $V = \{a, b, c, d, e, f\}$



example of a spanning tree:

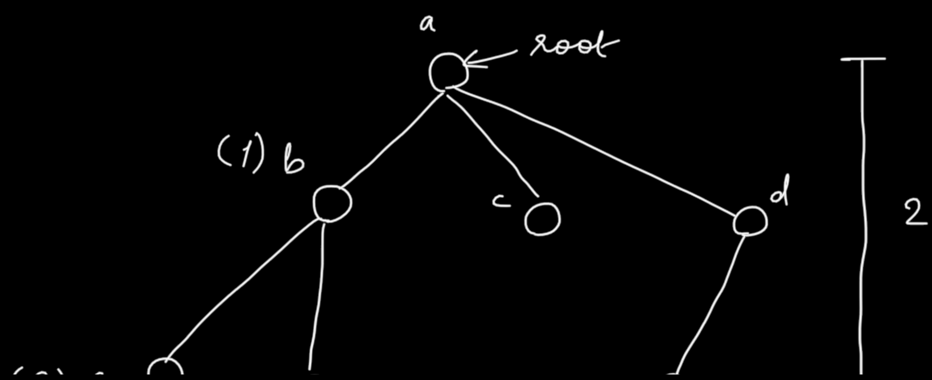
Every spanning subgraph is a spanning tree.
True or false?
<u>False</u>

Every spanning tree is a spanning subgraph
True or false? <u>True</u>

Rooted Trees

A rooted tree is a tree in which one vertex is distinguished from the others and is called the root

level :
height :
children :
parent :



$(2) \in \cup \qquad \circ_{f(2)} \qquad \circ_{g} \qquad \perp$

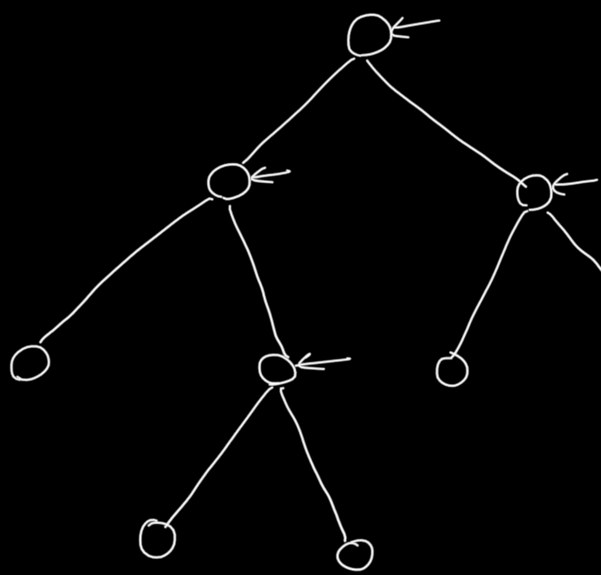
$b, c, d \rightarrow a$

children $\rightarrow$ parent

Binary Tree

A rooted tree in which every internal vertex has <u>at most two children</u>

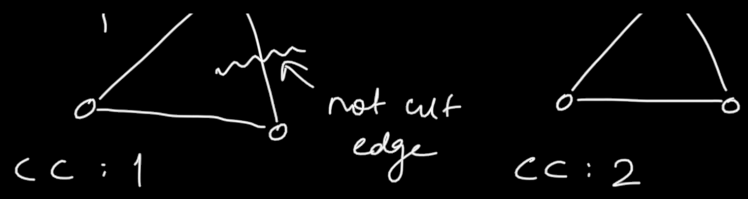
A full binary tree is a binary tree in which each internal vertex has exactly two children

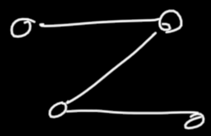


full binary tree

Practice Exam

5) Let G be a connected graph where all vertices are of even degree.
Prove that G has no cut edges. A cut edge is an edge, that if removed, would increase the number of connected components of a graph.



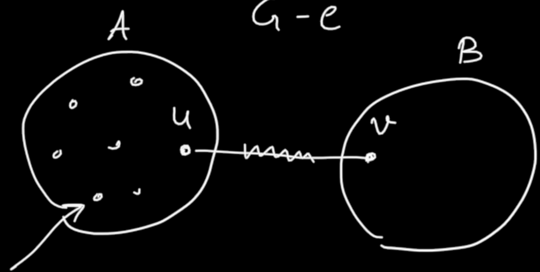


Proof: Assume for contradiction
G does have a cut edge $e = (u, v)$

G is connected
$G - e$ should have $\underline{2}$ connected components

Each vertex in $G - e$ will have the same degree as in G except for u & v
$d(u) - 1$
$d(v) - 1$

u is the only vertex with odd degree in component A



some vertex in A with an odd degree

A has all vertices with an even degree
so contradiction !

1) Prove that for any prime p, $\sqrt{p}$ is irrational.

<u>Proof</u>: Assume for contradiction

$$\sqrt{p} = \frac{a}{b}$$

$$p = \frac{a^2}{b^2}$$

$$b^2$$

$$a^2 = p b^2 \quad — \quad ①$$

$S(m) \rightarrow$ sum of the number of times each prime factor occurs in the unique factorization of m

$45 : 3 \times 3 \times 5$ $\quad$ $6 = 2 \times 3 \quad S(6) = 2$

$\quad\quad 2 \quad\quad 1$ $\quad$ $36 = 2 \times 2 \times 3 \times 3$

$S(m) = 2 + 1 = 3$ $\quad$ $S(36) = 4$

$S(a) = k$

$S(a^2) = 2\,S(a)$ $\quad\quad$ $S(b^2) = 2S(b)$

$S(p b^2) = 1 + 2S(b) \leftarrow$ odd

$\nearrow$
prime number

$S(a^2) = 2S(a) \leftarrow$ even
$S(p b^2) = 1 + 2S(b) \leftarrow$ odd $\Big]$ never be equal

$a^2 \neq p b^2$ $\quad$ contradiction

6) b) At all times H is an acyclic graph

Proof by induction on the number of edges added to H
(n)

BC : No edges in H, it is acyclic, $n = 0$

IH : H is acyclic $n = k$ edges for $k \geq 0$

IS : $n = k + 1$
edge (u, v) is $(k+1)^{th}$ edge
$\rightarrow$ edge (u, v) creates a cycle — assume for contradiction
There must be a path between u and v before adding edge (u, v)
u and v must have been in the same connected component. —

This is a contradiction

Q) Show that there exist irrational numbers x and y such that x^y is rational

Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational ← one of the possibilities

$\sqrt{2}$ is irrational

irrational

$x = \sqrt{2} \quad y = \sqrt{2}$

$x^y = \sqrt{2}^{\sqrt{2}} \rightarrow$ rational

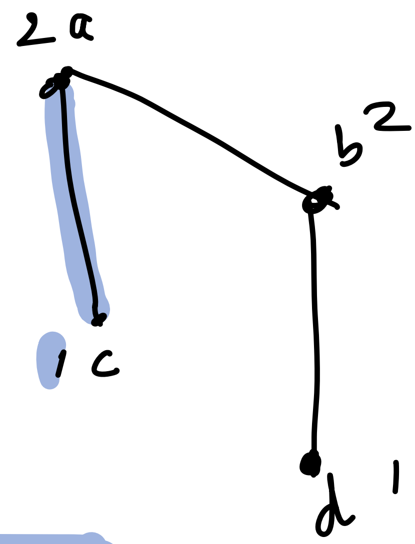
Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = (\sqrt{2})^2 = \boxed{2}$$

Graphs

Definition of Graphs

$$G = (V, E)$$



Degree of a vertex v:

$$\deg(v) = |N(v)|$$

$$\delta(G) = \min_v (\deg(v)) \qquad \Delta(G) = \max_v (\deg(v))$$

$$\sum_{v \in V} \deg(v) = 2m$$

Claim: Every graph with n vertices and m edges has at least $n-m$ connected components.

Subgraphs :

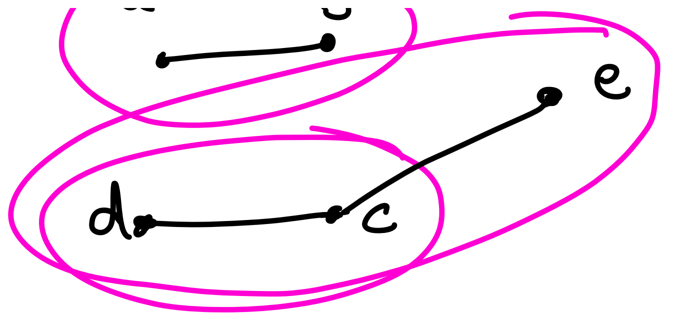
$H(V', E')$ is a subgraph of $G(V,E)$ if and only if

$$V' \subseteq V$$
$$E' \subseteq E$$

Connected Components

H is a connected component of G if

- H is connected
- H is a subgraph of G.
- H is <u>maximal</u>


e
d
c

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HW:1

Q1.) p: You get an A on the final exam

q: You do every exercise in the textbook

r: You get an A in the course

$p \to q \not\equiv q \equiv p$

c) To get an A in the course, it is necessary to get an A in the final exam.

p: its raining

q: wet

$p \to q$

$\to \quad \neg p \to \neg r \equiv r \to p$

Practise Exam

1) for a prime p, $\sqrt{p}$ is irrational.

$\to$ AFSOC $\sqrt{p}$ is rational

$\exists a, b \in \mathbb{Z}, b \neq 0 .\ \sqrt{p} = \frac{a}{b}$

$\Rightarrow p = \frac{a^2}{b^2} \Rightarrow a^2 = p b^2$

S(a) : highest power of p dividing a

S(b) : " " " " b

$S(a^2) = 2S(a) = \text{even}$

$S(b^2) = 2S(b) = \text{even}$

} unique prime factorization

$a^2 = p\, b^2$

highest pow = even

$1 + \text{even} = \text{odd}$

even = odd.

contradiction.

$p = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k} \quad e_i > 0$

$q = q_1^{f_1} q_2^{f_2} - q_l^{f_l} \quad f_j > 0$

Claim: If p and q are relatively prime integers then so are p^2 and q^2.

Proof. By the prime factorization theorem we know that p, q, p^2, and q^2 can be represented as a unique product of primes.

$$
\begin{aligned}
p &= p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k} \\
q &= q_1^{e_1} q_2^{e_2} \ldots q_\ell^{e_\ell} \\
p^2 &= p_1^{2e_1} p_2^{2e_2} \ldots p_k^{2e_k} \\
q^2 &= q_1^{2e_1} q_2^{2e_2} \ldots q_\ell^{2e_\ell}
\end{aligned}
$$

n is not a perfect

$$
\begin{aligned}
p &= p_1^{e_1} p_2^{e_2} \dots p_k^{e_k} \\
q &= q_1^{e_1} q_2^{e_2} \dots q_\ell^{e_\ell} \\
p^2 &= p_1^{2e_1} p_2^{2e_2} \dots p_k^{2e_k} \\
q^2 &= q_1^{2e_1} q_2^{2e_2} \dots q_\ell^{2e_\ell}
\end{aligned}
$$

Since p and q are relatively prime $p_i \neq q_j$, for all i and j. This implies that p^2 and q^2 are relatively prime.

Let n be an arbitrary but specific integer that is not a perfect square. Assume, for the sake of contradiction, that $\sqrt{n}$ is a rational number. Then there are numbers a and b ($b \neq 0$) with no common factors such that

$$\sqrt{n} = \frac{a}{b}$$

Squaring both sides of the above equation gives

$$n = \frac{a^2}{b^2}$$

Since a and b are relatively prime we know that a^2 and b^2 are relatively prime integers. This is only possible if $b^2 = 1$ which means that $b = 1$, which in turn implies that $a = \sqrt{n}$, which contradicts the fact that a is an integer.

n is not a perfect square.

$\Rightarrow \sqrt{n} \notin \mathbb{Z}$

AFSOC $\sqrt{n} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $(a,b) = 1$, $b \neq 0$

$n = \frac{a^2}{b^2}$ $\quad (\overset{p}{a^2}, \overset{q}{b^2}) = 1$

$n = \frac{p}{q}$

Since $n \in \mathbb{Z}$, $q = 1$

$\Downarrow$

$b^2 = 1$

$\therefore \sqrt{n} = \frac{a}{1} = a \in \mathbb{Z} \Rightarrow$ contradictn

$n = a^2$

2) $A \setminus B \subseteq C \cap D$, $x \in A$

Prove that if $x \notin D$ then $x \in B$

$\rightarrow$ Contrapositive: $x \notin B \Rightarrow x \in D$

$x \notin B$

We know $x \in A$

$\Rightarrow x \in A \setminus B$

$\Rightarrow x \in C \cap D \quad (\because A \setminus B \subseteq C \cap D)$

$\Rightarrow x \in D$

$\therefore$ Proved.

$A \setminus B$

$= \{e \mid e \in A, e \notin B\}$