CSC 402

Exam 1

February 20, 2023

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a. This exam contains 8 problems. You have 80 minutes to complete the exam.

b. The exam is closed-book and closed notes. You are not allowed to use a calculator.

c. Do not spend too much time on any one problem. It may be helpful to first glance through all of them and attack them in the order that allows you to make the most progress.

d. **Unless specified otherwise, you must justify all of your answers. Answers without justification may receive no points.**

e. You may use any result presented in the class/recitation, or from a homework as a building block for your solutions.

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[5] 1. Answer the following questions. No justification is necessary. No partial credit will be given for your answers.

a. State the negation of

If it is raining then the game is canceled.

b. State the contrapositive of

$$(\forall \epsilon \in \mathbb{R}^+(|a-b| < \epsilon)) \implies (a = b)$$

without using any negation symbols ($\neg$, !, $\sim$, or $\overline{p}$, for a proposition p).

c. **True or False**: A graph with n vertices and $n-1$ edges is a tree.

d. **True or False**: There exists an undirected graph G whose vertices have the following degrees: 1, 2, 3, 4, 5.

e. Consider a complete grapn on $2n$ vertices, i.e., a graph on $2n$ vertices in which there is an edge between every pair of vertices. What is the minimum number of edges that needed to be deleted to create two connected components of equal size?

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[6] 2. For all non-negative integers n, prove that $3^{3n+1} + 2^{n+1}$ is divisible by 5.

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[8] 3. Prove or disprove each of the following propositions.

a. Let n be an integer. If $n^3 - 5$ is an odd integer, then n is even.

b. For all rational numbers x and y, the number x^y is also rational.

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[6] 4. Prove that $\sqrt{8}$ is irrational.

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[5] 5. An acyclic graph is called a *forest*. Let G be a forest with n vertices and c components. Derive a formula for the number of edges in G.

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[8] 6. Consider a sequence S of *non-zero* real numbers $a_1, a_2, \ldots, a_n$ with the following constraints.

- $a_1 < 0$
- $a_n > 0$

Prove by induction on n that for any $n \geq 2$, the sequence S must have elements a_i and a_{i+1} such that $a_i < 0$ and $a_{i+1} > 0$.

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[5] 7. The *complement* of a graph G is a new graph formed by removing all the edges of G and replacing them by all possible edges that are not in G. Formally, consider a graph $G = (V, E)$. Then, the complement of the graph G is the graph $\overline{G} = (V, \overline{E})$, where

$$\overline{E} = \{\{x, y\} \mid x \neq y, \{x, y\} \notin E\}$$

Prove that for any graph G, G or $\overline{G}$ (or both) must be connected.

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[7] 8. Let G be a connected graph in which the average degree is less than 2. Prove that G is a tree.
