Rank of a Matrix: Let A be a non-zero matrix. Then the integer r is called the rank of a matrix A if,

i) There exists at least one non-zero minor order r of a matrix A. and

ii) Every minor of order greater than r is zero of a matrix A.

Or Order of any highest order non-zero minor of a matrix A is called order of a matrix.

Note: i) rank of matrix A and A^T are same.

ii) Any row or column transformation will not change the rank of the matrix

Normal Form: Using any row and column transform given matrix can be reduced to any one of the following form called normal form$N = I_r$, or $N = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$, or$N = [I_r \quad O]$, or$N = \begin{bmatrix} I_r \\ O \end{bmatrix}$,

Where I_r *is a unit matrix of order r and O is zero matrix of suitable order*

Echelon Form: A matrix is said to be in echelon form if it has the following two properties

1) If any row has all elements zero then such a row appears at the bottom of the matrix. If there are more such rows having all elements zero then they are grouped at the bottom.

2)If there are some rows which do not have all elements zero then they are arranged in such that they are arranged in such a way that the number of zeros before the first non-zero element to go an increasing as we move down the matrix

i.e. $A = \begin{bmatrix} 2 & * & * & * & * & * & * & * \\ 0 & 5 & * & * & * & * & * & * \\ 0 & 0 & 0 & 3 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Rank of matrix A is 4

$B = \begin{bmatrix} 3 & * & * & * & * \\ 0 & 0 & 5 & * & * \\ 0 & 0 & 0 & 4 & * \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Rank of matrix B is 4

$C = \begin{bmatrix} 2 & * & * & * & * \\ 0 & 3 & * & * & * \\ 0 & 0 & 0 & 4 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Rank of matrix C is 3

Example-1 Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 8 & 5 & 14 & 17 \\ 1 & 5 & 5 & 7 \end{bmatrix}$

Solution: By given $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 8 & 5 & 14 & 17 \\ 1 & 5 & 5 & 7 \end{bmatrix}$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 8R_1, R_4 \to R_4 - R_1$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & -11 & -10 & -15 \\ 0 & 3 & 2 & 3 \end{bmatrix}$

$R_4 \to R_4 + R_2, R_3 \to R_3 - 4R_2$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 \leftrightarrow R_3$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_3 \to R_3 + 3R_2$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Which is the required Echelon form of a matrix A, contain 3 non-zero rows

Therefore rank of a matrix A is 3 i.e.$\rho(A) = 3$

Example-2 Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 8 & 5 & 14 & 17 \\ 1 & 5 & 5 & 7 \end{bmatrix}$

Solution: By given $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

$R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - R_1$

$A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$

$R_2 \to R_2 - 2R_4, R_3 \to R_3 - 2R_4$

$A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$

$R_4 \to R_4 - 3R_2$

$A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -1 & -4 \end{bmatrix}$

$R_4 \to R_4 + R_3$

$A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Which is the required Echelon form of a matrix A, contain 3 non-zero rows

Therefore rank of a matrix A is 3 i.e.$\rho(A) = 3$

Example-3 Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Solution: By given $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$R_4 \to R_4 - (R_1 + R_2 + R_3), R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 \leftrightarrow R_3$

$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Which is the required Echelon form of a matrix A, contain 3 non-zero rows

Therefore rank of a matrix A is 3 i.e.$\rho(A) = 3$

Example-4 Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Solution: By given $A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 4 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

$R_3 \to R_3 - R_1, R_4 \to R_4 - R_1$

$A \sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 0 \\ 0 & 6 & 8 & 2 \end{bmatrix}$

$R_3 \to R_3 - R_2, R_4 \to R_4 - 2R_2$

$A \sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_2 \leftrightarrow R_3$

$A \sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_3 \to R_3 - 3R_2$

$A \sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Which is the required Echelon form of a matrix A, contain 3 non-zero rows

Therefore rank of a matrix A is 3 i.e.$\rho(A) = 3$

Example-5 Find the rank of the matrix $A=\begin{bmatrix}1 & 3 & 5 & 7\\ 4 & 6 & 8 & 10\\ 15 & 27 & 39 & 51\\ 6 & 12 & 18 & 24\end{bmatrix}$

Solution: By given $A=\begin{bmatrix}1 & 3 & 5 & 7\\ 4 & 6 & 8 & 10\\ 15 & 27 & 39 & 51\\ 6 & 12 & 18 & 24\end{bmatrix}$

$R_3 \to R_3 - 3(R_1+R_2), R_3 \to R_3 - (2R_1+R_2), R_3 \to R_3 - 4R_1$

$A\sim\begin{bmatrix}1 & 3 & 5 & 7\\ 0 & -6 & -12 & -18\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ Which is the required Echelon form of a matrix A, contain 3 non-zero rows

Therefore rank of a matrix A is 2 i.e.$\rho(A)=2$

Example-6 Find the rank of the matrix $A=\begin{bmatrix}25 & 31 & 17 & 43\\ 75 & 94 & 53 & 132\\ 75 & 94 & 54 & 134\\ 25 & 32 & 20 & 48\end{bmatrix}$

Solution: By given $A=\begin{bmatrix}25 & 31 & 17 & 43\\ 75 & 94 & 53 & 132\\ 75 & 94 & 54 & 134\\ 25 & 32 & 20 & 48\end{bmatrix}$

$R_4 \to R_4 - R_1, R_3 \to R_3 - R_2, R_2 \to R_2 - 3R_1$

$A=\begin{bmatrix}25 & 31 & 17 & 43\\ 0 & 1 & 2 & 3\\ 0 & 0 & 1 & 2\\ 0 & 1 & 3 & 5\end{bmatrix}$

$R_4 \to R_4 - R_2$

$A=\begin{bmatrix}25 & 31 & 17 & 43\\ 0 & 1 & 2 & 3\\ 0 & 0 & 1 & 2\\ 0 & 0 & 1 & 2\end{bmatrix}$

$R_4 \to R_4 - R_3$

$A=\begin{bmatrix}25 & 31 & 17 & 43\\ 0 & 1 & 2 & 3\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}$ Which is the required Echelon form of a matrix A, contain 3 non-zero rows

Therefore rank of a matrix A is 3 i.e.$\rho(A)=3$

Example-7 If a, b, c, d are unequal, find the rank of the matrix $A = \begin{bmatrix} 0 & b-a & c-a & b+c \\ a-b & 0 & c-b & c+a \\ a-c & b-c & 0 & a+b \\ b+c & c+a & a+b & 0 \end{bmatrix}$

Solution: By given $A = \begin{bmatrix} 0 & b-a & c-a & b+c \\ a-b & 0 & c-b & c+a \\ a-c & b-c & 0 & a+b \\ b+c & c+a & a+b & 0 \end{bmatrix}$

$R_1 \to R_1 + R_4, R_2 \to R_2 + R_4, R_3 \to R_3 + R_4$

$$A = \begin{bmatrix} b+c & b+c & b+c & b+c \\ c+a & c+a & c+a & c+a \\ a+b & a+b & a+b & a+b \\ b+c & c+a & a+b & 0 \end{bmatrix}$$

$R_1 \to \frac{1}{b+c} R_1, R_2 \to \frac{1}{c+a} R_2, R_3 \to \frac{1}{a+b} R_3$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ b+c & c+a & a+b & 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - (b+c)R_4$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a-b & a-c & -(b+c) \end{bmatrix}$$

$R_2 \to R_4$

$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & a-b & a-c & -(b+c) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Which is the required Echelon form of a matrix A, contain 3 non-zero rows

Therefore rank of a matrix A is 2 i.e.$\rho(A) = 2$

Find the rank of the following matrices using Echelon form

$$A = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & 7 \\ 4 & -1 & 7 \\ 2 & 1 & 5 \end{bmatrix}$$

Example on finding rank of matrix using Normal or Canonical form

Example-1 Find the rank of the matrix using normal form or canonical form of matrix

$$A=\begin{bmatrix}1 & -1 & -2 & -4\\ 2 & 3 & -1 & -1\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & 7\end{bmatrix}$$

Solution: By given

$$A=\begin{bmatrix}1 & -1 & -2 & -4\\ 2 & 3 & -1 & -1\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$

$$A\sim\begin{bmatrix}1 & -1 & -2 & -4\\ 0 & 5 & 3 & 7\\ 0 & 4 & 9 & 10\\ 0 & 9 & 12 & 17\end{bmatrix}$$

$C_2 \to C_2 + C_1, C_3 \to C_3 + 2C_1, C_4 \to C_4 + 4C_1$

$$A\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 5 & 3 & 7\\ 0 & 4 & 9 & 10\\ 0 & 9 & 12 & 17\end{bmatrix}$$

$R_4 \to R_4 - (R_2 + R_3), R_2 \to R_2 - R_3$

$$A\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -6 & -3\\ 0 & 4 & 9 & 10\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$R_3 \to R_3 - 4R_2$

$$A\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & -6 & -3\\ 0 & 0 & 33 & 22\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$C_3 \to C_3 + 6C_2, C_4 \to C_4 + 3C_2$

$$A\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 33 & 22\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$C_3 \to \frac{1}{33}C_3, C_4 \to \frac{1}{22}C_4$

$$A\sim\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$C_4 \to C_4 - C_3$

$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Which is of the form $A \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

Therefore rank of matrix A is 3 i.e. i.e.$\rho(A) = 3$

Example-2 Find the rank of the matrix using normal form or canonical form of matrix

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Solution By given

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$R_4 \to R_4 - (R_1 + R_2 + R_3), R_3 \to R_3 - R_2, R_2 \to R_2 - R_1$

$$A \sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & -2 & -4 & -6 \end{bmatrix}$$

$R_1 \leftrightarrow R_2, R_4 \to \frac{-1}{2} R_4$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_2 - C_1, C_3 \to C_3 - C_1, C_4 \to C_4 - C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - 2C_2, C_4 \to C_4 - 3C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{This is of the form} \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore rank of matrix A is 2 i.e. i.e.$\rho(A) = 2$

Example-3 Find the rank of the matrix using normal form or canonical form of matrix

$$A = \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 1 & 3 & -2 & 3 & 0 \\ 2 & 4 & -3 & 6 & 4 \\ 1 & 1 & -1 & 4 & 6 \end{bmatrix}$$

Solution: By given

$$A \sim \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 1 & 3 & -2 & 3 & 0 \\ 2 & 4 & -3 & 6 & 4 \\ 1 & 1 & -1 & 4 & 6 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - 2R_1, R_4 \rightarrow R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 5 \end{bmatrix}$$

$C_2 \rightarrow C_2 - 2C_1, C_3 \rightarrow C_3 + 2C_1, C_4 \rightarrow C_4 - 3C_1, C_5 \rightarrow C_5 - C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 5 \end{bmatrix}$$

$R_4 \rightarrow R_4 + R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

$C_5 \rightarrow C_5 + C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

$R_4 \rightarrow R_4 - R_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$C_5 \rightarrow C_5 - 4C_4$

$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ Which is of the form $[I_4 \quad 0]$

Therefore rank of matrix A is 4 i.e. i.e.$\rho(A) = 4$

Example-4 Find the rank of the matrix using normal form or canonical form of matrix

$$A = \begin{bmatrix} 3 & 2 & -4 & 3 & 6 \\ 1 & -2 & 3 & 4 & -3 \\ 2 & -4 & 6 & 8 & -6 \\ 3 & -6 & 9 & 12 & -9 \\ 5 & -2 & 2 & 11 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 & -3 \\ 3 & 2 & -4 & 3 & 6 \\ 2 & -4 & 6 & 8 & -6 \\ 3 & -6 & 9 & 12 & -9 \\ 5 & -2 & 2 & 11 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - 3R_1, R_5 \to R_5 - 5R_1$

$$A \sim \begin{bmatrix} 1 & -2 & 3 & 4 & -3 \\ 0 & 8 & -13 & -9 & 15 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & -13 & -9 & 15 \end{bmatrix}$$

$C_2 \to C_2 + 2C_1, C_3 \to C_3 - 3C_1, C_4 \to C_4 - 4C_1, C_5 \to C_5 + 3C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 8 & -13 & -9 & 15 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & -13 & -9 & 15 \end{bmatrix}$$

$R_5 \to R_5 - R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 8 & -13 & -9 & 15 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to \frac{1}{8} C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -13 & -9 & 15 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + 13C_2, C_4 \to C_4 + 9C_2, C_5 \to C_5 - 15C_2$

$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Which is of the form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ Therefore rank of matrix A is 2 i.e. i.e.$\rho(A) = 2$

Example-5 Find the rank of the matrix using normal form or canonical form of matrix

$$A=\begin{bmatrix}2&6&-2&6&10\\-3&3&-3&-3&-3\\1&-2&4&3&5\\2&0&4&6&10\\1&0&2&3&5\end{bmatrix}$$

Solution: By given

$$A=\begin{bmatrix}2&6&-2&6&10\\-3&3&-3&-3&-3\\1&-2&4&3&5\\2&0&4&6&10\\1&0&2&3&5\end{bmatrix}$$

$$R_1\to\frac{1}{2}R_1, R_2\to\frac{-1}{3}R_2, R_4\to\frac{1}{2}R_4$$

$$A\sim\begin{bmatrix}1&3&-1&3&5\\-1&1&-1&-1&-1\\1&-2&4&3&5\\1&0&2&3&5\\1&0&2&3&5\end{bmatrix}$$

$$R_2\to R_2+R_1, R_3\to R_3-R_1, R_4\to R_4-R_1, R_5\to R_5-R_1$$

$$A\sim\begin{bmatrix}1&3&-1&3&5\\0&4&-2&2&4\\0&-5&5&0&0\\0&-3&3&0&0\\0&-3&3&0&0\end{bmatrix}$$

$$C_2\to C_2+3C_1, C_3\to C_3+C_1, C_4\to C_4-3C_1, C_5\to C_5-5C_1$$

$$A\sim\begin{bmatrix}1&0&0&0&0\\0&4&-2&2&4\\0&-5&5&0&0\\0&-3&3&0&0\\0&-3&3&0&0\end{bmatrix}$$

$$R_2\to\tfrac{1}{2}R_2, R_2\to\tfrac{-1}{5}R_2, R_4\to\tfrac{-1}{3}R_4, R_5\to\tfrac{-1}{3}R_5$$

$$A\sim\begin{bmatrix}1&0&0&0&0\\0&2&-1&1&2\\0&1&-1&0&0\\0&1&-1&0&0\\0&1&-1&0&0\end{bmatrix}$$

$$R_2\leftrightarrow R_3$$

$$A\sim\begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 1 & -1 & 0 & 0\\0 & 2 & -1 & 1 & 2\\0 & 1 & -1 & 0 & 0\\0 & 1 & -1 & 0 & 0\end{bmatrix}$$

$R_3 \to R_3 - 2R_2, R_4 \to R_4 - R_2, R_5 \to R_5 - R_2$

$$A\sim\begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 1 & -1 & 0 & 0\\0 & 0 & 1 & 1 & 2\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$C_3 \to C_3 + C_2$

$$A\sim\begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 1 & 2\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$C_4 \to C_4 - C_3, C_5 \to C_5 - 2C_3$

$A\sim\begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$ Which is of the form $\begin{bmatrix}I_2 & 0\\0 & 0\end{bmatrix}$ Therefore rank of matrix A is 2 i.e. i.e.$\rho(A) = 2$

Homework: Find the rank of the following matrix using normal form or canonical form of matrix

1)$A = \begin{bmatrix}1 & 2 & 3 & 1\\2 & 4 & 3 & 2\\3 & 2 & 1 & 3\\6 & 8 & 7 & 6\end{bmatrix}$, 2)$A = \begin{bmatrix}6 & 1 & 3 & 6\\4 & 2 & 6 & 1\\10 & 3 & 9 & 7\\16 & 4 & 12 & 13\end{bmatrix}$,3)$A = \begin{bmatrix}-1 & 2 & 3 & -2\\2 & -5 & 1 & 2\\3 & -8 & 5 & 2\\5 & -12 & -1 & 6\end{bmatrix}$,

4)$A = \begin{bmatrix}3 & 4 & -2 & 1\\5 & 8 & 4 & 2\\8 & 12 & 2 & 3\\13 & 20 & 6 & 5\end{bmatrix}$, 5)$A = \begin{bmatrix}6 & 1 & 3 & 8\\3 & 2 & 6 & -1\\9 & 3 & 9 & 7\\15 & 4 & 12 & 15\end{bmatrix}$

Particular Examples:

1) Find the possible value of k for which the rank of the matrix A is 1, 2, 3 $A=\begin{bmatrix} k & 4 & 4 \\ 4 & k & 4 \\ 4 & 4 & k \end{bmatrix}$

Solution: By given $A=\begin{bmatrix} k & 4 & 4 \\ 4 & k & 4 \\ 4 & 4 & k \end{bmatrix}$

$R_3 \to R_3 - R_2, R_2 \to R_2 - R_1$

$A \sim \begin{bmatrix} k & 4 & 4 \\ 4-k & k-4 & 0 \\ 0 & 4-k & k-4 \end{bmatrix}$

Case-1 If k=4 then $A \sim \begin{bmatrix} k & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Therefore rank of A is 1

Case2 if $k \neq 4 \quad R_2 \to \frac{1}{4-k} R_2, R_3 \to \frac{1}{4-k} R_3 \; A \sim \begin{bmatrix} k & 4 & 4 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

Now $R_1 \to R_1 - kR_2, A \sim \begin{bmatrix} 0 & 4+k & 4 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

Now $R_1 \to R_1 - (4+k)R_3, A \sim \begin{bmatrix} 0 & 0 & 8+k \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

$Replace\ R_1 by R_2, Replace\ R_2\ by\ R_3, Replace\ R_3 by\ R_1$

$A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & k+8 \end{bmatrix}$

Case2 if $k \neq 4$, and k=-8 then $A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Case-3 if $k \neq 4$, and $k \neq -8$, $R_3 \to \frac{1}{k+8} R_3$ then $A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Then rank of matrix A is 3

Thus if k=4 rank of A is 1, if k=-8 rank of A is 2 and if $k \neq 4$, and $k \neq -8\ then\ rank\ of\ A\ is\ 3$

2) Find the possible value of p for which the rank of the matrix A is 1, 2, 3 $A = \begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$

Solution: By given $A = \begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$

$R_3 \to R_3 - R_2, R_2 \to R_2 - R_1$

$$A = \begin{bmatrix} 3 & p & p \\ p-3 & 3-p & 0 \\ 0 & p-3 & 3-p \end{bmatrix}$$

Case-1If P=3 then $A \sim \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ then rank of A is 1

Case-2 If $p \neq 3$,

$R_3 \to \frac{1}{p-3} R_3, R_2 \to \frac{1}{p-3} R_2$

$$A \sim \begin{bmatrix} 3 & p & p \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$Replace\ R_1 by R_2, Replace\ R_2\ by\ R_3, Replace\ R_3 by\ R_1$

$$A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 3 & p & p \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & p+3 & p \end{bmatrix}$$

$R_3 \to R_3 - (p+3)R_2$

$$A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2p+3 \end{bmatrix}$$

$R_3 \to \frac{1}{2} R_3$

$$A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & p+\frac{3}{2} \end{bmatrix}$$

Case-2 if $p = -\frac{3}{2}$, then $A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ then rank of A is 2

Case-3 if $p \neq 3, and\ p \neq -\frac{3}{2}$ then rank of A is 3

Homework:

3) Find the possible value of p for which the rank of the matrix A is 1, 2, 3 $A = \begin{bmatrix} p & p & 2 \\ 2 & p & p \\ p & 2 & p \end{bmatrix}$

4) Find the possible value of p for which the rank of the matrix A is 1, 2, 3 $A = \begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$

Solution: By given $A = \begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 3k & 3k+4 \\ 0 & k-8 & k-2 \\ 0 & k-2 & 2-k \end{bmatrix}$$

Case-1 If k=2 then $A \sim \begin{bmatrix} 1 & 6 & 10 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ then rank of A is 2

If $k \neq 2, R_3 \to \frac{1}{2-k} R_3$

$$A \sim \begin{bmatrix} 1 & 3k & 3k+4 \\ 0 & 4-3k & k-2 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 1 & 3k & 3k+4 \\ 0 & 1 & 0 \\ 0 & 4-3k & k-2 \end{bmatrix}$$

$R_3 \to R_3 - R_1$