① Find Eigen value and Eigen vector of a matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution: $\because$ A be a square matrix of order 3

$\therefore$ it's characteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad \text{—①}$$

where $S_1 = -1$

$$S_2 = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -12 + (-3) + (-6) = -21$$

$|A| = 45$

$$\therefore \lambda^3 + 1\lambda^2 - 21\lambda - 45 = 0$$

$\lambda = \lambda_1 = 5,\ \lambda = \lambda_2 = -3,\ \lambda = \lambda_3 = -3$ be the Eigen value of a matrix A

To find Eigen vector consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—②}$$

Case-1: If $\lambda = \lambda_1 = 5$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -4 & -6 \\ -2 & -5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & -6 \\ -1 & -5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & -4 \\ -1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{8} = \frac{-x_2}{-16} = \frac{x_3}{-8}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1} = k = 1$$

$$x_1 = 1,\ x_2 = 2,\ x_3 = -1$$

Thus for Eigen value $\lambda = \lambda_1 = 5$, Eigen vector $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Case-2: If $\lambda = \lambda_2 = -3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—③}$$

Now $O(A - \lambda I) = 3$, $S(A - \lambda I) = 1$ for $\lambda = -3$

$\therefore$ For $\lambda = -3$, $G.M. = O(A-\lambda I) - S(A-\lambda I) = 3 - 1 = 2$

$\therefore$ For Eigen value $\lambda = -3$, two Eigen vector exist

Now From equation ③ $1x_1+2x_2-3x_3=0$

$\therefore x_1 = -2x_2+3x_3$

$\because$ for Eigen value $\lambda=\lambda_2=-3$ G.M. = 2

$\therefore$ take $x_2=k_1,\ x_3=k_2$

$$\therefore x_1 = -2k_1+3k_2$$

$$x_2 = 1k_1+0k_2$$

$$x_3 = 0k_1+1k_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} k_1 + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} k_2$$

Thus for Eigen value $\lambda=\lambda_2=-3$, Eigen Vector $X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and

for Eigen value $\lambda=\lambda_3=-3$, Eigen vector $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

② Find Eigen value and Eigen vector of a matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Solution: $\because$ A be a square matrix of order 3

$\therefore$ it's characteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad - ①$$

where $S_1 = 7$

$$S_2 = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4+3+4 = 11$$

$|A| = 5$

$$\therefore \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$\therefore \lambda = \lambda_1 = 5, \lambda = \lambda_2 = 1, \lambda = \lambda_3 = 1$ be the Eigen values of a matrix A

To find Eigen vector consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad - ②$$

Case-1 If $\lambda = \lambda_1 = 5$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -2 & 1 \\ 2 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = k = 1$$

Thus $\therefore x_1 = 1, x_2 = 1, x_3 = 1$

For Eigen value $\lambda = \lambda_1 = 5$, Eigen vector $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case-2 If $\lambda = \lambda_2 = 1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad - ③$$

$\therefore \rho(A - \lambda I) = 1$, and $O(A - \lambda I) = 3$

Thus for $\lambda = \lambda_2 = 1$, G.M. $= O(A-\lambda I) - \rho(A - \lambda I) = 3 - 1 = 2$

Thus for Eigen value $\lambda = \lambda_2 = 1$ two Eigen vector exist.

Now from equation (1) $1x_1 + 2x_2 + 1x_3 = 0$

$\therefore \quad x_1 = -2x_2 - 1x_3$

$\because$ For Eigen value $\lambda = \lambda_2 = 1$, ~~the~~ geometric multiplicity is 2

$\therefore$ Take any value for two unknowns out of three

$\therefore \quad x_2 = k_1, \quad x_3 = k_2$

$$\therefore \quad x_1 = -2k_1 - 1k_2$$
$$x_2 = 1k_1 + 0k_2$$
$$x_3 = 0k_1 + 1k_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} k_1 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} k_2$$

Thus for Eigen value $\lambda = \lambda_1 = 1$, $X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ be Eigen vector

for Eigen value $\lambda = \lambda_2 = 1$, Eigen vector $X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

③ Find Eigen value and Eigen vector of a matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

Solution: ∵ A be a square matrix of order 3

∴ its characteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad -①$$

where $S_1 = 9$

$$S_2 = \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 6 + 5 + 4 = 15$$

$|A| = 7$

$$\therefore \lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$$

$\therefore \lambda = \lambda_1 = 7,\ \lambda = \lambda_2 = 1,\ \lambda = \lambda_3 = 1$ be the Eigen values of a matrix A

To find Eigen vectors consider $(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -②$$

Case 1: If $\lambda = \lambda_1 = 7$

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -4 & 2 \\ 3 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 2 \\ 3 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & -4 \\ 3 & 3 \end{vmatrix}}$$

$$\frac{x_1}{6} = \frac{-x_2}{-12} = \frac{x_3}{18}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3} = k = 1$$

$x_1 = 1,\ x_2 = 2,\ x_3 = 3$

Thus for Eigen value $\lambda = \lambda_1 = 7$, Eigen vector $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Case-2: If $\lambda = \lambda_2 = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, and $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -③$$

$\therefore\ S(A-\lambda I) = 1$ and $O(A-\lambda I) = 3$

$\therefore$ For $\lambda = \lambda_2 = 1$ Geometric Multiplicity $= O(A-\lambda I) - S(A-\lambda I)$

$\therefore$ G.M. $= 3-1 = 2$

Thus for Eigen value $\lambda = \lambda_2 = 1$ two Eigen vector exist

From (1) $1x_1 + 1x_2 + 1x_3 = 0$

$\therefore 1x_1 = -1x_2 - 1x_3 \quad - (4)$

$\because$ G.M. is 2 $\therefore$ we can select any value for two unknowns of any three unknown

$\therefore$ let $x_2 = k_1$ and $x_3 = k_2$

$$\therefore \quad x_1 = -k_1 - 1k_2$$
$$x_2 = 1k_1 + 0k_2$$
$$x_3 = 0k_1 + 1k_2$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} k_1 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} k_2$$

$\therefore$ For Eigen value of $\lambda = \lambda_2 = 1$ there exist corresponding Eigen vector $X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ & $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(4) Find Eigen value and Eigen vector of a matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

<u>Solution</u>: $\because$ A be a square matrix of order 3

$\therefore$ its characteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad -(1)$$

where $S_1 = 5$

$$S_2 = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2+2+3 = 7$$

$|A| = 3$

$$\begin{array}{c|cccc} 1 & 1 & -5 & 7 & -3 \\ & & 1 & -4 & 3 \\ \hline & 1 & -4 & 3 & \boxed{0} \end{array}$$

$\therefore \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

$\therefore (\lambda-1)(\lambda^2 - 4\lambda + 3) = 0$

$\therefore (\lambda-1)(\lambda-3)(\lambda-1) = 0$

$\therefore \lambda = \lambda_1 = 3,\ \lambda = \lambda_2 = 1$ and $\lambda = \lambda_3 = 1$ be the Eigen values of the matrix

To find Eigen vector consider $(A - \lambda I)X = 0$

i.e. $$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -(2)$$

<u>Case-1</u>: If $\lambda = \lambda_1 = 3$ $$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 0 & -2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x_1}{2} = \frac{-x_2}{-2} = \frac{x_3}{0}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0} = k = 1$$

$\therefore x_1 = 1,\ x_2 = 1,\ x_3 = 0$

Thus For Eigen value $\lambda = \lambda_1 = 3$, corresponding Eigen vector $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

<u>Case-2</u> If $\lambda = \lambda_2 = 1$ $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -(3)$$

$R_2 \leftrightarrow R_3$

$\therefore \rho(A - \lambda I) = 1$

$\because$ $O(A-\lambda I) = 3$ and $\rho(A-\lambda I) = 1$ for $\lambda = 1$

$\therefore$ For $\lambda = \lambda_2 = 1$, Geometric multiplicity $G.M. = O(A-\lambda I) - \rho(A-\lambda I) = 3-1 = 2$

$\therefore$ Two Eigen vector exist for Eigen value $\lambda = \lambda_2 = 1$

$\therefore$ From equation 3 we get $1x_1 + 1x_2 + 1x_3 = 0$

$$\therefore \quad x_1 = -1x_2 - 1x_3 \quad \text{--- (4)}$$

$\because$ $G.M. = 2$ $\therefore$ we can select any value for any two unknown out of three unknown

$$\therefore \; x_2 = k_1 \;\&\; x_3 = k_2$$

$\therefore$ From 4
$$x_1 = -1k_1 - 1k_2$$
$$x_2 = 1k_1 + 0k_2$$
$$x_3 = 0k_1 + 1k_2$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} k_1 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} k_2$$

Thus for Eigen value $\lambda = \lambda_2 = 1$, corresponding Eigen vectors are $X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

⑤ Find Eigen value and Eigen vector of matrix $A=\begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}$

<u>Solution:</u> ∵ A be a square matrix of order 3

∴ it's characteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad -①$$

$S_1 = 1$

$S_2 = \begin{vmatrix}3 & 4\\ 8 & 7\end{vmatrix} + \begin{vmatrix}-9 & 4\\ -16 & 7\end{vmatrix} + \begin{vmatrix}-9 & 4\\ -8 & 3\end{vmatrix} = -11 + (1) + (5) = -5$

$|A| = 3$

$\therefore 1\lambda^3 - 1\lambda^2 - 5\lambda - 3 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & -5 & -3 \\ & & -1 & 2 & 3 \\ \hline & 1 & -2 & -3 & \boxed{0} \end{array}$$

$\therefore (\lambda+1)(\lambda^2 - 2\lambda - 3) = 0$

$(\lambda+1)(\lambda-3)(\lambda+1) = 0$

$\therefore \lambda = \lambda_1 = 3,\ \lambda = \lambda_2 = -1,\ \lambda = \lambda_3 = -1$, be the Eigen values of a matrix A

To find Eigen vector consider $(A - \lambda I)X = 0$

$$\therefore \begin{bmatrix}-9-\lambda & 4 & 4\\ -8 & 3-\lambda & 4\\ -16 & 8 & 7-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \quad -②$$

<u>case 1</u> If $\lambda = \lambda_1 = 3$

$$\begin{bmatrix}-12 & 4 & 4\\ -8 & 0 & 4\\ -16 & 8 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix}4 & 4\\ 0 & 4\end{vmatrix}} = \frac{-x_2}{\begin{vmatrix}-12 & 4\\ -8 & 4\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}-12 & 4\\ -8 & 0\end{vmatrix}}$$

$$\frac{x_1}{16} = \frac{-x_2}{-16} = \frac{x_3}{32}$$

$$\therefore \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{2} = t = 1$$

$\therefore x_1 = 1,\ x_2 = 1,\ x_3 = 2$

Thus For Eigen value $\lambda = \lambda_1 = 3$, corresponding Eigen vector $X_1 = \begin{bmatrix}1\\ 1\\ 2\end{bmatrix}$

<u>Case-2</u> If $\lambda = \lambda_2 = -1$

$$\begin{bmatrix}-8 & 4 & 4\\ -8 & 4 & 4\\ -16 & 8 & 8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$R_2 \to R_2 - R_1$ $R_3 \to R_3 - 2R_1$ $R_1 \to -\frac{1}{4}R_1$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad - \text{(3)}$$

$\therefore \ \rho(A-\lambda I) = 1$

Thus $O(A-\lambda I) = 3$, $\rho(A-\lambda I) = 1$

$\therefore$ For $\lambda = \lambda_2 = -1$, G.M. $= O(A-\lambda I) - \rho(A-\lambda I) = 3-1 = 2$

$\therefore$ From (3) $2x_1 - 1x_2 - 1x_3 = 0$.

ie $2x_1 = 1x_2 + 1x_3$

$\therefore$ we can select any values for two unknow out of three unknown

$\therefore$ let $x_2 = 2k_1$, $x_3 = 2k_2$

$\therefore \ 2x_1 = 2k_1 + 2k_2$

$\therefore \ x_1 = 1k_1 + 1k_2$

$x_2 = 2k_1 + 0k_2$

$x_3 = 0k_1 + 2k_2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} k_1 + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} k_2$$

Thus for Eigen value $\lambda = \lambda_2 = -1$ there exist two Eigen vectors $X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ \& $X_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

⑥ Find Eigen Value and Eigen vector of Matrix

$$A=\begin{bmatrix}1 & -6 & -4\\ 0 & 4 & 2\\ 0 & -6 & -3\end{bmatrix}$$

Solution: ∵ A be a square matrix of order 3

∴ it's characteristic equation is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0 \quad -①$$

where $S_1 = 2$

$$S_2 = \begin{vmatrix}4 & 2\\ -6 & -3\end{vmatrix} + \begin{vmatrix}1 & -4\\ 0 & -3\end{vmatrix} + \begin{vmatrix}1 & -6\\ 0 & 4\end{vmatrix} = 0-3+4 = 1$$

$|A| = 0$

$$\therefore \lambda^3 - 2\lambda^2 + 1\lambda - 0 = 0$$

$$\therefore \lambda(\lambda^2 - 2\lambda + 1) = 0$$

$$\therefore \lambda(\lambda-1)(\lambda-1) = 0$$

$\therefore \lambda = \lambda_1 = 0,\ \lambda = \lambda_2 = 1,\ \lambda = \lambda_3 = 1$ be the Eigen value of a Matrix A

To find Eigen vector consider $(A-\lambda I)X = 0$

ie
$$\begin{bmatrix}1-\lambda & -6 & -4\\ 0 & 4-\lambda & 2\\ 0 & -6 & -3-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \quad -②$$

<u>Case-1</u> If $\lambda = \lambda_1 = 0$
$$\begin{bmatrix}1 & -6 & -4\\ 0 & 4 & 2\\ 0 & -6 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix}-6 & -4\\ 4 & 2\end{vmatrix}} = \frac{-x_2}{\begin{vmatrix}1 & -4\\ 0 & 2\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}1 & -6\\ 0 & 4\end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{-x_2}{2} = \frac{x_3}{4}$$

$$\therefore \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{2} = k = 1$$

$\therefore x_1 = 2,\ x_2 = -1,\ x_3 = 2$

Thus for Eigen value $\lambda = \lambda_1 = 0$, Eigen vector $X_1 = \begin{bmatrix}2\\ -1\\ 2\end{bmatrix}$

<u>Case-2</u> If $\lambda = \lambda_2 = 1$,
$$\begin{bmatrix}0 & -6 & -4\\ 0 & 3 & 2\\ 0 & -6 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$R_3 \to R_3 - R_1$ ~~$R_2 \to \frac{1}{2}R_2$ $R_1 \to \frac{1}{2}R_1$~~, $R_2 \to R_2 + \frac{1}{2}R_1$

$$\begin{bmatrix} 0 & -3 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -(3)$$

$$\cancel{R_2 \rightarrow 3R_2 + 2R_1}$$

$$\cancel{\begin{bmatrix} 0 & -3 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$\because \rho(A-\lambda I) = 1$ and $O(A-\lambda I) = 3$

$\therefore$ For $\lambda = \lambda_2 = 1$ G.M. $= O(A-\lambda I) - \rho(A-\lambda I) = 3 - 1 = 2$

$\therefore$ For Eigen value $\lambda = \lambda_2 = 1$, there exist ~~Ba~~ two Eigen vectors

Now from equation (3)

$$0x_1 - 3x_2 - 2x_3 = 0$$

$$0x_1 = +3x_2 + 2x_3 \quad \text{or} \quad 2x_3 = 0x_1 - 3x_2$$

$$x_3 = 0x_1 - \frac{3}{2}x_2$$

$\because$ G.M $= 2$

$\therefore$ we can select any value for two unknowns out of three unknown

$\therefore$ let $x_1 = k_1,\ x_2 = 2k_2$

$$\therefore x_1 = 1k_1 + 0k_2$$

$$x_2 = 0k_1 + 2k_2$$

$$x_3 = 0k_1 - 3k_2$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} k_1 + \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} k_2$$

$\therefore$ For $\lambda = \lambda_2 = 1$, we get two Eigen vector

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ \& \ X_2 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$