Mathematical Foundations of Computer Science
Lecture Outline
February 3, 2022

Example. Prove that every graph with n vertices and m edges has at least $n - m$ connected components.

Solution. We will prove this claim by doing induction on m.
Induction Hypothesis: Assume that for some $k \geq 0$, every graph with n vertices and k edges has at least $n - k$ connected components.
Base Case: $m = 0$. A graph with n vertices and no edges has n connected components as each vertex itself is a connected component. Hence the claim is true for $m = 0$.
Induction Step: We want to prove that a graph, G, with n vertices and $k + 1$ edges has at least $n-(k+1) = n-k-1$ connected components. Consider a subgraph G' of G obtained by removing any arbitrary edge, say $\{u, v\}$, from G. The graph G' has n vertices and k edges. By induction hypothesis, G' has at least $n - k$ connected components. Now add $\{u, v\}$ to G' to obtain the graph G. We consider the following two cases.
Case I: u and v belong to the same connected component of G'. In this case, adding the edge $\{u, v\}$ to G' is not going to change any connected components of G'. Hence, in this case the number of connected components of G is the same as the number of connected components of G' which is at least $n - k > n - k - 1$.
CaseII: u and v belong to different connected components of G'. In this case, the two connected components containing u and v become one connected component in G. All other connected components in G' remain unchanged. Thus, G has one less connected component than G'. Hence, G has at least $n - k - 1$ connected components.

Example. Prove that every connected graph with n vertices has at least $n - 1$ edges.

Solution. We will prove the contrapositive, i.e., a graph G with $m \leq n-2$ edges is disconnected. From the result of the previous problem, we know that the number of components of G is at least

$$n - m \geq n - (n - 2) = 2$$

which means that G is disconnected. This proves the claim.

One could also have proved the above claim directly by observing that a connected graph has exactly one connected component. Hence, $1 \geq n - m$. Rearranging the terms gives us $m \geq n - 1$.

Trees

A graph with no cycles is *acyclic*. A *tree* is a connected acyclic graph. A vertex of degree greater than 1 in a tree is called an *internal vertex*, otherwise it is called a *leaf*. A *forest* is an acyclic graph.

Example. Prove that every tree with at least two vertices has at least two leaves and deleting a leaf from an n-vertex tree produces a tree with $n-1$ vertices.

Solution. A connected graph with at least two vertices has an edge. In an acyclic graph, an endpoint of a maximal non-trivial path (a path that is not contained in a longer path) has no neighbors other than its only neighbor on the path. Hence, the endpoints of such a path are leaves.

Let v be a leaf of a tree T and let $T' = T - v$. A vertex of degree 1 belongs to no path connecting two vertices other than v. Hence, for any two vertices $u, w \in V(T')$, every path from u to w in T is also in T'. Hence T' is connected. Since deleting a vertex cannot create a cycle, T' is also acyclic. Thus, T' is a tree with $n-1$ vertices.

Example. For a n-vertex graph G, the following are equivalent and characterize trees with n vertices.

(1) G is a tree.
(2) G is connected and has exactly $n-1$ edges.
(3) G is minimally connected, i.e., G is connected but $G - \{e\}$ is disconnected for every edge $e \in G$.
(4) G contains no cycle but $G + \{x, y\}$ does, for any two non-adjacent vertices $x, y \in G$.
(5) Any two vertices of G are linked by a unique path in G.

Solution. **($1 \to 2$).** We can prove this by induction on n. The property is clearly true for $n = 1$ as G has 0 edges. Assume that any tree with k vertices, for some $k \geq 0$, has $k-1$ edges. We want to prove that a tree G with $k+1$ vertices has k edges. From the example we did in last class we know that G has a leaf, say v, and that $G' = G - \{v\}$ is connected. By induction hypothesis, G' has $k-1$ edges. Since $deg(v) = 1$, G has k edges.