An algorithm outline to solve a general linear system

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Assume $PA = LU$ is a row-permuted LU factorization of A (so $A = P^T LU$). Let b be given and consider $Ax = b$. Multiplying by P gives

$$LUx = Pb.$$

Introduce $y := Ux$. Then the system is equivalent to the triangular systems

$$Ly = Pb, \qquad Ux = y.$$

The columns of U fall into two classes: pivot (basic) columns and nonpivot (free) columns. Let the pivot column indices be $B \subset \{1, \dots, n\}$ and the complement (free) column indices be $F = \{1, \dots, n\} \setminus B$. If $|B| = r$, then the basic variables are the components of x indexed by B and the free variables are those indexed by F.

Important remark: the pivot columns of U are not, in general, the leftmost columns of U. In other words, U_B (the matrix formed from the pivot columns) is usually not equal to the left block of U unless a suitable column reordering has been applied. To work with a block partitioned form of U one therefore needs to perform a column permutation (or equivalently select columns by index). We now explain this in detail.

Define a permutation matrix $Q \in \mathbb{R}^{n\times n}$ that reorders the standard basis so that the pivot column indices B are moved to the first r positions and the free indices F occupy the remaining $n-r$ positions. Equivalently,

$$UQ = \begin{bmatrix} U(:,B) & U(:,F) \end{bmatrix} =: \begin{pmatrix} U_B & U_F \end{pmatrix},$$

where $U(:,B)$ denotes the submatrix of U formed by the columns indexed by B. (If one prefers not to form an explicit Q, one may work directly with the column-selection operators $U_B = U(:,B)$ and $U_F = U(:,F)$, but introducing Q makes the block formulas below concise.)

Correspondingly permute the unknowns via

$$x' := Q^T x,$$

so that $x' = \begin{pmatrix} x'_B \\ x'_F \end{pmatrix}$ with $x'_B \in \mathbb{R}^r$ the basic components and $x'_F \in \mathbb{R}^{n-r}$ the free components in the reordered variable vector. Then

$$Ux = U(QQ^T)x = (UQ)x' = \begin{pmatrix} U_B & U_F \end{pmatrix} \begin{pmatrix} x'_B \\ x'_F \end{pmatrix}.$$

As usual, restrict attention to the pivot rows (the rows that contain the leading entries of the selected pivot columns). After discarding any all-zero trailing rows of U we obtain an $r \times r$ upper triangular invertible submatrix formed by the pivot rows and pivot columns; this is the invertible U_B referred to above (viewed as $r \times r$ once restricted to pivot rows) and U_F is the corresponding $r \times (n-r)$ block of free columns restricted to the pivot rows. With these conventions the equation $Ux = y$ becomes, in the permuted variables,

$$U_B x'_B + U_F x'_F = y_{\text{pivot}},$$

where y_{pivot} denotes the entries of y corresponding to the pivot rows (equivalently, y_{pivot} is the leading r entries of y after discarding any zero trailing rows of U). Solving for the basic variables gives

$$x'_B = U_B^{-1}\big(y_{\text{pivot}} - U_F x'_F\big).$$

Hence the general solution in the permuted coordinates is

$$x' = \begin{pmatrix} x'_B \\ x'_F \end{pmatrix} = \begin{pmatrix} U_B^{-1} y_{\text{pivot}} \\ 0 \end{pmatrix} + \begin{pmatrix} -U_B^{-1} U_F \\ I_{n-r} \end{pmatrix} x'_F, \qquad x'_F \in \mathbb{R}^{n-r} \text{ arbitrary.}$$

Returning to the original variable ordering $x = Qx'$, the particular part and the null-space part of the solution in the original coordinates are obtained by applying Q to the two blocks above. Thus the null-space mapping N (which maps free parameters to full solutions in the original ordering) and the particular vector c are

$$N \;=\; Q \begin{pmatrix} -U_B^{-1} U_F \\ I_{n-r} \end{pmatrix} \in \mathbb{R}^{n\times(n-r)}, \qquad c \;=\; Q \begin{pmatrix} U_B^{-1} y_{\text{pivot}} \\ 0 \end{pmatrix}.$$

Finally, using invertibility of L we express y in terms of b:

$$y = L^{-1} P b,$$

and y_{pivot} is the restriction of this vector to the pivot rows. Substituting gives the particular vector for the original system $Ax = b$:

$$c \;=\; Q \begin{pmatrix} U_B^{-1} L^{-1} P b_{\text{pivot}} \\ 0 \end{pmatrix} = Q \begin{pmatrix} U_B^{-1} L^{-1} P b \\ 0 \end{pmatrix},$$

where the latter equality indicates that U_B^{-1} is applied to the entries of $L^{-1}Pb$ corresponding to the pivot rows (zero entries in trailing rows of U play no role).

Therefore the full solution of $Ax = b$ in the original variable ordering is

$$x = N x_F + c,$$

with

$$N = Q \begin{pmatrix} -U_B^{-1} U_F \\ I_{n-r} \end{pmatrix}, \qquad c = Q \begin{pmatrix} U_B^{-1} L^{-1} P b \\ 0 \end{pmatrix},$$

and $x_F \in \mathbb{R}^{n-r}$ arbitrary. Note that N depends only on the LU factor (and the pivot pattern encoded by Q), not on b.

In practice one may implement these steps either by (i) forming the column-permutation Q that brings pivot columns to the front and working with the left block U_B, or (ii) by working directly with indexed column selections $U(:, B)$ and $U(:, F)$ and mapping the components of x between the original and reduced orderings as required. The algebra above makes explicit how the required column operations (reordering or selection) enter into the construction of U_B, U_F, N and c.