Indian Institute of Science Education & Research Kolkata

MA1101: Mathematics I

Final Examination

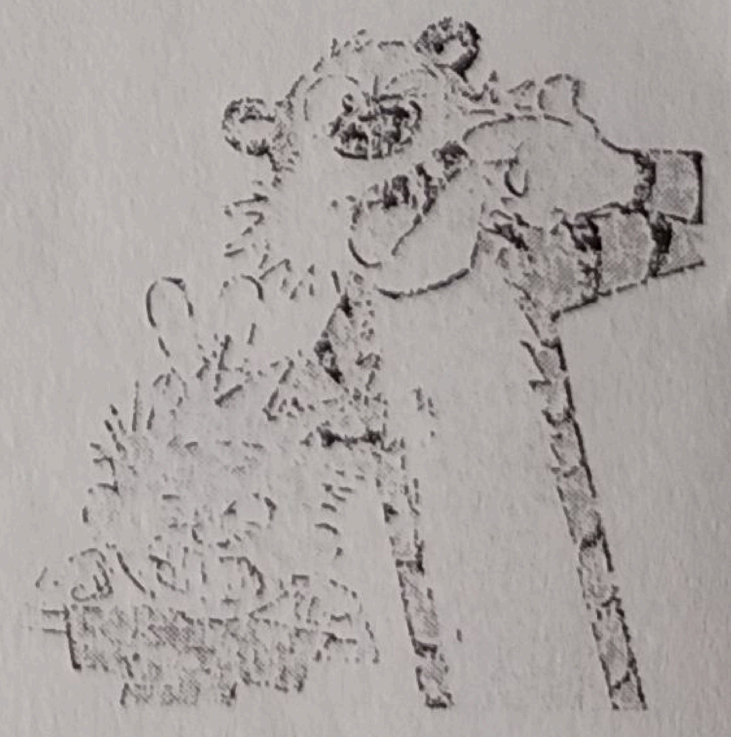
INSTRUCTIONS

This is a **closed-book** exam. 9th December 2024

You have 150 minutes.

- The examination is scored out of 50 points.
- Answers without justification will receive NO credit.
- If you are using a result/theorem proved in class, state it clearly before using it.

Good luck!



MA1101

(2024) Final Examination Samarjit Das

Problem 1 [6+6 points] (i) Let $S := \{f \mid f : \mathbb{R} \to \mathbb{R}\}$ be the set consisting of all real valued functions defined on $\mathbb{R}$. Define a relation $\sim$ on S as follows: declare $f \sim g$ if $f(0) = g(0)$. Show that $\sim$ is an equivalence relation.

(ii) Let $S/\sim$ denote the set of equivalence classes. Define a map

$$\Phi : S/\sim \longrightarrow \mathbb{R}, \ [f] \mapsto f(0).$$

Prove that Φ is a function and a bijection.

Problem 2 [9 points] Using the principle of mathematical induction, prove that if $n \in \mathbb{N}$, then

$$\frac{1}{n+1}\binom{2n}{n} < 4^n.$$

Problem 3 [5+5 points] (i) Let $f : \mathbb{R} \to \mathbb{R}, f(x) = [x]$ be the greatest integer function, i.e., $[x]$ is the greatest integer such that $[x] \leq x$. Show that f is continuous at $x_0 = \frac{1}{2}$.

(ii) Show that $L = 2$ is not a limit of the function $f(x) = x^2$ at $x_0 = 1$.

Note: *If you are using any of the following facts (if a limit exists, then it is unique OR the identity function $h(x) = x$ is continuous OR polynomials are continuous OR constant functions are continuous), then you must prove it before using it.*

Problem 4 [3+6 points] (i) Let $-\infty < a < b < \infty$ and let $f : (a, b) \to \mathbb{R}$ be a function. Define what we mean by "the function f is differentiable in (a, b)."

(ii) Consider the function $f : (-1, 1) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

Verify if f is differentiable in $(-1, 1)$

Problem 5 [6+[illegible] points] (i) Let $p : \mathbb{R} \to \mathbb{R}$ be the quartic defined by

$$p(x) = x^4 - 2x^3 + x^2 + 2024.$$

Compute, with justification, all the local maxima and minima of p.

Note: *You need not use the $\varepsilon - \delta$ definition to compute p' etc. You may use the standard/usual way to differentiate polynomials. For instance you are allowed to use that the derivative of the function $g(x) = x^n$ is the function $g'(x) = nx^{n-1}$*

(ii) With respect to your findings in (i), indicate, with justification, which ones are global maxima/minima and which ones are local.