IISER Kolkata

End-sem Exam

MA 1201 : Mathematics II

May 05, 2025

2:00 - 4:30 PM

INSTRUCTIONS

- The maximum marks you can score is **50**.
- Start each problem on a new page.
- Writing on the question paper is strictly prohibited.
- Answers without justification will receive NO credit.
- You may use any result/theorem proved in class (unless explicitly asked to prove it), BUT you must clearly state it while applying it.

Problem 1.

(a) Suppose that A is an uncountable set and B is countable subset of A. Show that A and $A \setminus B$ has same cardinality (i.e., A is equipotent with $A \setminus B$).

(b) Prove or disprove : The plane $\mathbb{R}^2$ is not a union of countable number of lines.

[5 + 4 = 9]

Problem 2.

(a) Assume that the complete solution to $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is given by $x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + a \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, where $a, b \in \mathbb{R}$. Find the matrix A.

(b) Prove that if V and W are three-dimensional subspaces of $\mathbb{R}^5$, then V and W must have a nonzero vector in common.

[5 + 4 = 9]

Problem 3.

(a) Use row-reduced echelon form to find the rank of the matrix associated with the following linear system of equations, and determine all x, y, z and w such that

$$\begin{bmatrix} x & 3 \\ y & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x,\, y,\, z) = (x,\, y + z).$$

Consider the bases $\mathcal{B} = \{v_1,\, v_2,\, v_3\}$ of $\mathbb{R}^3$ and $\mathcal{B}' = \{w_1,\, w_2\}$ of $\mathbb{R}^2$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad w_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ w_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

Find the matrix representation $[T]_{\mathcal{B}'}^{\mathcal{B}}$.

[4 + 5 = 9]

[Please turn over...]

Problem 4.

(a) Consider the matrix A with real entries :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 0 & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix}.$$

Prove that $\det(A) = 0$ always or provide an example of such a matrix A with $\det(A) \neq 0$.

(b) Diagonalize $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, that is, find a matrix P such that $P^{-1}AP = D$ is a diagonal matrix. Use it to prove that $A^k = \frac{1}{2}\begin{bmatrix} 3^k+1 & 3^k-1 \\ 3^k-1 & 3^k+1 \end{bmatrix}$ for any $k \in \mathbb{N}$.

[3 + 6 = 9]

Problem 5.

Consider the 2nd order linear ODE :

$$y'' + p(x)y' + q(x)y = 0, \quad x \in I, \tag{1}$$

where $I \subset \mathbb{R}$ is an interval and $p(x), q(x)$ are continuous functions on I.

(a) Can x^3 be a solution of (1) on $I = [-1, 1]$? Justify your answer.

(b) Let f and g be two linearly independent solutions of (1). Prove that any solution of the equation is a linear span of f and g.

[3 + 5 = 8]

Problem 6.

(a) Without solving, find an interval in which the solution is guaranteed to exist uniquely of the IVP :

$$ty'' - y' = t^2 + t, \quad y(1) = 1,\ y'(1) = 5.$$

Solve the ODE explicitly and find the largest interval of validity of the solution.

(b) Use the method of variation of parameters to find the general solution of the ODE :

$$y'' + 4y = 2\cos^2 x + 10e^x.$$

[5 + 5 = 10]